This is to certify that the

dissertation entitled

A QUANTITATIVE POLE-PLACEMENT APPROACH

FOR ROBUST TRACKING

presented by

Chang - Doo Kee

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FOR ROBUST TRACKING

By

Chang - Doo Kee

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ABSTRACT

A QUANTITATIVE POLE-PLACEMENT APPROACH FOR ROBUST TRACKING

By

Chang-Doo Kee

Uncertainty in plant(p) parameters and in disturbances (d) are among the principal reasons for using feedback in control systems. Often control systems are synthesized to satisfy qualitative measures of performance characterized by such popular choices as overshoot, rise time, and settling time. Here a framework is considered where the uncertainties are quantitatively defined in terms of a set of plants $\mathbf{P} = \{p\}$ and disturbances $\mathbf{D} = \{d\}$. Acceptable output sets $\mathcal{A}_{rd}$ are defined for the output y in response to command inputs r and disturbances d, to be achieved $\forall\ p \in \mathbf{P}$. This framework is a manifestation of the need to satisfy the design specifications directly.

The research described in here is based on a novel concept of tracking called "**tracking in the sense of input-output spheres**", first introduced by Barnard and Jayasuriya [4]. The main approach to the problem is based on methods of functional analysis motivated by the topological characterization of the said notion of tracking. Design

criteria formulated within this framework lead to an interesting pole-placement idea which is quantitative in character. Primary contributions of this work were motivated by this notion of "**quantitative pole-placement**".

In this thesis a previous formulation in $L_\infty^k[0, \infty)$ is extended to incorporate external disturbances. It is shown that for sector bounded nonlinearities computation of norms is somewhat simplified. L_∞^k formulation captures the physical nature of continuous tracking precisely. The broadness of this function space however, makes it difficult to obtain simple, yet accurate, criteria for the so called quantitative pole placement. A somewhat loose form of initializing an algorithm for this pole placement is developed however. It is shown that if a Butterworth configuration is chosen as the initial guess for eigenvalues then the numerical scheme "**converges**" to the desired solution.

In order to generate frequency domain interpretations the tracking problem was next embedded in $L_2^k[0, \infty)$. This allows a neat geometric interpretation of the design criteria in terms of the frequency response characterization of a certain linear operator norm for SISO systems. These L_2 interpretations are somewhat similar to the H^∞ - criteria recently developed in the literature for linear systems. Several applications to support the L_∞ theory and L_2 - theory are also included.

To my parents

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CHAPTER I

INTRODUCTION

1.1 Literature Survey

During recent years a number of references have appeared that deal with the design of controllers for uncertain dynamical systems, assuring the proper dynamical performance with respect to stability, regulation, and/or tracking. However, almost all of these deal with qualitative aspects of system behavior, and not quantitative ones. Here, on the other hand, the primary research objective is to further develop a controller synthesis procedure proposed for treating uncertain dynamical systems in a quantitative framework. In particular, controller synthesis for "**precision tracking**" in systems that are both uncertain and nonlinear is considered.

One of the popular methods of design incorporates state feedback controllers that force the closed-loop poles to be at suitable locations in the open complex left-half plane, depending on the design specifications. Typical criteria are relative stability, speed of response, accuracy, and insensitivity to disturbance inputs or parameter variations.

Various approaches have been pursued to develop controller criteria, whether pertaining to tracking or not, based on this state feedback concept. Some of these approaches are based on

(i) Lyapunov stability theory [1, 7, 9, 19, 20, 32, 49],

(ii) Adaptive control [8, 29, 39],

(iii) Classical control concepts [21, 22, 23 37],

(iv) Geometric notions [6, 48, 52],

(v) Servomechanism theory [12, 13, 46],

(vi) Functional analysis [3, 4, 5, 14, 16, 17, 24, 25, 26, 31, 41, 42, 43, 51, 54, 55, 56]

The state feedback schemes require the accessibility of system states. When they are not available, a nominal observer structure is used in the feedback loop to generate state estimates [2, 34, 44].

Criteria for uncertain systems that consider stability are derived by Barmish, Corless, Leitmann, and Thorp [7, 9, 20, 32, 49]. They are based on the constructive use of Lyapunov theory in which first a suitable Lyapunov matrix is generated by solving either a Lyapunov equation or a suitable optimal control problem, and then inventing a control action which admits a suitable Lyapunov function for all admissible uncertainties. The possible sizes of uncertain elements are assumed to be in prescribed compact sets. When information about the possible sizes of uncertain elements is not available , adaptive control strategies are employed with the estimates of uncertain bounds [8, 29, 39].

Criteria by Horowitz and coworkers [21, 22, 23, 37] are essentially in the frequency domain, and rely heavily on classical design concepts. These controller criteria are developed for assuring system performance specified by acceptable range of rise time, overshoot, and settling time in the presence of parameter variations and disturbance inputs. The Shauder's fixed-point theorem is used to establish the validity of these designs. For higher order systems, controller

criteria are often based on the existence of a few dominant poles and zeros which primarily determine the system transient response. Design techniques for third and fourth order dominant systems are investigated in [21, 37]. These procedures, although very intuitive, do not lend themselves readily to time domain analysis. Nevertheless, Horowitz was one of the early researchers to point out the importance of uncertainty in shaping appropriate controllers.

In the geometric approach [6, 48, 52], it is recognized that the properties of linear systems depend on certain linear subspaces of the state space. The design problem then is to generate a linear subspace which has desirable structure in the state space so that the design specifications are met. By describing the design specifications of a controlled system as a specific structure of the subspace generated by the feedback controller, the design is treated more intuitively and generally better insights are gained for the conditions of existence of solutions for decoupled control or disturbance localization.

In the servomechanism problem [12, 13, 46], the controller consists of two devices, namely, a servo-compensator and a stabilizing compensator. It is assumed that the disturbance is unknown and unmeasurable, but satisfies a certain ordinary differential equation. The reference outputs are also assumed to satisfy a similar differential equation. A servo-compensator is constructed according to the characteristic equation obtained from the differential equation, and then a stabilizing compensator is designed so that the resultant closed loop system becomes asymptotically stable.

The functional analysis approach which will play a central role in this research was pioneered by Zames and Sandberg. They studied the absolute stability of Lur'e type nonlinear systems [41, 42, 43, 53, 55]. Desoer and Wang [14] studied asymptotic tracking and disturbance rejection properties within this framework for general nonlinear multivariable systems which consist of input as well as output channel nonlinearities. They established the robustness of these controllers with respect to linear perturbation of the plant. Lecoq and Hopkin [31] developed a bounded-input bounded-output stability criterion for systems with nonlinearities that do not satisfy sector conditions. Recently quantitative controller criteria for precision tracking in nonlinear uncertain systems were developed by Barnard and Jayasuriya [3, 4, 5, 24, 25]. They formulated the tracking specifications in terms of topological neighbourhoods in normed function spaces and employed nonlinear state observers related to uncertain plants for implementing state feedback controllers. Their approach is based on the application of the Banach fixed-point theorem and equation comparison techniques.

1.2 General Objectives and Approach

The need for quantitative controller criteria stems from a wide spread demand for more stringent tracking specifications as opposed to the classic notion of asymptotic tracking, (i.e. the output vector $y(t) \to y_o(t)$ as $t \to \infty$), especially as required by high performance robot manipulators, automotive engine and clutch control, and missiles with ram jets where uncertainty becomes an important issue.

Although several attempts and approaches have been taken to solve this very realistic problem of controlling systems with parameter uncertainties, it is far from complete. The work of Horowitz and his coworkers and that of Barnard and Jayasuriya constitute a design theory for the direct satisfaction of design specifications. Although much recent work has been done on robustness, a theory for the direct satisfaction of design criteria for uncertain, nonlinear systems is yet to appear. All these works however are a step in the right direction. The research described in this thesis is an attempt to develop a formal procedure for the quantitative pole-allocation that is pivotal to the successful execution of the design methodology proposed by Barnard and Jayasuriya [4]. In particular, the work described in here addresses **"precision tracking"** in uncertain nonlinear plants with multiple inputs and outputs. Information available about the uncertainties of the model is restricted to be their possible sizes, i.e., the uncertainties are assumed to belong to certain prescribed compact sets. It is important to emphasize that these uncertainties are deterministic in nature, i.e., they do not fall into the usual category of random variables.

In their problem formulation Barnard and Jayasuriya employed

(i) weighted norms in L_∞(the space of essentially bounded functions) to measure tracking error and plant disturbances;

(ii) nonlinear observers of the Luenberger type to realize controllers;

(iii) operator equations in L_∞ to represent combined plants and controllers, and

(iv) nominal or average systems to define suitable command inputs and to compare the actual and specified plant responses.

The author's main design criterion is stated as a quantitative pole placement procedure for controlling the size of a certain linear operator norm. The idea of pole placement appears frequently in control theory and has been around for quite some time now. The main attempt here has been the arbitrary placement of closed loop eigenvalues. What is not clear, however, is how one should specify their location for the satisfaction of design specifications. Some guidelines are available, for example in terms of algebraic Riccati equations for a special class of linear systems (LQ problem). In the present work the satisfaction of the tracking specifications is directly related to the pole-placement. In particular, the tracking specifications are met if an eigenvalue placement can be found so that an operator which may be characterized by these eigenvalues falls within a set of linear operators whose L_∞ - induced norms are upper bounded by a threshold value. We therefore appropriately refer to this "**quantitative pole placement**" notion as a sufficient condition for "**trackability in the sense of spheres**". This is clearly a stronger notion than controllability.

The main contributions of this thesis are

(i) A pole-placement criterion developed through a generalized LQ formulation, for the satisfaction of the norm condition.

(ii) An algorithm based on eigenvalue-eigenvector placement for the computation of the operator norm in multi-input multi-output (MIMO) systems.

(iii) An L_2 - problem formulation for a class of uncertain systems, and a graphical design procedure for the resulting controller realization.

1.3 Organization

In Chapter II, the concept of tracking in the sense of input-output spheres is introduced. Then design criteria for the precise tracking for uncertain nonlinear systems are developed by employing the Banach fixed point theorem together with the comparison of equations. Finally, as a special case, a system with a sector bounded nonlinearity is considered where simpler algorithms for accomplishing the design are obtained.

In Chapter III, servo-tracking in Lur'e type nonlinear systems is considered. In particular, the design criteria formulated in an L_2 - setting for single-input single-output(SISO) systems with sector bounded nonlinearities offer a neat circle type geometric interpretation.

In Chapter IV, a pole placement based on a generalized LQ performance measure is formulated, using classical variational techniques. By means of a limiting process, an optimal pole pattern which is compatible with the Butterworth configuration is obtained. This provides a means for selecting a set of eigenvalues for the satisfaction of design criteria.

Computer oriented algorithms are developed in Chapter V in order to utilize the theorem developed in Chapter II. Especially an explicit expression for the operator norm of (II-16) is given in terms

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Chapter VI contains several examples including a 3 degree of freedom(DOF) robot manipulator, a synchronous machine, and a gyroscope. Computer simulation results confirming the validity of the theory are also given. Finally conclusions and some suggestions for future work are given in Chapter VII.

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CHAPTER II

PROBLEM FORMULATION FOR SERVO-TRACKING

In this chapter the design philosophy advanced by Barnard and Jayasuriya [4, 26] is revisited. The notion of tracking that is central to the formulation is carefully stated. Then the tracking problem is formulated for a class of nonlinear uncertain systems with external disturbances. The inclusion of the external disturbance explicitly in the plant model is a minor extension of the original formulations [4, 26]. The uncertainties allowed are assumed to be varying only within the boundaries of certain prescribed sets, i.e., they belong to certain pre-specified compact sets.

The controller structure employed to realize the tracking specifications is nonlinear and is of a feedback form. In order to estimate the inaccessible states a Luenberger type nonlinear observer is employed. The observer realization is based on the nominal model corresponding to the actual uncertain plant. Deviations in tracking that arise due to uncertainties are also quantified with respect to this nominal model. A sufficiency theorem guaranteeing tracking in the sense of spheres is derived. This theorem essentially captures the pole-placement nature of the primary design criterion.

2.1 Servo - Tracking

Conventionally, tracking is referred to as following a specified trajectory in an asymptotic sense. That is the actual trajectory y(t) $\epsilon\ \mathbf{R}^b$ approaches a reference or nominal trajectory $y_o(t)\ \epsilon\ \mathbf{R}^b$ as time t $\rightarrow \infty$. In this notion of tracking the initial deviations in transient performance such as large overshoots are not significant as long as the system exhibits a stable behavior and the actual output y(t) eventually approaches the nominal output $y_o(t)$ in spite of undesirable transients.

Opposed to this classic notion of asymptotic tracking, a "**precise**" servo-tracking in which the actual output y(t) follows the nominal output $y_o(t)$ within an error bound β_o for all time t ϵ $[0, \infty)$ is adopted in the work presented in this thesis. This concept of tracking is known as "**tracking in the sense of input-output spheres**". The precise mathematical definition of what this means is given below.

Definition 1 : A given output $\{y : T \rightarrow \mathbf{R}^b\}\ \epsilon\ L_p^b[0, \infty)$, is said to belong to an **output sphere** $\Omega(y : y_o, \beta_o)$ of radius $\beta_o > 0$ centered at $\{ y_o : T \rightarrow \mathbf{R}^b \}\ \epsilon\ L_p^b[0, \infty)$ if $||y - y_o|| \leq \beta_o$, where $||\cdot||$ is any norm associated with the output function space $T = \{y \mid y : T \rightarrow Y\}$. y_o is referred to as the nominal output and

$$\Omega(y : y_o, \beta_o) = \{y \mid ||y - y_o|| \leq \beta_o\}$$

Here $T = [0, \infty)$ is the time set.

Definition 2 : A given input $\{ r : T \rightarrow R^m \}\ \epsilon\ L_p^m[0, \infty)$ is said to belong to an **input sphere** $\Omega(r : r_o, \beta_i)$ of radius $\beta_i > 0$ centered at $\{ r_o : T \rightarrow \mathbf{R}^m\} \in L_p^m[0, \infty)$ if $||r - r_o|| \leq \beta_i$, where $||\cdot||$ is any norm associated with the input function space $U = \{r \mid r : T \rightarrow \xi\}$. r_o is referred to as the nominal input.

$$\Omega(r : r_o, \beta_i) = \{r \mid ||r - r_o|| \leq \beta_i \}.$$

With these concepts, we can now formalize the notion of tracking alluded to earlier as follows.

Consider a system described by the operator equation

$$y_o = \Phi_o r_o$$

where the nominal output $y_o \in T$, the nominal input $r_o \in U$ and the operator $\Phi_o : U(\xi) \rightarrow T$. Let $r : T \rightarrow \mathbf{R}^m$ be any other input function in the sphere $\Omega (r : r_o, \beta_i)$ which generates $y : T \rightarrow \mathbf{R}^b$ as the output satisfying the operator equation

$$y = \Phi_o r$$

with specified constants $\beta_i > 0$ and $\beta_o > 0$. If the system output $y \in \Omega (y : y_o, \beta_o)$ with any $r \in \Omega (r : r_o, \beta_i)$, then the system is said to track y_o in the sense of input - output spheres. This idea is illustrated in Figure 1.

In view of the above definitions, it seems appropriate to define a sphere accounting for allowable external disturbances as follows.

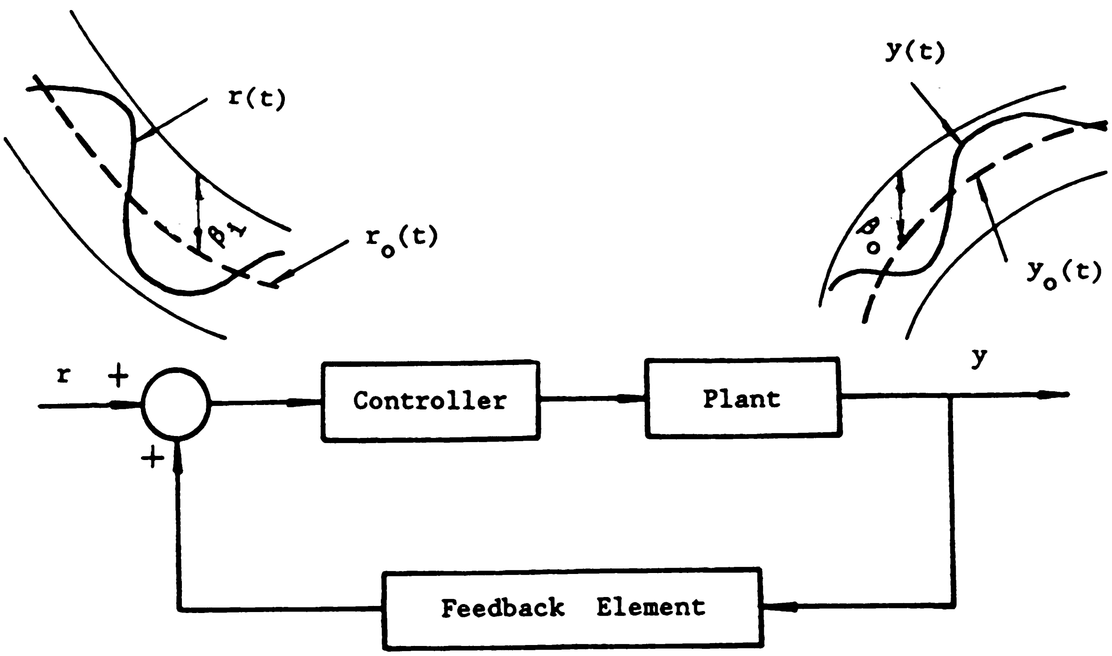


Figure 1 Illustration for precision tracking

Definition 3 : Let $w \in L_p^d[0, \infty)$ be an external disturbance with the specified constant $\beta_w > 0$ such that

$$\Omega\ (w : 0, \beta_w) = \{ w \in L_p^d[0, \infty) \mid \| w - 0 \| \leq \beta_w \}$$

where $\|\cdot\|$ denotes any L_p - norm associated with the function space $W = \{w \mid w : T \to \mathbf{R}^d \}$. Then the disturbance is said to be belongingto a compact set of radius β_w centered at the zero element $0 \in L_p^d[0, \infty)$.

2.2 System Formulation

2.2.1 Plant

Consider the MIMO systems governed by the state equations of the form

$$\dot{x}(t) = A\,x(t) + B\,u(t) + D\,w(t) + f(x(t),\gamma,t) \tag{II-1a}$$

$$y(t) = C\,x(t) \tag{II-1b}$$

where the state $x(t) \in \mathbf{R}^n$, the control $u(t) \in \mathbf{R}^m$, the uncertainty $\gamma \in \Gamma \subset \mathbf{R}^a$, the time $t \in T = [0, \infty)$, the external disturbance $w(t) \in W \subset \mathbf{R}^d$, the nonlinear function $f : \mathbf{R}^n \times \mathbf{R}^a \times T \rightarrow \mathbf{R}^n$, and the output $y(t) \in \mathbf{R}^b$. A, B, C and D are constant matrices of dimension n x n, n x m, b x n, and n x d respectively.

The following assumptions are made with regard to the plant description (II-1)

(i) The pair {A, B} is completely controllable

(ii) The pair {A, C} is completely observable

(iii) The uncertain elements $\gamma \in \Gamma \subset \mathbf{R}^a$ and external disturbances $w \in W \subset \mathbf{R}^d$ where Γ and W are compact sets with prescribed boundaries.

The control problem to be considered is the synthesis of a controller that assures tracking in the sense of spheres for the above system irrespective of the uncertainties. Specifically we seek a feedback controller which assures that every output y is in $\Omega(r : y_o, \beta_o)$ for every input r in $\Omega(r : r_o, \beta_i)$ and any disturbance w in $\Omega(w : w_o, \beta_w)$, no matter what specific value the vector parameter γ takes in the prescribed boundary.

2.2.2 Nominal Plant

By considering a hypothetical plant which is completely known (i.e., free of uncertain elements and external disturbances) we establish a nominal plant corresponding to equation (II-1) given by

$$\dot{x}(t) = A\,x(t) + B\,u(t) + f_o(x(t),t) \tag{II-2a}$$

$$y(t) = C\,x(t) \tag{II-2b}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, time $t \in T = [0, \infty)$, the nonlinear function $f_o : \mathbf{R}^n \times T \rightarrow \mathbf{R}^n$ and $y(t) \in \mathbf{R}^b$. A, B, and C are constant matrices of dimensions n x n, n x m, and b x n respectively. It is worth noting here that the nonlinear design function f_o may be chosen to be the nonlinear uncertain term, with the uncertain parameters replaced with certain nominal values.

Based on this nominal plant, a nonlinear observer is constructed next for estimating the system states, which may be inaccessible.

2.2.3 Observer and the Controller

For the implementation of any state feedback controller for the plant (II-1), a nonlinear observer based on the nominal plant is employed. This observer is synthesized according to the following equations

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + Bu(t) + f_o(\hat{x}(t),t) + GC(\hat{x}(t)-x(t)) + V_1 r(t) \tag{II-3}$$

where $\hat{x}(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $t \in T$ $[0, \infty)$, $r(t) \in \mathbf{R}^m$, $f_o : \mathbf{R}^n \times T \rightarrow \mathbf{R}^n$. G and V_1 are constant matrices of order n x b and n x m respectively. The state feedback control law given by

$$u(t) = V_2 r(t) + K \hat{x}(t) \tag{II-4}$$

is postulated as a possible controller for servoaction, where $u(t) \in \mathbb{R}^m$, $r(t) \in \mathbb{R}^m$ and $\hat{x}(t) \in \mathbb{R}^n$, and K and V_2 are constant design matrices of order m x n and m x m respectively.

The feedback system represented by equations (II-1), (II-3) and (II-4) are combined into the form

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -GC & A+BK+GC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BV_2 \\ BV_2+V_1 \end{bmatrix} r(t) + \begin{bmatrix} D \\ 0 \end{bmatrix} w(t)$$

$$+ \begin{bmatrix} f(x(t), \gamma, t) \\ f_o(\hat{x}(t), t) \end{bmatrix} \tag{II-5a}$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{II-5b}$$

This augmented system configuration is shown in Figure 2.

Rewriting (II-5) in a combined state form gives

$$\dot{z}(t) = R\, z(t) + B_o r(t) + B_1 w(t) + \vartheta(z(t), \gamma, t) \tag{II-6a}$$

$$y(t) = C_o\, z(t) \tag{II-6b}$$

where $$z = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$R = \begin{bmatrix} A & BK \\ -GC & A+BK+GC \end{bmatrix}$$

$$B_o = \begin{bmatrix} BV_2 \\ BV_2+V_1 \end{bmatrix}$$

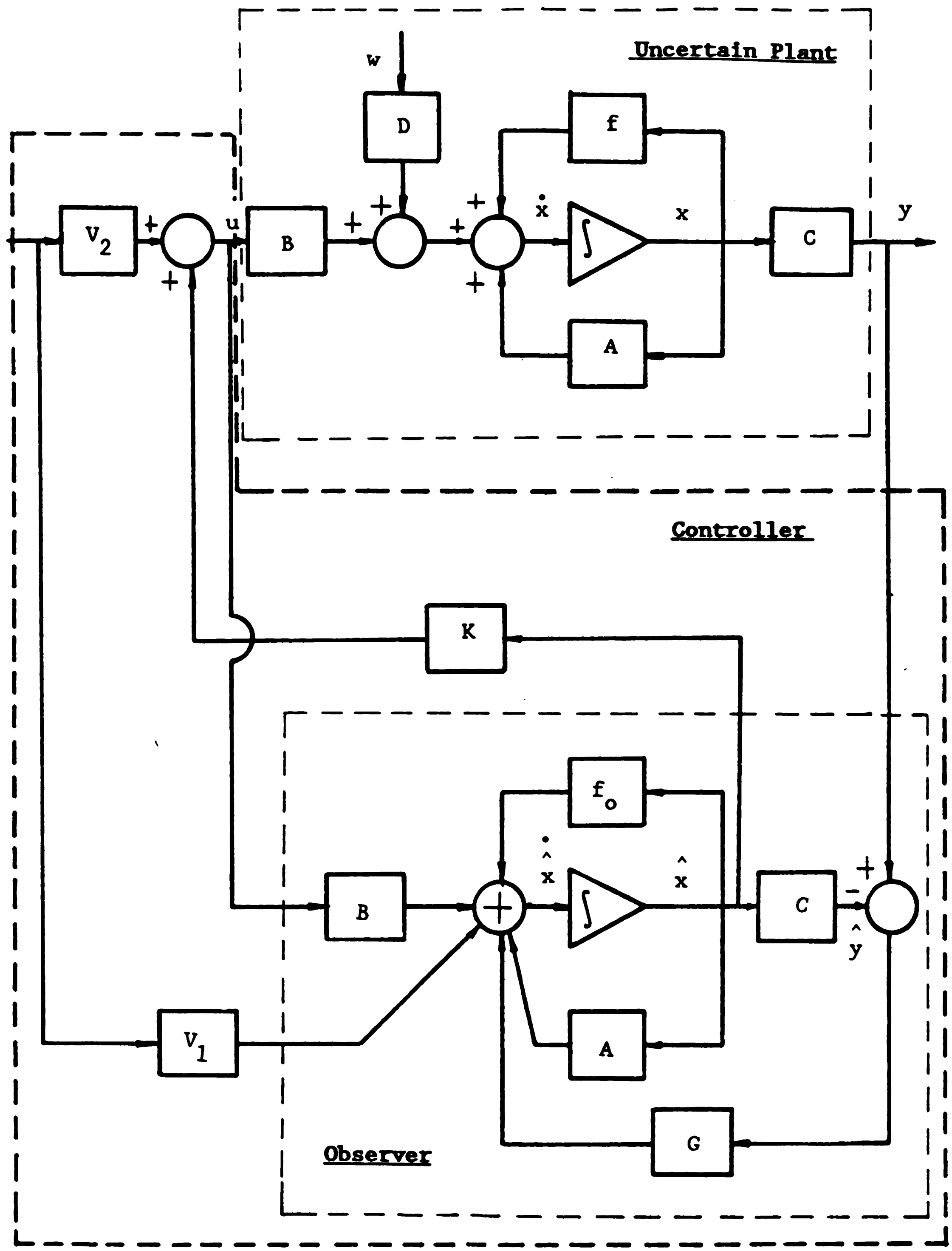


Figure 2 Controller system configuration

$$B_1 = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

$$C_o = [\ C \qquad 0\ ]$$

$$\vartheta(z(t), \gamma, t) = \begin{bmatrix} f(x(t), \gamma, t) \\ f_o(\hat{x}(t), t) \end{bmatrix}$$

2.2.4. Operator Representation

The augmented system incorporating the plant and the observer leads to an operator formulation that lends itself especially well to tracking analysis. Taking the Laplace transform of equation (II-6) yields

$$sZ(s) - z(0) = R\ Z(s) + B_oR(s) + B_1W(s) + \tilde{\vartheta} \qquad \text{(II-7)}$$

where $Z(s) = \pounds\ z(t)$

$R(s) = \pounds\ r(t)$

$W(s) = \pounds\ w(t)$

$\tilde{\vartheta} = \pounds\ v(z(t), \gamma, t)$

and $z(0) = \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix}$ is the initial combined state.

Rewriting (II-7) yields

$$[sI - R]\ Z(s) = B_oR(s) + B_1W(s) + \tilde{\vartheta} + z(0) \qquad \text{(II-8)}$$

where I denotes the 2n x 2n identity matrix.

Premultiplying equation (II-8) by $P^{-1}(s) = [sI - R]^{-1}$, we obtain

$$Z(s) = P^{-1}(s)B_oR(s) + P^{-1}(s)B_1W(s)+P^{-1}(s)\tilde{\vartheta} + P^{-1}(s)z(0) \qquad \text{(II-9)}$$

Taking the inverse Laplace transform of (II-9) yields

$$z(t) = \pounds^{-1}\{P^{-1}(s)B_oR(s)\} + \pounds^{-1}\{P^{-1}(s)B_1W(s)\} + \pounds^{-1}\{P^{-1}(s)\tilde{\vartheta}\}$$

$$+ \pounds^{-1}\{P^{-1}(s)z(0)\} \qquad \text{(II-10)}$$

By utilizing the convolution theorem, (II-10) can be written in the integral form

$$z(t) = \int_0^t e^{R(t - \tau)} [B_o r(\tau) + B_1 w(\tau) + \vartheta(z(\tau),\gamma,\tau)]d\tau + e^{Rt}z(0) \qquad \text{(II-11)}$$

where $\pounds^{-1}\{P^{-1}(s)\} = e^{Rt}$

Remarks :

(i) $\pounds$ stands for the Laplace transform operator and

$\pounds^{-1}$ stands for the inverse Laplace transform operator

(ii) If $\pounds\{f_1(t)\} = F_1(s)$, $\pounds\ \{f_2(t)\} = F_2(s)$ and the convolution

$$(f_1 * f_2) = \int_0^t f_1(t-\tau)\ f_2(\tau)\ d\tau\ , \quad \text{then}$$

$$\pounds\ \{f_1 * f_2\} = F_1(s) \cdot F_2(s)\ .$$

Assuming that the function $\vartheta(z(t), \gamma, t)$ is continuous, and the eigenvalues of the matrix R are in the open left half complex plane (i.e., the augmented system represents a stable behavior), we can write (II-11) in an equivalent operator form

$$z(t) = \Psi N_\gamma z(t) + \Psi B_o r(t) + \Psi B_1 w(t) + q(t) \qquad \text{(II-12)}$$

where Ψ and N_γ are respectively a linear and an uncertain nonlinear map from $L_\infty^{2n}(T)$ back into itself given by

$$(\Psi\, z)(t) = \int_0^t e^{R(t-\tau)}\, z(\tau)\, d\tau$$

$$(N_\gamma z)(t) = \vartheta(z(t),\, \gamma,\, t)$$

and $$q(t) = e^{Rt}\, z(0)$$

The closed loop system in this operator form is shown in Figure 3(a).

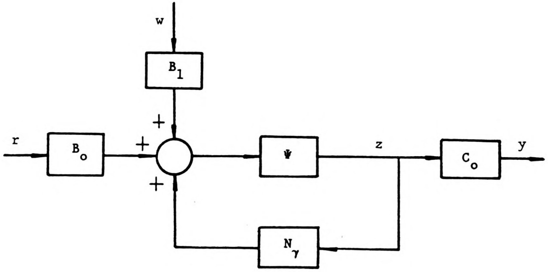


Figure 3(a) Combined uncertain system

Now by augmenting the nominal plant defined by (II-2) with the nonlinear observer given by (II-3), a set of nominal operator equations can be immediately defined as

$$z_o(t) = \Psi\, N_o z_o(t) + \Psi\, B_o r_o(t) + q(t) \qquad \text{(II-13a)}$$

$$y_o(t) = C_o z_o(t) \qquad \text{(II-13b)}$$

where $(r_o,\, y_o,\, z_o)$ is a completely known triple of a specified nominal output y_o, and corresponding solutions z_o, r_o, with r_o serving as a nominal command input relative to y_o. The nonlinear map N_o is represented by

$$(N_o z)(t) = \begin{bmatrix} f_o(x(t),t) \\ f_o(\hat{x}(t),t) \end{bmatrix} .$$

The nominal system charaterized by operator equation (II-13) is shown in Figure 3(b).

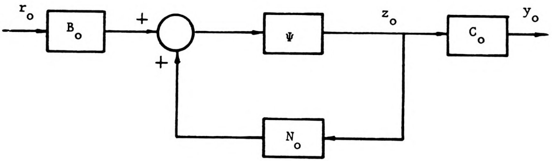


Figure 3(b) Nominal system

Remarks :

(i) It should be pointed out that the function space $L_\infty^b[0, \infty)$ is used rather than $L_1^b[0, \infty)$, or L_2^b $[0, \infty)$, because the norm denoted by $||\cdot||$ associated with $L_\infty^b[0, \infty)$ can represent more precisely and naturally the physical constraints usually attributed to tracking. Thus, throughout the thesis $||\cdot||$ denotes L_∞- norm defined in the Appendix A, unless otherwise specified.

(ii) For MIMO systems where the nominal outputs $y_{o1} \cdots\cdots y_{ob}$ are to be tracked within the spheres of radii $\beta_{o1} \cdots\cdots \beta_{ob}$, an effective sphere of radius β_o is considered such that

$$\beta_o = \min \{\beta_{o1} \cdots\cdots \beta_{ob}\}.$$

Then the effective output sphere is defined by

$$\| y - y_o \|_{L_\infty^b} \le \beta_o ,$$

so that the system can be treated via a single sphere condition. However due to the overly restrictive error bound β_o, the results are usually very conservative.

(iii) To obtain less conservative results, weighting factors $\dfrac{\beta_o}{\beta_{oi}}$, i = 1, ····, b, may be employed so that

$$\| \Sigma_o(y - y_o) \|_{L_\infty^b} \le \beta_o$$

where Σ_o is the $\mathbf{R}^{b \times b}$ nonsingular diagonal weighting matrix given by

$$\Sigma_o = \begin{bmatrix} \dfrac{\beta_o}{\beta_{o1}} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \dfrac{\beta_o}{\beta_{ob}} \end{bmatrix}.$$

2.3. Design Criteria

To develop controller criteria for servo-tracking in the sense of spheres, a combined equation comparison and the local form of the Banach fixed-point theorem are employed. First, the comparison of the actual and the nominal systems, (that is, the comparison of any uncertain combination (r, y, z, w) satisfying equation (II-12) and a completely known combination (r_o, y_o, z_o) satisfying equation (II-13)) leads to the error type operator equation

$$z - z_o = \Psi(N_\gamma z - N_o z_o) + \Psi B_o(r - r_o) + \Psi B_1 w \qquad \text{(II-14a)}$$

$$y - y_o = C_o(z - z_o) \qquad \text{(II-14b)}$$

The initial values of the actual and nominal systems are assumed to be identical.

