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NUMERICAL DERIVATION OF A MOMENTUM EQUATION
OF HIGH FLOW RATE FLOWS IN POROUS MEDIA

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NUMERICAL DERIVATION OF A MOMENTUM EQUATION OF HIGH FLOW RATE FLOWS IN POROUS MEDIA

By

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ABSTRACT

NUMERICAL DERIVATION OF A MOMENTUM EQUATION OF HIGH FLOW RATE FLOWS IN POROUS MEDIA

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The objective of this thesis was to develop a relationship between the specific discharge or Darcian velocity and the pressure drop in a porous material when inertia effects are important. An array of circular cylinders with uniform dimension was assumed as a model of the porous medium. By considering the viscous incompressible flow in the narrow gap between circular cylinders, the dimensionless form of conservation equations were developed.

These equations are solved in stream function form by quasi-linearizing, finite differencing, and applying successive over-relaxation to the resulting difference equations. The pressure drop across the cylinders is calculated from the stream function. It is found that the pressure drop is a quadratic function of specific discharge, in good agreement with experimental observations.

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Finally, I wish to express my greatest thanks to my father for his love, support, and understanding throughout the thesis work and all my years of study.

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NOMENCLATURE

A_s	: surface area of the pores
d	: diameter of grain
E_x	: the energy of flow per unit weight in x-direction
g	: gravity acceleration
h	: piezometric head
H_η	: differential step size of η coordinate
H_ξ	: differential step size of ξ coordinate
i,j	: integer term indices
J	: hydraulic gradient
J_1, J_2, J_3	: hydraulic gradients along the axes
K	: hydraulic conductivity
K	: permeability 1
L	: the average free distance between two grains
M	: permeability 2
p	: pressure
q	: specific discharge
Re, Re_A, Re^*	: Reynolds number
s	: total cross-sectional area
u,v,w	: the corresponding components of the velocity vector v
u^*, v^*, w^*	
u_s, v_s	

v_b : bulk volume

v_v : volume of void space

v_s : the volume of solids within v_b

X^*, Y^* : the axes of a cartesian system of coordinates

X,Y,Z

Greek letters

$\alpha, \beta, \beta_1, \beta_2$: numerical shape factors

ϵ : porosity

ρ : density of the mixture

μ : dynamic viscosity

γ : kinematic viscosity

Σ : specific surface

Σ_s : specific area

ψ : stream function

η, ξ : transformed coordinates

ω : the relaxation parameter

Subscripts

s : area

x,y,z : the axes of a cartesian coordinate

Superscript

-	:	average value
*	:	value with dimension, special index
k	:	value of previous iteration

CHAPTER 1

Introduction

Many engineering processes involve the flow of a fluid through a porous medium. Examples include packed bed chemical reactors, degraded nuclear reactor cores, thermal and chemical pollution of aquifers, and secondary oil recovery in a petroleum reservoir. Historically, one may trace the orgins of an engineering concern for flows in porous media to the ancient Romans and their beautiful fountains as evidenced by the Trevi Fountain in Rome. These fountains were once flow through devices which employed the pressurized water from the underground rivers of Rome. By standing the test of time these fountains demonstrate that the ancient Romans understood the engineering principles of flow through porous medium , but it was left to a French civil servant to express these principles in mathematical form. While designing the public water works for the village of Dijon, Darcy(1856) concluded that the specific discharge through a porous medium is directly proportional to the pressure drop across the medium. Darcy's law, as it has become known has, can be expressed mathematically for one dimensional flow in porous media as

$$- \frac{dP}{dX} = \frac{\mu}{K} q \qquad (1\text{-}1)$$

where q is the specified discharge and K is the permeability, a property of the porous medium which will be discussed later.

Though Darcy's law is a sound mathematical model for homogeneous flow through porous media there are a number of engineering applications when it is not appropriate. For non-homogeneous porous media Boussinesq(1904) and Irmay(1958) extended the linear equation of Darcy's. Similar extensions have been developed by Schneebeli(1957) and Ferrandon(1948) for non-isotropic media, by Muskat(1937) for compressible fluid flow, and by Anderson(1942) for solutions and adsorptive media. A detailed review of several modifications of Darcy's law is given by Muskat(1937). All of the physical processes listed above require some modification in Darcy's law. However the most serious breakdown of the mathematical model occurs when the flow rate is large. In Figure 1 the pressure drop-specific discharge data of Ahmed(1969) is plotted. It is seen that at small flow rates a linear relationship exists between pressure gradient and specific discharge. However, as the flow rate grows the data shows a marked deviation from the linear relationship of Darcy's law. In viewing this trend in his own experimental data, Forchheimer(1930) proposed a quadratic relationship between pressure gradient and specific discharge. The Forchheimer extension to Darcy's law can be expressed mathematically for one dimensional flow as

$$-\frac{dP}{dX} = \frac{\mu}{K} q + \frac{\rho}{M} q^2 \qquad (1\text{-}2)$$

where M has been called the Forchheimer coefficient and is dependent on the physical character of the porous medium. It is the theoretical basis of Eq(1-2) which is the main concern of this thesis. As it currently stands the Forchheimer extension may be considered semi-empirical and heuristic in nature.

In this thesis first principles will be used to verify the form of Eq(1-2) for high flow rate flows in porous media, to identify the

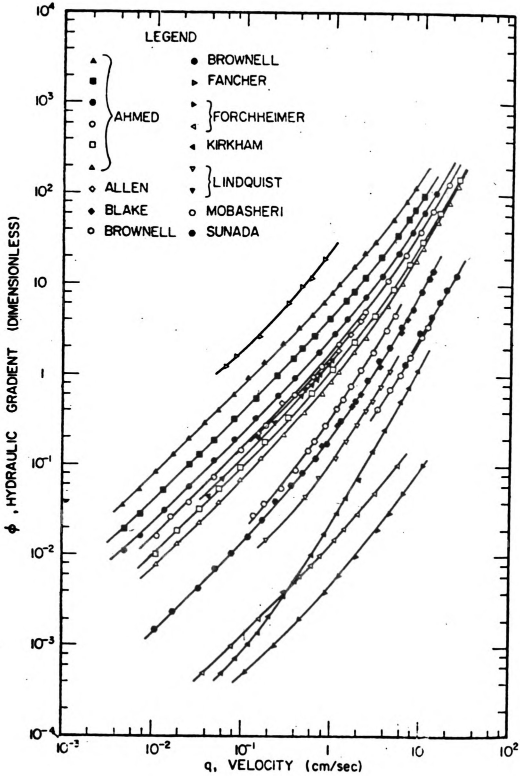

LEGEND
AHMED
ALLEN
BLAKE
BROWNELL
BROWNELL
FANCHER
FORCHHEIMER
KIRKHAM
LINDQUIST
MOBASHERI
SUNADA
φ, HYDRAULIC GRADIENT (DIMENSIONLESS)
q, VELOCITY (cm/sec)
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
10^{2}

Fig. 1

physical process giving rise to the quadratic term, and to develop an expression for the Forchheimer coefficient in terms of the geometric parameters of porous media.

This thesis continues with a discussion on the physical characteristics of porous media which will lead to the development of the physical model used in the thesis. The describing differential equations and boundary conditions of the physical model are then presented. The method of solution applied to the equations follows. The results of the solution are discussed and the thesis concludes by addressing the three points listed above.

CHAPTER 2

Development of the Physical Model

Before the physical model used in the thesis is presented a discussion of the physical characteristics of porous media is needed. The typical structure of a porous medium is shown in Fig.2.

At any point in the interstitial space the flow obeys the Navier-Stokes equations. Due to the obvious complexity of the boundaries of the interstitial space a three-dimensional viscous flow in the tortuous channels of the medium arises. This occurs in spite of the apparent one-dimensional nature of the flow in terms of the specific discharge. It is clear from these observations there are two perspectives from which to view fluid flow in porous media: an interstitial perspective and a global or Darcian perspective. These two perspectives can be coupled through the following definition of the specific discharge as related to the interstitial velocity,

$$q = \frac{1}{A_c} \int \bar{u}.\bar{n} \; dA_c \qquad (2\text{-}1)$$

where A_c is the total crossectional area, including pore space and solid, in the direction of the specific discharge and $\bar{n}$ is its unit normal vector.

The preferred perspective for engineering analysis is the Darcian perspective. The two principle reasons involve the ease of measuring a specific discharge rather than an interstitial velocity and the difficulty in characterizing a porous medium in the interstitial

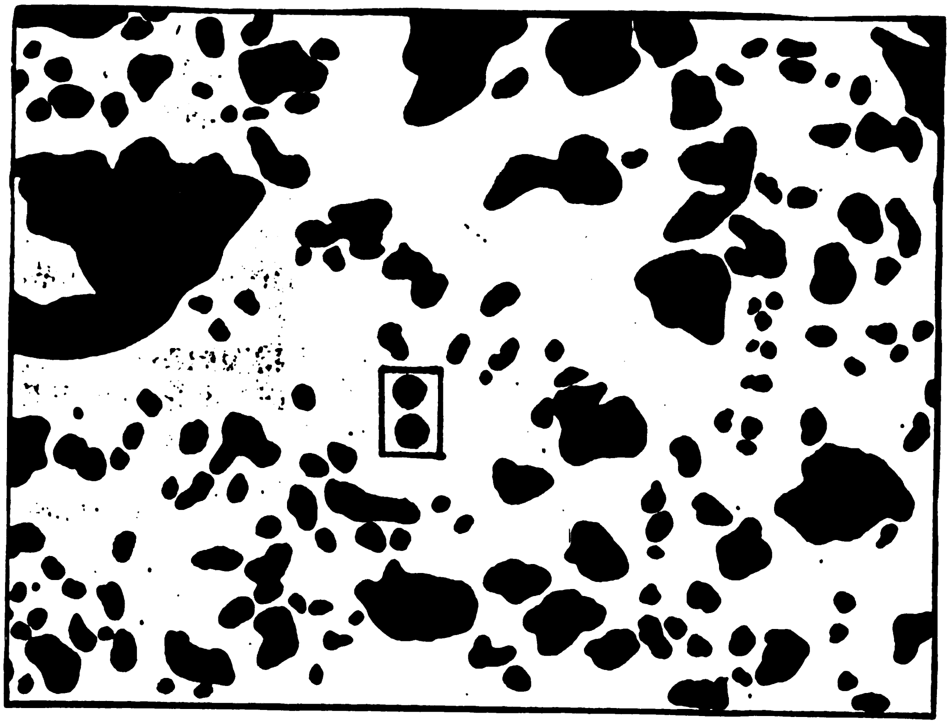


Fig. 2 Example of a porous medium

perspective. To fully characterize a porous medium from the interstitial perspective a complete description of the very complex boundaries of the interstitial space would be required. Clearly for a real porous medium this would be next to impossible. From the Darcian perspective average or bulk geometric properties are used to characterize the porous medium. It is generally agreed upon that three properties are necessary to fully characterize the medium. Perhaps the three most useful are the porosity, ϵ, the specific surface, Σ, and the tortuosity, τ.

The porosity is defined as the ratio of volume of void space to the bulk volume of the porous medium,

$$\epsilon = \frac{V_v}{V_b} \qquad (2\text{-}3)$$

Some representative values of porosity for a variety of media are given in Table 1. The specific surface is defined as the total interstitial surface area of the porous per unit bulk volume of the porous medium,

$$\Sigma = \frac{A_s}{V_b} \qquad (2\text{-}4)$$

Some typical values of specific surface are given in Table 2. The tortuosity is defined as the square of the ratio of the distance traveled by a fluid element as it passes through porous sample to the length of the sample,

TABLE 1

Representative Values of Porosity for Various Substances

Substance	Porosity range (porosity in %)	Literature reference
Berl saddles	68-83	Carman, (1937)
Raschig rings	56-65	Ballard and Piret, (1950)
Wire crimps	68-76	Carman, (1937)
Black slate powder	57-66	Carman, (1937)
Silica powder	37-49	Carman, (1937)
Silica grains(grains only)	65.4	Shapiro and Kolthoff(1948)
Sand	37-50	Carman, (1937)
Granular crushed rock	44-45	Bernard and Wilhelm, (1950)
Soil	43-54	Peerlkamp, (1948)
Coal	2-12	Bond et al., (1950)
Concrete	2-7	Berbeck, (1951)
Leather	56-59	Mitton, (1945)
Fibre glass	88-93	Wiggins et al., (1937)
Cigarette filters	17-49	Corte, (1955)
Hot-compacted copper powder	9-34	Arthur, (1956)

TABLE 2

Representative Values of Specific Surface for Various Substances

Substance	Specific surface range	Literature reference
	(specific surface in cm^{-1})	
Berl saddles	3.9-7.7	Carman,(1937)
Raschig rings	2.8-6.6	Ballard and Piret,(1950)
Wire Crimps	$2.9*10 - 4.0*10$	Carman,(1937)
Black slate powder	$7.0*10^{3} - 8.9*10^{3}$	Carman,(1937)
Silica powder	$6.8*10^{3} - 8.9*10^{3}$	Carman,(1937)
Catalyst	$5.6*10^{5}$	Spengler,(1936)
Sand	$1.5*10^{2} - 2.2*10^{2}$	Carman,(1937)
Leather	$1.2*10^{4} - 1.6*10^{4}$	Mitton et al.,(1945)
Fibre glass	$5.6*10^{2} - 7.7*10^{2}$	Wiggins et al.,(1939)

$$\tau = \left(\frac{L_{fe}}{L_s} \right)^2 \qquad (2\text{-}5)$$

The two properties of momentum transport in porous medium, the permeability and the Forchheimer coefficient, must depend on these geometric properties. Formally, it may be written

$$K = K(\epsilon, \Sigma, \tau) \qquad (2\text{-}6a)$$

$$M = M(\epsilon, \Sigma, \tau) \qquad (2\text{-}6b)$$

The exact form of the functions represented by Eqs(2-5) and (2-6) is difficult to specify for a real porous medium. A number of investigators have developed functional forms from their experimental data for the simple porous medium consisting of a bed of uniformed sized spheres. These forms are shown in Table3. It is important to note the variability in the models even with the simple geometry being considered. Several permeability models have been developed through theoretical derivations of Darcy's law. An excellent review of these models is given by Scheidegger (1960). Some of these derivations employ capillary-tube analogies, hydraulic radius theories by Terzaghi (1951) and a turbulent flow analogy by Yuhara (1954). These models have the following weak points:

i) they are mostly analogies,

ii) their applicability have not been proved, or

iii) they are based on the introduction of unspecified coefficients.

TABLE 3

Permeability Models

Source	K	M
Ergun(1952)	$\frac{\epsilon^3 d_p^2}{150(1-\epsilon)^2}$	$\frac{\epsilon^3 d_p}{1.75(1-\epsilon)}$
Schneebeli(1957)	$\frac{d_p^2}{1100}$	$\frac{d_p}{12}$
Carman(1953)	$\frac{\epsilon^3 d_p^2}{180(1-\epsilon)^2}$	$\frac{\epsilon^3 d_p^{1.1}}{2.87\nu^{0.1}(1-\epsilon)^{1.1}}$
Irmay(1958)	$\frac{\epsilon^3 d_p^2}{180(1-\epsilon)^2}$	$\frac{\epsilon^3 d_p}{0.5(1-\epsilon)}$

None of these models lead towards the Forchheimer extension.

