THE REDUCED KNOT FLOER COMPLEX

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ABSTRACT

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We define a “reduced” version of the knot Floer complex $CFK^{-}(K)$, and show that it behaves well under connected sums and retains enough information to compute Heegaard Floer d-invariants of manifolds arising as surgeries on the knot K. As an application to connected sums, we prove that if a knot in the three-sphere admits an L-space surgery, it must be a prime knot. As an application to the computation of d-invariants, we show that the Alexander polynomial is a concordance invariant within the class of L-space knots, and show the four-genus bound given by the d-invariant of $+1$-surgery is independent of the genus bounds given by the Ozsváth-Szabó τ invariant, the knot signature and the Rasmussen s invariant.

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Introduction

In the 1980's, Donaldson applied gauge theory to smooth four-dimensional topology, with remarkable results. Combining his results with those previously known in the topological case [16], Donaldson showed in particular that many simply connected, closed, topological four-manifolds admit no smooth structures [8]. In short, understanding the gap between topological and smooth four-manifolds required genuinely different techniques than those which had been useful in higher dimensions. By counting solutions to a PDE, up to gauge equivalence, Donaldson introduced invariants which were sensitive enough to distinguish different smooth structures on a topological four-manifold [9].

In [12], Floer defined the "instanton homology" of an integral homology three-sphere, and showed how in certain cases, the Donaldson invariants of a compact four-manifold could be computed from the instanton homology of a separating three-manifold. While counting instantons is still difficult in general, Floer further showed that the instanton homology groups of Y, Y' and Y'' fit into a "surgery exact triangle" [14], where Y' and Y'' are the manifolds obtained by doing $0-$ and $1-$framed surgery, respectively, on a knot in Y (this required enlarging the class of "admissable" three-manifolds on which instanton homology could be defined). This allows some computations to be determined algebraically, rather than analytically.

Still, one of the primary difficulties in computing Donaldson invariants in general is the fact that the moduli space of solutions is non-compact. A further revolution in topology using gauge theory came with the introduction of the Seiberg-Witten equations [69]. While it is conjectured that the smooth four-manifold invariants defined using the Seiberg-Witten equations are equivalent to the Donaldson invariants (for an appropriate class of four-manifolds; see [11] for partial results), the moduli space of solutions was shown to be compact, making the computation of invariants considerably simpler. It was tempting to then try to emulate Floer's approach, and define a homology theory of three-manifolds which could be used to compute the Seiberg-Witten invariants of smooth four-manifolds. This was the underlying motivation for the development of Heegaard Floer homology, which provides the context for the rest of this thesis. [1]

In [54], Ozsváth and Szabó define a collection of invariants of closed, oriented three-manifolds, known collectively as Heegaard Floer homology, which come in the form of various graded abelian groups. While one of the goals was to compute four-manifold invariants from invariants of separating three-manifolds, these three-manifold invariants themselves are defined by first considering a separating two-manifold. That is, the basic input is a Heegaard diagram, corresponding to a Heegaard splitting of a three-manifold [24]. In contrast to counting instantons, which involves solving PDE, enumerating the generators of the Heegaard Floer chain complexes becomes a simple combinatorial procedure. Defining the boundary map on this complex, however, still requires solving PDE, in the form of finding pseudo-holomorphic disks satisfying certain boundary conditions. Nevertheless, versions of Heegaard Floer homology have been shown in many cases to be algorithmically computable from just the combinatorics of the Heegaard diagram [36, 39, 40, 41, 66].

As in instanton homology, the Heegaard Floer groups satisfy surgery exact triangles,

[1] This was also the motivation for the development of Seiberg-Witten Floer homology, which is explained in detail in [30]. It has been shown in a series of papers [31] that the groups associated to a spin^c three-manifold by these two theories are isomorphic.

providing algebraic means of computation in some cases. In contrast, the invariants are defined for a manifold resulting from surgery with *any* rational slope. To further simplify computations, the theory can be extended to define an invariant of a knot [52, 64], and this knot invariant can be used to compute the three-manifold invariants of surgery along the knot, as explained in Section 1.3. The theory also extends to give invariants of links, four-manifolds and three-manifolds with boundary [58, 57, 28, 34, 35]. A partial summary of results in knot theory and three- and four-dimensional topology which have been proven using Heegaard Floer homology is given in Sections 1.1 and 1.2.

The aim of this thesis is to provide an efficient use of some of the algebraic structure provided by the theory. We will develop a reduced form of Heegaard Floer homology applied to knots, called the *reduced knot Floer complex*, whose primary objective is to simplify computations involving connected sums of knots. We will then show how it can be used to aid explicit computations of invariants, or to prove general results.

1.1 Heegaard Floer homology

Let Σ be a closed, connected oriented surface of genus g. Let $\vec{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ be a collection of g mutually disjoint simple closed curves on Σ, which are linearly independent when viewed as elements of $H_1(\Sigma)$, and $\vec{\beta} = \{\beta_1, \dots, \beta_g\}$ a collection with the same properties. This specifies a closed, connected oriented three-manifold in the following way.

Take $\Sigma \times [0,1]$, and attach a copy of $D^2 \times [0,1]$ along each α_i by identifying $\partial(D^2) \times [0,1]$ with a neighborhood of $\alpha_i \times \{0\}$. Similarly, attach a copy of $D^2 \times [0,1]$ by idenifying $\partial(D^2) \times [0,1]$ with a neighborhood of $\beta_i \times \{1\}$. Because of the linear independence of each family of curves, each boundary component of the resulting manifold is homeomorphic to

S^2, and up to diffeomorphism there is a unique way to attach two three-balls to get a closed manifold Y.

If z is any point in $\Sigma \setminus \left(\vec{\alpha} \cup \vec{\beta}\right)$, and $\vec{\alpha}$ and $\vec{\beta}$ intersect transversely, then the collection $\left(\Sigma, \vec{\alpha}, \vec{\beta}, z\right)$ is called a *pointed Heegaard diagram* for Y. In [54], given a pointed Heegaard diagram for Y, Ozsváth and Szabó consider the symmetric product $\mathrm{Sym}^g(\Sigma)$, which is the quotient of the product $\Sigma^{\times g}$ by the obvious action of the symmetric group S_g. When endowed with a symplectic form (which agrees with the product symplectic form away from the diagonal; see [60]), this manifold contains two Lagrangian submanifolds,

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g.$$

Heegaard Floer homology is then a filtered version of the Lagrangian Floer homology [13] of $\left(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta\right)$.

More precisely, assuming $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are transverse, their intersection is a finite set, written as $\mathfrak{G}$. An element in $\mathfrak{G}$ can be thought of as a g-tuple of intersection points on Σ, using each α and β curve exactly once. Ozsváth and Szabó define a chain complex freely generated over $\mathbb{F}[U, U^{-1}]$ by the elements of $\mathfrak{G}$, written as $CF^\infty(Y)$ (the original definition was over $\mathbb{Z}[U, U^{-1}]$, but in this thesis we will always take coefficients in $\mathbb{F}$, the field with two elements). The differential counts pseudo-holomorphic disks in $\mathrm{Sym}^g(\Sigma)$ of Maslov index one connecting two intersection points[2].

The basepoint z plays two roles in this construction. First, it partitions $\mathfrak{G}$ into subsets

[2]See [33] for a reformulation of the theory which involves counting disks in $\Sigma \times [0,1] \times \mathbb{R}$ rather than $\mathrm{Sym}^g(\Sigma)$.

corresponding with spin^c structures on Y, so that we get a splitting

$$CF^\infty(Y) = \bigoplus_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} CF^\infty(Y,\mathfrak{t}).$$

Second, it distinguishes a subvariety $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$, and the intersection number of a pseudo-holomorphic disk with this subvariety is what determines the exponent of the variable U in the differential. Since transverse holomorphic submanifolds of complimentary dimension can only intersect positively, the exponent of U can never be decreased by the differential of a homogeneous element. That is to say, the basepoint provides a *filtration* of CF^∞, by the exponent of the variable U. This extra structure provided by this filtration makes Heegaard Floer homology a powerful three-manifold invariant.

As a result, complexes CF^+, CF^- and $\widehat{CF}$ can be defined by taking coefficients in $\dfrac{\mathbb{F}[U,U^{-1}]}{U\mathbb{F}[U]}$, $\mathbb{F}[U]$ and $\dfrac{\mathbb{F}[U]}{U}$, respectively (alternatively, these are quotient-, sub- and subquotient complexes of CF^∞, respectively). We will denote this collection of chain complexes as CF° for brevity. The construction of these complexes depends on several choices, but it is shown in [54, Theorem 1.1] that their filtered chain homotopy types – and in particular, the homology groups, which are denoted HF^∞, HF^+, HF^- and $\widehat{HF}$ – are topological invariants of $(Y,\mathfrak{t})$.

In [51], it was shown that Heegaard Floer homology detects the Thurston norm, and in particular the Seifert genus of a knot. There is also an invariant of a contact structure on a three-manifold [55] which vanishes on overtwisted contact structures.

1.2 Heegaard Floer homology for knots

Suppose $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ is a pointed Heegaard diagram for S^3. Placing a second basepoint w on Σ specifies a knot $K \subset S^3$ in the following way: let K_α be an oriented embedded arc in $\Sigma \setminus \vec{\alpha}$ from z to w and K_β be an oriented embedded arc in $\Sigma \setminus \vec{\beta}$ from w to z. After pushing the interior of K_α into U_α and the interior of K_β into U_β, $K := K_\alpha \cup K_\beta$ is a knot in S^3. It was observed independently by Ozsváth-Szabó [52] and Rasmussen [64] that this additional basepoint can be used to put a second $\mathbb{Z}$-filtration on $CF^\infty(S^3)$, and that the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of this complex, denoted $CFK^\infty(K)$, is an invariant of $K \subset S^3$. Similarly, changing the coefficient module as in Section 1.1, we get filtered complexes CFK^+, CFK^- and $\widehat{CFK}$. Again, we may refer to this collection as CFK° when we do not wish to specify.

In particular, the filtration on $\widehat{CFK}(K)$ gives a sequence of subcomplexes

$$\cdots \subseteq \widehat{\mathcal{F}}(K, i-1) \subseteq \widehat{\mathcal{F}}(K, i) \subseteq \widehat{\mathcal{F}}(K, i+1) \subseteq \cdots ,$$

with associated graded complexes

$$\widehat{CFK}(K, i) := \frac{\widehat{\mathcal{F}}(K, i)}{\widehat{\mathcal{F}}(K, i-1)}.$$

The homology groups of the associated graded object,

$$\widehat{HFK}(K, i) := H_* \left(\widehat{CFK}(K, i) \right)$$

are called the *knot Floer homology* groups of K. These groups "categorify" the symmetrized

Alexander polynomial of K, in the sense that

$$\sum_i \chi\left(\widehat{HFK}(K,i)\right) \cdot T^i = \Delta_K(T). \tag{1.1}$$

Strictly stronger information can be obtained from knot Floer homology than from its Euler characteristic, however. For example, the lower bound on the Seifert genus

$$\deg \Delta_K(T) \leq g(K)$$

is strengthened to the result

$$g(K) = \max\{i | \widehat{HFK}(K,i) \neq 0\}. \tag{1.2}$$

Similarly, the fact that a fibered knot must have monic Alexander polynomial is strengthened to the statement [45]

$$K \text{ is fibered if and only if } \widehat{HFK}(K, g(K)) \cong \mathbb{F}. \tag{1.3}$$

Ignoring the knot filtration on $CFK^-(K)$ gives the Heegaard Floer complex $CF^-(S^3)$. Our method in this thesis will be to ignore the other filtration, the one which measures the exponent of the variable U, and obtain a simplified version of the knot Floer complex, called the *reduced knot Floer complex*, which we will denote $\underline{CFK}^-(K)$. This will be $\mathbb{Z}$-filtered chain homotopy equivalent to $CFK^-(K)$, and we will further require it to keep track of "multiplication by U", in a sense to be made precise in Chapter 3.

In [52, Theorem 7.1], it was shown that there is a tensor product formula for the knot

Floer complexes of connected sums,

$$CFK^{\infty}(K_1\#K_2) \cong CFK^{\infty}(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^{\infty}(K_2).$$

A corresponding statement holds for other versions of CFK, tensoring over the appropriate ring. These tensor product complexes become difficult to work with, however, even for simple sums. The essence of the following theorem is that, while the reduced complex is smaller, is still has a simple tensor product formula under connected sums.

Theorem 1.2.1. *If K_1 and K_2 are knots in S^3, then*

$$\underline{CFK}^{-}(K_1) \otimes_{\mathbb{F}[U]} CFK^{-}(K_2)$$

is a $(\mathbb{Z}, U)$-filtered chain deformation retract of $CFK^{-}(K_1\#K_2)$.

Much of Chapter 2 is devoted to making precise the notion of equivalence mentioned in Theorem 1.2.1. If we wish to connect sum a third knot, we can now reduce $CFK^{-}(K_1\#K_2)$ and iterate Theorem 1.2.1. As a result, this object can greatly simplify computations for sums of knots, and, as we will show, still retains enough information for these computations to be useful.

1.3 Dehn surgeries on knots

In addition to providing knot invariants, a crucial property of knot Floer complexes is their relation to the Heegaard Floer complexes of the three-manifolds obtained by doing Dehn surgery along K. It was observed in [52, 64] – roughly, due to the similarity between the

Heegaard diagrams for a knot and for manifolds obtained by integer surgery on the knot – that the Heegaard Floer homology groups of sufficiently large surgeries are the homology of certain subsets of $CFK^\infty(K)$. This idea was used in [63] to explicitly compute HF^+ for maniolfds obtained by large surgery on two-bridge knots, and in [52] for manifolds obtained by large surgery on a knot resulting from blowing down a component of a two-bridge link.

On the other hand, it had been shown in [53, Section 9] that, as expected in analogy with instanton Floer homology, the Heegaard Floer homology of manifolds differing by surgeries on the same knot fit into various surgery exact triangles. By using these exact sequences, in [59, 48], it was shown that the Heegaard Floer homology of any integer or rational Dehn surgery is the homology of the mapping cone of a map between two subsets of $CFK^\infty(K)$.

As a result, there is a deep connection between $CFK^\circ(K)$ and Dehn surgeries along K. This connection was used in [56], for example, to place strong restrictions on which knots in S^3 admit lens space surgeries. Because a lens space admits a genus one Heegaard diagram in which each intersection point represents a unique spin^c structure, it follows that lens spaces have the smallest possible Heegaard Floer homology; namely, if $Y = L(p, q)$,

$$\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{F} \quad \text{for all } \mathfrak{s} \in \mathrm{Spin}^c(Y). \tag{1.4}$$

Ozsváth and Szabó define an *L-space* as any rational homology sphere satisfying (1.4), and so their restrictions are more generally on which knots in S^3 admit L-space surgeries.

Roughly speaking, the only way to get such a simple result from the surgery formula is to start with a knot which has a simple knot Floer complex. Ozsváth and Szabó define an *L-space knot* as any knot which has a positive integral surgery which results in an L-space, and show that a knot $K \subset S^3$ is an L-space knot if and only if $CFK^\circ(K)$ has a specific form, described in Proposition (4.1.1). In particular, in light of (1.1), there are strong restrictions on the Alexander polynomial of such a knot.

Examples of L-space knots include positive torus knots (or any knot with a positive lens space surgery) and, for $n > 0$, the $P(-2, 3, 2n+1)$ pretzel knots [56], and more generally, a family of twisted torus knots [68]. By combining work of Hedden and Hom [21, 26], the (p, q)-cable of a knot K is an L-space knot if and only if K is an L-space knot and

$$\frac{q}{p} \geq 2g(K) - 1,$$

where g is the Seifert genus. In [68], Vafaee asks if there are any other satellite operations which can produce L-space knots. In this thesis, we give a negative answer for the simplest satellite operation, connected sums. After describing the reduced complexes of L-space knots, we use Theorem 1.2.1 to prove

Theorem 1.3.1. *A knot in S^3 which admits an L-space surgery must be a prime knot.*

We should remark here that it is easy to see that the sum of two non-trivial *L-space knots* is not an L-space knot; for example, by observing that the characterization of knot Floer complexes of L-space knots given in [56, Theorem 1.2] is not preserved under tensor products. However, our reduced complex will make this statement just as apparent for the sum of *any* non-trivial knots.

1.4 Correction terms

Suppose W is a cobordism from one three-manifold Y_1 to another Y_2, and $\mathfrak{s}$ a spin^c structure on W which restricts to $\mathfrak{t}_i$ on Y_i. In [57], given a handle decomposition of W, Ozsváth and

Szabó define chain maps from $CF^\circ(Y_1, \mathfrak{t}_1)$ to $CF^\circ(Y_2, \mathfrak{t}_2)$, which induce maps

$$F^\circ_{W,s} : HF^\circ(Y_1, \mathfrak{t}_1) \to HF^\circ(Y_2, \mathfrak{t}_2). \tag{1.5}$$

These maps are independent of handle decomposition, and of all choices involved in constructing $CF^\circ(Y_i)$. [3]

One result of this construction is a $\mathbb{Z}$-valued invariant for closed spin^c four-manifolds $(X, \mathfrak{s})$ with $b_2^+(X) > 1$, which is conjecturally equal to the Seiberg-Witten invariant. This is obtained by removing two four-balls from X and viewing the resulting manifold as a cobordism from S^3 to S^3, and using a "mixed" version of the cobordism invariant.