Rewriting equation (II-14) by introducing the nonsingular weighting matrix W_o yields

$$W_o z = W_o \Psi(N_\gamma W_o^{-1} W_o z - N_o W_o^{-1} W_o z_o) + W_o \Psi B_o(r - r_o) + W_o \Psi B_1 w + W_o z_o$$

and $y - y_o = C_o W_o^{-1}(W_o z - W_o z_o)$

which can be written as

$$\bar{z} = \Phi \bar{z}$$

$$= W_o \Psi(N_\gamma W_o^{-1} \bar{z} - N_o W_o^{-1} \bar{z}_o) + W_o \Psi B_o(r-r_o) + W_o \Psi B_1 w + \bar{z}_o \qquad \text{(II-15a)}$$

and $y - y_o = C_o W_o^{-1}(\bar{z} - \bar{z}_o)$ (II-15b)

where the nonlinear map $\Phi : L_\infty^{2n}(T) \to L_\infty^{2n}(T)$,

$$\bar{z} = W_o z$$

and

$$\bar{z}_o = W_o z_o \ .$$

Equation (II-15) is a fixed point equation. In essence the discussion of the tracking associated with the original problem has now been reduced to a study of the solutions of this fixed point equation.

By applying the Banach contraction mapping theorem to equation (II-15), we obtain the following result which gives sufficient condition on design elements G, K, V_1, V_2, and the design function f_o that assure servo-tracking in the sense of input-output spheres. This theorem is central to much of what is discussed in the remaining chapters. Computation of operator norms involving the nonlinearities and uncertainties are important to fix the linear operator norm pivotal for the satisfaction of design criteria as well as effective evaluation of this norm. This norm depends quite naturally on the eigenstructure of a related linear operator.

Theorem 1 : Let f and f_o be continuous, and let G and K be assigned so that the eigenvalues of matrix R are in the open left-hand complex plane. Let (r_o, y_o, z_o) be a known combination satisfying equation (II-13) and (r, y, z, w) be any combination satisfying equation (II-12). Then for any input r in the specified sphere

$$\Omega\ (r : r_o, \beta_i) = \{ r \in L_\infty^m \mid \| r - r_o \| \le \beta_i \}$$

and for any external disturbances w in the specified sphere

$$\Omega\ (w : 0, \beta_w) = \{ z \in L_\infty^d \mid \| w \| \le \beta_w \}$$

there exists a unique combined response z in the specified β_o-neighbourhood

$$\Omega\ (z : z_o, \beta_o) = \{ z \in L_\infty^{2n} \mid \| W_o(z - z_o) \| \le \beta_o \}$$

provided
$$\eta \le \frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} \qquad \text{(II-16)}$$

where
$$\eta = \| W_o \Psi Q_o^{-1} \|$$

$$\rho_o = \| Q_o B_o \|$$

$$\rho_1 = \| Q_o B_1 \|$$

$$\rho_2 = \sup_{\gamma \in \Gamma} \| Q_o [N_\gamma z_o - N_o z_o] \|$$

$$\rho_3 = \sup_{\substack{z, z' \in \Omega(z : z_o, \beta_o) \\ z \neq z' \\ \gamma \in \Gamma}} \frac{\| Q_o [N_\gamma z - N_\gamma z'] \|}{\| W_o (z - z') \|}$$

with respect to a nonsingular constant matrix Q_o.

In order to prove this theorem we invoke the following.

Lemma 1 : Let $(x, \|\cdot\|)$ be a Banach space and let $\Omega_\beta(x_o)$ be a closed sphere in $\mathbf{X}$ with center at $x_o \in \mathbf{X}$ and constant $\beta > 0$ such that

$$\Omega_\beta(x_o) = \{ x \mid \| x - x_o \| \le \beta \} .$$

Let $\Phi : \mathbf{X} \to \mathbf{X}$ be a linear operator satisfying the following conditions : with constant κ, $0 < \kappa < 1$

(i) $\| \Phi (x) - \Phi(y) \| \le \kappa \| x - y \|, \quad \forall \; x, y \in \Omega_\beta(x_o)$

(ii) $\| \Phi (x_o) - x_o \| \le \beta (1 - \kappa)$

Then it follows that

(i) Φ maps $\Omega_\beta(x_o)$ back into itself

(ii) Φ has a unique fixed point $x^* \in \Omega_\beta(x_o)$ such that

$$\Phi (x^*) = x^*$$

Moreover,

(i) $x^* = \lim_{n \to \infty} x_n = \Phi^n(x_o)$

where $x_{n+1} = \Phi(x_n)$, $n = 0, 1, 2, \cdots$

and x_o is any element in $\Omega_\beta(x_o)$

(ii) $|| x^* - x_n || \leq \frac{\kappa^n}{1-\kappa} || \Phi(x_o) - x_o ||$

Proof of Lemma 1 : See Martin ([33], Chapter 4)

We follow the steps given below to prove the theorem.

(i) First, we establish the conditions needed for the map Φ to be a contraction.

(ii) Next, we use the fact that Φ must be a contraction in the output sphere $\Omega(z: z_o, \beta_o)$ and that the closed sphere

$$\Omega_1 = \{ \bar{z} \in L_\infty^{2n}(T) \mid || \bar{z} - \bar{z}_o|| \leq \frac{|| \Phi \bar{z}_o - \bar{z}_o ||}{1 - \kappa} \}$$

must be entirely contained within $\Omega(z: z_o, \beta_o)$ which establishes the existence of a unique solution $\bar{z}^*$ in the output sphere $\Omega(z: z_o, \beta_o)$.

Proof of Theorem 1 :

Consider two arbitrary points $\bar{z}, \bar{z}' \in \Omega(z : z_o, \beta_o)$ and compute $|| \Phi \bar{z} - \Phi \bar{z}' ||$, yielding

$$\Phi \bar{z} - \Phi \bar{z}' = W_o \Psi Q_o^{-1} Q_o (N_\gamma W_o^{-1} \bar{z} - N_\gamma W_o^{-1} \bar{z}') \qquad \text{(II-17)}$$

which on taking the L_∞- norm on both sides yields

$$|| \Phi \bar{z} - \Phi \bar{z}' || \le$$

$$||W_o \Psi Q_o^{-1}|| \sup_{\substack{\bar{z},\bar{z}' \epsilon \Omega(\bar{z}:\bar{z}_o,\beta_o) \\ \bar{z} \neq \bar{z}' \\ \gamma \epsilon \Gamma}} \frac{|| Q_o[N_\gamma W_o^{-1}\bar{z} - N_\gamma W_o^{-1}\bar{z}' ||}{|| \bar{z} - \bar{z}' ||} ||\bar{z} - \bar{z}'|| \tag{II-18}$$

where $\Omega(\bar{z} : \bar{z}_o, \beta_o) = \{ \bar{z} \epsilon L_\infty^{2n}[0, \infty) \mid || \bar{z} - \bar{z}_o || \le \beta_o \}$.

Remark : It is easy to show that if $z \epsilon L_\infty^{2n}$ that $\bar{z} \epsilon L_\infty^{2n}$.

For the operator $\Phi : L_\infty^{2n} \to L_\infty^{2n}$ to be a contraction map, it is required that

$$|| \Phi \bar{z} - \Phi \bar{z}' || \le \kappa || \bar{z} - \bar{z}' ||, \quad 0 < \kappa < 1$$

where

$$\kappa = ||W_o \Psi Q_o^{-1}|| \sup_{\substack{\bar{z},\bar{z}' \epsilon \Omega(\bar{z}:\bar{z}_o,\beta_o) \\ \bar{z} \neq \bar{z}' \\ \gamma \epsilon \Gamma}} \frac{|| Q_o[N_\gamma W_o^{-1}\bar{z} - N_\gamma W_o^{-1}\bar{z}' ||}{|| \bar{z} - \bar{z}' ||} < 1$$

which leads to the condition

$$\kappa = ||W_o \Psi Q_o^{-1}|| \sup_{\substack{z,z' \epsilon \Omega(z:z_o,\beta_o) \\ z \neq z' \\ \gamma \epsilon \Gamma}} \frac{|| Q_o[N_\gamma z - N_\gamma z']||}{|| W_o(z - z') ||} < 1 \quad . \tag{II-19}$$

Next, we consider

$$\Phi\bar{z}_o - \bar{z}_o = W_o \Psi Q_o^{-1} Q_o (N_\gamma W_o^{-1}\bar{z}_o - N_o W_o^{-1}\bar{z}_o) + W_o \Psi Q_o^{-1} Q_o B_o (r - r_o)$$

$$+ W_o \Psi Q_o^{-1} Q_o B_1 w \quad . \tag{II-20}$$

Taking the norm on both sides of equation (II-20) together with the triangular inequality gives

$$|| \Phi \bar{z}_o - \bar{z}_o || \le || W_o \Psi Q_o^{-1} || \; (\sup_{\gamma \epsilon \Gamma} ||Q_o(N_\gamma W_o^{-1} \bar{z}_o - N_o W_o^{-1} \bar{z}_o)||$$

$$+ || Q_o B_o || \; ||r - r_o|| + ||Q_o B_1|| \; ||w||)$$

which can be written as

$$|| \Phi \bar{z}_o - \bar{z}_o || \le || W_o \Psi Q_o^{-1} || \; (\sup_{\gamma \epsilon \Gamma} ||Q_o(N_\gamma z_o - N_o z_o)||$$

$$+ || Q_o B_o || \; ||r - r_o|| + ||Q_o B_1|| \; ||w||) \; . \qquad \text{(II-21)}$$

Since the output is required to be in the sphere $\Omega(z : z_o, \beta_o)$, we require

$$|| \Phi \bar{z}_o - \bar{z}_o || \le \beta_o(1 - \kappa) \qquad \text{(II-22)}$$

which follows from the Lemma 1. Therefore, from equations (II-19), (II-21) and (II-22) we obtain the inequality of the theorem given by

$$\eta \le \frac{\beta_o}{\rho_o \beta_i + \rho_1 \beta_w + \rho_2 + \rho_3 \beta_o}$$

with β_o, ρ_1, ρ_2, and ρ_3 defined as in the statement of the theorem. This completes the proof of Theorem 1. □

Remarks : Inequality (II-16) will be referred to as the primary design criterion for precision tracking in the sense of spheres. Some important design features of this criterion are

(i) Design elements G, K, V_1, and V_2 must be chosen so that the eigenvalues of the matrix $R \epsilon \mathbf{R}^{2n \times 2n}$ are at suitable locations in the open left-half complex plane and that the inequality (II-16) of the Theorem 1 is satisfied. The upper bound on the operator norm $|| W_o \Psi Q_o^{-1} ||$ depends on the design specifications such as tracking accuracy, the extent of the disturbances and the size of the uncertainties. Thus for precise tracking in the presence of large disturbances and large plant uncertainties, a small value of operator norm $\| W_o \Psi Q_o^{-1} \|$ is typically needed which might result in high gain feedback. It should be noted however, that the norm bound requirement is only a sufficient condition.

(ii) A larger upper bound for the linear operator norm $\| W_o \Psi Q_o^{-1} \|$ can be allowed by a proper choice of a nonlinear design function f_o. The norm values ρ_2, ρ_3 and η play a vital role in the design procedure. ρ_2 can be regarded as a measure of the maximal difference of operator N_γ and operator N_o at z_o, ρ_3 is a measure of the severity of the nonlinearity and the uncertainty of the system, and η is a measure of the "trackability" of the system. The function f_o should be assigned so that the values of ρ_2 and ρ_3 will allow a larger upper bound for the linear operator norm $\| W_o \Psi Q_o^{-1} \|$.

(iii) A Quantitative pole-placement is defined by (II-16). That is, a proper selection of eigenvalues of the matrix R will potentially enable one to satisfy the operator norm condition. To achieve an optimum design the eigenvalues must be placed such that the operator norm is as close as possible to the threshold value,

$$\frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o}.$$

(iv) The nominal plant defined in equation (II-2) is an essential part of the design criteria. That is, the nominal input r_o must be determined so that (r_o, y_o) satisfies the nominal equations.

2.4 A Special Case

Although it is theoretically possible to compute the norms ρ_2 and ρ_3 in the inequality (II-16) for virtually any nonlinearity, it is very useful to have special classes of nonlinearities which can be handled rather easily by using simple algorithms. To this end, a special type of nonlinearity, i.e. a sector bounded nonlinearity, associated with the system given by (II-1) is considered in this section. It is defined in precise mathematical terms as follows.

Definition 4 : If a nonlinear function $\psi(\cdot, \cdot)$ satisfies

(i) $\psi(0, t) = 0$ for all $t \in [0, \infty)$

(ii) $\alpha \leq \dfrac{\psi(\varphi_1, t) - \psi(\varphi_2, t)}{\varphi_1 - \varphi_2} \leq \beta$, for all $\varphi_1 \neq \varphi_2$ and for some $\alpha, \beta \in \mathbf{R}^1$.

then ψ is said to belong to the sector $[\alpha, \beta]$, or to be confined to the sector $[\alpha, \beta]$.

The above definition implies that the nonlinear function $\psi(\cdot, \cdot)$ lies between two straight lines having slopes α and β respectively and passes through the origin.

Now consider the system given by (II-1)

$$\dot{x}(t) = A\ x(t) + B\ u(t) + D\ w + f(x(t),\ \gamma,\ t) \tag{II-22a}$$

$$y(t) = C\ x(t) \tag{II-22b}$$

where $f(x, \gamma, t) : \mathbf{R}^n \times \mathbf{R}^a \times T \rightarrow \mathbf{R}^n$ satisfies Definition 4, and the other variables are the same as those defined in (II-1).

Consequently we can establish a nominal plant corresponding to the above system (II-22) of the form

$$\dot{x}(t) = A\ x(t) + B\ u(t) + f_o(x(t)) \tag{II-23a}$$

$$y(t) = C\ x(t) \tag{II-23b}$$

where $f_o(x) : \mathbf{R}^n \rightarrow \mathbf{R}^n$.

Due to the characteristics of the sector bounded nonlinearity, the nonlinear design function $f_o(x)$ can be chosen to be linear with slope equal to the arithmetic mean of the lower and upper bounds α, β on the nonlinearity. Thus,

$$f_o(x) = \frac{1}{2}\ (\alpha + \beta)\ x(t) \tag{II-24}$$

For the state feedback design scheme, the same type of observer given by (II-3) based on the nominal plant (II-23) is employed.

Following the same procedure outlined in section 2.3, we can obtain simple formulae avoiding difficult computations for evaluating each norm in the inequality (II-16). Namely, ρ_2 , the measure of the maximum difference of N_γ and N_o at z_o , is

$$\rho_2 = \sup_{\gamma\epsilon\Gamma} \| Q_o [N_\gamma z_o - N_o z_o] \|$$

$$= \frac{1}{2} (\alpha + \beta) \sup \| Q_o z_o \|$$

and the measure of the severity of the nonlinearity, ρ_3 , is

$$\rho_3 = \sup_{\substack{z,z'\epsilon\Omega(z:z_o,\beta_o) \\ z \neq z' \\ \gamma \epsilon \Gamma}} \frac{\| Q_o[N_\gamma z - N_\gamma z'] \|}{\| W_o(z - z') \|}$$

$$= \max \{ |\alpha|, |\beta| \} \tag{II-25}$$

for a proper choice of the weighting matrices W_o and Q_o.

Remark : Consider, for example, a system with the nonlinear function

$f(x, \gamma) = \begin{bmatrix} 0 \\ \gamma \sin x_1 \end{bmatrix}$, where $|\gamma| \leq 0.2$, and choose the weighting

matrices W_o and Q_o as

$$W_o = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_{max}} \end{bmatrix}, \text{ and } Q_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Consequently,

$$Q_o[N_\gamma x - N_\gamma x'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \gamma(\sin x_1 - \sin x_1') \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \gamma(\sin x_1 - \sin x_1') \end{bmatrix}$$

and $$W_o[x - x'] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_{max}} \end{bmatrix} \begin{bmatrix} x_1 - x_1' \\ x_2 - x_2' \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_1' \\ \frac{1}{\lambda_{max}} (x_2 - x_2') \end{bmatrix}$$

Since N_γ is only a function of the state x_1, ρ_3 is the maximum gradient of N_γ with respect to x_1, or $\rho_3 = |\gamma|$. However, if Q_o and W_o are chosen as

$$W_o = \begin{bmatrix} \frac{1}{\lambda_{max}} & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } Q_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then $Q_o[N_\gamma x - N_\gamma x'] = \begin{bmatrix} 0 \\ \gamma(\sin x_1 - \sin x_1') \end{bmatrix}$

and $\qquad W_o[x - x'] = \begin{bmatrix} \frac{1}{\lambda_{max}} (x_1 - x_1') \\ x_2 - x_2' \end{bmatrix}$

Hence, $\rho_3 = \lambda_{max}|\gamma|$ which is not the same as $\max\{|\alpha|, |\beta|\}$.

Finally, inequality (II-16) reduces to the following

$$\eta \le \frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} \tag{II-26}$$

where
$$\eta = ||\, W_o \Psi Q_o^{-1} \,||$$

$$\rho_o = ||\, Q_o B_o \,||$$

$$\rho_1 = ||\, Q_o B_1 \,||$$

$$\rho_2 = \frac{1}{2}(\alpha + \beta) \sup ||\, Q_o z_o \,||$$

$$\rho_3 = \max\{\, |\alpha| \,,\, |\beta| \,\}$$

Inequality (II-26) is the same as (II-16), however, we can eliminate involved computations for evaluating the norms ρ_2 and ρ_3 in this special case. A reasonably large class of physical problems can be brought into this form. A single DOF example is considered in Chapter VI to illustrate the use of this special form.

CHAPTER III

SERVO-TRACKING IN A LUR'E TYPE SYSTEM

In Chapter II, we considered the input-output tracking problem from an L_∞ point of view. Design criteria were established for this servoproblem that were primarily numerical in character. It is in general difficult to generate explicit closed form results and/or elegant geometric interpretations in the L_∞ - setting. The latter observation was motivation for the study of servotracking in a so called Lur'e type system.

At a general level we are concerned with the servotracking in uncertain nonlinear systems in a L_2 - setting. The often considered asymptotic tracking problem can be captured in such a formulation. In asymptotic tracking one typically considers the behaviour of the error vector as time gets infinitely large. No global error measures are employed. In our present formulation however we impose a global tracking error bound in addition to the asymptotic requirement, i.e., if $y_{oi}(t)$ is a desired nominal trajectory and $y_i(t)$ is the actual trajectory, then we require

$$\text{(i)} \quad | \, y_i(t) - y_{oi}(t) \, | \to 0 \quad \text{as } t \to \infty, \quad \forall \quad i = 1, \ldots, n$$

$$\text{and} \quad \text{(ii)} \quad || \, y - y_o \, ||_{L_2} \leq \beta_o$$

The general servomechanism problem has been addressed by Desoer and Wang [14], Solomon and Davison [46], and Barnard et.al. [5].

Except in [5] the notion of conventional asymptotic tracking was treated . In [5] the notion of tracking employed is similar to our present work.

The class of systems whose forward loop has a linear, time-invariant subsystem and whose feedback loop contains a memoryless time varying nonlinearity is what is typically known as a Lur'e type system. This configuration, though simple, includes a fairly large class of important feedback systems and has been studied quite extensively from an absolute stability view-point [36, 41, 42, 50, 54, 55]. A classic example of a system in this class would be a set of coupled nonlinear oscillators where the restoring force is nonlinear. In what follows we formulate the servotracking problem for this class and give a methodology for the direct design for tracking specifications. We also give a geometric interpretation of the design criteria in the case of a SISO system where the nonlinearity is assumed to be sector bounded. This interpretation is given in the frequency domain and is similar to the Nyquist stability criterion and the circle criterion for the absolute stability problem.

3.1 Statement of Problem

We consider the system shown in Figure 4 given by the state equations

$$\dot{x}(t) = A\, x(t) - B\, \psi(y(t),t) + B\, u(t) \qquad \text{(III-1a)}$$

and the output equations

$$y(t) = C\, x(t) \qquad \text{(III-1a)}$$

where the state $x(t) \in \mathbf{R}^n$, the control $u(t) \in \mathbf{R}^m$, $t \in T = [0, \infty)$, the output $y(t) \in \mathbf{R}^b$, and the nonlinear function $\psi(\cdot, \cdot) \in \mathbf{R}^b \times T \to \mathbf{R}^m$ is

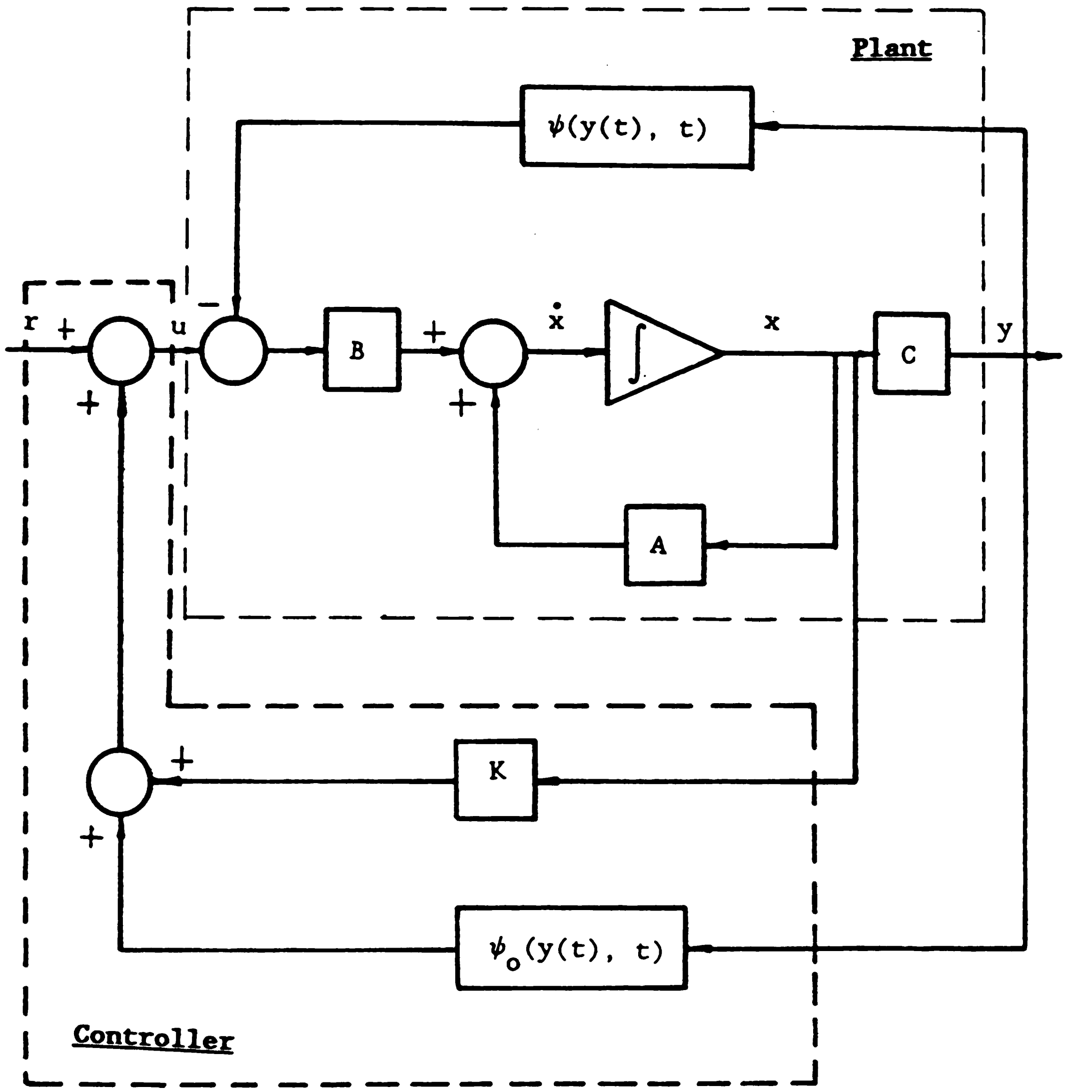


Figure 4 Configuration of control system

continuous in both its arguments. A, B, and C are constant matrices of order n x n, n x m, and b x n respectively.

Remark : Note that the above system description incorporates a linear, time-invariant subsystem and a nonlinear, time varying element in a feedback path.

We make the following assumptions regarding the system.

(i) The pair {A, B} is completely controllable

(ii) The pair {A, C} is completely observable

(iii) The nonlinearity $\psi(\cdot,\cdot)$ satisfies the memoryless condition

$$\psi(0, t) = 0 \in \mathbf{R}^m \quad \forall \ t \in [0, \infty)$$

and the generalized sector bound condition

$$||[\psi(\varphi_1, t) - \nu\varphi_1] - [\psi(\varphi_2, t) - \nu\varphi_2]|| \le \xi(\nu) \ ||\varphi_1 - \varphi_2|| \tag{III-2}$$

$\nu \in \mathbf{R}^1$, $\forall$ $t \in [0, \infty)$ and $\forall$ $\varphi_1, \varphi_2 \in \mathbf{R}^b$, where $\xi(\nu) = \{ \ |\beta - \nu|, \ |\nu - \alpha| \ \}$ for real numbers α, β satisfying $\beta \ge \alpha$ and $\beta > 0$.

Control Objective : The primary design objective here is to synthesize a controller that tracks a specified nominal output $y_o \in L_2^b \ [0, \infty)$ within a pre-specified tolerance β_o with respect to the L_2 - norm i.e., we require

$$|| \ y - y_o \ ||_{L_2} \le \ \beta_o$$

despite the uncertainties in the system, nonlinearity and the reference input. The input uncertainties are assumed to be of the form

$$\| r - r_o \|_{L_2} \leq \beta_i ,$$

and may be viewed as a disturbance with finite energy. We say it is of finite energy primarily because the measure employed is the L_2 - norm.

3.2 Problem Formulation

Control : We consider the state feedback control law

$$u(t) = r(t) + K\, x(t) + \psi_o(y(t),\, t) \tag{III-3}$$

as a means of satisfying the design specifications, where $u(t) \in \mathbf{R}^m$ is the control, $r(t) \in \mathbf{R}^m$ is the reference input, $\psi_o(y(t),\, t) \in \mathbf{R}^m$ is a nonlinear design function in the class of generalized sector bound functions and a constant feedback gain matrix K of order m x n . Combining the system equations (III-1) and (III-3) yields

$$\dot{x}(t) = (A + B\, K)x(t) - B\psi(y(t),t) + Br(t) + B\psi_o(y(t),t) \tag{III-4a}$$

$$y(t) = C\, x(t) \tag{III-4b}$$

which can be written as

$$\dot{x} = \bar{A}\, x - B\, \psi(y,t) + B\, r + B\, \psi_o(y,t) \tag{III-5a}$$

$$y = C\, x \tag{III-5b}$$

where $\bar{A} = [A + B\, K] \in \mathbf{R}^{n \times n}$.

Next, we formally transform equation (III-5) into the integral form following the procedure of Chapter II given by

$$x = \int_0^t e^{\bar{A}(t-\tau)}[-B\psi(y(\tau),\, \tau) + B\psi_o(y(\tau),\tau) + Br(\tau)]\, d\tau + q(t) \tag{III-6a}$$

with the output equation

$$y = C\, x \tag{III-6b}$$

where $q(t) = e^{\bar{A}t}\ x(0)$, and $x(0)$ is the initial state. Since the function $\psi(y, t)$ and $\psi_o(y, t)$ are continuous, equation (III-6a) can be written in the standard operator form

$$x = -\Psi BN_\gamma y + \Psi BN_o y + \Psi Br + q \qquad \text{(III-7)}$$

where Ψ is a linear operator and N_γ, N_o are nonlinear operators mapping $L_2^n[0, \infty)$ back into itself given by

$$(\Psi\ x)(t) = \int_0^t e^{\bar{A}(t-\tau)}\ x(\tau)\ d\tau$$

$$(N_\gamma y)(t) = \psi(y(t), t)$$

$$(N_o y)(t) = \psi_o(y(t), t)$$

To show that N_γ, N_o : $L_2^m[0, \infty) \rightarrow L_2^m[0, \infty)$ it suffices to consider

$$||N_\gamma y_1 - \nu y_1||_2 = \int_0^\infty |N_\gamma y_1 - \nu y_1|^2\ dt$$

$$= \int_0^\infty |\psi y_1 - \nu y_1|^2\ dt$$

From the sector bound condition it follows that

$$\int_0^\infty |\psi y_1 - \nu y_1|^2\ dt \le \max\ \{|\beta - \nu|,\ |\nu - \alpha|\}\ ||y_1||^2$$

Hence $(\psi y_1 - \nu y_1) \in L_2^m$ since $y_1 \in L_2^m[0, \infty)$. Now it follows that $\psi y_1 \in L_2^m[0, \infty)$ since $L_2^m[0, \infty)$ is a linear space and $\nu y_1 \in L_2^m[0, \infty)$. Thus

$$N_\gamma : L_2^m[0, \infty) \rightarrow L_2^m[0, \infty).$$

On premultiplying (III-7) by the matrix C, we obtain

$$C\ x = -C\Psi BN_\gamma y + C\Psi BN_o y + C\Psi Br + Cq \qquad \text{(III-8)}$$

which can be written as

$$y = -LN_{\gamma}y + LN_{o}y + Lr + q_1 \qquad \text{(III-9)}$$

where $L = C\Psi B$

$q_1 = Cq$.

The actual servosystem is now characterized by the input-output representation (III-9). In order to estimate the tracking error we now compare this system with a hypothetical reference model characterizing the nominal system corresponding to (III-9). This nominal system is considered to be given by

$$y_o = Lr_o + q_1 \qquad \text{(III-10)}$$

where (y_o, r_o) is a completely known combination of the prespecified nominal output y_o, and the corresponding solution r_o serving as a nominal command input.

Remark : In the above we have assumed the same initial conditions for the actual system and the nominal system . If they are not the same it can be accounted for by adjusting the radius of the output sphere. For a treatment of this notion see Jayasuriya [25].

3.3 Main Result

To establish design criteria under which servo-tracking in the sense of sphere is guaranteed, comparison of equations and the Banach fixed point theorem are employed. By comparison of any combination (r, y) satisfying equation (III-9) and known combination (r_o, y_o) satisfying equation (III-10), we obtain the difference

$$y - y_o = -LN_{\gamma}y + LN_{o}y + L(r - r_o) \qquad \text{(III-11)}$$

which governs the tracking error.

Now we rewrite equation (III-11) in the Hammerstein form

$$
\begin{aligned}
y &= -LN_\gamma y + LN_o y + L(r - r_o) + y_o \\
&= -LN_\gamma y + r_1 \qquad\qquad \text{(III-12)}
\end{aligned}
$$

where $r_1 = LN_o y + L(r - r_o) + y_o$.

Invoking the local form of the Banach Contraction mapping theorem with the fixed point equation (III-12) obtained via equation comparison, forms the basis of the following theorem for servotracking in the L_2 - sense. For the rest of this chapter $||\cdot||$ denotes the L_2-norm.

Theorem 1 : Let the eigenvalues of the matrix $\bar{A}$ be in the open left half complex plane. Let $r \in L_2^m[0,\infty)$ and $y \in L_2^b[0,\infty)$ be any combination satisfying (III-9) and $r_o \in L_2^m[0, \infty)$, $y_o \in L_2^b[0, \infty)$ be a known combination satisfying (III-10). Then for any input r(t) in the specified input sphere

$$\Omega(r: r_o, \beta_i) = \{ r \in L_2^m[0, \infty) \mid || r - r_o || \leq \beta_i \}$$

and for any nonlinearity in the sector $[\alpha, \beta]$, there exists a unique response $y(t) \in L_2^b[0, \infty)$ in the specified output sphere

$$\Omega(y: y_o, \beta_o) = \{ y \in L_2^b[0, \infty) \mid || W_o(y - y_o) || \leq \beta_o \}$$

provided $$\eta \leq \frac{\rho_o \beta_o}{\beta_i + \rho_1 + (\rho_2 + \rho_3)\beta_o} \qquad \text{(III-13)}$$

where $$\eta = ||W_o L Q_o^{-1}||$$

$$\rho_o = || Q_o ||^{-1}$$

$$\rho_1 = \rho_o \sup_{\gamma \in \Gamma} || Q_o (N_\gamma y_o - N_o y_o) ||$$

$$\rho_2 = \rho_o \sup_{\substack{y, y' \in \Omega(y : y_o, \beta_o) \\ y \neq y' \\ \gamma \in \Gamma}} \frac{|| Q_o (N_\gamma y - N_\gamma y') ||}{|| W_o (y - y') ||}$$

$$\rho_3 = \rho_o \sup_{\substack{y, y' \in \Omega(y : y_o, \beta_o) \\ y \neq y'}} \frac{|| Q_o (N_o y - N_o y') ||}{|| W_o (y - y') ||}$$

Proof of this theorem follows the same reasoning given for the main theorem of Chapter II.

Proof of Theorem 1 : Consider the operator equation (III-12) which can be written as the fixed point equation

$$\bar{y} = \Phi \bar{y} \tag{III-14}$$

$$= -W_o L Q_o^{-1} Q_o N_\gamma W_o^{-1} \bar{y} + W_o L Q_o^{-1} Q_o N_o W_o^{-1} \bar{y} + W_o L Q_o^{-1} Q_o (r - r_o) + \bar{y}_o$$

where $\bar{y} = W_o y$, $\bar{y}_o = W_o y_o$, and W_o, Q_o are nonsingular weighting matrices.