The physical source of the quadratic term in the Forchheimer extension has been argued considerably. One argument states that the term represents turbulence while another argument is that it represents inertia. As will be seen shortly the later argument seems to be correct. The only derivation of the Forchheimer term of any merit is that of Irmay(1958). Though not rigorous his derivation is provided here as background. Consider the packing of spherical particles shown in Fig.3. The Navier-Stokes equations for the flow of an incompressible Newtonian fluid through the interstitial space can be written as

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}.\nabla)\bar{u} = -\frac{1}{\rho} \nabla P + \mu \nabla^2 \bar{u} \qquad (2\text{-}7)$$

The continuity equation is

$$\nabla.\bar{u} = 0 \qquad (2\text{-}8)$$

Now introducing the following vector identity,

$$(\bar{u}.\nabla)\bar{u} = \frac{1}{2} \nabla(\bar{u}.\bar{u}) - (\bar{u}*(\nabla*\bar{u})) \qquad (2\text{-}9)$$

and substituting

$$\frac{\partial \bar{u}}{\partial t} + \frac{1}{2} \nabla (\bar{u}.\bar{u}) - (\bar{u}*(\nabla*\bar{u})) = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{u} \qquad (2\text{-}10)$$

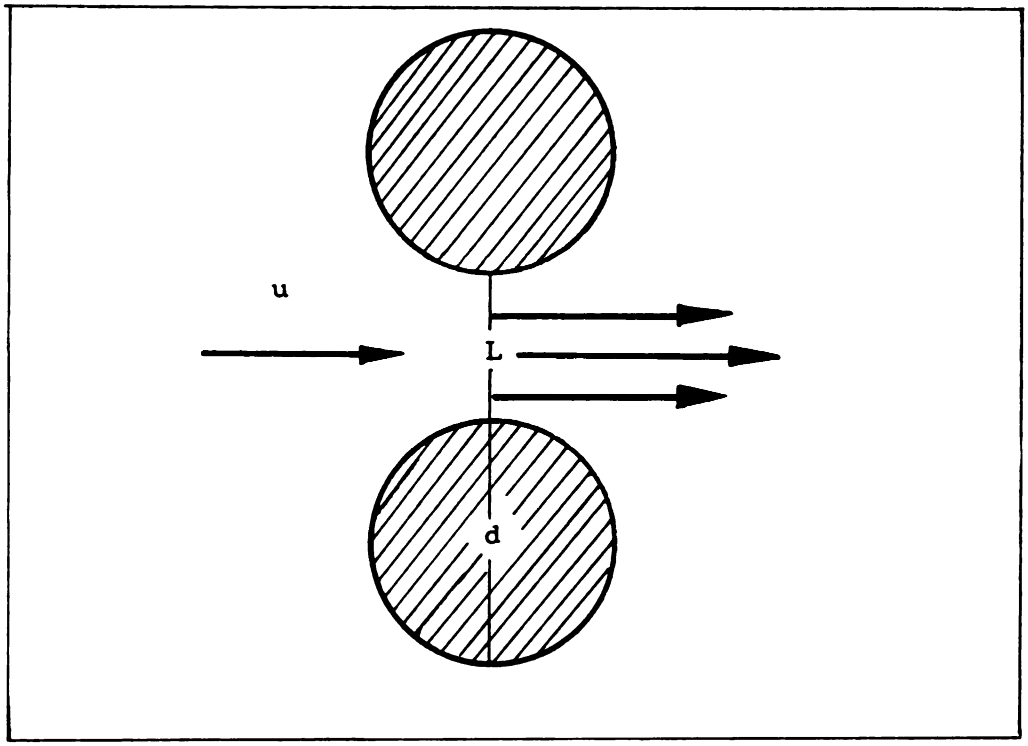


Fig.3-Intergranular flow

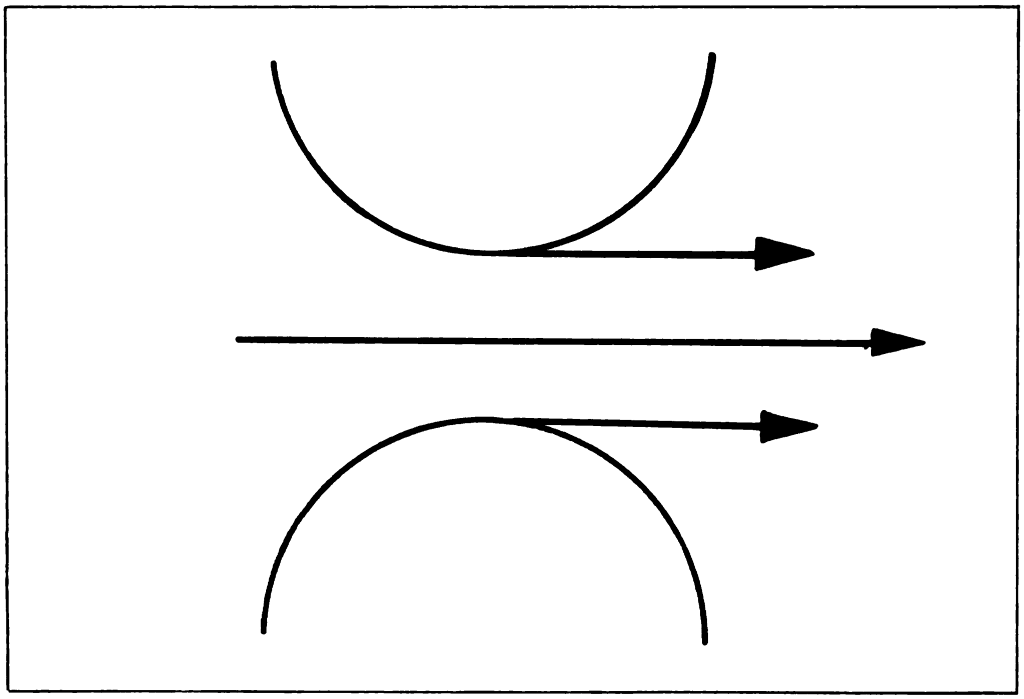


Fig.4-flow between modelled parabora

Consider the overall flow in the x-direction and assume that the interstitial flow is steady and two-dimensional. By introducing the total energy of flow per unit weight of fluid as

$$E = \frac{P}{\rho g} + \frac{(\bar{u}.\bar{u})}{2 g} \qquad (2\text{-}11)$$

its derivative in the X-direction may be written

$$\frac{\partial(gE)}{\partial X} = \frac{1}{2}\frac{\partial}{\partial x}(v^2) - \frac{\partial}{\partial Y}(uv) + u\frac{\partial v}{\partial Y} + \nu\left[\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right] \qquad (2\text{-}12)$$

This equation will be averaged vertical and denoting the vertical average with an overscore so that

$$\bar{F} = \frac{1}{L_y}\int_0^{L_y} F \, dy \qquad (2\text{-}13)$$

For the u velocity this will result in the specific discharge

$$q = \epsilon \, \bar{u} \qquad (2\text{-}14)$$

where

$$\bar{u} = \frac{1}{L_y}\int_0^{L_y} u \, dy \qquad (2\text{-}15)$$

By reason of homogeneity and isotropy the Y-component of velocity must vanish upon averaging. That is,

$$\bar{v}\ \nabla = 0 \qquad (2\text{-}16)$$

Assuming that no correlation exists among the velocity components yields,

$$\overline{\frac{\partial (uv)}{\partial Y}} = \overline{u\frac{\partial v}{\partial Y}} = 0 \qquad (2\text{-}17)$$

By applying continuity it follows that

$$\frac{\partial u}{\partial X} = -\frac{\partial v}{\partial Y} \qquad (2\text{-}18a)$$

$$\frac{\partial^2 u}{\partial X^2} = -\frac{\partial^2 v}{\partial X \partial Y} \qquad (2\text{-}18b)$$

$$\overline{\frac{\partial^2 u}{\partial X^2}} = -\overline{\frac{\partial^2 v}{\partial X \partial Y}} \qquad (2\text{-}18c)$$

The energy will simply become the average energy on a vertical line given by

$$\bar{E} = \frac{\bar{P}}{\rho\, g} + \frac{\bar{u}^2}{2g} \qquad (2\text{-}19)$$

where $\bar{P}$ is the average pressure along a vertical line. With these simplifications the momentum equation reduces to

$$\frac{\partial (g\bar{E})}{\partial X} = \frac{1}{2} \overline{\frac{\partial (v^2)}{\partial X}} + \nu \overline{\frac{\partial^2 u}{\partial Y^2}} \qquad (2\text{-}20)$$

The two remaining terms can be approximated from order of magnitude arguments which leads to the less than rigorous nature of the derivation. It is observed that in the interstitial space the u velocity will achieve a maximum as we travel along a vertical line so that we expect

$$\overline{\frac{\partial^2 u}{\partial X^2}} < 0 \qquad (2\text{-}21)$$

It seems reasonable to have

$$\overline{\frac{\partial^2 u}{\partial Y^2}} \;\alpha\; \frac{\bar{u}}{l^2} \qquad (2\text{-}22)$$

or introducing the concept of permeability

$$\overline{\frac{\partial^2 u}{\partial Y^2}} = \frac{q}{K} \qquad (2\text{-}23)$$

For the final term note that in the throat of the interstitial space

$$v = 0 \tag{2-24}$$

Hence, in the converging zone

$$\frac{\partial}{\partial X}(v^2) < 0 \tag{2-25}$$

in the diverging zone there will most probably be separation of flow at larger velocities, so that

$$\frac{\partial}{\partial X}(v^2) < 0 \tag{2-26}$$

is also valid there. Again an order of magnitude estimate can be employed using

$$\partial(v^2) = \bar{u}^2 \tag{2-27a}$$

$$\partial x = d \tag{2-27b}$$

so that

$$\overline{\frac{\partial(v^2)}{\partial X}} = \frac{\bar{u}^2}{d} \tag{2-28}$$

or introducing the Forchheimer term

$$\frac{\overline{\partial (v^2)}}{\partial X} = - \frac{q^2}{M} \tag{2-29}$$

With these expressions the momentum equation for porous media flow may be written as

$$\frac{1}{\rho}\frac{d\bar{P}}{dX} + \frac{1}{2}\frac{d(\bar{u}^2)}{dX} = - \frac{\nu}{K} q - \frac{q^2}{M} \tag{2-30}$$

But

$$\frac{d(\bar{u}^2)}{dX} = 2\,\bar{u}\,\frac{d\bar{u}}{dX} \tag{2-31}$$

And by continuity

$$\frac{d\bar{u}}{dX} = 0 \tag{2-32}$$

The resulting equation is

$$\frac{d\bar{P}}{dX} + \frac{\mu}{K} q + \frac{\rho}{M} q^2 = 0 \tag{2-33}$$

which is the Forchheimer extension of Darcy's law.

For the numerical derivation of a momentum equation for porous media flows presented in this thesis an appropriate model must be chosen. Though the spherical packing of Irmay's (1958) derivation looks attractive, it is not strictly a two-dimensional model. Shayesteh(1984) used an array of cylinders as a porous media model to obtain a heat transfer coefficient relationship. He also showed that by neglecting inertia terms Darcy's law could be derived. This model, shown in Fig.4, will be used in the thesis. To be consistent with the description of a porous medium the porosity and specific surface of this model can be obtained as

$$\epsilon = 1 - \frac{\pi}{(1+L/d)} \qquad (2\text{-}34)$$

$$\Sigma = \frac{\pi}{d(1+L/d)} \qquad (2\text{-}35)$$

These two expressions will allow one to represent the array of cylinders as a porous medium.

CHAPTER 3

Describing Equations and Boundary Conditions

The describing equations for this problem are discussed considering the viscous incompressible flow in the narrow gap between two circular cylinders, see Fig.3. The active forces are those due to pressure and shear resulting from the fluid viscosity. The mathematical analysis is intended to define the wall equation and non-dimensional form of momentum equations.

The continuity equation can be written

$$\frac{\partial u^*}{\partial X^*} + \frac{\partial v^*}{\partial Y^*} = 0 \qquad (3\text{-}1)$$

The momentum equations for the X^*- direction and Y^*- direction are:

$$\rho \left(u^* \frac{\partial u^*}{\partial X^*} + v^* \frac{\partial u^*}{\partial Y^*} \right) = -\frac{\partial p^*}{\partial X^*} + \mu \left(\frac{\partial^2 u^*}{\partial X^{*2}} + \frac{\partial^2 u^*}{\partial Y^{*2}} \right) \qquad (3\text{-}2)$$

and

$$\rho \left(u^* \frac{\partial v^*}{\partial X^*} + v^* \frac{\partial v^*}{\partial Y^*} \right) = -\frac{\partial p^*}{\partial Y^*} + \mu \left(\frac{\partial^2 v^*}{\partial X^{*2}} + \frac{\partial^2 v^*}{\partial Y^{*2}} \right) \qquad (3\text{-}3)$$

The boundary conditions are :

at the surface where $Y^* = H(X^*)$

$$u^* = 0 \ , \quad v^* = 0 \qquad (3\text{-}4)$$

at $Y^* = 0$ from symmetry

$$\frac{\partial u^*}{\partial Y^*} = 0 \quad , \quad v^* = 0 \qquad (3\text{-}5)$$

as $x \rightarrow -\infty$

$$u^* = U_0 \ , \quad v^* = 0 \qquad (3\text{-}6)$$

as $x \rightarrow \infty$

$$u^* = U_0 \ , \quad v^* = 0 \qquad (3\text{-}7)$$

To describe the boundary $H(x^*)$ consider a circle centered at $Y^* =$ A+T.

Beginning with the equation for a circle,

$$X^2 + Y^2 = A^2 \qquad (3\text{-}8)$$

and transforming X and Y to X^* and Y^*.

$$X = X^* \qquad (3\text{-}9)$$

$$Y = Y^* - T - A \qquad (3\text{-}10)$$

Substituting and rearranging gives

$$Y^{*2} - 2Y^*(A+T) + X^{*2} = - T^2 - 2TA \qquad (3\text{-}11)$$

as the equation of the upper boundary.

Perturbation and Asymptotic Expansion Method

To develop a simplified form of these equation the perturbation and asymptotic expansion method employed by Shayesteh(1984) is followed. Assume that there is some small parameter about which we can expand the dependent variables. It is found that

$$\Phi = T/A \qquad (3\text{-}12)$$

will serve well as the small parameter.

Thus the distance between the cylinders is considered small with respect to the diameter of the cylinder. Applying the following non-dimensionalization gives

$$X = \frac{X^*}{\alpha(\Phi)A} \qquad (3\text{-}13a)$$

$$Y = \frac{Y^*}{T} \qquad (3\text{-}13b)$$

$$u = \frac{u^*}{U_o} \qquad (3\text{-}13c)$$

$$v = \frac{v^*}{v_s} \qquad (3\text{-}13d)$$

$$p = \frac{p^*}{p_s} \qquad (3\text{-}13e)$$

Order of magnitude arguments are used to determine $\alpha(\Phi)$, p_s and v_s. Substituting the dimensionless variables the equation of the upper boundary becomes :

$$Y^2T^2 - 2TY(A+T) + \alpha^2A^2X^2 = -T^2 - 2TA \qquad (3\text{-}14)$$

dividing this equation by A^2,

$$Y^2\Phi^2 - 2Y(\Phi + \Phi^2) + \alpha^2X^2 = -\Phi^2 - 2\Phi \qquad (3\text{-}15)$$

The parameter α must be chosen to retain the physics of the problem.

To begin let $\alpha = 1.0$, then

$$X^2 + Y^2\Phi^2 - 2Y(\Phi + \Phi^2) = -\Phi^2 - 2\Phi \qquad (3\text{-}16)$$

Looking at terms of this equation in terms of the order of Φ

$$O(1) : X^2 = 0 \qquad (3\text{-}17)$$

Since using $X^2 = 0$ as the equation of the wall is meaningless, the choice for α must be wrong. It is desired to have an $\alpha^2 X^2$ in the order Φ terms so let

$$\alpha = \Phi^{1/2} \qquad (3\text{-}18)$$

Then

$$Y^2\Phi - 2Y(\Phi + \Phi^2) + \Phi X^2 = -\Phi^2 - 2\Phi \qquad (3\text{-}19)$$

The highest order terms in this equation are of $O(\Phi)$ which gives

$$-2Y + X^2 = -2 \qquad (3\text{-}20)$$

Then the equation for the wall becomes

$$Y = H(X) = X^2/2 + 1 = 1/2(X^2 + 2) \qquad (3\text{-}21)$$

Note that this is equivalent to transforming the flow between the cylinders to that through the parabolic channel shown in Fig.4.

Using the functional form of α the non-dimensionalized continuity equation becomes

$$\frac{U_0}{\Phi^{1/2} A} \cdot \frac{\partial u}{\partial X} + \frac{v_s}{T} \cdot \frac{\partial v}{\partial Y} = 0 \qquad (3\text{-}22)$$

Rearranging,

$$\frac{\partial u}{\partial X} \cdot \frac{U_0}{v_s} + \frac{1}{\Phi^{1/2}} \cdot \frac{\partial v}{\partial Y} = 0 \qquad (3\text{-}23)$$

The physics implies that $\frac{\partial u}{\partial X}$ and $\frac{\partial v}{\partial Y}$ should be of the same size to retain the equality, hence

$$\frac{v_s}{U_0} \cdot \frac{1}{\Phi^{1/2}} \approx O(1) \approx 1 \qquad (3\text{-}24)$$

$$v_s = \Phi^{1/2} U_0 \qquad (3\text{-}25)$$

Now considering the dimensionless form of the momentum equation in X^*- direction.

$$\frac{U_0^2}{\Phi^{1/2} A} \cdot u \frac{\partial u}{\partial X} + \frac{U_0^2 \Phi^{1/2}}{T} \cdot v \frac{\partial u}{\partial Y} = - \frac{1}{\rho} \frac{p_s}{\Phi^{1/2} A} \cdot \frac{\partial p}{\partial X} +$$

$$\nu \left[\frac{U_0}{A^2 \Phi} \cdot \frac{\partial^2 u}{\partial Y^2} + \frac{U_0}{T^2} \frac{\partial^2 u}{\partial Y^2} \right] \qquad (3\text{-}26)$$

The above equation can be written in the following form after being multiplied by $A^2 \Phi/\nu U_0$

$$\frac{AU_0\Phi^{1/2}}{\nu} \left[u \frac{\partial u}{\partial X} + v \frac{\partial u}{\partial Y} \right] = - \frac{p_s A \Phi^{1/2}}{\rho \nu U_0} \cdot \frac{\partial p}{\partial X} + \left[\frac{\partial^2 u}{\partial X^2} + \frac{1}{\Phi} \frac{\partial^2 u}{\partial Y^2} \right] \qquad (3\text{-}27)$$

$$\text{where} \qquad Re_A = \frac{U_0 A}{\nu} \qquad (3\text{-}28)$$

In words,

Inertia force = Driving force + Viscous Force

For a small Reynolds number the driving force must be balanced by the viscous force or

$$\frac{p_s A \Phi^{1/2}}{\rho \nu U_0} \approx \frac{1}{\Phi} \qquad (3\text{-}29)$$

$$\text{so that } p_s = \frac{\mu U_0}{A\Phi^{3/2}} \qquad (3\text{-}30)$$

Then in final form,

$$Re_A\Phi^{1/2}\left(u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y}\right) = -\frac{1}{\Phi}\frac{\partial p}{\partial X} + \frac{\partial^2 u}{\partial X^2} + \frac{1}{\Phi}\frac{\partial^2 u}{\partial Y^2} \quad (3\text{-}31)$$