A second result, which is of greater interest in this thesis, is that if $\mathfrak{t}$ is a torsion spin^c structure on Y (meaning $c_1(\mathfrak{t})$ is a torsion cohomology class), the relative $\mathbb{Z}$-valued homological grading on $HF^\circ(Y, \mathfrak{t})$ can be lifted to an absolute $\mathbb{Q}$-valued grading. Roughly, starting with a surgery diagram for a three-manifold Y, Ozsváth and Szabó construct a Heegaard triple diagram which specifies a four-manifold bounded by S^3, $\#^n S^2 \times S^1$ and Y. They then define the absolute grading of a class in $\xi \in \widehat{HF}(Y, \mathfrak{t})$, denoted $\widetilde{\mathrm{gr}}(\xi)$, by considering a Whitney triangle in the diagram connecting ξ to the generators of highest grading in $\widehat{HF}(S^3)$ and $\widehat{HF}(\#^n S^2 \times S^1)$ [57, Equation 12]. They then show [57, Theorem 7.1] that, for a cobordism W from Y_1 to Y_2, this grading satisfies

$$\widetilde{\mathrm{gr}}\left(F^\circ_{W,\mathfrak{s}}(\xi)\right) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1^2(\mathfrak{s}) - 2\chi(W) - 3\sigma(W)}{4} \tag{1.6}$$

for any homogeneous class $\xi \in HF^\circ(Y_1, \mathfrak{t}_1)$, where $\mathfrak{t}_i$ is the restriction of $\mathfrak{s}$ to Y_i.

[3] While the groups HF° are originally only defined *up to isomorphism*, it is shown in [29] that they can be defined as explicit groups, and so the map (1.5) can be explicitly defined.

The remaining applications of the reduced complex presented in this thesis will pertain to the Heegaard Floer *correction terms*, or *d-invariants*. In [49, Definition 4.1], Ozsváth and Szabó use the absolute grading to define the d-invariant of a spin^c rational homology three-sphere $(Y, \mathfrak{t})$ as

$$d(Y, \mathfrak{t}) = \min\{\widetilde{\mathrm{gr}}(x) |\ x \in \mathrm{Im}\left(\pi_* : HF^\infty(Y, \mathfrak{t}) \to HF^+(Y, \mathfrak{t})\right)\}. \tag{1.7}$$

We will work with the equivalent definition

$$d(Y, \mathfrak{t}) = \max\{\widetilde{\mathrm{gr}}(x) |\ x \in HF^-(Y, \mathfrak{t}), x \text{ is not } U\text{-torsion.}\} \tag{1.8}$$

The condition of x being non-torsion is equivalent to saying that x is not in the kernel of the induced map $HF^-(Y, \mathfrak{t}) \to HF^\infty(Y, \mathfrak{t})$.

Remark 1.4.1. *Our convention which makes these definitions agree is slightly different than that of Ozsváth and Szabó – we assume both CF^+ and CF^- to contain the element 1 in $\mathbb{F}[U, U^{-1}]$. This will be convenient for computing correction terms, but has the drawback that CF^+ is not quite the quotient complex corresponding to the subcomplex CF^-.*

These invariants have been used to answer questions related to Dehn surgery [6, 7, 46], the smooth knot concordance group [4, 22, 27, 38] and various notions of genera of knots [1, 17, 47].

The property of "keeping track of multiplication by U" mentioned in Section 1.2 is essential, since it will allow us to compute d-invariants with the reduced complex.

1.5 Knot concordance and four-genus bounds

In [15], Fox and Milnor investigated knotted embeddings $\iota : S^2 \to \mathbb{R}^4$. Of particular interest here, they considered the intersection of $\iota(S^2)$ with a generic $\mathbb{R}^3$-hyperplane slice of $\mathbb{R}^4$. A generic intersection would be a closed embedded one-manifold in $\mathbb{R}^3$. In the case that this intersection is a knot K, then because K is a simple closed curve on a manifold homeomorphic to S^2, it bounds a disk, which is embedded in $\mathbb{R}^4$. Since S^2 is embedded as a submanifold, it has a trivial normal bundle, and so the disk is in fact locally flatly embedded. This leads to the definition of a *slice knot* as a knot in S^3 which bounds a locally flatly, properly embedded disk in B^4.

Of course, bounding a disk is equivalent to cobounding an annulus with the unknot. So, the notion of a slice knot being "four-dimensionally trivial" can be generalized to an equivalence relation on knots: K and J are called *concordant* if there is a locally flatly embedded annulus in $S^3 \times [0,1]$ which is bounded by $K \subset S^3 \times \{0\}$ and $J \subset S^3 \times \{1\}$. If we consider only smooth embeddings, we get notions of smoothly slice knots and smooth concordance, which are distinct from the topological, locally flat case (see, for example, [5, 18, 10, 22]). For the remainder of this thesis, unless explicitly stated otherwise, we will always be working in the smooth category.

For any knot $K \subset S^3$, consider the product annulus $K \times [0,1] \subset S^3 \times [0,1]$. If we let "$-K$" denote the mirror image of K with reversed orientation, then, for $x \in K$, removing a neighborhood of $x \times [0,1]$ from the annulus yields a disk bounded by $K\#(-K)$, so it is clear that $K\#(-K)$ is a slice knot. As a result, the set of knots up to concordance forms an abelian group under the connected sum operation, called the *knot concordance group*.

If a knot is not slice – that is, it does not bound a smooth disk in B^4 – it is natural to ask:

what is the minimal genus of a smoothly, properly embedded surface in B^4 which *is* bounded by K? This minimum is called the four-genus of K, denoted $g_4(K)$. It is evident that g_4 is an invariant of the concordance class $[K]$, and also that $g_4(K) \leq g(K)$. Further, $g_4(K)$ gives a lower bound on the unknotting number $u(K)$, the minimum number of crossings which can be changed to turn a diagram of K into a diagram of the unknot. Both g_4 and u are hard to compute in general; using their definitions directly requires considering every possible surface bounded by K, or every possible choice of crossing changes in every possible diagram of K. For this reason, it is useful to have computable invariants which provide lower bounds on g_4 (and therefore u). In particular, such invariants also provide obstructions to a knot being slice.

The tools of Heegaard Floer homology can be used to define and compute several such invariants; two of which will be considered in this thesis. If $K \subset S^3$ is a knot, consider the integral homology sphere $S^3_1(K)$, the manifold resulting from Dehn surgery along K with slope 1. This manifold has a unique spin^c structure $\mathfrak{t}$, so we can define an invariant

$$d_1(K) := d(S^3_1(K), \mathfrak{t}). \tag{1.9}$$

This invariant was studied by Peters in [61], where he showed that it is in fact a concordance invariant. This can be seen by surgering the annulus which provides the concordance between K and J; the resulting four-manifold is a homology cobordism from $S^3_1(K)$ to $S^3_1(J)$, and the boundary components of a homology cobordism have equal d-invariants (see, for example, [61, Corollary 1.3]).

Further, by computing d-invariants of circle bundles over surfaces with Euler number ± 1, Peters is able to show that we get a four-genus bound,

$$0 \le -d_1(K) \le 2g_4(K).$$

Using the surgery formula provided by Ozsváth and Szabó, Peters then gives an algorithm to compute $d_1(K)$ from the knot Floer complex $CFK^\infty(K)$.

There is another concordance invariant which can be computed – and is in fact defined – using the knot Floer complex; namely, the Ozsváth-Szabó τ invariant. This is defined in [50], where it is also shown that

$$|\tau(K)| \le g_4(K).$$

In comparing the computations of these two invariants, Peters poses the question:

Question 1.5.1. *What is the relation between d_1 and τ? Is it necessarily true that*

$$|d_1(K)| \le 2|\tau(K)|?$$

Of course, if the answer were "yes", then d_1 would be a rather ineffective four-genus bound. At this point, however we should observe that τ defines a group homomorphism from the concordance group to $\mathbb{Z}$, while d_1 does not (it is always non-positive). One could hope to exploit this by finding a sum of knots for which τ is forced to be zero, but d_1 is not. Indeed, this will be the strategy to providing negative answers to Question 1.5.1[4]. In some sense, the role which the reduced complex plays is that it simplifies computations

[4]The first knot in a family of knots described in Chapter 4 which provides the negative answer was actually alluded to by Peters later in his paper.

when dealing with sums of knots, by virtue of Theorem 1.2.1. This makes computations manageable when dealing with an invariant which is not additive under connected sum. Using the reduced complex, we first show

Theorem 1.5.2. *Suppose that K_1 and K_2 are two knots in S^3 which admit L-space surgeries. If*

$$d_1(K_1\# - K_2) = d_1(-K_1\#K_2) = 0,$$

then

$$\Delta_{K_1}(T) = \Delta_{K_2}(T).$$

In particular, the Alexander polynomial is a concordance invariant of L-space knots.

Following from this, we have the corollary

Corollary 1.5.3. *If K_1 and K_2 are two L-space knots whose Alexander polynomials are distinct but have the same degree, then*

$$\tau(K_1\# - K_2) = \tau(-K_1\#K_2) = 0,$$

but either

$$d_1(K_1\# - K_2) \neq 0 \quad \textit{or} \quad d_1(-K_1\#K_2) \neq 0.$$

In particular, d_1 gives a stronger four-genus bound than τ for $K_1\# - K_2$ and its mirror.

As another consequence, this yields an alternate proof of a result of Lê.

Corollary 1.5.4 (Lê)**.** *Algebraic knots are concordant if and only if they are isotopic.*

In addition to τ, two other concordance invariants which have proven to give useful four-genus bounds are the knot signature σ, defined from a bilinear form corresponding to a

Seifert surface [44], and the Rasmussen s invariant which comes from a spectral sequence on Khovanov homology [65]. To strengthen the result of Corollary 1.5.3, and show the effectiveness of d_1 as a smooth four-genus bound, we give examples of knots for which $\tau(K) = \sigma(K) = s(K) = 0$, but $|d_1(K)|$ is arbitrarily large. It should be noted that in general the ω-signature, of which σ is a special case [67], gives a topological four-genus bound for any unit complex number $\omega \neq 1$. For the examples mentioned above, which involve sums of torus knots, the collection of ω-signatures give a better bound than d_1.

The remainder of this thesis is organized as follows. In Chapter 2, we begin by introducing the algebraic framework which will be necessary. In Chapter 3, we review the definition and properties of the knot Floer complex, and define its reduced form. Section 3.2 explains how the tensor product formula extends to the reduced complex. In Chapter 4, we apply the theory to L-space knots, prove Theorems 1.3.1 and 1.5.2, and provide examples.

Chapter 2

Algebraic preliminaries

2.1 Filtered complexes

Throughout this thesis, we will be working with coefficients in the field with two elements, which we will denote $\mathbb{F}$. Given a chain complex (C, ∂), and a partially ordered set S, a *(decreasing) S-filtration* on C is an exhaustive collection of subcomplexes indexed by S,

$$C = \bigcup_{i \in S} C^i,$$

such that

$$C^i \subseteq C^j \quad \text{if} \quad i \leq j.$$

A filtration is said to be *bounded* if $C^i = C$ for some $i \in S$.

If (C, ∂) and (C', ∂') are S-filtered complexes, then a map $f : C \to C'$ is a *filtered map* if, for all $i \in S$,

$$f(C^i) \subseteq (C')^i.$$

We will say that C and C' are *filtered chain homotopy equivalent* if there exist filtered chain maps $f : C \to C'$ and $g : C' \to C$, and filtered chain homotopies $h : C \to C$ and $h' : C' \to C'$ such that

$$f \circ g = I_{C'} + \partial' h' + h' \partial' \quad \text{and} \quad g \circ f = I_C + \partial h + h \partial.$$

We will further say that C' is a *filtered chain deformation retract of* C if the chain homotopy h' is trivial; i.e., if

$$f \circ g = I_{C'}.$$

The complexes dealt with in this thesis will be filtered by $\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}$, with partial ordering

$$(i,j) \leq (i',j') \quad \text{iff} \quad i \leq i' \quad \text{and} \quad j \leq j'.$$

In the case that (C, ∂) is $\mathbb{Z}$-filtered, a convenient way to describe the filtration is as a function $F : C \to \mathbb{Z} \cup \{-\infty\}$, where

$$F(x) = \begin{cases} \min \left\{ i \in \mathbb{Z} \middle| x \in C^i \right\} & \text{if the minimum exists,} \\ -\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

with the convention that $-\infty \leq i$ for all $i \in \mathbb{Z}$. Note that this function satisfies

$$F(x+y) \leq \max\{F(x), F(y)\} \quad \text{and} \quad F(\partial x) \leq F(x)$$

for all $x, y \in C$. In every case which will be considered here, we will further have that $F^{-1}(-\infty) = \{0\}$. We will call $F(x)$ the *filtration level of* x, and we will abuse terminology and sometimes refer to F as the filtration.

A $\mathbb{Z} \oplus \mathbb{Z}$-filtration, by complexes $C^{i,j}$, gives rise to two $\mathbb{Z}$-filtrations, by considering

$$C^i := \bigcup_j C^{i,j} \quad \text{and} \quad C^j := \bigcup_i C^{i,j}.$$

On the other hand, if collections $\{C^i\}$ and $\{C^j\}$ are two $\mathbb{Z}$-filtrations of C, then the complexes

$$C^{i,j} := C^i \cap C^j$$

are a $\mathbb{Z} \oplus \mathbb{Z}$-filtration, so in fact the two notions – a $\mathbb{Z} \oplus \mathbb{Z}$-filtration and a pair of $\mathbb{Z}$-filtrations – are equivalent. Therefore, we can describe a $\mathbb{Z} \oplus \mathbb{Z}$-filtration as a function $F(x) = (F_1(x), F_2(x))$, where F_1 and F_2 are the functions describing the two corresponding $\mathbb{Z}$-filtrations.

We present here a prototypical example of what will follow. Figure 2.1 represents a $\mathbb{Z}$-filtered complex C generated over $\mathbb{F}$, where the vertical height of each generator corresponds to its filtration level. We will denote by $\partial(x, y)$ the coefficient of y in ∂x. If $\partial(x, y) = 1$, we draw an arrow from x to y. Intuitively, we can "cancel" an arrow which is horizontal, while preserving the filtered chain homotopy type of C. For example, canceling the arrow from b to c gives a complex C' in the following way. The generators are obtained by deleting the generators b and c, and the differential on C' is given by

$$\partial'(x, y) = \partial(x, y) + \partial(x, c)\partial(b, y).$$

In other words, if an arrow went from x to c, and another arrow went from b to y, we add an arrow going from x to y. To put it precisely, we make the filtered change of basis $c \mapsto \partial b$, then take the quotient of C by the acyclic subcomplex which is generated by b and ∂b. If we define a homomorphism $h : C \to C$ by setting $h(c) = b$ and $h(x) = 0$ for all other generators (i.e., h is the inverse of the horizontal arrow we are canceling), then C' is seen to be a filtered

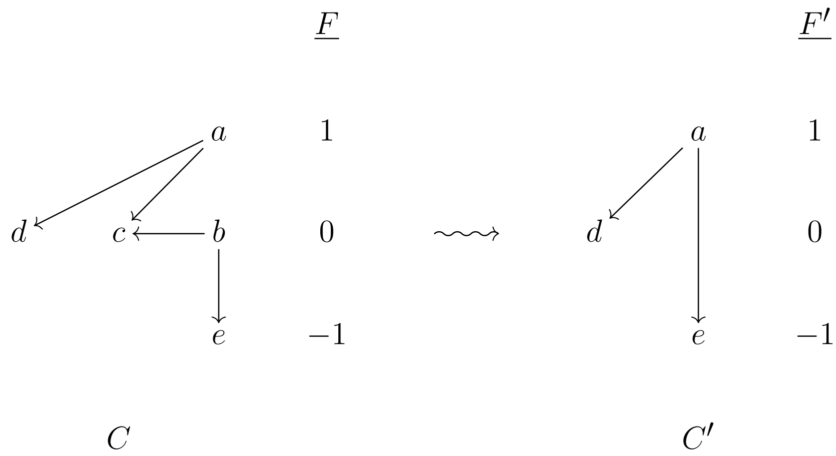


Figure 2.1: **A filtered chain deformation retraction.** Canceling the horizontal arrow from b to c yields the filtered chain deformation retract C'. The additional arrow from a to e is obtained by "traveling backward" through the canceled arrow. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this thesis.

chain deformation retract of C, via the maps

$$f = \pi \circ (I + \partial h), \quad g = (I + h\partial) \circ \iota$$

and the chain homotopy h. Details are explained well in [23, Lemma 4.1] and [64, Section 5.1], and will be worked out in Chapter 3.

