BOUNDARY VALUE PROBLEMS AND PERIODIC SOLUTIONS FOR CONTINGENT DIFFERENTIAL EQUATIONS

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ABSTRACT

BOUNDARY VALUE PROBLEMS AND PERIODIC SOLUTIONS
FOR CONTINGENT DIFFERENTIAL EQUATIONS

By

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In this thesis, we shall investigate contingent differential equations in which the orientor fields $F(t,x)$ satisfy the Carathéodory conditions, i.e. $F(t,x)$ is measurable in t for each fixed x, $F(t,x)$ is upper semi-continuous in x for each fixed t and $F(t,x)$ is integrably bounded on every compact subset of $R \times R^n$.

We begin our investigation with the fundamental theory of such equations. Two similar Kamke-type convergence theorems are proved. Following from the convergence theorems are the properties of continuous dependence of solutions on initial conditions and parameters.

We then study the general boundary value problems of contingent differential equations. An existence theorem like Fredholm's alternative is proved by using a fixed point theorem which we formulate with degree theory. As the boundary conditions require only linearity and continuity, applications

can be obtained on periodic solutions, Nicoletti problems and aperiodic boundary value problems. We observe also that the set of solutions is compact in the space of continuous functions. Therefore, optiomal solutions do exist with respect to any continuous (or semi-continuous) functionals.

In case the orientor fields are functional and T-periodic for some $T > 0$, we have contingent equations in which a finite time lag $r \geq 0$ is involved. A T-periodic set-valued transformation is set up from the space $\mathcal{C}_T$ of continuous T-periodic functions into the space of non-empty, compact and convex subsets of $\mathcal{C}_T$ so that the previous fixed point theorem can be applied. Thus we obtain an existence theorem of periodic solutions for contingent functional differential equations.

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By

Wei-Hwa Shaw

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DEDICATED TO MY MOTHER

AND

IN THE MEMORY OF MY FATHER

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INTRODUCTION

A relation of the form

$$\text{(E)} \qquad \dot{x}(t) = f(t,x(t))$$

where $x = x(t)$ is an n-dimensional vector valued function defined on a real interval and $f(t,x)$ is a function from a certain region of $R \times R^n$ into R^n, is called an <u>ordinary differential equation</u>. The function f is called a <u>vector field</u> and the <u>solutions</u> of (E) are curves with their tangents prescribed by the vector field f.

It may happen that the right-hand side of the system (E) is approximately known up to a given accuracy. If this is the case, then we have to consider differential systems with multi-valued right-hand sides.

Instead of a single-valued mapping, we consider a multi-valued mapping F, called an orientor field, which associates with every point (t,x) of a certain region $W \subset R \times R^n$ a non-empty, convex subset $F(t,x)$ of R^n. For any function $x(t)$ from an open subset $J \subset R$ into R^n, we have the following definition:

Definition 0.1. The set denoted by $D^*x(t)$ and defined by

$$D^*x(t) = \{u(t) \in R^n : \text{there exists a sequence } \{t_k\}_{k=1}^{\infty} \subset J,\ t_k \neq t \text{ and } t_k \xrightarrow{k} t \text{ such that } \frac{x(t_k)-x(t)}{t_k - t} \xrightarrow{k} u(t)\}$$

is called the contingent derivative of x at t.

Consider the differential system

$$(C_1) \qquad D^*x(t) \subset F(t,x(t)),$$

where $F(t,x)$ is an orientor field and $x(t)$ a mapping from some real interval J into R^n. Such an $x(t)$ will be called a solution of (C_1) if $x(t)$ satisfies (C_1) for t a.e. in J.

When F is bounded and continuous in the sense of Hausdorff metric with $F(t,x)$ a non-empty, compact and convex subset of R^n for each $(t,x) \in W$, T. Ważewski has shown (see [37]) that (C_1) can be written as

$$(C_2) \qquad \dot{x}(t) \in F(t,x(t)) \quad \text{a.e.,}$$

where $\dot{x}(t) = \frac{dx(t)}{dt}$ is the usual derivative of x at t. A mapping $x(t)$ from some real interval J into R^n will be called a solution of (C_2) if $x(t)$ satisfies (C_2) for t a.e. in J.

The systems (C_1) and (C_2) are called <u>contingent differential equations</u>. This more general theory of differential systems was developed independently by A. Marchand ([24],[25]) and S.K. Zaremba ([40],[41]) in the mid'30's. Under the assumptions that the orientor field is bounded and continuous with each image a non-empty, compact and convex subset, they succeeded in obtaining some fundamental properties of solutions, e.g. the existence of solutions for the Cauchy problems, the compactness of solution funnel, the Kneser property and the Hukuhara property.

Another importance of contingent differential equations is their close connection with control problems. This was observed by T. Ważewski.

Let $\mathcal{P}(R^n)$ denote the collection of non-empty subsets of R^n.

<u>Definition 0.2</u>. By a <u>control system</u> $S(f,C)$ we mean a pair: a function $f(t,x,u) : R \times R^n \times R^m \to R^n$ and a field $C(t,x) : R \times R^n \to \mathcal{P}(R^m)$. $C(t,x)$ is called the <u>control domain</u> of $S(f,C)$.

<u>Definition 0.3</u>. A function $x = x(t)$ from an interval $J \subset R$ into R^n is said to be a <u>trajectory</u> of $S(f,C)$, if $x(t)$ is absolutely continuous on J and there exists a function $u(t)$ such that

(i) $\dot{x}(t) = f(t,x(t),u(t))$ for almost all $t \in J$;

(ii) $u(t)$ is measurable on J; and

(iii) $u(t) \in C(t,x(t))$ for almost all $t \in J$.

$u(t)$ is called the <u>control function</u> corresponding to the trajectory $x(t)$.

Now, given a control system $S(f,C)$, one can eliminate the control term u and get an orientor field. That is, define a function $F(t,x) : R \times R^n \to \mathcal{P}(R^n)$ by

$$F(t,x) = \{v \in R^n : v = f(t,x,u),\ u \in C(t,x)\}.$$

This orientor field F is called the <u>orientor field associated with the control system</u> $S(f,C)$.

If a suitable implicit function theorem is provided, solving a control system is equivalent to solving the contingent differential equation in which the orientor field is the orientor field associated with this control system. That is, if we know a trajectory of the orientor field associated with some control system, then a suitable implicit function theorem will enable us to find its corresponding control function. Such implicit function theorems have been discussed in [35], [8] and [36].

A good survey of the early works is [38].

In this thesis, we shall investigate contingent differential equations in which the orientor fields $F(t,x)$

satisfy the Carathéodory conditions, i.e. $F(t,x)$ is measurable in t for each fixed x, $F(t,x)$ is upper semi-continuous in x for each fixed t and $F(t,x)$ is integrably bounded on every compact subset of $R \times R^n$.

Chapter I gives preliminaries. Here upper semi-continuity, measurability, integral and topological degree of set-valued mappings are discussed.

Chapter II deals with the fundamental theory. We prove two similar Kamke-type convergence theorems (compare with Theorem 3.2 of [12], p.14). Following from the convergence theorems are the properties of continuous dependence of solutions on initial conditions and parameters.

Chapter III is a study of general boundary value problems. An existence theorem like Fredholm's alternative is proved by using a fixed point theorem which we formulate with degree theory. Several applications are also included to show how our existence theorem could be applied to prove the existence of solutions satisfying periodic conditions, Nicoletti conditions or aperiodic conditions as well as the existence of optimal solutions.

In the last chapter, we conclude this thesis by using the same fixed point theorem of Chapter III to establish an existence theorem for periodic solutions of contingent functional differential equations in which a finite time lag $r \geq 0$ is involved.

For reading convenience, a hollow square, $\square$, is used to signal the end of a proof.

Chapter I

SET-VALUED MAPPINGS

§1. Set-valued mappings and upper semi-continuity:

Let X be a metric space, E be a Banach space and $\mathcal{P}(E)$ be the collection of all non-empty subsets of E.

Definition 1.1. A set-valued mapping with domain $A \subset X$ into E is a mapping

$$F : A \to \mathcal{P}(E).$$

The range of F is defined to be

$$F(A) = \bigcup\{F(x) : x \in A\}.$$

It is natural that the next thing we shall do is to define the concept of continuity of a set-valued mapping. If we restrict $\mathcal{P}(E)$ to Comp (E) - the collection of non-empty compact subsets of E, the Hausdorff continuity induced by the Hausdorff metric on Comp (E) can be imposed upon a set-valued mapping from A into Comp (E). However, this continuity is too strong for our purposes. Instead, we shall introduce a concept of upper semi-continuity which is weaker. Upper semi-continuity can be defined in many ways. But first, we begin with the limit inferior and limit superior of a sequence of sets.

<u>Definition 1.2</u>. Let $\{A_k\}_{k=1}^{\infty} \subset E$ be a sequence of subsets of a Banach space E. Then

$$\liminf_{k\to\infty} A_k = \{x \in E : \text{every neighborhood of } x \text{ intersects all the } A_k\text{'s with } k \text{ sufficiently large}\}$$

and

$$\limsup_{k\to\infty} A_k = \{x \in E : \text{every neighborhood of } x \text{ intersects infinitely many } A_k\text{'s}\}.$$

If $\liminf_{k\to\infty} A_k = \limsup_{k\to\infty} A_k = A$, then we say $\{A_k\}_{k=1}^{\infty}$ <u>converges</u> to A and write $\lim_{k\to\infty} A_k = A$ or $A_k \xrightarrow{k} A$.

<u>Remark 1.1</u>. If d denotes the metric induced by the norm in E, then it is equivalent to define $\liminf_{k\to\infty} A_k$ and $\limsup_{k\to\infty} A_k$ as

$$\liminf_{k\to\infty} A_k = \{x \in E : \lim d(x,A_k) = 0 \text{ for } k \to \infty\}$$

and

$$\limsup_{k\to\infty} A_k = \{x \in E : \liminf d(x,A_k) = 0 \text{ for } k \to \infty\}.$$

<u>Remark 1.2</u>. If $\{A_{k'}\}_{k'=1}^{\infty}$ is a subsequence of $\{A_k\}_{k=1}^{\infty}$, it is easy to see that

$$\liminf_{k\to\infty} A_k \subset \liminf_{k'\to\infty} A_{k'} \subset \limsup_{k'\to\infty} A_{k'} \subset \limsup_{k\to\infty} A_k.$$

Definition 1.3. A mapping $F : A \to \mathcal{P}(E)$ is said to be upper semi-continuous at $x_0 \in A$ in the sense of limit superior if

$$F(x_0) \supset \limsup_{n\to\infty} F(x_n)$$

for any sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \xrightarrow{n} x_0$.

F is upper semi-continuous in the sense of limit superior if F is upper semi-continuous at every $x \in A$ in the sense of limit superior.

Definition 1.4. A mapping $F : A \to \mathcal{P}(E)$ is said to be upper semi-continuous at $x_0 \in A$ in the sense of metric if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$F(x) \subset B_\epsilon(F(x_0))$$

for all $x \in A$ with $|x-x_0| < \delta$, where $B_\epsilon(D) = \{x \in E : d(x,D) < \epsilon\}$.

F is upper semi-continuous in the sense of metric if F is upper semi-continious at every $x \in A$ in the sense of metric.

Definition 1.5. A mapping $F : A \to \mathcal{P}(E)$ is said to be upper semi-continuous at $x_0 \in A$ in the sense of Kuratowski if $x_n \xrightarrow{n} x_0$, $y_n \xrightarrow{n} y_0$ and $y_n \in F(x_n)$ imply $y_0 \in F(x_0)$.

F is upper semi-continuous in the sense of Kuratowski if F is upper semi-continuous at every $x \in A$ in the sense of Kuratowski.

Remark 1.3. It is easy to see that F is upper semi-continuous in the sense of Definition 1.5 if and only if the graph of F, $\Gamma(F) = \{(x,y) : y \in F(x)\} \subset A \times E$, is closed in $A \times E$.

Definition 1.6. A mapping $F : A \to \mathcal{P}(E)$ is said to be upper semi-continuous at $x_0 \in A$ in the sense of topology if for each open set $U \supset F(x_0)$, there exists a $\delta > 0$ such that $|x-x_0| < \delta$ implies $F(x) \subset U$.

F is upper semi-continuous in the sense of topology if F is upper semi-continuous at every $x \in A$ in the sense of topology.

Proposition 1.1. Let $F : A \to \mathcal{P}(E)$. Assume that for $x_0 \in A$, there exists a neighborhood $N(x_0)$ of x_0 such that $\bigcup_{x \in N(x_0)} F(x)$ is relatively compact, i.e. $\overline{\bigcup_{x \in N(x_0)} F(x)}$ is compact. Then, F is upper semi-continuous at x_0 in the sense of Definition 1.3 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.4.

Proof: Let $\epsilon > 0$ be given. Suppose F is not upper semi-continuous at x_0 in the sense of Definition 1.4. Then, for each $\delta > 0$, there exists $x \in A$ such that $|x-x_0| < \delta$ but $F(x) \not\subset B_\epsilon(F(x_0))$. Hence, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \xrightarrow{n} x_0$ and there exists

$y_n \in F(x_n)$ with $d(y_n, F(x_0)) \geq \varepsilon$. Since $x_n \xrightarrow{n} x_0$, $x_n \in N(x_0)$ for all n sufficiently large. Hence, $y_n \in \bigcup_{x \in N(x_0)} F(x)$ for all n sufficiently large. It follows from the relative compactness of $\bigcup_{x \in N(x_0)} F(x)$ that $\{y_n\}_{n=1}^{\infty}$ has at least a limit point, say y_0. It is clear that $y_0 \in \limsup F(x_n)$. Let $\{y_j\}_{j=1}^{\infty}$ be a subsequence of $\{y_n\}_{n=1}^{\infty}$ such that $y_j \xrightarrow{j} y_0$. It follows from the triangle inequality of d that

$$d(y_0, F(x_0)) \geq |d(y_j, F(x_0)) - d(y_j, y_0)|$$

for all j. Letting $j \to \infty$, we have $d(y_0, F(x_0)) \geq \varepsilon$. Hence, $y_0 \notin F(x_0)$. Therefore, F is not upper semi-continuous at x_0 in the sense of Definition 1.3. $\square$

<u>Example 1.1</u>. The assumption that F be locally relatively compact at x_0 is necessary in Proposition 1.1.

Let R be the space of real numbers with usual metric and let ℓ^{∞} be the space of all bounded sequences of real numbers with the norm of each element $\{\xi_n\}_{n=1}^{\infty}$, $\|\{\xi_n\}\| = \sup_n |\xi_n|$. Let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset R$.

Consider $F : A \to \ell^{\infty}$ defined by

$$F(0) = \{\xi_k\}_{k=1}^{\infty} \text{ with } \xi_k = 0 \text{ for all } k$$

$$F(\tfrac{1}{n}) = \{\xi_k\}_{k=1}^{\infty} \text{ with } \xi_k = 0 \text{ if } k \neq n \text{ and } \xi_k = 1 \text{ if } k = n.$$

As we observe that the unit sphere of ℓ^∞ is not compact, F is not locally relatively compact at 0. F is upper semi-continuous at 0 in the sense of Definition 1.3 since $\limsup_{n\to\infty} F(\frac{1}{n}) = \emptyset \subset F(0)$. However, $d(F(\frac{1}{n}), F(0)) = 1$ for all $n \geq 1$. F is not upper semi-continuous at 0 in the sense of Definition 1.4.

<u>Proposition 1.2</u>. Let $F : A \to \mathcal{P}(E)$. If $F(x_0)$ is closed for some $x_0 \in A$, then F is upper semi-continuous at x_0 in the sense of Definition 1.4 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.3.

<u>Proof</u>: Let $\{x_n\}_{n=1}^\infty$ be any sequence in A such that $x_n \xrightarrow{n} x_0$. By Definition 1.4, for each $\varepsilon > 0$, there exists an $N > 0$ such that $F(x_n) \subset B_\varepsilon(F(x_0))$ for all $n \geq N$. Let $y \in \limsup F(x_n)$ be arbitrary, i.e. there exists a subsequence $\{x_j\}_{j=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ with $y_j \in F(x_j)$ such that $y_j \xrightarrow{j} y$. Thus $y \in B_\varepsilon(F(x_0))$. Since $\varepsilon > 0$ is arbitrary and $F(x_0)$ is closed, $y \in F(x_0)$. □

<u>Example 1.2</u>. The assumption that $F(x_0)$ be closed is necessary in Proposition 1.2.

Consider $F : R \to \mathcal{P}(R)$ defined by

$$F(x) = (x-1, x+1) = \{t \in R : x-1 < t < x+1\}.$$

Let $x_0 \in R$ be arbitrary. It is clear that F is upper semi-continuous at x_0 in the sense of Definition 1.4

(take $\delta = \varepsilon$). Let $x_n = x_0 + \frac{1}{n}$. Then $x_n \xrightarrow{n} x_0$. However, $\limsup F(x_n) = [x_0-1, x_0+1] \not\subset (x_0-1, x_0+1) = F(x_0)$.

<u>Proposition 1.3</u>. Let $F : A \to \mathcal{P}(E)$. If $F(x_0)$ is closed for some $x_0 \in A$, then F is upper semi-continuous at x_0 in the sense of Definition 1.4 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.5.

<u>Proof</u>: Let $x_n \xrightarrow{n} x_0$, $y_n \xrightarrow{n} y_0$ and $y_n \in F(x_n)$. By Definition 1.4, for each $\varepsilon > 0$, there exists an $N_1(\varepsilon) > 0$ such that $F(x_n) \subset B_\varepsilon(F(x_0))$ for all $n \geq N_1$. Since $y_n \xrightarrow{n} y_0$, there exists an $N_2(\varepsilon) > 0$ such that $d(y_0, y_n) < \varepsilon/2$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Then

$$d(y_0, F(x_0)) \leq d(y_1, y_n) + d(y_n, F(x))$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \geq N$. Hence, $y_0 \in \overline{F(x_0)} = F(x_0)$ since $F(x_0)$ is closed. $\square$

<u>Example 1.3</u>. Proposition 1.3 is not true without the assumption that $F(x_0)$ be closed.

Consider $F : R \to \mathcal{P}(R)$ defined by

$$F(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ \{1\} & \text{if } x > 0 \\ (-1,1) & \text{if } x = 0. \end{cases}$$

$F(0)$ is not closed. It is easy to see that F is upper semi-continuous at 0 in the sense of Definition 1.4 but not in the sense of Definition 1.5.

Proposition 1.4. Let $F : A \to \mathcal{P}(E)$. Assume that F is locally relatively compact at some $x_0 \in A$. Then F is upper semi-continuous at x_0 in the sense of Definition 1.5 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.4.

Proof: Suppose F is not upper semi-continuous at x_0 in the sense of Definition 1.4. Let $N(x_0)$ be a neighborhood of x_0 such that $\overline{\bigcup_{x \in N(x_0)} F(x)}$ is compact. Then there exists an $\varepsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset N(x_0)$ such that $x_n \xrightarrow{n} x_0$ and $F(x_n) \not\subset B_{\varepsilon}(F(x_0))$. Hence, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in F(x_n)$ and $d(y_n, F(x_0)) \geq \varepsilon$. Since $\{x_n\}_{n=1}^{\infty} \subset N(x_0)$, $\{y_n\}_{n=1}^{\infty}$ is contained in a compact subset of E by our assumption on $N(x)$. Therefore, there exists a subsequence $\{y_{n'}\}_{n'=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $y_{n'} \xrightarrow{n'} y_0$ for some $y_0 \in E$. Considering the inequality

$$d(y_0, F(x_0)) \geq |f(y_{n'}, f(x_0)) - d(y_{n'}, y_0)|$$

and letting $n' \to \infty$, we have $d(y_0, F(x_0)) \geq \varepsilon > 0$. Hence $y_0 \notin F(x_0)$. Therefore, F is not upper semi-continuous at x_0 in the sense of Definition 1.5. $\square$

Example 1.4. Proposition 1.4 is not true without the assumption that F be locally relatively compact at x_0 even if F is single-valued. Consider the following function

$$F : R \to R \text{ defined by}$$
$$F(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Evidently, F is upper semi-continuous in the sense of Definition 1.5 since its graph $\Gamma(F)$ is closed in R^2. However, F is not upper semi-continuous at 0 in the sense of Definition 1.4.

Proposition 1.5. Let $F : A \to \mathcal{P}(E)$ and $x_0 \in A$ be arbitrary. F is upper semi-continuous at x_0 in the sense of Definition 1.6 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.4.

Proof: For each $x_0 \in A$, $B_\epsilon(F(x_0))$ is an open set containing $F(x_0)$. Definition 1.4 follows immediately from Definition 1.6. □

Proposition 1.6. Let $F : A \to \mathcal{P}(E)$. Assume that $F(x_0)$ is compact for some $x_0 \in A$. Then, F is upper semi-continuous at x_0 in the sense of Definition 1.4 implies that F is upper semi-continuous at x_0 in the sense of Definition 1.6.