Consider two arbitrary points $\bar{y}$, $\bar{y}' \in \Omega(y : y_o, \beta_o)$ and compute

$$\Phi\bar{y} - \Phi\bar{y}' = -(W_o L Q_o^{-1}) (Q_o [N_\gamma W_o^{-1} \bar{y} - N_\gamma W_o^{-1} \bar{y}'])$$

$$+ (W_o L Q_o^{-1}) (Q_o [N_o W_o^{-1} \bar{y} - N_o W_o^{-1} \bar{y}']) \tag{III-15}$$

which on taking the L_2 - norm on both sides yields

$$||\Phi\bar{y} - \Phi\bar{y}'|| \leq ||W_o L Q_o^{-1}|| \Big\{ \sup_{\substack{\bar{y},\bar{y}' \epsilon \Omega(y:y_o,\beta_o) \\ \bar{y} \neq \bar{y}' \\ \gamma \epsilon \Gamma}} \frac{||Q_o(N_\gamma W_o^{-1}\bar{y} - N_\gamma W_o^{-1}\bar{y}')||}{|| \bar{y} - \bar{y}' ||}$$

$$+ \sup_{\substack{\bar{y},\bar{y}' \epsilon \Omega(y:y_o,\beta_o) \\ \bar{y} \neq \bar{y}'}} \frac{||Q_o(N_o W_o^{-1}\bar{y} - N_o W_o^{-1}\bar{y}')||}{|| \bar{y} - \bar{y}' ||} \Big\} ||\bar{y} - \bar{y}'|| \qquad \text{(III-16)}$$

Thus, for Φ to be a contraction we require that

$$|| \Phi\bar{y} - \Phi\bar{y}' || \leq \kappa || \bar{y} - \bar{y}' ||, \qquad 0 < \kappa < 1 \qquad \text{(III-17)}$$

Consequently from (III-16) and (III-17) we get

$$\kappa = ||W_o L Q_o^{-1}|| \Big\{ \sup_{\substack{y,y' \epsilon \Omega(y:y_o,\beta_o) \\ y \neq y' \\ \gamma \epsilon \Gamma}} \frac{|| Q_o (N_\gamma y - N_\gamma y') ||}{|| W_o(y - y') ||}$$

$$+ \sup_{\substack{y,y' \epsilon \Omega(y:y_o,\beta_o) \\ y \neq y'}} \frac{|| Q_o (N_o y - N_o y') ||}{|| W_o(y - y') ||} \Big\} < 1 \qquad \text{(III-18)}$$

Next, we consider

$$\Phi\bar{y}_o - \bar{y}_o = - (W_o L Q_o^{-1}) \{ Q_o(N_\gamma W_o^{-1}\bar{y}_o - N_o W_o^{-1}\bar{y}_o) + Q_o(r - r_o) \}$$

and take the L_2 - norm on both sides along with the triangular inequality to get

$$||\Phi\bar{y}_o - \bar{y}_o|| \leq ||W_o L Q_o^{-1}|| \{ \sup_{\gamma \epsilon \Gamma} ||Q_o(N_\gamma W_o^{-1}\bar{y}_o - N_o W_o^{-1}\bar{y}_o) ||$$

$$+ ||Q_o || \, || r - r_o || \} \qquad \text{(III-19)}$$

Since the output is required to be in the sphere $\Omega(y : y_o, \beta_o)$ we require

$$|| \Phi\bar{y}_o - \bar{y}_o || \leq \beta_o (1 - \kappa) \qquad \text{(III-20)}$$

Finally, from (III-18), (III-19), and (III-20), we obtain the inequality (III-13)

$$|| W_o L Q_o^{-1} || \le \frac{\rho_o \beta_o}{\beta_i + \rho_1 + (\rho_2 + \rho_3)\beta_o}$$

with ρ_o, ρ_1, ρ_2, and ρ_3 as given in the statement of the theorem. This completes the proof of Theorem 1. □

3.4 A Special Case

The above result when specialized to the SISO case admits a very elegant geometric interpretation. This geometric interpretation is brought about by minimizing the contraction coefficient that results from the application of the fixed-point theorem. This minimization in a sense leads to the least conservative design obtainable via the operator methods employed here.

In what follows we will first establish conditions for the minimum contraction coefficient of a Hammerstein type equation with the general nonlinearity considered above. Then, we will interpret this minimum contraction coefficient in terms of the frequency response of the linear time-invariant portion of the overall system. Finally, we will combine this with the SISO version of the main result given earlier to yield the geometric interpretation.

First write the general Hammerstein equation given by (III-12)

$$y = - LN_\gamma y + r_1$$

where L is a linear operator mapping a Banach space Y into itself and r_1 is a term accounting for both independent energy sources and initial conditions . We assume the nonlinearity to be of the wider class

$N_\gamma : Y \to Y$ such that for any real constant ν and for all $y_1, y_2 \in Y$

$$||N_\gamma y_1 - \nu y_1 - (N_\gamma y_2 - \nu y_2)|| \le \xi(\nu) \; ||y_1 - y_2|| \qquad \text{(III-21)}$$

where $\xi(\nu) = \max \{ |\beta - \nu|, |\nu - \alpha| \}$ for real numbers α , β satisfying $\beta \ge \alpha$ and $\beta > 0$ as defined earlier. Sector bounded nonlinearities clearly belong to the class (III-21). This can be shown by considering

$$\begin{aligned} \psi &= N_\gamma y \\ &= \psi(y(t), t) : L_2[0, \infty) \to L_2[0, \infty) \end{aligned}$$

satisfying the following conditions

(i) $\psi(y(t), t)$ is measurable when $y(t)$ is

(ii) $\psi(0, t) = 0$ for all $t \in [0, \infty)$

(iii) For two real numbers α and β

$$\alpha \le \frac{\psi\,(\varphi_1(t), t) - \psi(\varphi_2(t), t)}{\varphi_1(t) - \varphi_2(t)} \le \beta$$

where $\varphi_1 \neq \varphi_2$, and $t \in T = [0, \infty)$.

Condition (iii) implies that the nonlinearity is confined to a sector $[\alpha, \beta]$ whose lower and upper bound are α and β respectively. At any fixed $t \in [0, \infty)$, the function $\psi(y(t), t) - \nu\, y : \mathbf{R}^b \times T \to \mathbf{R}^m$ has a slope with magnitude not exceeding $\max \{ | \beta - \nu|, |\nu - \alpha| \}$. Hence, with $y_1, y_2 \in L_2[0, \infty)$

$$\begin{aligned} ||N_\gamma y_1 - \nu y_1 - (N_\gamma y_2 - \nu y_2)||^2 &= \int_0^\infty |\psi(y_1, t) - \nu y_1 - (\psi(y_2, t) - \nu y_2)|^2 \, dt \\ &\le (\max\{|\beta-\nu|, |\nu-\alpha|\})^2 \int_0^\infty |y_1 - y_2|^2 \, dt \\ &= (\max\{|\beta-\nu|, |\nu-\alpha|\})^2 \; ||y_1 - y_2||^2 \end{aligned}$$

so that (III-21) is satisfied.

Next, we assume that the linear operator L can be defined by the convolution

$$\begin{aligned} y(t) &= \int_0^{\infty} h(t - \tau)\, u(\tau)\, d\tau \\ &= L\, u \end{aligned}$$

where $h \in L_1[0, \infty)$ and $L : L_2[0, \infty) \to L_2[0, \infty)$.

Remark : This is obviously true if the linear time-invariant subsystem of the forward path is asymptotically stable, which is equivalent to requiring that $\bar{A}$ be a Hurwitz matrix. The latter condition be clearly met since $\bar{A} = A + BK$ where $\{A, B\}$ is controllable.

Moreover, the Fourier transform of h, $\tilde{h}(j\omega)$, is assumed to satisfy

$$[1 + \frac{1}{2}(\alpha + \beta)\, \tilde{h}(j\omega)] \neq 0 \quad , \quad \forall\, \omega \in \mathbf{R}^1$$

In Theorem 2 we develop the minimum contraction constant for the Hammerstein equation.

Theorem 2 : Let the eigenvalues of the matrix $\bar{A}$ be in the left half complex plane, then the minimum contraction constant of the equation (III-12) for the SISO case is

$$\frac{1}{2}(\beta - \alpha) \sup_{\omega \in \mathbf{R}^1} \left| [\, 1 + \frac{1}{2}(\alpha + \beta)\, \tilde{h}(j\omega)]^{-1}\, \tilde{h}(j\omega) \right| < 1 \qquad \text{(III-22)}$$

It is worth noting that $(\beta - \alpha)$ is a measure of the deviation of the nonlinearity from linearity. When $\alpha = \beta$, the contraction constant becomes zero and the Nyquist stability criterion is recovered.

To prove Theorem 2, the following lemmas are needed.

Lemma 1 : Let ν be a real number such that $(I + \nu L)^{-1}$ exists and suppose that

$$\kappa(\nu) = \|(I + \nu L)^{-1} L\| \; \xi(\nu) < 1 .$$

Then for any $r_1 \in Y$, there exists a unique $y \in Y$ satisfying (III-12). Note that $\kappa(\nu)$ is the contraction coefficient.

Proof of Lemma 1 : Before applying the contraction mapping theorem to (III-12), it is modified to another mapping whose fixed point is also the fixed point of the original mapping. This modification is used to facilitate the idea of minimizing the contraction coefficient.

Let the operator $\tilde{N}_\gamma$ be defined by

$$N_\gamma y = \nu y + \tilde{N}_\gamma y . \qquad \text{(III-23)}$$

Combining (III-12) and (III-23) yields

$$y = -L (\nu y + \tilde{N}_\gamma y) + r_1$$

which can be written as

$$(I + \nu L) y = -L\tilde{N}_\gamma y + r_1 . \qquad \text{(III-24)}$$

Since ν can be chosen so that $(I + \nu L)^{-1}$ exists, (III-24) becomes

$$y = -(I + \nu L)^{-1} L\tilde{N}_\gamma y + (I + \nu L)^{-1} r_1 \qquad \text{(III-25)}$$

$$= \phi\, y$$

whose solution is clearly the same as that of (III-12).

Now consider

$$\phi y_1 - \phi y_2 = -(I + \nu L)^{-1} L\tilde{N}_\gamma (y_1 - y_2) \qquad \text{(III-26)}$$

where y_1, $y_2 \in Y$.

Taking the L_2 - norm on both sides of (III-26) yields

$$\begin{aligned} ||\phi y_1 - \phi y_2|| &= ||(I + \nu L)^{-1} L(\tilde{N}_\gamma y_1 - \tilde{N}_\gamma y_2) \,|| \\ &= ||(I + \nu L)^{-1} L(N_\gamma y_1 - N_\gamma y_2 - \nu(y_1 - y_2)|| \\ &\le ||(I + \nu L)^{-1} L \left(\frac{N_\gamma y_1 - N_\gamma y_2}{y_1 - y_2} - \nu\right)||\ ||y_1 - y_2|| \\ &\le ||(I + \nu L)^{-1} L||\ \xi(\nu)\ ||y_1 - y_2|| \,. \qquad \text{(III-27)} \end{aligned}$$

If the coefficient $\kappa(\nu) = ||\ (I + \nu L)^{-1} L\ ||\ \xi(\nu) < 1$ in (III-27), then the operator ϕ is a contraction. Thus there exists a unique $y \in Y$ satisfying (III-12) and (III-25), according to the contraction mapping theorem. □

Lemma 2 : Let S be the set of real ν such that $(I + \nu L)^{-1}$ exists, i.e.,

$$S = \{\ \nu \mid \nu \in R,\ (I + \nu L)^{-1} \text{ exists }\}$$

and if there is a real number $\nu \in S$ such that $\kappa(\nu) < 1$, then

$$\kappa(\nu_o) = \inf \kappa(\nu)$$

i.e., $\kappa(\nu_o) \le \kappa(\nu)$

where $\nu_o = \frac{1}{2}(\alpha + \beta)$.

Proof of Lemma 2 : We first show that $(I + \nu_o L)^{-1}$ exists. If $L = 0$, the inverse obviously exists. If $L \neq 0$ and $\nu \in S$, then $||(I + \nu L)^{-1}L|| \neq 0$.

$$I + \nu_o L = I + \nu L + (\nu_o - \nu)L$$

$$= (I + \nu L)\ [\ I + (\nu_o - \nu)(I + \nu L)^{-1}L]\ . \qquad \text{(III-28)}$$

In equation (III-28), $(I+ \nu_o L)^{-1}$ exists if $|\nu_o - \nu|\ ||(I + \nu L)^{-1}L|| <1$. This condition is satisfied because $|\nu_o - \nu| \leq \xi(\nu)$ (See Figure 5(a)) and $||(I + \nu L)^{-1}L||\ \xi(\nu) < 1$. Since $\xi(\nu_o) = \xi(\nu) - |\nu_o - \nu|$ as shown in Figure 5(b).

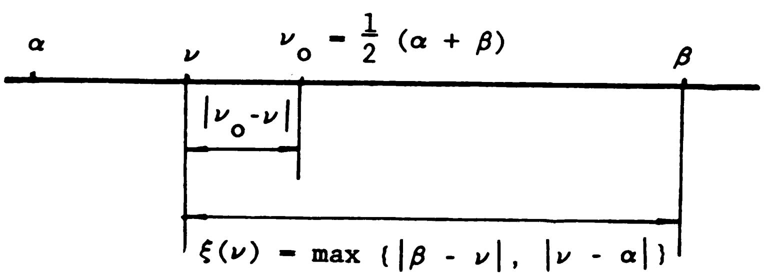


Figure 5(a) Illustration of α, β, ν, and ν_o

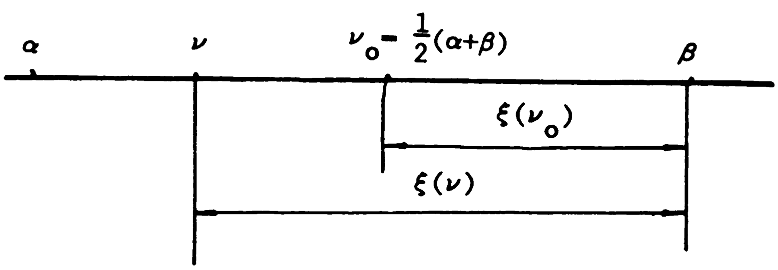


Figure 5(b) Illustration of α, β, $\xi(\nu_o)$, and $\xi(\nu)$

Now consider

$$||(I+\nu_o L)^{-1}L||\ \xi(\nu_o)$$

$$= ||(I+\nu_o L)^{-1}L||\xi(\nu) - |\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L||$$

$$= ||(I+\nu L)^{-1}L+(I+\nu_o L)^{-1}L-(I+\nu L)L||\xi(\nu) - |\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L||$$

$$\leq ||(I+\nu L)^{-1}L||\xi(\nu) + ||(I+\nu_o L)^{-1}L-(I+\nu L)^{-1}L||\xi(\nu)$$

$$-|\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L|| \ . \qquad \text{(III-29)}$$

Here, $(I+\nu_o L)^{-1} - (I+\nu L)^{-1} = (I+\nu_o L)^{-1}[(I+\nu L)-(I+\nu_o L)](I+\nu L)^{-1}$

$$= (I+\nu_o L)^{-1}(\nu L-\nu_o L)(I+\nu L)^{-1}$$

$$= (\nu-\nu_o)\ (I+\nu_o L)^{-1}L(I + \nu L)^{-1}. \qquad \text{(III-30)}$$

From (III-29) and (III-30), we obtain

$$||(I+\nu_o L)^{-1}L-(I+\nu L)^{-1}L||\xi(\nu) = ||((I+\nu_o L)^{-1}-(I+\nu L)^{-1})L||\xi(\nu)$$

$$= |\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L(I+\nu L)^{-1}L||\xi(\nu)$$

$$\leq |\nu-\nu_o|\ ||(I+\nu_o L^{-1}L||\ ||(I+\nu L)^{-1}L||\xi(\nu)$$

$$= |\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L||\ \kappa(\nu) \qquad \text{(III-31)}$$

Finally, substituting (III-31) into (III-29) yields

$$\kappa(\nu_o) = ||(I + \nu_o L)^{-1}L||\ \xi(\nu_o)$$

$$\leq \kappa(\nu) - (I-\kappa(\nu))\ |\nu-\nu_o|\ ||(I+\nu_o L)^{-1}L\ ||$$

Since $\kappa(\nu) < 1$, $\kappa(\nu_o) \leq \kappa(\nu)$. This completes the proof of Lemma 2. □

Hence, the minimum contraction coefficient is obtained when the real number ν is the arithmetic mean of the maximum and minimum slopes of the nonlinearity $\psi(y(t),t)$.

Lemma 3 : The L_2 - induced norm of the linear operator is given by

$$||L||_2 = \sup_{\omega \in \mathbf{R}^1} |\tilde{h}(j\omega)|$$

Proof of Lemma 3 : Let $u \in L_2[0, \infty)$ and $y = L u$, then with Fourier transform of u, $\tilde{u}(j\omega)$,

$$||y||^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{h}(j\omega)|^2 \, |\tilde{u}(j\omega)|^2 \, d\omega \quad \text{(from Parseval's theorem)}$$

$$\leq \sup_{\omega \in \mathbf{R}^1} |h(j\omega)|^2 \, ||u||^2$$

which shows that

$$||L|| \leq \sup_{\omega \in \mathbf{R}^1} |\tilde{h}(j\omega)| \ . \tag{III-32}$$

Since $\lim_{\omega \to \pm\infty} \tilde{h}(j\omega) = 0$ (Riemann-Lesbegue Lemma), and from the continuity of $|\tilde{h}(j\omega)|$, there exists an ω_o such that

$$|\tilde{h}(j\omega_o)| = \sup_{\omega \in \mathbf{R}^1} |\tilde{h}(j\omega)| \ .$$

Now consider a sequence of functions

$$u_n(t) = \sqrt{\pi n} \;\; 2 \cos(\omega_o t) \sin(\frac{1}{2n} \; t) \; / \; (\pi t)$$

whose Fourier transforms are

$$F_n(j\omega) = \Bigg\lceil \sqrt{\pi n} \qquad \text{for } \omega \in [\ \omega_o - \frac{1}{2n} \ , \ \omega_o + \frac{1}{2n}\ ]$$

$$\begin{cases} & \text{and} \quad \text{for } \omega \in [-\omega_o - \frac{1}{2n}, -\omega_o + \frac{1}{2n}] \\ 0 & \text{otherwise,} \end{cases}$$

and $||u_n|| = 1$.

Then for $y_n = L\, u_n$

$$| \, |\tilde{h}(j\omega_o)|^2 - ||y_n||^2 \, | = | \int_{\omega_o - \frac{1}{2n}}^{\omega_o + \frac{1}{2n}} n \left(|\tilde{h}(j\omega_o)|^2 - |h(j\omega)|^2 \right) d\omega \, |$$

$$\leq \max_{\omega \in [\omega_o - \frac{1}{2n}, \, \omega_o + \frac{1}{2n}]} | \, |h(j\omega_o)|^2 - |h(j\omega)|^2 \, | \qquad \text{(III-33)}$$

In equation (III-33), $||y_n||$ can be made arbitrarily close to $|\tilde{h}(j\omega_o)|$ by choosing n large enough. Since $||y|| \leq ||L||$, the inequality (III-32) can be replaced by equality. □

Proof of Theorem 2 : From Lemmas (1), (2) and (3) it follows quite easily that

$$\kappa(\nu) = \frac{1}{2} (\beta - \alpha) \sup_{\omega \in \mathbf{R}^1} [\, 1 + \frac{1}{2} (\alpha + \beta) \, \tilde{h}(j\omega) \,]^{-1} \, \tilde{h}(j\omega) < 1$$

Having established the minimum contraction coefficient we now invoke our main result to develop a theorem for the SISO case leading to a geometric interpretation.

Theorem 3 : Let K be assigned so that the eigenvalues of the matrix $\bar{A}$ are in the open left half complex plane. Let h(s) be the frequency response transfer function of the linear, time invariant portion of the forward path. Let $(r,y) \in L_2[0,\infty)$ be any combination satisfying equation (III-9) and $(r_o,y_o) \in L_2[0, \infty)$ be a known combination satisfying (III-10). Then , for any input r(t) in the specified input sphere

$$\Omega\ (r: r_o, \beta_i) = \{\ r \in L_2[0, \infty) \mid \ \|r - r_o\|_2 \le \beta_i\ \}$$

and for any nonlinearity in the sector $[\alpha, \beta]$, there exists a unique response $y(t) \in L_2[0, \infty)$ in the specified β_o-neighbourhood such that

$$\Omega\ (y: y_o, \beta_o) = \{\ y \in L_2[0, \infty) \mid \ \|y - y_o\|_2 \le \beta_o\ \}$$

if

$$\sup_{\omega \in \mathbf{R}^1} |h(j\omega)|\ \delta \le 1 - \sup_{\omega \in \mathbf{R}^1} \left| \frac{h(j\omega)}{1 + \frac{1}{2}\ (\alpha + \beta)\ h(j\omega)} \right| \frac{1}{2}(\beta - \alpha) \qquad \text{(III-34)}$$

where $\delta = \dfrac{\rho_o + \beta_i}{\beta_o}$

$$\rho_o = \sup_{\gamma \in \Gamma} \|\ N_\gamma y_o - N_o y_o\ \|_2$$

$$\beta_i = \|\ r - r_o\ \|_2$$

<u>Proof of Theorem 3</u> : Consider the operator equation (III-12) which can be written as

$$y = -LN_\gamma y + L\ N_o y + L(\ r - r_o\ ) + y_o$$

$$= \Phi y \qquad \text{(III-35)}$$

From Theorem 1, the minimum contraction constant κ of (III-35) is

$$\kappa = \sup_{\omega\ \in\ \mathbf{R}^1} \left| \frac{h(j\omega)}{1 + \frac{1}{2}\ (\alpha + \beta)\ h(j\omega)} \right| \frac{1}{2}\ (\beta - \alpha)\ \ < \ \ 1 \qquad \text{(III-36)}$$

Now from equation (III-35)

$$\Phi y_o - y_o = -(LN_\gamma y_o - LN_o y_o) + L(r - r_o) \qquad \text{(III-37)}$$

Taking the norm on both sides of equation (III-37) gives

$$||\Phi y_o - y_o|| \le ||L|| \; [\sup_{\omega \, \epsilon \, \mathbf{R}^1} ||N_\gamma y_o - N_o y_o|| + ||r - r_o||] \qquad \text{(III-38)}$$

Since output y(t) is required to be within the β_o - sphere , from Lemma 1 of Chapter II, we require

$$||\Phi y_o - y_o \,|| \le \beta_o \, (1 - \kappa) \qquad \text{(III-39)}$$

Now, by using equations (III-38) and (III-39), inequality (III-34) is obtained, thus establishing Theorem 3. □

Geometric Interpretation :

The design criteria advanced in Theorem 3 have an interpretation similar to the circle criterion known for absolute stability . First of all note that the constant ρ_o is a measure of the deviation of the nonlinearity from linearity. Thus the proper selection of the nonlinear design function ψ_o which eliminates or averages out as much of the nonlinearity will allow a larger upper bound for sup $|h(j\omega)|$ in (III-34). Since the nonlinearity is confined to a sector $[\alpha, \beta]$, a favourable design function ψ_o is simply the arithmetic mean of the upper and the lower bounds of the nonlinearity $\psi(y(t), t)$.

Next,the minimum contraction coefficient κ

$$\kappa = \sup_{\omega \, \epsilon \, \mathbf{R}^1} \left| \frac{h(j\omega)}{1 + \frac{1}{2} (\alpha + \beta) \, h(j\omega)} \right| \frac{1}{2} (\beta - \alpha) \; < \; 1$$

is evaluated similar to the M - circle concept known in the design of classical compensators.

First write

$$h(j\omega) = \sigma + j\omega \quad \text{where } \sigma, \, \omega \, \epsilon \, \mathbf{R}^1.$$

Then M is given by

$$M = \left| \frac{\sigma + j\omega}{1 + \frac{1}{2}(\alpha + \beta)(\sigma + j\omega)} \right|$$

and $$M^2 = \frac{\sigma^2 + \omega^2}{\{1 + \frac{1}{2}(\alpha + \beta)\sigma\}^2 + \{\frac{1}{2}(\alpha + \beta)\omega\}^2} \qquad \text{(III-40)}$$

From equation (III-40), we obtain

$$\sigma^2\{1 - \frac{M^2}{4}(\alpha+\beta)^2\} - M^2(\alpha+\beta)\sigma - M^2 + \omega^2\{1 - \frac{M^2}{4}(\alpha+\beta)^2\} = 0 \,. \qquad \text{(III-41)}$$

If $\frac{M}{4}(\alpha + \beta)^2 - 1 = 0$, we obtain $\sigma = \frac{-1}{\alpha + \beta}$ which is a straight line parallel to the ω - axis and passing through the point $(\frac{-1}{\alpha + \beta}, 0)$.

If $\frac{M}{4}(\alpha + \beta)^2 - 1 \neq 0$, from equation (III-41), we obtain an equation of a circle which is given by

$$\{ \sigma - \frac{2M^2(\alpha + \beta)}{4 - M^2(\alpha + \beta)^2} \}^2 + \omega^2 = (\frac{4M}{4 - M^2(\alpha + \beta)^2})^2 \,. \qquad \text{(III-42)}$$

This circle has a radius of $\left| \frac{4M}{4 - M^2(\alpha + \beta)^2} \right|$ and is centered at

$$(\frac{2M^2(\alpha + \beta)}{4 - M^2(\alpha + \beta)^2}, 0).$$

In interpreting the design criterion

$$\sup_{\omega \in \mathbf{R}^1} |h(j\omega)| \, \delta \le 1 - \sup_{\omega \in \mathbf{R}^1} \left| \frac{h(j\omega)}{1 + \frac{1}{2}(\alpha + \beta) h(j\omega)} \right| \frac{\beta - \alpha}{2}$$

the quantity on the right hand side of the inequality can be readily evaluated by the M - circle approach. Consequently the above condition reduces to a simple circle condition

$$\sup_{\omega \in \mathbf{R}^1} |h(j\omega)| \leq \delta_r$$

where δ_r depends on M, β, α, and δ .

We also require that

$$\sup_{\omega \in \mathbf{R}^1} \left| \frac{h(j\omega)}{1 + \frac{1}{2}(\alpha + \beta)\, h(j\omega)} \right| \frac{\beta - \alpha}{2} < 1 \qquad \text{(III-43)}$$

which may be interpreted as outlined in Appendix B.

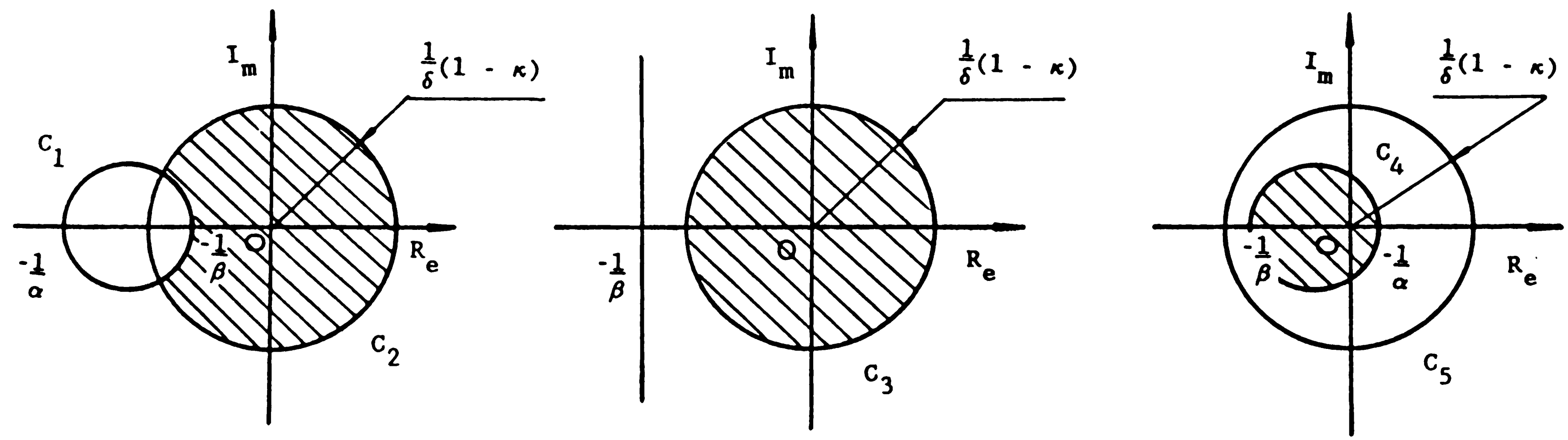
The overall design criterion therefore has the following ultimate interpretation. The design criterion is met if one of the following three conditions is satisfied.

Case (a) : $\underline{\alpha > 0}$

The locus of $h(j\omega)$ for $\omega \in (-\infty, \infty)$ lies outside the circle C_1 of radius $\frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$ centered in the complex plane at $\left[-\frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), 0\right]$ and inside the circle C_2 of radius $\frac{1}{\delta}(1 - \kappa)$ centered at the origin. This is depicted in Figure 6(a)..

Case (b) : $\underline{\alpha = 0}$

$\text{Re}\,[h(j\omega)] > -\frac{1}{\beta}$ for all real ω and therefore $h(j\omega)$ should be inside the circle C_3 of radius $\frac{1}{\delta}(1 - \kappa)$ centered at the origin as shown in Figure 6(b).



(a) $\alpha > 0$ (b) $\alpha = 0$ (c) $\alpha < 0$

Figure 6 Geometric interpretation

Case (c) : $\underline{\alpha < 0}$

The locus of $h(j\omega)$ for $\omega \in (-\infty, \infty)$ is contained within the intersection of the circle C_4 of radius $\frac{1}{2}(\frac{1}{\beta} - \frac{1}{\alpha})$ centered at $[-\frac{1}{2}(\frac{1}{\beta} + \frac{1}{\alpha}), 0]$ and the circle C_5 of radius $\frac{1}{\delta}(1 - \kappa)$ centered at the origin. This is shown in Figure 6(c).

<u>Remark</u> : Unlike the L_∞ - problem formulation where an algorithm for determining the feedback gain matrices is difficult to obtain , here we can utilize the Butterworth pole-patterns for satisfying the design criteria. It should be reemphasized that L_2 design criterion too is a quantitative pole placement. This however can be achieved quite easily by monitoring the frequency response.

3.5 Design Procedure

The magnitude of the closed loop frequency response $|h(j\omega)|$ depends on the augmented system matrix $\bar{A}$, thus an algorithmic procedure is proposed for the satisfaction of the design criteria. The following procedure is found to be effective.

(i) Specify α, β, β_o, β_i, and y_o .

(ii) Select ψ_o as the arithmetic mean of the sector $[\alpha, \beta]$ and evaluate ρ_o.

(iii) Evaluate δ .

(iv) Find K based on the algebraic Riccati equation and evaluate r_o. The input weighting matrix is taken to be the identity and the state weighting matrix is iteratively adjusted.

(v) Find κ by using the M - circle diagram .

(vi) Check the inequality (III-34) based on the circle interpretation .

(vii) Iterate steps (iv) through (vi) until the inequality is satisfied.

3.6 An Illustrative Example

To demonstrate the applicability of the design criteria above, consider the linear, time invariant system with a nonlinear feedback element given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \psi(y(t),t)$$

$$y = [\,1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where the nonlinearity $\psi(y(t), t)$ is confined to a sector $[\alpha, \beta]$. We consider two cases below.

Case 1 : $\alpha = 0.5$, $\beta = 3.5$

Consider the above system with the nonlinearity

$$\psi(y(t), t) = 2\, y(t) + 1.5\, y(t) \sin(5\, t)$$

which certainly belongs to the sector [0.5, 3.5].

Let the input sphere $\beta_i = 1.0$, the output sphere $\beta_o = 0.1$, and the desired output $y_o = t^2 e^{-t}$.

Design procedure :

Choosing the nonlinear design function ψ_o as

$$\psi_o = \frac{1}{2} (\alpha + \beta)\ y(t)$$

$$= 2\ y(t)$$

we get, $\rho_o = \sup \| N_\gamma y_o - N_o y_o \|$

$$= \max \| 2\ y_o + 1.5\ y_o \sin(t) - 2\ y_o \|$$

$$= \max \| 1.5\ y_o \sin(t) \|$$

$$\leq 1.5 \| y_o \|$$

According to the definition of L_2 - norm and the nominal output y_o , the norm of y_o is

$$\| y_o \| = [\int_0^\infty | t^2 e^{-t} |^2 dt]^{1/2}$$

$$= 0.866$$

which yields $\rho_o = 1.299.$

Consequently δ can be computed as

$$\delta = \frac{\rho_o + \beta_i}{\beta_o}$$

$$= 22.99$$

Now the eigenvalues of matrix $\bar{A} = (A + BK)$ are obtained via the Butterworth pole pattern to be

$$\lambda_{1,2} = -3.535 \pm j\ 3.464$$

yielding the feedback gain K

$$K = [\ -24.495 \quad -6.07\ ]$$

Thus the transfer function h(s) becomes, with $s = \sigma + j\omega$,

$$h(s) = C\,(sI - R)^{-1}\,B$$

$$= \frac{1}{s^2 + 7.07\ s + 24.495}$$

where I is the n x n identity matrix .

With K and the nominal output y_o, the nominal command input r_o can now be evaluated as

$$r_o = e^{-t}\,(2. + 10.14\ t + 18.424\ t^2)$$

Next consider the contraction coefficient κ and $\sup_{\omega \in \mathbf{R}^1} |h(j\omega)|$. Using the M - circle diagram as shown in Figure 7, we obtain

$$M = 0.0383$$

and $$\max\ |h(j\omega)| = 0.0408$$

yielding

$$\kappa = \frac{1}{2}\,(\beta - \alpha)\,(M)$$

$$= 0.0287\ .$$

Finally, the inequality (III-34) is computed as

$$\sup_{\omega\ e\ \mathbf{R}^1} |h(j\omega)| \le \frac{1}{\delta}\,(1 - \kappa)$$

$$= 0.04225.$$

Thus the design criteria for servo-tracking are satisfied.

Remark : According to the geometric interpretation (Case (a) of section 3.4), this result can be easily checked directly as in Figure 7. Since the frequency response $h(j\omega)$ remains outside the circle C_1 of radius $\frac{1}{2}(\frac{1}{\alpha} - \frac{1}{\beta}) = 0.857$ centered at $(-\frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta}),\ 0) = (-1.142,\ 0)$ (this circle is not shown in Figure 7, since it is located to the far left of

the locus $h(j\omega)$) and inside the circle C_2 of radius $\frac{1}{\delta}(1 - \kappa) = 0.04225$ centered at the origin, the same conclusion as above is drawn .