A vertical arrow could similarly be canceled, but the map h would not be filtered, and so the result would be a chain homotopy equivalent, but not filtered chain homotopy equivalent, complex. The idea of reduction, put simply, is that as long as we have horizontal arrows (terms in the differential which preserve the filtration level), we can reduce the number of generators of a chain complex, while maintaining its filtered chain homotopy type.

2.2 $(\mathbb{Z}, U)$-filtered complexes

Suppose C is a $\mathbb{Z}$-filtered complex which comes equipped with a specified filtered chain map U. Then we will call the pair (C, U) a $(\mathbb{Z}, U)$-*filtered chain complex*. It will be convenient to view such complexes as modules over $\mathbb{F}[U]$, so that we will often refer to composition with the map U as "multiplication by U". Two $(\mathbb{Z}, U)$-filtered chain complexes (C, U) and (C', U') will be called $(\mathbb{Z}, U)$-*filtered chain homotopy equivalent* if they are filtered chain homotopy equivalent, and the maps f and g respect multiplication by U, in the sense that

$$gU' \sim Ug \quad \text{and} \quad fU \sim U'f,$$

where $\sim$ signifies that the maps are filtered chain homotopic.

Remark 2.2.1. *The notion of $(\mathbb{Z}, U)$-filtered chain homotopy equivalence is an equivalence relation.*

Finally, we will also want to consider complexes which arise as products of other complexes. If (C_1, F_1) and (C_2, F_2) are two $\mathbb{Z}$-filtered chain complexes which are freely generated over $\mathbb{F}$, we can define a filtration $F^\times$ on the direct product $C_1 \times C_2$ in the following way. Suppose that $\{x_i\}$ and $\{y_i\}$ are generating sets for C_1 and C_2, respectively, so that any element in $C_1 \times C_2$ can be written as $\sum_{i,j} \varepsilon_{ij} x_i y_j$, where $\varepsilon_{ij} \in \mathbb{F}$, and all but finitely many of the ϵ_{ij} are zero. Then,

$$F^\times\left(\sum_{i,j} \varepsilon_{ij} x_i y_j\right) = \max\{F_1(x_i) + F_2(y_j) | \varepsilon_{ij} = 1\}$$

defines a filtration on $C_1 \times C_2$.

If each C_i comes with a filtered chain map U_i, they can naturally be thought of as modules over $\mathbb{F}[U]$, so that we will refer to the maps U_i as "multiplication by U". In this case, we can also consider the complex

$$C_1 \otimes_{\mathbb{F}[U]} C_2.$$

An element in $C_1 \otimes_{\mathbb{F}[U]} C_2$ is an equivalence class of elements in $C_1 \times C_2$, and we define a filtration F on the tensor product by simply taking the minimum over each equivalence class. That is, for any $s \in C_1 \times C_2$,

$$F([s]) = \min \left\{ F^{\times}(t) \middle| t \in [s] \right\}. \tag{2.2}$$

Actually, for the tensor products we will consider in this thesis, we can describe the product filtration more concretely. In our case, we will consider a $(\mathbb{Z}, U)$-filtered complex C_1 for which U is not necessarily homogeneous, but always decreases the filtration level by at least 1; that is,

$$F_1(Ux) \leq F_1(x) - 1 \quad \text{for all } x \in C_1. \tag{2.3}$$

We will then consider a second $(\mathbb{Z}, U)$-filtered complex, C_2, on which U is homogeneous of degree 1, so

$$F_2(Ux) = F_2(x) - 1 \quad \text{for all } x \in C_2,$$

and further, C_2 is *free* when viewed as an $\mathbb{F}[U]$-module. In this case, for a homogeneous element $x \in C_1$ and a generator $y \in C_2$, the filtration on $C_1 \otimes_{\mathbb{F}[U]} C_2$ given by (2.2) is

$$F(x \otimes U^n y) = F_1(U^n x) + F_2(y), \tag{2.4}$$

for all $n \geq 0$. In other words, to avoid ambiguity, we can think of U as always being applied to the first component (the module on which U is not necessarily homogeneous). In this case, we can prove the following lemma, which we will use in Section 3.2.

Lemma 2.2.2. *Suppose that there is a $(\mathbb{Z}, U)$-filtered chain homotopy equivalence between C_1 and C_1', and C_2 is a $(\mathbb{Z}, U)$-filtered chain complex which is freely generated over $\mathbb{F}[U]$. Suppose also that the maps U on C_1 and C_1' always decrease the filtration level by at least k, and the map U on C_2 is homogeneous of degree k. Then there is a $(\mathbb{Z}, U)$-filtered chain homotopy equivalence between $C_1 \otimes_{\mathbb{F}[U]} C_2$ and $C_1' \otimes_{\mathbb{F}[U]} C_2$.*

PROOF. The idea is that, since C_2 is freely generated, we can define a map on $C_1 \otimes C_2$, for example, by extending a map defined on C_1. Further, because the map U on C_1 decreases the filtration level by at least as much as the map U on C_2, the extended map will still be filtered.

More precisely, let $f : C_1 \to C_1'$ and $g : C_1' \to C_1$ be the chain maps which give the equivalence, and let h and h' be the chain homotopies from $g \circ f$ to I_{C_1} and from $f \circ g$ to $I_{C_1'}$, respectively. We define chain maps $f : C_1 \otimes C_2 \to C_1' \otimes C_2$ and $g : C_1' \otimes C_2 \to C_1 \otimes C_2$ as follows. Suppose first that y is a generator of C_2 as an $\mathbb{F}[U]$-module, then we set

$$f(x \otimes U^n y) = f(U^n x) \otimes y \quad \text{and} \quad g(x \otimes U^n y) = g(U^n x) \otimes y,$$

and extend the maps bilinearly over $\mathbb{F}$. Similarly, for y a generator of C_2, we define maps $h : C_1 \otimes C_2 \to C_1 \otimes C_2$ and $h' : C_1' \otimes C_2 \to C_1' \otimes C_2$ by setting

$$h(x \otimes U^n y) = h(U^n x) \otimes y \quad \text{and} \quad h'(x \otimes U^n y) = h'(U^n x) \otimes y,$$

and extending bilinearly.

Now we have

$$
\begin{aligned}
g \circ f(x \otimes U^n y) &= g \circ f(U^n x) \otimes y \\
&= (I + \partial h + h\partial)(U^n x) \otimes y \\
&= U^n x \otimes y + \partial h(U^n x) \otimes y + h(\partial U^n x) \otimes y \\
&= U^n x \otimes y + \partial h(U^n x) \otimes y + h(\partial U^n x) \otimes y \\
&\qquad + [h(U^n x) \otimes \partial y + h(U^n x) \otimes \partial y] \\
&= U^n x \otimes y + (\partial h(U^n x) \otimes y + h(U^n x) \otimes \partial y) \\
&\qquad + (h(\partial U^n x) \otimes y + h(U^n x) \otimes \partial y) \\
&= (I + \partial h + h\partial)(x \otimes U^n y),
\end{aligned}
$$

and, since the maps are bilinear, we see that $g \circ f$ is chain homotopic to $I_{C_1 \otimes C_2}$ via the chain homotopy h. By a symmetric argument, $f \circ g \sim I_{C_1' \otimes C_2}$ via h'.

Note also that, for example,

$$
\begin{aligned}
F_{C_1' \otimes C_2}(f(x \otimes U^n y)) &= F_{C_1' \otimes C_2}(f(U^n x) \otimes y) \\
&= F_{C_1'}(f(U^n x)) + F_{C_2}(y) \\
&\leq F_{C_1}(U^n x) + F_{C_2}(y) \\
&= F_{C_1 \otimes C_2}(x \otimes U^n y).
\end{aligned}
$$

In other words, because the maps f, g, h and h' are filtered on C_1 and C_1', the maps f, g, h and h' are filtered on $C_1 \otimes C_2$ and $C_1' \otimes C_2$. So, the two tensor product complexes are $\mathbb{Z}$-filtered chain homotopy equivalent.

Finally, we verify that this equivalence is in fact $(\mathbb{Z}, U)$-filtered. By assumption, $fU \sim Uf$ as maps from C_1 to C_1', so there exists a map $\phi : C_1 \to C_1'$ such that

$$fU = Uf + \partial_{C_1'}\phi + \phi\partial_{C_1}.$$

In a similar fashion as before, we define

$$\phi : C_1 \otimes C_2 \to C_1' \otimes C_2$$

by setting

$$\phi(x \otimes U^n y) = \phi(U^n x) \otimes y$$

when y is a generator of C_2, and extending bilinearly over $\mathbb{F}$.

We then check that

$$\begin{aligned}
f(U(x \otimes U^n y)) &= f(U^{n+1}x) \otimes y \\
&= (Uf + \partial\phi + \phi\partial)(U^n x) \otimes y \\
&= (Uf + \partial\phi + \phi\partial)(U^n x) \otimes y \\
&\qquad + [\phi(U^n x) \otimes \partial y + \phi(U^n x) \otimes \partial y] \\
&= (Uf + \partial\phi + \phi\partial)(x \otimes U^n y).
\end{aligned}$$

That is, $fU \sim Uf$, and by the same reasoning, $Ug \sim gU$. □

This lemma will be the key to simplifying computations for connected sums of knots, for the complexes which we will define in Chapter 3.

Chapter 3

Reducing the knot Floer complex

3.1 The reduced complex

We now turn to the complexes of interest in this thesis. Given a pointed Heegaard diagram for a three-manifold Y (with k α-curves and k β-curves), Ozsváth and Szabó [54] define a $\mathbb{Z}$-filtered chain complex $CF^\infty(Y, \mathfrak{t})$ for each Spin^c structure $\mathfrak{t}$ on Y. The complex is freely generated over $\mathbb{F}[U, U^{-1}]$ by k-tuples of intersection points on the Heegaard diagram. We will denote the set of generators by $\mathfrak{G}$. The filtration level of the homogeneous element $U^n x$ is $-n$, for any $x \in \mathfrak{G}$ and any $n \in \mathbb{Z}$. Adding a second basepoint to the Heegaard diagram specifies a knot K in Y. Using this additional basepoint (and fixing a Seifert surface for K), each $x_i \in \mathfrak{G}$ can be assigned an integer $A(x_i)$, called the *Alexander grading*. If we let y_i denote a homogeneous element, then after setting

$$A(U^n x_i) = A(x_i) - n, \quad A\left(\sum_i y_i\right) = \max_i\{A(y_i)\}, \tag{3.1}$$

A defines an additional filtration on $CF^\infty(Y, \mathfrak{t})$, discovered in [52], and independently by Rasmussen [64]. The $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of this complex is an invariant of $K \subset Y$, and in the case $Y \cong S^3$, this invariant is denoted $CFK^\infty(K)$.

We can write the $\mathbb{Z} \oplus \mathbb{Z}$-filtration level of the homogeneous element $U^n x$ as

$$F(U^n x) = (-n, A(x) - n)$$

for any $x \in \mathfrak{G}$ and $n \in \mathbb{Z}$. It will be convenient to represent these complexes graphically in the (i, j)-plane, where $U^n x$ will be represented by a dot with coordinates $(-n, A(x) - n)$. If x and y are two homogeneous elements such that $\partial(x, y) = 1$, then we will draw an arrow from the dot representing x to the dot representing y. We should also point out here that $CFK^\infty(K)$ comes with a homological $\mathbb{Z}$-grading M, called the *Maslov grading*, and that multiplication by U decreases M by 2; i.e.,

$$M(Ux) = M(x) - 2. \tag{3.2}$$

The difference between the Maslov gradings of two generators can be read from the Heegaard diagram, and to fix an absolute Maslov grading, we declare that the element 1 in

$$H_*(CFK^\infty(K)) \cong HF^\infty(S^3) \cong \mathbb{F}[U, U^{-1}]$$

has Maslov grading zero.

Following convention, for a subset $S \subset \mathbb{Z} \oplus \mathbb{Z}$, we will denote by $C\{S\}$ the elements of C whose (i, j)-coordinates are contained in S, along with the arrows between these elements. We will often consider the subcomplex $C\{i \leq 0\}$, which is written as $CFK^-(K)$ (using the convention described in Remark 1.4.1). Figure 3.1 shows the complex $CFK^-(K)$ in the case where K is the right-handed trefoil.

Saying that A defines a filtration means in particular that $A(\partial x) \leq A(x)$ for any x, and

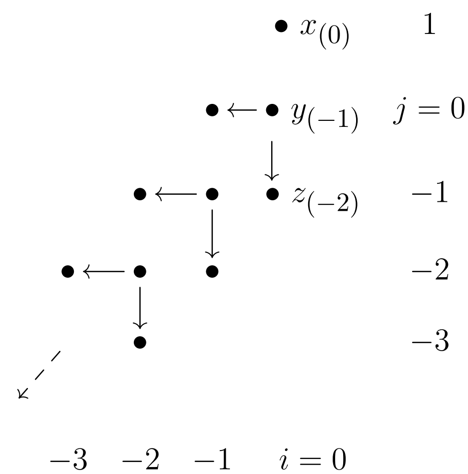


Figure 3.1: **The knot Floer complex $CFK^-(T(2,3))$.** The generators are represented by dots in the $i = 0$ column with their Maslov grading in parentheses, and multiplication by U translates an element one row down and one column to the left. The arrows represent the nonzero terms of the differential.

we will call the part of the differential which preserves the Alexander grading the *horizontal differential*, denoted ∂_H. Diagramatically, ∂_H simply consists of those arrows which are horizontal. For example, in Figure 3.1, we have $\partial y = Ux + z$, and $\partial_H y = Ux$. If we restrict our attention to a single Alexander grading – that is, a single row – of CFK^∞, we get a filtered chain complex which has homology isomorphic to $\mathbb{F}$. The "simplest" such filtered complex would be one which has generators paired into acyclic summands, and a single isolated generator of homology, which has no arrow going into or out of it. It was shown in [35, Proposition 11.52] that one can always find such a basis for $CFK^\infty(K)$, called a *horizontally simplified* basis. In other words, we can choose a generating set $\mathfrak{G} = \{x_i, y_i, z, |1 \leq i \leq N\}$, such that

- $\partial_H(y_i) = U^{r_i} x_i$, for some $r_i > 0$,
- $\partial_H(x_i) = \partial_H(z) = 0$.

With respect to this basis, the homology of each subquotient complex $C\{j = k\}$ (each row) is generated by the class $U^l z$ (where l differs from k by a constant). All other generators are paired by horizontal arrows, and can therefore be canceled as in Chapter 2.

We refer back to Figure 3.1 to make one more observation. The basis $\{x, y, z\}$ shown there is horizontally simplified, but because we are considering the subcomplex $CFK^-(K)$, rather than all of $CFK^\infty(K)$, it is not only $U^k z$ which generates homology: x is also homologically nontrivial (the horizontal arrow which would cancel it has been "cut off"). Starting with any nontrivial knot K, if we cancel horizontal arrows in $CFK^-(K)$, we are left with some elements which are eventually canceled for high enough powers of U, but not for lower powers of U.

We are now ready to construct the object of interest in this section, the reduced knot Floer complex. It will be convenient to think of $CFK^-(K)$ as being $(\mathbb{Z}, U)$-filtered, by the Alexander grading A, rather than $\mathbb{Z} \oplus \mathbb{Z}$-filtered. So, when we refer to the filtration level of an element, we will mean its j-coordinate in the diagram (although, for aesthetic reasons, we will maintain the appearance of different i-coordinates). In this case, the map U is a filtered chain map of degree 1; i.e., $A(Ux) = A(x) - 1$.

Let K be a knot in S^3. To simplify notation, let us define $C_0 := CFK^-(K)$. After choosing a horizontally simplified basis

$$\{x_i, y_i, z | 1 \leq i \leq N\},$$

we will reduce the complex by "canceling" all of the horizontal arrows, as in Chapter 2. More precisely, let h_1 be the $\mathbb{F}$-linear map on C_0 which inverts the horizontal differential going from y_1 to x_1, and all of its U-translates, so that $h_1(U^{r_1+n} x_1) = U^n y_1$ for all $n \geq 0$, and h_1 is zero on all other homogeneous elements.