For the Y-momentum equation non-dimensionalizing gives

$$Re_A\Phi^{1/2}\left[u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y}\right] = -\frac{1}{\Phi^2}\frac{\partial p}{\partial Y} + \left[\frac{\partial^2 v}{\partial X^2} + \frac{1}{\Phi}\frac{\partial^2 v}{\partial Y^2}\right] \quad (3\text{-}32)$$

In summary the dimemsionless governing equations for the flow between the cylinders are given as :

continuity equation :

$$u_X + v_Y = 0 \quad (3\text{-}33)$$

momentum equation in X - direction :

$$Re_A\Phi^{1/2}(u\, u_X + v\, u_Y) = -\frac{p_X}{\Phi} + u_{XX} + \frac{u_{YY}}{\Phi} \quad (3\text{-}34)$$

momentum equation in Y - direction :

$$Re_A\Phi^{5/2}(u\, v_X + v\, v_Y) = - p_Y + \Phi^2 v_{XX} + \Phi\, v_{YY} \quad (3\text{-}35)$$

Assuming that the parameter Φ is small and taking only the highest order terms in the momentum equations gives

$$Re_A \Phi^{3/2}(u\, u_X + v\, u_Y) = - p_X + u_{YY} \qquad (3\text{-}36)$$

for the X-momentum equation. Since the analysis requires the retention of the inertia terms (otherwise one would expect to derive Darcy's law as Shayesteh(1984) did) it is required that

$$Re_A \Phi^{3/2} = O(1) \qquad (3\text{-}37)$$

or

$$Re_A = O(\Phi^{-3/2}) \qquad (3\text{-}38)$$

Then

$$Re^*(u\, u_X + v\, u_Y) = - p_X + u_{YY} \qquad (3\text{-}39)$$

and

$$p_Y = 0 \qquad (3\text{-}40)$$

will serve as the momentum equation where

$$Re^* = Re_A \Phi^{3/2} = O(1) \qquad (3\text{-}41)$$

CHAPTER 4

Method of Solution

In the previous chapter the describing differential equations appropriate for the present problem were derived from the Navier-Stokes equations. To summarize the describing differential equations are

$$u_X + v_Y = 0 \qquad (4\text{-}1)$$

$$Re^* (u\, u_X + v\, u_Y) = - p_X + u_{YY} \qquad (4\text{-}2)$$

$$p_Y = 0 \qquad (4\text{-}3)$$

which must be solved subject to the boundary conditions ;

at $Y = 0$

$$v = 0 \quad , \quad u_Y = 0 \qquad (4\text{-}4)$$

at $Y = h(X)$

$$u = 0 \quad , \quad v = 0 \qquad (4\text{-}5)$$

as $X \to -\infty$

$$u = 1 \quad , \quad v = 0 \tag{4-6}$$

as $X \to \infty$

$$u = 1 \quad , \quad v = 0 \tag{4-7}$$

Equations (4-1) through (4-3) represent a system of coupled, partial differential equations for the dependent variables u, v and p. To apply a numerical technique in solving these equations, manipulations are carried forth to yield a single partial differential equation of a single variable, the stream function.

To begin, the pressure is eliminated by operating with $\dfrac{\partial}{\partial Y}$ on Eqn (4-2) and noting

$$p_{XY} = (p_Y)_X = 0 \tag{4-8}$$

from the Y - momentum equation, then

$$Re^* \left(u_Y\, u_X + u\, u_{XY} + v_Y\, u_Y + v\, u_{YY} \right) = u_{YYY} \tag{4-9}$$

Now defining the stream function as

$$u = \psi_Y \quad , \quad v = -\psi_X \tag{4-10}$$

so that continuity is satisfied, substitution into Eq (4-9) gives

$$\psi_{YYYY} + Re^{*} (\psi_X \psi_{YYY} - \psi_Y \psi_{YYX}) = 0 \qquad (4\text{-}11)$$

which is a nonlinear partial differential equation for the stream function. The boundary conditions on the stream function are

at $Y = 0$

$$\psi_X = 0 \quad , \quad \psi_{YY} = 0 \qquad (4\text{-}12)$$

at $Y = h(X)$

$$\psi_X = 0 \quad , \quad \psi_Y = 0 \qquad (4\text{-}12)$$

as $X \to -\infty$

$$\psi_X = 0 \quad , \quad \psi_Y = 1 \qquad (4\text{-}13)$$

as $X \to \infty$

$$\psi_X = 0 \quad , \quad \psi_Y = 1 \qquad (4\text{-}13)$$

To handle the nonlinear terms, quasi-linearization is employed. The dependent variable is expanded by a Taylor's series about a previous various value (from a previous iteration). Thus,

$$\psi_X = \overset{o}{\psi}_X + \Delta \psi_X \qquad (4\text{-}14a)$$

$$\psi_{YYY} = \overset{o}{\psi}_{YYY} + \Delta \psi_{YYY} \qquad (4\text{-}14b)$$

$$\psi_Y = \overset{o}{\psi}_Y + \Delta \psi_Y \qquad (4\text{-}14c)$$

$$\psi_{XYY} = \overset{o}{\psi}_{XYY} + \Delta \psi_{XYY} \qquad (4\text{-}14d)$$

The superscript ^{o} denotes the value of the variable at the previous iteration. Substiting the expansions into Eq(4-10) and dropping higher order terms (that is , products of Δ terms),

$$\psi_{YYYY} + Re^{*}(\overset{o}{\psi}_X \psi_{YYY} + \overset{o}{\psi}_{YYY}\psi_X - \overset{o}{\psi}_Y \psi_{XYY} - \overset{o}{\psi}_{XYY}\psi_Y) = Re^{*}(\overset{o}{\psi}_{YYY}\overset{o}{\psi}_X - \overset{o}{\psi}_{XYY}\overset{o}{\psi}_Y) \qquad (4\text{-}15)$$

The current X,Y coordinate system is somewhat burdensome, especially in light of finite differencing, due to the irregular boundary. To give a more regular computational domain the following coordinate transformation is employed.

$$\eta = \frac{Y}{h(X)} \qquad (4\text{-}16)$$

$$\xi = X \qquad (4\text{-}17)$$

$$\text{where, } h(X) = \frac{X^2}{2} + 1 \qquad (4\text{-}18)$$

Therefore

$$\eta = \frac{Y}{\xi^2/2 + 1} \qquad (4\text{-}19)$$

The derivatives can then be written

$$\frac{\partial}{\partial Y} = \frac{\partial \eta}{\partial Y}\frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial Y}\frac{\partial}{\partial \xi} \qquad (4\text{-}20a)$$

$$= \frac{1}{(\xi^2/2 + 1)}\frac{\partial}{\partial \eta} \qquad (4\text{-}20b)$$

$$\frac{\partial}{\partial X} = \frac{\partial \eta}{\partial X}\frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial X}\frac{\partial}{\partial \xi} \qquad (4\text{-}21a)$$

$$= -\frac{\xi Y}{(\xi^2/2 + 1)^2}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \qquad (4\text{-}21b)$$

$$= -\frac{\xi \eta}{(\xi^2/2 + 1)} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \quad (4\text{-}21c)$$

Define $b = (\xi^2/2 + 1)^{-1}$ (4-22)

$$\psi_X = - \xi \eta b \psi_\eta + \psi_\xi \quad (4\text{-}23a)$$

$$\psi_Y = b \psi_\eta \quad (4\text{-}23b)$$

$$\psi_{XY} = - \xi \eta b^2 \psi_{\eta\eta} + b \psi_{\xi\eta} - \xi b^2 \Psi_\eta \quad (4\text{-}23c)$$

$$\psi_{XYY} = - \xi \eta b^3 \psi_{\eta\eta\eta} + b^2 \psi_{\xi\eta\eta} - 2 \xi b^3 \Psi_{\eta\eta} \quad (4\text{-}23d)$$

$$\psi_{YY} = b^2 \psi_{\eta\eta} \quad (4\text{-}23e)$$

$$\psi_{YYY} = b^3 \psi_{\eta\eta\eta} \quad (4\text{-}23f)$$

$$\psi_{YYYY} = b^4 \psi_{\eta\eta\eta\eta} \quad (4\text{-}23g)$$

Substituting into the momentum equation gives

$$b^4 \psi_{\eta\eta\eta\eta} + Re^* \left(b^3 \psi^0_\xi \psi_{\eta\eta\eta} - b^3 \psi^0_\eta \psi_{\xi\eta\eta} + b^3 \psi^0_{\eta\eta\eta} \psi_\xi - b^3 \psi_{\xi\eta\eta} \psi_\eta + 2 \xi b^4 \Psi^0_\eta \Psi_{\eta\eta} \right) = Re^* \left(b^3 \psi^0_{\eta\eta\eta} \psi^0_\xi - b^3 \psi^0_\eta \psi^0_{\xi\eta\eta} + 2 \xi b^4 \Psi^0_\eta \Psi^0_{\eta\eta} \right) \qquad (4\text{-}24)$$

The boundary conditions in the ξ and η coordinate system become

at $\eta = 0$

$$\psi_{\eta\eta} = 0 \quad , \quad \psi_\xi = 0 \qquad (4\text{-}25)$$

at $\eta = 1$

$$\Psi_\eta = 0 \quad , \quad \Psi_\xi = 0 \qquad (4\text{-}26)$$

as $\xi \rightarrow -\infty$

$$\psi_\eta = \frac{1}{b} \quad , \quad \psi_\xi = \xi \ \eta \qquad (4\text{-}27)$$

as $\xi \rightarrow \infty$

$$\psi_\eta = \frac{1}{b} \quad , \quad \psi_\xi = \xi \ \eta \qquad (4\text{-}28)$$

A finite difference method will be used to solve Eq(4-24) subject to boundary conditions. Using a central difference approach gives

$$\psi_\eta = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\,H_\eta} \tag{4-29a}$$

$$\psi_\xi = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\,H_\xi} \tag{4-29b}$$

$$\psi_{\eta\eta} = \frac{\psi_{i,j+1} - 2\Psi_{i,j} + \psi_{i,j-1}}{H_\eta^{2}} \tag{4-29c}$$

$$\psi_{\eta\eta\eta} = \frac{\psi_{i,J+2} - 2\psi_{i,j+1} + 2\psi_{i,j-1} - \psi_{i,j-2}}{2\,H_\eta^{3}} \tag{4-29d}$$

$$\psi_{\eta\eta\eta\eta} = \frac{\psi_{i,j+2} - 4\psi_{i,j+1} + 6\psi_{i,j} - 4\psi_{i,j-1} + \psi_{i,j-2}}{H_\eta^{4}} \tag{4-29e}$$

$$\psi_{\xi\eta\eta} = \frac{\psi_{i+1,j+1} - 2\psi_{i+1,j} + \psi_{i+j-1} - \psi_{i-1,j+1} + 2\psi_{i-1,j} - \psi_{i-1,j-1}}{2H_\eta^{2}\;H_\xi} \tag{4-29f}$$

Substitution and rearrangement gives a difference equation of the form

$$C_0\, \psi_{i-1,j-1} + C_1\, \psi_{i-1,j} + C_2\, \psi_{i-1,j+1} + C_3\, \psi_{i,j-2} + C_4\, \psi_{i,j-1} + C_5\, \psi_{i,j} + C_6\, \psi_{i,j+1} + C_7\, \psi_{i,j+2} + C_8\, \psi_{i+1,j-1} + C_9\, \psi_{i+1,j} + C_{10}\, \psi_{i+1,j+1} = S_{i,j} \tag{4-30}$$

where,

$$C_0 = \frac{Re^*\, \psi^0_\eta}{2\, H^2_\eta\, H_\xi} \tag{4-31a}$$

$$C_1 = -\frac{Re^*\, \psi^0_\eta}{H^2_\eta\, H_\xi} - \frac{Re^*\, \psi^0_{\eta\eta\eta}}{2\, H_\xi} \tag{4-31b}$$

$$C_2 = \frac{Re^*\, \psi^0_\eta}{2\, H^2_\eta\, H_\xi} \tag{4-31c}$$

$$C_3 = \frac{b}{H^4_\xi} - \frac{Re^*\, \psi^0_\xi}{2\, H^3_\eta} \tag{4-31d}$$

$$C_4 = -\frac{4\, b}{H^4_\eta} + \frac{Re^*\, \psi^0_\xi}{H^3_\eta} + \frac{Re^*\, \psi^0_{\xi\eta\eta}}{2\, H_\eta} + \frac{2\, b\, \xi\, Re^*\, \psi^0_\eta}{H^2_\eta} \tag{4-31e}$$

$$C_5 = \frac{6\, b}{H^4_\eta} - \frac{4\, \xi\, b\, Re^*\, \psi^0_\eta}{H^2_\eta} \tag{4-31f}$$

$$C_6 = -\frac{4\,b}{H_\eta^4} - \frac{Re^* \psi_\xi^o}{H_\eta^3} - \frac{Re^* \psi_{\xi\eta\eta}^o}{2\,H_\eta} + \frac{2\,\xi\,b\,Re^* \psi_\eta^o}{H_\eta^2} \qquad (4\text{-}31g)$$

$$C_7 = \frac{b}{H_\eta^4} + \frac{Re^* \psi_\xi^o}{2\,H_\eta^3} \qquad (4\text{-}31h)$$

$$C_8 = -\frac{Re^* \psi_\eta^o}{2\,H_\eta^o\,H_\xi} \qquad (4\text{-}31i)$$

$$C_9 = \frac{Re^* \psi_\eta^o}{H_\eta^2\,H_\xi} + \frac{Re^* \psi_{\eta\eta\eta}^o}{2\,H_\xi} \qquad (4\text{-}31j)$$

$$C_{10} = -\frac{Re^* \psi_\eta^o}{2\,H_\eta^2\,H_\xi} \qquad (4\text{-}31k)$$

$$S_{i,j} = Re^* \left(\psi_{\eta\eta\eta}^o \psi_\xi^o - \psi_\eta^o \psi_{\xi\eta\eta}^o + 2\,\xi\,b\,\psi_\eta^o \psi_{\eta\eta}^o \right) \qquad (4\text{-}31l)$$

The above equation will be valid for $3 \le i \le n-2$ and $4 \le j \le m-3$. To handle the other nodes the boundary conditions must be applied. Numerically the boundary conditions as $\xi \to \pm \infty$ are handled by defining L_ξ as half of the channel length. The correct size for L_ξ

will have to be determined from numerical testing. Thus the boundary conditions as $\xi \rightarrow \pm \infty$ become boundary conditions at $\xi = \pm L_\xi$.

Writing the boundary conditions in difference form gives

at i = 1

$$\frac{\psi_{1,j+1} - \psi_{1,j-1}}{2 H_\eta} = \frac{L_\xi^2}{2} + 1 \tag{4-32a}$$

$$\frac{\psi_{2,j} - \psi_{0,j}}{2 H_\xi} = - L_\xi \eta_j \tag{4-32b}$$

at i = M

$$\frac{\psi_{M,j+1} - \psi_{M,j-1}}{2 H_\eta} = \frac{L_\xi^2}{2} + 1 \tag{4-33a}$$

$$\frac{\psi_{M+1,j} - \psi_{M-1,j}}{2 H_\xi} = L_\xi \eta_j \tag{4-33b}$$

at j = 1

$$\psi_{i,0} - 2\psi_{i,1} + \psi_{i,2} = 0 \tag{4-34a}$$

$$\psi_{i+1,1} + \psi_{i-1,1} = 0 \tag{4-34b}$$

at $j = N$

$$\psi_{i,N+1} - \psi_{i,N-1} = 0 \tag{4-35a}$$

$$\psi_{i+1,N} - \psi_{i-1,N} = 0 \tag{4-35b}$$

Since all of the boundary conditions are on the derivatives of the stream function we must choose to apply a boundary condition on the stream function itself. We choose

$$\psi\ (\xi,0) = 0 \tag{4-36}$$

so that in difference form

$$\psi_{i,1} = 0 \tag{4-37}$$

Next consider the i=1 boundary where

$$\psi_{1,3} = \psi_{1,1} + 2\ H_\eta\ (\ L_\xi^2 / \ 2 + 1\ ) \tag{4-38a}$$

but

$$\psi_{1,1} = 0 \tag{4-38b}$$

so that

$$\psi_{1,3} = 2 H_\eta (L_\xi^2 / 2 + 1) \tag{4-38c}$$

Similarly

$$\psi_{1,5} = \psi_{1,3} + 2 H_\eta (L_\xi^2 / 2 + 1) \tag{4-38d}$$

or substituting

$$\psi_{1,5} = 4 H_\eta (L_\xi^2 / 2 + 1) \tag{4-38e}$$

The trend is obvious so that

$$\psi_{1,N} = (N - 1) H_\eta (L_\xi^2 / 2 + 1) \tag{4-38f}$$

but

$$L_\eta = (N - 1) H_\eta = 1 \tag{4-38g}$$

so that

$$\psi_{1,N} = L_\xi^2 / 2 + 1 \tag{4-39}$$

Equations (4-37) and (4-39) now replace Eqs. (4-34b) and (4-35b) at the boundaries.