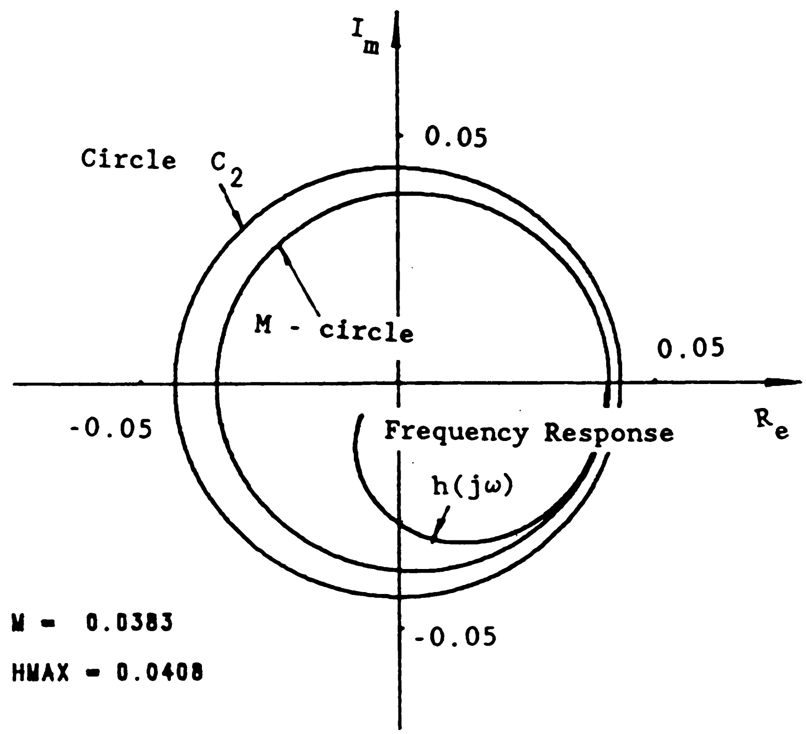


Figure 7 M - Circle diagram for $\alpha = 0.5$, $\beta = 3.5$

Computer Simulation :

In order to verify that the design specifications are satisfied, the resulting system was simulated on a digital computer. Figure 8(a) shows that the output $y(t)$ and the nominal output $y_o(t)$ resulting from the command input $r(t)$ shown in Figure 9(a) and given explicitly by

$$r(t) = \begin{cases} r_o(t) & \text{if } r_o(t) \le 12.1 \\ 12.1 & \text{if } r_o(t) > 12.1 \end{cases}$$

so $||r - r_o|| \le 1.0$. The input disturbance $||r - r_o||$ is shown in Figure 9(b) and is clearly in the input sphere of radius $\beta_i = 1.0$.

The error $||y - y_o||$ computed from Figure 8(b) is of the order of 10^{-2} which satisfies the output sphere $\beta_o = 0.1$. The control $u = r + K x + \psi_o$ is shown in Figure 10.

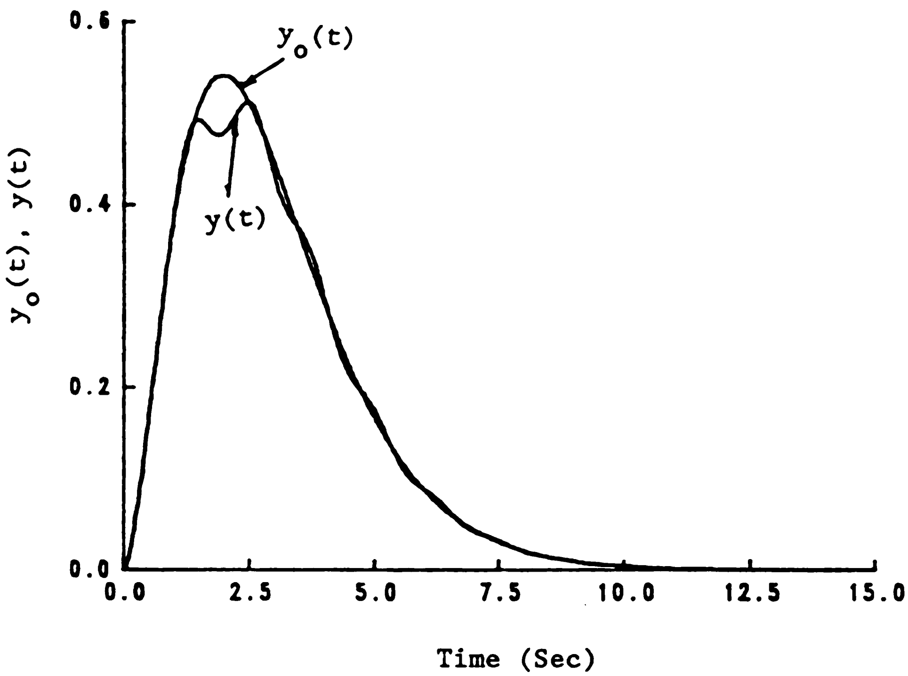


Figure 8(a) $y(t)$ and $y_o(t)$ for $\alpha = 0.5$, $\beta = 3.5$

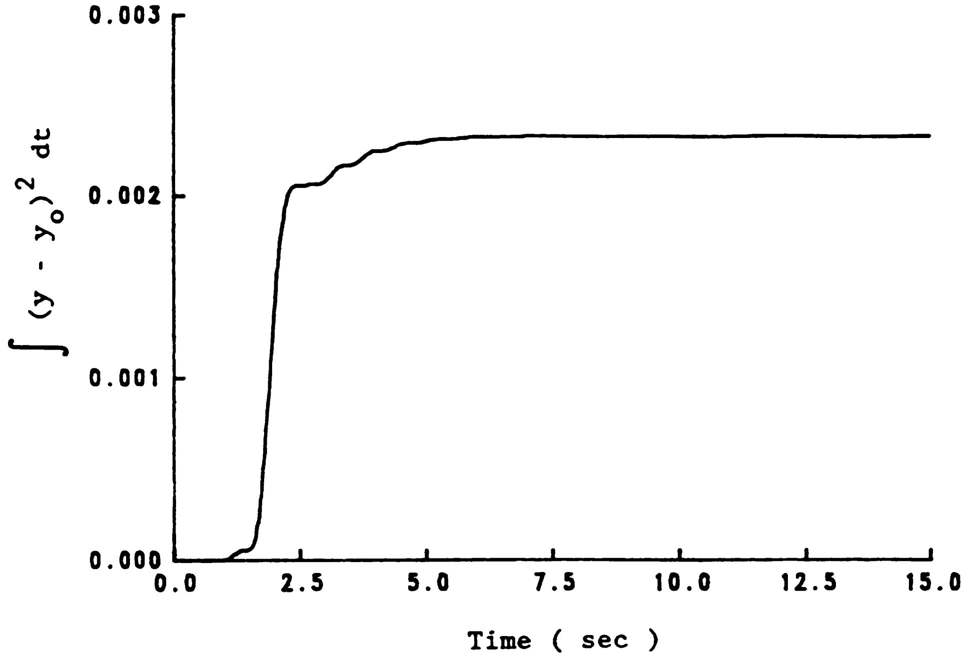


Figure 8(b) The tracking error $\int (y - y_o)^2\, dt$ for $\alpha = 0.5$, $\beta = 3.5$

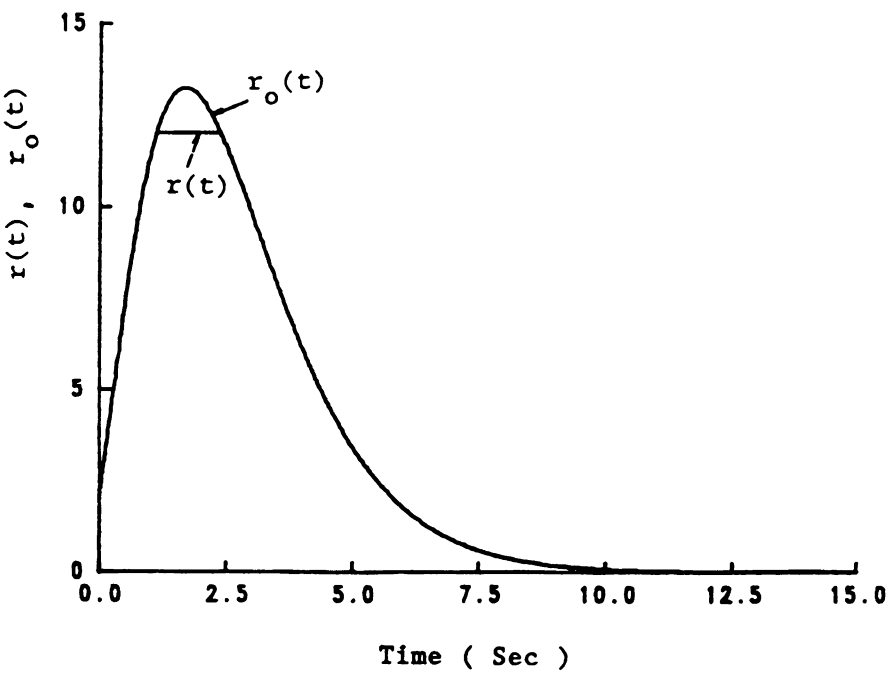


Figure 9(a) r(t) and $r_o(t)$ for $\alpha = 0.5$, $\beta = 3.5$

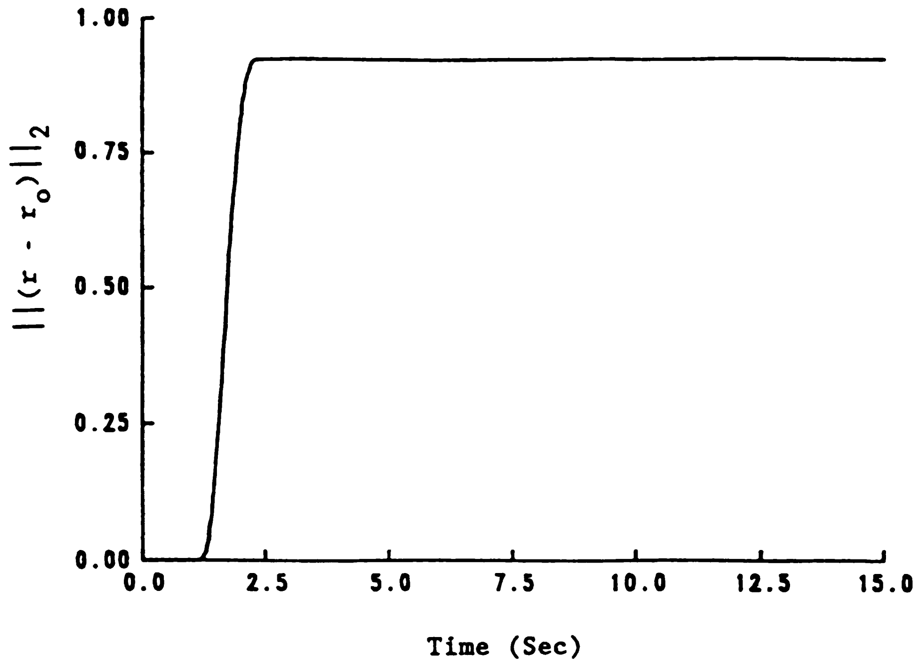


Figure 9(b) Input disturbance in $||r - r_o||_2$ for $\alpha = 0.5$, $\beta = 3.5$

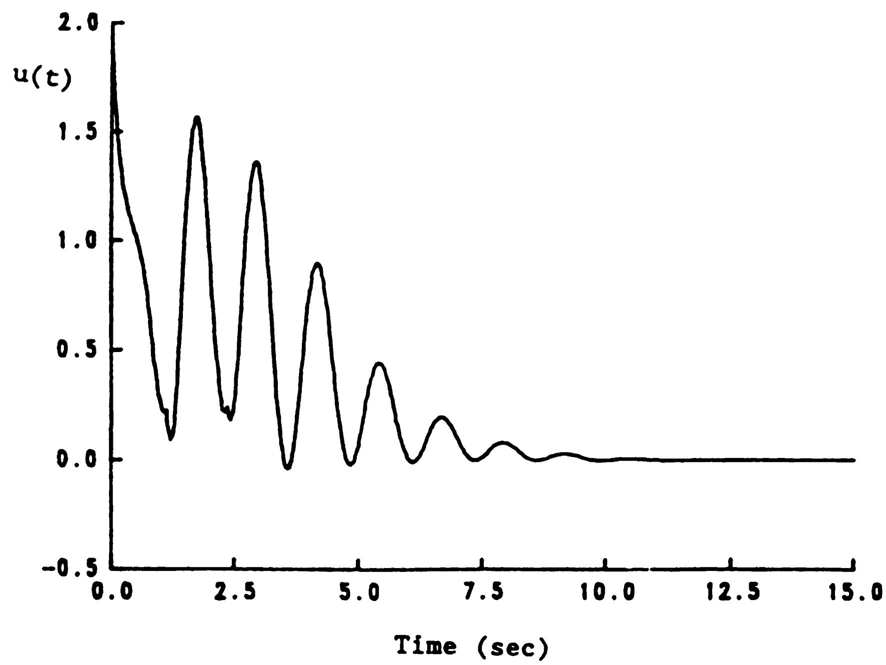


Figure 10 Control effort u(t) for $\alpha = 0.5$, $\beta = 3.5$

Case 2 : $\alpha = -0.5, \beta = 0.5$

Now consider the same system as in Case 1 , but with the nonlinearity

$$\psi = \frac{1}{2} y(t) \sin(t)$$

which is confined to a sector [-0.5, 0.5], i.e., $\alpha = -0.5$, $\beta = 0.5$.

Let $\beta_i = 1.0$, $\beta_o = 0.1$, and $y_o = t^2 e^{-t}$ as in the previous case.

Design Procedure :

Similar to the previous case, the nonlinear design function $\psi_o = 0$ is chosen. Hence the norm ρ_o can be evaluated

$$\rho_o = \sup || N_\gamma y_o - N_o y_o ||$$

$$= \max || \frac{1}{2} y_o \sin(t) ||$$

$$= \frac{1}{2} || y_o ||$$

Here, $||y_o|| = [\int_0^\infty |t^2 e^{-t} |^2 dt]^{1/2}$

$$= 0.866$$

Therefore, $\rho_o = 0.433$

Consequently, $\delta = \dfrac{\beta_i + \rho_o}{\beta_o} = 14.33$

Now we choose the eigenvalues of $\bar{A}$ as

$$\lambda_{1,2} = -2.985 \pm j\ 2.900$$

from the Butterworth pole pattern yielding

$$K = [\ -17.32,\ -4.97\ ]$$

and the frequency response transfer function

$$h(s) = \frac{1}{s^2 + 5.97\, s + 17.32}$$

With K and y_o, r_o is evaluated as

$$r_o = e^{-t} (2. + 7.94\, t + 12.35\, t^2)$$

From Figure 11, M = 0.059 and $\sup_{\omega \in \mathbf{R}^1} |h(j\omega)| = 0.0577$.

Consequently, with the contraction coefficient κ

$$\kappa = \frac{1}{2} (\beta - \alpha)\ (M)$$

$$=0.0295$$

the design criterion (III-34) is verified, yielding

$$\sup_{\omega \in \mathbf{R}^1} |h(j\omega)| \le \frac{1}{\delta} (1 - \kappa)$$

$$= 0.0677$$

This result is also verified directly by the geometric interpretation given in the previous section (Case (c)) which shows that the locus of $h(j\omega)$ lies in the intersection of the circle C_4 of radius 2.0 centered at the origin (this circle is not shown in Figure 11) and circle C_5 of radius 0.0667 centered at the origin.

Computer Simulation :

The command input r(t) which satisfies the input sphere condition $||r - r_o|| \le 1.0$ given by

$$r(t) = r_o(t) + 1.1\, e^{-0.3\, t} \sin (10.\, t)$$

is used. The reference command input $r_o(t)$, actual command input r(t), and corresponding input disturbance $||r - r_o||$ are shown in

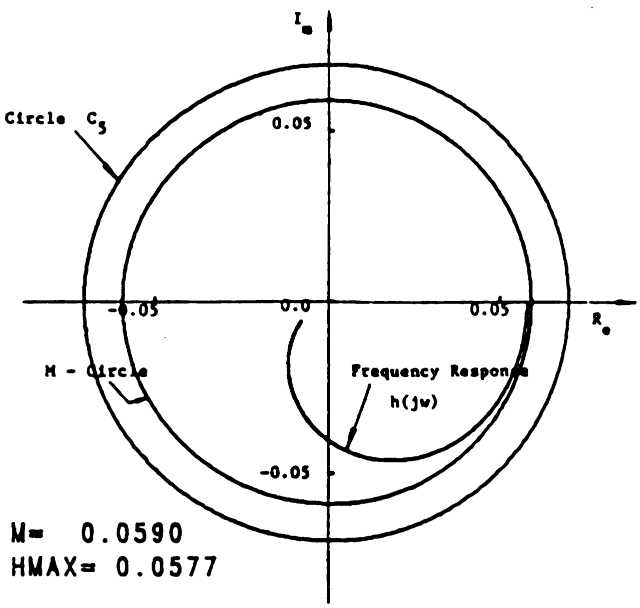


Figure 11 M - circle diagram for α = -0.5, β = 0.5

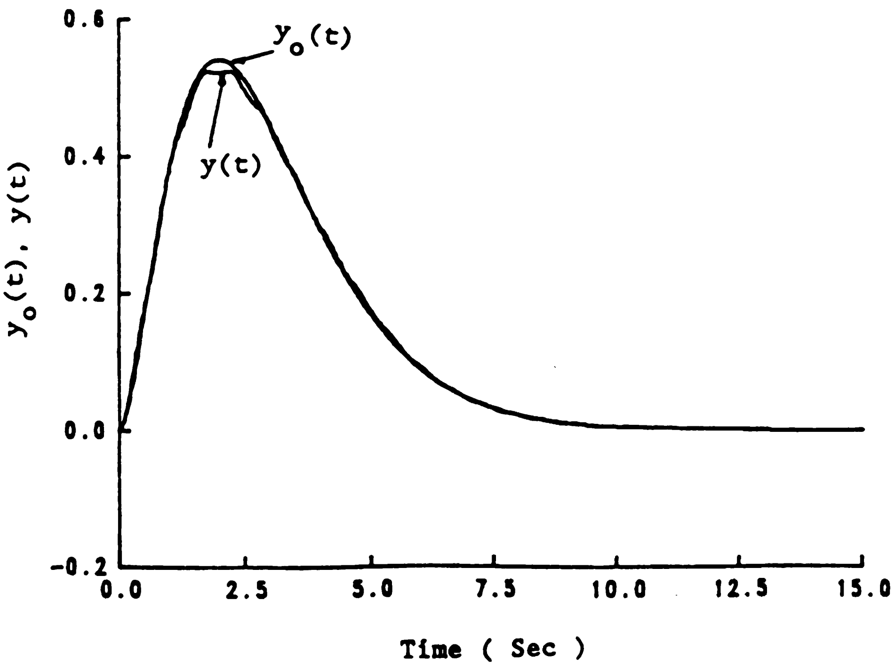


Figure 12(a) y(t) and $y_o(t)$ for α = -0.5, β = 0.5

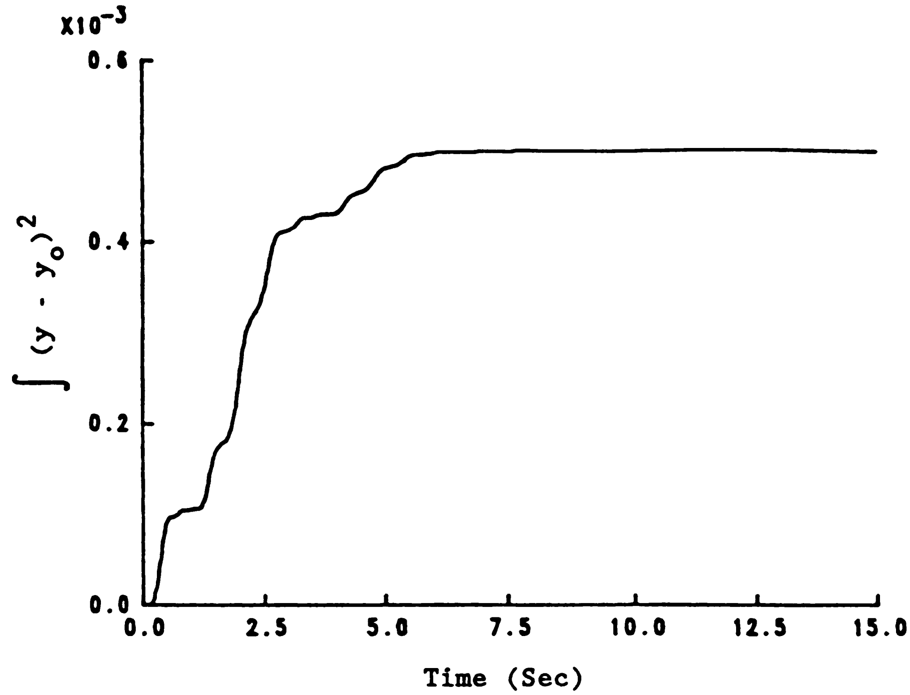


Figure 12(b) The tracking error $\int (y - y_o)^2 \, dt$ for $\alpha = -0.5$, $\beta = 0.5$

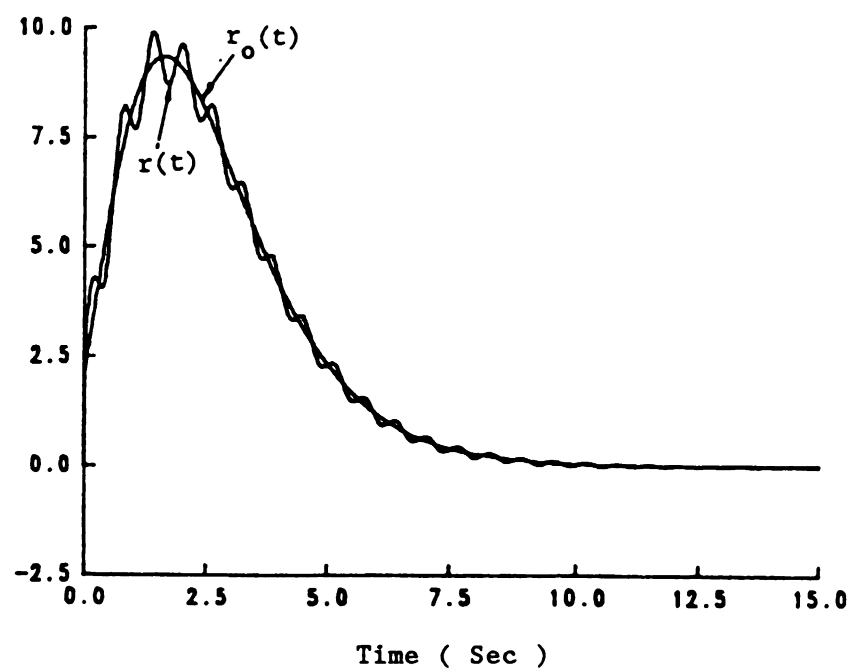


Figure 13(a) $r(t)$ and $r_o(t)$ for $\alpha = -0.5$, $\beta = 0.5$

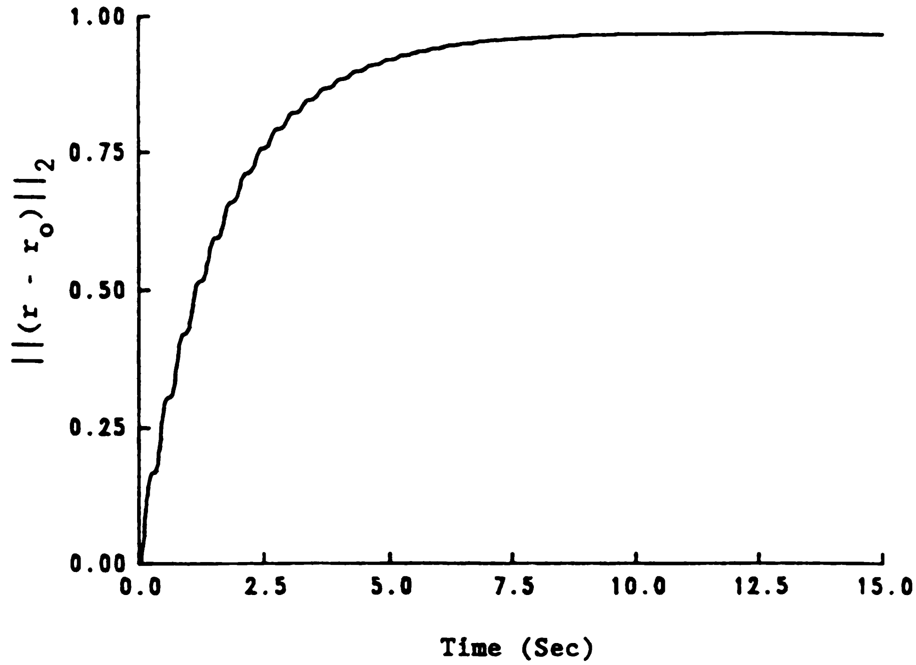


Figure 13(b) Input disturbance in $||r - r_o||_2$ for $\alpha = -0.5$, $\beta = 0.5$

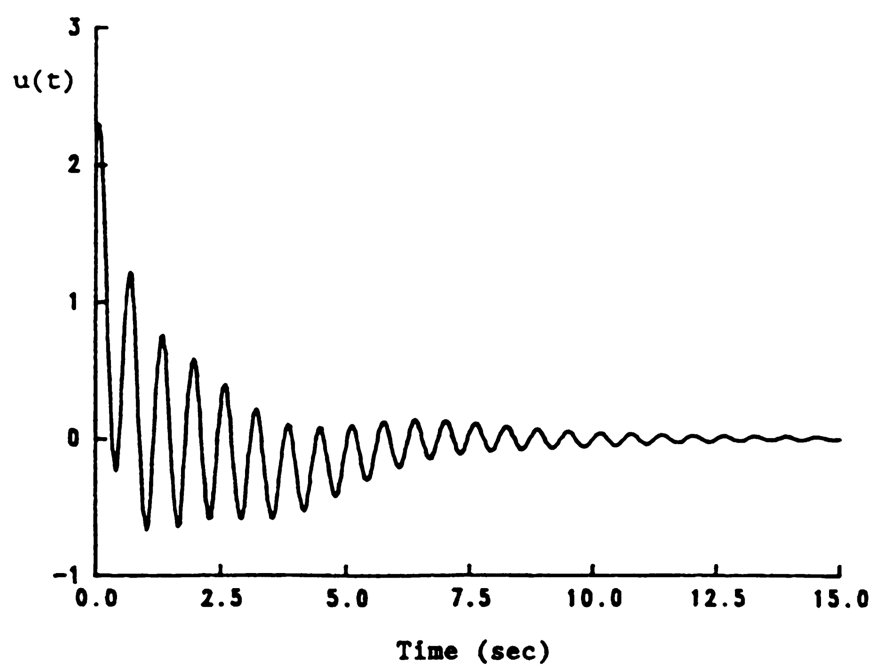


Figure 14 Control effort u(t) for $\alpha = -0.5$, $\beta = 0.5$

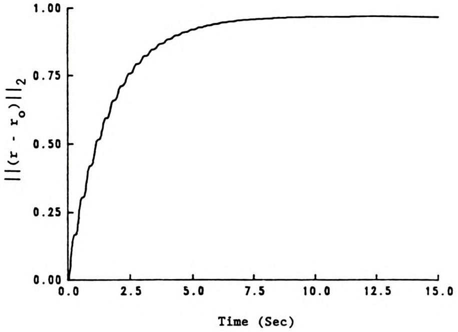


Figure 13(b) Input disturbance in $||r - r_o||_2$ for $\alpha = -0.5$, $\beta = 0.5$

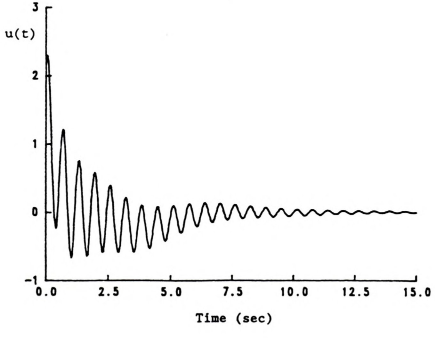


Figure 14 Control effort u(t) for $\alpha = -0.5$, $\beta = 0.5$

Figure 13(a) and Figure 13(b) respectively. Figure 12(a) shows the reference input $y_o(t)$ and the actual output $y(t)$. The error $||y - y_o||$ is shown in Figure 12(b). This clearly satisfies the output sphere specification $\beta_o = 0.1$. On comparing the actual output error which is of order 10^{-2} with the design specification which is of order 10^{-1}, it is clear that the result is conservative. This characteristic is primarily due to the generality of inputs and the nonlinearities that are admissible in $L_2[0, \infty)$. The required control $u(t)$ is shown in Figure 14. This control effort is smaller than that of the previous case for $\alpha = 0.5$ and $\beta = 3.5$ shown in Figure 10. It is expected, since this nonlinearity is less severe than the previous nonlinear element.

CHAPTER IV

AN APPROACH FOR SELECTION OF EIGENVALUES

In Chapter II, design criteria for precision tracking were developed by embedding the problem in L_∞- space of functions. With respect to the satisfaction of those design criteria, a quantitative pole-placement was identified. In this chapter an algorithm is developed for the selection of eigenvalues which makes the operator norm η a minimum. The approach to this eigenvalue placement is via a generalized LQ formulation.

Although the idea of pole-placement is central to much of linear control theory, it is not very well understood how eigenvalues should be selected to satisfy performance specifications. One of the interesting features associated with the design criteria contained in here is that the placement of eigenvalues is directly related to the ability to effect specified tracking. Namely, the tracking specifications can be met if the eigenvalues can be placed in such a way as to obtain minimal values for η. Hence it would be appropriate to refer to this quantitative pole-placement idea as a sufficient condition for "**trackability**". Thus it is clear why it would be important to seek algorithms for pole-placement that yield minimal values for η. A criterion which resembles the Butterworth pole configuration is developed for the SISO case. MIMO case, however, still remains an open issue.

4.1 Conventional Pole - Placement Problem

Consider a linear time invariant dynamical system of the form

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

where, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and A and B are constant matrices of appropriate dimensions. Applying the state feedback $u(t) = K\,x(t)$ to the system above, a closed loop system with a design parameter K is obtained. The question then is can K be adjusted so that this closed loop system exhibits desirable features such as stability and reasonable transients. It is well known that if {A, B} is controllable then arbitrary eigenvalues for the closed loop system can be achieved. This idea of arbitrary placement of closed loop eigenvalues is what is typically referred to as pole - placement.

Thus if {A, B} is controllable then the design problem reduces to the question whether it is possible to capture the performance specifications in terms of specific pole locations. Typically this is achieved by a trial and error procedure : First a set of eigenvalues is selected and the design completed followed by an actual simulation study. If simulations show undesirable performance pole locations are changed and the design carried out once more. This process is repeated until a satisfactory performance is achieved.

The LQ formulation however selects appropriate eigenvalues once a performance index(PI) is chosen. In this case although the eigenvalues are not picked arbitrarily their locations depend on the weighting matrices of the PI. The problem with this is that it is not known how one should select the weightings to satisfy a specified performance. Nevertheless it affords a way of selecting the closed loop poles in somewhat of a definitive form. This is the spirit in which we develop the generalized LQ problem.

4.2 Generalized LQ problem

We observe that the operator norm $\eta = || W_o \Psi Q_o^{-1} ||$ depends only on the linear structure of the uncertain nonlinear plant. The linear part of the plant (II-1) is

$$\dot{x}(t) = A\, x(t) + B\, u(t) \qquad \text{(IV-1a)}$$

$$y(t) = C\, x(t) \qquad \text{(IV-1b)}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^b$, and A, B , C are constant matrices of order n x n, n x m and b x n respectively.

Since L_∞ - measures are used to describe tracking errors the quadratic performance index of the LQ problem is modified as

$$J_p = \int_0^{T_f} [\frac{1}{2p} \{(y(t)-y_o(t))^T Q_1 (y(t)-y_o(t))\}^p + \frac{1}{2p} \{u(t)^T Q_2 u(t)\}^p] dt \qquad \text{(IV-2)}$$

to reflect such a measure, where the nominal output $y_o(t) \in \mathbf{R}^b$, $t \in [0., T_f]$, and Q_1, Q_2 are, respectively, symmetric positive semidefinite and symmetric positive definite weighting matrices of order b x b and m x m. In (IV-2) the limit as the positive integer $p \to \infty$ corresponds to the L_∞ - case.

Next, we determine the optimal control $\tilde{u}(t)$ that steers the system output given by (IV-1) so as to track the nominal output $y_o(t)$ which simultaneously minimizes the performance measure given by (IV-2). By augmenting the performance measure (IV-2) with the state equations via Lagrange multipliers $v(t) \in \mathbf{R}^n$, we obtain

$$J_g = \int_0^{T_f} [\frac{1}{2p} \{(y(t)-y_o(t))^T Q_1 (y(t)-y_o(t))\}^p + \frac{1}{2p} \{u(t)^T Q_2 u(t)\}^p$$

$$+ v^T(t) (A\, x(t) + B\, u(t) - \dot{x}(t))]\, dt$$

which may be written as

$$J_g = \int_0^{T_f} [\frac{1}{2p} \{(C\, x - y_o)^T Q_1 (C\, x - y_o)\}^p + \frac{1}{2p} \{ u^T Q_2 u \}^p$$

$$+ v^T (A\, x + B\, u - \dot{x})]\, dt$$

$$= \int_0^{T_f} \phi (x, \dot{x}, u, v, t)\, dt \qquad \text{(IV-3)}$$

where $\phi (x, \dot{x}, u, v, t) \overset{\Delta}{=} \frac{1}{2p}\{(C\, x - y_o)^T Q_1 (C\, x - y_o)\}^p$

$$+ \frac{1}{2p}\{ u^T Q_2 u \}^p + v^T(A\, x + B\, u - \dot{x})$$

Now we consider the extrema of the functional (IV-3) under the following assumptions.

(i) The state and the control are not constrained.

(ii) The end point is free, i.e., T_f is free.