<u>Proof</u>: Let U be an open set containing $F(x_0)$. For each $y \in F(x_0)$, let $B_{r(y)}(y)$ be an open ball in E centered at y with radius $r(y)$ and $B_{r(y)}(y) \subset U$. Since $F(x_0)$ is compact, the covering $\{B_{\frac{r(y)}{2}}(y) : y \in F(x_0)\}$ has a finite subcovering, say $\{B_{\frac{r_1}{2}}(y_1), B_{\frac{r_2}{2}}(y_2), \ldots, B_{\frac{r_n}{2}}(y_n)\}$.

Let $r = \min\{r_1, \ldots, r_n\}$. We claim that $B_{\frac{r}{2}}(F(x_0)) \subset U$. Let $y \in F(x_0)$ be arbitrary and $y' \in B_{\frac{r}{2}}(y)$. In the finite subcovering, there exist y_i, $i \leq i \leq n$, such that $y \in B_{\frac{r_i}{2}}(y_i)$. Hence

$$\begin{aligned} d(y', y_i) &\leq d(y', y) + d(y, y_i) \\ &< \frac{r}{2} + \frac{r_i}{2} \\ &\leq r_i. \end{aligned}$$

Therefore, $y' \in B_{r_i}(y_i) \subset U$. Since y' is arbitrary in $B_{\frac{r}{2}}(y)$, $B_{\frac{r}{2}}(y) \subset U$. Also, $y \in F(x_0)$ is arbitrary. It follows that $B_{\frac{r}{2}}(F(x_0)) \subset U$ as desired. By Definition 1.4, there exists a $\delta = \delta(\frac{r}{2}) > 0$ such that

$$F(x) \subset B_{\frac{r}{2}}(F(x_0)) \subset U$$

for any $x \in A$ with $|x - x_0| < \delta$. Hence, F is upper semi-continuous at x_0 in the sense of Definition 1.6. $\square$

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Example 1.5. Without the assumption of the compactness of $F(x_0)$, Proposition 1.6 is not true even if $F(x_0)$ is closed.

Consider $F : R \to \mathcal{P}(R^2)$ defined by

$$F(x) = \{(y,z) : z = \frac{1}{y-x} > 0\}.$$

It is evident that F is upper semi-continuous in the sense of Definition 1.4 since $F(x_1)$ is only a translation of $F(x_2)$ for any $x_1, x_2 \in R$. $F(x)$ is closed for every $x \in R$. Let $x_0 \in R$ be arbitrary. Define

$$U = \{(y,z) : y > x_0,\ (y-x_0)z > 0\}.$$

Then U is open in R^2 and $F(x_0) \subset U$. For each $\delta > 0$, take x such that $0 < x_0 - x < \delta$. We find that

$$(x + \tfrac{1}{2}(x_0-x),\ \frac{1}{x + \frac{1}{2}(x_0-x) - x}) = (\tfrac{1}{2}(x_0+x),\ \frac{1}{\frac{1}{2}(x_0-x)}) \in F(x) \setminus U.$$

Hence, F is not upper semi-continuous at x_0 in the sense of Definition 1.6.

The following simple example also shows that Definition 1.6 does not imply Definition 1.5 in general:

Example 1.6. Consider $F : R \to \mathcal{P}(R)$ defined by

$$F(x) = (0,1) \quad \text{for all } x \in R.$$

From these examples, we see that the four definitions of upper semi-continuity are quite different. However, from the previous propositions, we find also that under certain conditions they are indeed equivalent.

Theorem 1.1. Let A be an open subset of a metric space and E be a Banach space. Suppose that $F : A \to \mathcal{P}(E)$ is a set-valued mapping satisfying:

(i) $F(x)$ is compact for every $x \in A$; and

(ii) for each bounded subset $D \subset A$, $F(D)$ is relatively compact.

Then, Definitions 1.3-1.6 of upper semi-continuity are all equivalent.

From now on, the set-valued mappings that we shall consider in our contingent equations satisfy the hypotheses of Theorem 1.1. Therefore, when we say upper semi-continuity, we mean any of the Definitions 1.3-1.6 with no ambiguilty.

Remark 1.3. Suppose we restrict $\mathcal{P}(E)$ to Comp (E) and consider mappings from A into Comp (E). It follows from Propositions 1.2, 1.3 and 1.6 that Definition 1.4 is the strongest among all.

For the sake of completeness, we introduce the following definitions of lower semi-continuity.

Definition 1.7. A mapping $F : A \to \text{Comp}(E)$ is said to be *lower semi-continuous at* $x_0 \in A$ *in the sense of limit inferior* if

$$F(x_0) \subset \liminf_{n\to\infty} F(x_n)$$

for any sequence $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \overset{n}{\to} x_0$.

F is *lower semi-continuous in the sense of limit inferior* if F is lower semi-continuous at every $x \in A$ in the sense of limit inferior.

Definition 1.8. A mapping $F : A \to \text{Comp}(E)$ is said to be *lower semi-continuous at* $x_0 \in A$ *in the sense of metric* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$F(x_0) \subset B_\varepsilon(F(x))$$

for all $x \in A$ with $|x-x_0| < \delta$.

F is *lower semi-continuous in the sense of metric* if F is lower semi-continuous at every $x \in A$ in the sense of metric.

Remark 1.4. In view of Remark 1.1, it is easy to see that Definitions 1.7 and 1.8 are equivalent.

Definition 1.9. A mapping $F : A \to \text{Comp}(E)$ is said to be *continuous at* $x_0 \in A$ *in the sense of metric* if F is both upper and lower semi-continuous at x_0 in the sense of metric.

F is <u>continuous in the sense of metric</u> if F is continuous at every $x \in A$ in the sense of metric.

<u>Proposition 1.7</u>. (Hukuhara [13]). Let E be any Banach space. The operations $+, \cdot,$ and $\cap$ have the following properties of continuity with respect to the product topology of the Hausdorff topologies induced by the Hausdorff metrics of R and Comp (E):

(i) the addition $+$: Comp $(E) \times$ Comp $(E) \to$ Comp (E) defined by

$$A + B = \{a+b : a \in A \text{ and } b \in B\}$$

is continuous;

(ii) the scalar multiplication $\cdot$: $R \times$ Comp $(E) \to$ Comp (E) defined by

$$\alpha A = \{\alpha a : a \in A\}$$

is continuous;

(iii) the intersection $\cap$: Comp $(E) \times$ Comp $(E) \to$ Comp (E) defined by

$$A \cap B = \{c : c \in A \text{ and } c \in B\}$$

is upper semi-continuous.

Let A be a bounded subset of a Banach space E. The norm of A, $|A|$, is defined to be $|A| = \sup\{|x| : x \in A\}$.

<u>Proposition 1.8</u>. Let F be a mapping from a subset A of a metric space X into Comp (E) and let D be a

closed, bounded and convex subset of the Banach space E such that $0 \in D$. If F is upper semi-continuous in the sense of Definition 1.4, then the mapping G from A into the space of non-empty, closed, bounded and convex subsets of E defined by

$$G(x) = |F(x)|D \quad \text{for all} \quad x \in A$$

is also upper semi-continuous in the sense of Definition 1.4.

Proof: Let F be upper semi-continuous in the sense of Definition 1.4 and $x_0 \in A$ be arbitrary. Since D is bounded, there exists a positive integer n such that $\frac{1}{n}|D| < 1$. Given $\varepsilon > 0$, by Definition 1.4, there exists a $\delta > 0$ such that $F(x) \subset B_{\frac{\varepsilon}{n}}(F(x_0))$ for all x with $|x-x_0| < \delta$. For any $x \in D$,

$$\begin{aligned}|F(x)|x &= (|F(x_0)| + \tfrac{\varepsilon}{n}) \cdot \frac{|F(x)|}{(|F(x_0)| + \frac{\varepsilon}{n})} x \\ &\in (|F(x_0)| + \tfrac{\varepsilon}{n})D\end{aligned}$$

since D is convex and contains 0. Hence, we have

$$\begin{aligned}|F(x)|D &\subset (|F(x_0)| + \tfrac{\varepsilon}{n})D \\ &\subset |F(x_0)|D + B_{\varepsilon}(0) \\ &= B_{\varepsilon}(|F(x_0)|D).\end{aligned}$$

Therefore, G is upper semi-continuous in the sense of Definition 1.4. $\square$

Let X, X', and X'' be complete metric spaces, F be a mapping from X into $\operatorname{Comp} X'$ and F' be a mapping from $\operatorname{Comp} X'$ into $\operatorname{Comp} X''$. The composite function G, denoted by $G = F'F$, is a mapping from X into $\operatorname{Comp} X''$ defined by

$$G(x) = F'(F(x)) \quad \text{for each} \quad x \in X.$$

In the following, we shall consider the continuity properties of composite functions. But first,

<u>Definition 1.10</u>. Let X and X' be two complete metric spaces. A mapping $F : \operatorname{Comp}(X) \to \operatorname{Comp}(X')$ is said to be <u>increasing</u> if for any $A, B \in \operatorname{Comp}(X)$, $A \subset B$ implies $F(A) \subset F(B)$.

<u>Example 1.7</u>. Let E be a Banach space. The mappings F and G from $\operatorname{Comp}(E) \times \operatorname{Comp}(E)$ into $\operatorname{Comp}(E)$ defined by

$$F(A,B) = \lambda A + \mu B \quad \text{where} \quad \lambda, \mu \in R$$

and $$G(A,B) = A \cap B$$

are all increasing.

<u>Proposition 1.9</u>. (Hukuhara [13]). If F and F' are upper semi-continuous (resp. lower semi-continuous) and moreover, if F' is increasing, then the composite function $G = F'F$ is also upper semi-continuous (resp. lower semi-continuous).

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In the above proposition, the assumption that F' be increasing is superfluous if F is continuous.

<u>Proposition 1.10.</u> (Hukuhara [13]). If F is continuous and F' is upper semi-continuous (resp. lower semi-continuous), then the composite function $G = F'F$ is upper semi-continuous (resp. lower semi-continuous).

§2. Measurability and integrals of set-valued mappings:

The set-valued mappings that we shall consider in this section are mappings from a subset A of R^m into Comp (R^n). It is known that for any finite dimensional Euclidean space R^m we can define a Lebesgue measure on R^m (see [33]: pp.49-53). Then with no difficulty, one can generalize the measurability of a single-valued mapping f from A into R^n to a set-valued mapping F from A into Comp (R^n).

<u>Definition 1.11</u>. A set-valued mapping F from a measurable subset A of R^m into Comp (R^n) is said to be <u>measurable</u> if the set $\{x \in A : F(x) \subset B\}$ is measurable for every closed subset B of R^n.

<u>Proposition 1.11</u>. (Hukuhara [14]). If $F : A \subset R^m \to \text{Comp}\ (R^n)$ is upper (resp. lower) semi-continuous in the sense of Definition 1.4 (resp. Definition 1.8), then F is measurable.

The following characterization of measurability of set-valued mappings is due to Plis:

<u>Theorem 1.2</u>. (Plis [30]). Let F be a mapping from a bounded measurable subset A of R^m into Comp (R^n). In order that F be measurable, it is necessary and sufficient that for each $\epsilon > 0$, there exists a closed subset A' of A such that F is continuous on A' and the measure of $A \setminus A'$ is less than ϵ.

The measurability of set-valued mappings can also be defined in the following way:

Let (S, Σ, μ) be a finite and positive measure space. And let d denote the Hausdorff metric on Comp (E).

<u>Definition 1.12</u>. A set-valued mapping $F : S \to$ Comp (R^n) is called <u>μ-simple</u> if it assumes only a finite number of values $K_1, K_2, \ldots, K_r \in$ Comp (R^n) and each of them on a μ-measurable set.

<u>Definition 1.13</u>. A set-valued mapping $F : S \to$ Comp (R^n) is called <u>μ-measurable</u> if and only if there exists a sequence F_k of μ-simple functions converging in μ-measure to F; that is

$$\rho(F_k, F) = \inf_{\alpha > 0}(\alpha + \mu^*\{s \in S : d(F_k(s), F(s)) > \alpha\}),$$

where $\mu^*(D) = \inf\{\mu(E) : E \in \Sigma$ and $D \subset E\}$, converges to 0 as $k \to \infty$.

Let $\xi = \{x_1, x_2, \ldots, x_n\}$ be an orthonormal basis of R^n. If $x \in R^n$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes the coordinates of x with respect to the basis ξ; that is $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$.

Definition 1.14. Let $K \in \text{Comp}\ (R^n)$. A point $x_0 \in K$ is called the lexicographic maximum of K if $x_0 = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0)$ such that

(i) $\alpha_1^0 = \max\{\alpha_1 : x = (\alpha_1, \ldots, \alpha_n) \in K\}$

(ii) $\alpha_k^0 = \max\{\alpha_k : x = (\alpha_1, \ldots, \alpha_n) \in K$ and $\alpha_i = \alpha_i^0$ for $i < K\}$.

We shall use $e(K, \xi)$ to denote the lexicographic maximum of K with respect to the basis ξ.

Clearly, for any compact K, $e(K, \xi) \in K$. The following selection theorem is due to Olech:

Theorem 1.3. (Olech [29]). Let (S, Σ, u) be a finite measure space. If $F : S \to \text{Comp}\ (R^n)$ is μ-measurable, then the mapping $e(F(s), \xi)$ of S into R^n is μ-measurable in s for each fixed orthonormal basis ξ of R^n.

Let T be a Lebesgue measurable subset of R and F be a set-valued mapping from T into $\mathcal{P}(R^n)$. We use $\mathcal{J}$ to denote the family of all point-valued mappings f

from T into R^n such that f is Lebesgue measurable over T and $f(t) \in F(t)$ for every $t \in T$. Then the following definitions of the integral of F is a natural generalization of the integral of a point-valued mapping:

<u>Definition 1.15</u>. Let $F : T \to \mathcal{P}(R^n)$, we define the <u>set-valued integral</u> of F over T, $(S)\int_T F(t)dt$, by

$$(S)\int_T F(t)dt = \{\int_T f(t)dt : f \in \mathcal{J}\}.$$

The following are some fundamental theorems which will be useful in our later development:

<u>Theorem 1.4</u>. Let F be a Lebesgue measurable function from a measurable subset $T \subset R$ into Comp (R^n) such that the measure of T is finite and F is integrably bounded; i.e. there exists a point-valued function f which is integrable over T and $|F(t)| \leq f(t)$ for all $t \in T$. Then,

$$(S)\int_T F(t)dt \neq \emptyset.$$

<u>Proof</u>: In order to get a measurable selection of F, it suffices to show that Definition 1.11 implies Definition 1.13 when μ is the Lebesgue measure m defined on T so that we can apply Theorem 1.3.

Let F be Lebesgue measurable. Given any positive integer $k > 0$, by Theorem 1.2, there is a closed subset $T' = T(k) \subset T$ such that F is Hausdorff continuous on T' and $m(T \setminus T') < \frac{1}{2k}$. As $m(T) < \infty$, there exist $a,b \in R$ such that $T \subset [a,b]$. Since T' is compact, F is uniformly continuous on T'. Let $a = a_0 < a_1 < \ldots < a_m = b$ be a subdivision of $[a,b]$ with $a_j - a_{j-1} = \frac{1}{m}(b-a)$ for $j = 1,\ldots,m$. Let $I_j = [a_j, a_{j-1}]$, $T_j = T \cap I_j$ and $T_j' = T' \cap I_j$ for $j = 1,\ldots,m$. We choose m so large such that $d(F(t_1),F(t_2)) < \frac{1}{2k}$ if $t_1, t_2 \in T_j'$ for some j.

Define $F_k : T \to \text{Comp}\,(R^n)$ by

$$F_k(t) = \begin{cases} F(t_j) & \text{for all } t \in T_j \text{ if there exists a } t_j \in T_j' \\ \{0\} & \text{for all } t \in T_j \text{ if } T_j' = \emptyset. \end{cases}$$

It is clear that F_k is a simple Lebesgue measurable function and

$$\begin{aligned} \rho(F_k,F) &= \inf_{\alpha > 0}(\alpha + m^*\{t \in T : d(F_k(t),F(t)) > \alpha\}) \\ &\leq \frac{1}{2k} + m^*\{t \in T : d(F_k(t),F(t)) > \frac{1}{2k}\} \\ &\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \end{aligned}$$

which converges to 0 as $k \to \infty$. Hence, F is Lebesgue measurable in the sense of Definition 1.13. □

<u>Theorem 1.5</u>. (Aumann [1]). Let $F : T \to \mathcal{P}(R^n)$ such that F is integrably bounded and $F(t)$ is closed for all $t \in T$. Then, $(S)\int_T F(t)\,dt$ is compact.

The following theorem is an analogue of Fatou's lemma:

<u>Theorem 1.6</u>. (Aumann [1]). If $\{F_k\}_{k=1}^{\infty}$ is a sequence of set-valued functions that are all defined and bounded by the same integrable point-valued function h on a measurable set $T \subset R$, then

$$(S)\int_T \limsup_{k\to\infty} F_k \supset \limsup_{k\to\infty} \int_T F_k.$$

<u>Theorem 1.7</u>. (Hukuhara [14]). Let F be a measurable set-valued function defined on a set $T \subset R$ with $m(T) < \infty$. If $T = T_1 \cup T_2$ such that T_1 and T_2 are disjoint and measurable, then

$$(S)\int_T F(t)dt = (S)\int_{T_1} F(t)dt + (S)\int_{T_2} F(t)dt.$$

§3. Topological degree of set-valued mappings:

One of the important theories in non-linear analysis is that of the degree of a mapping as developed by Leray and Schauder in 1934 (see [20]). Their work not only generalized the Brouwer degree to a certain class of mappings in Banach space but also made it possible to formulate more powerful fixed point theorems. As our main interest is the set-valued mappings, we shall omit the lengthy development of the degree theory of point-valued mappings which was first defined on finite dimensional linear spaces and then extended to normed and locally convex linear spaces. An extensive treatment of this subject may be found in [32],[26],[27], and [19].

The extension of the topological degree from point-valued mappings to set-valued mappings has been established also by many mathematicians, first by Granas [10], then by Hukuhara [13] and recently by Cellina and Lasota [4]. Each of their approaches is different from the others. The way we present here follows the approach of Granas since it is convenient for our future purposes. For simplicity, our topological degree will be defined for a class of set-valued mappings in a Banach space E with domains solid spheres in E.

The following notations are needed:

E_α : an arbitrary Banach space.

$P_\alpha = E_\alpha \setminus \{0\}$.

$V_\alpha(x_0, \rho) = \{x \in E_\alpha : |x-x_0| < \rho\}$ where $x_0 \in E_\alpha$ and $\rho > 0$.

$S_{\alpha-1}(x_0, \rho) = \{x \in E_\alpha : |x-x_0| = \rho\}$ where $x_0 \in E_\alpha$ and $\rho > 0$.

$cf(E_\alpha)$ = the collection of all non-empty closed convex subsets of E_α

Definition 1.16. A mapping $\Phi : A \subset E_\alpha \to cf(E_\alpha)$ is said to be compact if for each subset D of A, $\Phi(D)$ is relatively compact in E_α, i.e. $\overline{\Phi(D)}$ is compact. Φ is said to be completely continuous on A if Φ is compact and upper semi-continuous in the sense of Definition 1.5 on A.

<u>Definition 1.17</u>. A mapping $\varphi : A \subset E_\alpha \to cf(E_\alpha)$ is said to be a <u>completely continuous multi-valued vector field</u> on A if it can be expressed in the form

$$\varphi(x) = x - \Phi(x) \quad \text{for all} \quad x \in A,$$

where $\Phi(x)$ is completely continuous on A.

<u>Definition 1.18</u>. We say that a completely continuous multi-valued field $\varphi(x) = x - \Phi(x)$, $x \in A$, <u>does not vanish</u> and denote it by $\varphi : A \to cf(P_\alpha)$ if the point 0 does not belong to the set $\varphi(x)$ for any $x \in A$.

Let $S_{\alpha-1} = S_{\alpha-1}(x_0,\rho)$, $V_\alpha = V_\alpha(x_0,\rho)$ and $f,\varphi : S_{\alpha-1} \to cf(P_\alpha)$ be completely continuous (resp. single and multi-valued) vector fields. f is called a <u>selection</u> of φ if $f(x) \in \varphi(x)$ for each $x \in S_{\alpha-1}$. By [32], for every such f, an integer $\nu(f,S_{\alpha-1})$ is defined and is called the <u>characteristic</u> of f on $S_{\alpha-1}$. The following theorems are all due to Granas:

<u>Theorem 1.8</u>. (Granas [10]). To every non-vanishing completely continuous multi-valued vector field $\varphi : S_{\alpha-1} \to cf(P_\alpha)$ we can assign an integer $\nu(\varphi,S_{\alpha-1})$ called the <u>characteristic</u> of the field φ on $S_{\alpha-1}$ such that

$$\nu(\varphi,S_{\alpha-1}) = \nu(f,S_{\alpha-1})$$

for every selection f of φ.