Similarly it may be shown that at $\xi = \pm L_\xi$

$$\psi = (\xi^2 / 2 + 1) \eta \tag{4-40}$$

Then Eqs (4-32a) and (4-33a) can be replaced with

$$\psi_{1,i} = \eta_j (L_\xi^2 / 2 + 1) \tag{4-41}$$

$$\psi_{M,j} = \eta_j (L_\xi^2 / 2 + 1) \tag{4-42}$$

By way of Eqs(4-37), (4-39), (4-41) and (4-42) the stream function is known at all boundary nodes. For the inner nodes ($2 \le i \le M-1$) and $3 \le j \le N-2$) Eq(4-30) will be appropriat. This leaves the nodes $j=2$ and $j=N-1$ for $2 \le i \le M-1$ left to be addressed. Applying Eqs.(4-34a) and (4-35a)

$$\psi_{i,0} = 2 \psi_{i,1} - \psi_{i,2} \tag{4-43a}$$

$$\psi_{i,N+1} = \psi_{i,N-1} \tag{4-43b}$$

into Eq(4-30) gives the final two equations,

$$C_0 \psi_{i-1,1} + C_1 \psi_{i-1,2} + C_2 \psi_{i-1,3} + (C_4 + 2 C_3) \psi_{i,1} + (C_5 - C_3) \psi_{i,2} + C_6 \psi_{i,3} + C_7 \psi_{i,4} + C_8 \psi_{i+1,1} + C_9 \psi_{i+1,2} + C_{10} \psi_{i+1,3} = S_{i,2} \quad (4\text{-}44)$$

and

$$C_0 \psi_{i-1,N-2} + C_1 \psi_{i-1,N-1} + C_2 \psi_{i-1,N} + C_3 \psi_{i,N-3} + C_4 \psi_{i,N-2} + (C_5 - C_7) \psi_{i,N-1} + (C_6 + 2 C_7) \psi_{i,N} + C_8 \psi_{i+1,N-2} + C_9 \psi_{i+1,N-1} + C_{10} \psi_{i+1,N} = S_{i,N-1} \quad (4\text{-}45)$$

which are valid for $2 \le i \le M-1$. A summary of the appropriate difference equation at each node point is given in Table 4.

Table 4 represents a system of algebraic equations for the discretized stream function, $\psi(i,j)$. Solution to this system will require iteration since the source term, $S_{i,j}$, contains derivatives of the stream function. Though this system could be solved in matrix form, this proves to be impractical due to the large size of the matrix and iterative nature of the problem. Each node equation could be written in such a form so that only the stream function at the node $\psi_{i,j}$ is at the current iteration while the remaining stream functions are evaluated at the previous iteration. For example, Eq.(4-30) may be rewritten,

$$\psi_{i,j} = \frac{1}{C_5} (S_{i,j} - C_0 \psi^{0}_{i-1,j-1} - C_1 \psi^{0}_{i-1,j} - C_2 \psi^{0}_{i-1,j+1} - C_3$$

$$\psi^{0}_{i,j-2} - C_4 \psi^{0}_{i,j-1} - C_5 \psi^{0}_{i,j} - C_6 \psi^{0}_{i,j+1} - C_7 \psi^{0}_{i,j+2} - C_8 \psi^{0}_{i+1,j-1} -$$

$$C_8 \psi^{0}_{i+1,j-1} - C_9 \psi^{0}_{i+1,j} - C_{10} \psi^{0}_{i+1,j+1})$$

where once again the superscript ^{0} refers to the previous iteration. The method of successive over-relaxation (SOR) is used to iteratively solve this system of equations.

Table 4

Difference Equations

Node Points	Equation
$i=1 \quad 1 \le j \le N$	(4-41)
$i=m \quad 1 \le j \le N$	(4-42)
$1 < i < M \quad j=1$	(4-37)
$1 < i < M \quad j=N$	(4-39)
$2 \le i \le M-1 \quad 3 \le j \le N-2$	(4-30)
$2 \le i \le M-1 \quad j=2$	(4-44)
$2 \le i \le M-1 \quad j=N-1$	(4-45)

Successive over-relaxation is a technique which can be used in an attempt to accelerate any iterative procedure. As Gauss-Seidel iteration is applied to a system of simultaneous algebraic equations, it is expected to make several recalculations or iterations will be required before convergence to an acceptable level is achieved.

Suppose that during this process a change in the value of the unknown at a point between two successive iterations, not the direction of change, is observed and that it is anticipated the same trend will continue on to the next iteration. Why not go ahead and make a correction to the variable in the anticipated direction before the next iteration thereby, hopefully, accelerating the convergence? An arbitrary correction to the intermediate values of the unknowns from any iterative procedure (Gauss-Seidel iteration is of most interest at this point so it will be used as the representative iterative scheme) according to the form

$$u_{i,j}^{k+1'} = u_{i,j}^{k'} + \omega (u_{i,j}^{k+1} - u_{i,j}^{k'}) \qquad (4\text{-}46)$$

is known as over-relaxation or successive over-relaxation (**SOR**). Here, k denotes iteration level and $u_{i,j}^{k+1}$ is the most recent value of $u_{i,j}$ calculated from the Gauss-Seidel procedure, $u_{i,j}$ calculated from the Gauss-seidel procedure, $u_{i,j}^{k'}$ is the value from the previous iteration as adjusted by previous application of this formula if the over-relaxation is being applied successively (at each iteration) and $u_{i,j}^{k+1'}$ is the newly adjusted or better guess for $u_{i,j}$ at the k+1 iteration level. That is, $u_{i,j}^{k+1'}$ should be closer to the final solution than the unaltered value $u_{i,j}^{k+1}$ from the Gauss-Seidel calculation. The formula is applied immediately at each point after

$u_{i,j}^{k+1}$ has been obtained and $u_{i,j}^{k+1'}$ replaces $u_{i,j}^{k+1}$ in all subsequent calculations in the cycle.

The vairable ω is the relaxation parameter and when $1 \leq \omega \leq 2$ over-relaxation is being employed. Over-relaxation is similar to a linear extrapolation based on values $u_{i,j}^{k'}$ and $u_{i,j}^{k+1}$. In some problems under-relaxation, $0 < \omega < 1$, is employed. Under-relaxation appears to be most appropriate when the convergence at a point is taking on an oscillatory pattern and tending to " overshoot" the apparent final solution under-relaxation is sometimes called for in elliptic problems, it seems, when the equations are non-linear. Occasionally, for non linear problems, under-relaxation is observed to be neccessary for convergence. In this thesis under-relaxation method is used to approach more exact value of ψ(stream function) point by point. The relaxation parameter,ω, is chosen by trial and error.

A computer program to employ this solution method to the equations represented by Table4 has been written and a copy of it may be found in Appendix A. A flow chart of the program is shown in Fig.5, Fig.6 provides flow charts for the four principal subroutines.

FLOWCHART 1 (Main Program)

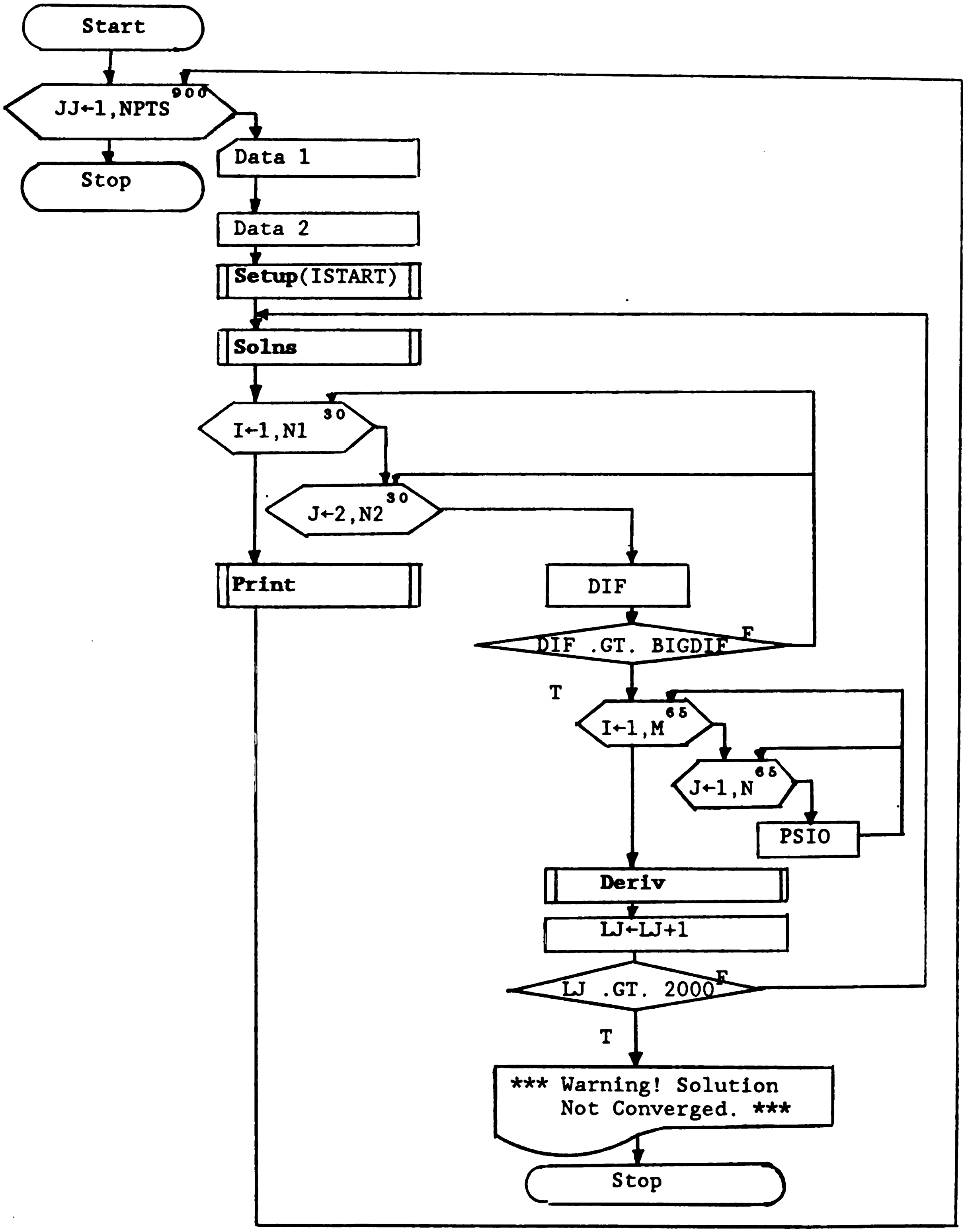
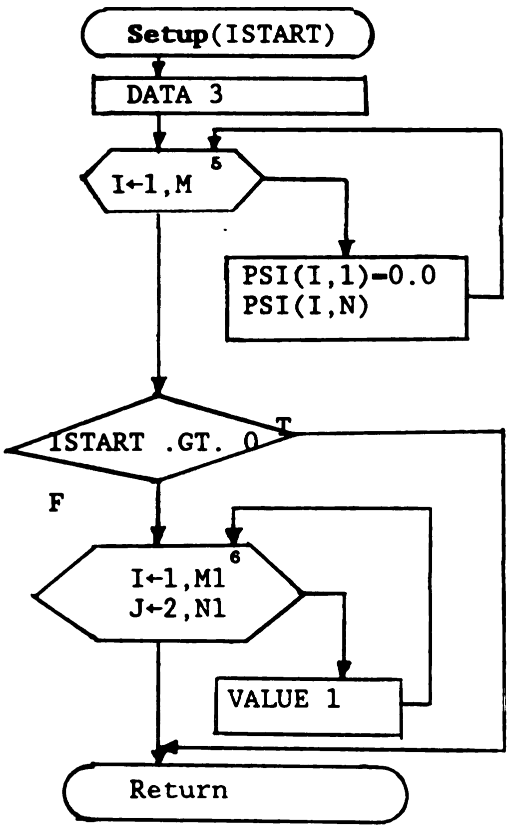
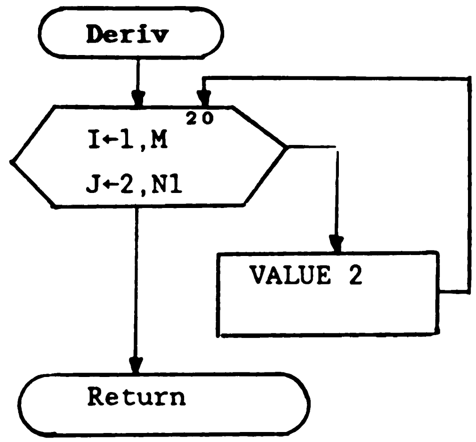


Figure 5

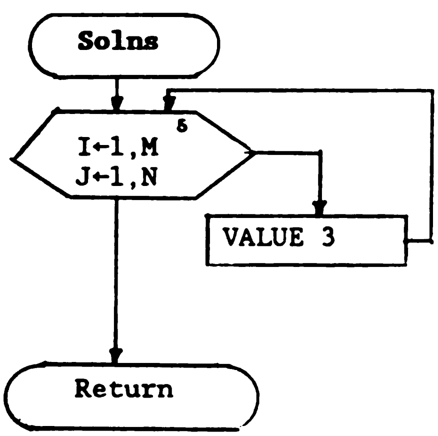
FLOWCHART 2 (Subroutine 1)



FLOWCHART 3 (Subroutine 2)



FLOWCHART 4 (Subroutine 3)



FLOWCHART 5 (Subroutine 4)

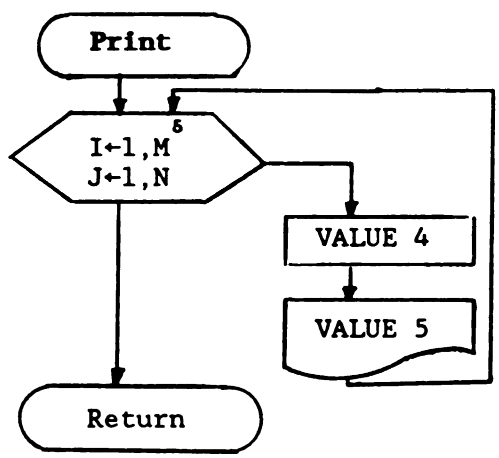


Figure 6

CHAPTER 5

Results and Discussion

Before presenting the results of this thesis, it is important to verify the correctness of the numerical solution. This is done by first considering the sum of the squares of the residuals. Since the difference equation which is solved is only an approximation of the differential equation an error in the solution will occur. The solution of the finite difference method is substituted into the differential equation and the residual, or difference that occurs, is evaluated at every node point. Each residual is then squared and the squares are summed. The sum of the squares of the residuals gives some measure of the correct choice of convergence criteria. For this thesis a point-wise convergence criteria is used by defining

$$\epsilon_{ij} = \left| \frac{\psi_{ij}^{K} - \psi_{ij}^{K-1}}{\psi_{ij}^{K-1}} \right| \qquad (5\text{-}1)$$

and iterating until at every node point ϵ_{ij} is less than some prescribed value. It is found that by imposing

$$\epsilon_{ij} < 10^{-4} \qquad (5\text{-}2)$$

at every node gives a sum of the squares of the residuals between 10^{-17} and 10^{-32} depending on Reynolds number. This gives some confidence that the difference equation is being solved correctly and is a good approximation for the differential equation.