(iii) The initial condition of the state $x(0) = x_o$. The final state $x(T_f) = x_T$ is free.

With the above assumptions, we now take the first variation of J_g, denoted by ΔJ_g yielding

$$\Delta J_g = \int_0^{T_f} \{\phi(x+\delta x, \dot{x}+\delta\dot{x}, u+\delta u, v+\delta v, t) - \phi(x, \dot{x}, u, v, t)\}\, dt \qquad \text{(IV-4)}$$

By employing the Taylor expansion theorem, equation (IV-4) becomes

$$\Delta J_g = \int_0^{T_f} \{ [\frac{\partial \phi}{\partial x} (x, \dot{x}, u, v, t)]^T \delta x + \{ [\frac{\partial \phi}{\partial \dot{x}} (x, \dot{x}, u, v, t)]^T \delta \dot{x}$$

$$+ [\frac{\partial \phi}{\partial u} (x, \dot{x}, u, v, t)]^T \delta u + [\frac{\partial \phi}{\partial v} (x, \dot{x}, u, v, t)]^T \delta v$$

$$+ [\text{ higher order terms in } (\delta x, \delta \dot{x}, \delta u, \delta v)] \} dt \qquad \text{(IV-5)}$$

Neglecting the higher order terms of $(\delta x, \delta \dot{x}, \delta u, \delta v)$, integrating by parts the term involving $\delta \dot{x}$, and relating $\delta x(T_f)$ to δx_T and δT_f by (See Figure 15)

$$\delta x(T_f) = \delta x_T - \dot{\tilde{x}}(T_f) \, \delta T_f$$

yield

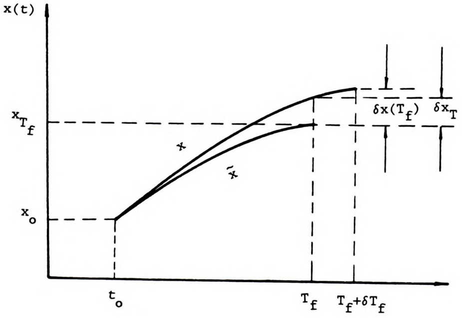


Figure 15 Boundary condition for free end point

$$\Delta J_g = \int_0^{T_f} \{[[\frac{\partial \phi}{\partial x} (\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t)]^T$$

$$- \frac{d}{dt} [\frac{\partial \phi}{\partial \dot{x}} (\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t)]^T]\ \delta x(t)$$

$$+ [\ \frac{\partial \phi}{\partial u} (\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t)]^T\ \delta u(t)$$

$$+ [\ \frac{\partial \phi}{\partial v} (\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t)]^T\ \delta v(t)\ \}\ dt$$

$$+ [\ \frac{\partial \phi}{\partial \dot{x}} (\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]^T\ \delta \dot{x}_T$$

$$+ \{[\ \phi(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]$$

$$- [\frac{\partial \phi}{\partial \dot{x}} (\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]^T\ \dot{\tilde{x}}(T_f)\}\ \delta T_f$$

(IV-6)

where $\tilde{x}$, $\dot{\tilde{x}}$, $\tilde{u}$, and $\tilde{v}$ are respectively the state, the time derivative of the state, the control and the Lagrange multipliers along an extremal. Since the variation ΔJ_g vanishes on an extremal, each coefficient of the independent variables must be zero . Therefore from equation (IV-6), we obtain the following governing equations.

(i) The differential constraints

$$\frac{\partial \phi}{\partial v} (\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t)] = A\ \tilde{x} + B\ \tilde{u} - \dot{\tilde{x}}$$

$$= 0.$$

are the state equations

$$\dot{\tilde{x}} = A\ \tilde{x} + B\ \tilde{u} \qquad \text{(IV-7)}$$

(ii) The coefficients of $\delta x(t)$ are

$$[\frac{\partial \phi}{\partial x}(\tilde{x}(t),\dot{\tilde{x}}(t),\tilde{u}(t),\tilde{v}(t),t)]^T - \frac{d}{dt}[\frac{\partial \phi}{\partial \dot{x}}(\tilde{x}(t),\dot{\tilde{x}}(t),\tilde{u}(t),\tilde{v}(t),t)]^T = 0$$

which leads to costate equations

$$\dot{\tilde{v}} = \{(C\tilde{x} - y_o)^T Q_1 (C\tilde{x} - y_o)\}^{p-1} C^T Q_1 (C\tilde{x} - y_o) - A^T \tilde{v} \qquad \text{(IV-8)}$$

(iii) The coefficients of $\delta u(t)$ becomes

$$\frac{\partial \phi}{\partial u}(\tilde{x}(t),\ \dot{\tilde{x}}(t),\ \tilde{u}(t),\ \tilde{v}(t),\ t) = \{\tilde{u}^T Q_2 \tilde{u}\}^{p-1} Q_2 \tilde{u} + B^T \tilde{v}$$

$$= 0. \qquad \text{(IV-9)}$$

(iv) The coefficients of δT_f are

$$\frac{\partial \phi}{\partial \dot{x}}(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]^T \delta \dot{x}_T$$

$$+ \{[\ \phi(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]$$

$$- [\frac{\partial \phi}{\partial \dot{x}}(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)]^T \dot{\tilde{x}}(T_f)\}\ \delta T_f$$

$$= 0$$

which give the boundary conditions

$$\tilde{v}^T(T_f)\ \delta x_T + H(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f)^T\ \delta T_f = 0$$

where the Hamiltonian

$$H \overset{\Delta}{=} \frac{1}{2p}\{(C\ x - y_o)^T Q_1 (C\ x - y_o)\}^p + \frac{1}{2p}\{\ u^T Q_2 u\ \}^p + v^T (A\ x + B\ u).$$

Therefore, the boundary conditions for the free end point become

$$\tilde{v}(T_f) = 0 \qquad \text{(IV-10a)}$$

$$H(\tilde{x}(T_f),\ \dot{\tilde{x}}(T_f),\ \tilde{u}(T_f),\ \tilde{v}(T_f),\ T_f) = 0. \qquad \text{(IV-10b}$$

Equation (IV-7) - (IV-10b) constitute a set of necessary conditions for an extremal of the generalized LQ performance index (IV-2) If $p = 1$ we recover the LQ results in the form of a state feedback wit gains given by the Riccatti differential equation. In order to obtai a perturbed form of this LQ solution or equivalently the LQ pole patterns we start by defining the positive quantities

$$\{(C\tilde{x} - y_o)^T Q_1 (C\tilde{x} - y_o)\}^{p-1} = \epsilon_1 \qquad \text{(IV-11a}$$

$$\{\tilde{u}^T Q_2 \tilde{u}\}^{p-1} = \epsilon_2 \qquad \text{(IV-11b}$$

which are then substituted in (IV-8) and (IV-9).
Then the costate equations (IV-8) become

$$\dot{\tilde{v}} = - \epsilon_1 C^T Q_1 (C\tilde{x} - y_o) - A^T \tilde{v} \qquad \text{(IV-12}$$

and the control $\tilde{u}(t)$ from (IV-9) is

$$\tilde{u}(t) = - \frac{1}{\epsilon_2} Q_2^{-1} B^T \tilde{v} \qquad \text{(IV-13}$$

Substituting (IV-13) into (IV-7) and augmenting it with (IV-12) yield

$$\dot{\tilde{x}} = A\tilde{x} - \frac{1}{\epsilon_2} B Q_2^{-1} B^T \tilde{v} \qquad \text{(IV-14a}$$

$$\dot{\tilde{v}} = - \epsilon_1 C^T Q_1 C\tilde{x} - A^T \tilde{v} + \epsilon_1 C^T Q_1 y_o \qquad \text{(IV-14b}$$

Rewriting equation (IV-14) in a matrix form, gives

$$\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}\tilde{u}_c \qquad \text{(IV-15}$$

where $\tilde{z} = \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} \epsilon\ \mathbf{R}^{2n}$

$$\tilde{A} = \begin{bmatrix} A & -\frac{1}{\epsilon_2} B Q_2^{-1} B^T \\ -\epsilon_1 C^T Q_1 C & -A^T \end{bmatrix} \epsilon \mathbf{R}^{2n \times 2n}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix} \epsilon \mathbf{R}^{2n}$$

$$\tilde{u}_c = \epsilon_1 C^T Q_1 y_o \quad \epsilon \mathbf{R}^m .$$

Remark : In (IV-11) when $p \to \infty$, we consider $\epsilon_1 \to 0$ to be a situation when tracking occurs quite accurately and $\epsilon_1 \to \infty$ to correspond to a situation where no tracking is apparent. This motivates the limiting analysis given below in which one would initiate the design by letting $\epsilon_1 \to \infty$ to correspond to the worst case and then subsequently decreasing ϵ_1 so as to correspond to the specifications dictated by the operator norms mentioned previously. These limiting values would basically guide the selection of pole-configurations for initial start up of the primary quantitative pole-placement algorithm.

4.3 Optimal Pole Configuration

Now we investigate the pole patterns of the augmented system (IV-14) with respect to parameters ϵ_1, ϵ_2 that determine respectively how much significance is attributed to the output error or the control effort. From an algebraic Riccati type of equation for a steady state operation, the optimal feedback gains associated with the optimal pole configuration may be determined. The present method, however, allows us to study the structure of the optimal pole configuration more easily as a function of ϵ_1 and ϵ_2 . Namely, by using the root locus method,

we gain more insights to the solution. Results are obtained in the limiting cases for SISO systems. We first derive the characteristic polynomials of $\tilde{A}$ as a function of ϵ_1 and ϵ_2, observing that the optimal closed loop poles which are the eigenvalues of the matrix $\tilde{A}$ should lie in the open left half complex plane. Next, the migration of poles with respect to parameters ϵ_1, ϵ_2 is studied. Since the tracking performance is considered most important, we arbitrarily set $\frac{1}{\epsilon_2} Q_2^{-1} = 1$ for simplicity, i.e., the control effort is allowed to take any value.

The characteristic polynomial of $\tilde{A}$ is computed by employing Lemmas C1, C2 in Appendix C.

$$\det(sI - \tilde{A}) = \det \begin{bmatrix} sI - A & B B^T \\ \epsilon_1 C^T Q_1 C & sI + A^T \end{bmatrix}$$

$$= \det(sI - A) \det\{(sI + A^T) - \epsilon_1 C^T Q_1 C (sI - A)^{-1} B B^T\} \qquad \text{(by Lemma C1)}$$

$$= \det(sI - A) \det\{(sI + A^T)(I - \epsilon_1 C^T Q_1 C (sI - A)^{-1} B B^T (sI + A^T)^{-1}\}$$

$$= \det(sI - A) \det(sI + A^T) \det\{I - \epsilon_1 B^T (sI + A^T)^{-1} C^T Q_1 C (sI - A)^{-1} B\}$$

$$\text{(by Lemma C2)}$$

$$= (-1)^n \det(sI - A) \det(-sI - A) \det\{I + \epsilon_1 (C(-sI - A)^{-1} B)^T Q_1 C (sI - A)^{-1} B\}$$

$$= (-1)^n \, a(s) \, a(-s) \det(I + \epsilon_1 \, h(-s)^T \, Q_1 \, h(s)) \qquad \text{(IV-16)}$$

where $a(s) = \det(sI - A)$

$h(s) = C(sI - A)^{-1} B$

and I's are the identity matrices of appropriate dimensions.

Thus the eigenvalues of the closed loop system are the zeros of (IV-16) which are in the open left half complex plane. Let the open loop transfer function h(s) of a SISO system be represented by

$$h(s) = \frac{b(s)}{a(s)}$$

$$= \frac{b_o \prod_{i=1}^{r} (s - \varphi_i)}{\prod_{i=1}^{n} (s - \lambda_i)} \qquad \text{(IV-17)}$$

where b_o is a nonzero constant, φ_i, $i = 1 \cdots r$, are the zeros of the open loop system and λ_i, $i = 1 \cdots n$, are the poles of open loop system. Then with $Q_1 = 1$ for simplicity, (IV-16) becomes

$$\prod_{i=1}^{n} (s - \lambda_i)(s + \lambda_i) + (-1)^{n-r} \epsilon_1 b_o^2 \prod_{i=1}^{r} (s - \varphi_i)(s + \varphi_i) = 0 \qquad \text{(IV-18)}$$

The asymptotic behavior of the closed loop poles given in (IV-18) as a function of ϵ_1 is outlined in the following theorem .

<u>**Theorem 1**</u> : Suppose that the open loop system whose transfer function is represented by (IV-17) is controllable and observable then

(i) for $\epsilon_1 \to 0$, the eigenvalues of the closed loop system approach asymptotically the numbers $\tilde{\lambda}_i$, $i = 1 \cdots n$, where

$$\tilde{\lambda}_i = \begin{cases} \lambda_i & \text{if } \operatorname{Re}(\lambda_i) \le 0 \\ -\lambda_i & \text{if } \operatorname{Re}(\lambda_i) > 0 \end{cases}$$

(ii) for $\epsilon_1 \to \infty$, r eigenvalues of the closed loop system approach asymptotically the values $\tilde{\varphi}_i$, $i = 1 \cdots r$, where

$$\tilde{\varphi}_i = \begin{cases} \varphi_i & \text{if} \quad \mathrm{Re}\,(\varphi_i) \le 0 \\ -\varphi_i & \text{if} \quad \mathrm{Re}\,(\varphi_i) > 0 \end{cases}$$

and the remaining (n - r) eigenvalues approach the asymptotes through the origin and make angles θ with the negative real axis of

(a) $\theta = \pm \dfrac{\ell\pi}{n-r}$, $\ell = 0 \cdots \dfrac{n-r-1}{2}$, for (n - r) is odd

(b) $\theta = \pm \dfrac{(\ell+\frac{1}{2})\pi}{n-r}$, $\ell = 0 \cdots \dfrac{n-r-2}{2}$, for (n - r) is even.

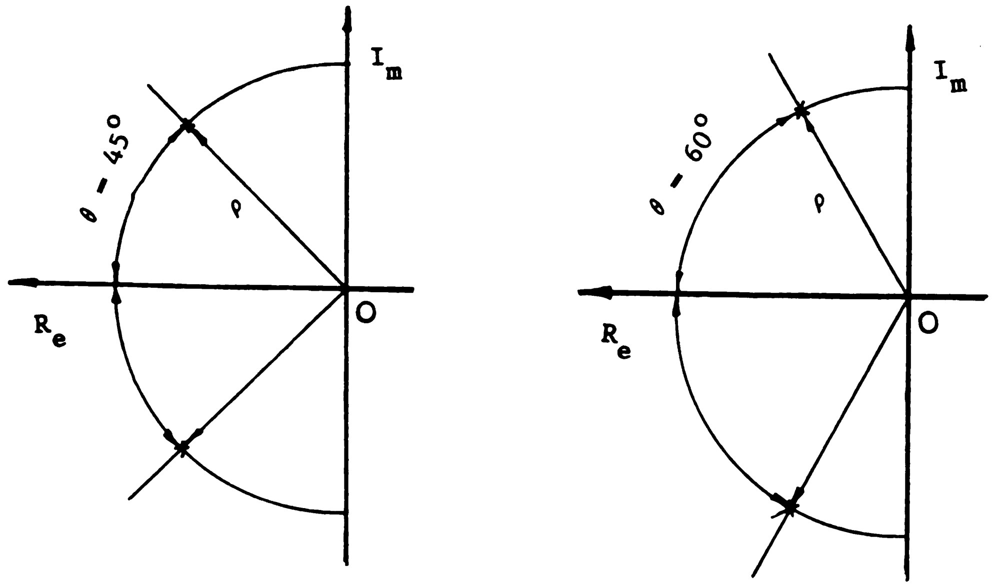
The distance from the origin for the far away eigenvalues are asymptotically $(\epsilon_1 b_o^2)^{\frac{1}{2(n-r)}}$.

Figure 16(a) and 16(b) show the pole configurations for (n-r) = 2, 3 respectively.

Illustrative examples : In order to verify the theorem above, the sensitivity of the norm η with respect to eigenvalue selection is computed for the following second and third order systems.

(i) Second order system

A system given by (IV-19) is simulated to compute the norm η.



(a) n - r = 2 (b) n - r = 3

Figure 16 Optimal pole configuration

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.23 & 0. \\ -0.57 & 1.42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0. \\ 1. \end{bmatrix} u(t) \qquad \text{(IV-19)}$$

The set of eigenvalues for the closed loop system are chosen so that the distance from the origin ρ, and the angle θ measured from the negative real axis shown in Figure 17(a) are

ρ : 0.5 1.0 1.5 2.0 3.0 5.0 10.0 20.0

θ : 10. 20. 30. 40. 50. 50. 70. 80.

The results shown in Figure 17(b) verify that the minimum norm values are obtained when the set of eigenvalues for the closed loop system are chosen in the vicinity of a 45^{o} line. This is quite consistent with the predictions of the theorem given previously.

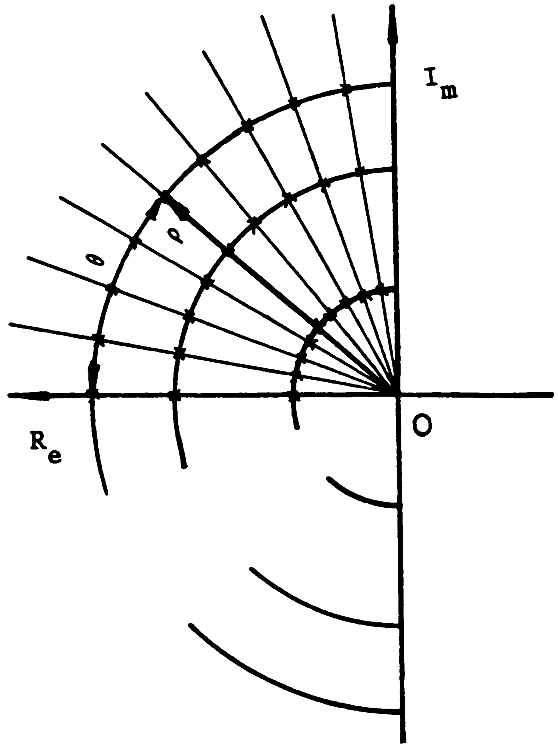


Figure 17(a) Eigenvalue placement for $n - r = 2$

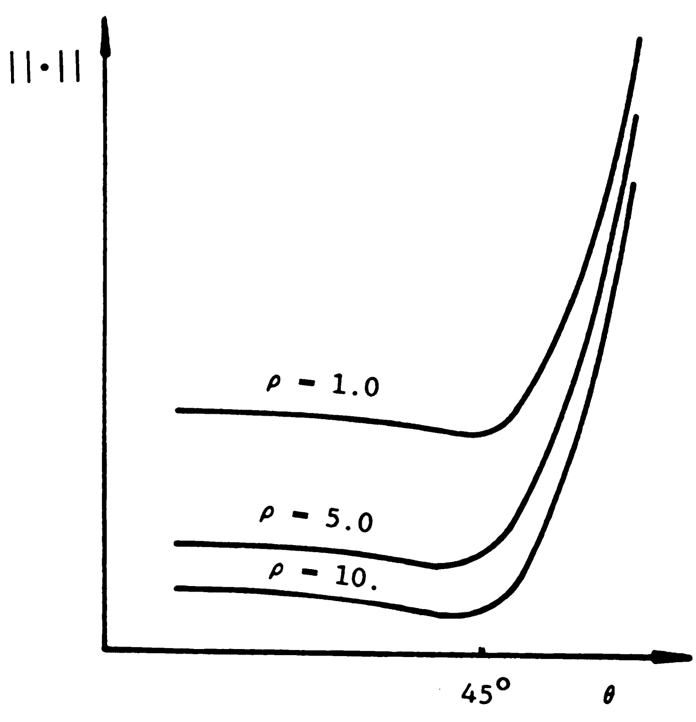


Figure 17(b) Norm configuration for $n = 2$

(ii) Third order system

The system given by (IV-20) is simulated for computation of the operator norm η with the set of eigenvalues, that is

ρ = 1.0 5.0 10.0

θ = 0.0 30.0 45.0 60.0 85.0

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -72.464 \\ 0 & 0.027 & -10. \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad \text{(IV-20)}$$

Figure 18 shows the norm configuration we expect. Namely, the minimum norm value of η is obtained when a pair of complex conjugate eigenvalues lie in the vicinity of a 60° line together with a real pole.

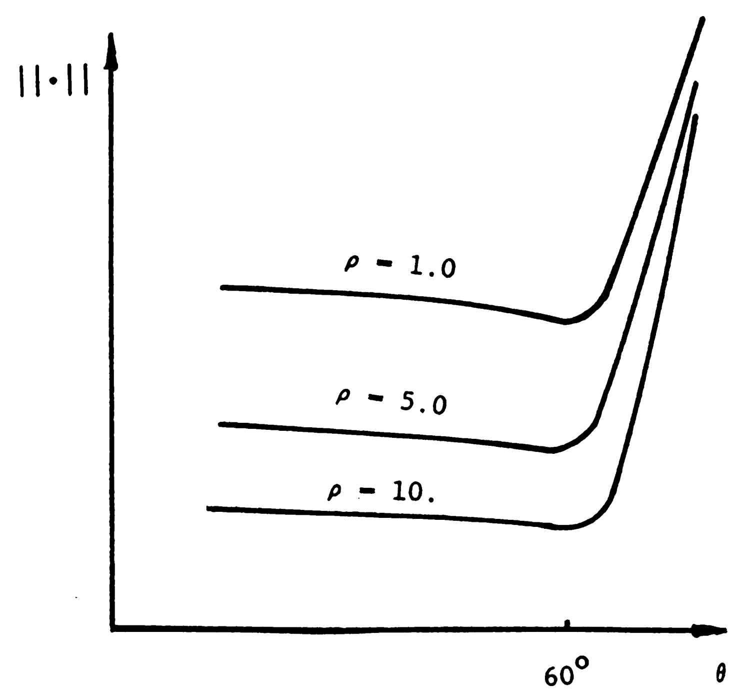


Figure 18 Norm configuration for n = 3

For MIMO systems too, the asymptotic behavior of the closed loop eigenvalues can be obtained from equation (IV-16). However, it is not as simple to determine the pole configuration because the eigenvalues that tend to infinity generally form several clusters of different order and radii. For the limiting cases, the theorem applies with varying asymptotic distances from the origin.

CHAPTER V

COMPUTER ALGORITHMS

In order to accomplish the design task computational algorithms are required so that the operator norm needed in the design inequality (II-16) can be evaluated. In this chapter, explicit computer oriented formulae necessary for the execution of Theorem 1 of Chapter II are developed. In particular, algorithms for computing the design matrices G and K that specify the closed loop eigenvalues of R and for computing the critical operator norm $|| W_o \Psi Q_o^{-1} ||$ are given. Several programming concerns associated with the numerical schemes are also discussed in here.

5.1 The Operator Norm $|| W_o \Psi Q_o^{-1} ||$

In this section computer oriented formulae for evaluating the operator norm $|| W_o \Psi Q_o^{-1}||$ are developed. The norm of the linear operator Ψ

$$(\Psi z)(t) = \int_0^t e^{R(t - \tau)} z(\tau) d\tau \qquad \text{(V-1)}$$

is first considered.

<u>Theorem 1</u> : Let $\Psi : L_\infty^{2n}[0, \infty) \to L_\infty^{2n}[0, \infty]$ be a linear operator given by

$$(\Psi z)(t) = \int_0^t H(t - \tau) \quad z(\tau)\, d\tau \quad ,$$

and let each element H_{ij} of the matrix H be such that $H_{ij} \in L_1[0 , \infty)$

Then,
$$||\Psi|| = \sup_{\substack{z \in L_\infty^{2n}[0,\infty) \\ z \neq 0}} \frac{||\Psi z||_{L_\infty^{2n}}}{||z||_{L_\infty^{2n}}} \leq \int_0^t || H(\tau) ||\, d\tau$$

where $||H(\tau)|| = \max_i \sum_j^{2n} | H_{ij}(\tau) |$.

Proof of Theorem 1 : Let
$$g(t) = \int_0^t H(t - \tau) \quad z(\tau)\, d\tau \tag{V-2}$$

where $g, z \in L_\infty^{2n}$ and $H_{ij} \in L_1[0 , \infty)$.

Equation (V-2) can be written in component form as

$$g_i(t) = \int_0^t \sum_j H_{ij}(t - \tau) \quad z_j(\tau)\, d\tau \tag{V-3}$$

which on taking absolute values on both sides yield

$$| g_i(t)| \leq \int_0^t \sum_j | H_{ij}(t - \tau)| \quad | z_j(\tau)| \quad d\tau \tag{V-4}$$

By defining the induced matrix norm

$$|| H(t - \tau) ||_{L_\infty} = \max_i [\sum_j | H_{ij}(t - \tau) |]$$

we get
$$\sum_j | H_{ij}(t -\tau) | \leq || H(t - \tau) ||_{L_\infty} \tag{V-5}$$

and similarly,
$$| z_j(\tau) | \leq \max_j | z_j(\tau) | = || z(\tau) ||_{L_\infty} \tag{V-6}$$

By substituting (V-5) and (V-6) into (V-4) , we obtain

$$| g_i(t)| \leq ||z|| \int_0^t || H(t - \tau)||\, d\tau$$

$$= ||z|| \int_0^{\infty} || H(\tau)|| \, d\tau$$

which implies

$$|| g ||_{L_{\infty}^{2n}} \leq ||z|| \int_0^{\infty} || H(\tau)|| \, d\tau \qquad \text{(V-7)}$$

Equation (V-7) can now be written as

$$\frac{|| g ||}{|| z ||} \leq \int_0^{\infty} || H(\tau)|| \, d\tau \ , \ \text{for } z \neq 0$$

which implies $|| \Psi || \leq \int_0^{\infty} || H(\tau)|| \, d\tau$.

This completes the proof of Theorem 1 . □

From Theorem 1, an upper bound of the operator norm $|| \Psi ||$ can be estimated by evaluating an integral if $H(t) = e^{Rt}$ can be explicitly represented. In what follows , an explicit expression for e^{Rt} is given.

Lemma 1 : Suppose matrix R has <u>distinct eigenvalues</u> λ_i, i = 1 ••• 2n , and p_i, i = 1 ••• 2n, are the corresponding eigenvectors, then the matrix R can be diagonalized in the form of

$$P^{-1} R P = \Lambda = \begin{bmatrix} \lambda_i & & O \\ & \ddots & \\ O & & \lambda_{2n} \end{bmatrix}$$

where P is the modal matrix such that

$$P = [p_1, p_2, \cdots , p_{2n}] .$$

Proof of Lemma 1 : See Strang ([47], Chap 5)

Remark : Distinct eigenvalues are assumed for the matrix R since a specific spectral decomposition corresponding to this case is used later in the chapter.

Lemma 2 : If a matrix R is diagonalizable, then its exponential

$$e^{Rt} = P^{-1} e^{\Lambda} P$$

$$= P^{-1} \begin{bmatrix} e^{\lambda_1 t} & & \bigcirc \\ & \ddots & \\ \bigcirc & & e^{\lambda_{2n} t} \end{bmatrix} P$$

where λ_i, $i = 1 \cdots 2n$, are the eigenvalues of R, and P is the corresponding modal matrix.

Proof of Lemma 2 : See Strang ([47], Chap 5)

In order to facilitate the evaluation of the operator norm $||\Psi||$, e^{Rt} is modified as follows. By defining the matrix E_i, $i = 1 \cdots 2n$, of order 2n x 2n which has 1 in the i^{th} diagonal position and 0's everywhere else, or

$$[\, E_i \,]_{jk} = \begin{cases} 1 \text{ , if } i = j = k \\ 0 \text{ , otherwise} \end{cases}$$

the diagonal matrix $e^{\Lambda t}$ can be written as

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \bigcirc \\ & \ddots & \\ \bigcirc & & e^{\lambda_{2n} t} \end{bmatrix}$$

$$= \sum_{i=1}^{2n} E_i \, e^{\lambda_i t} \tag{V-8}$$

where $\Lambda = \begin{bmatrix} \lambda_i & & \bigcirc \\ & \ddots & \\ \bigcirc & & \lambda_{2n} \end{bmatrix}$.

Thus, using Lemmas 1, 2 and (V-8), for matrix R with distinct eigenvalues λ_i, $i = 1 \cdots 2n$,

$$e^{Rt} = P^{-1} e^{\Lambda t} P$$

$$= P^{-1} \sum_{i=1}^{2n} E_i \, e^{\lambda_i t} P$$

$$= \sum_{i=1}^{2n} P^{-1} E_i \, P \, e^{\lambda_i t}$$

$$= \sum_{i=1}^{2n} S_i \quad e^{\lambda_i t}$$

where $S_i = P^{-1} E_i P$.

From spectral theory in a finite dimensional space ([15], Chap VII) , the matrix S_i can be represented by

$$S_i = \frac{\prod_{j \neq i}^{2n} (R - \lambda_j I)}{\prod_{j \neq i}^{2n} (\lambda_i - \lambda_j)}$$

which leads to $\quad H(t) = e^{Rt}$

$$= \frac{\sum_{i=1}^{2n} e^{\lambda_i t} \prod_{j \neq i}^{2n} (R - \lambda_j I)}{\prod_{j \neq i}^{2n} (\lambda_i - \lambda_j)} \tag{V-9}$$

Now following Theorem 1, an upper bound of the weighted operator norm $|| W_o \Psi Q_o^{-1} ||$ may be computed as given in Theorem 2.

Theorem 2 : Suppose the eigenvalues λ_i, $i = 1 \cdots 2n$, of matrix R, are distinct. Then the weighted operator norm

$$|| W_o \Psi Q_o^{-1} || \le \int_0^\infty \max_i [\sum_j | \hat{H}_{ij}(t) |] dt$$

where $(\Psi z)(t) = \int_0^t e^{R(t - \tau)} z(\tau) d\tau$,

and $\hat{H}(t) = W_o e^{Rt} Q_o^{-1} = \sum_{i=1}^{2n} e^{\lambda_i t} W_o S_i Q_o^{-1}$.

Proof of Theorem 2 : By Theorem 1,

$$|| W_o \Psi Q_o^{-1} || \le \int_0^\infty || \hat{H}(\tau) || d\tau \qquad \text{(V-10)}$$

where $\hat{H}(\tau) = W_o e^{R\tau} Q_o^{-1}$.

$\hat{H}$ can be written in the matrix form

$$\begin{aligned} \hat{H}(\tau) &= W_o e^{R\tau} Q_o^{-1} \\ &= W_o \sum_{i=1}^{2n} e^{\lambda_i \tau} S_i Q_o^{-1} \\ &= \sum_{i=1}^{2n} e^{\lambda_i \tau} W_o S_i Q_o^{-1} \end{aligned}$$

Finally combining (V-10) with the induced matrix norm

$$||\hat{H}(\tau)|| = \max_i [\sum_j |\hat{H}_{ij}(\tau)|] \qquad \text{(V-11)}$$

we obtain

$$\| W_o \Psi Q_o^{-1} \| \le \int_0^\infty [\max_i \sum_j | \hat{H}_{ij}(t) |] \, dt . \qquad \square$$

Consequently the weighted operator norm $\| W_o \Psi Q_o^{-1} \|$ is bounded above as follows.

$$\| W_o \Psi Q_o^{-1} \| \le \int_0^\infty [\max_i \sum_j | \hat{H}_{ij}(t) |] \, dt \qquad \text{(V-12)}$$

where $$\hat{H}(t) = \frac{\sum_{i=1}^{2n} e^{\lambda_i t} \; W_o \, [\prod_{j \ne i}^{2n} (R - \lambda_j I) \, Q_o^{-1}}{\prod_{j \ne i}^{2n} (\lambda_i - \lambda_j)}$$

It must be noted that the formula (V-12) is valid only when the eigenvalues λ_i, $i = 1 \cdots 2n$, of the matrix R are distinct. This suffices for our purposes since in Chapter IV it was argued that the optimal pole locations were of a Butterworth form which does not allow repeated eigenvalues. Next the true norm of the operator, is developed in the following theorem.

Theorem 3 : Let $\Psi : L_\infty^{2n} [0, \infty) \to L_\infty^{2n} [0 , \infty)$ be the linear operator given by

$$(\Psi z)(t) = \int_0^t H(t - \tau) \; z(\tau) \, d\tau$$

and let each element H_{ij} of the matrix H be in $L_1[0, \infty)$.

Then, $$\| \Psi \| = \max_i \int_0^\infty \sum_j | H_{ij}(\tau) | \, d\tau .$$

Proof of Theorem 3 : This proof consists of two parts. The first part is to show the norm $\| \Psi \|$ is bounded from above by

$$\max_i \int_0^\infty \sum_j | H_{ij}(\tau) | d\tau ,$$

and the second is to show that $\| \Psi \|$ is bounded from below by

$$\max_i \int_0^\infty \sum_j | H_{ij}(\tau) | d\tau .$$

From Theorem 1, we know that

$$\| \Psi \| \leq \max_i \int_0^\infty \sum_j | H_{ij}(\tau) | d\tau . \qquad \text{(V-13)}$$

So it suffices to show that

$$\| \Psi \| \geq \max_i \int_0^\infty \sum_j | H_{ij}(\tau) | d\tau .$$

Consider $$g_i(t) = \int_0^t \sum_j H_{ij}(t - \tau)\, z_j(\tau)\, d\tau \qquad \text{(V-14)}$$

where the index i is arbitrary but fixed.

Next, define $$z_{j,t_0}(t) \overset{\Delta}{=} \mathrm{Sgn}\, [\, H_{ij}(t_0 - t)] \qquad \text{(V-15)}$$

and $\| z_{j,\, t_0}(t) \| = \max_j \| z_{j,\, t_0}(t) \| = 1$

where t_0 is arbitrary but fixed.