We now define a filtered chain complex C_1 which is freely generated over $\mathbb{F}$ as follows. The generators for C_1 over $\mathbb{F}$ are obtained from the homogeneous elements of C_0 by removing $U^{r_1+n}x_1$ and $U^n y_1$ for all $n \geq 0$. Define maps $f_1 : C_0 \to C_1$ and $g_1 : C_1 \to C_0$ by

$$f_1 = \pi \circ (I + \partial h_1), \quad g_1 = (I + h_1 \partial) \circ \iota,$$

where π and ι are the natural projection and inclusion maps. Further, we define a differential and U map on C_1:

$$\partial_1 := \pi \circ (\partial + \partial h_1 \partial) \circ \iota, \qquad U_1 := \pi \circ (U + \partial h_1 U) \circ \iota. \tag{3.3}$$

The filtration A_1 on C_1 is induced by inclusion, $A_1(x) := A(\iota x)$. Since the maps ∂, U and h_1 are all filtered, so are the maps f_1, g_1, ∂_1, and U_1 defined above. Let us consider the map U_1 in more detail. Recall that on the complex C_0, U is a homogeneous map of degree 1. The map U_1, however, will not be homogeneous. There are two cases to consider. For the generator $U^{r_1-1}x_1$ (the highest remaining U-power which was not canceled by a horizontal arrow),

$$\begin{aligned} U_1(U^{r_1-1}x_1) &= \left(\pi \circ (U + \partial h_1 U) \circ \iota\right)(U^{r_1-1}x_1) \\ &= \pi \circ \left(U^{r_1}x_1 + \partial h_1(U^{r_1}x_1)\right) \\ &= \pi \circ (U^{r_1}x_1 + \partial y_1). \end{aligned} \tag{3.4}$$

Since we chose a basis which is horizontally simplified, $U^{r_1}x_1$ is the only term in ∂y_1 which has filtration level equal to that of y_1. That is,

$$A(U^{r_1}x_1 + \partial y_1) < A(y_1).$$

It follows from (3.4) then, that

$$A_1\big(U_1(U^{r_1-1}x_1)\big) < A(y_1) = A_1(U^{r_1-1}x_1) - 1, \tag{3.5}$$

so the map U_1 decreases the filtration level by more than 1. On all other generators of C_1, however, the map U_1 is equal to $\pi \circ U \circ \iota$, and therefore still decreases the filtration level by exactly 1. Therefore, the complex C_1 is of the type mentioned in Equation (2.3); it is $(\mathbb{Z}, U)$-filtered, and U is a filtered map which decreases the filtration level by *at least* one.

Remark 3.1.1. *By its definition, the map h_1 increases the Maslov grading by 1, and as a result, the map U_1 still lowers the Maslov grading by exactly 2.*

Lemma 3.1.2. *C_1 is a $(\mathbb{Z}, U)$-filtered chain deformation retract of C_0.*

Proof. We first verify the chain homotopy equivalence. For any homogeneous element x for which $\pi(x) \neq 0$,

$$\begin{aligned}
f_1 \circ g_1(\pi(x)) &= \pi \circ (I + \partial h_1)(I + h_1\partial) \circ \iota(\pi(x)) \\
&= \pi \circ (I + \partial h_1 + h_1\partial)(x) \\
&= \pi \circ (x + \partial(0) + h_1(\partial x)) \\
&= \pi(x),
\end{aligned}$$

since the image of h_1 projects to zero. So, $f_1 \circ g_1 = I_{C_1}$.

Next, we consider the composition $g_1 \circ f_1$, in three distinct cases. First, if x is any homogeneous element for which $\pi(x) \neq 0$, then $h_1(x) = 0$, so

$$\begin{aligned} g_1 \circ f_1(x) &= ((I + h_1\partial) \circ \iota) \circ (\pi \circ (I + \partial h_1))(x) \\ &= (I + h_1\partial)(x) \\ &= (I + h_1\partial + \partial h_1)(x). \end{aligned}$$

Second, for $n \geq 0$, we have

$$\begin{aligned} g_1 \circ f_1(U^{r_1+n}x_1) &= ((I + h_1\partial) \circ \iota) \circ (\pi \circ (I + \partial h_1))(U^{r_1+n}x_1) \\ &= (I + h_1\partial) \circ \iota \circ \pi \circ (U^{r_1+n}x_1 + \partial(U^n y_1)). \end{aligned} \tag{3.6}$$

Recall that multiplication by U lowers the Maslov grading by 2. Since $\partial(U^n y_1, U^{r_1+n}x_1) = 1$, it follows that $\partial(U^n y_1, U^k x_1) = 0$ for any $k \neq r_1 + n$. So, the expression $(U^{r_1+n}x_1 + \partial(U^n y_1))$ in (3.6) has no terms of the form $U^k x_1$. Similarly, by considering Maslov gradings, it also contains no elements of the form $U^k y_1$. Because of this, the composition $\iota \circ \pi$ in (3.6) is the identity, so we again get

$$g_1 \circ f_1(U^{r_1+n}x_1) = (I + h_1\partial + \partial h_1)(U^{r_1+n}x_1).$$

Finally, $g_1 \circ f_1(U^n y_1) = 0$ for all $n \geq 0$ (since $f_1(U^n y_1) = 0$). Also,

$$(I + h_1\partial + \partial h_1)(U^n y_1) = U^n y_1 + U^n y_1 + 0 = 0.$$

So, we have verified that in all cases,

$$g_1 \circ f_1 = I + h_1 \partial + \partial h_1,$$

which is to say, $g_1 \circ f_1$ is chain homotopic to the identity.

This shows that C_1 is a $\mathbb{Z}$-filtered chain deformation retract of C_0. It remains to check that this equivalence respects multiplication by U. In most cases, the fact that f_1 commutes with U is immediate, because, in most cases, all maps in the definition of f_1 commute with U. In fact, this is true for every homogeneous element except $U^{r_1-1}x_1$, on which h_1 and U do not commute. We verify the claim directly in this case,

$$\begin{aligned}
f_1 U(U^{r_1-1}x_1) &= \pi \circ (U + \partial h_1 U)(U^{r_1-1}x_1) \\
&= \pi \circ (U + \partial h_1 U) \circ \iota \circ \pi \circ (U^{r_1-1}x_1) \\
&= U_1 \circ \pi \circ (U^{r_1-1}x_1) \\
&= U_1 \circ \pi \circ (I + \partial h_1)(U^{r_1-1}x_1) \\
&= U_1 f_1(U^{r_1-1}x_1).
\end{aligned}$$

Similarly, one can verify that $g_1 U_1$ is chain homotopic to $U g_1$ via the (filtered) chain homotopy $h_1 U g_1$. $\square$

Beginning with C_0, we have now reduced the number of horizontal arrows and obtained a $(\mathbb{Z}, U)$-filtered chain homotopy equivalent complex C_1. The differential ∂_1 is nearly just $\pi \circ \partial$. The exception being that, if there was an arrow going from a homogeneous element to $U^{r_1+n}x_1$, the differential ∂_1 adds an arrow from that element to the remaining image of $\partial(U^n y_1)$ (see the discussion in Chapter 2 and Figure 2.1). Note though, that these additional arrows must always decrease the filtration level (in fact, by more than 1). In particular, the basis given for C_1 is still horizontally simplified, with the horizontal differential being $\pi \circ \partial_H$. This means we can iterate the above process, at each step moving from C_i to C_{i+1} by canceling the horizontal arrows $U^n y_i \to U^{r_i+n} x_i$, and obtaining a $(\mathbb{Z}, U)$-filtered chain homotopy equivalent complex with a filtered chain map U_i. If we begin with a basis $\mathfrak{G}$ for C_0 consisting of $2N+1$ elements, then C_N will have no horizontal arrows, and will be said to be a "reduced" version of $CFK^-(K)$. An example of this process of reduction is shown for the $(2,7)$-torus knot in Figure 3.2.

The process described above explicitly obtains a reduced complex after a choice of an ordered, horizontally simplified basis. More generally, we have the following definition.

Definition 3.1.3. *Let K be a knot in S^3, and C be a $(\mathbb{Z}, U)$-filtered chain complex. If C is $(\mathbb{Z}, U)$-filtered chain homotopy equivalent to $CFK^-(K)$, and the differential on C strictly decreases the filtration level, then C is called the reduced $CFK^-(K)$, denoted $\underline{CFK}^-(K)$.*

The condition that the differential strictly decreases filtration level says precisely that there are no horizontal arrows. Of course, this complex is only well-defined up to $(\mathbb{Z}, U)$-filtered chain homotopy equivalence of complexes with no horizontal arrows.

Note that, since $CFK^-(K)$ is $\mathbb{Z}$-filtered by the Alexander grading, there is naturally a spectral sequence whose (E_0, d_0) page is $(CFK^-(K), \partial_H)$, which converges to

$$H_*(CFK^-(K), \partial) \cong \bigoplus_{i \geq 0} \mathbb{F}_{(-2i)} \tag{3.7}$$

as a graded group, where, as in Figure 3.1, the subscript denotes the homological grading of each generator (the filtration on this group depends on K). An alternative view of the above construction, then, is that $\underline{CFK}^-(K)$ is the E_1 page of this spectral sequence. Further, the

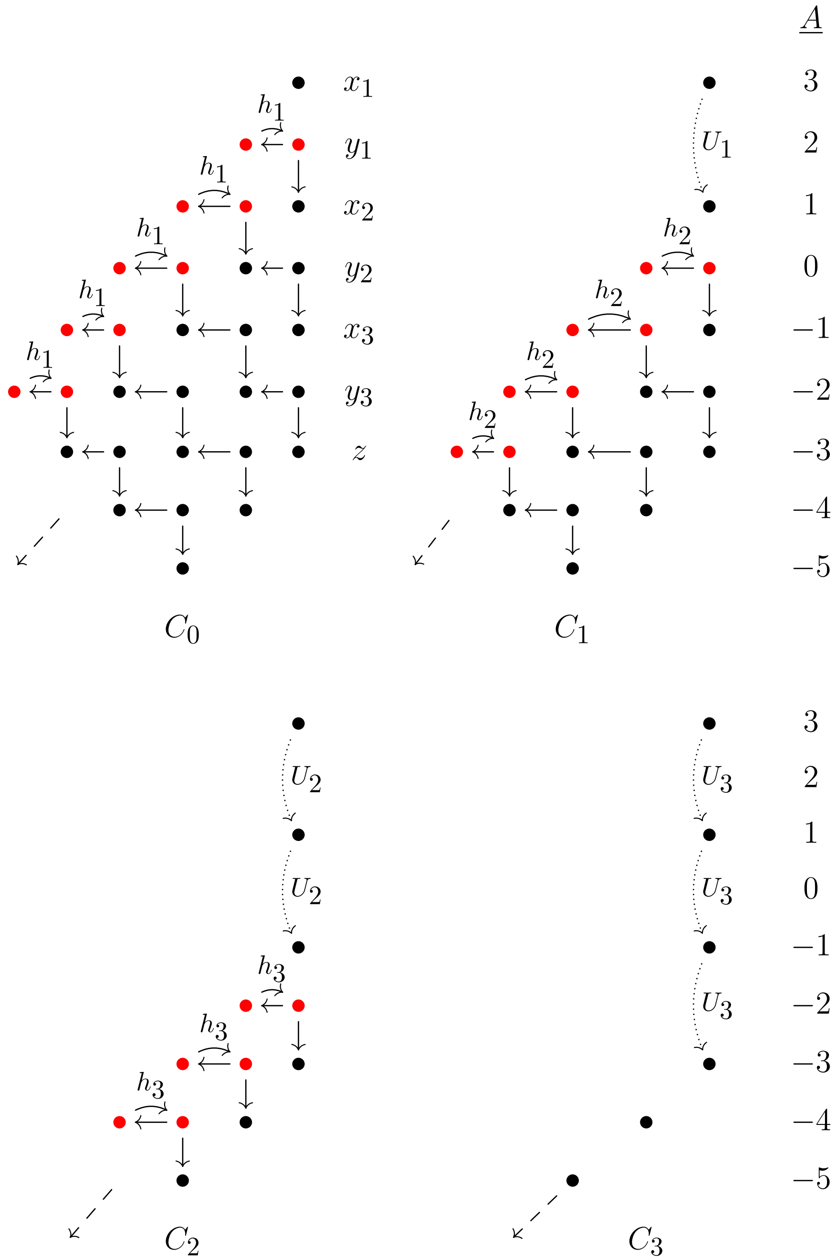


Figure 3.2: **Reducing a knot Floer complex.** The complex $C_0 = CFK^-(T(2,7))$, and the process of reducing to $C_3 = \underline{CFK}^-(T(2,7))$. At each step, the map h_i provides the chain homotopy necessary to cancel the horizontal arrows from $U^n y_i$ to $U^{n+1}x_i$ (the dots colored red). Multiplication by U is always taken to be translation one down and one to the left, unless otherwise shown with a dotted arrow.

filtered endomorphism U on E_0 induces a filtered endomorphism on E_1, and that is precisely the map U we have defined on $\underline{CFK}^-(K)$. It is the additional information given by this induced U map which will be useful for computing d-invariants in Chapter 4.

3.2 Sums of knots

As will be seen in this Chapter and Chapter 4, the reduced complex $\underline{CFK}^-(K)$ retains much of the information contained in $CFK^-(K)$, including, by definition, the homology of its associated graded complex, which is denoted $HFK^-(K)$. It was shown by Ozsváth and Szabó in [52, Theorem 7.1] that CFK^- behaves simply under connected sums of knots; namely,

$$CFK^-(K_1 \# K_2) \cong CFK^-(K_1) \otimes_{\mathbb{F}[U]} CFK^-(K_2). \tag{3.8}$$

However, these tensor product complexes are inconvenient to deal with by hand, even for sums of knots with small knot Floer homology. Many of the applications of this thesis are to sums of knots, and so it will be convenient to be able to reduce a complex before taking a tensor product, in order to decrease the size of the product. The following theorem ensures that this is possible.

Theorem 3.2.1. *If K_1 and K_2 are knots in S^3, then*

$$\underline{CFK}^-(K_1) \otimes_{\mathbb{F}[U]} CFK^-(K_2) \tag{3.9}$$

is a $(\mathbb{Z}, U)$-filtered chain deformation retract of $CFK^-(K_1 \# K_2)$.

Proof. After noting the relationship in equation (3.8), the result will follow from Lemma 2.2.2. To verify that the lemma applies, however, we must verify the following.

First, the complex $CFK^-(K_2)$ is, by definition, freely generated over $\mathbb{F}[U]$, and multiplication by U decreases the filtration level by 1. Second, recall that the maps U_i defined as in equation (3.3), decrease the filtration level by at least 1. It follows that the map U on the reduced complex $\underline{CFK}^-(K_1)$ also decreases the filtration level by at least 1. In other words, we define a filtration on the tensor product by formula (2.4), and the result follows from Lemma 2.2.2. $\square$

In order to use this effectively, we would like to further reduce the product complex (3.9), which we will call C for brevity, to get $\underline{CFK}^-(K_1 \# K_2)$. The method of reduction described explicitly above was for complexes which are freely generated over $\mathbb{F}[U]$, but we should point out here that a complex such as C can be handled similarly. This is because for sufficiently negative i, the subcomplex

$$A_i = \{x \in C | A(x) \leq i\}$$

is freely generated over $\mathbb{F}[U]$, and the corresponding quotient

$$C/A_i = \{x \in C | A(x) > i\}$$

is finitely generated over $\mathbb{F}$. So, after canceling the finite number of horizontal arrows with filtration level greater than i, we can use the same method as before.

Remark 3.2.2. *Suppose K_1 and K_2 are knots in S^3, and we wish to reduce $CFK^-(K_1 \# K_2)$. Then we can first reduce $CFK^-(K_1)$, tensor the reduced complex with $CFK^-(K_2)$, and then further reduce the product.*

To give an idea of how this facilitates computation, we consider a simple example, the sum of the right-handed and left-handed trefoils, $T(2,3) \# - T(2,3)$. Figure 3.3 shows the knot Floer complexes of these two knots. To obtain $\underline{CFK}^-(T(2,3)\# -T(2,3))$, we first compute $\underline{CFK}^-(T(2,3))$ and then tensor it with $CFK^-(-T(2,3))$. To provide contrast, we also show in Figure 3.4 the tensor product complex $CFK^-(T(2,3)) \otimes_{\mathbb{F}[U]} CFK^-(-T(2,3))$.