<u>Definition 1.19</u>. We say that two non-vanishing completely continuous multi-valued fields $\varphi_1, \varphi_2 : A \to cf(P_\alpha)$, $\varphi_1(x) = x - \Phi_1(x)$, $\varphi_2(x) = x - \Phi_2(x)$ are <u>homotopic</u> and denoted by $\varphi_1 \simeq \varphi_2$, if there exists an upper semi-continuous (in the sense of Definition 1.5) function $\Psi(x,t) : A \times [0,1] \to cf(E_\alpha)$ such that the following conditions are satisfied:

(i) the point 0 does not belong to any set $\psi(x,t) = x - \Psi(x,t)$ for all $x \in A$ and $t \in [0,1]$;

(ii) $\Psi(x,0) = \Phi_1(x)$, $\Psi(x,1) = \Phi_2(x)$ for all $x \in A$;

(iii) the set $\Psi(A,[0,1])$ is relatively compact in E_α.

<u>Theorem 1.9</u>. (Granas [10]). If two non-vanishing completely continuous multi-valued vector fields $\varphi_1, \varphi_2 : S_{\alpha-1} \to cf(P_\alpha)$ are homotopic, $\varphi_1 \simeq \varphi_2$, then their characteristics are equal, i.e. $\nu(\varphi_1, S_{\alpha-1}) = \nu(\varphi_2, S_{\alpha-1})$.

<u>Theorem 1.10</u>. (Fixed point property, Granas [10]). Let $\varphi(x) = x - \Phi(x)$ be a completely continuous multi-valued vector field

$$\varphi : \bar{V}_\alpha \to cf(E_\alpha)$$

defined on a full sphere $\bar{V}_\alpha = \overline{V_\alpha(x_0,\rho)}$ into $cf(E_\alpha)$. If the restriction of φ on $S_{\alpha-1}$ does not vanish,

$\varphi_0 = \varphi|S_{\alpha-1} : S_{\alpha-1} \to cf(P_\alpha)$ and $\nu(\varphi_0, S_{\alpha-1}) \neq 0$, then there exists a point $x \in V_\alpha$ such that $0 \in \varphi(x)$, i.e. there exists $x \in V_\alpha$ such that $x \in \Phi(x)$.

Although we do not include them here, there are many important fixed point theorems, e.g. the Kakutani-Ky Fan theorem (see [15] and [7]), that can also be obtained without much difficulty from the view of topological degree. We shall conclude this chapter with the following extension of the well-known theorem of Borsuk on antipodes (see [3]):

Theorem 1.11. (Granas [11]). If a non-vanishing completely continuous multi-valued vector field $\varphi : S_{\alpha-1} \to cf(P_\alpha)$, defined on a sphere $S_{\alpha-1} \subset E_\alpha$, is odd, that is

$$\varphi(x) = -\varphi(-x) \quad \text{for all} \quad x \in S_{\alpha-1},$$

then its characteristic $\nu(\varphi, S_{\alpha-1})$ is odd.

Chapter II

CONTINGENT DIFFERENTIAL EQUATIONS

In an ordinary differential equation, the tangent at each point is prescribed by a point-valued function. This gives a vector field. In a contingent differential equation, the tangent is prescribed by a set-valued function. This direction field is usually called an orientor field. This more general class of equations was developed independently by A. Marchard and S.K. Zaremba in the mid 30's and has then been intensively investigated by many other mathematicians. In this chapter, we shall study the fundamental theory of such differential systems.

§1. Existence and continuation of solutions:

Definition 2.1. Let I be an interval in R. A mapping F from $I \times R^n$ into $\text{Comp}\,(R^n)$ is called an orientor field.

Definition 2.2. Let $x(t)$ be a function from an interval $I \subset R$ into R^n. For each $t_0 \in I$, let

$$P(t) = \frac{x(t)-x(t_0)}{x-t_0} .$$

The set $D^*x(t_0)$ defined by

$$D^*x(t_0) = \{c \in R^n : \text{there exists a sequence } \{t_k\}_{k=1}^{\infty} \subset I \text{ such that } t_k \neq t_0,\ t_k \overset{k}{\to} t_0 \text{ and } P(t_k) \overset{k}{\to} c\}$$

is called the <u>contingent derivative</u> of $x(t)$ at t_0.

<u>Definition 2.3</u>. Let $x(t) : I \to R^n$ and $F(t,x) : I\times R^n \to \text{Comp}\,(R^n)$. The relation

$$\text{(C)} \qquad D^*x(t) \subset F(t,x(t))$$

is called a <u>contingent differential equation</u>.

Let J be an interval in R and $D \subset I\times R^n$. We shall use the following notations:

$\text{Proj}_1\ D = \{t \in I$: there exists an $x \in R$ such that $(t,x) \in D\}$

$\text{Proj}_2\ D = \{x \in R^n$: there exists a $t \in R$ such that $(t,x) \in D\}$

$L^1(J)$ = the collection of Lebesgue integrable functions from J into R^n

$C(J)$ = the collection of continuous functions from J into R^n

$AC(J)$ = the collection of absolutely continuous functions from J into R^n

$cc(R^n)$ = the collection of non-empty, compact and convex subsets of R^n.

For each $x(t) \in AC(J)$, we denote by $\dot{x}(t)$ the usual derivative of $x(t)$. The abbreviation a.e. J means almost everywhere in J.

Definition 2.4. A function $x = x(t)$ defined on an interval $J \subset I$ into R^n will be called a solution of (C) in the sense of Marchand if

$$x(t) \in C(J) \quad \text{and}$$
$$D^* x(t) \subset F(t, x(t)) \quad \text{a.e.} \quad J.$$

Definition 2.5. A function $x = x(t)$ defined on an interval $J \subset I$ into R^n will be called a solution of (C) in the sense of Ważewski if

$$x(t) \in AC(J) \quad \text{and}$$
$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.} \quad J.$$

Definition 2.6. A function $x = x(t)$ defined on an interval $J \subset I$ into R^n will be called a solution of (C) if

$$x(t) \in AC(J) \quad \text{and}$$
$$x(t) \in x(t_0) + (S)\int_{t_0}^{t} F(s, x(s))ds$$

for each $t, t_0 \in J$.

The contingent differential equation that we shall study in this thesis are equations in which the orientor fields are of the type defined as follow:

<u>Definition 2.7</u>. We shall say that an orientor field $F(t,x)$ from $I \times R^n$ into $cf(R^n)$ satisfies the <u>Carathéodory conditions</u> provided

(i) $F(t,x)$ is measurable in t for each fixed $x \in R^n$;

(ii) $F(t,x)$ is upper semi-continuous (in the sense of Definition 1.4) in x for each fixed $t \in I$; and

(iii) for each compact subset D of $I \times R^n$, there exists a function $m(t) = m_D(t)$ which is integrable over $\text{Proj}_1 D$ such that

$$|F(t,x)| \leq m(t)$$

for all $(t,x) \in D$.

A contingent differential equation is said to be of <u>Carathéodory type</u> if its orientor field satisfies the Carathéodory conditions.

<u>Proposition 2.1</u>. (Pliś [31]). Let $F(t,x)$ be an orientor field from $I \times R^n$ into $cc(R^n)$ such that F satisfies conditions (i) and (ii) in Definition 2.7. Then, there exists a orientor field $H(t,x)$ such that

(i) $H(t,x) \subset F(t,x)$ for every $(t,x) \in J \times R^n$;

(ii) $H(t,x)$ is upper semi-continuous (in the sense of Definition 1.4) in x for each fixed $t \in I$;

(iii) $H(t,x(t))$ is measurable in t for any measurable function $x(t)$ from I into R^n;

(iv) $H(t,x)$ is measurable in (t,x);

(v) there exists a countable dense subset B of R^n such that

$$H(t,x) = F(t,x)$$

for $(t,x) \in I \times B$.

Remark 2.1. For an orientor field $F(t,x)$ satisfying conditions (i) and (ii) of Definition 2.7, it may happen that $F(t,x(t))$ is not measurable for a measurable function $x(t)$ as the following example shows. This example shows an error in Proposition 5 of [17] by Kikuchi and also shows the need for a theorem of the Pliś type.

Example 2.1. Let S be a non-measurable subset of $[0,1]$. Define $F(t,x) : [0,1] \times [0,1] \to cc(R)$ by

$$F(t,x) = \begin{cases} [1,2] & \text{if } t = x \in S \\ [0,1] & \text{if } t = x \notin S \\ [1] & \text{if } t \neq x. \end{cases}$$

It is clear that $F(t,x)$ is upper semi-continuous (hence measurable) in t for each fixed x and $F(t,x)$ is upper semi-continuous in x for each fixed t. Moreover, $F(t,x)$ is measurable in (t,x) since the set $\{(t,x) : t = x \in [0,1]\}$ is a set of measure 0 in $[0,1] \times [0,1]$.

Let $x(t) : [0,1] \to [0,1]$ defined by $x(t) = t$ which is continuous (hence measurable). However, $\{t : F(t,x(t)) \subset [1,2]\} = S$ is non-measurable.

Remark 2.2. If $H(t,x) \subset F(t,x)$, then every trajectory of $H(t,x)$ is a trajectory of $F(t,x)$.

Remark 2.3. Let $F(t,x)$ be an orientor field satisfying the Carathéodory conditions. It follows from Theorem 1.4 and (iii) of Proposition 2.1 that

$$(S)\int_I F(t,x(t))dt \neq \emptyset$$

for any measurable function $x(t)$ from I into R^n with $m(I) < \infty$.

The following proposition will allow us to express a contingent differential equation of Carathéodory type in the ordinary differential form as well as the integral form:

Proposition 2.2. Let $F(t,x)$ be an orientor field from $I \times R^n$ into $cc(R^n)$ such that F satisfies the Carathéodory conditions. Then, Definitions 2.4-2.6 are all equivalent.

Proof: The equivalence of Definition 2.5 and 2.6 is evident. It is clear that Definition 2.5 implies Definition 2.4. When $F(t,x) \in cc(R^n)$, Definition 2.4 implies Definition 2.5 which is due to Ważewski (see [37]). $\square$

When a contingent differential system is defined, the immediate questions that one may ask are:

(i) When does a solution exist?

(ii) How can a solution be continued?

(iii) Does the family of solutions have certain properties of convergence and continuous dependence upon initial conditions?

(iv) What can we say about the family of solutions emanating from a given initial point?

The next theorem which is due to Pliś gives a complete answer to the first two questions.

<u>Definition 2.8</u>. Let S be an open subset of $R \times R^n$ and $F(t,x)$ be an orientor field from S into $cc(R^n)$. A solution $x(t)$ defined on an open interval I is called a <u>non-continuable solution</u> of (C) if $\lim_{n\to\infty} (t_n, x(t_n)) \in S$ implies $\lim_{n\to\infty} t_n \in I$. In this case, I is called a <u>maximal interval of existence</u> of $x(t)$.

<u>Remark 2.4</u>. (i) From the existence theorem of Plis, we can see that a maximal interval I exists and is unique and we shall denote it by $I = (\omega^-, \omega^+)$.

(ii) If (ω^-, ω^+) is the maximal interval of existence of a solution, then $(t,x(t))$ tends to the boundary ∂S of S as $t \to \omega-$ or $\omega+$. To say $(t,x(t))$ tends to ∂S

as $t \to \omega^+$ (resp. $t \to \omega^-$), we mean that either $\omega^+ = \infty$ (resp. $\omega^- = -\infty$) or for any compact subset D of S, $(t,x(t)) \notin D$ when t is sufficiently close to ω^+ (resp. ω^-).

Theorem 2.1. (Pliś [31]). Let $F(t,x)$ be an orientor field from an open subset $S \subset R \times R^n$ into $cc(R^n)$ such that $F(t,x)$ satisfies the Carathéodory conditions. Then, for each point $(t_0,x_0) \in S$, there exists at least one non-continuable solution $x(t)$ of (C) such that $x(t_0) = x_0$.

Remark 2.5. Actually, Pliś proves this theorem under weaker assumptions. Instead of conditions (i) and (ii) of Definition 2.7, he only requires that $F(t,x)$ is upper semi-continuous for almost all t and that it contains an orientor field $G(t,x)$ which is densely measurable in t on S.

§2. Convergence properties of solutions:

The convergence property is one of the most important properties in differential equations. As it was shown by Strauss and Yorke (see [34]), much of the fundamental theory in ordinary differential equations follows directly from a convergence theorem. Kamke has given a convergence theorem (see [12]) in which the vector fields are assumed to be continuous. In [34], this theorem was proved with vector fields of Caratheodory type. For contingent differential equations, Zaremba has shown a convergence theorem (see [41]). And later, Bebernes and Schuur also established a Kamke-type convergence theorem of initial values for contingent differential equations in which the orientor fields are assumed to be upper semi-continuous (see [2]). In this section, we shall consider contingent differential equations of Carathéodory type with certain perturbation and investigate the convergence properties of their solutions.

<u>Proposition 2.3</u>. Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two sequences of subsets of R^n. Then,

(i) if $A_n \subset B_n$ for all $n = 1,2,\ldots,$ then $\limsup_{n\to\infty} A_n \subset \limsup_{n\to\infty} B_n$;

(ii) if $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are bounded, then $\limsup_{n\to\infty} \{A_n+B_n\} \subset \limsup_{n\to\infty} A_n + \limsup_{n\to\infty} B_n$.

<u>Proof</u>: (i) follows immediately from the definition of limit superior and (ii) follows from the Bolzano-Weierstrass property of R^n. □

For any $A,B \in \text{Comp}(R^n)$, the <u>escape</u> of A from B is defined and denoted by

$$\rho(A,B) = \sup_{a\in A} d(a,B)$$

where $d(x,B) = \inf_{b\in B} |x-b|$. It is obvious that ρ is not symmetric but does satisfy the triangle inequality.

<u>Definition 2.9</u>. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a family of orientor fields from $I \times R^n$ into Comp (R^n). We say $\{F_\lambda\}_{\lambda\in\Lambda}$ is an <u>equi-upper-semi-continuous family at</u> $x_0 \in R^n$ if for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x_0) > 0$ such that $|x-x_0| < \delta$ implies

$$\rho(F_\lambda(t,x), F_\lambda(t,x_0)) < \epsilon$$

for all $t \in I$ and for all $\lambda \in \Lambda$.

If $\{F_\lambda\}_{\lambda\in\Lambda}$ is equi-upper-semi-continuous at every $x \in R^n$, we say $\{F_\lambda\}_{\lambda\in\Lambda}$ is <u>equi-upper-semi-continuous</u> on R^n.

<u>Proposition 2.4</u>. Suppose that

(i) $F_n(t,x) : I \times R^n \to$ Comp (R^n) such that $\{F_n\}_{n=1}^{\infty}$ is an equi-upper-semi-continuous family on R^n;

(ii) there exists a $F_0(t,x) : I \times R^n \to$ Comp (R^n) such that

$$\rho(F_n(t,x), F_0(t,x)) \overset{n}{\to} 0$$

for all $(t,x) \in I \times R^n$;

(iii) $\varphi_n(t) : I \to R^n$ such that $\varphi_n(t) \overset{n}{\to} \varphi_0(t)$ for all $t \in I$.

Then, $\limsup_{n\to\infty} F_n(t,\varphi_n(t)) \subset F_0(t,\varphi_0(t))$ for all $t \in I$.

<u>Proof</u>: For any $t \in I$, let $y(t) \in \limsup_{n\to\infty} F_n(t,\varphi_n(t))$ be an arbitrary element. By definition, there exists a subsequence $y_{n_j}(t) \in F_{n_j}(t,\varphi_{n_j}(t))$ such that $y_{n_j}(t) \overset{j}{\to} y(t)$.

Now, consider the sequences $\{F_{n_j}(t,\varphi_k(t))\}_{k=1}^{\infty}$ and $j = 1,2,\cdots$. Since $\varphi_k(t) \xrightarrow{k} \varphi_0(t)$, it follows from the equi-upper-semi-continuity of $\{F_{n_j}\}_{j=1}^{\infty}$ that for any $\varepsilon > 0$, there exists a $K = K(\varepsilon,t) > 0$ such that $k > K$ implies

$$(1) \qquad \rho(F_{n_j}(t,\varphi_k(t)),\ F_{n_j}(t,\varphi_0(t))) < \frac{\varepsilon}{2}$$

for $j = 1,2,\cdots$. It follows from $y_{n_j}(t) \xrightarrow{j} y(t)$ and $y_{n_j}(t) \in F_{n_j}(t,\varphi_{n_j}(t))$ that for any $\varepsilon > 0$, there exists a $J = J(\varepsilon,t) > 0$ such that $j > J$ implies

$$(2) \qquad \rho(y,F_{n_j}(t,\varphi_{n_j}(t))) < \frac{\varepsilon}{2}\ .$$

Now, for each $\varepsilon > 0$ and $t \in I$, we pick $r > \max\{J,K\}$. Then, by (1) and (2) we have

$$\begin{aligned} d(y,F_{n_r}(t,\varphi_0(t)) &= \rho(y,F_{n_r}(t,\varphi_0(t))) \\ &\leq \rho(y,F_{n_r}(t,\varphi_{n_r}(t))) \\ &\quad + \rho(F_{n_r}(t,\varphi_{n_r}(t)),F_{n_r}(t,\varphi_0(t))) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\rho(y,F_{n_r}(t,\varphi_0(t)) \xrightarrow{r} 0$. By assumption (ii), we have

$$\begin{aligned} d(y,F_0(t,\varphi_0(t))) &= \rho(y,F_0(t,\varphi_0(t))) \\ &\leq \rho(y,F_{n_r}(t,\varphi_0(t))) \\ &\quad + \rho(F_{n_r}(t,\varphi_0(t)),F_0(t,\varphi_0(t))) \xrightarrow{r} 0. \end{aligned}$$

It follows from the compactness of $F_0(t,\varphi_0(t))$ that $y(t) \in F_0(t,\varphi_0(t))$ as desired. $\square$

Example 2.2. Proposition 2.4 is not true without assumption (i) even in the case of point-valued functions. Consider

$F_n(t,x) : [1,2] \times R \to R$ defined by

$$F_n(t,x) = \frac{ntx}{t^2+n^2x^2} \quad \text{for} \quad n = 1,2,\ldots$$

and $F_0(t,x) \equiv 0$.

$\varphi_n(t) : [1,2] \to R$ defined by

$$\varphi_n(t) = \frac{1}{n}t \quad \text{for} \quad n = 1,2,\ldots$$

and $\varphi_0(t) = 0$.

It is easy to see that F_n and φ_n are all continuous for $n = 0,1,2,\cdots$. Also, $\rho(F_n(t,x),F_0(t,x)) \overset{n}{\to} 0$ and $\varphi_k \overset{k}{\to} \varphi_0$ uniformly on $[1,2]$. However, $\{F_n\}_{n=1}^{\infty}$ is not an equi-upper-semi-continuous family. Let $\varepsilon = \frac{1}{5}$ and take $x_0 = 0$. For any $\delta > 0$, choose N so large such that $\frac{1}{N} < \delta$. Let $x_1 = \frac{1}{n_1}$ where $n_1 > N$. Then $|x_1-x_0| = |x_1| < \delta$ and

$$F_{n_1}(t,x_1) = \frac{t}{t^2+1} \geq \frac{1}{t^2+1} = \frac{1}{5} .$$

Hence, $\rho(F_{n_1}(t,x_1), F_{n_1}(t,0)) \geq \frac{1}{5} = \varepsilon$. Therefore, $\{F_n\}_{n=1}^{\infty}$ is not equi-upper-semi-continuous at 0. By a simple calculation, we have

$$\limsup_{n\to\infty} F_n(t,\varphi_n(t)) = \{\tfrac{1}{2}\} \not\subset \{0\} = F_0(t,\varphi_0(t)).$$

<u>Proposition 2.5</u>. Suppose that

(i) $\{F_n\}_{n=1}^{\infty}$ is a family of orientor fields from $I \times R^n$ into $\mathrm{Comp}\,(R^n)$;

(ii) there exists an orientor field $F_0(t,x)$ from $I \times R^n$ into $\mathrm{Comp}\,(R^n)$ such that $F_0(t,x)$ is upper semi-continuous (in the sense of Definition 1.4) in x for each fixed $t \in I$;

(iii) $\rho(F_n(t,x), F_0(t,x)) \overset{n}{\to} 0$ uniformly with respect to x;

(iv) $\varphi_n(t) : I \to R^n$ such that $\varphi_n(t) \overset{n}{\to} \varphi_0(t)$ for each $t \in I$.

Then, $\limsup\limits_{n\to\infty} F_n(t,\varphi_n(t)) \subset F_0(t,\varphi_0(t))$ for any $t \in I$.