The second aspect that needs to be considered with respect to the correctness of the numerical solution is the choice of grid spacing, $\Delta\xi$ and $\Delta\eta$. Since the pressure drop will prove to be the key parameter to use the numerical model in the investigation of porous media, its dependence on grid spacing is investigated. In Fig.7, the influence of ξ grid spacing on the pressure drop is shown. The pressure drop is determined at the same locations for each point and the ratio of $\Delta\eta/\Delta\xi$ is also kept constant. It is seen that over a good range of $\Delta\xi$ the pressure drop is relatively invariant with grid size. From this analysis grid spacings of

$$\Delta\xi = 0.4 \qquad (5\text{-}3)$$

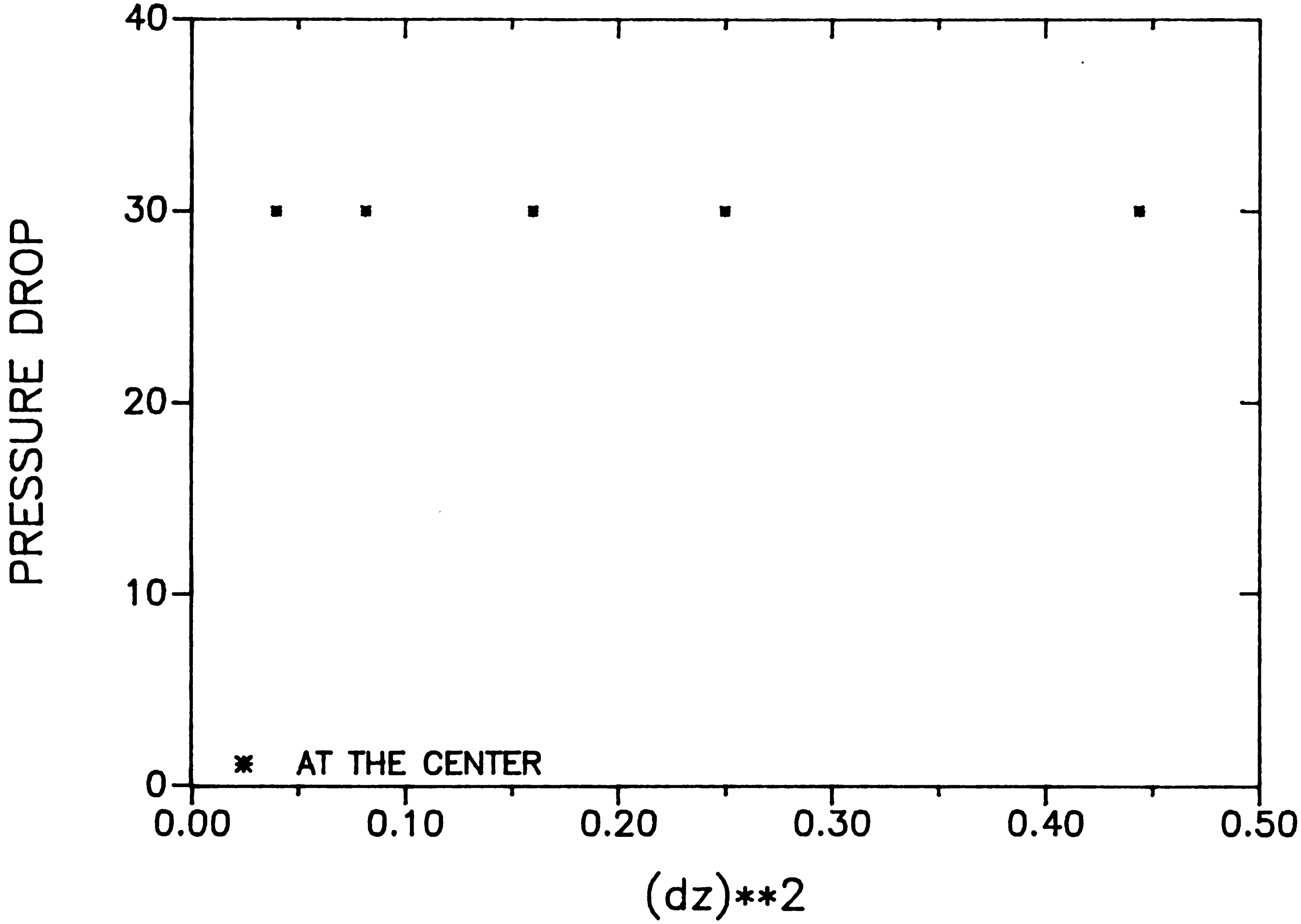
$$\Delta\eta = 0.1 \qquad (5\text{-}4)$$

were chosen and used for all of the remaining computer runs. As was stated above the key parameter for the porous media model is the pressure. The pressure is calculated from the momentum equations. In ξ and η coordinates the ξ momentum equation is written

$$P_{\xi} - \xi\eta b P_{\eta} = b^{3}\psi_{\eta\eta\eta} + Re^{*}(b^{2}\psi_{\eta\eta}\psi_{\xi} - b^{2}\psi_{\xi\eta}\psi_{\eta} + \xi b^{3}\psi_{\eta}^{3}) \qquad (5\text{-}5)$$

while the η-momentum equation becomes

$$P_{\eta} = 0 \qquad (5\text{-}6)$$



EFFECT OF GRID SIZE ON NUMERICAL SOLUTION TO PRESSURE DROP FOR REYNOLDS NUMBER OF 0.5
FIG. 7

Then the pressure as a function of streamwise position (ξ-direction) is obtained by integrating Eq(5-5). A typical pressure profile is shown in Fig.8. The shape is what could be anticipated from the analytical results of Shayesteh(1984) when Re=0.

Recalling that the primary thrust of this thesis is to determine the validity of the Forchheimer term for flows in porous media, it may be written

$$\frac{\Delta P^*}{L} = \frac{\mu}{K} \bar{u}^* + \frac{\rho}{m} \bar{u}^{*2} \qquad (5\text{-}7)$$

where $\bar{u}^*$ is the averaged or Darcian velocity. Using the non-dimensionalization of Chapter 3 gives

$$\frac{\Delta P}{2L_\xi} = \frac{A^2 \Phi^2}{K} \bar{u} + \frac{A}{m} \Phi^2 Re_A \bar{u}^2 \qquad (5\text{-}8)$$

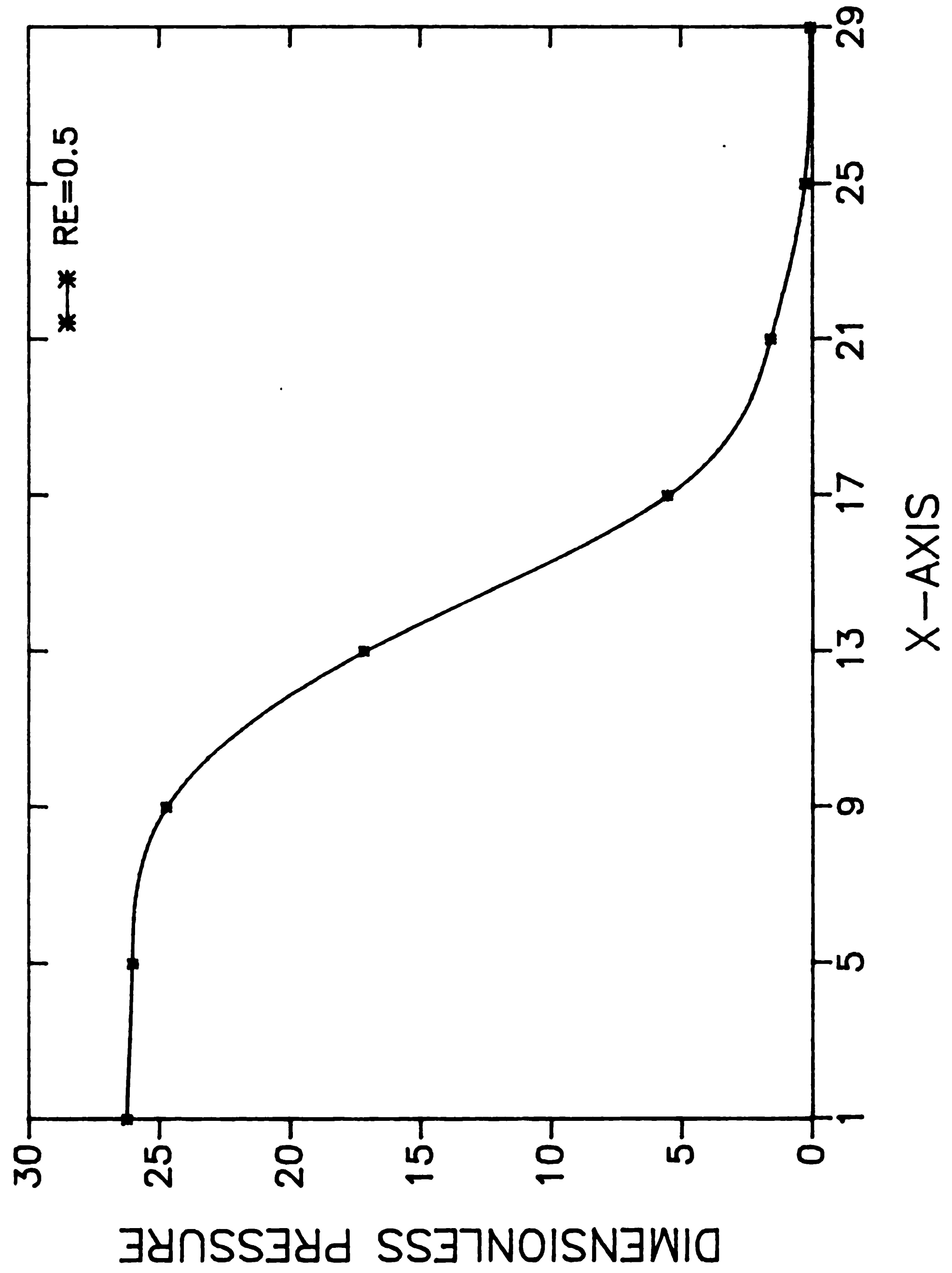
Due to the non-dimensionalization

$$\bar{u} = 1.0 \qquad (5\text{-}9)$$

so that Eq(5-8) could be written as

$$\Delta P = a + b\, Re_A \qquad (5\text{-}10)$$

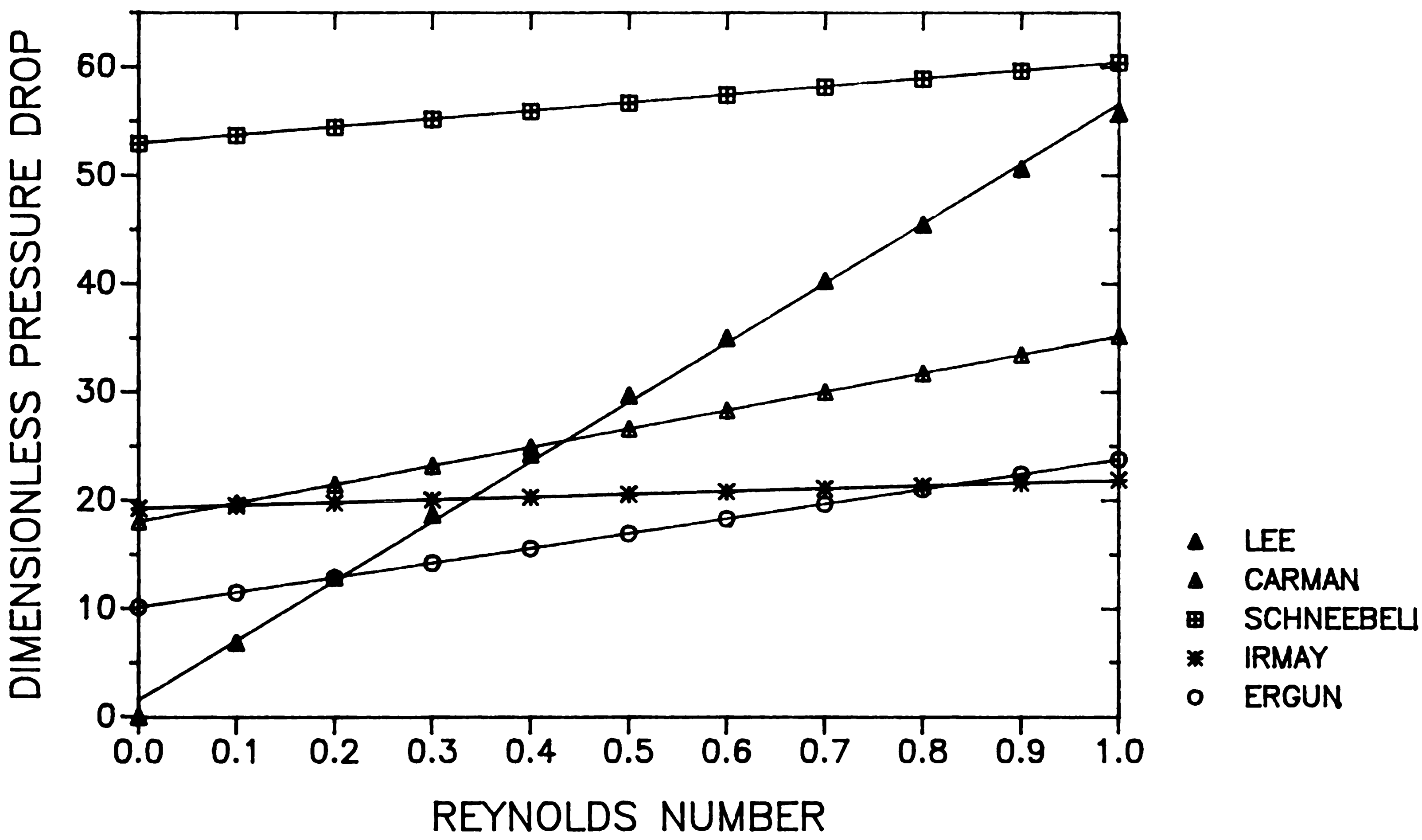
Hence a linear relationship between the numerically determined pressure drop and the Reynolds number would be expected if the



DIMENSIONLESS PRESSURE DISTRIBUTION

FIG.**8**

Forchheimer term appropriate. In Fig.9 the numerical results of this thesis in terms of pressure drop versus Reynolds number are presented. It is seen that, in fact, a linear relationship is predicted. Also presented in Fig.9 are relationships which have been developed by a number of investigators from experimental data. First note the considerable variations among these empirical relationships. The numerical work of this thesis somewhat agrees with the empirical relationships, but clearly the slope of the numerical curve is considerably different. This is not unexpected since the empirical relationships were developed from experiments involving beds of packed spheres which will have porosities and specific surfaces considerably different from those of the cylinder array of the numerical work. Probably the most important trend in the numerical work is the linearity of the pressure drop and Reynolds which not only validates the use of Forchheimer's term but confirms that the quadratic term represents inertia forces.



DIMENSIONLESS PRESSURE DROP VERSUS REYNOLDS NUMBER FOR EMPIRICAL AND NUMERICAL CASES
FIG. 9

CHAPTER 6

Conclusions and Recommendations

The following conclusions can be drawn from this thesis ;

1. An array of cylinders proves to be an acceptable model as a porous medium

2. The Forchheimer relation appears to be valid in the context of high velocity flows in porous media.

3. The quadratic term of the Forchheimer relation is seen to represent the inertia forces of momentum transport.

Several recommendations follow which have been generated from this thesis ;

1. The slope and intercept of the pressure drop-Reynolds number curve need to be investigated as the physical configuration of the model is changed.

2. In considering the slope, a differential equation for $\partial(\Delta P)/\partial Re$ in the limit of Re=0 should be developed to see the sensitivity the physical configuration has on the slope.

3. The influence of turbulent flow in porous media can be studied using the present model by introducing a Reynolds stress term.

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APPENDIX

```
      PROGRAM FORCHHEIMER

C*****************************************************************
C*                                                              *
C*      DETERMINE PSI(Z,E) IN REGION I AM INTERESTED AND        *
C*      THE RELATION BETWEEN PRESSURE DROP AND REYNOLDS NUMBER  *
C*                                                              *
C*                                                              *
C*****************************************************************

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LE,LZ

      COMMON/SOLS/PSI(100,50),PSIO(100,50)
      COMMON/DER1/PSIOE(100,50),PSIOZ(100,50),PSIOZZ(100,50)
      COMMON/DER2/PSIOEE(100,50),PSIOZEE(100,50)
      COMMON/DER3/PSIOEEE(100,50),S(100,50),PREZ(100,50)
      COMMON/DER4/U(100,50),V(100,50)
      COMMON/DER5/PSIOZE(100,50),PREE(100,50)
      COMMON/DER6/PSIE(100,50),PSIZ(100,50)
      COMMON/GRID/M,N,M1,N1,N2,HE,HZ,LJ
      COMMON/PPARM/RE,BETHA,BIGDIF,DELP,LE,LZ
      COMMON/SQU/SQ(100,50)
      DIMENSION A3(100),H(100)

C*****************************************************************
C*      LE=LENGTH OF ETHA                                       *
C*      LZ=LENGTH OF ZETHA                                      *
C*****************************************************************

      OPEN(UNIT=30,FILE='MIKYOUNG.INP3',STATUS='OLD')
      READ(30,21) NPTS
21    FORMAT(I2)
      DO 900 JJ=1,NPTS
      READ(30,22) RE,DELP,ISTRT,LSTJOB
22    FORMAT(F3.1,2X,F4.2,2X,I2,2X,I)
      LJ=0
C*****************************************************************
C*   IF ISTRT IS LESS THAN 0,CALCULATE INITIAL GUESS IN SETUP   *
C*   IF ISTRT IS GREATER THAN 0,USE PREVIOUS CALCULATION AT     *
C*   LAST REYNOLD'S NUMBER INITIAL GUESSES.                     *
C*   IF LSTJOB IS GREATER THAN 0, ONLY READING IN ONE SET OF    *
C*   DATA.                                                      *
C*****************************************************************

      OPEN(UNIT=35,FILE='ANSWER3',STATUS='NEW')

      BIGDIF=.01
      M=22
      N=15
      LZ=3.0
      LE=1.0
      HZ=2.0*LZ/(M-1)
      HE=LE/(N-1)
      N1=N-1
      N2=N-2
      M1=M-1
```

```
              BETHA=.309
              WRITE(35,110)
      110     FORMAT(' ',3X,'DELP',5X,'BIGDIF',6X,'REYNS #')
              WRITE(35,120)DELP,BIGDIF,RE
      120     FORMAT(' ',1F7.2,3X,1F7.3,5X,1F7.3)
              WRITE(35,130)
      130     FORMAT(' ',3X,'HE',7X,'HZ',10X,'BETHA')
              WRITE(35,140)HE,HZ,BETHA
      140     FORMAT(' ',1X,1F7.3,2X,1F7.3,5X,1F7.3)


              CALL SETUP(ISTRT)

      18      CONTINUE

              CALL SOLNS

              DO 30 I=1,M
              DO 30 J=1,N

              DIF=DABS((PSI(I,J)-PSIO(I,J))/PSIO(I,J))
              IF(DIF.GT.1.D-4)  GOTO 50
      30      CONTINUE

      C***************************************************************
      C*      OUTPUT RESULT                                          *
      C***************************************************************

              CALL PRINT

              GOTO 998
      50      CONTINUE

      C***************************************************************
      C*      NEW GUESSES BY S.O.R.                                  *
      C***************************************************************

              DO 65 I=1,M
              DO 65 J=1,N
              PSIO(I,J)=PSIO(I,J)-DELP*(PSIO(I,J)-PSI(I,J))
      65      CONTINUE

              CALL DERIV

              LJ=LJ+1
              IF(LJ.GE.50000) GOTO 30

              GOTO 18
      C1000   WRITE(35,90)
      C90     FORMAT(' ','*** WARNING!  SOLUTION NOT CONVERGED.***')

      998     CONTINUE
      900     CONTINUE
              CLOSE(UNIT=30)
              CLOSE(UNIT=35)
              STOP
              END
```

```
      SUBROUTINE SETUP(ISTRT)

C*******************************************************************
C*    SET STREAM FUNCTION AS INITIAL VALUES                        *
C*******************************************************************

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LE,LZ

      COMMON/SOLS/PSI(100,50),PSIO(100,50)
      COMMON/DER1/PSIOE(100,50),PSIOZ(100,50),PSIOZZ(100,50)
      COMMON/DER2/PSIOEE(100,50),PSIOZEE(100,50)
      COMMON/DER3/PSIOEEE(100,50),S(100,50),PREZ(100,50)
      COMMON/DER4/U(100,50),V(100,50)
      COMMON/DER5/PSIOZE(100,50),PREE(100,50)
      COMMON/GRID/M,N,M1,N1,N2,HE,HZ,LJ
      COMMON/PPARM/RE,BETHA,BIGDIF,DELP,LE,LZ
      COMMON/SQU/SQ(100,50)
      DIMENSION A3(100),H(100)

      A1=-2.0*((2.0)**(.5))/9.0/3.14
      B1=2.0*((2.0)**(.5))/3.0/3.14

      DO 5 I=1,M
      PSI(I,1)=0.01
5     CONTINUE

      DO 8 I=1,M
      PSI(I,N)=LE*(.5*((LZ)**2)+1.0)
8     CONTINUE

      IF (ISTRT.GT.0) GOTO 999

      DO 6 I=1,M
      DO 6 J=1,N
      PSIO(I,J)=(A1*(((J-1)*HE)**3)+B1*(J-1)*HE)*45+0.01
6     CONTINUE

      DO 7I=1,M
      DO 7 J=1,N
      PSIOE(I,J)=(-B1*(((J-1)*HE)**2-1))*45
7     CONTINUE

      DO 11 I=1,M
      DO 11 J=1,N
      PSIOZ(I,J)=0.0
11    CONTINUE

      DO 9 I=1,M
      DO 9 J=1,N
      PSIOEE(I,J)=-B1*(2.0*(J-1)*HE)*45
9     CONTINUE

      DO 10 I=1,M
      DO 10 J=1,N
      PSIOEEE(I,J)=-2.0*B1*45
10    CONTINUE
```

```
        DO 12 I=1,M
        DO 12 J=1,N
        PSIOZEE(I,J)=0.0
12      CONTINUE

999     CONTINUE
        RETURN
        END
```