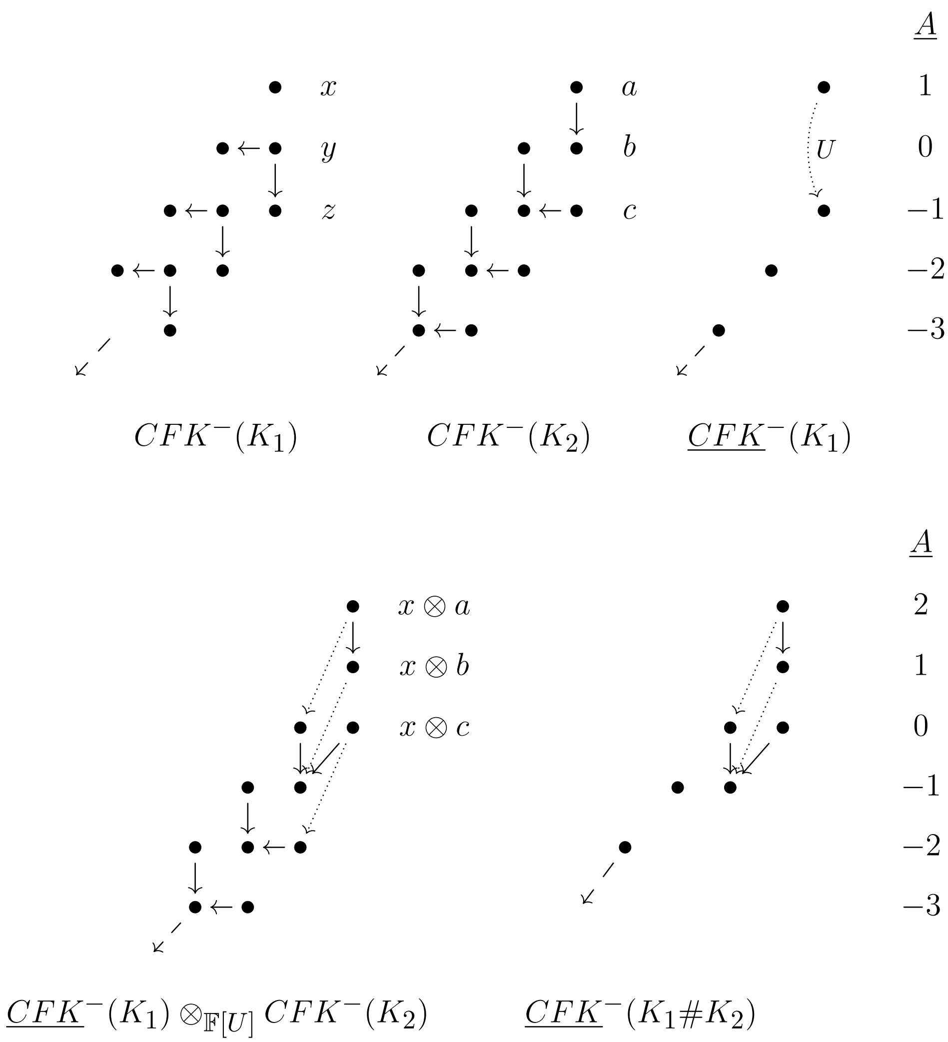


Figure 3.3: **A connected sum with a reduced complex.** In this example, K_1 is the right-handed trefoil and K_2 is the left-handed trefoil. We can first reduce to get $\underline{CFK}^-(K_1)$, then tensor this complex with $CFK^-(K_2)$. One more reduction gives the complex $\underline{CFK}^-(K_1 \# K_2)$.

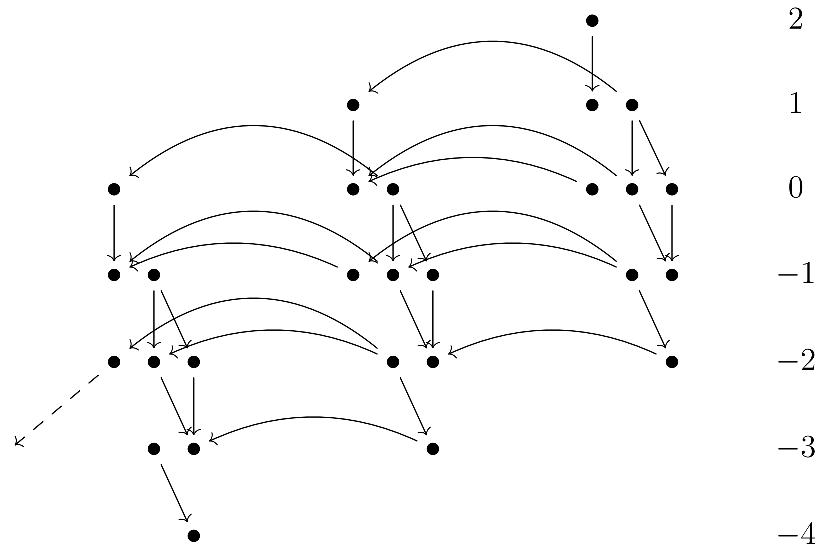


Figure 3.4: **A tensor product without using the reduced complex.** The tensor product complex corresponding to the sum of the right- and left-handed trefoils. Even for knots with the simplest nontrivial knot Floer complexes, computing with tensor product complexes becomes tedious.

Chapter 4

Applications to L-space knots

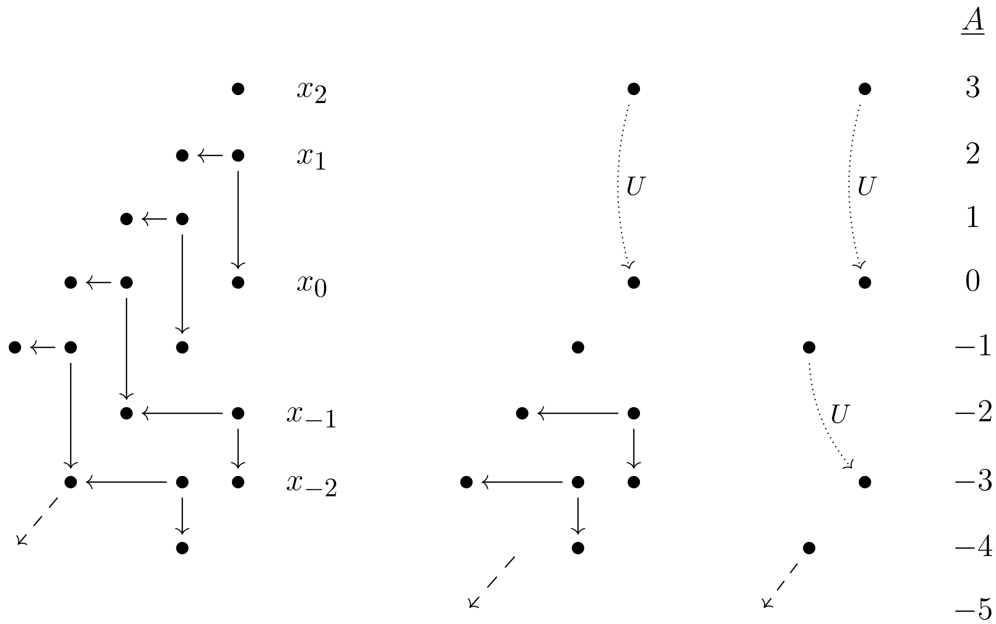
4.1 Reduced complexes of L-space knots

In this chapter, we show how the reduced complex $\underline{CFK}^-(K)$ can be used to elucidate some properties of L-space knots. Recall that a rational homology 3-sphere Y is called an *L-space* if – like a lens space – it has the "smallest possible" Heegaard Floer homology; i.e., for each spin^c structure $\mathfrak{t}$, $\widehat{HF}(Y, \mathfrak{t}) \cong \mathbb{F}$. A knot K in S^3 is called an *L-space knot* if n-surgery on S^3 along K is an L-space, for some positive integer n. It was shown in [56, Theorem 1.2 and Corollary 1.6], and restated more conveniently for our purposes in [25, Remark 6.6], that L-space knots have knot Floer complexes of a particular form, which we describe here.

Proposition 4.1.1 (Ozsváth-Szabó). *If K admits a positive L-space surgery, then $CFK^-(K)$ has a basis $\{x_{-k}, \cdots, x_k\}$ with the following properties:*

- $A(x_i) = n_i$, *where* $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$
- $n_i = -n_{-i}$
- *If* $i \equiv k \mod 2$, *then* $\partial(x_i) = 0$
- *If* $i \equiv k + 1 \mod 2$, *then* $\partial(x_i) = x_{i-1} + U^{n_{i+1}-n_i} x_{i+1}$ $\square$

Diagramatically, the knot Floer complex of an L-space knot has a "staircase" shape, as shown, for example, in Figure 4.1. The basis described in Proposition 4.1.1 is, in particular,



$CFK^-(T(3,4))$ $\qquad$ $\underline{CFK}^-(T(3,4))$

Figure 4.1: **Reducing an L-space knot complex.** At the left is the staircase-shaped knot Floer complex for the (3,4)-torus knot, and at the right is its reduction (with an intermediate step shown in between). Each time we cancel the horizontal arrow from x_{2-2i-1} to x_{2-2i}, we see that U takes the last remaining power of x_{2-2i} to x_{2-2i-2}, so, in fact, every generator which remains in $\underline{CFK}^-(T(3,4))$ can be written as $U^k x_2$ for some $k \geq 0$.

horizontally simplified, and so the method of reduction will proceed exactly as in Chapter 3. We include a proof of the corollary below, although the result should be more readily evident by seeing the reduction in Figure 4.1.

Corollary 4.1.2. *If K is an L-space knot, then $\underline{CFK}^-(K)$ has exactly one generator of Maslov grading $-2i$ for each $i \geq 0$, and no other generators. Further, if x is the generator with Maslov grading $-2i$, and y is the generator with Maslov grading $-2i-2$, then $Ux = y$.*

Proof. We will first show that if we use a basis as in Proposition 4.1.1, and proceed with the reduction as described in Chapter 3, then we get a representative of $\underline{CFK}^-(K)$ with the desired properties. We will then show that, in fact, any representative must be isomorphic to this one.

First, we consider the subquotient complex

$$\widehat{CFK}(K) := C\{i = 0\}.$$

This is a filtered complex which is chain homotopy equivalent to $\widehat{CF}(S^3)$, and so its homology is generated by a single element. To fix an absolute Maslov grading, this generator of the homology of $\widehat{CF}(S^3)$ is declared to have Maslov grading zero. It is clear from the explicit differential given in Propostion 4.1.1 that, when K is an L-space knot, x_k generates the homology of $\widehat{CFK}(K)$, and so $M(x_k) = 0$.

With this as our starting point, we now consider what happens through reduction. To simplify notation, we will define

$$r_i := n_i - n_{i-1}$$

to be the difference in the Alexander gradings of x_i and x_{i-1}. We first cancel the horizontal arrows which form the top steps of the staircases, those corresponding to

$$\partial_H(U^m x_{k-1}) = U^{m+r_k} x_k$$

for $m \geq 0$. Note that after canceling $U^{r_k} x_k$, we have

$$\begin{aligned} U_1(U^{r_k-1}x_k) &= \left(\pi \circ (U - \partial h_1 U) \circ \iota\right)(U^{r_k-1}x_k) \\ &= \pi \circ (U^{r_k}x_k - \partial h_1(U^{r_k}x_k)) \\ &= \pi \circ (U^{r_k}x_k - \partial x_{k-1}) \\ &= x_{k-2} \end{aligned}$$

That is, after canceling higher U-powers of x_k, the map U takes the highest remaining power, $U^{r_k - 1} x_k$, to the next generator, x_{k-2}. The exact same argument applies to the cancellation of the horizontal arrows

$$\partial_H(x_{k-3}) = U^{r_{k-2}} x_{k-2},$$

and we see that

$$U_2(U^{r_{k-2}-1} x_{k-2}) = x_{k-4}.$$

We proceed in this fashion, until finally we see that

$$U_k(U^{r_{-k+2}-1} x_{-k+2}) = x_{-k}.$$

It follows that each of these generators of $\underline{CFK}^-(K)$ can be written as

$$U^i x_k$$

for some $i \geq 0$. Since multiplication by U lowers the Maslov grading by 2, the result follows.

Now let us call the reduced complex just constructed C, and suppose that C' is a $(\mathbb{Z}, U)$-filtered chain homotopy equivalent complex which also has no horizontal arrows (i.e., a different representative of $\underline{CFK}^-(K)$). Since it is filtered chain homotopy equivalent, each subquotient complex $C'\{j = k\}$ must have homology isomorphic to that of $C\{j = k\}$, which is either 0 or $\mathbb{F}$, depending on k. Since there are no horizontal arrows, these subquotient complexes have trivial differential, so they either have no generators or they have exactly 1 generator, with even homological grading. Therefore, C' also has trivial differential, so it is isomorphic to C as a filtered chain complex. The fact that the equivalence is $(\mathbb{Z}, U)$-filtered implies that U also takes generator to generator for C' as it does for C, so in fact they are isomorphic as $(\mathbb{Z}, U)$-filtered complexes. $\square$

A concise way of stating Corollary 4.1.2 is that, for an L-space knot K_1,

$$\underline{CFK}^-(K_1) \cong \mathbb{F}[U]_{(0)}, \tag{4.1}$$

where the subscript here means that the generator has Maslov grading zero. Necessarily, this complex also has trivial differential. This means that a tensor product of the form

$$\underline{CFK}^-(K_1) \otimes_{\mathbb{F}[U]} CFK^-(K_2) \tag{4.2}$$

will be isomorphic to $CFK^-(K_2)$ as a chain complex, which we will make use of below. It is important to note however, that the isomorphism (4.1) is not filtered (when K_1 is not the unknot). As a result, the tensor product (4.2) is not $(\mathbb{Z}, U)$-filtered chain homotopy equivalent to $CFK^-(K_2)$.

Knowing that L-space knots must have this particularly simple reduced knot Floer complex, we make use of the behavior under connected sums to record the following observation.

Theorem 4.1.3. *A knot in S^3 which admits an L-space surgery must be a prime knot.*

The general argument, a proof by contradiction using Corollary 4.1.2, will be suggested by considering an example, so we refer back to Figure 3.3. Consider the generator of maximal Alexander grading in $CFK^-(K_1)$, which we labeled x. We have $A(x) = 1$, and so

$$A(Ux) = 0 = A(x) - 1.$$

But Ux is canceled by a horizontal arrow, so when we move to $\underline{CFK}^-(K_1)$, we have

$$A(Ux) = -1 < A(x) - 1$$

That is, multiplication by U "jumps" down by more than 1. When we take the tensor product

$$\underline{CFK}^-(K_1) \otimes_{\mathbb{F}[U]} CFK^-(K_2),$$

this has the effect of "bending downward" the horizontal arrow from c to Ub. That is, the arrow from $x \otimes c$ to $Ux \otimes b$ is not horizontal. Therefore, when we reduce this tensor product complex, we cannot cancel these generators, and so they both remain. But, being connected by an arrow, their Maslov gradings differ by exactly 1, so the complex $\underline{CFK}^-(K_1 \# K_2)$ does not have the form described in Corollary 4.1.2. With this example as motivation, we provide the details.

Proof of Theorem 4.1.3. We begin by noting that if *negative* surgery on a knot produces an L-space, then positive surgery on its mirror image produces an L-space, so by definition its mirror image is an L-space knot. Since a knot is a nontrivial connected sum if and only if its mirror image is, it will be sufficient to show that *positive* surgery on a connected sum can never produce an L-space. That is, we will show that no nontrivial connected sum is an L-space knot.

Suppose K_1 and K_2 are two nontrivial knots in S^3. Let C_i denote the complex $CFK^-(K_i)$. Choose a horizontally simplified basis for C_1, where y_1 and x_1 are generators such that $\partial_H(y_1) = U^{r_1} x_1$. Likewise, choose a horizontally simplified basis for C_2, where Y_1 and X_1 are generators such that $\partial_H(Y_1) = U^{R_1} X_1$. Without loss of generality, assume that

$r_1 \leq R_1$. Our goal is to show by contradiction that $\underline{CFK}^-(K_1 \# K_2)$ is a complex which cannot correspond to an L-space knot, by Corollary 4.1.2.

We will make use of Remark 3.2.2, and begin by reducing C_1 to get a complex $\underline{C_1}$. We will denote the filtration on this reduced complex by F_1 and the filtration on C_2 by F_2. Finally, the filtration on the tensor product $\underline{C_1} \otimes_{\mathbb{F}[U]} C_2$, defined as in equation (2.4), will be denoted F.