<u>Proof</u>: For any $t \in I$, let $y(t) \in \limsup\limits_{n\to\infty} F_n(t,\varphi_n(t))$ be arbitrary. By definition, there exists a subsequence $\{F_{n_j}(t,\varphi_{n_j}(t))\}_{j=1}^{\infty}$ of $\{F_n(t,\varphi_n(t))\}_{n=1}^{\infty}$ such that

(1) $$\rho(y(t), F_{n_j}(t,\varphi_{n_j}(t))) \overset{j}{\to} 0.$$

Given $\varepsilon > 0$, by (1), there exists an integer $J = J(\varepsilon,t) > 0$ such that $j > J$ implies

(2) $$\rho(y(t), F_{n_j}(t,\varphi_{n_j}(t))) < \frac{\varepsilon}{2}.$$

By assumption (iii), there exists an integer $N = N(\varepsilon,t) > 0$ such that $n > N$ implies

(3) $$\rho(F_n(t,x),\ F_0(t,x)) < \frac{\varepsilon}{2}$$

for all $x \in R^n$. Now, let us choose $k = k(\varepsilon,t)$ so large that $k > J$ and $n_k > N$. By (2) and (3), we have

$$\begin{aligned} d(y(t), F_0(t, \varphi_{n_k}(t))) &= \rho(y(t), F_0(t, \varphi_{n_k}(t))) \\ &\leq \rho(y(t), F_{n_k}(t, \varphi_{n_k}(t))) \\ &\quad + \rho(F_{n_k}(t, \varphi_{n_k}(t)), F_0(t, \varphi_{n_k}(t))) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence,

(4) $$d(y(t), F_0(t, \varphi_{n_k}(t))) \xrightarrow{k} 0.$$

Applying (4) with assumptions (ii) and (iv), we have

$$\begin{aligned} d(y(t), F_0(t, \varphi_0(t))) &= \rho(y(t), F_0(t, \varphi_0(t))) \\ &\leq \rho(y(t), F_0(t, \varphi_{n_k}(t))) \\ &\quad + \rho(F_0(t, \varphi_{n_k}(t)), F_0(t, \varphi_0(t))) \xrightarrow{k} 0. \end{aligned}$$

It follows from the compactness of $F_0(t, \varphi_0(t))$ that $y \in F_0(t, \varphi_0(t))$ as desired. □

Remark 2.6. The previous example shows also that it is essential that the convergence in assumption (iii) of Proposition 2.5 must be uniform in x. In Example 2.2, take $t=1$ and let $x_n = \frac{1}{n}$. Then, $F_n(1, x_n) = \frac{1}{2}$ for all $n = 1, 2, \cdots$. Therefore, if we choose $\varepsilon = \frac{1}{2}$, it is clear that we must take $N > n$ so that $\rho(F_N(1, x_n), F_0(1, x_n)) < \frac{1}{2} = \varepsilon$.

The following two examples will show that Propositions 2.4 and 2.5 are generally two different sufficient conditions for the limit superior of a sequence of orientor fields to be contained in their limit function.

Example 2.3. Consider the orientor fields

$F_n(t,x) : R\times R \to cc(R)$ defined by

$$F_n(t,x) = [\sin \tfrac{tx}{n}, 1] \quad \text{for} \quad n = 1,2,...$$

and $F_0(t,x) = [0,1]$.

It is clear that F_n is upper semi-continuous for $n = 0,1,2,...$ and $\rho(F_n(t,x), F_0(t,x)) \overset{n}{\to} 0$ for all $(t,x) \in R\times R$.

(i) $\{F_n\}_{n=1}^{\infty}$ is equi-upper-semi-continuous in x on R : For any $(t,x) \in R\times R$, since sine function is continuous, given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, (t,x)) > 0$ such that $|y-x| < \delta$ implies $|\sin ty - \sin tx| < \varepsilon$. For any positive integer n, $|y-x| < \delta$ implies $|\frac{y}{n} - \frac{x}{n}| < \frac{\delta}{n} \leq \delta$ which implies $|\sin \frac{ty}{n} - \sin \frac{tx}{n}| < \varepsilon$. Therefore, for each $\varepsilon > 0$ and $x \in R$, we have $\rho(F_n(t,y),F_n(t,x)) < \varepsilon$ for all y with $|y-x| < \delta$ and $n = 1,2,\cdots$.

(ii) $\rho(F_n(t,x),F_0(t,x)) \overset{n}{\to} 0$ is not uniform with respect to x : Given $t \in R$, for every positive integer n, choose $x_n = \frac{3\pi n}{2t}$. We have

$$\rho(F_n(t,x_n),F_0(t,x_n)) = \rho([-1,1],[0,1]) = 1.$$

Hence, the convergence is not uniform.

Example 2.4. Consider the orientor fields

$F_n(t,x) : [0,1] \times R \to cc(R^n)$ defined by

$$F_n(t,x) = [-1, \sin nx + \tfrac{t}{n} \sin x] \text{ for } n = 1,2,...$$

and $F_0(t,x) = [-1,1]$.

It is clear that F_n is continuous for all $n = 0,1,2,\cdots$.

(i) $\rho(F_n(t,x),F_0(t,x)) \overset{n}{\to} 0$ uniformly with respect to x : Clearly, $\rho(F_n(t,x),F_0(t,x)) \le \frac{t}{n} \overset{n}{\to} 0$ is independent of x.

(ii) $\{F_n\}_{n=1}^{\infty}$ is not equi-upper-semi-continuous : Consider $(t,0) \in [0,1] \times R$. Let $0 < \epsilon < 1$ be given. For any $\delta > 0$, choose n so large such that $\frac{\pi}{2n} < \delta$. We have

$$\rho(F_n(t,\tfrac{\pi}{2n}),F_n(t,0))$$
$$= \rho([-1, \sin\tfrac{\pi}{2} + \tfrac{t}{n}\sin\tfrac{\pi}{2}],[-1, \sin 0 + \tfrac{t}{n}\sin 0])$$
$$= \rho([-1,1 + \tfrac{t}{n}], [-1,0]) = 1 + \tfrac{t}{n} \ge 1 = \epsilon .$$

Hence, $\{F_n\}_{n=1}^{\infty}$ is not equi-upper-semi-continuous at 0.

Definition 2.10. Let $\{\varphi_n\}_{n=0}^{\infty}$ be a sequence of functions with domains $D_{\varphi_n} \subset R$. We say φ_n converges compactly to φ_0, denoted by $\varphi_n \overset{c}{\to} \varphi_0$, if for any compact subset K of D_{φ_0}, $K \subset D_{\varphi_n}$ except at most a finite number of D_{φ_n} and $\varphi_n \overset{n}{\to} \varphi_0$ uniformly on K.

For any open subset U of $R \times R^n$, let $\mathcal{M}_U$ denote the family of all orientor fields of Carathéodory type from U into $cc(R^n)$. We shall consider the following contingent differential equations with initial conditions:

(1) $\dot{x}(t) \in A(t,x(t)) + F_n(t,x(t)) \qquad x(t_n) = x_n$

(2) $\dot{x}(t) \in A(t,x(t)) + F_O(t,x(t)) \qquad x(t_O) = x_O$

<u>Theorem 2.2</u> (convergence). Suppose that

(i) U and U_n are open subsets of $R \times R^n$ for $n = 0,1,2,\ldots$ such that $U_O \subset U_n \subset U$ except for a finite number of U_n's;

(ii) $A \in \mathcal{M}_U$ and $F_n \in \mathcal{M}_{U_n}$ for $n = 0,1,2,\ldots$;

(iii) $\{F_n\}_{n=1}^{\infty}$ is an equi-upper-semi-continuous family on $\text{Proj}_2\, U_O$ such that $\rho(F_n(t,x),F_O(t,x)) \overset{n}{\to} 0$ for all $(t,x) \in U_O$;

(iv) for each compact subset $Q \subset U_O$, there exists a function $m(t) = m_Q(t)$ such that $m(t)$ is integrable over $\text{Proj}_1\, Q$ and $|F_n(t,x)| \leq m(t)$ for all $(t,x) \in Q$ and for all $n = 0,1,2,\ldots$;

(v) $(t_n,x_n) \in U_n$ for $n = 0,1,2,\ldots$ such that $(t_n,x_n) \overset{n}{\to} (t_O,x_O)$.

Then, for every sequence $\{\varphi_n\}_{n=1}^{\infty}$ of non-continuable solutions of (1), there exists a subsequence $\{\varphi_{n_j}\}_{j=1}^{\infty}$ and there exists a non-continuable solution φ_O of (2) such that $\varphi_{n_j} \overset{c}{\to} \varphi_O$.

<u>Proof</u>: Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of non-continuable solutions of (1) with interval domains $\{D_{\varphi_n} = (\omega_n^-, \omega_n^+)\}_{n=1}^{\infty}$.

(i) For any compact subset $Q \subset U_0$ with $(t_0, x_0) \in \text{Int } Q$, there exists an open interval I_Q such that $t_0 \in I_Q \subset (\omega_n^-, \omega_n^+)$ except for a finite number of n: By assumption (i), $Q \subset U_n$ for all n large enough. Define

$$L(t) = \int_{t_0}^{t} \ell(s)\,ds \quad \text{and} \quad M(t) = \int_{t_0}^{t} m(s)\,ds$$

where $\ell(t) = \ell_Q(t)$ and $m(t) = m_Q(t)$ are integrable functions that bound $A(t,x)$ and $F_n(t,x)$, respectively, on Q. We note that $L(t), M(t) \in AC$ on $\text{Proj}_1 Q$. Let $\alpha > 0$ be such that the set W_α

$$W_\alpha = \{(t,x) : |t-t_0| < \alpha, |x-x_0| < \tfrac{1}{2} d(t_0,x_0),\partial Q)$$

is contained in Q, where ∂Q denotes the boundary of Q. Let $0 < \beta < \alpha$ be such that $|t-t_0| < \beta$ implies $|L(t)| + |M(t)| < \frac{1}{8} d((t_0,x_0),\partial Q)$. Define

$$W_\beta = \{(t,x) : |t-t_0| < \beta, |x-x_0| < \tfrac{1}{4} d((t_0,x_0),\partial Q)\}.$$

Clearly, we have $W_\beta \subset W_\alpha \subset Q \subset U_0$. By assumption (v), there exists an integer $N > 0$ such that $(t_n,x_n) \in W_\beta$ for all $n > N$. Since φ_n is a solution of (1), we have

$$\begin{aligned}
|\varphi_n(t)-x_n| &\le \left|\int_{t_n}^{t} \ell(s)\,ds\right| + \left|\int_{t_n}^{t} m(s)\,ds\right| \\
&= |L(t)-L(t_n)| + |M(t)-M(t_n)| \\
&\le |L(t)| + |M(t)| + |L(t_n)| + |M(t_n)| \\
&< \tfrac{1}{8} d((t_0,x_0),\partial Q) + \tfrac{1}{8} d((t_0,x_0),\partial Q) \\
&= \tfrac{1}{4} d((t_0,x_0),\partial Q)
\end{aligned}$$

for all $n > N$ and all t such that $|t-t_0| < \beta$. Hence,

$$\begin{aligned}|\varphi_n(t)-x_0)| &\leq |\varphi_n(t)-x_n| + |x_n-x_0| \\ &< \frac{1}{4}\, d((t_0,x_0),\partial Q) + \frac{1}{4}\, d((t_0,x_0),\partial Q) \\ &= \frac{1}{2}\, d((t_0,x_0),\partial Q)\end{aligned}$$

for all $n > N$ and all t such that $|t-t_0| < \beta$. This shows that $(t,\varphi_n(t)) \in W_\alpha$ for all $n > N$ and all t such that $|t-t_0| < \beta$. Since φ_n is non-continuable, this shows that $\varphi_n(t)$ exists at least on the interval $(t_0-\beta, t_0+\beta)$ for all $n > N$. This proves (i).

(ii) For any compact interval $J \subset I_Q$ with $t_0 \in \text{Int } J$, $\{\varphi_n(t)\}_{n=1}^{\infty}$ has a subsequence $\{\varphi_{n_j}(t)\}_{j=1}^{\infty}$ which converges uniformly to some function $\varphi_0(t)$ on J; Without loss of generality, let $J = [a,b] \subset I_Q$ and $t_0 \in (a,b)$. Clearly, φ_n is defined on $[a,b]$ for $n > N$. In part (i) of our proof, we have shown that $\{(t,\varphi_n(t)) : t \in I_Q, n > N\} \subset W_\alpha \subset Q$. Since Q is compact, $\{\varphi_n\}_{n=N+1}^{\infty}$ is uniformly bounded on $[a,b]$. Also, by Proposition 2.2, the solution φ_n of (1) can be expressed as

$$\varphi_n(t) \in x_n + (S)\int_{t_n}^{t}[A(s,\varphi_n(s)) + F_n(s,\varphi_n(s))]ds$$

for $t \in [a,b]$ and $n > N$. It follows from Theorem 1.7 that

$$\varphi_n(t) - \varphi_n(s) \in (S)\int_{s}^{t}[A(r,\varphi_n(r)) + F_n(r,\varphi_n(r))]dr$$

for any $t, s \in [a,b]$ and $n > N$. Hence, we have

$$|\varphi_n(t) - \varphi_n(s)| \leq \left|\int_s^t [\ell(r) + m(r)]dr\right|$$

$$\leq |L(t)-L(s)| + |M(t)-M(s)|$$

for any $t, s \in [a,b]$ and $n > N$. As $L(t)$, $M(t) \in AC([a,b])$ and are independent of n, $\{\varphi_n\}_{n=N+1}^{\infty}$ is equi-continuous on $[a,b]$. It follows from Ascoli's that there exists a subsequence $\{\varphi_{n_j}(t)\}_{j=1}^{\infty}$ which converges uniformly to some function $\varphi_0(t)$ on $[a,b]$.

(iii) $\varphi_0(t)$ is a solution of (2) on $[a,b]$: Consider the limit superior of both sides of the equation

$$\varphi_{n_j}(t) \in x_{n_j} + (S)\int_{t_{n_j}}^{t} [A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s)]ds$$

where $t \in [a,b]$. By Proposition 2.3 (i), we have

$$\limsup_{j\to\infty} \varphi_{n_j}(t) \subset \limsup_{j\to\infty}\{x_{n_j} + (S)\int_{t_{n_j}}^{t} [A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s))]ds\}.$$

Since $A(t,\varphi_{n_j}(t))$ and $F_{n_j}(t,\varphi_{n_j}(t))$ are integrably bounded by $\ell(t)$ and $m(t)$ respectively for all $t \in [a,b]$, we can apply Proposition 2.3 (ii) and get

$$\begin{aligned}\limsup_{j\to\infty} \varphi_{n_j}(t) &\subset \limsup_{j\to\infty} x_{n_j} + \limsup_{j\to\infty}(S)\int_{t_{n_j}}^{t} [A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s))]ds \\ &\subset \limsup_{j\to\infty} x_{n_j} + \limsup_{j\to\infty}\{(S)\int_{t_0}^{t} [A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s))]ds \\ &\quad + B(\int_{t_0}^{t_{n_j}} [\ell(s)+m(s)]ds)\}\end{aligned}$$

$$\subset \limsup_{j\to\infty} x_{n_j} + \limsup_{j\to\infty} (S)\int_{t_0}^{t} [A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s))]ds + \limsup_{j\to\infty} B\left(\int_{t_0}^{t_{n_j}} [\ell(s)+m(s)]ds\right)$$

where $B(\int_{t_0}^{t} [\ell(s)+m(s)]ds)$ is the solid sphere in R^n centered at 0 with radius $|\int_{t_0}^{t} [\ell(s)+m(s)]ds|$. Since $\varphi_{n_j}(t) \overset{j}{\to} \varphi_0(t)$ uniformly on $[a,b]$, $t_{n_j} \overset{j}{\to} t_0$, $x_{n_j} \overset{j}{\to} x_0$ and by Theorem 1.6, we have

$$\varphi_0(t) \in x_0 + (S)\int_{t_0}^{t} \limsup_{j\to\infty}[A(s,\varphi_{n_j}(s)) + F_{n_j}(s,\varphi_{n_j}(s))]ds + \{0\}.$$

Define $G_{n_j}(t,x) = A(t,x) + F_{n_j}(t,x)$ for $j = 1,2,\ldots$ and

$$G_0(t,x) = A(t,x) + F_0(t,x).$$

By assumption (iii), $\{G_{n_j}\}_{j=1}^{\infty}$ is equi-upper-semi-continuous on R^n and $\rho(G_{n_j}(t,x),G_0(t,x)) \overset{j}{\to} 0$. It follows from Proposition 2.4 that

$$\limsup_{j\to\infty} G_{n_j}(t,\varphi_{n_j}(t)) \subset G_0(t,\varphi_0(t)).$$

Hence,

$$\varphi_0(t) \in x_0 + (S)\int_{t_0}^{t} [A(s,\varphi_0(s)) + F_0(s,\varphi_0(s))]ds.$$

Therefore, $\varphi_0(t)$ is a solution of (2) on $[a,b]$.

(iv) $\varphi_0(t)$ can be extended to $I_Q = (a_0,b_0)$ defined in part (i) such that $\varphi_0(t)$ is a solution of (2) on I_Q and $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence $\{\varphi_{(k,k)}\}_{k=1}^{\infty}$ such that $\varphi_{(k,k)} \overset{c}{\to} \varphi_0$ on I_Q: From what we have shown above, we know that for every compact interval $[a,b]$ such that $t_0 \in (a_0,b_0)$ and $[a,b] \subset (a_0,b_0)$, $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence which converges compactly to a solution φ_0 of (2) on $[a,b]$. Let $\{[a_k,b_k]\}_{k=1}^{\infty}$ be a sequence of compact intervals such that $t_0 \in (a_k,b_k)$, $[a_k,b_k] \subset [a_{k+1},b_{k+1}]$ and $(a_0,b_0) = \bigcup_{k=1}^{\infty} [a_k,b_k]$.

Let $\{\varphi_{(1,r)}\}_{r=1}^{\infty}$ be a subsequence of $\{\varphi_n\}_{n=1}^{\infty}$ such that $\varphi_{(1,r)} \xrightarrow{r} \varphi_{(0,1)}$ uniformly on $[a_1,b_1]$. If $\{\varphi_{(k,r)}\}_{r=1}^{\infty}$ is chosen and $\varphi_{(k,r)} \to \varphi_{(0,k)}$ uniformly on $[a_k,b_k]$, we pick a subsequence $\{\varphi_{(k+1,r)}\}_{r=1}^{\infty}$ of $\{\varphi_{(k,r)}\}_{r=1}^{\infty}$ such that $\varphi_{(k+1,r)} \xrightarrow{r} \varphi_{(0,k+1)}$ uniformly on $[a_{k+1},b_{k+1}]$. Clearly, $\varphi_{(0,k+1)}$ is a solution of (2) on $[a_{k+1},b_{k+1}]$ and $\varphi_{(0,k+1)}|[a_k,b_k] = \varphi_{(0,k)}$ for $k = 1,2,\cdots$. Define $\varphi_0(t) : (a_0,b_0) \to R^n$ by $\varphi_0(t) = \varphi_{(0,k)}(t)$ where k is so large that $t \in [a_k,b_k]$. From our selection process, it is clear that $\varphi_0(t)$ is a solution of (2) on (a_0,b_0) and $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence $\{\varphi_{(k,k)}\}_{k=1}^{\infty}$ such that $\varphi_{(k,k)} \xrightarrow{c} \varphi_0$ on (a_0,b_0).