PROGRAM SECTIONS

	Name	Bytes	Attributes						
0	$CODE	742	PIC	CON	REL	LCL	SHR	EXE	R
2	$LOCAL	1608	PIC	CON	REL	LCL	NOSHR	NOEXE	R
3	SOLS	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
4	DER1	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
5	DER2	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
6	DER3	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
7	DER4	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
8	DER5	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
9	GRID	40	PIC	OVR	REL	GBL	SHR	NOEXE	R
10	PPARM	48	PIC	OVR	REL	GBL	SHR	NOEXE	R
11	SQU	40000	PIC	OVR	REL	GBL	SHR	NOEXE	R
	Total Space Allocated	602438							

ENTRY POINTS

Address	Type	Name
0-00000000		SETUP

VARIABLES

Address	Type	Name	Address	Type	Name	Address	T
**	R*8	A1	2-00000640	R*8	B1	10-00000008	
10-00000018	R*8	DELP	9-00000014	R*8	HE	9-0000001C	
AP-00000004@	I*4	ISTRT	**	I*4	J	10-00000020	
10-00000028	R*8	LZ	9-00000000	I*4	M	9-00000008	
9-0000000C	I*4	N1	9-00000010	I*4	N2	10-00000000	

ARRAYS

Address	Type	Name	Bytes	Dimensions
2-00000000	R*8	A3	800	(100)
2-00000320	R*8	H	800	(100)
8-00009C40	R*8	PREE	40000	(100, 50)

```
      SUBROUTINE DERIV

C*****************************************************************
C*    ALL DERIVATIVE OF STREAM FUNCTION ARE DERIVED BY           *
C*    DIFFERENCE METHOD.                                         *
C*****************************************************************

      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 LE,LZ

      COMMON/SOLS/PSI(100,50),PSIO(100,50)
      COMMON/DER1/PSIOE(100,50),PSIOZ(100,50),PSIOZZ(100,50)
      COMMON/DER2/PSIOEE(100,50),PSIOZEE(100,50)
      COMMON/DER3/PSIOEEE(100,50),S(100,50),PREZ(100,50)
      COMMON/DER4/U(100,50),V(100,50)
      COMMON/DER5/PSIOZE(100,50),PREE(100,50)
      COMMON/GRID/M,N,M1,N1,N2,HE,HZ,LJ
      COMMON/PPARM/RE,BETHA,BIGDIF,DELP,LE,LZ
      DIMENSION A3(100),H(100)

      DO 20 I=1,M

      Z=(I-1)*HZ-LZ
      A3(I)=1.0/(.5*(Z**2)+1.0)
      H(I)=(I-1)*HZ-LZ
20    CONTINUE

      DO 21 I=2,M1
      DO 21 J=3,N2
      PSIOE(I,J)=(PSIO(I,(J+1))-PSIO(I,(J-1)))/2.0/HE
21    CONTINUE

      DO 22 I=2,M1
      DO 22 J=3,N2
      PSIOZ(I,J)=(PSIO((I+1),J)-PSIO((I-1),J))/2.0/HZ
22    CONTINUE


      DO 24 I=2,M1
      DO 24 J=3,N2
      PSIOZE(I,J)=(PSIO((I+1),(J+1))-PSIO((I-1),(J+1))-PSIO((
     1             I+1),(J-1))+PSIO((I-1),(J-1)))/4.0/HE/HZ
24    CONTINUE

      DO 25 I=2,M1
      DO 25 J=3,N2
      PSIOEE(I,J)=(PSIO(I,(J+1))-2.0*PSIO(I,J)+PSIO(I,(J-1)))
     1             /((HE)**2)
25    CONTINUE

      DO 26 I=2,M1
      DO 26 J=3,N2
      PSIOEEE(I,J)=(PSIO(I,(J+2))-2.0*PSIO(I,(J+1))+2.0*PSIO
     1              (I,(J-1))-PSIO(I,(J-2)))/2.0/((HE)**3)
26    CONTINUE

      DO 27 I=2,M1
```

```
            DO 27 J=3,N2
            PSIOZEE(I,J)=(PSIO((I+1),(J+1))-2.0*PSIO((I+1),J)+PSIO
     1                  ((I+1),(J-1))-PSIO((I-1),(J+1))+2.0*PSIO(
     2                  (I-1),J)-PSIO((I-1),(J-1)))/2.0/((HE)**2)/HZ
 27         CONTINUE

            DO 28 I=2,M1
            PSIOE(I,1)=(-PSIO(I,3)+4*PSIO(I,2)-3*PSIO(I,1))/2/HE
 28         CONTINUE

            DO 29 I=2,M1
            PSIOZ(I,1)=(PSIO((I+1),1)-PSIO((I-1),1))/2.0/HZ
 29         CONTINUE


            DO 31 I=2,M1
            PSIOZE(I,1)=(-(PSIO((I+1),1)-PSIO((I-1),1))/2/HZ+2*(PSIO((I+1),
     1                 2)-PSIO((I-1),2))/HZ-3*(PSIC((I+1),3)-PSIO((I-1),3)
     2                 )/2/HZ)/2/HE
 31         CONTINUE

            DO 32 I=2,M1
            PSIOEE(I,1)=(-PSIO(I,4)+4*PSIO(I,3)-5*PSIO(I,2)+2*PSIO(I,1))
     1                 /(HE)**2
 32         CONTINUE

            DO 33 I=2,M1
            PSIOEEE(I,1)=(-3*PSIO(I,5)+14*PSIO(I,4)-24*PSIO(I,3)+18*PSIO
     1                  (I,2)-5*PSIO(I,1))/2/(HE)**3
 33         CONTINUE

            DO 34 I=2,M1
            PSIOZEE(I,1)=(-(PSIO((I+1),4)-PSIO((I-1),4))/2/HZ+2*(PSIO((I
     1                  +1),3)-PSIO((I-1),3))/HZ-2.5*(PSIO((I+1),2)-PSI
     2                  O((I-1),2))/HZ+(PSIO((I+1),1)-PSIO((I-1),1))/HZ
     3                  )/(HE)**2
 34         CONTINUE

            DO 35 I=2,M1
            PSIOE(I,N)=(3*PSIO(I,N)-4*PSIO(I,N1)+PSIO(I,N2))/2/HE
 35         CONTINUE

            DO 36 I=2,M1
            PSIOZ(I,N)=(PSIO((I+1),N)-PSIO((I-1),N))/HZ/2.0
 36         CONTINUE


            DO 38 I=2,M1
            PSIOZE(I,N)=(1.5*(PSIO((I+1),N)-PSIO((I-1),N))/HE-2*(PSIO((I-1
     1                 ),N1)-PSIO((I-1),N1))/HZ+(PSIO((I+1),N2)-PSIO((I-1
     2                 ),N2))/2/HZ)/2/HE
 38         CONTINUE

            DO 39 I=2,M1
            PSIOEE(I,N)=(2*PSIO(I,N)-5*PSIO(I,N1)+4*PSIO(I,N2)-PSIO(I,(N-3
     1                 )))/(HE)**2
 39          CONTINUE
```

```
      DO 40 I=2,M1
      PSIOEEE(I,N)=(5*PSIO(I,N)-18*PSIO(I,N1)+24*PSIO(I,N2)-14*PSIO
     1              (I,(N-3))+3*PSIO(I,(N-4)))/2/(HE)**3
   40 CONTINUE

      DO 41 I=2,M1
      PSIOZEE(I,N)=((PSIO((I+1),N)-PSIO((I-1),N))/HZ-2.5*(PSIO((I+1
     1              ),N1)-PSIO((I-1),N1))/HZ+2*(PSIO((I+1),N2)-PSIO(
     2              (I-1),N2))/HZ-(PSIO((I+1),(N-3))-PSIO((I-1),(N-3
     3              )))/2/HZ)/(HE)**2
   41 CONTINUE

      DO 42 I=2,M1
      PSIOE(I,2)=(PSIO(I,3)-PSIO(I,1))/HE/2.0
   42 CONTINUE

      DO 43 I=2,M1
      PSIOZ(I,2)=(PSIO((I+1),2)-PSIO((I-1),2))/2.0/HZ
   43 CONTINUE


      DO 45 I=2,M1
      PSIOZE(I,2)=(PSIO((I+1),3)-PSIO((I-1),3)-PSIO((I+1),1)+PSIO((I-1
     1             1))/4.0/HE/HZ
   45 CONTINUE

      DO 46 I=2,M1
      PSIOEE(I,2)=(PSIO(I,3)-2.0*PSIO(I,2)+PSIO(I,1))/((HE)**2)
   46 CONTINUE

      DO 47 I=2,M1
      PSIOEEE(I,2)=(-3*PSIO(I,6)+14*PSIO(I,5)-24*PSIO(I,4)+18*PSIO(I,3
     1              -5*PSIO(I,2))/2/(HE)**3
   47 CONTINUE

      DO 48 I=2,M1
      PSIOZEE(I,2)=(PSIO((I+1),3)-PSIO((I-1),3)-2.0*PSIO((I+1),2)+
     1             2.0*PSIO((I-1),2)+PSIO((I+1),1)-PSIO((I-1),1))
     2              /((HE)**2)/HZ/2.0
   48 CONTINUE

      DO 49 I=2,M1
      PSIOE(I,N1)=(PSIO(I,N)-PSIO(I,N2))/HE/2.0
   49 CONTINUE

      DO 50 I=2,M1
      PSIOZ(I,N1)=(PSIO((I+1),N1)-PSIO((I-1),N1))/HZ/2.0
   50 CONTINUE


      DO 52 I=2,M1
      PSIOZE(I,N1)=(PSIO((I+1),N)-PSIO((I-1),N)-PSIO((I+1),N2)+PSIO((I
     1              N2))/4.0/HE/HZ
   52 CONTINUE

      DO 53 I=2,M1
```

```
            PSIOEE(I,N1)=(PSIO(I,N)-2.0*PSIO(I,N1)+PSIO(I,N2))/((HE)**2)
   53       CONTINUE

            DO 54 I=2,M1
            PSIOEEE(I,N1)=(5*PSIO(I,N1)-18*PSIO(I,N2)+24*PSIO(I,(N-3))-14*PS
         1                (I,(N-4))+3*PSIO(I,(N-5)))/2/(HE)**3
   54       CONTINUE

            DO 55 I=2,M1
            PSIOZEE(I,N1)=(PSIO((I+1),N)-PSIO((I-1),N)-2.0*PSIO((I+1),N1)
         1                +2.0*PSIO((I-1),N1)+PSIO((I+1),N2)-PSIO((I-1),N2))
         2                /2.0/HZ/((HE)**2)
   55       CONTINUE

            DO 56 J=3,N2
            PSIOE(1,J)=(PSIO(1,(J+1))-PSIO(1,(J-1)))/2.0/HE
   56       CONTINUE

            DO 57 J=3,N2
            PSIOZ(1,J)=(-PSIO(3,J)+4*PSIO(2,J)-3*PSIO(1,J))/2/HZ
   57       CONTINUE

            DO 1 J=3,N2
            PSIOZE(1,J)=(-PSIO(3,(J+1))+4*PSIO(2,(J+1))-3*PSIO(1,(J+1))+
         1              PSIO(3,(J-1))-4*PSIO(2,(J-1))+3*PSIO(1,(J-1)))
         2              /4/HE/HZ
    1       CONTINUE
            DO 58 J=3,N2
            PSIOEE(1,J)=(PSIO(1,(J+1))-2.0*PSIO(1,J)+PSIO(1,(J-1)))/((HE)**2
   58       CONTINUE

            DO 59 J=3,N2
            PSIOZEE(1,J)=((-PSIO(3,(J+1))+4*PSIO(2,(J+1))-3*PSIO(1,(J+1)))
         1               /2/HZ-(-PSIO(3,J)+4*PSIO(2,J)-3*PSIO(1,J))/HZ