After reducing C_1, we have

$$F_1(U^{r_1-1}x_1) = F_1(x_1) - (r_1 - 1),$$

but the map U on $U^{r_1-1}x_1$ lowers the filtration level by at least two, so

$$F_1(U^{r_1}x_1) < F_1(x_1) - r_1.$$

As a consequence, since $R_1 \geq r_1$, and U always lowers the filtration level by at least 1,

$$F_1(U^{R_1}x_1) < F_1(x_1) - R_1.$$

Note also that, on the freely generated complex C_2, U is a homogeneous map of degree 1, so

$$F_2(Y_1) = F_2(X_1) - R_1.$$

It follows that

$$\begin{aligned} F(U^{R_1}x_1 \otimes X_1) &= F_1(U^{R_1}x_1) + F_2(X_1) \\ &< F_1(x_1) - R_1 + F_2(X_1) \\ &= F_1(x_1) + F_2(Y_1) \\ &= F(x_1 \otimes Y_1). \end{aligned}$$

Because of this, every term in

$$\partial(x_1 \otimes Y_1) = \partial x_1 \otimes Y_1 + x_1 \otimes \partial Y_1$$

has filtration level strictly less than that of $x_1 \otimes Y_1$. That is to say, $\partial_H(x_1 \otimes Y_1) = 0$. When we reduce the tensor product complex, there is no horizontal differential to cancel $x_1 \otimes Y_1$, so it will project to a nonzero homogeneous element in $\underline{CFK}^-(K_1\#K_2)$. Of course, $x_1 \otimes X_1$ also projects to a nonzero homogeneous element in $\underline{CFK}^-(K_1\#K_2)$. But,

$$\begin{aligned} M(x_1 \otimes Y_1) &= M(x_1) + M(Y_1) \\ &= M(x_1) + M(U^{R_1}X_1) + 1 \\ &= M(x_1) + M(X_1) + 1 - 2R_1 \\ &= M(x_1 \otimes X_1) + 1 - 2R_1, \end{aligned}$$

so $\underline{CFK}^-(K_1\#K_2)$ has two elements with Maslov gradings of opposite parity. By Corollary 4.1.2, $K_1\#K_2$ cannot be an L-space knot. $\square$

4.2 Correction terms and concordance

We now turn to our second application, pertaining to the Heegaard Floer correction terms, or d-invariants. Given a rational homology three-sphere Y and spin^c structure $\mathfrak{t}$, we obtain a chain complex $CF^\infty(Y,\mathfrak{t})$ which is freely generated over $\mathbb{F}[U,U^{-1}]$, as described in Chapter 3, and its associated subcomplex $CF^-(Y,\mathfrak{t})$. The homology of this subcomplex, denoted $HF^-(Y,\mathfrak{t})$, consists of a direct summand isomorphic to $\mathbb{F}[U]$, and possibly other terms which are U-torsion. The correction term associated to $(Y,\mathfrak{t})$, denoted $d(Y,\mathfrak{t})$, is simply the maximal Maslov grading of any nontorsion generator in $HF^-(Y,\mathfrak{t})$.

Given a knot K in S^3, let $S^3_1(K)$ denote the integer homology sphere obtained from S^3 by doing Dehn surgery along K with slope 1. We can associate to this manifold a number, $d(S^3_1(K),\mathfrak{t})$ (where $\mathfrak{t}$ is the unique spin^c structure on $S^3_1(K)$), which we will abbreviate as $d_1(K)$. This invariant was studied by Peters in [61], where it was shown to have the following properties.

Proposition 4.2.1 (Theorem 1.5 and Proposition 2.1 of [61]). *For any knot K in S^3,*

- *$d_1(K)$ is an even integer*
- *$d_1(K)$ is a concordance invariant of K*
- *If we denote by $g_4(K)$ the smooth four-dimensional genus of K,*

$$0 \leq -d_1(K) \leq 2g_4(K).$$

In addition, Peters gave an algorithm to compute $d_1(K)$ from $CFK^\infty(K)$, using the fact that $CFK^\infty(K)$ contains all the information needed to compute the Heegaard Floer homology of manifolds arising from surgery on K. We briefly recount the idea here. For details, see [61].

In [49, Lemma 7.11], the degrees of the maps in the integer surgery exact sequence

$$\cdots \to HF^+(S^3_0(K)) \to HF^+(S^3_N(K)) \to HF^+(S^3) \to \cdots$$

were computed, from which it was concluded in [61, Sec. 5] that[1]

$$d_{1/2}(S^3_0(K)) = d(S^3_N(K), \mathfrak{s}_0) - \frac{N-3}{4}. \tag{4.3}$$

In particular, we have

$$d_1(K) = d(S^3_N(K), \mathfrak{s}_0) - \frac{N-1}{4}, \tag{4.4}$$

so the invariant $d_1(K)$ is determined by $d(S^3_N(K), \mathfrak{s}_0)$. For N sufficiently large, this can be computed directly from $CFK^\infty(K)$.

Let A_0^+ denote the quotient complex

$$C\{i \geq 0 \text{ or } j \geq 0\}$$

of $CFK^\infty(K)$ (recall that $C\{S\}$ denotes the elements with (i,j)-coordinates in S, and the arrows between these elements). Ozsváth and Szabó [52, Corollary 4.2] (c.f., [64]) show that,

[1] For three-manifolds with $H_1(Y) \cong \mathbb{Z}$, Ozsváth and Szabó define $d_{\pm 1/2}(Y)$ to be the minimal grading of an element in $HF^+(Y, \mathfrak{s}_0)$ which is in the image of U^k for all $k > 0$ whose grading is additionally congruent to $\pm 1/2 \mod 2$, where $\mathfrak{s}_0$ is the unique spin^c-structure for which $c_1(\mathfrak{s}_0) = 0$.

for any sufficiently large positive integer N,

$$HF^+_{l+\left(\frac{N-1}{4}\right)}(S^3_N(K), \mathfrak{s}_0) \cong H_l(A_0^+). \tag{4.5}$$

That is, up to a shift in grading which depends on N, the homology of this complex is the Heegaard Floer homology of the three-manifold obtained by surgery. Combining equations (4.4) and (4.5), we see that the grading shifts cancel nicely, and $d_1(K)$ is equal to the minimum grading of a generator of $H_*(A_0^+)$ which is in the image of U^k for all $k > 0$.

Remark 4.2.2. *It should be pointed out that, by* (4.3)*, we could get the same information from the invariant* $d(S^3_N(K), \mathfrak{s}_0)$*, which is also a concordance invariant, as we get from* $d_1(K)$*. The choice* $N = 1$ *is a matter of convenience, because it gives a four-genus bound without any shift.*

We will find it convenient to work with the subcomplex $CFK^-(K)$ rather than the quotient $CFK^+(K)$. From this point of view, $d_1(K)$ is the *maximum* grading of a non-torsion generator of homology of the subcomplex

$$C\{i \leq 0 \ \text{ and } \ j \leq 0\}.$$

Remark 4.2.3. *To justify this this point of view, we first point out that Ozsváth and Szabó define* $d^-(Y, \mathfrak{t})$ *to be the maximal grading of a non-torsion generator in* $HF^-(Y, \mathfrak{t})$*, and observe in the proof of [49, Prop. 4.2] that*

$$d^-(Y, \mathfrak{t}) = d(Y, \mathfrak{t}) - 2$$

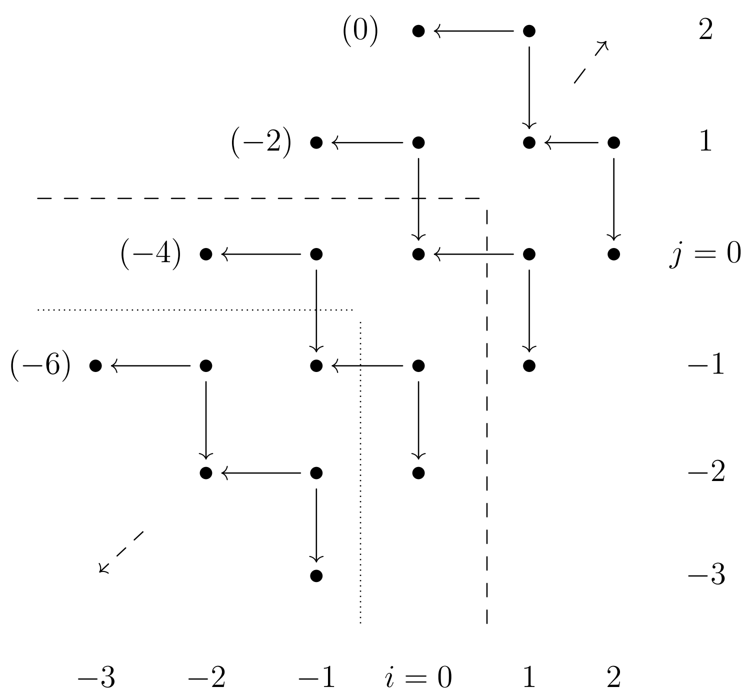


Figure 4.2: **Computing d_1 from the knot Floer complex.** The complex $CFK^\infty(T(2,5))$. The invariant $d_1(T(2,5))$ can be seen to equal -2, either by considering the *maximal* grading of a generator of homology of the *sub*complex $C\{i \leq 0$ and $j \leq 0\}$ (below and to the left of the dashed line); or by considering the *minimal* grading of a generator of homology of the *quotient* complex $C\{i \geq 0$ or $j \geq 0\}$ (above or to the right of the dotted line). We will find it easier to work with the subcomplex through the rest of this thesis.

(recalling that U lowers grading by 2). But our definition of CF^- differs from Ozsváth and Szabó's by a shift by U^{-1} (see Remark 1.4.1). So, the maximal grading of a non-torsion generator of our $CF^-(Y, \mathfrak{t})$ is

$$d'(Y, \mathfrak{t}) = d^-(Y, \mathfrak{t}) + 2 = d(Y, \mathfrak{t}),$$

so we will think of the d-invariant this way.

Figure 4.2 shows how d_1 can be computed from the knot Floer complex in the case of the (2,5)-torus knot.

There is another concordance invariant which comes from the knot Floer complex, $\tau(K)$, which was introduced by Ozsváth and Szabó in [50], where they showed that it also gives a lower bound for the smooth four-dimensional genus of K,

$$|\tau(K)| \leq g_4(K).$$

Given a knot Floer complex, this invariant is easily computed, and yet has been shown to be a quite powerful four-genus bound. For example, its value on torus knots, shown in [50, Corollary 1.7] to be

$$\tau(T(p,q)) = \frac{(p-1)(q-1)}{2}, \tag{4.6}$$

was used to provide an alternate proof of the Milnor conjecture, which says that this is in fact the four-genus of $T(p,q)$ (see [42]). The invariant is defined from $CFK^{\infty}(K)$ by considering the subquotient complex

$$\widehat{CFK}(K) := C\{i=0\}.$$

If we let ι_k be the inclusion map

$$\iota_k : C\{i=0, j \leq k\} \to C\{i=0\},$$

we get an induced map on homology

$$(\iota_k)_* : H_*(C\{i=0, j \leq k\}) \to H_*(C\{i=0\}).$$

This map is clearly an isomorphism for large enough k, and the zero map for sufficiently negative values of k (since the complex is finitely generated). We can then define

$$\tau(K) := \min\{k | (\iota_k)_* \text{ is non-trivial}\}.$$

This quantity is additive under tensor products of complexes, and therefore τ defines a homomorphism from the smooth concordance group to $\mathbb{Z}$.

It follows from Proposition 4.1.1 that, for an L-space knot K,

$$\tau(K) = A(x_k) = \max\{j | \widehat{HFK}(K, j) \neq 0\},$$

(which is also the Seifert genus of K). In general, it was shown in [52] that

$$\sum_j \chi\left(\widehat{HFK}(K, j)\right) \cdot T^j = \Delta_K(T), \tag{4.7}$$

where $\Delta_K(T)$ is the symmetrized Alexander polynomial of K. Since, for an L-space knot, we can choose a basis for which the rank of $\widehat{CFK}(K, j)$ is either 0 or 1 for each j, the rank of each subcomplex is determined by its Euler characteristic, so, by Proposition 4.1.1, the knot Floer complex contains the same amount of information as the Alexander polynomial.

In particular, $\tau(K) = \deg \Delta_K(T)$. That, however, is all the information τ can give in this case. The statement that $\tau(K_1\# - K_2) = 0$ for two L-space knots K_1 and K_2 is precisely the statement that their Alexander polynomials have equal degree. In contrast, the next theorem gives a sense in which the invariant d_1 is more sensitive.

Theorem 4.2.4. *Suppose that K_1 and K_2 are two knots in S^3 which admit positive L-space surgeries. If*

$$d_1(K_1\# - K_2) = d_1(-K_1\#K_2) = 0,$$

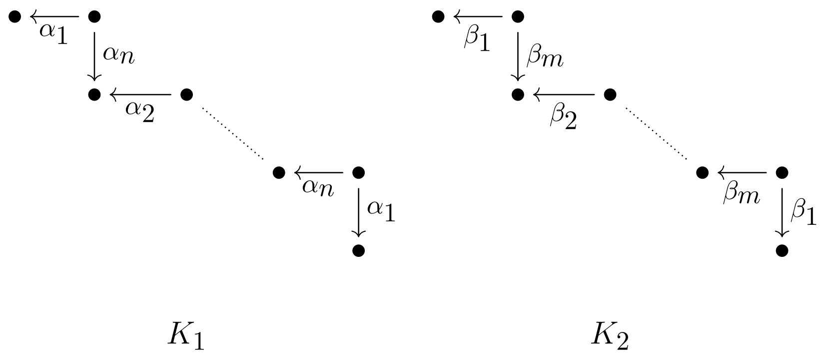


Figure 4.3: **Labeling staircase complexes.** The staircases for K_1 and K_2.

then

$$\Delta_{K_1}(T) = \Delta_{K_2}(T).$$

In particular, the Alexander polynomial is a concordance invariant of L-space knots.

PROOF. As mentioned above, in light of Proposition 4.1.1, the Alexander polynomial of K_i gives equivalent information to the knot Floer complex of K_i, which we will represent by its staircase shape. We can represent a staircase by listing the horizontal lengths in order from left to right. By the symmetry of the Alexander polynomial, this list is also the list of vertical lengths, in order from bottom to top. Suppose that K_1 has staircase $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, and K_2 has staircase $\{\beta_1, \beta_2, \dots, \beta_m\}$, as shown in Figure 4.3, and also that

$$d_1(K_1\# - K_2) = d_1(-K_1\#K_2) = 0.$$

The proof will proceed by showing first that the Alexander polynomials must have equal degrees, then, one step at a time, that $\alpha_i = \beta_i$ for all i (and consequently, that $m = n$).

The complex $CFK^-(-K_2)$ will take the shape of an "upside down staircase", since $-K_2$ is the mirror image of an L-space knot (see Figure 4.4). In this case, $\tau(-K_2) = -\deg \Delta_{K_2}(T)$. We record here a particularly useful property of these complexes, which

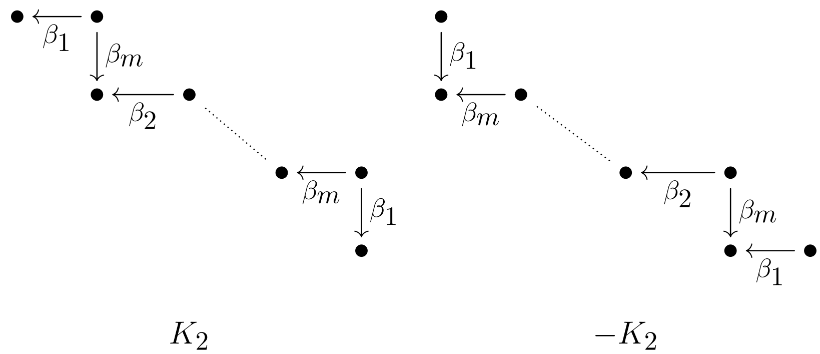


Figure 4.4: **An upside-down staircase.** The knot Floer complex of the L-space knot K_2, and that of its mirror image.

follows from direct inspection.

Remark 4.2.5. *If K is an L-space knot, then the knot Floer complex of its mirror, $CFK^-(-K)$, has a basis for which it satisfies the following:*

- *$CFK^-(-K)$ splits into a direct sum of complexes C_{2i}, for each integer i, where C_{2i} consists of the homogeneous elements of Maslov gradings $2i$ and $2i-1$*
- *for $i > 0$, the complex C_{2i} is acyclic*
- *for $i \leq 0$, the complex C_{2i} has homology isomorphic to $\mathbb{F}$, generated by the sum of all homogeneous elements of Maslov grading $2i$*

If K_1 is an L-space knot, then, by Corollary 4.1.2, the complex $\underline{CFK}^-(K_1)$ is "almost" the knot Floer complex of the unknot. More precisely, each is isomorphic to $\mathbb{F}[U]$, supported in grading zero, with the only difference being that, for the unknot, U is homogeneous of degree one, while for K_1, U is a non-homogeneous map which decreases the Alexander grading by *at least* one. It follows that the tensor product complex

$$\underline{CFK}^-(K_1) \otimes_{\mathbb{F}[U]} CFK^-(-K_2),$$

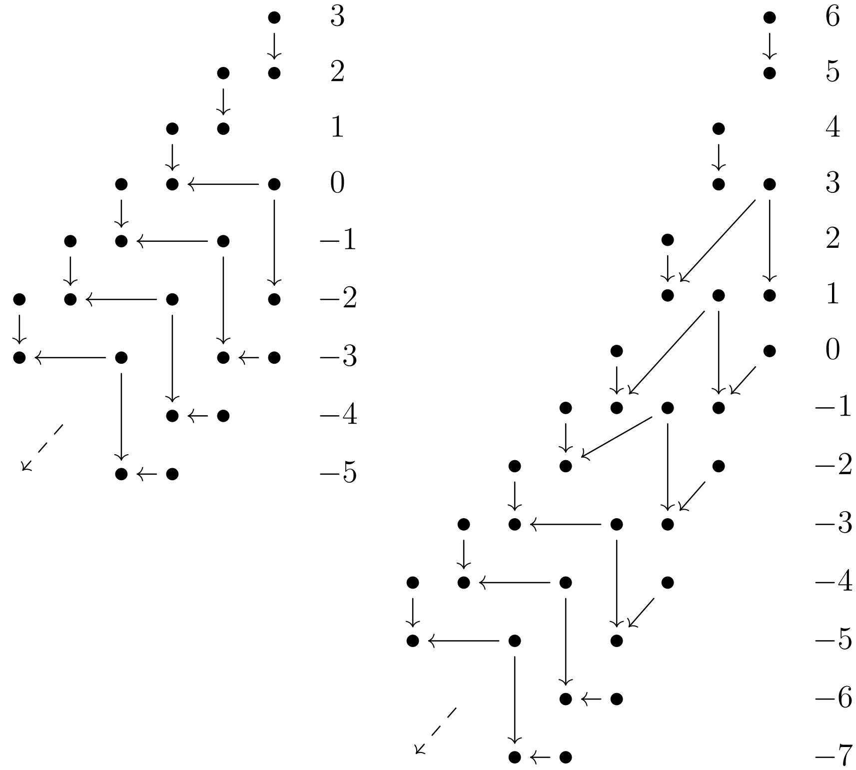


Figure 4.5: **The difference of two L-space knots.** On the left, the complex of the mirror image knot $CFK^-(-T(3,4))$. On the right, the complex $\underline{CFK}^-(T(2,7)) \otimes_{\mathbb{F}[U]} CFK^-(-T(3,4))$. Note that it retains the same "staircase" form, but some horizontal arrows get bent downward.

which we will denote by C, is "almost" an upside down staircase; the only difference being that some of its "stairs" have been bent. Figure 4.5 shows an example of this, the complex $\underline{CFK}^-(T(2,7)) \otimes_{\mathbb{F}[U]} CFK^-(-T(3,4))$ (recall that the reduced complex $\underline{CFK}^-(T(2,7))$ is shown in Figure 3.2).