(v) Let $\tilde{I}_Q$ be the maximal interval which is contained in $\text{Proj}_1\, Q$ and $t_0 \in \text{Int}\, \tilde{I}_Q$ such that a solution $\varphi_0(t)$ of (2) is defined on $\tilde{I}_Q$ and $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence which converges compactly to φ_0 on $\tilde{I}_Q$. Then, $\tilde{I}_Q$ is closed. Moreover, if $\tilde{a}$ and $\tilde{b}$ are the left and right end points of $\tilde{I}_Q$ respectively, then $(t,\varphi_0(t))$ tends to ∂Q as $t \to \tilde{a} + 0$ or $t \to \tilde{b} - 0$: We shall consider the right end point $\tilde{b}$ only. The left end point $\tilde{a}$ can be proved similarly. Since φ_n is a solution, for $\tilde{a} < t_1 < t_2 < \tilde{b}$ and $n = 0,1,2,\ldots,$ we have

$$|\varphi_n(t_1)-\varphi_n(t_2)| \le \left|\int_{t_1}^{t_2} \ell(s)\,ds\right| + \left|\int_{t_1}^{t_2} m(s)\,ds\right|$$
$$\le |L(t_2)-L(t_1)| + |M(t_2)-M(t_1)|.$$

As $L(t)$, $M(t) \in AC((\tilde{a},\tilde{b}))$, $\varphi_n(t_1) - \varphi_n(t_2) \to 0$ as $t_1, t_2 \to \tilde{b} - 0$. By the Cauchy criterion for convergence, $\varphi_n(\tilde{b}-0)$ exists for all $n = 0,1,2,\cdots$. Also $\lim_{t\to\tilde{b}-0} (t,\varphi_n(t)) \in Q$ since Q is compact and $(t,\varphi_n(t)) \in Q$ for all $t \in (\tilde{a},\tilde{b})$ and $n = 0,1,2,\cdots$. Define

$$\tilde{\varphi}_n : (\tilde{a},\tilde{b}] \to \mathrm{Proj}_2\ Q \text{ by}$$
$$\tilde{\varphi}_n(t) = \varphi_n(t) \quad \text{if } t \in (\tilde{a},\tilde{b})$$
$$\tilde{\varphi}_n(t) = \varphi_n(\tilde{b}-0) \quad \text{if } t = \tilde{b}$$

where $n = 0,1,2,\cdots$. By the existence theorem of Pliś, φ_n can be extended so that $(t,\varphi_n(t)) \notin Q$. And $\tilde{\varphi}_n$ is a continuous extension of φ_n in Q. Hence, $\tilde{\varphi}_n$ is a solution of (1) for $n \ge 1$ and of (2) for $n=0$. Consider the subsequence $\{\varphi_{(k,k)}\}_{k=1}^{\infty}$ defined in part (iv). By a similar proof as in part (ii), $\{\tilde{\varphi}_{(k,k)}\}_{k=1}^{\infty}$ has a subsequence $\{\tilde{\varphi}_{(r,r)}\}_{r=1}^{\infty}$ such that $\tilde{\varphi}_{(r,r)} \xrightarrow{r} \tilde{\varphi}$ uniformly on $[\tilde{c},\tilde{b}]$, where $\tilde{a} < \tilde{c} < \tilde{b}$. Since $\varphi_{(r,r)} \xrightarrow{r} \varphi_0$ on $[\tilde{c},\tilde{b})$, $\tilde{\varphi} = \tilde{\varphi}_0$. It follows that $\tilde{I}_Q$ must contain $\tilde{b}$. Hence, $\tilde{I}_Q$ is closed. Clearly, $\tilde{\varphi}_{(r,r)}(\tilde{b}) = \xi_r \xrightarrow{r} \xi_0 = \tilde{\varphi}_0(\tilde{b})$. Suppose $\lim_{t\to\tilde{b}-0} (t,\varphi_0(t)) \in \mathrm{Int}\ Q$, i.e. $(\tilde{b},\tilde{\varphi}_0(\tilde{b})) \in \mathrm{Int}\ Q$. Then, following our previous proof, we can use $\{(\tilde{b},\xi_n)\}_{n=0}^{\infty}$ as our new sequence of initial points and extend $\tilde{I}_Q$ to the right which contradicts the maximality of $\tilde{I}_Q$.

(vi) Let $\tilde{I}$ be the maximal interval of the $\tilde{I}_Q$'s where Q is a compact subset of U_0 and $(t_0,x_0) \in \text{Int } Q$. Then $\tilde{I}$ is always open, say, $I = (\omega^-,\omega^+)$. $(\omega^-,\omega^+) \subset (\limsup_{n\to\infty} \omega_n^-$ $\liminf_{n\to\infty} \omega_n^+)$. Furthermore, there exists a non-continuable solution φ_0 of (2) defined on (ω^-,ω^+) and $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence $\{\varphi_{n_j}\}_{j=1}^{\infty}$ such that $\varphi_{n_j} \overset{c}{\to} \varphi_0$ on (ω^-,ω^+): Let $\{Q_k\}_{k=1}^{\infty}$ be a sequence of compact subsets of U_0 such that $(t_0,x_0) \in \text{Int } Q_k$, $Q_k \subset Q_{k+1}$ and $U_0 = \bigcup_{k=1}^{\infty} Q_k$. For each Q_k, we can find a maximal interval $\tilde{I}_{Q_k}$ and a subsequence $\{\varphi_{(k,r)}\}_{r=1}^{\infty}$ of $\{\varphi_n\}_{n=1}^{\infty}$ such that $\varphi_{(k,r)} \overset{r}{\to} \varphi_{(0,k)}$ uniformly on $\tilde{I}_{Q_k}$. Choosing $\{\varphi_{(k,r)}\}_{r=1}^{\infty}$ inductively and taking the diagonal process as in part (iv), we get the desired result. $\square$

The following is another convergence theorem which is parallel to the previous one. The proof is analogous. The only difference is we shall apply Proposition 2.5 instead of Proposition 2.4. We give the statement as follow:

<u>Theorem 2.3</u> (convergence). Suppose that

(i) U and U_n are open subsets of $R \times R^n$ for $n = 0,1,2,\ldots$ such that $U_0 \subset U_n \subset U$ except for a finite number of U_n's;

(ii) $A \in \mathcal{M}_U$ and $F_n \in \mathcal{M}_{U_n}$ for $n = 0,1,2,\ldots;$

(iii) $\rho(F_n(t,x), F_0(t,x)) \overset{n}{\to} 0$ uniformly with respect to x for $x \in \mathrm{Proj}_2\ U_0;$

(iv) for each compact $Q \subset U_0$, there exists a function $m(t) = m_Q(t)$ such that $m(t)$ is integrable over $\mathrm{Proj}_1\ Q$ and $|F_n(t,x)| \leq m(t)$ for all $(t,x) \in Q$ and all $n = 0,1,2,\ldots;$

(v) $(t_n,x_n) \in U_n$ for $n = 0,1,2,\ldots$ such that $(t_n,x_n) \overset{n}{\to} (t_0,x_0)$.

Then, for every sequence $\{\varphi_n\}_{n=1}^{\infty}$ of non-continuable solutions of (1), there exists a subsequence $\{\varphi_{n_j}\}_{j=1}^{\infty}$ and there exists a non-continuable solution φ_0 of (2) such that $\varphi_{n_j} \overset{c}{\to} \varphi_0$.

§3. Continuous dependence of solutions on initial conditions and parameters:

Let U be an open subset of $R \times R^n$ and Λ be a domain, $\Lambda = \{\lambda : |\lambda - \lambda_0| < c,\ c > 0\} \subset R$. We define an orientor field with a parameter $F(t,x,\lambda) : U \times \Lambda \to cc(R^n)$ by

$$F(t,x,\lambda) = A(t,x) + F_\lambda(t,x)$$

and we assume that

(i) for each $\lambda \in \Lambda$, U_λ is an open subset of R^{n+1} such that $U_{\lambda_0} \subset U_\lambda \subset U$ for all $\lambda \in \Lambda;$

(ii) $A \in \mathcal{M}_U$ and $F_\lambda \in \mathcal{M}_{U_\lambda}$ for all $\lambda \in \Lambda;$

(iii) $\{F_\lambda\}_{\lambda\in\Lambda}$ is an equi-upper-semi-continuous family on $\mathrm{Proj}_2\ U_{\lambda_0}$ such that $\rho(F_\lambda(t,x), F_{\lambda_0}(t,x)) \to 0$ as $\lambda \to \lambda_0$ for all $(t,x) \in U_{\lambda_0}$;

(iv) for each compact subset $Q \subset U_{\lambda_0}$, there exists a function $m(t) = m_Q(t)$ such that $m(t)$ is integrable over $\mathrm{Proj}_1\ Q$ and $|F_\lambda(t,x)| \leq m(t)$ for all $(t,x) \in Q$ and all $\lambda \in \Lambda$.

We shall call this family of orientor fields $\{F(t,x,\lambda) : \lambda \in \Lambda\}$ family (P) and consider the following contingent differential equation with parameter:

(P_λ) $$\dot{x}(t) \in F(t,x,\lambda), \quad x(t_\lambda) = x_\lambda.$$

Definition 2.11. For each $(t_{\lambda_0}, x_{\lambda_0}) \in U_{\lambda_0}$, by the positive solution funnel through $(t_{\lambda_0}, x_{\lambda_0})$ we mean the set

$$Z^+ = \{(t,\varphi(t)) : t \geq t_{\lambda_0} \text{ and } \varphi \text{ is a solution of } (P_{\lambda_0})\}.$$

For any interval I, we define $Z^+(I) = Z^+ \cap (I \times R^n)$. The negative solution funnel Z^- and $Z^-(I)$ can be defined similarly for $t \leq t_{\lambda_0}$. And we define the solution funnel $Z = Z^+ \cup Z^-$ and $Z(I) = Z^+(I) \cup Z^-(I)$.

The next theorem gives some answer to question (iv) in §1. This theorem though it follows immediately from Theorem 2.2 is actually a special case of Theorem 5 of [6] in which the equations are considered in Banach spaces:

Theorem 2.4. (Chow and Schuur: [6]). If I is a compact interval on which all solutions of (P_{λ_0}) exist, then $Z^+(I)$, $Z^-(I)$ and $Z(I)$ are all compact.

Theorem 2.5. (continuous dependence). Suppose that all solutions of (P_{λ_0}) exist on $[a,b]$. Then, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $(t_\lambda, x_\lambda, \lambda)$ satisfying

$$\rho((t_\lambda, x_\lambda), Z([a,b])) + |\lambda - \lambda_0| < \delta,$$

each non-continuable solution $\varphi_\lambda(t)$ of (P_λ) exists at least on $[a,b]$ and there exists a solution $\varphi_{\lambda_0}(t)$ of (P_{λ_0}) such that

$$|\varphi_\lambda(t) - \varphi_{\lambda_0}(t)| < \varepsilon \quad \text{for all} \quad t \in [a,b].$$

Proof: Suppose the first conclusion is false. Since $Z([a,b])$ is compact, there exists a sequence $(t_{\lambda_n}, x_{\lambda_n}, \lambda_n) \overset{n}{\to} (\tau_0, \xi_0, \lambda_0)$ such that $(\tau_0, \xi_0) \in Z([a,b])$ and a sequence of non-continuable solutions $\varphi_n(t)$ of (P_{λ_n}) with maximal interval of existence (ω_n^-, ω_n^+) and an integer $N > 0$ such that $[a,b] \not\subset (\omega_n^-, \omega_n^+)$ for all $n > N$. By Theorem 2.2, there exists a subsequence $\{\varphi_{n_j}(t)\}_{j=1}^\infty$ of $\{\varphi_n(t)\}_{n=1}^\infty$ and a non-continuable solution $\varphi_0(t)$ of

$$(P'_{\lambda_0}) \qquad \dot{x}(t) \in F(t, x(t), \lambda_0) \qquad x(\tau_0) = \xi_0$$

with maximal interval of existence (ω_0^-, ω_0^+) such that $\varphi_{n_j} \overset{c}{\to} \varphi_0$ on (ω_0^-, ω_0^+). Since (P'_{λ_0}) is (P_{λ_0}) with only the initial condition changed and this initial point $(\tau_0, \varphi_0(\tau_0)) = (\tau_0, \xi_0) \in Z([a,b])$, φ_0 is actually a non-continuable solution of (P_{λ_0}). By our assumption, φ_0 exists at least on $[a,b]$. Hence, $[a,b] \subset (\omega_{n_j}^-, \omega_{n_j}^+)$ for j large enough. Choosing j so large that $n_j > N$, we get a contradiction.

The second conclusion is evidently true. Suppose not. We consider the same sequence of solutions $\{\varphi_n\}_{n=1}^{\infty}$ as before and claim that there exists an integer $M > 0$ such that for each $n > M$ there exists a $t_n \in [a,b]$ with $|\varphi_n(t_n) - \varphi_0(t_n)| \geq \varepsilon$ for some $\varepsilon > 0$. However, by Theorem 2.2, $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence $\{\varphi_{n_j}\}_{j=1}^{\infty}$ which converges uniformly to φ_0 on $[a,b]$. Hence, $|\varphi_{n_j}(t) - \varphi_0(t)| < \varepsilon$ for all $t \in [a,b]$ and for all j sufficiently large. Choosing j so large that $n_j > M$, we get a contradiction again. □

<u>Remark 2.7</u>. Theorem 2.5 is an extension of Theorem 4 and Corollary 4.2 in [34] to contingent differential equations.

<u>Remark 2.8</u>. It is clear that if we replace assumption (iii) of family (P) by

(iii)′ $\rho(F_\lambda(t,x), F_{\lambda_0}(t,x)) \to 0$ uniformly with respect to x, for $x \in \mathrm{Proj}_2\, U_{\lambda_0}$ as $\lambda \to \lambda_0$,

then Theorem 2.5 also holds.

Next, we shall define another family called <u>family (Q)</u> of orientor fields. Let U be an open subset of

$R \times R^n$, $[a,b] \subset \mathrm{Proj}_1\ U$ and $\Lambda = \{\lambda : |\lambda-\lambda_0| < c,\ c > 0\} \subset R$. Let

$F(t,x,\lambda) : U\times\Lambda \to cc(R^n)$ defined by

$$F(t,x,\lambda) = F(t,x) + G_\lambda(t)$$

and assume that

(i) $F \in \mathcal{M}_U$;

(ii) for each $\lambda \in \Lambda$, $G_\lambda(t)$ is a continuous function from $\mathrm{Proj}_1\ U$ into $cc(R^n)$;

(iii) $\int_a^b |G_\lambda(t)|dt \to 0$ as $\lambda \to \lambda_0$;

(iv) $\{G_\lambda(t)\}_{\lambda\in\Lambda}$ is bounded on $[a,b]$.

Then we shall consider the following contingent differential equations with parameter:

(Q_λ) $\qquad \dot{x}(t) \in F(t,x,\lambda) \qquad x(t_\lambda) = x_\lambda.$

It is easy to see that family (Q) satisfies all the conditions in family (P). Therefore, Theorem 2.5 holds true for family (Q). We have the following theorem:

<u>Theorem 2.6</u>. (continuous dependence). Let $F(t,x) : U \to cc(R^n)$ satisfy the Carathéodory conditions on an open subset U of $R\times R^n$ such that all solutions of

(E) $\qquad \dot{x}(t) \in F(t,x(t)) \qquad x(t_0) = x_0$

exist on $[a,b]$. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every continuous G from $[a,b]$ into $cc(R^n)$

which satisfies

$$\rho((\tau,\xi),Z([a,b])) + \int_a^b |G(t)|dt < \delta,$$

each solution of $\varphi(t)$ of

$$\dot{x}(t) \in F(t,x(t)) + G(t)$$

through (τ,ξ) can be extended to $[a,b]$ and there exists a solution $\psi(t)$ of (E) such that

$$|\varphi(t)-\psi(t)| < \varepsilon \text{ for all } t \in [a,b].$$

<u>Remark 2.9</u>. Theorem 2.6 extends a result of Yoshizawa (see [39] p.22) and also Corollary 4.3 of Strauss and Yorke (see [34]) to contingent differential equations.

Chapter III

GENERAL BOUNDARY VALUE PROBLEMS FOR CONTINGENT DIFFERENTIAL EQUATIONS

It is well known that fixed point theorems play a main role in the proof of existence theorems of differential equations. The papers of Granas (see [10] and [11]) have extended the notion of topological degree to set-valued mappings and the fixed point theorems of Rothe and Borsuk have also been successfully established for the set-valued case. In this chapter we shall prove an existence theorem of contingent differential equations by using the degree theory described in §3 of the first chapter and we shall see some of its applications. The results obtained here are motivated by Theorem 2.1 and Theorem 2.2 of [12] (see p.413) and are also generalizations of the results in [23].

§1. A fixed point theorem:

Let E be a real Banach space with norm $|\cdot|$.

<u>Definition 3.1</u>. A mapping $F : E \to cc(E)$ is called homogeneous if $F(\lambda x) = \lambda F(x)$ for every real λ and every $x \in E$.

Lemma 3.1. (Chow and Lasota [5]). Let $F : E \to cc(E)$ be homogeneous and completely continuous with the property that

$$x \in F(x) \Rightarrow x = 0.$$

Then, there exists a constant $\alpha = \alpha(F) > 0$ such that

$$x \in F(x) + b \Rightarrow |x| \leq \alpha |b|$$

for each $x \in E$.

Lemma 3.2. Let $F(x) = x - V(x)$ and $G(x) = x - W(x)$ be two non-vanishing completely continuous vector fields mapping a bounded subset A of E into $cc(E)$ such that $G(x) \subset F(x)$ for every $x \in A$. Then, F and G are homotopic on A.

Proof: Define $\Phi(\lambda, x) : [0,1] \times A \to cc(E)$ by

$$\Phi(\lambda, x) = [\lambda W(x) + (1-\lambda) |V(x)| U] \cap V(x)$$

where U is the closed unit ball centered at 0 in E. Then we define

$$\varphi(\lambda, x) : [0,1] \times A \to cc(E) \quad \text{by}$$

$$\varphi(\lambda, x) = x - \Phi(\lambda, x).$$

Clearly, Φ (hence φ) is well-defined. It follows from Propositions 1.7, 1.8 and 1.9 that Φ (hence φ) is upper semi-continuous. The following properties of Φ and φ are immediate:

(i) $\Phi(\lambda, x) \subset \Phi(\lambda', x)$ if $\lambda \geq \lambda'$;

(ii) $\Phi(0, x) = V(x)$ and $\Phi(1, x) = W(x)$;

(iii) for each $\lambda \in [0,1]$, $\varphi(\lambda,x)$ is non-vanishing on A : This follows from the fact that $\varphi(\lambda,x) \subset \varphi(0,x) = F(x)$ which is non-vanishing on A.

(iv) $\Phi([0,1],A)$ is relatively compact : Since $V(x)$ is a compact mapping, $V(A)$ is relatively compact. It follows from

$$\Phi([0,1],A) \subset \Phi(0,A) = V(A)$$

that $\Phi([0,1],A)$ is relatively compact.

Therefore, F and G are homotopic on A with homotopy φ. □

<u>Theorem 3.1</u>. (fixed point property). Let $F,G,$ H and $J : E \to cc(E)$ be completely continuous such that

(i) F is homogeneous with the property : $x \in F(x) \Rightarrow x = 0$;

(ii) G is bounded, i.e. $\|G\| = \sup\limits_{x \in E} |G(x)| \le K$ for some K;

(iii) there exist a $\sigma > 0$ and an $\epsilon = \epsilon(F,\sigma) > 0$ such that

$$|H(x)| \le \epsilon |x|$$

for all x with $|x| \ge \sigma > 0$;

(iv) $J(x) \subset F(x) + G(x) + H(x)$ for all $x \in E$ with $|x| \ge \sigma > 0$.

Then, there exists at least one $x \in E$ such that $x \in J(x)$ provided ϵ is small enough.

<u>Proof</u>: Consider the following completely continuous multi-valued vector fields from E into $cc(E)$:

$$\varphi_1(x) = x - F(x);$$
$$\varphi_2(x) = x - F(x) - H(x);$$
$$\varphi_3(x) = x - F(x) - G(x) - H(x);$$
$$\varphi_4(x) = x - J(x).$$

(i) Let $\Phi(\lambda,x) : [0,1] \times E \to cc(E)$ defined by

$$\Phi(\lambda,x) = x - F(x) - \lambda H(x).$$

Let $S_\rho = \{x : x \in E \text{ and } |x| = \rho\}$ where $\rho > \sigma$. Clearly, $\Phi(\lambda,x)$ is upper semi-continuous and $\Phi([0,1],S_\rho)$ is relatively compact. Moreover, $\Phi(0,x) = \varphi_1(x)$ and $\Phi(1,x) = \varphi_2(x)$. It follows from assumption (i) that φ_1 does not vanish on S_ρ.

We claim that there exists a positive number $r > 0$ such that $|x-F(x)| \geq r > 0$ for all $x \in S_\rho$. Suppose not. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset S_\rho$ such that $|x_n-F(x_n)| \overset{n}{\to} 0$. One can see easily that there exists $y_n \in F(x_n)$ such that $|x_n-y_n| \overset{n}{\to} 0$. It is clear that $\{y_n\}_{n=1}^\infty$ is contained in $\overline{F(S_\rho)}$ which is compact. Hence, $\{y_n\}_{n=1}^\infty$ has a subsequence $\{y_{n_j}\}_{j=1}^\infty$ such that $y_{n_j} \overset{j}{\to} y_0 \in \overline{F(S_\rho)}$. Since $|x_{n_j}-y_{n_j}| \overset{j}{\to} 0$, we have $|x_{n_j}-y_0| \overset{j}{\to} 0$, $y_0 \in S_\rho$ since $\{x_{n_j}\}_{j=1}^\infty \subset S_\rho$ which is closed. Now, $y_{n_j} \in F(x_{n_j})$, $y_{n_j} \overset{j}{\to} y_0$ and $x_{n_j} \overset{j}{\to} y_0$. It follows from the upper semi-continuity of F that $y_0 \in F(y_0)$. However, $|y_0| = \rho > 0$. This contradicts assumption (i).