From (V-14) and (V-15) , we obtain

$$\begin{aligned} g_i(t_0) &= \int_0^t \sum_j | H_{ij}(t_0 - \tau)\, z_j(\tau) | d\tau \\ &= | g_i(t_0) | \end{aligned}$$

but $| g_i(t_0) | \leq \| H_{ij} \| \leq \| g \|_{L_\infty^{2n}}$

Therefore, $\int_0^{t_0} \sum_j \mid H_{ij}(t_0 - \tau) \mid d\tau \le \dfrac{\| g \|}{\| z \|}$

$$= \| \Psi \| ,$$

which yields $\int_0^{t_0} \sum_j \mid H_{ij}(\tau) \mid d\tau \quad \le \| \Psi \|$ (V-16)

Since t_0 in (V-16) is arbitrary, it can be written as

$$\int_0^{\infty} \sum_j \mid H_{ij}(\tau) \mid d\tau \quad \le \| \Psi \|$$

and $\max_i \int_0^{\infty} \sum_j \mid H_{ij}(\tau) \mid d\tau \quad \le \| \Psi \|$ (V-17)

Finally, by (V-13) and (V-17), it follow that

$$\| \Psi \| = \max_i \int_0^{\infty} \sum_j \mid H_{ij}(\tau) \mid d\tau$$

This completes the proof of Theorem 3. □

With Theorem 3 and (V-12), the operator norm $\| W_o \Psi Q_o \|$ is defined as follows :

$$\| W_o \Psi Q_o \| = \max_i \int_0^{\infty} [\sum_j \mid \hat{H}_{ij}(t) \mid dt] \tag{V-18}$$

where $\hat{H}(t) = \dfrac{\sum_{i=1}^{2n} e^{\lambda_i t} \; W_o \; [\prod_{j \ne i}^{2n} (R - \lambda_j \; I)] \; Q_o^{-1}}{\prod_{j \ne i}^{2n} (\lambda_i - \lambda_j)}$

Now we develop a lower bound of the norm $\| W_o \Psi Q_o^{-1} \|$. From equation (V-18),

$$|| W_o \Psi Q_o^{-1} || = \max_i \int_0^\infty [\sum_j | \hat{H}_{ij}(t) | dt]$$

$$\geq \max_i \int_0^\infty [| \sum_j \hat{H}_{ij}(t) | dt]$$

$$\geq \max_i | \int_0^\infty \sum_j \hat{H}_{ij}(t) dt] |$$

Hence, the norm $|| W_o \Psi Q_o^{-1} ||$ is bounded from above and below according to

$$\max_i | \int_0^\infty \sum_j \hat{H}_{ij}(t) dt] | \leq || W_o \Psi Q_o^{-1} || \leq \frac{\beta_o}{\rho_o\beta_i+\rho_1\beta_w+\rho_2+\rho_3\beta_o} .$$

In the above derivations the operator norm $|| W_o \Psi Q_o^{-1} ||$ is computed in terms of the closed loop eigenvalues of R which makes it clear that $|| W_o \Psi Q_o^{-1} ||$ depends on the assignment of the spectrum of the overall closed loop system. This reemphasizes that in a design context it is useful to know special pole locations in the open left half complex plane which minimize this operator norm. The pole patterns developed in Chapter III provide a means by which such an assessment can be made.

5.2 Design Matrices G and K

In equation (V-18), the norm $|| W_o \Psi Q_o^{-1} ||$ is expressed in terms of the matrix R and its distinct eigenvalues. Thus, it is convenient to select a set of eigenvalues for matrix R first and compute

$$R = \begin{bmatrix} A & B\,K \\ -G\,C & A + B\,K + G\,C \end{bmatrix}$$

in order to evaluate the operator norm. G and K are design matrices of orders n x b and m x n respectively. These design matrices can be computed by exploiting the separation property associated with the linear portion of the overall closed loop system represented by R. That is the characteristic polynomial of the overall system Δ_R is

$$\Delta_R = \Delta_{A+BK} \times \Delta_{A\,+\,GC} \,.$$

The pole placement of R can be viewed as two subproblems since

$$\sigma(\ R\ ) = \sigma(\ A + BK\ ) \cup \sigma(\ A + GC\ )$$

namely,

(i) For given $\sigma(A + BK)$ or $\Delta_{A\,+\,BK}$, compute the controller gain K

(ii) For given $\sigma(A + GC)$ or $\Delta_{A\,+\,GC}$, compute the observer gain G .

To this end, the eigenvalue - eigenvector placement algorithm for MIMO systems given in [35] is employed.

5.2.1 Eigenvalue - Eigenvector Placement Algorithm

Unlike in the SISO systems, specifying the closed loop eigenvalues for MIMO systems does not define a set of unique closed loop feedback gains. Given below is the algorithm [35] used for evaluating the feedback K. By considering an augmented matrix (A + B K) and an associated matrix R_{λ_i} given by

$$R_{\lambda_i} = [\ \lambda_i\ I - A : B\ ],\ \text{for } i = 1 \ldots n$$

we can define a matrix Q_{λ_i} whose columns constitute a basis for the kernel of R_{λ_i} denoted by ker [R_{λ_i}] such that

$$Q_{\lambda_i} = \begin{bmatrix} S_{\lambda_i} \\ T_{\lambda_i} \end{bmatrix}$$

where the partitioned matrices S_{λ_i} and T_{λ_i} are of order n x m and m x m respectively. In the following theorem [35], necessary and sufficient conditions are given for the existence of the feedback matrix K for a given set of distinct eigenvalues.

Theorem 4 : Let λ_i, i = 1 ... n, be a self conjugate set of distinct complex numbers. Then, there exists a matrix K of real numbers such that

$$(A + B K) p_i = \lambda_i p_i , \quad i = 1 \ldots n \tag{V-19}$$

if and only if the following three conditions are satisfied for i = 1, ..., n .

(i) Vectors p_i are linearly independent vectors in $\mathbf{C}^n$

(ii) $p_i = p_j^*$ whenever $\lambda_i = \lambda_j^*$ where * denotes complex conjugate

(iii) $p_i \in$ span $\{ S_{\lambda_i} \}$.

If (i) - (iii) hold and rank B = m, then K is unique.

Proof of Theorem 4 : See Moore [35]

Based on Theorem 4, the following steps provide a systematic way of computing the feedback gain K.

Step 1 : Select a self conjugate set of desired <u>distinct eigenvalues</u> λ_i, i = 1 ... n for the closed loop system, that is $\lambda_i \neq \lambda_j$ if $i \neq j$.

Step 2 : Form the matrix $R_{\lambda_i} = [\ \lambda_i I - A : B\ ]$ of order n x (n + m) for every λ_i, i = 1 ... n .

Step 3 : Find the matrix $Q_{\lambda_i} = [q_1, q_2, \ldots, q_m]$ of order n x (n + m), where q_i, i = 1 ... n , are linearly independent vectors and span the null space of R_{λ_i} , and partition Q_{λ_i} as

$$Q_{\lambda_i} = \begin{bmatrix} S_{\lambda_i} \\ \\ T_{\lambda_i} \end{bmatrix}$$

where S_{λ_i} and T_{λ_i} are of orders n x m and m x m respectively.

Step 4 : Select a corresponding set of closed loop eigenvectors p_i, i = 1, ..., n which meet the three conditions defined in Theorem 4. That is,

(i) Vectors p_i are linearly independent vectors in C^n

(ii) $p_i = p_j^*$ whenever $\lambda_i = \lambda_j^*$ where * denotes complex conjugate

(iii) $p_i \in$ span $\{ S_{\lambda_i} \}$.

Step 5 : Compute a vector $\zeta_i \in \mathbf{C}^m$ such that

$$p_i = S_{\lambda_i} \ \zeta_i$$

and form $w_i = T_{\lambda_i} \ \zeta_i$, for every λ_i , $i = 1 \ldots n$

Step 6 : Form the matrices P_λ and W_λ as

$$P_\lambda = [p_1, \ \ldots \ , p_n] \text{ and } W_\lambda = [w_1, \ \ldots \ , w_n]$$

Step 7 : Compute the feedback gain K

$$K = W_\lambda \ P_\lambda^{-1}$$

Remark : In step 3, to find a basis for the null space of Q_{λ_i} , a systematic procedure is given below.

(i) Form the augmented matrix R_i given by

$$R_i = \begin{bmatrix} \lambda_i I_a - A & : & B \\ & & \\ & I_e & \end{bmatrix}, \text{ for } \lambda_i \ , \ i = 1 \ldots n$$

where R_i is of order $(2n + m) \times (n + m)$, and I_a and I_e are identity matrices of order $n \times n$ and $(n + m) \times (n + m)$ respectively .

(ii) Obtain the m zero columns of $[\ \lambda_i I - A \ : \ B \ ]$ by performing elementary column operations on the matrix R_i.

(iii) The columns below the m zero columns of R_i span the null space R_{λ_i} .

5.2.2. Computation of the Matrix G

In order to compute matrix G the same logic outlined in section 5.2.1 can be used by exploiting the duality of controllability and observability.

Since $$\sigma(A + GC) = \sigma(A + GC)^T = \sigma(A^T + C^T G^T)$$

in Theorem 4, we can replace (V-19) by

$$(A^T + C^T G^T)\, p_i = \lambda_i\, p_i\ .$$

That is, the algorithm outlined in previous section is valid for computation of observer gains G by replacing A by A^T, B by C^T, and K by G^T of order m x m, n x b and b x n respectively.

Based on the eigenvalue - eigenvector placement algorithm outlined above a general computer program is written for evaluating the design matrices G and K , and for forming the matrix R. Finally the operator norm $|| W_o \Psi Q_o^{-1} ||$, for a given set of distinct eigenvalues is computed as outlined earlier. The flow chart of Figure 19 outlines the solution procedure employed for executing the design criteria of the main result of Chapter II. Given below is a brief description of the computer program. The main program **"CONPT"** (<u>C</u>omputation of <u>O</u>perator <u>N</u>orm for <u>P</u>recision <u>T</u>racking) has six high level routines.

Subroutine **EIGSEL** computes the spectra $\sigma(A + BK)$ and $\sigma(A + GC)$ based on the optimal pole configuration developed in Chapter IV.

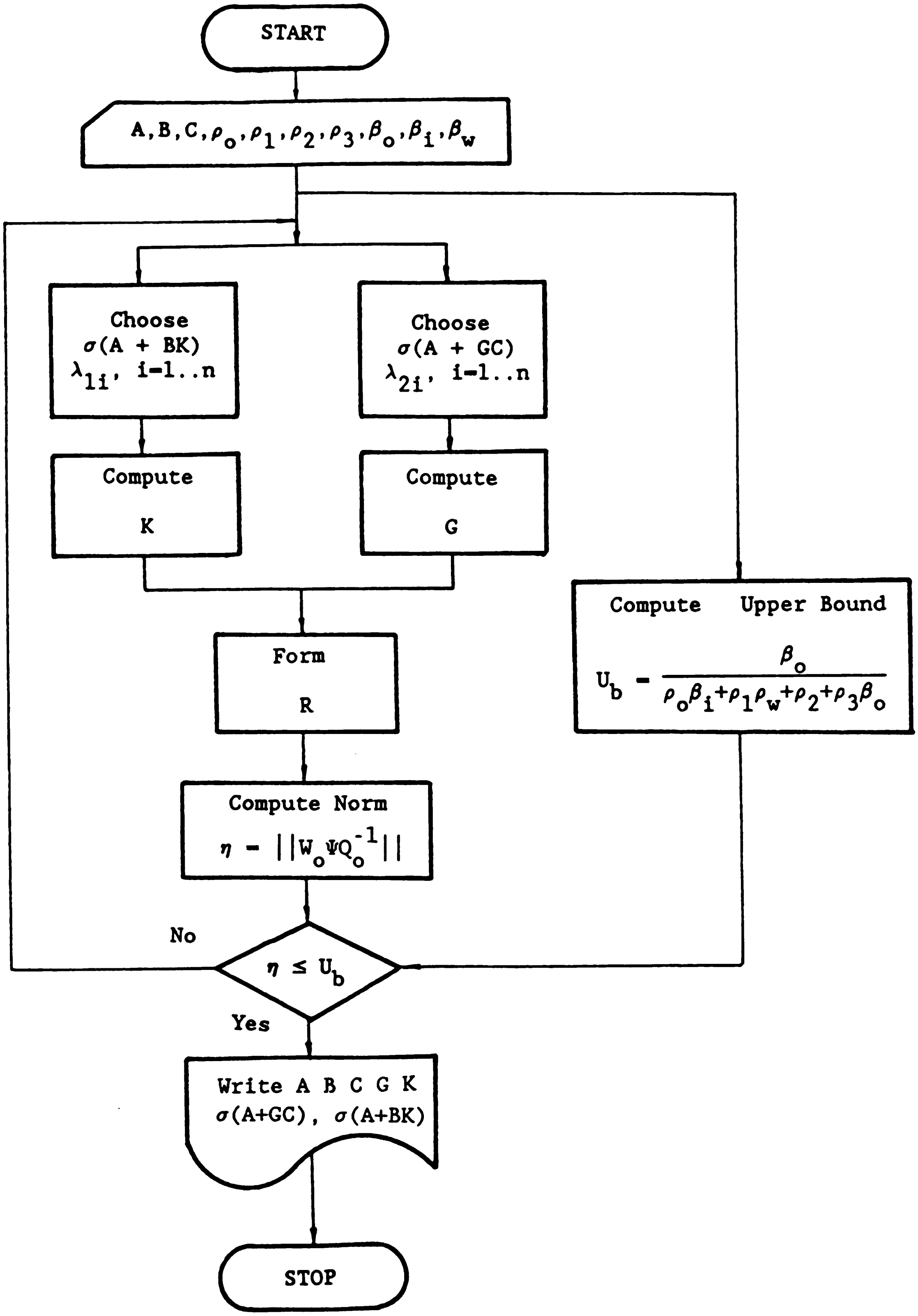


Figure 19 Flow chart

Subroutine **SPHERE** calculates the upper bound $\dfrac{\beta_o}{\rho_o\beta_i+\rho_1\beta_w+\rho_2+\rho_3\beta_o}$

on the operator norm $|| W_o \Psi Q_o^{-1} ||$.

Subroutine **PLFEED** calculates the feedback gain matrices K and G.

Subroutine **EIGCC** computes the eigenvalues of the augmented matrix R to validate the accuracy of K and G.

Subroutine **PROJ** computes the projection given by

$$e^{Rt} = \frac{\sum_{i=1}^{2n} e^{\lambda_i t} \; W_o \; [\prod_{j\neq i}^{2n} (R - \lambda_j I) \; Q_o^{-1}}{\prod_{j\neq i}^{2n} (\lambda_i - \lambda_j)}$$

Subroutine **OPNORM** computes the operator norm by employing the Trapezoidal rule for integration [10].

5.3 Numerical Integration Scheme

The operator norm represented by the integral

$$I = \max_i \int_0^\infty [\sum_j | H_{ij}(t) |] \, dt \qquad \text{(V-20)}$$

poses three difficulties. The first is the range of integration, which is infinite; the second is the oscillatory nature of the integrand due to complex conjugate pairs of eigenvalues; and the third difficulty is the stiffness of the set of ODE , i.e., the eigenvalues of the system matrix may be widely separated. If the equations are stiff, then very small time steps need to be used for integration when standard

algorithms such as Simpson's scheme or the trapezoidal integration scheme are employed.

(i) Infinite range of integration : In order to evaluate the integral it is necessary to write

$$I = \int_0^\infty = \int_0^{T_f} + \int_{T_f}^\infty$$

where the time interval $[0, \infty)$ is truncated at T_f so that the contribution of $\int_{T_f}^\infty$ is negligible. In order to estimate T_f , the settling time for a first or second order dominant pole may be used depending on whether the slowest mode is due to a real pole or a complex conjugate pair. The following conservative estimate for T_f is used.

$$T_f = \frac{5}{\max \{ \operatorname{Re} (\lambda_i) \}}$$

Then the integration is carried in two parts, i.e.,

$$I \overset{\Delta}{=} \int_0^{T_f} + \int_{T_f}^{1.5\ T_f} .$$

If the integral $\int_{T_f}^{1.5\ T_f}$ is very small, I is taken to be $I = \int_0^{T_f}$, however, if $\int_{T_f}^{1.5\ T_f}$ is not negligible, then the remaining interval is again divided into subintervals , until the contribution from the tail end of the integral is insignificant.

(ii) Oscillatory nature of the integrand : This creates a major problem, when using the standard schemes mentioned above. In particular, if only a few function values are considered, or a large time step is used, the integrand sometimes appears to be quite a different function than what it actually is. Thus, the usage of small time steps is inevitable with these schemes, which is aggravated if the eigenvalues have large oscillation frequencies ω. A time step T_s such that

$$T_s \leq \left(\frac{2\pi}{\omega} \right) \left(\frac{1}{20} \right)$$

is typically used when standard integration schemes are employed.

Here we use the time step T_s,

$$\begin{aligned} T_s &= 0.001 && \text{if} \quad \left(\frac{2\pi}{\omega} \right) \left(\frac{1}{20} \right) \geq 0.001 \\ &= \left(\frac{2\pi}{\omega} \right) \left(\frac{1}{20} \right) && \text{if} \quad \left(\frac{2\pi}{\omega} \right) \left(\frac{1}{20} \right) \leq 0.001 \quad . \end{aligned}$$

Remark : To overcome this difficulty, the subprogram **DCADRE** of the IMSL package that is based on adaptive integration [11] may be used. Here, however, due to some programming difficulties in linking DCADRE to our main program **CONPT**, the trapezoidal rule is employed, thus making small time steps inevitable for accurate results.

(iii) Stiffness of the Equations : The eigenvalues of R play a crucial role in evaluating the integral given by (V-20). For example, if we let λ_ℓ and λ_s be two eigenvalues of R such that

$$\lambda_\ell = \max \{ \mid Re(\lambda_i) \mid \; \mid \lambda_i \in \sigma(R) \; , \; i = 1 \; .... \; 2n \}$$

$$\lambda_s = \min \{ \mid Re(\lambda_i) \mid \; \mid \lambda_i \in \sigma(R) \; , \; i = 1 \; .... \; 2n \}$$

then the numerical integration must be carried out until the slowest decaying exponential in the transient part, (i.e., the one corresponding to $e^{\lambda_s t}$) is negligible. Thus the smaller the $| Re \lambda_s |$, the longer will be the range of integration. On the other hand, if λ_ℓ is far out into the left half complex plane, then the usage of excessively small time steps are prompted. If $| Re \lambda_\ell | \gg | Re \lambda_s |$, then the highly undesirable computational situation occurs, that is integration over a long range using a small time step which is everywhere excessively small relative to the interval. This suggests a procedure for selecting the eigenvalues of the closed loop system to avoid numerical problems. Namely, the eigenvalues for the closed loop system may be chosen to be in one or two clusters where the separation is small enough.

CHAPTER VI

APPLICATIONS

In this chapter, we give several examples to illustrate the applicability of the theory developed in Chapter II. These examples include a synchronous machine, tracking in a robot manipulator problem, and a single DOF gyroscope. In a synchronous machine only uncertainty in the input disturbance is considered, whereas in the robot example uncertainties in both the input and disturbances are allowed. The gyroscope problem has a sector bounded nonlinearity and input disturbance.

6.1 A Synchronous Machine

A synchronous machine with a conventional velocity governor connected to an infinite bus as shown in Figure 20, is considered. The dynamic equations of this machine represented in state space form are

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{P}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \dfrac{-C_e}{M} & \dfrac{-1}{M} \\ 0 & \dfrac{K_g}{\omega_o T_g} & \dfrac{-1}{T_g} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ P_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ f(\theta) \\ 0 \end{bmatrix} \tag{VI-1a}$$

$$\theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ P_m \end{bmatrix} \tag{VI-1b}$$

where θ : rotor angle

$\dot{\theta}$: rotor angular velocity

P_m: input power (p.u.)

C_e: machine damping coefficient (p.u. power second per electrical radian)

M : generator inertia constant (p.u. power second^2 per electrical radian)

T_g: equivalent time constant (p.u. second)

K_g: loop gain of governor system

$$f(\theta) = \frac{-1}{M} \{ 0.317 \sin(\theta+63.7) - 0.035 \sin 2(\theta+63.7) - 0.15 \}$$

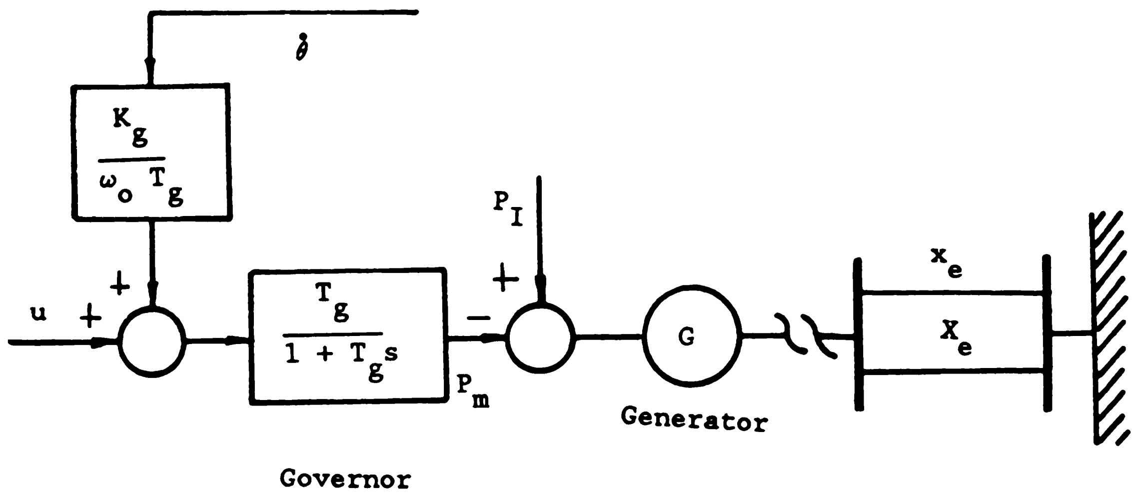


Figure 20 A synchronous machine

To obtain numerical results the following parameter values given in [53] are used. They are :

$M = 0.0138$

$C_e = 0.0138$

$T_g = 0.1$

$K_g = 1.0$

$\omega_o = 120 \pi$.

With these parameter values the equation (VI-1) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -72.464 \\ 0 & 0.027 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ f(x) \\ 0 \end{bmatrix} \tag{VI-2a}$$

$$y = [\ 1 \quad 0 \quad 0\ ] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{VI-2b}$$

where $x = [\theta, \dot{\theta}, P_m]^T \in \mathbf{R}^3$, $u \in \mathbf{R}^1$, $y \in \mathbf{R}^1$, and

$$f(x) = -9.928 \sin(x_1 + 63.7) + 2.536 \sin 2(x_1 + 63.7) + 10.915 \ .$$

The problem is to achieve a specified tracking performance for the rotor angle θ against an input disturbance.

Let the desired behaviour of the rotor angle θ be given by

$$y_o = 25 \ (\ 1. - \cos 2t \ )$$

with the acceptable error bound $\pm$ 1.0 % . This gives the output sphere specification

$$| \ y - y_o | \le 0.5 \ , \text{ or } \beta_o = 0.5 \ .$$

Next, let the input sphere specification be

$$| \ u - u_o \ | \le 1.0 \ , \text{ or } \beta_i = 1.0 \ .$$

The weighting matrices chosen primarily to yield favourable norm values are

$$W_o = Q_o = \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$

where $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda_{max}} & 0 \\ 0 & 0 & \frac{1}{\lambda_{max}} \end{bmatrix}$, $\lambda_{max} = \max \{ |\lambda| \mid \lambda \in \sigma(R) \}$,

and 0's are the zero matrices of appropriate dimension.

Design Execution :

It is easy to compute

$$\rho_o = \frac{1}{\lambda_{max}}$$

and $\rho_1 = 0$.

Due to the absence of any uncertainty in the nonlinear term, the nonlinear design function can be chosen as

$$f_o(x) = f(x)$$

yielding $N_\gamma z_o = N_o z_o$.

This gives

$$\rho_2 = \sup_{\gamma \in \Gamma} || N_\gamma z_o - N_o z_o ||$$

$$= 0$$

ρ_3 can be computed by finding the maximum gradient of f(x) with respect to each state variable. Thus,

$$\rho_3 = \max \{ \nabla_x f(x) \}$$

$$= \max \{ 0., |-9.928 \cos(x_1+63.7)+4.114 \cos 2(x_1+63.7)|. 0 \}$$

$$= 15.0$$

Now with the above norm values, the upper bound given by (II-16) is

$$\frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} = \frac{0.5}{\frac{1}{\lambda_{max}} + (0.5)(15.0)} \qquad \text{(VI-3)}$$

That is if a set of system eigenvalues is chosen so that the operator norm $\| W_o \Psi Q_o \|$ is less than the upper bound given in (VI-3), the conditions of Theorem 1 of Chapter II are satisfied.

Based on the pole configurations of Chapter IV and algorithms of Chapter V, we obtain the set of system eigenvalues

$$\sigma(A + BK) = \{ -90. , \quad -45. \pm j\, 77.94 \}$$

yielding $K = [\; 10059.73 \quad 221.057 \quad -169.0 \;]$

and $\sigma(A + GC) = \{ -150. \quad -75. \pm j\, 129.90 \}$

yielding $G = [\; -289.0 \quad -41808.05 \quad 40755.61 \;]^T$

With the set of spectra above, we obtain the threshold

$$\frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} = 0.066$$

and the critical value of the operator norm

$$\| W_o \Psi Q_o \| = 0.064$$

which satisfies the inequality.

Next, with the above feedback gain K the nominal command input $r_o(t)$ for the closed loop system is computed. It is

$$r_o(t) = -251493.017 + 251244.69 \cos 2t - 11174.955 \sin 2t$$

Computer Simulation : The augmented system for computer simulation is

Plant :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -72.464 \\ 0 & 0.027 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ f(x) \\ 0 \end{bmatrix}$$

$$y = [\; 1 \quad 0 \quad 0 \;] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $f(x) = -9.928 \sin(x_1+63.7) + 2.536 \sin 2(x_1+63.7) + 10.915$

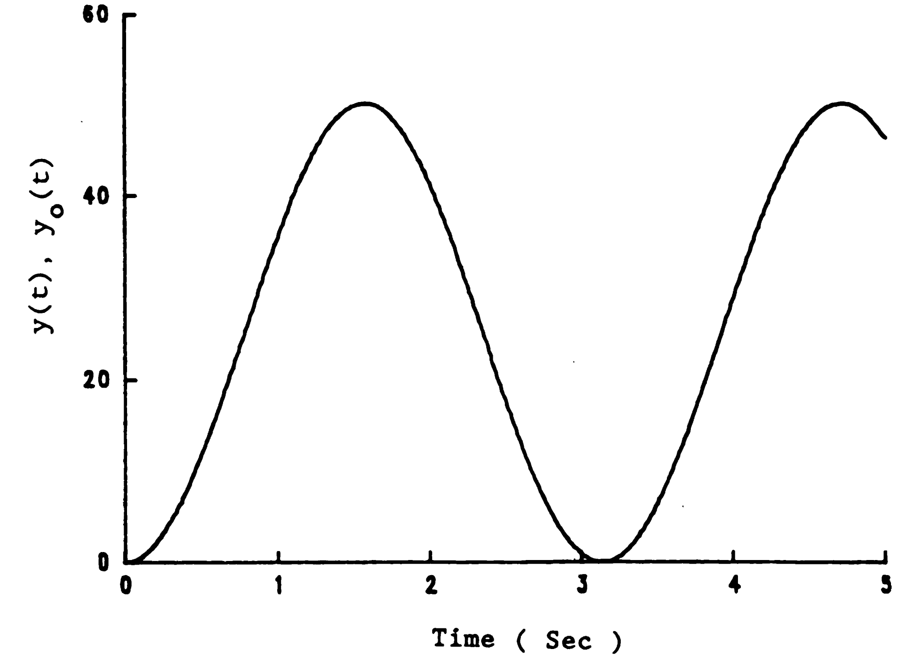


Figure 21 Actual output y(t) and nominal output $y_o(t)$

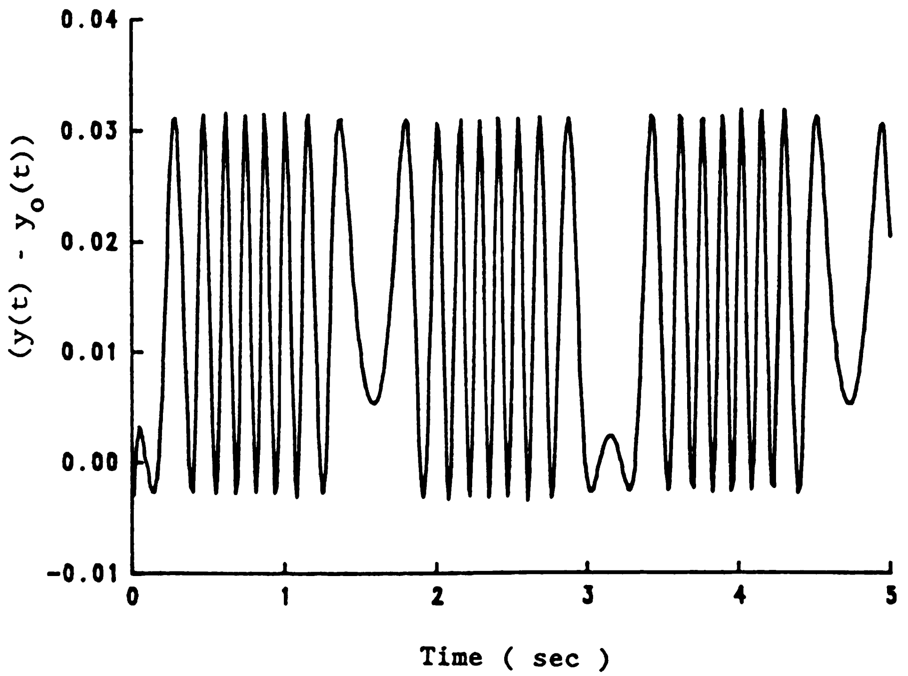


Figure 22 Output deviation $(y(t) - y_o(t))$

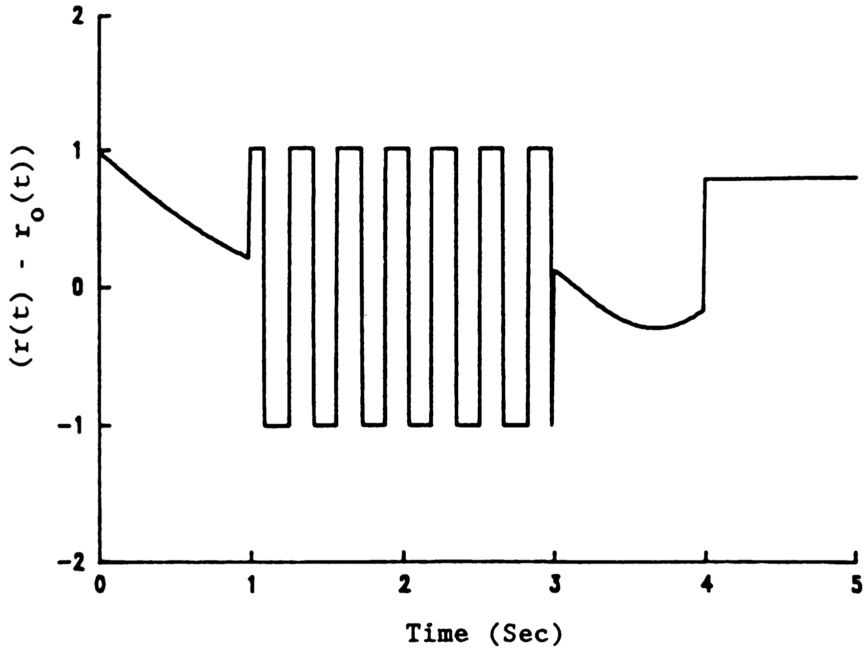


Figure 23 Input disturbance $(r(t) - r_o(t))$

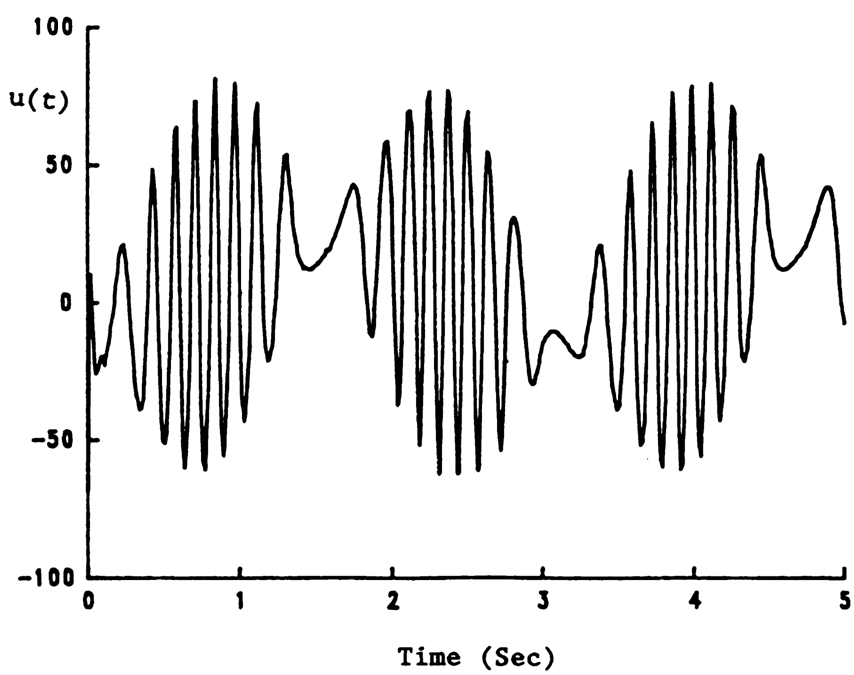


Figure 24 Control effort u(t)

Observer :

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -72.464 \\ 0 & 0.027 & -10 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ f(x) \\ 0 \end{bmatrix} + G\,[1 \quad 0 \quad 0]\,[\hat{x} - x]$$

where $\hat{x} \in \mathbf{R}^3$ and

$$G = [\ -289.0 \qquad -41808.06 \qquad 40755.61\ ]^T.$$

Control : $u(t) = r(t) + K\,\hat{x}(t)$

$$K = [\ 10059.73 \qquad 221.057 \qquad -169.0\ ]$$

with the initial conditions $x(0) = \hat{x}(0) = 0 \in \mathbf{R}^3$.