Remark 4.2.6. *In particular, C still has the properties in Remark 4.2.5, splitting into summands C_{2i}. Since the generator of homology of C_0 is the sum of all homogeneous elements with Maslov grading 0, its Alexander grading is the maximum of the Alexander gradings of all of these elements (see Equation* (3.1)*). So, in this case, the d_1-invariant is zero if and only if all of the elements with Maslov grading zero have Alexander grading less than or equal* *to zero.*

We will now see how the Alexander grading on the C_0 summand is determined from the shape of the staircases; i.e., from the α_i's and β_i's. Let us choose generators $\{x_{-n}, \dots, x_n\}$ for $CFK^-(K_1)$ as in Proposition 4.1.1. Then the generators for $\underline{CFK}^-(K_1)$ are

$$x_{n-2i}, Ux_{n-2i}, \dots, U^{\alpha_{i+1}-1}x_{n-2i}$$

for every $0 \leq i \leq n-1$, and $U^k x_{-n}$, for all $k \geq 0$. Further, as in the proof of Corollary 4.1.2, we have that

$$U(U^{\alpha_{i+1}-1}x_{n-2i}) = x_{n-2(i+1)} \quad \text{for all } \ 0 \leq i \leq n-1$$

We should also point out that if $0 \leq k < \alpha_{i+1}$,

$$A(U^k x_{n-2i}) = \tau(K_1) - k - \sum_{1\leq j\leq i} (\alpha_j + \alpha_{n+1-j}), \tag{4.8}$$

and

$$M(U^k x_{n-2i}) = -2k - \sum_{1\leq j\leq i} 2\alpha_j.$$

For $CFK^-(-K_2)$, we choose a basis $\{y_{-m}, \dots, y_m\}$, so that

$$A(y_{-m+2k}) = \tau(-K_2) + \sum_{1\leq j\leq k} (\beta_{m+1-j} + \beta_j)$$

and

$$M(y_{-m+2k}) = \sum_{1\leq j\leq k} 2\beta_j.$$

We now consider each generator in C_0 which has Maslov grading zero, and see what restrictions we get on the α_i's and β_i's by assuming it has Alexander grading less than or equal to zero. The first generator we consider is $x_n y_{-m}$, and we have that

$$\begin{aligned} A(x_n y_{-m}) &= A(x_n) + A(y_{-m}) \\ &= \tau(K_1) + \tau(-K_2) \\ &= \tau(K_1) - \tau(K_2). \end{aligned}$$

In order for this to be less than or equal to zero, we must have $\tau(K_1) \leq \tau(K_2)$. On the other hand, considering instead the knot $-K_1 \# K_2$, the same argument says we must also have $\tau(K_2) \leq \tau(K_1)$, so $\tau(K_1) = \tau(K_2)$, and the Alexander polynomials of K_1 and K_2 must have equal degree.

The rest of the proof proceeds similarly. We next consider the generator $U^{\beta_1} x_n y_{-m+2}$. If $\alpha_1 > \beta_1$, then

$$A(U^{\beta_1} x_n y_{-m+2}) = \tau(K_1) - \beta_1 + \tau(-K_2) + \beta_1 + \beta_m > 0,$$

so, in order to have $d_1 = 0$, we must have $\alpha_1 \leq \beta_1$. Again, considering $-K_1 \# K_2$, we must also have $\beta_1 \leq \alpha_1$, so $\alpha_1 = \beta_1$. Since $\alpha_1 = \beta_1$, $U^{\beta_1} x_n = x_{n-2}$, so

$$\begin{aligned} A(U^{\beta_1} x_n y_{-m+2}) &= A(x_{n-2} y_{-m+2}) \\ &= \tau(K_1) - \alpha_1 - \alpha_n + \tau(-K_2) + \beta_1 + \beta_m \\ &= -\alpha_n + \beta_m. \end{aligned}$$

This means we must also have $\alpha_n \geq \beta_m$; and once again considering $-K_1\#K_2$, we see that in fact $\alpha_n = \beta_m$.

We have to this point shown that the first elements of the lists representing these two staircases agree, and also that the last elements agree. Taking this as our base case, we will now work our way inductively toward the middle.

To that end, assume that $\alpha_i = \beta_i$ and $\alpha_{n+1-i} = \beta_{m+1-i}$, for all $1 \leq i \leq k$. Then consider the generator

$$U^{\beta_1+\beta_2+\cdots+\beta_{k+1}} x_n y_{-m+2k+2} = U^{\beta_{k+1}} x_{n-2k} y_{-m+2k+2}.$$

If $\alpha_{k+1} > \beta_{k+1}$, then

$$\begin{aligned} A(U^{\beta_{k+1}} x_{n-2k} y_{-m+2k+2}) &= A(U^{\beta_{k+1}} x_{n-2k}) + A(y_{-m+2k+2}) \\ &= \tau(K_1) - \beta_{k+1} - \sum_{1 \leq j \leq k} (\alpha_j + \alpha_{n+1-j}) \\ &\quad - \tau(K_2) + \sum_{1 \leq j \leq k+1} (\beta_j + \beta_{m+1-j}) \\ &= \beta_{m-k} \\ &> 0, \end{aligned}$$

so it must be that $\alpha_{k+1} \leq \beta_{k+1}$; considering $-K_1\#K_2$ gives $\alpha_{k+1} = \beta_{k+1}$.

Since $\alpha_{k+1} = \beta_{k+1}$, $U^{\beta_{k+1}} x_{n-2k} = x_{n-2k-2}$, so

$$
\begin{aligned}
A(U^{\beta_{k+1}} x_{n-2k} y_{-m+2k+2}) &= A(x_{n-2k-2} y_{-m+2k+2}) \\
&= \tau(K_1) - \sum_{1 \leq j \leq k+1} (\alpha_j + \alpha_{n+1-j}) \\
&\quad - \tau(K_2) + \sum_{1 \leq j \leq k+1} (\beta_j + \beta_{m+1-j}) \\
&= \ - \alpha_{n-k} + \beta_{m-k}
\end{aligned}
$$

This means $\alpha_{n-k} \geq \beta_{m-k}$, and as before, we see that $\alpha_{n-k} = \beta_{m-k}$, which completes the inductive step. *A priori*, n may not be equal to m, but this induction can be continued for all i until either α_i or β_i does not exist. That is, until we exceed the minimum of n and m. Assume, without loss of generality, that it is n. Upon reaching that point, we have $\alpha_i = \beta_i$, for $1 \leq i \leq n$. But since the Alexander polynomials have equal degree,

$$
\sum_{1 \leq i \leq n} \alpha_i = \sum_{1 \leq i \leq m} \beta_i,
$$

so n and m must be equal. $\square$

Figure 4.5 shows the example of the sum $T(2,7)\# -T(3,4)$. In Figure 4.6, it is exhibited that

$$
d_1(T(2,7)\# -T(3,4)) = -2,
$$

although

$$
\tau(T(2,7)\# -T(3,4)) = 0.
$$

This is an instance of a general fact which follows from Theorem 4.2.4.

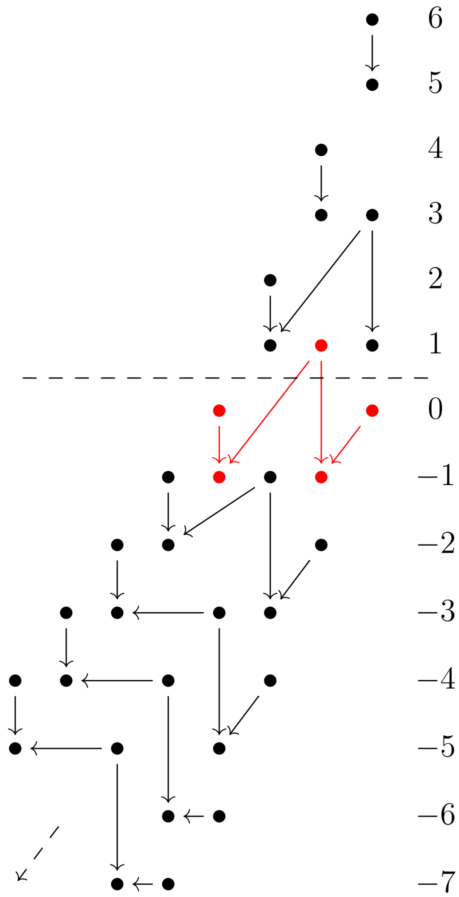


Figure 4.6: **Computing d_1 of the difference of two L-space knots.** The complex $\underline{CFK}^-(T(2,7)) \otimes_{\mathbb{F}[U]} CFK^-(-T(3,4))$. Multiplication by U takes staircase to staircase, but we suppress the dotted arrows to avoid obscuring the picture. Although $\tau\big(T(2,7)\# - T(3,4)\big) = 0$, the summand C_0 (shaded red) has its generator of homology with Alexander grading 1 (above the dashed line), so $d_1\big(T(2,7)\# - T(3,4)\big) \neq 0$ (in fact, $d_1 = -2$).

Corollary 4.2.7. *If K_1 and K_2 are two L-space knots whose Alexander polynomials are distinct but have the same degree, then*

$$\tau(K_1\# -K_2) = \tau(-K_1\# K_2) = 0,$$

but either

$$d_1(K_1\# -K_2) \neq 0 \quad \textit{or} \quad d_1(-K_1\# K_2) \neq 0.$$

In particular, d_1 gives a stronger four-genus bound than τ for $K_1\# -K_2$ and its mirror.

For a particular class of L-space knots, Theorem 4.2.4 can be strengthened. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be an irreducible polynomial such that $f(0) = 0$. Let Z be the curve in $\mathbb{C}^2$ through the origin which is the zero set of f, and let B be a small ball containing the origin in $\mathbb{C}^2$. Then $K = Z \cap \partial B$ is called an *algebraic knot*. Algebraic knots are torus knots or, more generally, iterated cables with cabling parameters (p_k, q_k) satisfying $q_k > p_k p_{k-1} q_{k-1}$ [32]. It was shown by Lê [32] that algebraic knots are determined up to isotopy by their Alexander polynomial. With this result, we can provide an altenate proof of a corollary proved by Lê.

Corollary 4.2.8 (Lê)**.** *Algebraic knots are concordant if and only if they are isotopic.*

Proof. It was shown by Moser [43] that torus knots are lens space knots, and more generally it follows from work of Hedden [21] that the iterated cables specified by Lê are L-space knots. By Theorem 4.2.4, if two L-space knots are concordant, they have equal Alexander polynomials. Lê showed that the Alexander polynomial determines the knot type for algebraic knots. □

4.3 Examples

Example 1. The example illustrated in Figure 4.6 can be generalized to the knots

$$K_p := T\left(2, p(p-1)+1\right) \# -T(p, p+1).$$

By examining the Alexander polynomials of torus knots, it can be seen that the lengths of the staircase for $T(p, p+1)$ are $\{1, 2, \cdots, p-1\}$, whereas the staircase for $T(2, q)$ has $\frac{q-1}{2}$ steps, all of length 1.

From this it can be seen (see Figure 4.7 for an example) that the generator of homology of the complex C_0 has Alexander grading

$$\begin{aligned} A = & \max_k \sum_{i=1}^{k} (p-i) - i \\ = & \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} p - 2i \\ = & \left\lfloor \frac{p}{2} \right\rfloor \left(p - \left\lfloor \frac{p}{2} \right\rfloor - 1\right). \end{aligned} \tag{4.9}$$

In general, showing that the Alexander grading of this generator is positive only shows that $d_1 \leq 0$, but in this case we can get an explicit value with relative ease. Roughly speaking, this is because the map U on $\underline{CFK}^-(T(2, p(p-1)+1))$ decreases the Alexander grading by 2 (at least on elements with high enough Maslov grading), and of course also decreases the

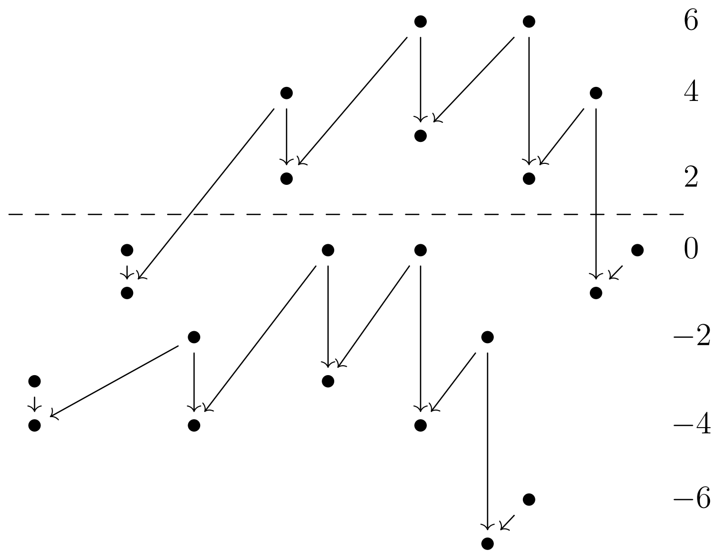


Figure 4.7: **Complexes for the knot K_p.** A portion of the complex $\underline{CFK^-}(T(2,31)) \otimes_{\mathbb{F}[U]} CFK^-(-T(6,7))$. The upper summand shown is C_0, and its generator of homology has Alexander grading 6. The generators of homology for C_{-2} and C_{-4} (not shown) are also above the dashed line. The generator of homology of the summand C_{-6}, the lower summand shown, has the maximal Maslov grading of any generator below the dashed line, so $d_1(K_6) = -6$.

Maslov grading by 2. So, in fact, the grading in (4.9) is exactly $-d_1$. That is, for any $p > 1$,

$$\tau(K_p) = 0, \qquad \text{but} \qquad d_1(K_p) = \begin{cases} -\frac{p^2-2p}{4} & p \ \text{ even}, \\ -\left(\frac{p-1}{2}\right)^2 & p \equiv 1 \mod 4, \\ -\left(\frac{p-1}{2}\right)^2 - 1 & p \equiv 3 \mod 4. \end{cases} \tag{4.10}$$

It should be pointed out that while d_1 has more to say than τ for these knots, the knot signature σ gives an even better topological four-genus bound (at least for $p > 5$).

Example 2. Even among sums of torus knots, however, we can find examples for which τ, σ and Rasmussen's s invariant defined using Khovanov homology [65] are all equal to zero, and $|d_1|$ is arbitrarily large (for all knots discussed in this thesis, $s = 2\tau$). Define

$$J_p^+ := T(2, 8p+1) \# T(4p, 4p+1),$$

and then let

$$J_p := J_p^+ \# - T(4p+1, 4p+2).$$

A direct computation using (4.6) and, for example, [19, Theorem 5.2] shows that

$$\tau(J_p) = \sigma(J_p) = 0 \quad \text{for all } p > 0.$$

The staircases of the individual torus knot summands here have the type described in Example 1. The sum of two L-space knots, as we have seen, is not an L-space knot. However, its reduced complex has an acyclic subcomplex which is U-torsion, and the corresponding quotient complex is isomorphic to $\mathbb{F}[U]$. Since U-torsion elements are not relevant to the computation of d-invariants, this means we can treat the sum of staircases as a staircase, if it is only the d-invariants we are interested in (see [2, Section 5] and [3, Section 2.4], where Borodzik and Livingston discuss the gap functions of connected sums of algebraic knots, for an alternate point of view). That is to say, this quotient complex is filtered isomorphic to the reduced complex corresponding to some staircase, which we may call the "representative staircase".