Choosing $\varepsilon < \frac{r}{2\rho}$ and applying assumption (iii), we have

$$|x-F(x)-\lambda H(x)| \geq |x-F(x)| - |\lambda H(x)|$$
$$\geq r - \varepsilon|x| > r - \frac{r}{2} > 0$$

for all $\lambda \in [0,1]$ and all $x \in S_\rho$. Hence $\Phi(\lambda,x)$ does not vanish on S_ρ for all $\lambda \in [0,1]$. Therefore, φ_1 is homotopic to φ_2 on S_ρ.

(ii) Let $\Psi : [0,1] \times E \to cc(E)$ defined by

$$\Psi(\lambda,x) = x - F(x) - \lambda G(x) - H(x).$$

Clearly, $\Psi(\lambda,x)$ is upper semi-continuous and $\Psi([0,1],S_\rho)$ is relatively compact. Moreover, $\Psi(0,x) = \varphi_2(x)$ and $\Psi(1,x) = \varphi_3(x)$. We claim that $\Psi(\lambda,x)$ does not vanish on S_ρ for all $\lambda \in [0,1]$ if ρ is large enough. Suppose not. Then there exists an $x \in S_\rho$ such that $x \in F(x) + \lambda G(x) + H(x)$. By Lemma 3.1, there exists an $\alpha = \alpha(F) > 0$ such that

$$|x| \leq \alpha|\lambda G(x) + H(x)| \leq \alpha\lambda K + \alpha\varepsilon|x|.$$

Choose $\varepsilon < \frac{1}{2\alpha}$. We have $|x| \leq \alpha\lambda K + \frac{1}{2}|x|$, i.e. $|x| \leq 2\alpha\lambda K$. This is a contradiction since $|x| = \rho$ which can be chosen arbitrarily large. Hence, we have shown that φ_2 and φ_3 are homotopic on S_ρ for sufficiently large ρ.

(iii) From part (i) and part (ii), φ_1 and φ_3 are homotopic on some large sphere S_ρ. By Lemma 3.2, φ_3 and φ_4 are homotopic on S_ρ.

Hence, φ_1 and φ_4 are homotopic on S_ρ. By Theorem 1.9, the characteristics of φ_1 and φ_4 on S_ρ are equal. Since F is homogeneous, we have

$$\varphi_1(x) = x - F(x) = -(-x+F(x)) = -(-x-F(-x)) = -\varphi_1(-x).$$

It follows from Theorem 1.11 that the characteristic of φ_1 (hence φ_4) on S_ρ is odd. By Theorem 1.10, there exists an $x \in E$ with $|x| < \rho$ such that $x \in J(x)$. $\square$

Remark 3.1. In Theorem 3.1, condition (iii) and (iv) can be replaced by

(iii)' $|H(x)| \leq \varepsilon|x|$ for some $\varepsilon = \varepsilon(F,\rho)$ and all x with $|x| \leq \rho$;

and (iv)' $J(x) \subset F(x) + G(x) + H(x)$ for all x with $|x| \leq \rho$, where $\rho > 0$ is sufficiently large.

Remark 3.2. In Theorem 3.1, if the condition $|H(x)| \leq \varepsilon|x|$ holds for all $x \in E$, then ε depends on F only.

The following corollary is clear from the proof of Theorem 3.1:

<u>Corollary 3.1</u>. Let F, H and $J : E \to cc(E)$ be completely continuous and such that

(i) F is homogeneous with the property :
$x \in F(x) \Rightarrow x = 0$;

(ii) for any positive number $\rho > 0$ and all x with $|x| < \rho$, $|H(x)| \leq \varepsilon|x|$ for some $\varepsilon = \varepsilon(F,\rho)$;

(iii) $J(x) \subset F(x) + H(x)$ for all $x \in E$ with $|x| < \rho$.

Then, there exists at least one $x \in E$ with $|x| < \rho$ such that $x \in J(x)$ provided ε is sufficiently small.

Instead of a Lipschitz type condition, Corollary 3.1 still holds if $|H(x)|$ is small in a neighborhood of $0 \in E$:

<u>Corollary 3.2</u>. Let F, H and $J : E \to cc(E)$ be completely continuous such that

(i) F is homogeneous with the property :
$x \in F(x) \Rightarrow x = 0$;

(ii) for some positive number $\rho > 0$, there exists an $\varepsilon = \varepsilon(F,\rho)$ such that
$$|H(x)| \leq \varepsilon\rho$$
for all $x \in E$ with $|x| \leq \rho$;

(iii) $J(x) \subset F(x) + H(x)$ for all $x \in E$ with $|x| \leq \rho$.

Then, there exists at least one $x \in E$ with $|x| < \rho$ such that $x \in J(x)$ provided ε is sufficiently small.

<u>Proof</u>: Let us use the same notations as in Theorem 3.1. It suffices for us to show that $\Phi(\lambda,x)$ does not vanish on S_ρ and the rest of the proof is clear from Theorem 3.1.

Let $r > 0$ be a number such that $|x-F(x)| \geq r > 0$ for all $x \in S_\rho$. Choosing $\varepsilon < \frac{r}{2\rho}$ and applying assumption (ii), we have

$$\begin{aligned} |\Phi(\lambda,x)| = |x-F(x)-\lambda H(x)| &\geq |x-F(x)| - |\lambda H(x)| \\ &\geq r - \lambda\varepsilon\rho > r - \frac{r}{2} > 0 \end{aligned}$$

for all $\lambda \in [0,1]$ and all $x \in S_\rho$. $\square$

The next corollary is an immediate consequence of Theorem 3.1:

<u>Corollary 3.3</u>. Let F, G, H and $J : E \to cc(E)$ be completely continuous such that

(i) F is homogeneous with the property: $x \in F(x) \Rightarrow x = 0$;

(ii) G is bounded;

(iii) $\frac{|H(x)|}{|x|} \to 0$ as $|x| \to \infty$;

(iv) $J(x) \subset F(x) + G(x) + H(x)$ for all $x \in E$.

Then, there exists at least one $x \in E$ such that $x \in J(x)$.

§2. An existence theorem like Fredholm's alternative:

Let Δ be a compact interval in R and let C^n be the Banach space of all continuous functions from Δ into R^n with the topology of uniform convergence.

Lemma 3.3 (Lasota [21]). If $\{y_k\}_{k=1}^{\infty}$ is a sequence of measurable functions from Δ into R^n such that there is an integrable function $\varphi(t)$ from Δ into R and $|y_k(t)| \leq \varphi(t)$ for all $k = 1,2,\ldots$ and t a.e. in Δ, then there exist a sequence of indices α_m and a system of coefficients λ_{km} $(m \leq k \leq \alpha_m,\ m = 1,2,\ldots)$ such that

$$\sum_{k=m}^{\alpha_m} \lambda_{km} = 1, \quad \alpha_m \geq m, \quad \lambda_{km} \geq 0$$

and that the sequence

$$z_m(t) = \sum_{k=m}^{\alpha_m} \lambda_{km}\, y_k(t)$$

converges to a function $z_0(t)$ a.e. in Δ.

Proposition 3.1. Let $F(t,x) : [a,b] \times R^n \to cc(R^n)$ be an orientor field which satisfies the Carathéodory conditions. Then the set-valued mapping $G(x) : C^n \to cc(C^n)$ defined by

$$G(x) = \{g(t) : g(t) = \int_a^t f_x(s)ds, \text{ where } f_x \in L^1([a,b]) \text{ and } f_x(s) \in F(s,x(s)) \text{ for all } s \in [a,b]\}$$

is completely continuous.

Proof: (i) G is well-defined: Let $x \in C^n$ be arbitrary. It follows from Proposition 2.1 and Theorem 1.5 that $F(t,x(t))$ has an integrable selection $f_x(t)$. Hence, $G(x) \neq \emptyset$. Let $g_1, g_2 \in G(x)$, say $g_1(t) = \int_a^t f_1(s)ds$ and $g_2(t) = \int_a^t f_2(s)ds$ where $f_1, f_2 \in L^1$ and $f_1(s)$,

$f_2(s) \in F(s,x(s))$ for $s \in [a,b]$. Then for any λ, $0 \leq \lambda \leq 1$, we have

$$\lambda g_1(t) + (1-\lambda)g_2(t) = \int_a^t [\lambda f_1(s) + (1-\lambda)f_2(s)]ds.$$

Clearly, $\lambda f_1 + (1-\lambda)f_2 \in L^1([a,b])$ and $\lambda f_1(s) + (1-\lambda)f_2(s) \in F(s,x(s))$ for $s \in [a,b]$ since $F(s,x(s))$ is convex. Hence, $\lambda g_1 + (1-\lambda)g_2 \in G(x)$, $G(x)$ is convex.

Let $\{g_n\}_{n=1}^{\infty} \subset G(x)$ be any sequence of functions from $G(x)$. Since $x(t)$ is bounded on $[a,b]$ and F satisfies the Carathéodory conditions, we have

$$|g_n(t)| \leq \int_a^b |F(s,x(s))|ds \leq \int_a^b m(s)ds$$
$$\leq M(b) - M(a)$$

for all $t \in [a,b]$ and $n = 1,2,\cdots$. Hence $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded. Moreover, by Theorem 1.7,

$$g_n(t) - g_n(s) \in \int_s^t F(r,x(r))dr$$

for all $t,s \in [a,b]$. Hence,

$$|g_n(t)-g_n(s)| \leq |\int_s^t m(r)dr| \leq |M(t)-M(s)|$$

for any $t,s \in [a,b]$. Since $M \in AC([a,b])$, $\{g_n\}_{n=1}^{\infty}$ is equi-continuous. It follows from the Ascoli Lemma that $\{g_n\}_{n=1}^{\infty}$ has a subsequence $\{g_k\}_{k=1}^{\infty}$ such that $g_k \xrightarrow{k} g_0$ uniformly on $[a,b]$ for some g_0. Since the convergence is uniform, $g_0 \in C^n$. We claim that $g_0 \in G(x)$. Each g_k can be written as $g_k(t) = \int_a^t f_k(s)ds$ where

$f_k(s) \in F(s,x(s))$ for $k = 1,2,\cdots$. $\{f_k\}_{k=1}^{\infty}$ is integrably bounded by $m(t)$. By Lemma 3.3, there exist a sequence of indices α_m and a system of coefficients λ_{km} $(m \leq k \leq \alpha_m)$ such that

$$\sum_{k=m}^{\alpha_m} \lambda_{km} = 1, \quad \alpha_m \geq m, \quad \lambda_k \geq 0$$

and that the sequence

$$h_m(t) = \sum_{k=m}^{\alpha_m} \lambda_{km} f_k(t)$$

converges to a function $h_0(t)$ a.e. in $[a,b]$. Since $f_k(t) \in F(t,x(t))$ which is convex, $h_m(t) \in F(t,x(t))$ and $h_m(t)$ is measurable for $m = 1,2,\cdots$. As $F(t,x(t))$ is closed, $h_0(t) \in F(t,x(t))$ and $h_0(t)$ is measurable. Clearly,

$$\int_a^t h_m(s)ds = \sum_{k=m}^{\alpha_m} \lambda_{km} \int_a^t f_k(s)ds = \sum_{k=m}^{\alpha_m} \lambda_{km}\, g_k(t)$$

converges to $\int_0^t h_0(s)ds$ by the Lebesgue dominated convergence theorem. Recall that $\{g_k\}_{k=1}^{\infty}$ is picked so that $\{g_k\}_{k=1}^{\infty}$ converges uniformly to g_0. Its finite convex combinations also converge to g_0. That is

$$\sum_{k=m}^{\alpha_m} \lambda_{km}\, g_k(t) = \int_a^t h_m(s)ds \overset{m}{\rightarrow} g_0 .$$

Hence, $g_0(t) = \int_a^t h_0(s)ds$. We have $g_0 \in G(x)$. $G(x)$ is therefore compact.

(ii) G is compact: Let $D = \{x \in C^n : |x| \leq K\}$ be any bounded subset of C^n. We want to show that $G(D)$ has compact closure. It is equivalent for us to show that $\overline{G(D)}$ is sequentially compact. Let $\{g_n\}_{n=1}^{\infty} \subset G(D)$. Then

$$g_n(t) = \int_a^t f_n(s)ds$$

where $f_n(s) \in F(s,x(s))$ for $s \in [a,b]$ and some $x \in D$. Since D is bounded and F satisfies the Carathéodory conditions, $\{f_n\}_{n=1}^{\infty}$ is bounded by an integrable function $m(t)$. It follows easily from Ascoli's that $\{g_n\}_{n=1}^{\infty}$ has a subsequence $\{g_k\}_{k=1}^{\infty}$ which converges uniformly to some function $g_0 \in C^n$. Since $\{g_k\}_{k=1}^{\infty} \subset G(D)$, $g_0 \in \overline{G(D)}$. $\overline{G(D)}$ is therefore compact.

(iii) G is upper semi-continuous: From what we have shown above, we know that G is a compact mapping from $C^n \to cc(C^n)$. By Theorem 1.1, all the definitions of upper semi-continuity are equivalent. Let $x_0 \in C^n$ be arbitrary. Let $\{x_n\}_{n=1}^{\infty} \subset C^n$ and $\{g_n\}_{n=0}^{\infty} \subset C^n$ be such that $x_n \overset{n}{\to} x_0$, $g_n \overset{n}{\to} g_0$ and $g_n \in G(x_n)$ for $n \geq 1$. Then

$$g_n(t) = \int_a^t f_n(s)\,ds$$

where $f_n \in L^1([a,b])$ and $f_n(s) \in F(s,x_n(s))$ for all $s \in [a,b]$ and $n = 1,2,\cdots$. Since $x_n \overset{n}{\to} x_0$, $|x_n(s)| \leq M$ for all $s \in [a,b]$ and $n \geq 1$. Hence, $\{f_n\}_{n=1}^{\infty}$ is bounded by an integrable function $m(t)$. By Lemma 3.3, there exist a sequence of indices α_m and a system of coefficients λ_{km} $(m \leq k \leq \alpha_m)$ such that

$$\sum_{k=m}^{\alpha_m} \lambda_{km} = 1, \quad \alpha_m \geq m, \quad \lambda_k \geq 0$$

and that the sequence

$$h_m(t) = \sum_{k=m}^{\alpha_m} \lambda_{km}\, f_k(t)$$

converges to a function $h_0(t)$ a.e. in $[a,b]$. Clearly, as we saw in part (ii), $g_0(t) = \int_a^t h_0(s)ds$ for all $t \in [a,b]$. Since $f_n(t) \in F(t,x_n(t))$, we have

$$h_m(t) \in \sum_{k=m}^{\alpha_m} \lambda_{km} F(t,x_k(t))$$

for all $t \in [a,b]$. For each t fixed, $F(t,x(t))$ is upper semi-continuous. Hence, given $\epsilon > 0$, there exists a $K > 0$ such that

$$F(t,x_k(t)) \subset F(t,x_0(t)) + B_\epsilon$$

for all $k \geq K$, where $B_\epsilon = \{x \in R^n : |x| \leq \epsilon\}$. Hence,

$$\begin{aligned}\sum_{k=m}^{\alpha_m} \lambda_{km} F(t,x_k(t)) &\subset \sum_{k=m}^{\alpha_m} \lambda_{km}(F(t,x_0(t)) + B_\epsilon) \\ &= F(t,x_0(t)) + B_\epsilon\end{aligned}$$

for all $m \geq K$. This shows that

$$\rho(h_m(t), F(t,x_0(t)) \overset{m}{\to} 0.$$

As $F(t,x_0(t))$ is closed and $h_m(t) \to h_0(t)$ a.e. in $[a,b]$, we have $h_0(t) \in F(t,x_0(t))$ a.e. in $[a,b]$. That is $g_0 \in G(x_0)$. Therefore, G is upper semi-continuous in the sense of Definition 1.5 (hence in the sense of all others). □

<u>Lemma 3.4</u>. Let $F,G : \Delta \to cc(R^n)$ be measurable and integrably bounded and $K(t)$ be a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over Δ. If $F(t) \subset G(t) + K(t)$ for all $t \in \Delta$, then

$$(S)\int_\Delta F \subset (S)\int_\Delta G + (S)\int_\Delta K.$$

Proof: Let $f(t)$ be a measurable selection of $F(t)$. Then the function $H(t) : \Delta \to cc(R^n)$ defined by

$$H(t) = [f(t)+K(t)] \cap G(t)$$

is well-defined and measurable. It follows from Theorem 1.4 that $H(t)$ has a measurable selection $h(t)$. Clearly, $h(t)$ is also a measurable selection of $G(t)$. And $h(t) = f(t) + k(t)$ where $k(t) \in K(t)$ is measurable. Hence,

$$f(t) = h(t) - k(t) = h(t) + (-k(t))$$

where $h(t) \in G(t)$ and $-k(t) \in K(t)$ are both measurable. □

Remark 3.3. If G and K are defined as in Lemma 3.4, it is easy to see that

$$(S)\int_\Delta (G+K) = (S)\int_\Delta G + (S)\int_\Delta K$$

Consider the following contingent differential equations with general boundary conditions:

(1) $\dot{x}(t) \in A(t,x(t))$, $L\,x(t) = 0$; and

(2) $\dot{x}(t) \in Q(t,x(t))$, $L\,x(t) = a$

where $Q(t,x) \subset A(t,x) + B(t,x) + P(t,x)$.

Theorem 3.2. (Fredholm's alternative). Let A,B,P and Q: $\Delta \times R^n \to cc(R^n)$ satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x, that is

$$A(t,\lambda x) = \lambda A(t,x)$$

for all real λ;

(ii) $B(t,x) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over Δ;

(iii) there exists $\varepsilon = \varepsilon(A,\sigma,\delta) > 0$ such that

$$\alpha(t,r) \leq \varepsilon r \quad \text{for any } r \geq \sigma > 0$$

where $\sigma > 0$ is arbitrary, $\alpha(t,r) = \sup_{|x| \leq r} |P(t,x)|$ and $\delta = m(\Delta)$ is the measure of Δ;

(iv) $L : C^n \to R^n$ is linear and continuous.

Then, (1) has unique solutions $x(t) \equiv 0$ implies that (2) has at least one solution for any $a \in R^n$, provided ε is small enough.

Proof: Without loss of generality, we may assume $\Delta = [0,T]$. Define F,G,H and $J : C^n \to cc(C^n)$ by

$$F(x) = \{\int_0^t u(s)ds + Lx + x(0) : u(s) \in A(s,x(s))\},$$

$$G(x) = \{\int_0^t u(s)ds - a : u(s) \in K(s)\},$$

$$H(x) = \{\int_0^t u(s)ds : |u(s)| \leq \varepsilon|x(s)|\}$$

$$J(x) = \{\int_0^t u(s)ds + Lx - a + x(0) : u(s) \in Q(s,x(s))\},$$

where $t \in [0,T]$ and $u(t) \in L^1([0,T])$. Then the existence of the solutions of (1) and (2) are equivalent to the existence of fixed points of F and J respectively.
It follows from Proposition 3.1 and the fact that the continuous linear operator L is bounded that F,G,H and J are all completely continuous.

Clearly, F is homogeneous with only $x(t) \equiv 0$ as its fixed point. G is bounded by assumption (ii). Let $\tilde{P} : R^n \to cc(R^n)$ defined by $\tilde{P}(x) = \{u \in R^n : |u| \leq \epsilon|x|\}$. Then for any $x(t) \in C^n$, $\tilde{P}(x(t))$ is a ball in R^n centered at 0 with radius $\epsilon|x(t)|$ which is integrable over $[0,T]$. From the way we define H, one sees easily that

$$H(x) = \{\int_0^t u(s)ds : u(s) \in \tilde{P}(x(s))\}$$

and

$$|H(x)| \leq \int_\Delta |\tilde{P}(x(s))|ds \leq \int_\Delta \epsilon|x(s)|ds \leq \epsilon T|x|.$$

Moreover, $\qquad |P(t,x)| \leq \alpha(t, |x|) \leq \epsilon|x|$

for all x, $|x| = r \geq \sigma > 0$. Hence, $P(t,x) \subset \tilde{P}(t,x)$ for all x, $|x| \geq \sigma$. It follows from Lemma 3.4 and Remark 3.3 that

$$\begin{aligned}
J(x) &= \{\int_0^t u(s)ds + Lx - a + x(0) : u(s) \in Q(s,x(s))\} \\
&\subset \{\int_0^t u(s)ds + Lx - a + x(0) : u(s) \in A(s,x(s)) \\
&\qquad + B(s,x(s)) + P(s,x(s))\} \\
&\subset \{\int_0^t u(s)ds + Lx - a + x(0) : u(s) \in A(s,x(s)) \\
&\qquad + K(s) + \tilde{P}(x(s))\} \\
&= \{\int_0^t u(s)ds + Lx + x(0) + \int_0^t v(s)ds - a + \int_0^t w(s)ds : \\
&\qquad u(s) \in A(s,x(s)),\ v(s) \in K(s) \text{ and } w(s) \in \tilde{P}(x(s))\} \\
&= F(x) + G(x) + H(x)
\end{aligned}$$

for all $x \in C^n$ with $|x| \geq \sigma > 0$. Hence, by Theorem 3.1, there exists at least one $x \in C^n$ such that $x \in J(x)$. $\square$

Remark 3.4. As in Remark 3.1, condition (iii) of Theorem 3.2 can be replaced by

(iii)′ there exists an $\varepsilon = \varepsilon(A, \rho, \delta) > 0$ such that

$$\alpha(t,r) \leq \varepsilon r \quad \text{for all} \quad r \leq \rho$$

where $\alpha(t,r) = \sup_{|x| \leq r} |P(t,x)|$, $\delta = m(\Delta)$ and $\rho > 0$ is sufficiently large.