         2               +(-PSIO(3,(J-1))+4*PSIO(2,(J-1))-3*PSIO(1,(J-1)))
         3               /2/HZ)/(HE)**2
   59       CONTINUE

            DO 60 J=3,N2
            PSIOEEE(1,J)=(PSIO(1,(J+2))-2.0*PSIO(1,(J+1))+2.0*PSIO(1,(J-1))
         1               -PSIO(1,(J-2)))/2.0/((HE)**3)
   60       CONTINUE

            DO 61 J=3,N2
            PSIOE(M,J)=(PSIO(M,(J+1))-PSIO(M,(J-1)))/2.0/HE
   61       CONTINUE

            DO 62 J=3,N2
            PSIOZ(M,J)=(3*PSIO(M,J)-4*PSIO(M1,J)+PSIO((M-2),J))/2/HZ
   62       CONTINUE

            DO 2 J=3,N2
            PSIOZE(M,J)=(3*PSIO(M,(J+1))-4*PSIO(M1,(J+1))+PSIO((M-2),(J+1))
         1              -3*PSIO(M,(J-1))+4*PSIO(M1,(J-1))-PSIO((M-2),(J-1))
         2              )/4/HE/HZ
    2       CONTINUE
            DO 63 J=3,N2
```

```
      PSIOEE(M,J)=(PSIO(M,(J+1))-2.0*PSIO(M,J)+PSIO(M,(J-1)))/((HE)**2
63    CONTINUE

      DO 64 J=3,N2
      PSIOZEE(M,J)=((3*PSIO(M,(J+1))-4*PSIO(M1,(J+1))+PSIO((M-2),(J+1)
     1            )/2/HZ-(3*PSIO(M,J)-4*PSIO(M1,J)+PSIO((M-2),J))/HZ
     2            +(3*PSIO(M,(J-1))-4*PSIO(M1,(J-1))+PSIO((M-2),(J-1)
     3            ))/2/HZ)/(HE)**2
64    CONTINUE

      DO 65 J=3,N2
      PSIOEEE(M,J)=(PSIO(M,(J+2))-2.0*PSIO(M,(J+1))+2.0*PSIO(M,(J-1))
     1            -PSIO(M,(J-2)))/2.0/((HE)**3)
65    CONTINUE
      PSIOE(1,1)=(-PSIO(1,3)+4*PSIO(1,2)-3*PSIO(1,1))/2/HE
      PSIOZ(1,1)=(-PSIO(3,1)+4*PSIO(2,1)-3*PSIO(1,1))/2/HZ
      PSIOZE(1,1)=(-(-PSIO(3,3)+4*PSIO(3,2)-3*PSIO(3,1))+4*(-PSIO(2,3)
     1           +4*PSIO(2,2)-3*PSIO(2,1))-3*(-PSIO(1,3)+4*PSIO(1,2)
     2           -3*PSIO(1,1)))/4/HZ/HE
      PSIOEE(1,1)=(-PSIO(1,4)+4*PSIO(1,3)-5*PSIO(1,2)+2*PSIO(1,1))/(
     1           HE)**2
      PSIOZEE(1,1)=(-(-PSIO(3,4)+4*PSIO(2,4)-3*PSIO(1,4))/2/HZ-5*(-
     1            PSIO(3,2)+4*PSIO(2,2)-3*PSIO(1,2))/2/HZ+(-PSIO(3,1
     2            )+4*PSIO(2,1)-3*PSIO(1,1))/HZ+2*(-PSIO(3,3)+4*PSIO(
     3            2,3)-3*PSIO(1,3))/HZ)/(HE)**2
      PSIOEEE(1,1)=(-3*PSIO(1,5)+14*PSIO(1,4)-24*PSIO(1,3)+18*PSIO(1
     1            ,2)-5*PSIO(1,1))/2/(HE)**3

      PSIOE(1,2)=(PSIO(1,3)-PSIO(1,1))/HE/2.0
      PSIOZ(1,2)=(-PSIO(3,2)+4*PSIO(2,2)-3*PSIO(1,2))/2/HZ
      PSIOZE(1,2)=(-(PSIO(3,3)-PSIO(3,1))+4*(PSIO(2,3)-PSIO(2,1))-3*
     1           (PSIO(1,3)-PSIO(1,1)))/4/HE/HZ
      PSIOEE(1,2)=(PSIO(1,3)-2*PSIO(1,2)+PSIO(1,1))/((HE)**2)
      PSIOEEE(1,2)=(-3*PSIO(1,6)+14*PSIO(1,5)-24*PSIO(1,4)+18*PSIO(1,
     1            3)-5*PSIO(1,2))/2/(HE)**3
      PSIOZEE(1,2)=((-PSIO(3,3)+4*PSIO(2,3)-3*PSIO(1,3))/2/HZ-(-PSIO
     1            (3,2)+4*PSIO(2,2)-3*PSIO(1,2))/HZ+(-PSIO(3,1)+4*PSI
     2            (2,1)-3*PSIO(1,1))/2/HZ)/(HE)**2

      PSIOE(1,N)=(3*PSIO(1,N)-4*PSIO(1,N1)+PSIO(1,N2))/2/HE
      PSIOZ(1,N)=(-PSIO(3,N)+4*PSIO(2,N)-3*PSIO(1,N))/2/HZ
      PSIOZE(1,N)=(-(3*PSIO(3,N)-4*PSIO(3,N1)+PSIO(3,N2))/2/HE+4*(3*
     1           PSIO(2,N)-4*PSIO(2,N1)+PSIO(2,N2))/2/HE-3*(3*PSIO(
     2           1,N)-4*PSIO(1,N1)+PSIO(1,N2))/2/HE)/2/HZ
      PSIOEE(1,N)=(2*PSIO(1,N)-5*PSIO(1,N1)+4*PSIO(1,N2)-PSIO(1,(N-3
     1            )))/(HE)**2
      PSIOEEE(1,N)=(5*PSIO(1,N)-18*PSIO(1,N1)+24*PSIO(1,N2)-14*PSIO(1
     1            ,(N-3))+3*PSIO(1,(N-4)))/2/(HE)**3
      PSIOZEE(1,N)=(2*(-PSIO(3,N)+4*PSIO(2,N)-3*PSIO(1,N))-5*(-PSIO(
     1            3,N1)+4*PSIO(2,N1)-3*PSIO(1,N1))+4*(-PSIO(3,N2)+4
     2            *PSIO(2,N2)-3*PSIO(1,N2))-(-PSIO(3,(N-3))+4*PSIO(
     3            2,(N-3))-3*PSIO(1,(N-3))))/2/HZ/(HE)**2

      PSIOE(1,N1)=(PSIO(1,N)-PSIO(1,N2))/HE/2.0
      PSIOZ(1,N1)=(-PSIO(3,N1)+4*PSIO(2,N1)-3*PSIO(1,N1))/2/HZ
      PSIOZE(1,N1)=(-(PSIO(3,N)-PSIO(3,N2))/HE/2.0+2*(PSIO(2,N)-PSIO
     1            (2,N2))/HE-3*(PSIO(1,N)-PSIO(1,N2))/HE/2.0)/2/HZ
```

```
      PSIOEE(1,N1)=(PSIO(1,N)-2*PSIO(1,N1)+PSIO(1,N2))/((HE)**2)
      PSIOEEE(1,N1)=(5*PSIO(1,N1)-18*PSIO(1,N2)+24*PSIO(1,(N-3))-
     1              14*PSIO(1,(N-4))+3*PSIO(1,(N-5)))/2/(HE)**3
      PSIOZEE(1,N1)=((-PSIO(3,N)+4*PSIO(2,N)-3*PSIO(1,N))/2/HZ-
     1               -(-PSIO(3,N1)+4*PSIO(2,N1)-3*PSIO(1,N1))/HZ+
     2               (-PSIO(3,N2)+4*PSIO(2,N2)-3*PSIO(1,N2))/2/HZ
     3               )/(HE)**2

      PSIOE(M,1)=(-PSIO(M,3)+4*PSIO(M,2)-3*PSIO(M,1))/2/HE
      PSIOZ(M,1)=(3*PSIO(M,1)-4*PSIO(M1,1)+PSIO((M-2),1))/2/HZ
      PSIOZE(M,1)=(3*(-PSIO(M,3)+4*PSIO(M,2)-3*PSIO(M,1))-4*(-PS
     1            IO(M1,3)+4*PSIO(M1,2)-3*PSIO(M1,1))+(-PSIO((M-
     2            2),3)+4*PSIO((M-2),2)-3*PSIO((M-2),1)))/4/HE/HZ
      PSIOEE(M,1)=(-PSIO(M,4)+4*PSIO(M,3)-5*PSIO(M,2)+2*PSIO(M,1)
     1            )/(HE)**2
      PSIOEEE(M,1)=(-3*PSIO(M,5)+14*PSIO(M,4)-24*PSIO(M,3)+18*PSI
     2             O(M,2)-5*PSIO(M,1))/2/(HE)**3
      PSIOZEE(M,1)=-(-(3*PSIO(M,4)-4*PSIO(M1,4)+PSIO((M-2),4))/2/HZ
     1             +2*(3*PSIO(M,3)-4*PSIO(M1,3)+PSIO((M-2),3))/HZ-2
     2            .5*(3*PSIO(M,2)-4*PSIO(M1,2)+PSIO((M-2),2))/HZ+(3*
     3             PSIO(M,1)-4*PSIO(M1,1)+PSIO((M-2),1))/HZ)/(HE)**2

      PSIOE(M,2)=(PSIO(M,3)-PSIO(M,1))/HE/2.0
      PSIOZ(M,2)=(3*PSIO(M,2)-4*PSIO(M1,2)+PSIO((M-2),2))/2/HZ
      PSIOZE(M,2)=(3*(PSIO(M,3)-PSIO(M,1))-4*(PSIO(M1,3)-PSIO(M1,1)
     1            )+PSIO((M-2),3)-PSIO((M-2),1))/4/HE/HZ
      PSIOEE(M,2)=(PSIO(M,3)-2*PSIO(M,2)+PSIO(M,1))/((HE)**2)
      PSIOEEE(M,2)=(-3*PSIO(M,6)+14*PSIO(M,5)-24*PSIO(M,4)+18*PSIO(
     1             M,3)-5*PSIO(M,2))/2/(HE)**3
      PSIOZEE(M,2)=((3*PSIO(M,3)-4*PSIO(M1,3)+PSIO((M-2),3))/2/HZ-
     1             (3*PSIO(M,2)-4*PSIO(M1,2)+PSIO((M-2),2))/HZ+(
     2             3*PSIO(M,1)-4*PSIO(M1,1)+PSIO((M-2),1))/2/HZ)/
     3             (HE)**2

      PSIOE(M,N1)=(PSIO(M,N)-PSIO(M,N2))/HE/2.0
      PSIOZ(M,N1)=(3*PSIO(M,N1)-4*PSIO(M1,N1)+PSIO((M-2),N1))/2/(
     1            HZ)
      PSIOZE(M,N1)=(3*(PSIO(M,N)-PSIO(M,N2))-4*(PSIO(M1,N)-PSIO(M
     1             1,N2))+PSIO((M-2),N)-PSIO((M-2),N2))/4/HE/HZ
      PSIOEE(M,N1)=(PSIO(M,N)-2*PSIO(M,N1)+PSIO(M,N2))/((HE)**2)
      PSIOEEE(M,N1)=(5*PSIO(M,N1)-18*PSIO(M,N2)+24*PSIO(M,(N-3))-
     1              14*PSIO(M,(N-4))+3*PSIO(M,(N-5)))/2/(HE)**3
      PSIOZEE(M,N1)=((3*PSIO(M,N)-4*PSIO(M1,N)+PSIO((M-2),N))/2/HZ
     1               -(3*PSIO(M,N1)-4*PSIO(M1,N1)+PSIO((M-2),N1))/
     2               HZ+(3*PSIO(M,N2)-4*PSIO(M1,N2)+PSIO((M-2),N2)
     3               )/2/HZ)/(HE)**2

      PSIOE(M,N)=(3*PSIO(M,N)-4*PSIO(M,N1)+PSIO(M,N2))/2/HE
      PSIOZ(M,N)=(3*PSIO(M,N)-4*PSIO(M1,N)+PSIO((M-2),N))/2/HZ
      PSIOZE(M,N)=(3*(3*PSIO(M,N)-4*PSIO(M,N1)+PSIO(M,N2))-4*(3*PSIO
     1            (M1,N)-4*PSIO(M1,N1)+PSIO(M1,N2))+3*PSIO((M-2),N)-
     2            4*PSIO((M-2),N1)+PSIO((M-2),N2))/4/HE/HZ
      PSIOEE(M,N)=(2*PSIO(M,N)-5*PSIO(M,N1)+4*PSIO(M,N2)-PSIO(M,(N-3)
     1            ))/(HE)**2
      PSIOEEE(M,N)=(5*PSIO(M,N)-18*PSIO(M,N1)+24*PSIO(M,N2)-14*PSIO
     1             (M,(N-3))+3*PSIO(M,(N-4)))/2/(HE)**3
      PSIOZEE(M,N)=((3*PSIO(M,N)-4*PSIO(M1,N)+PSIO((M-2),N))/HZ-2.5*
```

```
     1                  (3*PSIO(M,N1)-4*PSIO(M1,N1)+PSIO((M-2),N1))/HZ+
     2                  2*(3*PSIO(M,N2)-4*PSIO(M1,(N-2))+PSIO((M-2),N2))
     3                  /HZ-(3*PSIO(M,(N-3))-4*PSIO(M1,(N-3))+PSIO((M-2),
     4                  (N-3)))/HZ/2)/(HE)**2


        DO 77 I=1,M
        DO 77 J=1,N

        PREZ(I,J)=((A3(I))**3)*PSIOEEE(I,J)-RE*(-H(I)*(A3(I))*
     1              *3*((PSIOE(I,J))**2)+(A3(I))**2*PSIOE(I,J)*PSIOZE
     2              (I,J)-(A3(I))**2*PSIOZ(I,J)*PSIOEE(I,J))

 77     CONTINUE


        RETURN
        END
```