If one of the summands is $T(2, n)$, the resulting representative staircase can be obtained relatively simply. In the case at hand, the representative staircase for J_p^+ is given by

$$\{\overbrace{1, \cdots, 1}^{2p^2+5p}, 3, \overbrace{1, \cdots, 1}^{2p-2}, 5, \overbrace{1, \cdots, 1}^{2p-3}, \cdots, 4p-5, 1, 1, 4p-3, 1, 4p-1\}.$$

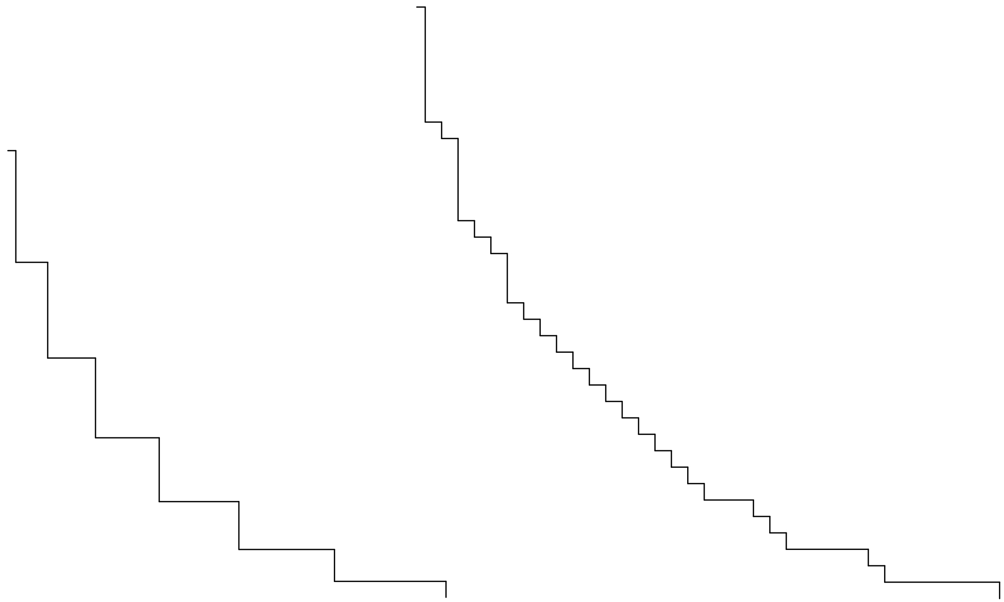


Figure 4.8: **A representative staircase.** On the left is the staircase for the torus knot $T(8,9)$ (the horizontal lengths range from 1 to 7, in order). On the right is the representative staircase corresponding to the knot $T(2,17)\#T(8,9)$, which we have called J_2^+. This representative staircase contains all of the generators which are relevant for computing d-invariants. Here, every point where two line segments meet is assumed to be a generator (a "dot"), and every line segment is assumed to be an arrow pointing down or to the left.

An example of this staircase is shown in Figure 4.8, for the case $p = 2$. Recall that, by the symmetry of the Alexander polynomials of the summands of J_p^+, this is also the list of vertical lengths, from bottom to top. With this, and also knowing the upside-down staircase shape of $CFK^-(-T(4p+1, 4p+2))$, we can compute the Alexander gradings as we did above. This allows us to see that, for all $p > 0$,

$$d_1(J_p) = -2p. \tag{4.11}$$

We suppress the explicit computations here, but instead show the $p = 2$ case in Figure 4.9.

There is, more generally, a family of knots whose knot Floer complexes are the direct sum of a staircase and an acyclic complex. In addition to L-space knots, Petkova [62, Lemma

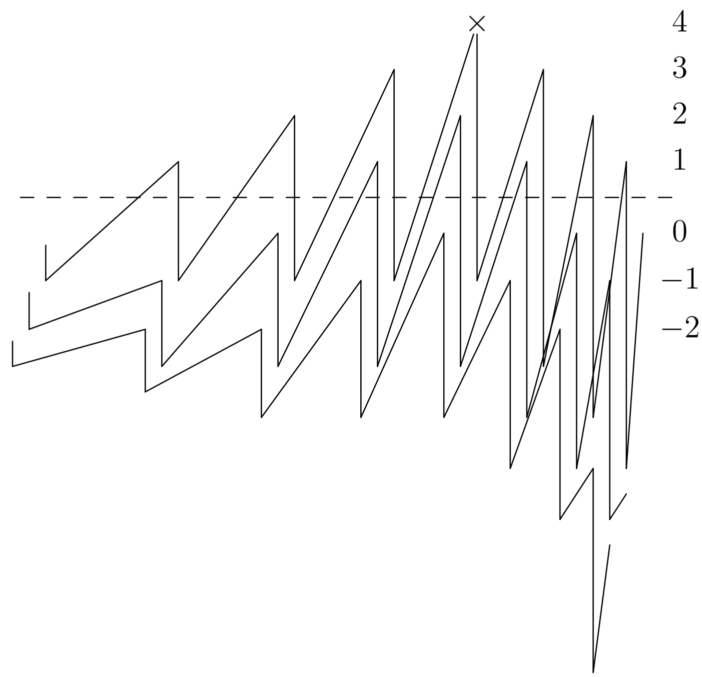


Figure 4.9: **Complexes for the knot J_p.** The portion of the complex $\underline{CFK}^-(J_p)$ which is relevant for computing $d_1(J_p)$, in the case where $p = 2$. The line segments are interpreted as in Figure 4.8. The uppermost summand is C_0. From right to left, its generators start with an Alexander grading of 0, and increase by 1 until reaching an Alexander grading of $2p$ (the generator marked with an $\times$). Multiplying by U takes staircase to staircase, and notice that the Alexander grading of the $\times$ generator decreases by 2 each time. This is the basic idea behind showing that $d_1(J_p) = -2p$.

7] showed that "Floer homologically thin" knots – which include alternating and quasi-alternating knots, as well as a family of hyperbolic knots found in [20] – are in this family (the staircase for a Floer homologically thin knot is the same as that of some $(2, n)$-torus knot). Since d-invariants are defined in terms of non-torsion generators of homology, the acyclic summands have no effect on d_1; it is determined solely by the staircase summand. Therefore, if K_1 and K_2 are knots in this family, then $d_1(K_1\# - K_2)$ can be computed as it was in the above examples for torus knots. In particular, if the staircases of K_1 and K_2 have different shapes, these knots are not concordant.

As an interesting further application of these ideas, one could investigate the linear independence of a family of knots in the smooth concordance group. As an example, if we let $T(r, s)_{p,q}$ denote the (p, q)-cable of $T(r, s)$, the knots

$$K_1 = T(2, 3)_{2,3}\#T(2, 5) \quad \text{and} \quad K_2 = T(2, 3)_{2,5}\#T(2, 3)$$

can be shown to be linearly independent using d_1, although $\Delta_{K_1}(T) = \Delta_{K_2}(T)$ (showing that Corollary 4.2.7 does not extend to *sums* of L-space knots). In contrast, the knots

$$T(2, 3)_{2,13}\#T(2, 15) \quad \text{and} \quad T(2, 3)_{2,15}\#T(2, 13)$$

cannot be distinguished in the concordance group using d_1 (that is, d_1 cannot obstruct the sliceness of the "Livingston-Melvin" knot [37]). So, while Theorem 4.2.4 settles the question of when two L-space knots are concordant, it would be interesting to understand which families of L-space knots can be shown to be independent using the d_1 invariant.

-BIBLIOGRAPHY

-BIBLIOGRAPHY

[1] J. Batson, *Nonorientable four-ball genus can be arbitrarily large*, preprint, available at http://arxiv.org/abs/1204.1985.

[2] M. Borodzik and C. Livingston, *Heegaard Floer homology and rational cuspidal curves*, preprint, available at http://arxiv.org/abs/1304.1062.

[3] ———, *Semigroups, d-invariants and deformations of cuspidal singular points of plane curves*, preprint, available at http://arxiv.org/abs/1305.2868.

[4] J. C. Cha and D. Ruberman, *Concordance to links with unknotted components*, Algebr. Geom. Topol. **12** (2012), no. 2, 963–977. MR 2928901

[5] T. Cochran and R. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology \$3\$-spheres and property \$P\$*, Topology **27** (1988), no. 4, 495–512.

[6] M. I. Doig, *Finite knot surgeries and Heegaard Floer homology*, preprint, available at http://arxiv.org/abs/1201.4187.

[7] ———, *Obstructing finite surgery*, preprint, available at http://arxiv.org/abs/1302.6130.

[8] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), no. 2, 279–315.

[9] ———, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **26** (1987), no. 1, 141–168. MR 892034 (88j:57035)

[10] H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology and its Applications **63** (1995), no. 3, 257–262.

[11] P. Feehan and T Leness, *Witten's conjecture for many four-manifolds of simple type*, preprint, available at http://arxiv.org/abs/math/0609530.

[12] A. Floer, *An instanton-invariant for \$3\$-manifolds*, Communications in Mathematical Physics **118** (1988), no. 2, 215–240.

[13] ———, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), no. 3, 513–547.

[14] ———, *Instanton homology, surgery, and knots*, Geometry of low-dimensional manifolds, 1 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 150, Cambridge Univ. Press, Cambridge, 1990, pp. 97–114. MR 1171893 (93h:57049)

[15] R. Fox and J. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka Journal of Mathematics **3** (1966), 257–267.

[16] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

[17] P. M. Gilmer and C. Livingston, *The nonorientable 4-genus of knots*, J. Lond. Math. Soc. (2) **84** (2011), no. 3, 559–577. MR 2855790 (2012j:57010)

[18] R. Gompf, *Smooth concordance of topologically slice knots*, Topology **25** (1986), no. 3, 353–373. MR 842430 (87i:57004)

[19] C. McA. Gordon, R. A. Litherland, and K. Murasugi, *Signatures of covering links*, Canad. J. Math. **33** (1981), no. 2, 381–394. MR 617628 (83a:57006)

[20] J. E. Greene and L. Watson, *Turaev torsion, definite 4-manifolds, and quasi-alternating knots*, Bull. Lond. Math. Soc. **45** (2013), no. 5, 962–972. MR 3104988

[21] M. Hedden, *On knot Floer homology and cabling. II*, International Mathematics Research Notices (2009), no. 12, 2248–2274.

[22] M. Hedden, C. Livingston, and D. Ruberman, *Topologically slice knots with nontrivial Alexander polynomial*, Adv. Math. **231** (2012), no. 2, 913–939. MR 2955197

[23] M. Hedden and Y. Ni, *Khovanov module and the detection of unlinks*, preprint, available at http://arxiv.org/abs/1204.0960.

[24] P. Heegaard, *Sur l' "Analysis situs"*, Bull. Soc. Math. France **44** (1916), 161–242. MR 1504754

[25] J. Hom, *The knot floer complex and the smooth concordance group*, preprint, available at http://arxiv.org/abs/1111.6635.

[26] ———, *A note on cabling and L-space surgeries*, Algebraic & Geometric Topology **11** (2011), 219–223.

[27] S. Jabuka and S. Naik, *Order in the concordance group and Heegaard Floer homology*, Geom. Topol. **11** (2007), 979–994. MR 2326940 (2009d:57008)

[28] A. Juhász, *Holomorphic discs and sutured manifolds*, Algebraic & Geometric Topology **6** (2006), 1429–1457.

[29] A. Juhász and D. Thurston, *Naturality and mapping class groups in Heegaard Floer homology*, preprint, available at http://arxiv.org/abs/1210.4996.

[30] P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs, vol. 10, Cambridge University Press, Cambridge, 2007. MR 2388043 (2009f:57049)

[31] C. Kutluhan, Y. Lee, and C. Taubes, *HF=HM I : Heegaard Floer homology and Seiberg–Witten Floer homology*, preprint, available at http://arxiv.org/abs/1007.1979.

[32] Lê D. T., *Sur les noeuds algébriques*, Compositio Math. **25** (1972), 281–321. MR 0320000 (47 #8541)

[33] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geometry & Topology **10** (2006), 955–1097.

[34] R. Lipshitz, P. Ozsváth, and D. Thurston, *Bimodules in bordered Heegaard Floer homology*, Tech. report.

[35] ———, *Bordered Heegaard Floer homology: Invariance and pairing*, Tech. report.

[36] ———, *Heegaard Floer homology as morphism spaces*, Quantum Topol. **2** (2011), no. 4, 381–449. MR 2844535

[37] C. Livingston and P. Melvin, *Algebraic knots are algebraically dependent*, Proc. Amer. Math. Soc. **87** (1983), no. 1, 179–180. MR 677257 (84a:57004)

[38] C. Manolescu and B. Owens, *A concordance invariant from the Floer homology of double branched covers*, Int. Math. Res. Not. IMRN (2007), no. 20, Art. ID rnm077, 21. MR 2363303 (2008k:57003)

[39] C. Manolescu, P. Ozsváth, and S. Sarkar, *A combinatorial description of knot Floer homology*, Annals of Mathematics. Second Series **169** (2009), no. 2, 633–660.

[40] C. Manolescu, P. Ozsváth, Z. Szabó, and D. Thurston, *On combinatorial link Floer homology*, Geom. Topol. **11** (2007), 2339–2412. MR 2372850 (2009c:57053)

[41] C. Manolescu, P. Ozsváth, and D. Thurston, *Grid diagrams and Heegaard Floer invariants*, preprint, available at http://arxiv.org/abs/0910.0078.

[42] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968. MR 0239612 (39 #969)

[43] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.

[44] K. Murasugi, *On the signature of links*, Topology **9** (1970), 283–298. MR 0261585 (41 #6198)

[45] Y. Ni, *Knot Floer homology detects fibred knots*, Inventiones Mathematicae **170** (2007), no. 3, 577–608.

[46] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in s^3*, preprint, available at http://arxiv.org/abs/1009.4720.

[47] ———, *Heegaard floer correction terms and rational genus bounds*, preprint, available at http://arxiv.org/abs/1205.7053.

[48] P. Ozsváth and Z. Szabó, *Knot Floer homology and rational surgeries*, Tech. report.

[49] ———, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics **173** (2003), no. 2, 179–261.

[50] ———, *Knot Floer homology and the four-ball genus*, Geometry & Topology **7** (2003), 615–639.

[51] ———, *Holomorphic Disks and Genus Bounds*, Geometry & Topology **8** (2004), 311–334.

[52] ———, *Holomorphic disks and knot invariants*, Advances in Mathematics **186** (2004), no. 1, 58–116.

[53] ——, *Holomorphic disks and three-manifold invariants: properties and applications*, Annals of Mathematics. Second Series **159** (2004), no. 3, 1159–1245.

[54] ——, *Holomorphic disks and topological invariants for closed three-manifolds*, Annals of Mathematics. Second Series **159** (2004), no. 3, 1027–1158.

[55] ——, *Heegaard Floer homology and contact structures*, Duke Mathematical Journal **129** (2005), no. 1, 39–61.

[56] ——, *On knot Floer homology and lens space surgeries*, Topology. An International Journal of Mathematics **44** (2005), no. 6, 1281–1300.

[57] ——, *Holomorphic triangles and invariants for smooth four-manifolds*, Advances in Mathematics **202** (2006), no. 2, 326–400.

[58] ——, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebraic & Geometric Topology **8** (2008), no. 2, 615–692.

[59] ——, *Knot Floer homology and integer surgeries*, Algebraic & Geometric Topology **8** (2008), no. 1, 101–153.

[60] T. Perutz, *Hamiltonian handleslides for Heegaard Floer homology*, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 15–35. MR 2509747 (2010m:57048)

[61] T. D. Peters, *A concordance invariant from the floer homology of +/- 1 surgeries*, preprint, available at http://arxiv.org/abs/1003.3038.

[62] I. Petkova, *Cables of thin knots and bordered Heegaard Floer homology*, Quantum Topol. **4** (2013), no. 4, 377–409. MR 3134023

[63] J. Rasmussen, *Floer homology of surgeries on two-bridge knots*, Algebr. Geom. Topol. **2** (2002), 757–789 (electronic). MR 1928176 (2003g:57049)

[64] ——, *Floer homology and knot complements*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)–Harvard University. MR 2704683

[65] ——, *Khovanov homology and the slice genus*, Inventiones Mathematicae **182** (2010), no. 2, 419–447.

[66] S. Sarkar and J. Wang, *An algorithm for computing some Heegaard Floer homologies*, Ann. of Math. (2) **171** (2010), no. 2, 1213–1236. MR 2630063 (2012f:57032)

[67] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264. MR 0248854 (40 #2104)

[68] F. Vafaee, *On the knot Floer homology of twisted torus knots*, preprint, available at http://arxiv.org/abs/1311.3711.

[69] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. MR 1306021 (96d:57035)