The time responses of the system are shown in Figure (21 - 24). Figure 21 shows the actual output $y(t)$ and the nominal output $y_o(t)$ to a square pulse input disturbance shown in Figure 23. Figure 24 shows the control $u(t)$. The error $(y - y_o)$ shown in Figure 22 is within the given tolerance for all time, and it is much less than the desired error bound 0.5. Although the output sphere specification has been satisfied it is clear that this scheme is quite conservative. This is primarily due to the extent of the disturbance that can be allowed within the specifications. Namely, an infinite number of admissible functions in $L_\infty[0, \infty)$ are allowed for the disturbances.

6.2 A 3 DOF Manipulator

We consider the three DOF manipulator shown in Figure 25. This manipulator has a rotational joint and a translational joint in the (x, y) plane. Moreover the arm can be lifted along the vertical z-axis thus defining the third degree of freedom. The dynamic equations for

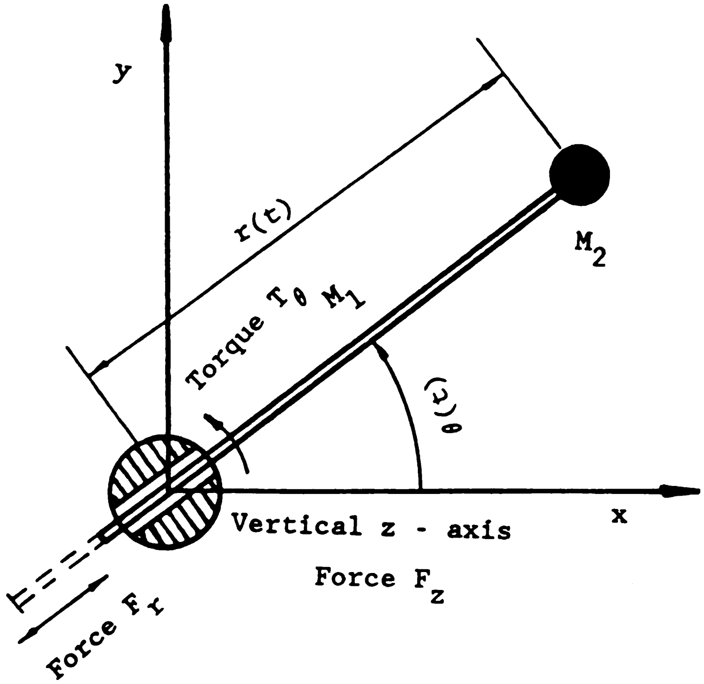


Figure 25 A three DOF robot manipulator

this robot configuration follow directly from an application of Lagrange's equations and take the following form [18]

$$M(\psi(t), \gamma)\ \ddot{\psi}(t) = -f(\psi(t), \dot{\psi}(t), \gamma) + u(t) \tag{VI-4}$$

where $\psi(t) = [r(t), \theta(t), z(t)]^T$ specifies the configuration at time t in a cylindrical frame of reference, γ is the payload uncertainty and the dot denotes time derivatives. u(t) represents the generalized forces and is given by

$$(t) = [F_r, T_\theta, F_z]^T$$

where F_r is the radial force, T_θ is the torque and F_z is the vertical force associated with the coordinates r, θ, and z respectively. $M(\psi(t),\gamma)$ and $f(\psi(t),\psi(t),\gamma)$ are given below.

$$M(\psi(t),\gamma)=\begin{bmatrix}(M_2+M_1) & 0 & 0\\ 0 & (M_1+M_2)\, r^2(t)-M_1\,\ell\, r(t)+J & 0\\ 0 & 0 & (M_1+M_2)\end{bmatrix} \quad \text{(VI-5a)}$$

$$f(\psi(t),\dot\psi(t),\gamma)=\begin{bmatrix}\left(-(M_1+M_2)\, r(t)+\frac{1}{2}\, M_1\,\ell\right)\dot\theta^2(t)\\ \left(2(M_1+M_2)\, r(t)-M_1\,\ell\right)\dot r(t)\,\dot\theta(t)\\ 0\end{bmatrix} \quad \text{(VI-5b)}$$

where M_1 and M_2 are the arm mass and the payload mass respectively, and ℓ is the length of the arm AB. The net moment of inertia of the arm and the swivel joint J is given by

$$J = J_{M_1} + J_{M_3}$$

$$= \frac{1}{2} M_3\, r_z^2 + \frac{1}{3} M_1\, \ell^2$$

where, J_{M_1} and J_{M_3} are the moments of inertia of the swivel and the arm, respectively, about the z-axis. M_3 and r_z are the mass and the radius of the swivel.

Equations (VI-5) depict a highly coupled nonlinear set of equations. By employing the state dependent transformation

$$u(t) = M(\psi(t),\gamma)\cdot u_t(t) \quad \text{(VI-6)}$$

on the input u(t), the equations of motion (VI-4) are transformed into

$$\ddot\psi(t) = -M(\psi(t),\gamma)\cdot f(\psi(t),\dot\psi(t),\gamma) + u_t(t) \quad \text{(VI-7)}$$

The inertia matrix $M(\psi(t), \gamma)$ is clearly invertible for all $t \in [0, \infty)$, which follows from the positive definiteness of the mass matrix of a manipulator.

Now equations (VI-7) can be rewritten in the usual state space [illegible] yielding

$$\dot{x}(t) = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_t(t) + \begin{bmatrix} 0 \\ f_N(x(t),\gamma) \end{bmatrix} \qquad \text{(VI-8a)}$$

$$y(t) = [\ I_3 \quad 0\ ]\ x(t) \qquad \text{(VI-8b)}$$

where, I_3 and 0 are 3 x 3 the identity and the null matrices respectively,

$$x(t) = \begin{bmatrix} \psi(t) \\ \dot{\psi}(t) \end{bmatrix}$$

$$= [\ r,\ \theta,\ z,\ \dot{r},\ \dot{\theta},\ \dot{z}\ ]^T \ \epsilon\ \mathbf{R}^6,$$

$u_t(t) = M^{-1}(\psi(t),\ \gamma)\ u(t)\ \epsilon\ \mathbf{R}^3$, and the nonlinear term

$$f_N = -\ M^{-1}(\psi(t),\gamma)\cdot f(\psi(t),\dot{\psi}(t),\gamma)$$

$$= \begin{bmatrix} (M_2 + M_1)^{-1} \quad (-(M_1+M_2)\ \ r(t) \ + \ \frac{1}{2}\ M_1\ \ell\ )\ \ \dot{\theta}^2(t) \\ ((M_1 + M_2)\ r^2(t) - M_1\ \ell\ r(t) + J)^{-1}(2(M_1+M_2)\ r(t)\ - M_1\ \ell\ )\dot{r}(t)\dot{\theta}(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_{N1} \\ f_{N2} \\ f_{N3} \end{bmatrix}$$

Equations (VI-8) are decoupled with respect to the linear parts and are used in executing the design procedure previously outlined in Chapter II. This form clearly allows the arbitrary placement of eigenvalues of each decoupled subsystem.

Design Objective :

Our basic design objective is to synthesize a control u(t) in order to achieve the tracking performance specified by the output constraints

$$|| y_i - y_{oi} || \le \beta_{oi}, \quad \beta_{oi} > 0, \; i = 1, 2, 3$$

despite the input disturbance and the payload uncertainty. y_{oi}, i = 1, 2, 3 are the three nominal outputs to be tracked and y_i, i= 1, 2, 3, are the three actual outputs.

Input Sphere :

Let the input sphere be given by

$$\beta_i = 1.0 \quad , i = 1, 2, 3.$$

Nominal Output :

The nominal outputs to be tracked are

$$y_{o1} = 0.8 - 0.8\, e^{-3t}(\cos(t) + 3.\, \sin(t))$$

for the radial displacement of the arm,

$$y_{o2} = t^2\, e^{-t}$$

for the angular rotation of the arm, and

$$y_{o3} = 0.5 - 0.5 \cos(t)$$

for the vertical motion of the arm.

Output Spheres :

Output sphere specifications are

$$\beta_{o1} = \beta_{o2} = \beta_{o3} = 0.1$$

Thus the tracking specifications call for precise tracking of the nominal outputs given above upto an accuracy of 0.1 m in y_1, 0.1 rad in y_2 and 0.1 m in y_3.

Bounded Uncertainty :

We consider the payload M_2 to be the primary uncertainty and assume that

$$M_2 \in \Gamma = [0,\ 20]\ \text{kg}.$$

In order to design a controller as outlined previously, the threshold value specified in equation (II-16) needs to be computed first. This requires the computation of several norm quantities as described in the main theorem of Chapter II. We use the following data for all computations.

$$M_1 = 40\ \text{kg}$$

$$M_2 = [0,\ 20\ ]\ \text{kg}$$

$$M_3 = 100\ \text{kg}$$

$$\ell = 1\ \text{m}$$

$$r(t) = [0.0,\ 1.0\ ]\ \text{m}$$

$$z(t) = [\ 0.0,\ 1.0\ ]\ \text{m}$$

$$\theta(t) = [0.,\ \pi\ ]\ \text{rad}$$

$$r_z = 0.1\ \text{m}.$$

Let the design matrices $V_1 = 0_3$ and $V_2 = I_3$, then

$$B_o = [\ 0_3 \quad I_3 \quad 0_3 \quad I_3\ ]^T$$

The weighting matrices W_o and Q_o chosen primarily to yield favorable norm values are

$$W_o = Q_o = \begin{bmatrix} \Sigma & 0_6 \\ 0_6 & \Sigma \end{bmatrix}$$

where $\Sigma = \begin{bmatrix} I_3 & 0_3 \\ 0_3 & \frac{1}{|\lambda_i|_{max}} I_3 \end{bmatrix}$ and $|\lambda_i|_{max}$ is the maximum absolute value of the eigenvalues of matrix R. I_3 is the 3 x 3 Identity matrix, 0_3 and 0_6 respectively are 3 x 3 and 6 x 6 null matrices.

Remark : The selection of the weighting matrices is rather arbitrary. For example they may be set to the identity matrix. This however will not yield favorable norm values.

Then, $\rho_o = || Q_o B_o ||$

$$= || \begin{bmatrix} \Sigma & 0_6 \\ 0_6 & \Sigma \end{bmatrix} \cdot [\ 0_3 \ \ I_3 \ \ 0_3 \ \ I_3\ ]^T ||$$

$$= \frac{1}{|\lambda_i|_{max}} \quad , \ i = 1,\cdots\cdot,12$$

and $\rho_1 = || Q_o B_1 ||$

$$= 0.$$

since $B_1 = 0$ due to the absence of external disturbances.

To calculate

$$\rho_2 = \sup_{\gamma \epsilon \Gamma} || Q_o(N_\gamma z_o - N_o z_o) ||$$

we need to select a nominal nonlinear function $f_o(x)$ to cancel as much as possible the uncertain effects of f_N. We choose $f_o(x)$ to be of the same form as $f_N(x, \gamma)$ with γ replaced by γ_o, where γ_o are the arithmetic means of the uncertain parameters .

In this case

$$\gamma = M_2 = [0,\ 20]\ \text{kg}$$

thus yielding

$$\gamma_o = \bar{\bar{M}}_2 = \frac{1}{2}(\underline{M}_2 + \bar{M}_2) = 10.\ \text{kg}$$

where $\bar{\bar{M}}_2$, $\underline{M}_2$ and $\bar{M}_2$ respectively are the mean value, lower bound and the upper bound of M_2.

Thus, $f_o(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ 0 \end{bmatrix}$

where $f_1(x) = \left(x_1 - \dfrac{M_1 \ell}{2(M_1 + \bar{\bar{M}}_2)}\right) x_5^2$

and $\quad f_2(x) = \dfrac{-2\,(M_1 + \bar{\bar{M}}_2)\, x_1 - M_1 \ell}{J - M_1 \ell x_1 + (M_1 + \bar{\bar{M}}_2)\, x_1^2}\ x_4\, x_5$

$$\begin{aligned}
\text{Thus,}\quad \rho_2 &= \sup \| Q_o(N_\gamma z_o - N_o z_o) \| \\
&= \sup \left\| \begin{bmatrix} I_3 & 0_3 \\ 0_3 & \dfrac{1}{|\lambda_i|_{max}} I_3 \end{bmatrix} \begin{bmatrix} 0_3 \\ f_N - f_o(x) \end{bmatrix} \right\| \\
&= \max \left\{ \left| \left(\frac{-M_1 \ell}{2(M_1 + M_2)} + \frac{M_1 \ell}{2(M_1 + \bar{\bar{M}}_2)} \right) x_5^2 \right| \frac{1}{|\lambda_i|_{max}}, \right.
\end{aligned}$$

$$\left| \left(\frac{-2(M_1 + M_2)\, x_1 - M_1 \ell}{J - M_1 \ell x_1 + (M_1 + M_2)\, x_1^2} + \frac{2(M_1 + \bar{\bar{M}}_2)\, x_1 + M_2 \ell}{J - M_1 \ell x_1 + (M_1 + \bar{\bar{M}}_2)\, x_1^2} \right) x_4\, x_5 \right|$$

$$\frac{1}{|\lambda_i|_{max}} \Big\}$$

On substitution of numerical values, it follows that

$$\rho_2 = \frac{6.0}{|\lambda_i|_{max}}$$

Computation of ρ_3 involves the calculation of gradients of the nonlinearity with respect to the state vector x, and is given by

$$\rho_3 = \max \{ G_1, G_2 \}$$

where,
$$G_1 = \max \| \nabla_x f_{N1}{}^T \|$$

$$= \max \left\{ |x_5| \frac{1.}{|\lambda_i|_{max}}, \quad \left|2 \left(x_1 - \frac{M_1 \ell}{2(M_1 + M_2)}\right)\right| |x_5| \right\}$$

$$= 1.34$$

$$G_2 = \max \| \nabla_x f_{N2}^T \|$$

$$= \max \left\{ \left| \frac{-2((J-M_1 \ell x_1+(M_1+M_2)x_1^2)(M_1+M_2)+(2(M_1+M_2)x_1+M_1 \ell)(-M_1 \ell+2(M_1+M_2)x_1}{(J - M_1 \ell x_1 + (M_1 + M_2) x_1^2)^2} \right| \cdot \right.$$

$$| x_4 x_5 | \frac{1.}{|\lambda_i|_{max}}, \quad \left| \frac{-2(M_1 + M_2) x_1 - M_1 \ell}{J - M_1 \ell x_1 + (M_1 + M_2) x_1^2} \right| \cdot |x_5| ,$$

$$\left. \left| \frac{-2(M_1 + M_2) x_1 - M_1 \ell}{J - M_1 \ell x_1 + (M_1 + M_2) x_1^2} \right| \cdot |x_4| \right\}$$

$$= 21.0$$

Hence, we obtain $\rho_3 = 21.0$

Remark : In computing ρ_2 and ρ_3 as above it is implicitly assumed that

$$\frac{1}{|\lambda_i|_{max}} < 1.$$ At the end of the design this condition needs to be verified. It will clearly be satisfied in this case.

Now assembling all of the above computations we compute the threshold given in equation (II-16)

$$\frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} = \frac{0.1}{\frac{1.}{|\lambda_i|_{max}} + \frac{6.0}{|\lambda_i|_{max}} + (21.0)(0.1)} \qquad \text{(VI-9)}$$

Now it only remains to find a set of eigenvalues for the system matrix

$$R = \begin{bmatrix} A & B\,K \\ -\,G\,C & A + B\,K + G\,C \end{bmatrix}$$

so that the norm $|| W_o \Psi Q_o^{-1} ||$ is less than the upper bound (VI-9).

Based on the numerical scheme previously outlined, we obtain the spectra

$$\sigma(A + B\,K) = \{ -47.0 \pm j\,49.0\,,\ -50.0 \pm j\,53.0\,,\ -53.0 \pm j\,51.0 \}$$

and

$$\sigma(A + GC) = \{ -110.0 \pm j\,111.0,\ -113.0 \pm j\,114.0,\ -115.0 \pm j\,113.0 \}$$

yielding

$$K = \begin{bmatrix} -4160. & 0. & 0. & -94. & 0. & 0. \\ 0. & -5309. & 0. & 0. & -100. & 0. \\ 0. & 0. & -5410. & 0. & 0. & -106. \end{bmatrix}$$

and

$$G = \begin{bmatrix} -220. & 0. & 0. & -24421. & 0. & 0. \\ 0. & -226. & 0. & 0. & -25765. & 0. \\ 0. & 0. & -230. & 0. & 0. & -25994. \end{bmatrix}^T$$

With the above spectra, we obtain the upper bound

$$\frac{\beta_o}{\rho_o\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_o} = 0.047$$

and the critical norm of the operator

$$||W_o \Psi Q_o^{-1}|| = 0.041$$

which clearly satisfies inequality (II-16).

With K known we can now compute the nominal command input functions $r_{oi}(t)$, $i = 1, 2, 3$, as follows.

$$r_{o1}(t) = 3688. - e^{-3t} (3680. \cos(t) + 10336. \sin(t))$$

$$r_{o2}(t) = e^{-t}(2. + 196. t + 5210. t^2)$$

$$r_{o3}(t) = 2705. - 2704.5 \cos(t) + 53. \sin(t).$$

Thus it follows that the design specified by matrices G, K, the nonlinear function $f_o(x)$ and the nominal inputs r_{oi}, $i = 1, 2, 3$, guarantee the rec[illegible] tracking performance according to the theorem. The validity of th[illegible]rem is also confirmed by simulation results.

Simulation of the Closed Loop System : The system dynamics for simulation are

Plant :

$$\dot{x}(t) = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_t(t) + \begin{bmatrix} 0 \\ f_N(x(t),\gamma) \end{bmatrix}$$

$$y(t) = [\ I_3 \quad 0\ ]\ x(t)$$

Observer :

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_t(t) + \begin{bmatrix} 0 \\ f_o(\hat{x}(t)) \end{bmatrix} + GC\,[\hat{x}(t) - x(t)]$$

where $$G = \begin{bmatrix} -220. & 0. & 0. & -24421. & 0. & 0. \\ 0. & -226. & 0. & 0. & -25765. & 0. \\ 0. & 0. & -230. & 0. & 0. & -25994. \end{bmatrix}^T$$

Control :

$$u_t(t) = r(t) + K\,\hat{x}(t)$$

where $$K = \begin{bmatrix} -4160. & 0. & 0. & -94. & 0. & 0. \\ 0. & -5309. & 0. & 0. & -100. & 0. \\ 0. & 0. & -5410. & 0. & 0. & -106. \end{bmatrix}$$

and the initial conditions $x(0) = \hat{x}(0) = 0. \in \mathbf{R}^6$.

Figure (26 - 29) show simulations for $M_2 = 20$ kg. Figure 26 shows the nominal output y_{o1} and the actual output y_1. There is hardly any difference in the two graphs. This clearly demonstrates the tracking accuracy. Figure 27(a), (b) and (c) show the errors $e_i = y_i - y_{oi}$, $i = 1, 2, 3$, respectively resulting from the input disturbance shown in Figure 29. These errors are of the order of 10^{-3} which is quite conservative in comparison with the imposed output sphere $\beta_o = 0.1$. This conservativeness is not surprising due to the generality of the inputs and the nonlinearity admissible in $L_\infty[0., \infty)$. The required control inputs are shown in Figure 28(a), (b), and (c). Figure (30 - 32) show simulations for a sinusoidally varying uncertainty $M_2 = 10. + 10 \sin(10t)$. Figure 30 shows the nominal output y_{o2} and the actual output y_2 . Figure 31 shows the error $y_2 - y_{o2}$ and the control

input u_2 is shown in Figure 32. The latter uncertainty is considered just for the sake of demonstrating that the methodology is valid for any uncertainty in a given band.