Remark 3.5. As in Remark 3.2, in Theorem 3.2, if (iii) holds for all $x \in E$, then ε depends on A and δ only.

In (2), when $B(t,x) \equiv \{0\}$ and $a = 0$, we have the equation:

(3) $$\dot{x}(t) \in Q(t,x(t)), \quad Lx(t) = 0$$

where $Q(t,x) \subset A(t,x) + P(t,x)$.

In this case, $G(x) \equiv \{0\}$. Therefore, following the same proof of the above theorem and applying Corollary 3.1 instead of Theorem 3.1, we obtain

Corollary 3.4. Let A, P and $Q : \Delta \times R^n \to cc(R^n)$ satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x;

(ii) for any $\rho > 0$, there exists an $\varepsilon = \varepsilon(A, \rho, \delta) > 0$ such that

$$\alpha(t,r) \leq \varepsilon r \quad \text{for all} \quad r \leq \rho$$

where $\alpha(t,r) = \sup_{|x| \leq r} |P(t,x)|$ and $\delta = m(\Delta)$ is the measure of Δ;

(iii) $L : C^n \to R^n$ is linear and continuous.

Then, (1) has unique solution $x(t) \equiv 0$ implies that (3) has at least one solution $\varphi(t)$ with $|\varphi| < \rho$ provided ϵ is small enough.

Similarly, applying Corollary 3.2, we have

Corollary 3.5. Let A, P and $Q : \Delta \times R^n \to cc(R^n)$ satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x;

(ii) for some positive number $\rho > 0$, there exist an $\epsilon = \epsilon(A, \rho, \delta) > 0$ such that

$$\alpha(t,\rho) = \sup_{|x| \leq \rho} |P(t,x)| \leq \epsilon\rho$$

where $\delta = m(\Delta)$ is the measure of Δ;

(iii) $L : C^n \to R^n$ is linear and continuous.

Then, (1) has unique solution $x(t) \equiv 0$ implies that (3) has at least one solution $\varphi(t)$ with $|\varphi| < \rho$ provided ϵ is small enough.

From Corollary 3.3 and the way we prove Theorem 3.2, there follows immediately

Corollary 3.6. Let A, B, P and $Q : \Delta \times R^n \to cc(R^n)$ satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x;

(ii) $B(t,x) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over Δ;

(iii) $\frac{\alpha(t,r)}{r} \to 0$ uniformly in t as $r \to \infty$,

where $\alpha(t,r) = \sup_{|x|\leq r} |P(t,x)|$;

(iv) $L : C^n \to R^n$ is linear and continuous.

Then, (1) has unique solution $x(t) \equiv 0$ implies that (2) has at least one solution for any $a \in R^n$.

§3. Some applications:

(a) Periodic solutions: Consider the following contingent differential equations:

(4) $\dot{x}(t) \in A(t,x(t))$;

(5) $\dot{x}(t) \in A(t,x(t)) + B(t,x(t))$; and

(6) $\dot{x}(t) \in A(t,x(t)) + B(t,x(t)) + P(t,x(t))$.

<u>Theorem 3.3</u>. Let A,B and $P : R \times R^n \to cc(R^n)$ be T-periodic in R with $T > 0$ and satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x;

(ii) $B(t,x) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over $[s,s+T]$ for any $s \in R$;

(iii) $P(t,x)$ is Lipschitzian at $0 \in R^n$ with Lipschitz content θ, i.e. $|P(t,x)| \leq \theta|x|$ for any $(t,x) \in R \times R^n$.

Then, (4) has only trivial T-periodic solution implies that (6) has at least one T-periodic solution provided θ is small enough.

Proof: Without loss of generality, we can restrict our consideration of orientor fields on $\Delta = [0,T]$. Define $L : C^n \to R^n$ by $Lx = x(0) - x(T)$. Clearly, L is linear and continuous. Let $\alpha(t,r) = \sup_{|x|\leq r} |P(t,x)|$ as before. By assumption (iii), we have

$$\alpha(t,r) = \sup_{|x|\leq r} |P(t,x)| \leq \sup_{|x|\leq r} \theta|x| = \theta r$$

for any $r > 0$. It follows from Theorem 3.2 with $a=0$ and $Q(t,x) = A(t,x) + B(t,x) + P(t,x)$ that (5) has at least one solution $\varphi(t)$ defined on $[0,T]$ and satisfying $\varphi(0) - \varphi(T) = 0$ provided θ is small enough.

Define $\tilde{\varphi}(t) : R \to R^n$ by $\tilde{\varphi}(t) = \varphi(s)$ where $s = t+kT$, $s \in [0,T)$ and k is some integer. $\tilde{\varphi}$ is continuous since φ is continuous and $\varphi(0) = \varphi(T)$. $\tilde{\varphi}$ is T-periodic by the way we define it. Clearly, $\tilde{\varphi}$ is a solution of (5) since $\tilde{\varphi}$ is a solution of (5) on $[0,T]$ and A,B and P are T-periodic in R. $\square$

Following the same way of proof of the above theorem and applying Corollary 3.5, we have

Corollary 3.7. Let A and $P : R \times R^n \to cc(R^n)$ be T-periodic in R with $T > 0$ and satisfy the Carathéodory conditions. Suppose that

(i) $A(t,x)$ is homogeneous with respect to x;

(ii) there exists a $\theta > 0$ such that

$|P(t,x)| \leq \theta\rho$ for all $t \in R$ and $|x| \leq \rho$

where ρ is some positive number.

Then, (4) has only trivial T-periodic solution implies that (5) has at least one T-periodic solution $\varphi(t)$ with $|\varphi| < \rho$ provided θ is small enough.

Remark 3.6. Theorem 3.3 and Corollary 3.7 generalize Theorem 2.1 and Theorem 2.2, respectively, in [12] (see p.413) from perturbed linear ordinary differential equations to perturbed homogeneous contingent differential equations. However, in Theorem 3.3, we lose the uniqueness.

(b) Optimal solutions:

Proposition 3.2. Let A,B,P,Q and L be defined and satisfy all the conditions (i)-(iv) as in Theorem 3.2 with ϵ sufficiently small and A having unique solution $x(t) \equiv 0$. Then the set of all solutions of (2) is a non-empty compact set in C^n.

Proof: The non-emptiness of the set of solutions of (2) is guaranteed by Theorem 3.2. To show it is compact, it is equivalent to show that it is sequentially compact since C^n is a metric space.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of solutions of (2). Then,

$$\begin{aligned} x_n(t) &\in (S)\int_0^t Q(s,x_n(s))ds + Lx_n - a + x_n(0) \\ &\subset (S)\int_0^t A(s,x_n(s))ds + Lx_n - x_n(0) + (S)\int_0^t K(s)ds \\ &\qquad + (S)\int_0^t \tilde{P}(x_n(s))ds - a \end{aligned}$$

for all $t \in [0,T]$, where $\tilde{P}$ is defined as in the proof of Theorem 3.2. By Lemma 3.1, we have

$$\begin{aligned} |x_n| &= \sup_{t\in[0,T]} |x_n(t)| \\ &\leq \alpha\left(\int_0^T |K(s)|ds + \int_0^T |\tilde{P}(x_n(s))|ds + |a|\right) \\ &\leq \alpha(\tilde{K} + \epsilon T|x_n| + |a|) \end{aligned}$$

for some $\alpha > 0$. Choosing $\epsilon < \frac{1}{2\alpha}$, we have

$$|x_n| \leq 2\alpha(\tilde{K} + |a|)$$

for all $n = 1,2,\cdots$. Hence, $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded.

Moreover,

$$x_n(t) - x_n(s) \in (S)\int_s^t Q(r,x_n(r))dr$$

for any $t,s \in [0,T]$. Since $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded and Q satisfies the Carathéodory conditions, there exists a a function $m(r)$ which is integrable over $[0,T]$ such that

$$|Q(r,x_n(r))| \leq m(r)$$

for all $r \in [0,T]$ and $n = 1,2,\cdots$. Hence,

$$|x_n(t)-x_n(s)| \leq |\int_s^t m(r)dr| \leq |M(t)-M(s)|.$$

As $M \in AC([0,T])$, $\{x_n\}_{n=1}^{\infty}$ is equi-continuous on $[0,T]$.

It follows from the Ascoli lemma that $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $x_{n_j} \xrightarrow{j} x_0 \in C^n$ uniformly on $[0,T]$. Now, for any $t \in [0,T]$,

$$\begin{aligned}
x_0(t) &= \limsup_{j\to\infty} x_{n_j}(t) \\
&\in \limsup_{j\to\infty}((S)\int_0^t Q(s,x_{n_j}(s))ds + Lx_{n_j} - a + x_{n_j}(0)) \\
&\subset \limsup_{j\to\infty}(S)\int_0^t Q(s,x_{n_j}(s))ds + Lx_0 - a + x_0(0) \\
&\subset (S)\int_0^t \limsup_{j\to\infty} Q(s,x_{n_j}(s))ds + Lx_0 - a + x_0(0) \\
&\subset (S)\int_0^t Q(s,x_0(s))ds + Lx_0 - a + x_0(0).
\end{aligned}$$

Therefore, x_0 is indeed a solution of (2). □

As we know that any real valued lower (resp. upper) semi-continuous function assumes minimum (resp. maximum) on a compact set, from Proposition 3.2 there follows immediately

<u>Theorem 3.4</u>. (existence of optimal solutions). Let A,B,P,Q and L be defined and satisfy all the conditions as in Theorem 3.2. Furthermore, a lower (resp. upper) semi-continuous functional $T : C^n \to R$ is given. Then, if (1) has unique solution $x(t) \equiv 0$, for any $a \in R^n$ there exists a solution of (2) which minimizes (resp. maximizes) T provided ϵ is small enough.

<u>Remark 3.7</u>. Similar existence theorems of optimal solutions can be formulated corresponding to Theorem 3.3 and Corollaries 3.4-3.7.

Before discussing further applications, we give

<u>Definition 3.2</u>. An orientor field $F : \Delta \times R^n \to cc(R^n)$ is said to satisfy the strong Carathéodory conditions if

(i) $F(t,x)$ is upper semi-continuous in t in the sense of Definition 1.4 for each fixed $t \in \Delta$;

(ii) $F(t,x)$ is measurable in t for each fixed $x \in R^n$;

(iii) there exist functions $\varphi(t)$ and $\psi(t)$ which are integrable over Δ such that

$$|F(t,x)| \leq \varphi(t)|x| + \psi(t)$$

for all $(t,x) \in \Delta \times R^n$.

<u>Remark 3.8</u>. It is clear that a mapping satisfying the strong Carathéodory conditions must satisfy the Caratheodory conditions. The converse is not true even when the orientor field is compact. Consider $F(t,x) : \Delta \times R \to R$ defined by $F(t,x) = te^x$. F satisfies the Carathéodory conditions and is compact but it does not satisfy the strong Carathéodory conditions. Therefore, all the results in this chapter hold true for A, B, P and Q satisfying the strong Carathéodory conditions.

(C) Nicoletti problem: Given $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ and $a = (a_1, a_2, \ldots, a_n) \in R^n$, we shall consider the existence of a solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of (6) which satisfies the Nicoletti conditions [28]:

(7) $$x_i(t_i) = a_i\ , \quad i = 1,2,\ldots,n.$$

Also, we consider the functional $T : C^n \to R$ defined by

(8) $$T(x) = \int_0^T |x(t)|dt.$$

<u>Lemma 3.5</u>. (Lasota and Olech: [22]). Suppose the function φ from $[0,T]$ into R is Lebesgue integrable and non-negative. Consider the differential inequality for an n vector valued function

(9) $$|\dot{x}(t)| \leq \varphi(t)|x(t)|\ , \quad 0 \leq t \leq T$$

and the boundary value conditions

(10) $$x_i(t_i) = 0, \quad 0 \leq t_i \leq T, \quad i = 1,2,\ldots,n.$$

If $\int_0^T \varphi(t)dt < \frac{\pi}{2}$, then $x(t) \equiv 0$ is the unique solution of (9) and (10).

<u>Theorem 3.5</u>. Let A, B and $P : [0,T] \times R^n \to cc(R^n)$ satisfy the strong Carathéodory conditions. Suppose that

(i) $|A(t,x)| \leq \varphi(t)|x| + \psi(t)$ with $\int_0^T \varphi(t)dt < \frac{\pi}{2}$;

(ii) $B(t,x) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over $[0,T]$;

(iii) $\frac{|P(t,x)|}{|x|} \to 0$ uniformly in t as $|x| \to \infty$.
Then, (6) has a solution which satisfies the Nicoletti conditions (7) and also minimizes (or maximizes) the functional (8).

<u>Proof</u>: Define $\tilde{A}, \tilde{B}$ and $Q : [0,T] \times R^n \to cc(R^n)$ by

$$\tilde{A}(t,x) = \{u \in R^n : |u| \leq \varphi(t)\,|x|\},$$

$$\tilde{B}(t,x) = B(t,x) + \{u \in R^n : |u| \leq \psi(t)\},$$

and $$Q(t,x) = A(t,x) + B(t,x) + P(t,x).$$

Clearly, $Q(t,x) \subset \tilde{A}(t,x) + \tilde{B}(t,x) + P(t,x)$. Let $L : C^n \to R^n$ defined by $Lx = (x_1(t_1), x_2(t_2), \ldots, x_n(t_n))$. One can see easily that $\tilde{A}, \tilde{B}, P, Q$ and L satisfy all the assumptions of Corollary 3.6. Since T is a continuous functional, our proof follows immediately from Lemma 3.5, Theorem 3.4, and Remark 3.7. □

(d) Aperiodic boundary value problem: Here, we shall consider the existence of a solution $x(t)$ of (6) which satisfies the aperiodic boundary condition:

(11) $$x(0) + \lambda x(T) = 0 \quad \text{where} \quad \lambda > 0$$

and also minimizes (or maximizes) the functional (8).

Another lemma of differential inequality is needed:

<u>Lemma 3.6</u>. (Kasprzyk and Myjak: [16]). If $\varphi(t) \geq 0$ and $\int_0^T \varphi(t)\,dt < (\pi^2 + \log^2 \lambda)^{1/2}$, then $x(t) \equiv 0$ is the unique solution of (9) and (11).

If we set $Lx = x(0) + \lambda x(1)$ and $a=0$, a result analogous to Theorem 3.5 follows from Lemma 3.6 and Theorem 3.4. As the proof is similar, we give the statement as follows:

Theorem 3.6. Let A, B and $P : [0,T] \times R^n \to cc(R^n)$ satisfy the strong Carathéodory conditions. Suppose that

(i) $|A(t,x)| \leq \varphi(t)|x| + \psi(t)$ with $\int_0^T \varphi(t)dt < (\pi^2 + \log^2 \lambda)^{1/2}$;

(ii) $B(t,x) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over $[0,T]$;

(iii) $\dfrac{|P(t,x)|}{|x|} \to 0$ uniformly in t as $|x| \to \infty$.

Then, (6) has a solution which satisfies the aperiodic condition (11) and also minimizes (or maximizes) the functional (8).

Chapter IV

PERIODIC SOLUTIONS OF CONTINGENT FUNCTIONAL EQUATIONS

In control problems, it may happen that the control system is described by a functional differential equation. Therefore, by eliminating the control term, we obtain a contingent functional differential equation. In this chapter, we shall consider the periodic solutions of such equations and formulate an existence theorem for Fredholm's alternative analogous to Theorem 3.2.

Suppose $r \geq 0$ is a given real number, $R = (-\infty, \infty)$ and R^n is the n-dimensional Euclidean space with norm $|\cdot|$. Let $C_r = C([-r,0],R^n)$ be the Banach space of all continuous functions from $[-r,0]$ into R^n with the norm of each element φ, $\|\varphi\| = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|$. For any function $x \in C(R,R^n)$ and $t \in R$, we define a function $x_t : [-r,0] \to R^n$ by

$$x_t(\theta) = x(t+\theta)$$

where $-r \leq \theta \leq 0$. Clearly, $x_t \in C_r$. The function x_t can be considered as the segment of $x(\tau)$ defined on $[t-r,t]$ and translated to $[-r,0]$.

Definition 4.1. A mapping $F : R \times C_r \to \text{Comp}\,(R^n)$ is called a functional orientor field. And a relation of the form

(F) $$\dot{x}(t) \in F(t,x_t)$$

is called a contingent functional differential equation.

Definition 4.2. For a fixed $x \in C(R,R^n)$, we say a function $F(t,x_t) : R \times C_r \to \text{Comp}\,(R^n)$ is measurable in t if the function $\tilde{F} : R \to \text{Comp}\,(R^n)$ defined by

$$\tilde{F}(t) = F(t,x_t)$$

is measurable.

Definition 4.3. For a fixed $t \in R$, we say a function $F(t,x_t)$ from $R \times C_r$ into $\text{Comp}\,(R^n)$ is upper semi-continuous in x (in the sense of metric) if, for any $\varepsilon > 0$ and any $x \in C(R,R^n)$, there exists a $\delta > 0$ such that

$$F(t,y_t) \subset B_\varepsilon(F(t,x_t))$$

for all $y \in C(R,R^n)$ with $\|y_t - x_t\| < \delta$.

Definition 4.4. We say a functional orientor field $F(t,x_t) : R \times C_r \to \text{Comp}\,(R^n)$ satisfies the Carathéodory conditions if

(i) $F(t,x_t)$ is measurable in t for each fixed $x \in C(R,R^n)$;

(ii) $F(t,x_t)$ is upper semi-continuous in x for each fixed $t \in R$;

(iii) for any closed and bounded subset D of $R \times C_r$, $|F(D)|$ is bounded.

The equation (F) is said of the <u>Carathéodory type</u> if its functional orientor field F satisfies the Carathéodory conditions.

<u>Remark 4.1</u>. It follows from condition (iii) of Definition 4.4 that a functional orientor field satisfying the Carathéodory conditions must be compact.

<u>Definition 4.5</u>. A function $x(t)$ is said to be a <u>solution of (F) in the sense of Marchaud</u> if there exist $t_0 \in R$ and $A > 0$ such that

$$x(t) \in C([t_0-r, t_0+A], R^n) \quad \text{and}$$
$$D^* x(t) \subset F(t, x_t) \quad \text{for almost every} \quad t \in [t_0, t_0+A].$$

<u>Definition 4.6</u>. A function $x(t)$ is said to be a <u>solution of (F) in the sense of Ważewski</u> if there exist $t_0 \in R$ and $A > 0$ such that

$$x(t) \in C([t_0-r, t_0+A], R^n),$$
$$x(t) \in AC([t_0, t_0+A]) \quad \text{and}$$
$$\dot{x}(t) \in F(t, x_t) \quad \text{for almost every} \quad t \in [t_0, t_0+A].$$

<u>Definition 4.7</u>. A function $x(t)$ is said to be a <u>solution of (F)</u> if there exist $t_0 \in R$ and $A > 0$ such that

$$x(t) \in C([t_0-r, t_0+A], R^n),$$
$$x(t) \in AC([t_0, t_0+A]) \quad \text{and}$$
$$x(t) \in x(t_0) + (S)\int_{t_0}^{t} F(s, x_s)ds \text{ for } t \in [t_0, t_0+A].$$

The following proposition shows that under certain conditions the solutions of (F) defined above are all equivalent:

Proposition 4.1. (Kikuchi [18]). Let $P(F)$, $Y(F)$ and $T(F)$ be the collection of all solutions of (F) with respect to Definitions 4.5, 4.6 and 4.7 respectively. Suppose that

(i) $F(t, x_t)$ satisfies the Carathéodory conditions;

and

(ii) $F(t, x_t) \in cc(R^n)$ for each $(t, x_t) \in R \times C_r$.

Then, $P(F) = Y(F) = T(F)$.

We shall consider the periodic solution of the following contingent functional differential equations of retarded type:

(1) $$\dot{x}(t) \in A(t, x_t)$$

(2) $$\dot{x}(t) \in Q(t, x_t)$$

where $Q(t, x_t) \subset A(t, x_t) + B(t, x_t) + P(t, x_t)$.