PROGRAM SECTIONS

	Name	Bytes	Attributes						
0	$CODE	12408	PIC	CON	REL	LCL	SHR	EXE	R
2	$LOCAL	1600	PIC	CON	REL	LCL	NOSHR	NOEXE	R
3	SOLS	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
4	DER1	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
5	DER2	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
6	DER3	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
7	DER4	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
8	DER5	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
9	GRID	40	PIC	OVR	REL	GBL	SHR	NOEXE	R
10	PPARM	48	PIC	OVR	REL	GBL	SHR	NOEXE	R
	Total Space Allocated	574096							

ENTRY POINTS

Address	Type	Name
0-00000000		DERIV

VARIABLES

Address	Type	Name	Address	Type	Name	Address	T
10-00000008	R*8	BETHA	10-00000010	R*8	BIGDIF	10-00000018	
9-0000001C	R*8	HZ	**	I*4	I	**	
9-00000024	I*4	LJ	10-00000028	R*8	LZ	9-00000000	
9-00000004	I*4	N	9-0000000C	I*4	N1	9-00000010	
**	R*8	Z					

```
      SUBROUTINE SOLNS

C********************************************************************
C*      TO GET STREAM FUNCTION IN GIVEN CONDITION                  *
C********************************************************************

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LE,LZ

      COMMON/SOLS/PSI(100,50),PSIO(100,50)
      COMMON/DER1/PSIOE(100,50),PSIOZ(100,50),PSIOZZ(100,50)
      COMMON/DER2/PSIOEE(100,50),PSIOZEE(100,50)
      COMMON/DER3/PSIOEEE(100,50),S(100,50),PREZ(100,50)
      COMMON/DER4/U(100,50),V(100,50)
      COMMON/DER5/PSIOZE(100,50),PREE(100,50)
      COMMON/GRID/M,N,M1,N1,N2,HE,HZ,LJ
      COMMON/PPARM/RE,BETHA,BIGDIF,DELP,LE,LZ
      COMMON/SQU/SQ(100,50)
      DIMENSION A3(100),H(100)

      DO 1 I=1,M
      DO 1 J=1,N
      H(I)=(I-1)*HZ-LZ
1     CONTINUE
      DO 10 J=2,N1

      A3(1)=1.0/((LZ)**2/2.0+1.0)
      FF=(HE)**4*HZ*(.00001)

      C0=(RE*PSIOE(1,J)/2.0/(HE)**2/HZ)*FF
      C1=-2.0*C0-(RE*PSIOEEE(1,J)/2.0/HZ)*FF
      C2=C0
      C3=(A3(1)/(HE)**4-RE*PSIOZ(1,J)/2.0/(HE)**3)*FF
      C4=(-4.0*A3(1)/(HE)**4+RE*PSIOZ(1,J)/(HE)**3+RE*
     1   PSIOZEE(1,J)/2.0/HE+RE*PSIOE(1,J)*2.0*H(1)*A3
     2   (1)/(HE)**2)*FF
      C5=(6.0*A3(1)/(HE)**4-4*RE*H(1)*A3(1)*PSIOE(1,J)
     1   /(HE)**2)*FF
      C6=(-4.0*A3(1)/(HE)**4-RE*PSIOZ(1,J)/(HE)**3-RE*
     1   PSIOZEE(1,J)/2.0/HE+2*RE*H(1)*A3(1)*PSIOE(1,J)
     2   /(HE)**2)*FF
      C7=(A3(1)/(HE)**4+RE*PSIOZ(1,J)/2.0/(HE)**3)*FF
      C8=-C0
      C9=-C1
      C10=C8
      S(1,J)=RE*(PSIOEEE(1,J)*PSIOZ(1,J)-PSIOE(1,J)*PSIOZEE(1,J)
     1       +2*H(1)*A3(1)*PSIOE(1,J)*PSIOEE(1,J))*FF

      PSI(1,J)=(S(1,J)-2.0*(C0*(J-2)+C1*(J-1)+C2*J)*HE*HZ*LZ
     1         -(C7-C3-C4)*HE*2.0/A3(1)-(C4+C6)*PSIO(1,(J+1)
     2         )-(C8+C0)*PSIO(2,(J-1))-(C9+C1)*PSIO(2,J)-(C10+C2)*PSIO
     3         (2,(J+1)))/(C5+C3+C7)
C     SQ(1,J)=(PSI(1,J)*(C5+C3+C7)+2*(C0*(J-2)+C1*(J-1)+C2*J)*HE*HZ*LZ
C     1        +(C7-C3-C4)*HE*2/A3(1)+(C4+C6)*PSIO(1,(J+1))+(C8+C0)*
C     2        PSIO(2,(J-1))+(C9+C1)*PSIO(2,J)+(C10+C2)*PSIO(2,(J+1))
C     3        -S(1,J))**(2)

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```
   10       CONTINUE

            A3(M)=A3(1)
            DO 11 J=2,N1
            C0=(RE*PSIOE(M,J)/2.0/(HE)**2/HZ)*FF
            C1=-2.0*C0-(RE*PSIOEEE(M,J)/2.0/HZ)*FF
            C2=C0
            C3=(A3(1)/(HE)**4-RE*PSIOZ(M,J)/2.0/(HE)**3)*FF
            C4=(-4.0*A3(1)/(HE)**4+RE*PSIOZ(M,J)/(HE)**3+RE*PSIOZEE(M,J)
         1     /2.0/HE+2*RE*H(M)*A3(M)*PSIOE(M,J)/(HE)**2)*FF
            C5=(6.0*A3(1)/(HE)**4-4*RE*H(M)*A3(M)*PSIOE(M,J)/(HE)**2)*FF
            C6=(-4.0*A3(1)/(HE)**4-RE*PSIOZ(M,J)/(HE)**3-RE*PSIOZEE(M,J)
         1     /2.0/HE+2*RE*H(M)*A3(M)*PSIOE(M,J)/(HE)**2)*FF
            C7=(A3(1)/(HE)**4+RE*PSIOZ(M,J)/2.0/(HE)**3)*FF
            C8=-C0
            C9=-C1
            C10=C8
            S(M,J)=RE*(PSIOEEE(M,J)*PSIOZ(M,J)-PSIOE(M,J)*PSIOZEE(M,J)
         1          +2*H(M)*A3(M)*PSIOE(M,J)*PSIOEE(M,J))*FF

            PSI(M,J)=(S(M,J)-(C8*(J-2)+C9*(J-1)+C10*J)*2.0*HE*HZ*LZ
         1          -(C7-C3-C4)*HE*2.0/A3(1)-(C0+C8)*PSIO((M-1),(J-
         2          1))-(C1+C9)*PSIO((M-1),J)-(C2+C10)*PSIO((M-1),(J+1))-
         3          (C4+C6)*PSIO(M,(J+1)))/(C5+C3+C7)
C           SQ(M,J)=(PSI(M,J)*(C5+C3+C7)+(C8*(J-2)+C9*(J-1)+C10*J)*2*HE*HZ*
C        1          LZ+(C7-C3-C4)*HE*2/A3(1)+(C0+C8)*PSIO((M-1),(J-1))+(C1
C        2          +C9)*PSIO((M-1),J)+(C2+C10)*PSIO((M-1),(J+1))+(C4+C6)*
C        3          PSIO(M,(J+1))-S(M,J))**(2)
   11       CONTINUE


            DO 15 I=2,M1
            Z=(I-1)*HZ-LZ
            A3(I)=1.0/(.5*(Z**2)+1.0)

            C0=(RE*PSIOE(I,2)/2.0/(HE)**2/HZ)*FF
            C1=-2.0*C0-(RE*PSIOEEE(I,2)/2.0/HZ)*FF
            C2=C0
            C3=(A3(I)/(HE)**4-RE*PSIOZ(I,2)/2.0/(HE)**3)*FF
            C4=(-4.0*A3(I)/(HE)**4+RE*PSIOZ(I,2)/(HE)**3+RE*PSIOZEE(I,2)
         1     /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,2)/(HE)**2)*FF
            C5=(6.0*A3(I)/(HE)**4-4*RE*H(I)*A3(I)*PSIOE(I,2)/(HE)**2)*FF
            C6=(-4.0*A3(I)/(HE)**4-RE*PSIOZ(I,2)/(HE)**3-RE*PSIOZEE(I,2)
         1     /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,2)/(HE)**2)*FF
            C7=(A3(I)/(HE)**4+RE*PSIOZ(I,2)/2.0/(HE)**3)*FF
            C8=-C0
            C9=-C1
            C10=C8
            S(I,2)=RE*(PSIOEEE(I,2)*PSIOZ(I,2)-PSIOE(I,2)*PSIOZEE(I,2)
         1          +2*H(I)*A3(I)*PSIOE(I,2)*PSIOEE(I,2))*FF

            PSI(I,2)=(S(I,2)-C1*PSIO((I-1),2)-C2*PSIO((I-1),3)
         1          -(C4+2.0*C3)*PSIO(I,1)-C6*PSIO(I,3)-C7*PSIO(I,4)
         2          -(C8+C0)*PSIO((I+1),1)-C9*PSIO((I+1),2)-C10*PSIO((I
         3          +1),3))/(C5-C3)
C           SQ(I,2)=(PSI(I,2)*(C5-C3)+C1*PSIO((I-1),2)+C2*PSIO((I-1),3)+
C        1          (C4+2*C3)*PSIO(I,1)+C6*PSIO(I,3)+C7*PSIO(I,4)+(C8+C0
```

```
C      2           )*PSIO((I+1),1)+C9*PSIO((I+1),2)+C10*PSIO((I+1),3)-
C      3           S(I,J))**(2)
   15     CONTINUE

          DO 16 I=2,M1
          C0=(RE*PSIOE(I,N1)/2.0/(HE)**2/HZ)*FF
          C1=-2.0*C0-(RE*PSIOEEE(I,N1)/2.0/HZ)*FF
          C2=C0
          C3=(A3(I)/(HE)**4-RE*PSIOZ(I,N1)/(HE)**3/2.0)*FF
          C4=(-4.0*A3(I)/(HE)**4+RE*PSIOZ(I,N1)/(HE)**3+RE*PSIOZEE(I,N1)
     1      /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,N1)/(HE)**2)*FF
          C5=(6.0*A3(I)/(HE)**4-4*RE*H(I)*A3(I)*PSIOE(I,N1)/(HE)**2)*FF
          C6=(-4.0*A3(I)/(HE)**4-RE*PSIOZ(I,N1)/(HE)**3-RE*PSIOZEE(I,N1)
     1      /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,N1)/(HE)**2)*FF
          C7=(A3(I)/(HE)**4+RE*PSIOZ(I,N1)/2.0/(HE)**3)*FF
          C8=-C0
          C9=-C1
          C10=C8
          S(I,N1)=RE*(PSIOEEE(I,N1)*PSIOZ(I,N1)-PSIOE(I,N1)*PSIOZEE(I,N1)
     1           +2*H(I)*A3(I)*PSIOE(I,N1)*PSIOEE(I,N1))*FF
          PSI(I,N1)=(S(I,N1)-C0*PSIO((I-1),N2)-C1*PSIO((I-1),N1)
     1           -C3*PSIO(I,(N-3))-C4*PSIO(I,(N2))-C6*PSIO(I,N)
     2           -C8*PSIO((I+1),(N2))-C9*PSIO((I+1),N1)-(C10+C2)*PSIO(
     3           (I+1),N))/(C5+C7)
C         SQ(I,N1)=(PSI(I,N1)*(C5+C7)+C0*PSIO((I-1),N2)+C1*PSIO((I-1),N1)
C      1           +C3*PSIO(I,(N-3))+C4*PSIO(I,N2)+C6*PSIO(I,N)+C8*PSIO(
C      2           (I+1),N2)+C9*PSIO((I+1),N1)+(C10+C2)*PSIO((I+1),N)-
C      3           S(I,N1))**(2)
   16     CONTINUE

          DO 20 I=2,M1
          DO 20 J=3,N2
          Z=(I-1)*HZ-LZ
          A3(I)=1.0/(.5*(Z**2)+1.0)

          C0=(RE*PSIOE(I,J)/2.0/(HE)**2/HZ)*FF
          C1=-2.0*C0-(RE*PSIOEEE(I,J)/2.0/HZ)*FF
          C2=C0
          C3=(A3(I)/(HE)**4-RE*PSIOZ(I,J)/(HE)**3/2.0)*FF
          C4=(-4.0*A3(I)/(HE)**4+RE*PSIOZ(I,J)/(HE)**3+RE*PSIOZEE(I,J)
     1      /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,J)/(HE)**2)*FF
          C5=(6.0*A3(I)/(HE)**4-4*RE*H(I)*A3(I)*PSIOE(I,J)/(HE)**2)*FF
          C6=(-4.0*A3(I)/(HE)**4-RE*PSIOZ(I,J)/(HE)**3-RE*PSIOZEE(I,J)
     1      /2.0/HE+2*RE*H(I)*A3(I)*PSIOE(I,J)/(HE)**2)*FF
          C7=(A3(I)/(HE)**4+RE*PSIOZ(I,J)/2.0/(HE)**3)*FF
          C8=-C0
          C9=-C1
          C10=C8
          S(I,J)=RE*(PSIOEEE(I,J)*PSIOZ(I,J)-PSIOE(I,J)*PSIOZEE(I,J)
     1          +2*H(I)*A3(I)*PSIOE(I,J)*PSIOEE(I,J))*FF
          PSI(I,J)=(S(I,J)-C0*PSIO((I-1),(J-1))-C1*PSIO((I-1),J)-C2*
     1           PSIO((I-1),(J+1))-C3*PSIO(I,(J-2))-C4*PSIO(I,(J-1))
     2           -C6*PSIO(I,(J+1))-C7*PSIO(I,(J+2))-C8*PSIO((I+1),(J
     3           -1))-C9*PSIO((I+1),J)-C10*PSIO((I+1),(J+1)))/C5
C         SQ(I,J)=(PSI(I,J)*C5+C0*PSIO((I-1),(J-1))+C1*PSIO((I-1),J)+C
C      1           2*PSIO((I-1),(J+1))+C3*PSIO(I,(J-2))+C4*PSIO(I,(J-1)
C      2           )+C6*PSIO(I,(J+1))+C7*PSIO(I,(J+2))+C8*PSIO((I+1),(J
```

```
C       3              -1))+C9*PSIO((I+1),J)+C10*PSIO((I+1),(J+1))-S(I,J))*
C       4              *(2)
20            CONTINUE
C             SUM=0.0
C             DO 101 I=1,M
C             DO 101 J=1,N
C             SUM=SUM+SQ(I,J)
C101          CONTINUE
C             WRITE(20,*) RE,LJ,SUM
              RETURN
              END
```

PROGRAM SECTIONS

	Name	Bytes	Attributes						
0	$CODE	5392	PIC	CON	REL	LCL	SHR	EXE	R
2	$LOCAL	1704	PIC	CON	REL	LCL	NOSHR	NOEXE	R
3	SOLS	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
4	DER1	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
5	DER2	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
6	DER3	120000	PIC	OVR	REL	GBL	SHR	NOEXE	R
7	DER4	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
8	DER5	80000	PIC	OVR	REL	GBL	SHR	NOEXE	R
9	GRID	40	PIC	OVR	REL	GBL	SHR	NOEXE	R
10	PPARM	48	PIC	OVR	REL	GBL	SHR	NOEXE	R
11	SQU	40000	PIC	OVR	REL	GBL	SHR	NOEXE	R
	Total Space Allocated	607184							

ENTRY POINTS

Address	Type	Name
0-00000000		SOLNS

VARIABLES

Address	Type	Name	Address	Type	Name	Address	T
10-00000008	R*8	BETHA	10-00000010	R*8	BIGDIF	2-00000648	
2-00000698	R*8	C10	2-00000658	R*8	C2	2-00000660	
2-00000670	R*8	C5	2-00000678	R*8	C6	2-00000680	
2-00000690	R*8	C9	10-00000018	R*8	DELP	2-00000640	
9-0000001C	R*8	HZ	2-000006A0	I*4	I	2-000006A4	
9-00000024	I*4	LJ	10-00000028	R*8	LZ	9-00000000	
9-00000004	I*4	N	9-0000000C	I*4	N1	9-00000010	
**	R*8	Z					

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```
      SUBROUTINE PRINT

C*********************************************************************
C*    PRINT OUT THE RESULTS WE CONCERNED.                            *
C*********************************************************************

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LZ,LE
      COMMON/SOLS/PSI(100,50),PSIO(100,50)
      COMMON/DER1/PSIOE(100,50),PSIOZ(100,50),PSIOZZ(100,50)
      COMMON/DER2/PSIOEE(100,50),PSIOZEE(100,50)
      COMMON/DER3/PSIOEEE(100,50),S(100,50),PREZ(100,50)
      COMMON/DER4/U(100,50),V(100,50)
      COMMON/DER5/PSIOZE(100,50),PREE(100,50)
      COMMON/DER6/PSIE(100,50),PSIZ(100,50)
      COMMON/GRID/M,N,M1,N1,N2,HE,HZ,LJ
      COMMON/PPARM/RE,BETHA,BIGDIF,DELP,LE,LZ
      DIMENSION A3(100),H(100),PRE(100,50),PRE1(100,50)

C     WRITE(35,200)
C200   FORMAT(' ',1X,'RE',2X,'I',2X,'J',2X,'PSI(I,J)')

C     DO 20 I=2,M1
C     DO 20 J=3,N2

C     PSIE(I,J)=(PSI(I,(J+1))-PSI(I,(J-1)))/2.0/HE
C20   CONTINUE

C     DO 21 I=2,M1
C     DO 21 J=3,N2
C     PSIZ(I,J)=(PSI((I+1),J)-PSI((I-1),J))/2.0/HZ
C21   CONTINUE

C     DO 22 I=2,M1
C     PSIE(I,1)=(2*PSI(I,2)-1.5*PSI(I,1)-.5*PSI(I,3))/HE
C22   CONTINUE

C     DO 23 I=2,M1
C     PSIZ(I,1)=(PSI((I+1),1)-PSI((I-1),1))/2.0/HZ
C23   CONTINUE

C     DO 24 I=2,M1
C     PSIE(I,N)=(1.5*PSI(I,N)-2*PSI(I,N1)+.5*PSI(I,N2))/HE
C24   CONTINUE

C     DO 25 I=2,M1
C     PSIZ(I,N)=(PSI((I+1),N)-PSI((I-1),N))/HZ/2.0
C25   CONTINUE

C     DO 26 I=2,M1
C     PSIE(I,2)=(PSI(I,3)-PSI(I,1))/HE/2.0
C26   CONTINUE

C     DO 27 I=2,M1
C     PSIZ(I,2)=(PSI((I+1),2)-PSI((I-1),2))/2.0/HZ
C27   CONTINUE

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```
C       DO 28 I=2,M1
C       PSIE(I,N1)=(PSI(I,N)-PSI(I,N2))/HE/2.0
C28     CONTINUE
C       DO 29 I=2,M1
C       PSIZ(I,N1)=(PSI((I+1),N1)-PSI((I-1),N1))/HZ/2.0
C29     CONTINUE
C       DO 30 J=3,N2
C       PSIE(1,J)=(PSI(1,(J+1))-PSI(1,(J-1)))/2.0/HE
C30     CONTINUE
C       DO 31 J=3,N2
C       PSIZ(1,J)=(PSI(2,J)-PSI(1,J))/HZ
C31     CONTINUE
C       DO 32 J=3,N2
C       PSIE(M,J)=(PSI(M,(J+1))-PSI(M,(J-1)))/2.0/HE
C32     CONTINUE
C       DO 33 J=3,N2
C       PSIZ(M,J)=(PSI(M,J)-PSI((M-1),J))/HZ
C33     CONTINUE
C       PSIE(1,1)=(PSI(1,2)-PSI(1,1))/HE
C       PSIZ(1,1)=(PSI(2,1)-PSI(1,1))/HZ
C       PSIE(1,2)=(PSI(1,3)-PSI(1,1))/HE/2.0
C       PSIZ(1,2)=(PSI(2,2)-PSI(1,2))/HZ
C       PSIE(1,N)=(PSI(1,N)-PSI(1,N1))/HE
C       PSIZ(1,N)=(PSI(2,N)-PSI(1,N))/HZ
C       PSIE(1,N1)=(PSI(1,N)-PSI(1,N2))/HE/2.0
C       PSIZ(1,N1)=(PSI(2,N1)-PSI(1,N1))/HZ
C       PSIE(M,1)=(PSI(M,2)-PSI(M,1))/HE
C       PSIZ(M,1)=(PSI(M,1)-PSI(M1,1))/HZ
C       PSIE(M,2)=(PSI(M,3)-PSI(M,1))/HE/2.0
C       PSIZ(M,2)=(PSI(M,2)-PSI(M1,2))/HZ
C       PSIE(M,N1)=(PSI(M,N)-PSI(M,N2))/HE/2.0
C       PSIZ(M,N1)=(PSI(M,N1)-PSI(M1,N1))/HZ
C       PSIE(M,N)=(PSI(M,N)-PSI(M,N1))/HE
C       PSIZ(M,N)=(PSI(M,N)-PSI(M1,N))/HZ
        DO 6 I=1,M
        Z=(I-1)*HZ-LZ
        A3(I)=1/(Z**2/2.0+1.0)
6       CONTINUE
C       DO 7 I=1,M
C       DO 7 J=1,N
C       U(I,J)=A3(I)*PSIE(I,J)
C7      CONTINUE
C       DO 8 I=1,M
C       DO 8 J=1,N
C       V(I,J)=(I-1)*HZ*(J-1)*HE*A3(I)*PSIE(I,J)-PSIZ(I,J)
C8      CONTINUE
```

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```
C       DO 9 I=1,M
C       DO 9 J=1,N
C       WRITE(35,210) RE,I,J,PSI(I,J)
C210    FORMAT(' ',1F3.1,1X,I2,2X,I2,3X,F17.6)
C       WRITE(35,211) I,J,PREZ(I,J)
C211    FORMAT(' ',I2,2X,I2,2X,1F7.3)
C9       CONTINUE

C       DO 99 I=1,M
C       DO 99 J=1,N1
C       PRE(I,J)=HE*(PREE(I,J)+PREE(I,(J+1)))/2.0
C99     CONTINUE

        DO 100 I=1,M1
        DO 100 J=1,N
        PRE1(I,N)=HZ*(PREZ(I,N)+PREZ((I+1),N))/2.0
100     CONTINUE

C       DO 101 J=1,N1
C       SUM1=SUM1+PRE(1,J)
C101    CONTINUE
C       DO 102 J=1,N1
C       SUM2=SUM2+PRE(M,J)
C102     CONTINUE

C       DO 79 I=1,M
C       SUM1=0.0
C       DO 101 J=1,N1
C       SUM1=SUM1+PRE(I,J)
C101    CONTINUE
        OPEN(UNIT=31,FILE='PRESSURE',STATUS='NEW')
C       WRITE(31,13)
C13      FORMAT(' ','RE',5X,'I',3X,'PRESSURE')
C        WRITE(31,14) RE,I,SUM1
C14      FORMAT(' ',1F3.1,3X,I2,2X,1F17.9)
C79     CONTINUE
        DO 103 I=1,M1
        DO 103 J=1,N
        WRITE(35,104) I,PRE1(I,N)
104     FORMAT(' ',I2,2X,1F17.6)
103     CONTINUE

C       DO 105 I=1,M1
C       WRITE(31,106)RE,PSIOE(I,1),PSIOZEE(I,1),PSIOZE(I,1),PSIOEE(I,1)
C106    FORMAT(' ',1F3.1,2X,1F17.6,2X,1F7.3,2X,1F7.3,2X,1F7.3)
C105    CONTINUE
        RETURN
        END
```