Theorem 4.1. (Fredholm's alternative). Let A, B, P and $Q : R \times C_r \to cc(R^n)$ satisfy the Carathéodory conditions and be T-periodic in R for some $T > 0$. Suppose that

(i) $A(t,x_t)$ is homogeneous with respect to x; i.e.

$$A(t,\lambda x_t) = \lambda A(t,x_t)$$

for all $\lambda \in R$ and $x_t \in C_r$;

(ii) $B(t,x_t) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over any T-interval $[t,t+T]$;

(iii) there exists an $\varepsilon = \varepsilon(A,\rho,T) > 0$ such that

$$\alpha(t,m) \leq \varepsilon m \quad \text{for all} \quad m \geq \rho > 0$$

where $\rho > 0$ is arbitrary and

$$\alpha(t,m) = \sup_{\|x_t\| \leq m} |P(t,x_t)|.$$

Then, (1) has $x(t) \equiv 0$ as a unique T-periodic solution implies that (2) has at least one T-periodic solution, provided ε is small enough.

<u>Proof</u>: Let $\mathcal{P}_T$ denote the set of all continuous T-periodic functions from R into R^n, $K(C^n)$ denote the set of all non-empty convex subsets of C^n and $L_1^{loc}(R)$ denote the set of all functions from R into R^n which are integrable over any finite interval in R.

(i) Let $P^*(t,x_t) : R \times C_r \to cc(R^n)$ be defined by

$$P^*(t,x_t) = \{a \in R^n : |a| \leq \varepsilon\|x_t\|\}.$$

Clearly, P^* is well-defined and $P(t,x_t) \subset P^*(t,x_t)$ for any $(t,x_t) \in R \times C_r$. One can check easily that P^* satisfies the Carathéodory conditions.

(ii) Define the operators F, G, H and $J : \mathcal{P}_T \to K(C^n)$ by

$$F(x) = \{\int_{-\infty}^{t} e^{-(t-s)}[x(s)+u(s)]ds : u \in L_1^{loc}(R) \text{ and } u(s) \in A(s,x_s) \text{ for all } s \in R\},$$

$$G(x) = \{\int_{-\infty}^{t} e^{-(t-s)} u(s)ds : u \in L_1^{loc}(R) \text{ and } u(s) \in K(s) \text{ for all } s \in R\},$$

$$H(x) = \{\int_{-\infty}^{t} e^{-(t-s)} u(s)ds : u \in L_1^{loc}(R) \text{ and } u(s) \in P^*(s,x_s) \text{ for all } s \in R\},$$

and

$$J(x) = \{\int_{-\infty}^{t} e^{-(t-s)}[x(s)+u(s)]\, ds : u \in L_1^{loc}(R) \text{ and } u(s) \in Q(s,x_s) \text{ for all } s \in R\}.$$

From our hypothesis that $B(t,x_t)$ is T-periodic in R, we can assume without loss of generality that the ball $K(t)$ which contains $B(t,x_t)$ is also T-periodic. All the improper integrals defined above converge since x, A, K, P^* and Q are periodic (hence bounded by (iii) of Definition 4.4). For each $x \in \mathcal{P}_T$, $A(t,x_t)$ is measurable in t by (i) of Definition 4.3 and is integrably bounded by (iii) of Definition 4.4 and the periodicity of $A(t,x_t)$. Hence, $F(x)$ is not empty. The convexity follows immediately from the fact that A is convex valued. Therefore, F is well-defined. Similarly, G, H and J are well-defined.

(iii) Let us define $\tilde{F}, \tilde{G}, \tilde{H}$ and $\tilde{J} : \mathcal{O}_T \to cc(\mathcal{O}_T)$ by

$$\tilde{F}(x) = F(x) \cap \mathcal{O}_T, \qquad \tilde{G}(x) = G(x) \cap \mathcal{O}_T$$
$$\tilde{H}(x) = H(x) \cap \mathcal{O}_T \quad \text{and} \quad \tilde{J}(x) = J(x) \cap \mathcal{O}_T.$$

Since for each $x \in \mathcal{O}_T$ $A(t,x_t)$ is T-periodic, measurable and bounded, there exists a T-periodic function $f_x(t) \in L_1^{loc}(R)$ such that $f_x(t) \in A(t,x_t)$ for all $t \in R$. This is possible because we can have a measurable selection $f_x(t)$ on $[0,T]$ first with $f_x(0) = f_x(T)$ and then duplicate it on the intervals $[kT,(k+1)T]$ where k is a non-zero integer.

Let $\tilde{f}_x(t) : R \to R^n$ defined by

$$\tilde{f}_x(t) = \int_{-\infty}^{t} e^{-(t-s)}[x(s)+f_x(s)]ds.$$

It is clear that $\tilde{f}_x \in F(x) \cap \mathcal{O}_T$. Hence, $\tilde{F}(x)$ is not empty. Since $\mathcal{O}_T$ is a convex subspace of C^n, $\tilde{F}(x)$ is convex. By using the Ascoli lemma and Lemma 3.3 as we did in the proof of Proposition 3.1, one finds that the set

$$F(x)|_{[0,T]} = \{f_x(t)|_{[0,T]} : f_x \in F(x)\}$$

is compact in $C([0,T],R^n)$. However, since $\tilde{F}(x)$ is a family of T-periodic continuous functions, a sequence of functions in $\tilde{F}(x)$ converges uniformly in $[0,T]$ implies that it converges uniformly in R and the limit function is also T-periodic. Hence, for each $x \in \mathcal{O}_T$, $\tilde{F}(x)$ is indeed compact in $\mathcal{O}_T$. Therefore, $\tilde{F}$ is well-defined. Similarly, $\tilde{G}, \tilde{H}$ and $\tilde{J}$ are all well-defined.

(iv) $\tilde{F}, \tilde{G}, \tilde{H}$ and $\tilde{J}$ are compact: Let $D = \{x \in \mathcal{C}_T : |x| \le K\}$ be a bounded subset of $\mathcal{C}_T$. We want to show that $\tilde{F}(D)$ is relatively compact. In a metric space, it is equivalent to show that $\overline{\tilde{F}(D)}$ is sequentially compact. Let $\{y_n\}_{n=1}^{\infty} \subset \tilde{F}(D) \subset F(D)$. Then,

$$y_n(t) \in (S)\int_{-\infty}^{t} e^{-(t-s)}[x_n(s)+A(s,x_{n_s})]ds$$

where $x_n \in D$. Since D is bounded and $A(t,x_t)$ is periodic in t, it follows from Remark 4.1 that $|A(s,x_{n_s})| \le \tilde{K}$ for all $s \in R$ and $n = 1,2,\cdots$. Hence,

$$|y_n(t)| \le \int_{-\infty}^{t} e^{-(t-s)}(K+\tilde{K})ds = (K+\tilde{K})e^{-t}\int_{-\infty}^{t} e^{s}\, ds$$
$$= K + \tilde{K}$$

for all $t \in R$ and $n = 1,2,\cdots$. Therefore, $\{y_n\}_{n=1}^{\infty}$ is uniformly bounded.

For any $t_1, t_2 \in R$, without loss of generality, we may assume $t_1 < t_2$. Let $B_{K+\tilde{K}}$ be the ball in R^n centered at 0 with radius $K + \tilde{K}$. Clearly,

$$x_n(s) + A(s,x_{n_s}) \subset B_{K+\tilde{K}}$$

for all $n = 1,2,...$ and $s \in R$. It is easy to see that

$$y_n(t) \in e^{-t}(S)\int_{-\infty}^{t} e^{s}\, B_{K+\tilde{K}}\, ds$$

for all $n = 1,2,...$ and $s,t \in R$. For any n, let

$$u_1(s) : (-\infty,t_1] \to R^n$$
$$u_2(s) : (-\infty,t_2] \to R^n$$

be integrable selections of $B_{K+\tilde{K}}$ and $u_2(S)|(-\infty,t_1] = u_1(s)$

such that

$$y_n(t_1) = e^{-t_1}\int_{-\infty}^{t_1} e^s\, u_1(s)ds \quad\text{and}\quad y_n(t_2) = e^{-t_2}\int_{-\infty}^{t_2} e^s\, u_2(s)ds.$$

Then

$$\begin{aligned} y_n(t_1)-y_n(t_2) &= e^{-t_1}\int_{-\infty}^{t_1} e^s\, u_1(s)ds - e^{-t_2}\int_{-\infty}^{t_1} e^s\, u_1(s)ds \\ &\qquad - e^{-t_2}\int_{t_1}^{t_2} e^s\, u_2(s)ds \\ &\le (e^{-t_1}-e^{-t_2})\int_{-\infty}^{t_1} e^s(K+\tilde{K})ds + e^{-t_2}\int_{t_1}^{t_2} e^s(K+\tilde{K})ds \\ &= (K+\tilde{K})[(e^{-t_1}-e^{-t_2})(e^{t_1}-1)+(1-e^{t_1-t_2})] \\ &= (K+\tilde{K})[2(1-e^{t_1-t_2})+(e^{-t_2}-e^{-t_1})]. \end{aligned}$$

Hence,

$$|y_n(t_1)-y_n(t_2)| \le (K+\tilde{K})[2|1-e^{t_1-t_2}|+|e^{-t_2}-e^{-t_1}|].$$

Therefore, $\{y_n\}_{n=1}^{\infty}$ is equi-continuous in R.

It follows from Ascoli's that $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_j\}_{j=1}^{\infty}$ such that $y_j \overset{j}{\to} y_0$ uniformly on $[0,T]$. Since the y_j's are T-periodic, $y_j \overset{j}{\to} y_0$ uniformly on R and $y_0 \in \theta_T$. Therefore, $\tilde{F}$ is compact. Similarly, one can show that $\tilde{G},\tilde{H}$ and $\tilde{J}$ are all compact.

(v) $\tilde{F},\tilde{G},\tilde{H}$ and $\tilde{J}$ are upper semi-continuous: From parts (iii) and (iv) of our proof, we know that $\tilde{F},\tilde{G},\tilde{H}$ and $\tilde{J}$ are compact mappings from θ_T into $cc(\theta_T)$. By Theorem 1.1, all the definitions of upper semi-continuity are equivalent. Let $x_0 \in \theta_T$ be arbitrary. Let $\{x_n\}_{n=1}^{\infty} \subset \theta_T$ and $\{y_n\}_{n=0}^{\infty} \subset \theta_T$ such that $x_n \overset{n}{\to} x_0$, $y_n \overset{n}{\to} y_0$ and $y_n \in \tilde{F}(x_n) \subset F(x_n)$ for $n = 1,2,\cdots$. Then,

$$y_n(t) = \int_{-\infty}^{t} e^{-(t-s)}[x_n(s)+u_n(s)]ds$$

where $u_n \in L_1^{loc}(R)$ and $u_n(s) \in A(s,x_{n_s})$ for $s \in R$.

Applying Lemma 3.3 as we did in the proof of Proposition 3.1, one can see easily that

$$y_0(t) = \int_{-\infty}^{t} e^{-(t-s)}[x_0(s)+u_0(s)]ds$$

where $u_0 \in L_1^{loc}(R)$ and $u_0(s) \in A(s,x_{0_s})$ for $s \in R$. We have $y_0 \in \tilde{F}(x_0)$. Therefore, $\tilde{F}$ is upper semi-continuous in the sense of Definition 1.5 (hence in the sense of all others). The upper semi-continuity of $\tilde{G},\tilde{H}$ and $\tilde{J}$ can be proved similarly.

(vi) By direct computations, we know that any fixed point of $\tilde{F}$ (resp. $\tilde{J}$) is a T-periodic solution of (1) (resp. (2)). Therefore, it is equivalent for us to show that the operator $\tilde{J}$ has at least one fixed point for small ε if $\tilde{F}$ has only $x(t) \equiv 0$ as its fixed point.

(vii) It follows easily from assumption (i) that F (hence $\tilde{F}$) is homogeneous.

(viii) Consider the function $\beta : R \to R$ defined by

$$\beta(t) = \int_{-\infty}^{t} e^{-(t-s)} |K(s)|ds.$$

Since $K(t)$ is T-periodic, by using the transformation $s' = s-T$, we have

$$\begin{aligned}\beta(t+T) &= \int_{-\infty}^{t+T} e^{-(t+T-s)} |K(s)|ds \\ &= \int_{-\infty}^{t} e^{-(t-s')} |K(s'+T)|ds' \\ &= \int_{-\infty}^{t} e^{-(t-s')} |K(s')|ds' = \beta(t)\end{aligned}$$

for any $t \in R$. Hence, $\beta(t)$ is T-periodic. Clearly, $\beta(t)$ is continuous. Hence, $\beta(t)$ is bounded on $[0,T]$. Since $\beta(t)$ is T-periodic, $\beta(t)$ is bounded on R. Let

$$M = \|\beta\| = \sup_{t\in R} |\beta(t)| = \sup_{t\in[0,T]} \beta(t) < \infty.$$

Then

$$\|G\| = \sup_{x\in\mathscr{O}_T} |G(x)| \leq \sup_{t\in R} \int_{-\infty}^{t} e^{-(t-s)} |K(s)| ds = \sup_{t\in R} \beta(t)$$
$$= M.$$

Since $\tilde{G}(x) \subset G(x)$ for all $x \in \mathscr{O}_T$, we have $\|\tilde{G}\| \leq M$.

(ix) For each $x \in \mathscr{O}_T$ such that $\|x\| = m \geq \rho > 0$, by the way we define $H(x)$ and P^*, one has

$$\begin{aligned}
|H(x)| &\leq \sup_{t\in R} \int_{-\infty}^{t} e^{-(t-s)} |P^*(s,x_s)| ds \\
&\leq \sup_{t\in R} \int_{-\infty}^{t} e^{-(t-s)} \epsilon\|x_s\| ds \\
&\leq \sup_{t\in R} \int_{-\infty}^{t} e^{-(t-s)} \epsilon m\, ds \\
&= \epsilon m \sup_{t\in R} e^{-t} \int_{-\infty}^{t} e^{s} ds = \epsilon\|x\|.
\end{aligned}$$

Since $\tilde{H}(x) \subset H(x)$, it follows that $|\tilde{H}(x)| \leq \epsilon\|x\|$ for all $x \in \mathscr{O}_T$ with $\|x\| \geq \rho > 0$.

(x) For every $(t,x_t) \in R\times C_r$, we have

$$\begin{aligned}
Q(t,x_t) &\subset A(t,x_t) + B(t,x_t) + P(t,x_t) \\
&\subset A(t,x_t) + K(t) + P^*(t,x_t).
\end{aligned}$$

For each $x \in \mathscr{O}_T$, $K(t)$ and $P^*(t,x_t)$ are balls in R^n centered at 0 with radii $|K(t)|$ and $\epsilon\|x_t\|$ respectively.

Hence, by Lemma 3.4, we have

$$J(x) \subset F(x) + G(x) + H(x)$$

for any $x \in \mathcal{P}_T$. It follows that for any $j_x \in J(x) \cap \mathcal{P}_T$ we have

$$j_x(t) = f_x(t) + g_x(t) + h_x(t)$$

where $f_x \in F(x)$, $g_x \in G(x)$, $h_x \in H(x)$ and $t \in R$. Since $j_x \in \mathcal{P}_T$, we can restrict our consideration on $[0,T]$. Let

$$f_x(t) = \int_{-\infty}^{t} e^{-(t-s)}[x(s)+u(s)]ds,$$

$$g_x(t) = \int_{-\infty}^{t} e^{-(t-s)} v(s)ds,$$

and $$h_x(t) = \int_{-\infty}^{t} e^{-(t-s)} w(s)ds,$$

where $u,v,w \in L^1([0,T])$ and $u(s) \in A(s,x_s)$, $v(s) \in K(s)$ and $w(s) \in P^*(s,x_s)$ for all $s \in [0,T]$. Without loss of generality, we may assume that $u(0) = u(T)$, $v(0) = v(T)$ and $w(0) = w(T)$.

Define u^*, v^* and $w^* : R \to R^n$ by

$$u^*(t) = u(s) \quad \text{for} \quad t = kT + s$$

$$v^*(t) = v(s) \quad \text{for} \quad t = kT + s$$

$$w^*(t) = w(s) \quad \text{for} \quad t = kT + s$$

where k is an integer and $s \in [0,T)$. Clearly, u^*, v^* and w^* are all T-periodic by the way we define them and $u^*,v^*,w^* \in L_1^{loc}(R)$. It is also evident that $u^*(t) \in A(t,x_t)$, $v^*(t) \in K(t)$ and $w^*(t) \in P^*(t,x_t)$ for all $t \in R$. Now, define $\tilde{f}_x$, $\tilde{g}_x$ and $\tilde{h}_x : R \to R^n$ by

$$\tilde{f}_x(t) = \int_{-\infty}^{t} e^{-(t-s)}[x(s)+u^*(s)]ds,$$

$$\tilde{g}_x(t) = \int_{-\infty}^{t} e^{-(t-s)} v^*(s)ds,$$

and
$$\tilde{h}_x(t) = \int_{-\infty}^{t} e^{-(t-s)} w^*(s)ds.$$

One can see esaily that $\tilde{f}_x \in \tilde{F}(x)$, $\tilde{g}_x \in \tilde{G}(x)$ and $\tilde{h}_x \in \tilde{H}(x)$. It is also clear that

$$j_x(t) = \tilde{f}_x(t) + \tilde{g}_x(t) + \tilde{h}_x(t)$$

for all $t \in R$. Hence, $\tilde{J}(x) \subset \tilde{F}(x) + \tilde{G}(x) + \tilde{H}(x)$ for all $x \in \mathcal{O}_T$.

(xi) From parts (iii)-(x), we have shown that $\tilde{F}, \tilde{G}, \tilde{H}$ and $\tilde{J}$ satisfy all the hypotheses of Theorem 3.1. Hence, if (1) has only trivial T-periodic solution (i.e. $\tilde{F}$ has only 0 as its fixed point). Then it follows from Theorem 3.1 that $\tilde{J}$ has at least one fixed point $x \in \mathcal{O}_T$ which is a T-periodic solution of (2) provided ε is small enough. □

<u>Remark 4.2</u>. In view of Remark 3.1, assumption (iii) of Theorem 4.1 can be replaced by

(iii)′ there exists an $\varepsilon = \varepsilon(A,\rho) > 0$ such that

$$\alpha(t,m) \leq \varepsilon m \quad \text{for all} \quad m \leq \rho$$

where ρ is a sufficiently large number

and $\alpha(t,m) = \sup_{\|x_t\| \leq m} |P(t,x_t)|$.

From Corollary 3.3 and the way we prove Theorem 4.1, there follows immediately

Corollary 4.1. Let A, B, P and $Q : R \times C_r \to cc(R^n)$ satisfy the Carathéodory conditions and be T-periodic in R for some $T > 0$. Suppose that

(i) $A(t,x_t)$ is homogeneous with respect to x;

(ii) $B(t,x_t) \subset K(t)$ where $K(t)$ is a ball in R^n centered at 0 with radius $|K(t)|$ such that $|K(t)|$ is integrable over any T-interval $[t,t+T]$;

(iii) $\frac{\alpha(t,m)}{m} \to 0$ uniformly in t as $m \to \infty$, where $\alpha(t,m) = \sup_{\|x_t\| \leq m} |P(t,x_t)|$.

Then, (1) has unique T-periodic solution $x(t) \equiv 0$ implies that (2) has at least one T-periodic solution.

Remark 4.3. Corollary 4.1 is a generalization of a recent result by R. Funnel (see [9]).

In (2), when $B(t,x_t) \equiv 0$, we have the following equation:

(3) $\dot{x}(t) \in Q(t,x_t)$ where $Q(t,x_t) \subset A(t,x_t) + P(t,x_t)$.

In this case, it is clear that $\tilde{G}(x) \equiv G(x) \equiv \{0\}$. Therefore, following the same proof of Theorem 4.1 and applying Corollary 3.5 instead of Theorem 3.1, we obtain

Corollary 4.2. Let A, P and $Q : R \times C_r \to cc(R^n)$ satisfy the Carathéodory conditions and be T-periodic in R for some $T > 0$. Suppose that

(i) $A(t,x_t)$ is homogeneous with respect to x;

(ii) there exists a $\theta > 0$ such that

$$|P(t,x_t)| \leq \theta\rho$$

for all $t \in R$ and $\|x_t\| \leq \rho$, where ρ is some positive number.

Then, (1) has only trivial T-periodic solution implies that (3) has at least one T-periodic solution $\varphi(t)$ with $|\varphi| < \rho$ provided θ is small enough.

Remark 4.4. Theorem 4.1 and Corollary 4.2 generalize Theorem 2.1 and Theorem 2.2, respectively, in [12] (p.413) to perturbed homogeneous contingent functional differential equations. However, for the generalization of Theorem 2.1, we lose the uniqueness.

Remark 4.5. It is easy to observe that the results of this chapter become the periodic case of the results of the previous chapter when the time lag $r = 0$.

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