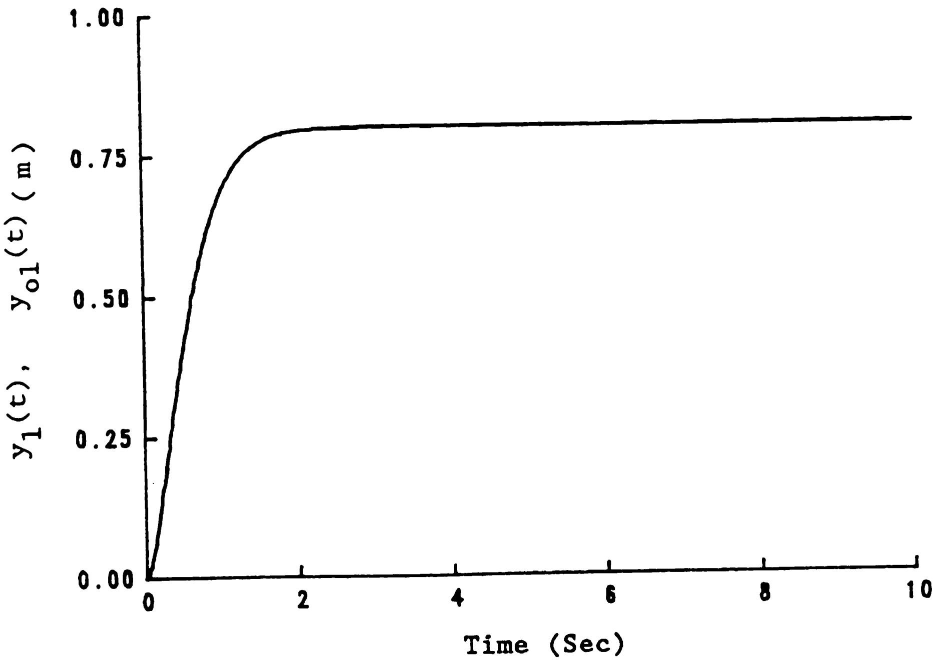


Figure 26 $y_1(t)$ and $y_{o1}(t)$

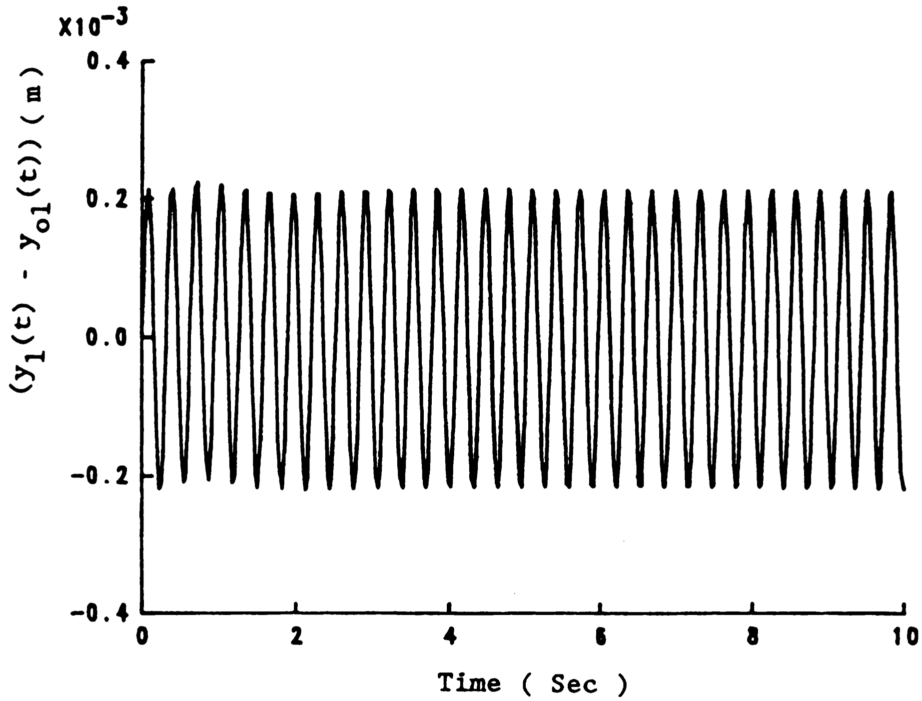


Figure 27(a) Output deviation $(y_1(t) - y_o(t))$

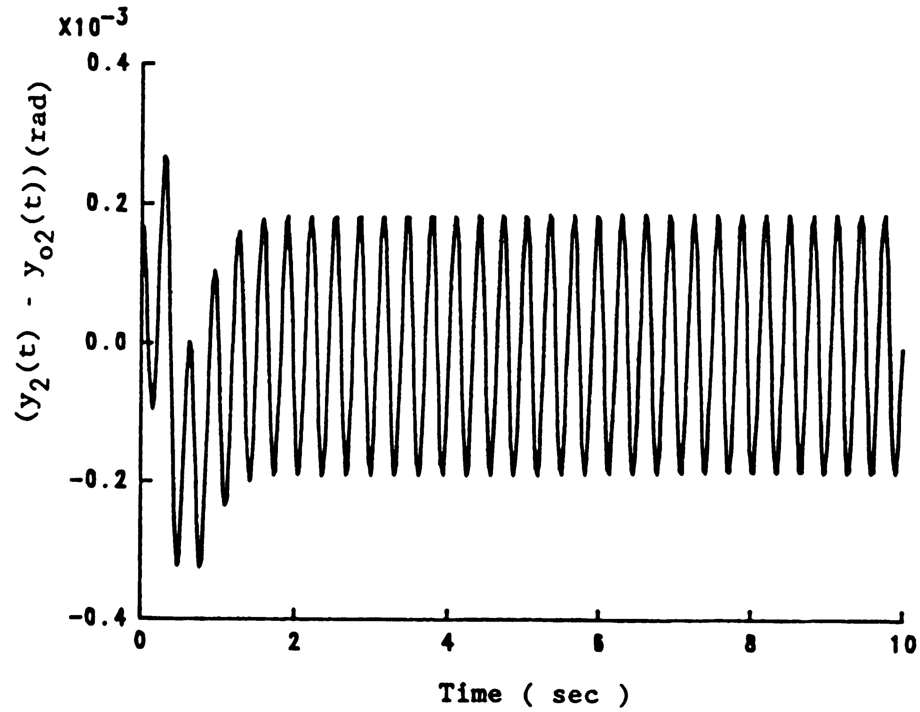


Figure 27(b) Output deviation $(y_2(t) - y_{o2}(t))$

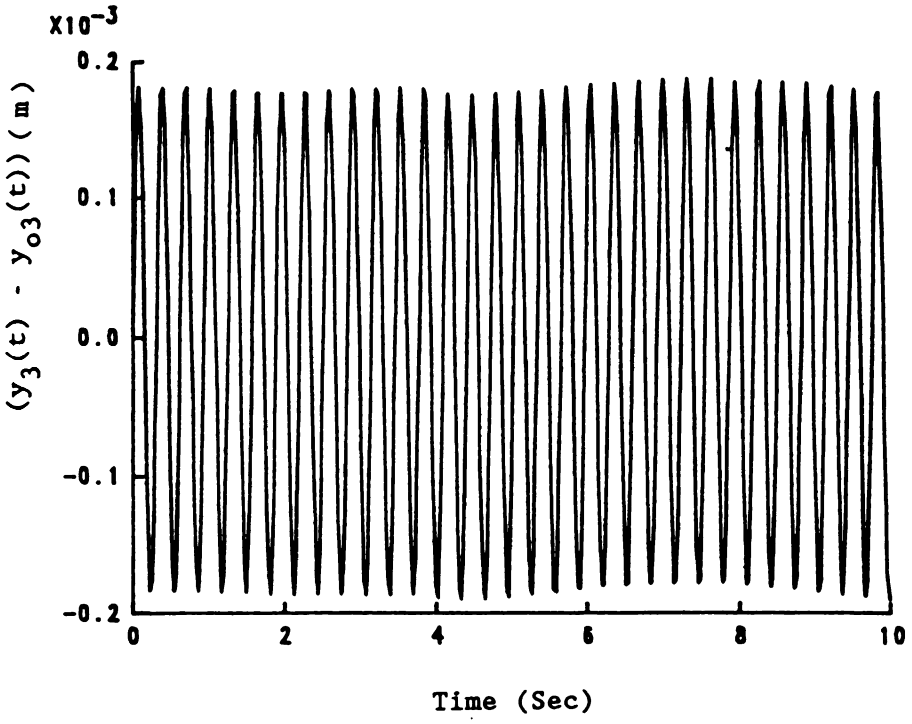


Figure 27(c) Output deviation $(y_3(t) - y_{o3}(t))$

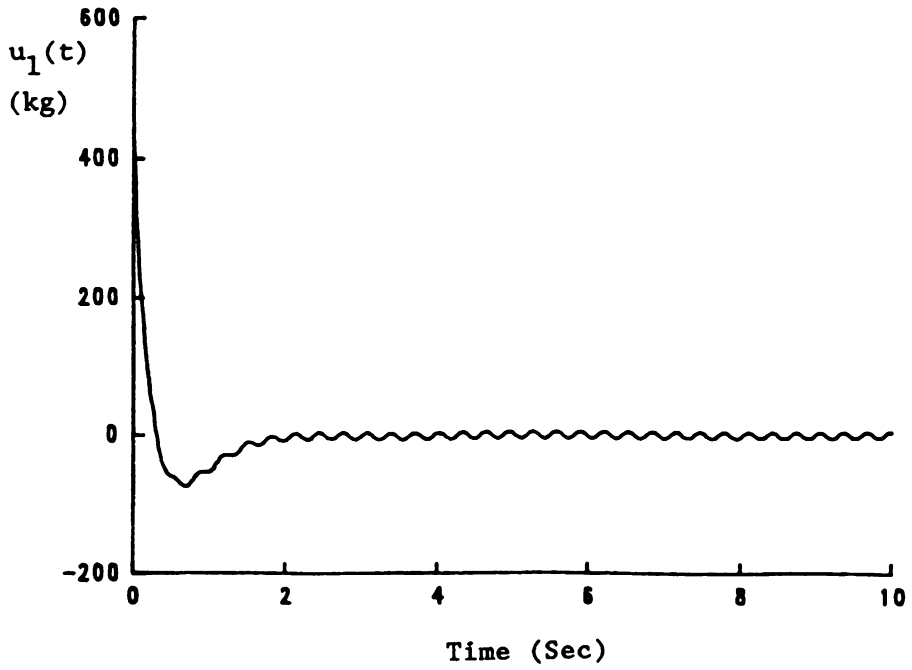


Figure 28(a) Control effort $u_1(t)$

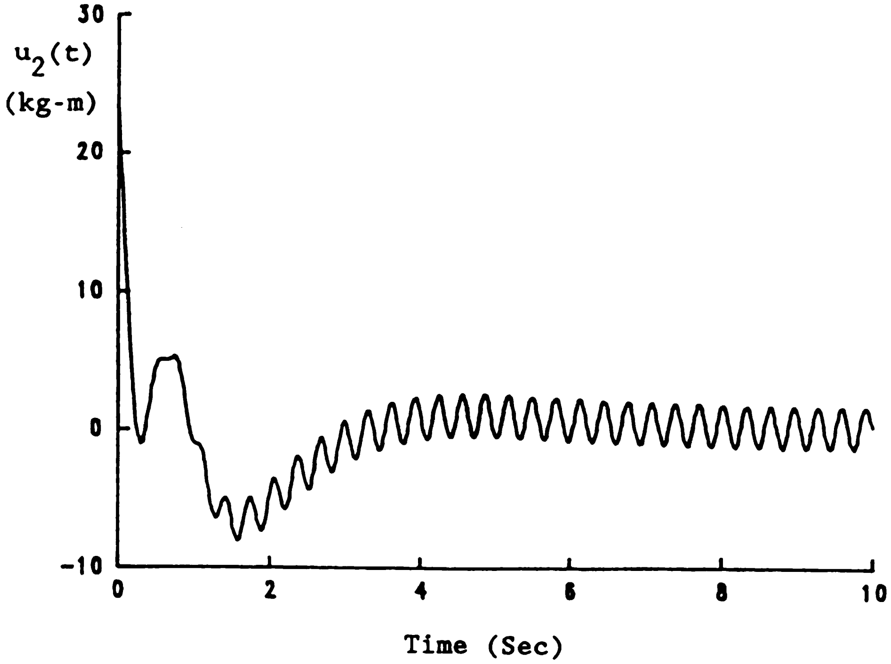


Figure 28(b) Control effort $u_2(t)$

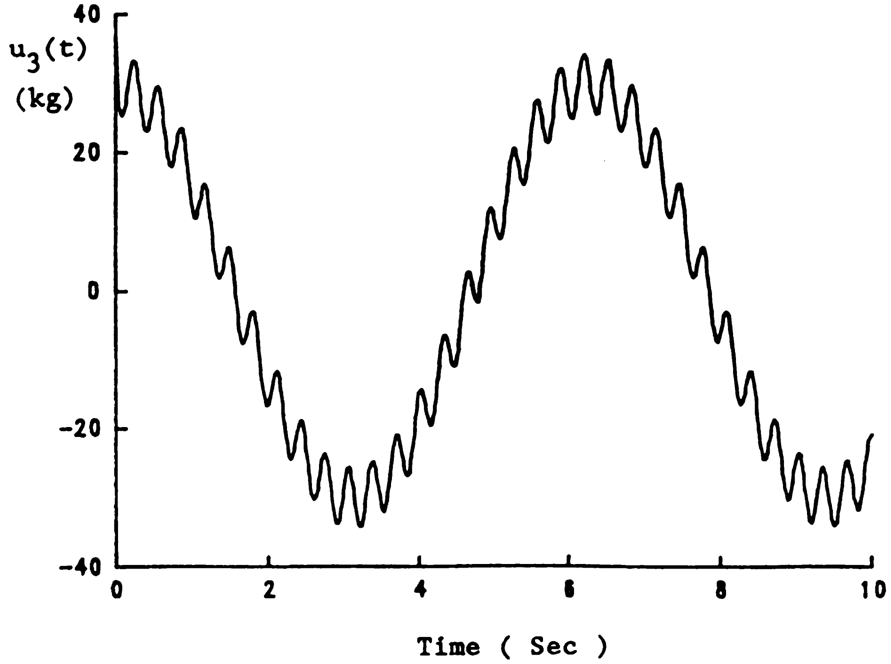


Figure 28(c) Control effort $u_3(t)$

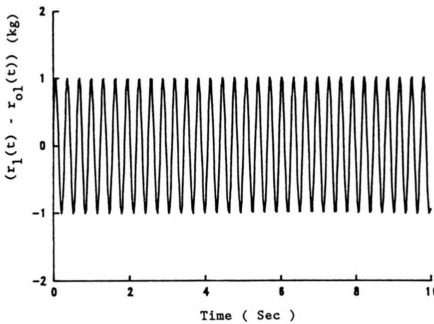


Figure 29 Input disturbance $(r_1(t) - r_{o1}(t))$

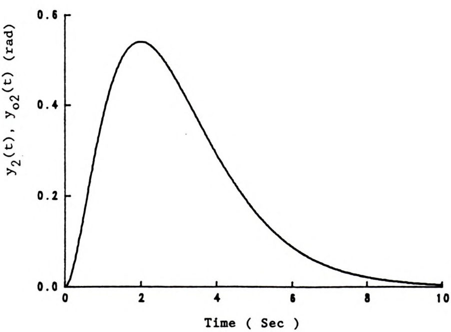


Figure 30 $y_2(t)$ and $y_{o2}(t)$

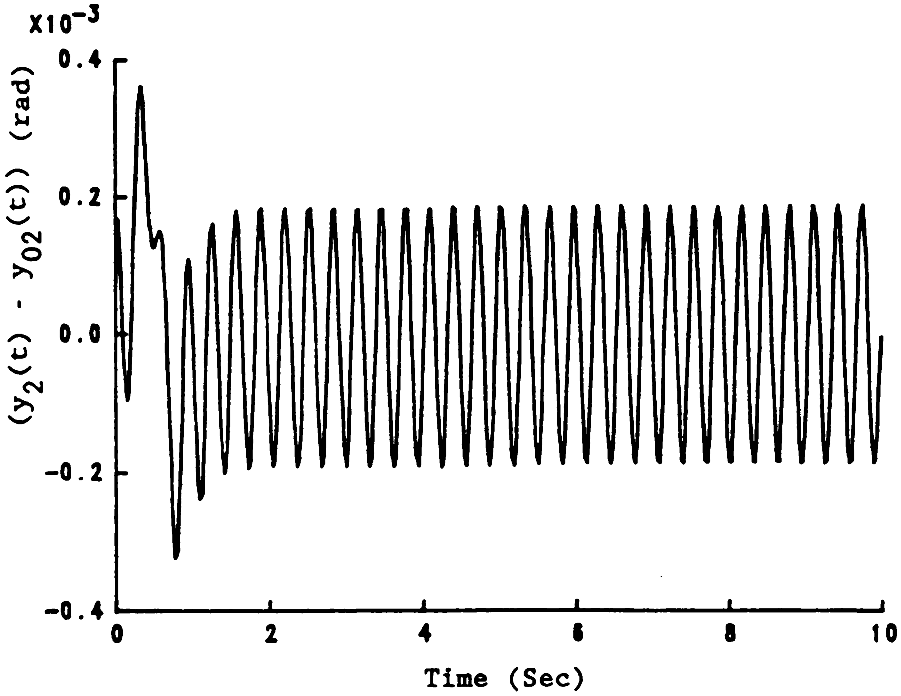


Figure 31 The tracking error $(y_2(t) - y_{o2}(t))$

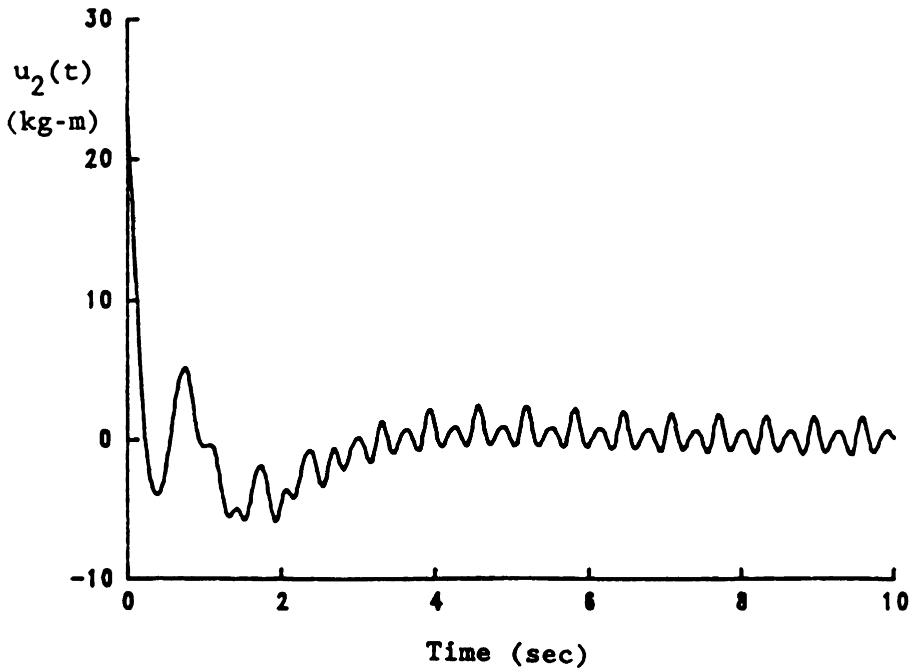


Figure 32 Control effort $u_2(t)$

6.3 A Single DOF Gyroscope

In this section a structurally rigid model of a gyro as shown in Figure 33 is considered to illustrate the special case of Chapter II.

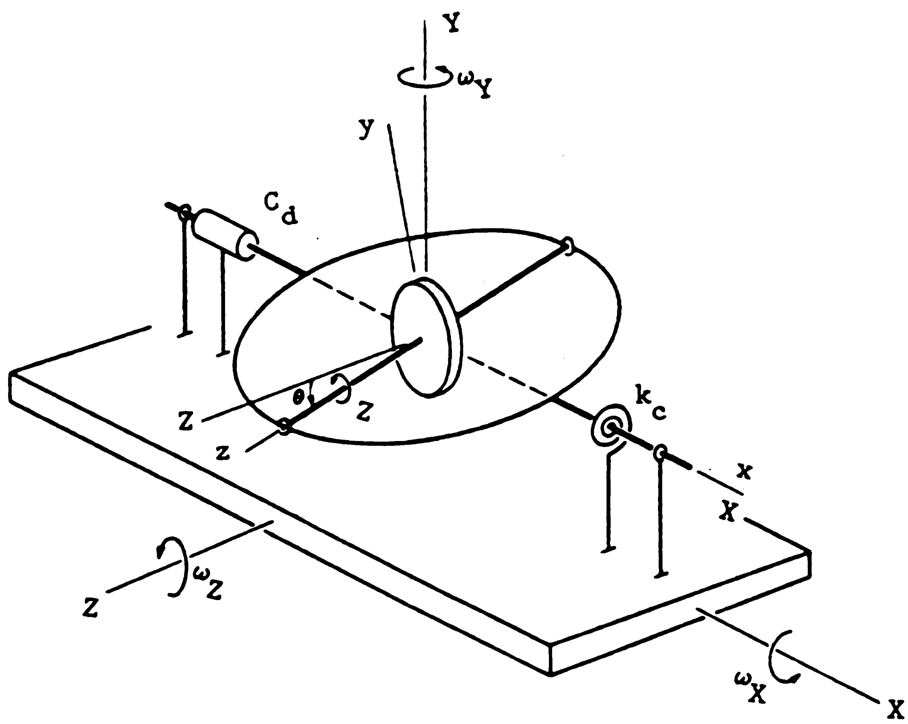


Figure 33 A single DOF gyroscope

Let X, Y, Z be a set of axes attached to the vehicle. The rotor is mounted in a single gimbal, in which it can turn about axes, so that a rotation about that axis leads to the gimbal axis x, y, z. By employing the Lagrangian approach, for small θ and $\frac{d\omega_X}{d\tau} = 0$, the equation of motion for gimbal rotation θ about the output axis is obtained :

$$(J_r + J_g) \frac{d^2\theta}{d\tau^2} + C_d \frac{d\theta}{d\tau} + (K_c + C_n\omega_Z)\, \theta = - C_n\omega_Y \tag{VI-10}$$

where J_r and J_g denote the moments of inertia of rotor and gimbal, respectively, about the axis x, ω_X, ω_Y and ω_Z denote the angular velocity components of vehicle along X, Y, Z, and K_c and C_d represent the torsional spring constant and torsional damping coefficient of the torsional spring and dashpot, respectively. Defining new variables relating the gimbal rotation θ and the time scale τ in (VI-10) by

$$x = (K_c \,/\, C_n)\, \theta$$

$$t = (K_c \,/(J_r + J_g))^{1/2}\, \tau$$

yields $\ddot{x} + 2\,\delta\,\dot{x} + (1 + \gamma)\, x = u$ (VI-11)

where $\cdot$ denotes the derivative of x with respect to t and

$$\gamma = (C_n\, \omega_Z)\, K_c$$

$$\delta = \frac{C_d\, (K_c\, (J_r + J_g))^{-1/2}}{2}$$

$$u = -\,\omega_Y$$

Rearranging equation (V-11) in the state space form yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ \gamma\, x_1 \end{bmatrix} \qquad \text{(VI-12a)}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(VI-12b)}$$

For computer simulation the numerical values of the parameters of the gyro given in [45] are used. They are :

$$(J_r + J_g) = 54 \text{ dyne-cm-sec}^2$$

$$C_n = 10.8 \times 10^4 \text{ dyne-cm-sec}$$

$$K_c = 54 \times 10^4 \text{ dyne-cm-rad}^{-1}$$

$$C_d = 324 \text{ dyne-cm-sec}$$

It is assumed that the uncertain angular velocity ω_Z is bounded, and that the uncertainty $|\gamma| \le 0.2$.

The problem is to determine a control u(t) which guarantees that the system output is within a given tolerance, (that is $|y - y_o| \le \beta_o$), despite the uncertain parameter and the input disturbance.

Nominal output : The nominal output of the gimbal rotation to be tracked is

$$y_o = 0.25 (1 - \cos 3t)$$

Output sphere specification : Let the controller requirement be to maintain the fluctuation in gimbal rotation θ within ± 0.7 % of $y_o(t)$ for all $t \in [0., \infty)$. That is

$$|y - y_o| \le 0.0035 \text{ , or } \beta_o = 0.0035 \text{ .}$$

Input sphere specification : Let the input ω_Y be of uncertainty such that

$$|u - u_o| \le 0.1 \text{ , or } \beta_i = 0.1 \text{ .}$$

The weighting matrices W_o and Q_o which give favourable norm values are

$$W_o = \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}, \text{ and}$$

$$Q_o = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_{max}} \end{bmatrix}$, $\lambda_{max} = \max \{ |\lambda| \mid \lambda \in \sigma(R) \}$, and

I is the identity matrix of order 2 x 2 .

Then $\rho_o = ||Q_o \, B_o|| = 1.0$

Since there is no external disturbance

$$\rho_1 = ||Q_o \, B_1|| = 0.$$

The uncertain nonlinear term

$$f(x, \gamma, t) = \begin{bmatrix} 0 \\ \gamma \, x_1 \end{bmatrix}$$

can be considered as a sector bounded nonlinearity with respect to state x_1 . Since the uncertain element $|\gamma| \leq 0.2$ defines the lower and the upper bounds of a sector bound nonlinearity with $\alpha = -0.2$ and $\beta = 0.2$, the nominal nonlinear function $f_o(x)$ can be chosen as the zero function.

Consequently, $\rho_2 = \frac{1}{2} (\beta - \alpha) \, ||Q_o \, Z_o||$

$=0.15$

and $\rho_3 = \max \{ |\beta|, |\alpha| \}$

$= 0.2$

Now with all of the above computations, the threshold given in (II-26) becomes

$$\frac{\beta_o}{\rho_o \beta_i + \rho_1 \beta_w + \rho_2 + \rho_3 \beta_o} = 0.014$$

Next, solving the pole - placement problem using the algorithms of Chapter IV, we get

$$\sigma(A + BK) = \{ \ -9.0 \pm j \ 9.0 \ \}$$

$$\sigma(A + GC) = \{ \ -18.0 \pm j \ 18.0 \ \}$$

yielding
$$K = [\ -161.0 \qquad -17.94 \ ]$$

$$G = [\ -35.94 \qquad -644.84 \ ]$$

and the critical norm of the operator

$$||W_o \Psi Q_o^{-1}|| = 0.013$$

$$< 0.014$$

satisfies the inequality (II-26).

Now, with K the nominal command input $r_o(t)$ is computed as

$$r_o = 40.5 - 38.25 \cos 3t + 13.5 \sin 3t$$

Computer Simulation : The overall closed loop is

Plant :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ \gamma \ x_1 \end{bmatrix}$$

$$y = [\ 1 \qquad 0 \ ] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $|\gamma| \leq 0.2$

Observer :

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + G [\ 1 \qquad 0 \ ] [\ \hat{x} - x \ ]$$

where $\hat{x} \in \mathbf{R}^2$, and $G = [\ -35.94 \qquad -644.84 \ ]^T$

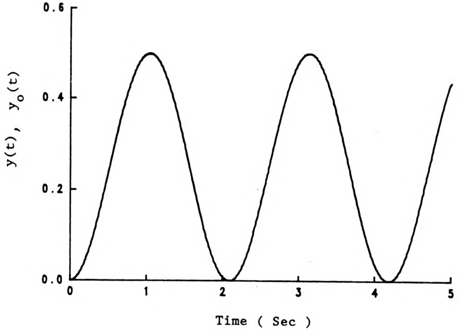


Figure 34 $y(t)$ and $y_o(t)$

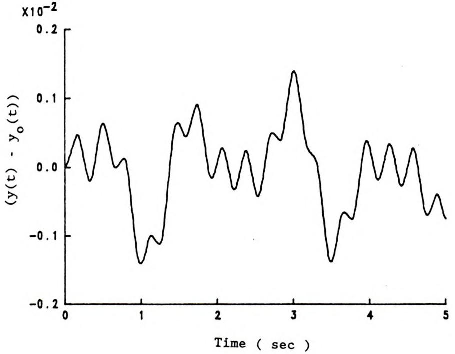


Figure 35 The tracking error $(y(t) - y_o(t))$

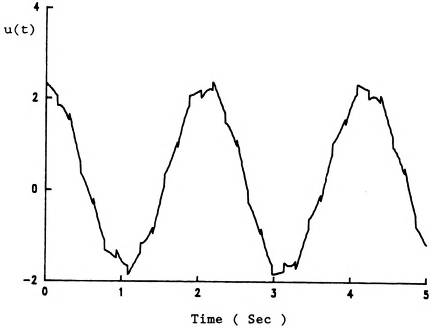


Figure 36 Required control u(t)

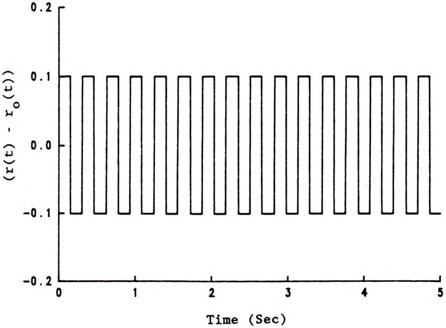


Figure 37 Input uncertainty $(r(t) - r_o(t))$

Control :

$$u(t) = r(t) + K\,\hat{x}(t)$$

where $K = [\ -161.0 \quad -17.94\ ]$

with the initial conditions $x(0) = \hat{x}(0) = 0. \ \epsilon\ \mathbf{R}^2$.

For computer simulation the uncertain element γ is assumed to be a function of time given by $\gamma = 0.2 \sin 5t$ and the input uncertainty is a square pulse with magnitude $\pm\ 0.1$ and in Figure 37. Figures (34 - 36) show the simulation results. The actual output y(t) and the nominal output $y_o(t)$ are shown in Figure 34, and control u(t) is shown in Figure 36. The error $(y - y_o)$ shown in Figure 35 has a maximum of 0.002 rad, and is less than the given tolerance 0.0035 rad.

Remark : It is worth noting that systems with sector bounded non-linearities lend themselves well to straightforward norm calculations as outlined in Chapter II and employed in the above example.

CHAPTER VII

CONCLUSIONS, DISCUSSION AND FUTURE WORK

The research presented in this thesis in a broad sense addressed the issue of "robustness" of control systems. In particular, precision tracking of specified outputs in the presence of uncertain parameters and external disturbance was considered. Specifically there were three objectives for this research :

(a) To study different formulation of the tracking problem for uncertain systems incorporating a more global concept of tracking than is typically considered in asymptotic considerations. This global form allows one to precisely capture the physical nature of tracking.

(b) Consider different controller structures that allow precision tracking and are simple. In particular, investigate a pole-placement technique that is quantitative in nature.

(c) Apply the results of (a) and (b) to the currently active and important area of robotic manipulators.

Given below is a discussion of the contributions made under each objective.

(a) Problem Formulation :

Two basic formulations based on functional analysis of poorly defined systems were studied. In the first formulation given in Chapter II the tracking problem was embedded in the Banach space of essentially bounded functions $L_{\infty}^{k}[0, \infty)$. In this formulation the physical requirements of tracking are quite naturally captured and the necessary design criteria are given in the time domain.

In the second formulation developed in Chapter III the problem was embedded in the Hilbert space $L_2^k(T)$ and frequency domain interpretations were sought for the precision tracking problem. This formulation allows one to capture asymptotic features of tracking with an L_2 bound on the overall error function. A circle type criterion with a transparent geometric interpretation was developed. The latter was possible due to the frequency domain interpretations manifested by L_2 - functions. The latter interpretation however is valid only for SISO systems.

(b) Controller Structure :

The basic controller structure investigated consisted of a non-linear Luenberger observer based state feedback control. This essentially gives rise to a design criterion that requires nothing but a clever way of assigning the spectrum of a linear operator so that a certain upper bound on a crucial operator norm is satisfied. This is what is referred to in here as a "**quantitative pole-placement**".

For the L_{∞} - formulation a design procedure leading to a certain perturbed form of the well known Butterworth pole-patterns was

developed. These were arrived at by setting up a minimization problem in $L_p(T)$ and considering the limit as $p \to \infty$. To obtain the pole patterns a modified Riccatti type equation needs to be solved. To get a quick idea however we start with the Butterworth patterns for ∞ radius and then gradually decrease the radius maintaining the patterns until design criteria are satisfied in an "optimal" sense. Admittedly this approach using Butterworth pole-patterns lacks rigour but accomplishes the task adequately.

It is also interesting to note that the results contained in here seem to suggest that high - gain plays an important role in uncertain problems. It should be pointed out however that the results here provide a rational scheme for selecting such high gains.

(c) Applications :

Several examples including a robotic manipulator were reported in Chapter VI. Algorithms needed for design execution were developed in Chapter V.

The work presented in here together with the "Quantitative Feedback Theory" of Isaac Horowitz [22, 23] constitute the only design theory available to the best of our knowledge for directly satisfying the design specifications for uncertain systems. Despite much recent work on robustness (especially H^{∞} point of view) with significant contributions, direct design for specifications remains incomplete.

The conservative nature of the results obtained thus far remains a major concern. Consideration of time dependent weighting schemes may

prove useful to answer the latter. Future work may also include feedback linearization techniques in the problem formulation. The geometric interpretation given for the SISO sector bounded case appears to be extendable to nonlinearities with more structure, for example, monotone nonlinearities. Different classes of nonlinearities and more structured uncertainties should also be considered in future investigations. Also it seems plausible that frequency domain interpretations leading to transparent design criteria similar to the one obtained for the L_2 - case can be obtained for the L_∞ - tracking problem, by introducing the notion of exponential weighting which is predicated on the two facts given below :

(1) If $y(t) = g(t)*u(t)$

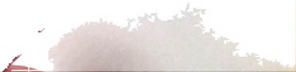
then $\forall a \in \mathbb{R}$

$$y(t)\exp(at) = g(t)\exp(at) * u(t)\exp(at)$$

(2) $\mathcal{L}[f(t)\exp(at)] = \hat{f}(s-a)$ when $\hat{f}(s) = \mathcal{L}[f(t)]$

(* denotes convolution)

To gain additional insight to the problem future studies should also include linear problems with uncertainties for which more explicit results can be anticipated. Recently Vidyasagar [51] reported some interesting results based on factorization methods for a somewhat related problem dealing with linear systems. Specifically, regulation and asymptotic tracking in the presence of persisting disturbances were studied. The notion of persisting disturbances is quite naturally captured when the problem is embedded in the $L_\infty^k[0, \infty)$ space of functions as done in our work. Same thought should also be given to



incorporate the factorization approach to investigate continuous tracking studied in this thesis. Establishing a connection between the L_2 - theory presented in here and H^{∞} methods appears feasible.

APPENDIX A

MATHEMATICAL PRELIMINARIES

The study of nonlinear differential equations in general requires rather sophisticated mathematical tools. The work reported in this thesis is primarily based on properties of linear operators. Some useful definitions are collected in this appendix to give the reader an idea of what is involved. Any standard text on functional analysis such as Rudin (Functional Analysis, McGraw-Hill, 1973) can be consulted for details.

<u>**Definition A1**</u> : A set **X** over a field **F** (**R** or **C** suffices for our purposes) together with two operations + termed addition, and • termed scalar multiplication is a **linear vector space** if the following axioms hold :

(i) $x_1 + x_2 = x_2 + x_1 \quad , \quad \forall \; x_1, x_2 \in \mathbf{X}$

(Commutativity of addition)

(ii) $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3), \quad \forall \; x_1, x_2 \in \mathbf{X}$

(Associativity of addition)

(iii) There is an element $0 \in \mathbf{X}$ such that

$$x + 0 = 0 \quad , \quad \forall \; x \in \mathbf{X}$$

(iv) For each $x \in \mathbf{X}$ there is an element $-x \in \mathbf{X}$ such that

$$x + (-x) = 0 \, , \quad \forall \; x \in \mathbf{X}$$

(v) $\alpha (\beta x) = (\alpha \beta) x$, for each $\alpha, \beta \in \mathbf{F}$ and for each $x \in \mathbf{X}$

(vi) $(\alpha + \beta) x = \alpha x + \beta x$, for each $\alpha, \beta \in \mathbf{F}$ and for each $x \in \mathbf{X}$

(vii) $1 x = x$, for each $x \in \mathbf{X}$

The set of all n-tuples of real numbers denoted by $\mathbf{R}^n$ is a real linear vector space, similarly, $\mathbf{C}^n$ consisting of all n-tuples of complex numbers is a complex linear vector space. In what follows we assume that the field $\mathbf{F}$ is the reals $\mathbf{R}$.

<u>**Definition A2**</u> : A nonempty subset $\mathbf{X}_s$ $\mathbf{X}$ is a **linear subspace** if $\mathbf{X}_s$ is closed under the operations of addition and scalar multiplication in $\mathbf{X}$. That is, (i) $x_1 + x_2 \in \mathbf{X}_s$, $\forall$ $x_1, x_2 \in \mathbf{X}_s$

(ii) $\alpha x \in \mathbf{X}_s$, $\forall$ $x \in \mathbf{X}_s$ and $\alpha \in \mathbf{R}^1$

One of the important subspaces is the kernel of a linear map

$$\Psi : \mathbf{X}_1 \to \mathbf{X}_2$$

where $\mathbf{X}_1$, $\mathbf{X}_2$ are vector spaces. The kernel also termed the nullspace of Ψ is the set given by

$$\mathrm{Ker}\ \Psi = \{ x \in \mathbf{X}_1 \mid \Psi x = 0 \}$$

<u>**Definition A3**</u> : A **normed linear space** $(\mathbf{X}, ||\cdot||)$ where $\mathbf{X}$ is a linear vector space and $||\cdot||$ is a real valued function in $\mathbf{X}$ called the norm such that

(i) $|| x || \geq 0$, $\forall$ $x \in X$, and $||x|| = 0$ if and only if $x=0$.

(ii) $|| \alpha x || \leq |\alpha|\ ||x||$ if $x \in \mathbf{X}$ and α is a scalar.

(iii) $|| x_1 + x_2 || \le ||x_1|| + ||x_2||$, $\forall\ x_1, x_2 \in \mathbf{X}$

The "norm" on the linear space $\mathbf{X}$ is used to denote the real valued function that maps x to $||x||$. Hence, it can be considered as a generalization of the concept of the length of a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$. Namely, given a vector x in a normed linear space, the nonnegative number $|| x ||$ can be thought of as the length of the vector x. Similarly, given two elements x_1 and x_2 in $\mathbf{X}$, $|| x_1 - x_2 ||$ can be considered distance in a sense between the two points x_1 and x_2.

Definition A4 : A sequence $\{ x_n \}$ in a normed linear space ($\mathbf{X}$, $|| \cdot ||$) is said to be a **Cauchy Sequence** if, for every $\epsilon > 0$, there is an integer $N(\epsilon)$ such that

$$|| x_n - x_m || < \epsilon \quad \text{whenever } n, m \ge N(\epsilon).$$

Definition A5 : A normed linear space is said to be **complete** if every Cauchy sequence in the space converges to a point in the space. That is, if for each Cauchy sequence $\{ x_n \}$ in the space, there is an element x in the space such that $x_n \to x$. A complete normed linear space is called a **Banach space**.

Some examples of Banach spaces are (i) continuous functions defined on compact intervals C[a, b], (ii) Lebesgue spaces L_p, (iii) $\mathbf{R}^n$ $(\mathbf{C}^n)$, and (iv) sequence spaces ℓ_p.

Definition A6 : Let **X** and **Y** be linear spaces, then $\Psi : \mathbf{X} \rightarrow \mathbf{Y}$ is said to be a **linear map** if

(i) $\Psi(\alpha x_1 + \beta x_2) = \alpha\Psi x_1 + \beta\Psi x_2$,

(ii) $\Psi(\alpha x) = \alpha\Psi x_1$, $\forall\ x_1, x_2, \in \mathbf{X}$, and $\forall\ \alpha, \beta \in \mathbf{R}^1$.

The notation $\Psi : \mathbf{X} \rightarrow \mathbf{Y}$ implies that Ψ is mapping from **X** into **Y**. Hence a linear mapping can be thought of as a function whose domain and range are both linear spaces. Linear mappings from **X** into the scalar field **R** are known as functionals.

Definition A7 : For all $p \in [1, \infty)$, $\mathbf{L_p}\ \mathbf{[0, \infty)}$ is defined as the set containing all measurable functions $f(\cdot) : [0, \infty) \rightarrow [0, \infty)$ such that

$$\int_0^\infty |\ f(t)\ |^p\, dt < \infty$$

$\mathbf{L_\infty [0, \infty)}$ denotes the set of all measurable functions $f(\cdot) : [0, \infty) \rightarrow [0, \infty)$ that are **essentially bounded** on $[0, \infty)$.

Remark : A function is said to be essentially bounded if it is bounded everywhere except possibly on a subset of measure zero.

Thus, for $1 \leq p < \infty$, $L_p[0, \infty)$ denotes the set of measurable functions whose p^{th} powers are absolutely integrable over $[0, \infty)$, whereas $L_\infty[0, \infty)$ denotes the set of essentially bounded measurable functions. Furthermore, for all $p \in [1, \infty]$, the set $L_p[0, \infty)$ is a real vector space in the sense of Definition A3

Definition A8 : For $p \in [1, \infty)$, the norm function $||\cdot|| : L_p[0, \infty) \to [0, \infty)$ is defined by

$$||f(\cdot)||_p = [\int_0^\infty | f(t) |^p dt]^{1/p} \tag{A-1}$$

And the function $||\cdot||_\infty : L_\infty[0, \infty) \to [0, \infty)$ is defined by

$$||f(\cdot)||_\infty = \operatorname*{ess.\ sup}_{t \in [0, \infty)} |f(t)| \tag{A-2}$$

The function $||\cdot||_p$, for $1 \le p \le \infty$, maps the linear space $L_p[0, \infty)$ into the interval $[0, \infty)$. According to Definition A7, the right hand sides of (A-1) and (A-2) are well defined and finite. For each $p \in [1, \infty]$, the normed linear space $(L_p[0, \infty) \to ||\cdot||)$ is complete and hence a Banach space. For $p = 2$, the norm $||\cdot||_2$ is an inner product. As a matter of fact L_2 is a complete inner product space and therefore is a Hilbert space.

Definition A9 : Let $L_p^n[0, \infty)$ denotes the set of all n-tuples $f = [f_1, \cdots\cdots, f_n]^T$, where $f_i \in L_p[0, \infty)$ for $i = 1\cdots\cdots n$. Then the L_p - norm of $f \in L_p^n[0, \infty)$ is defined by

$$|| f(\cdot) ||_p = [\sum_{i=1}^{n} || f_i(\cdot) ||_p^2]^{1/2} , \ p = [1, \infty)$$

In other words, the norm of a vector valued function $f(\cdot)$ is the square root of the sum of the squares of the norms of the component function $f_i(\cdot)$ for $i = 1 \cdots n$. For $f \in L_\infty^n[0, \infty)$ however

$$||f(\cdot)||_\infty = \max_i \{ \text{ess. sup } |f_i(t)| \}$$

Definition A10: Let $||\cdot||$ be a given norm on C^n, then for each matrix $A \in C^{n \times n}$, the quantity $||A||_p$ defined by

$$||A||_p = \sup_{\substack{x \neq 0 \\ x \in C^n}} \frac{||Ax||_p}{||x||_p}$$

$$= \sup_{||x||=1} ||Ax||_p$$

$$= \sup_{||x||\leq 1} ||Ax||_p .$$

is called the (matrix) norm of A induced by the corresponding vector norm $||\cdot||$.

APPENDIX B

DERIVATION OF A CIRCLE TYPE CRITERION

Consider the term

$$\left| \frac{h(j\omega)}{1 + \frac{1}{2}(\alpha + \beta)\, h(j\omega)} \right| \frac{1}{2}(\alpha - \beta) < 1 \tag{B-1}$$

appearing in the inequality (III-34)

The complex quantity $h(j\omega)$ can be written as

$$h(j\omega) = \sigma + j\omega \text{ , where } \sigma, \omega \in \mathbf{R}^1 \tag{B-2}$$

Combining (B-1) and (B-2) yields

$$\left| \frac{\sigma + j\omega}{1 + \frac{1}{2}(\alpha + \beta)(\sigma + j\omega)} \right| \frac{1}{2}(\alpha - \beta) < 1 \tag{B-3}$$

which on rewriting explicitly gives

$$\frac{(\sigma^2 + \omega^2)^{1/2}}{[\{1+\frac{1}{2}(\alpha + \beta)\sigma\}^2 + \{\frac{1}{2}(\alpha + \beta)\omega\}^2]^{1/2}} \frac{1}{2}(\beta - \alpha) < 1 \tag{B-4}$$

Now, squaring both sides of (B-4) and rearranging it gives

$$\frac{(\alpha\sigma + 1)(\beta\sigma + 1) + \alpha\beta\,\omega^2}{\{1+\frac{1}{2}(\alpha+\beta)\sigma\}^2 + \{\frac{1}{2}(\alpha+\beta)\omega\}^2} > 0 \tag{B-5}$$

If $1 + \frac{1}{2}(\alpha + \beta)\, h(j\omega) \neq 0$, then (B-5) can be simplified further to yield

$$(\alpha\sigma + 1)(\beta\sigma + 1) + \alpha\beta\,\omega^2 > 0 , \quad \forall\, \omega \in \mathbf{R}^1 \tag{B-6}$$

from which we get the following :

(i) When $\alpha > 0$, the equation (B-6) becomes

$$(\sigma + \frac{1}{\alpha}) (\sigma + \frac{1}{\beta}) + \omega^2 > 0 , \forall \omega \in \mathbf{R}^1 .$$

(ii) When $\alpha = 0$, (B-6) reduces to

$$\beta\sigma + 1 > 0 , \forall \omega \in \mathbf{R}^1 .$$

(iii) When $\alpha < 0$, equation (B-6) yields

$$(\sigma + \frac{1}{\alpha}) (\sigma + \frac{1}{\beta}) + \omega^2 < 0 , \forall \omega \in \mathbf{R}^1 .$$

Next, by considering the inequality (III-34)

$$\sup_{\omega \in \mathbf{R}^1} |h(j\omega)| \delta \le (1 - \kappa) \tag{B-7}$$

it is clear that $h(j\omega)$ should lie inside the circle of radius $\frac{1}{\delta}(1 - \kappa)$ centered at the origin. Cases (i) - (iii) and (B-7) form the basis of the geometric interpretation given in section 3.4 .

APPENDIX C

TWO USEFUL LEMMAS

Lemma C1 : Let P be a square partitioned matrix of the form

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{11} \in \mathbf{R}^m \times \mathbf{R}^m$, $P_{12} \in \mathbf{R}^m \times \mathbf{R}^n$, $P_{21} \in \mathbf{R}^n \times \mathbf{R}^m$, and $P_{22} \in \mathbf{R}^n \times \mathbf{R}^n$.

Then

$$(\text{ i}) \ \det(P) = \det(P_{11}) \det(P_{22} - P_{21}P_{11}^{-1}P_{12}) \quad \text{if } \det(P_{11}) \neq 0$$

$$(\text{ii}) \ \det(P) = \det(P_{22}) \det(P_{11} - P_{12}P_{22}^{-1}P_{21}) \quad \text{if } \det(P_{22}) \neq 0$$

Proof of Lemma C1 : This proof is straight forward by using elementary row and column operations.

(i) When $\det(P_{11}) \neq 0$, P can be written as

$$P = \begin{bmatrix} I_m & 0 \\ P_{21}P_{11}^{-1} & I_n \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22}-P_{21}P_{11}^{-1}P_{12} \end{bmatrix} \begin{bmatrix} I_m & P_{11}P_{12}^{-1} \\ 0 & I_n \end{bmatrix}$$

Thus $\det(P) = \det(I_{m+n}) \det(P_{11}) \text{ et}(P_{22} - P_{21}P_{11}^{-1}P_{12}) \det(I_{m+n})$

$$= \det(P_{11}) \det(P_{22} - P_{21}P_{11}^{-1}P_{12})$$

(ii) When $\det(P_{22}) \neq 0$, then

$$P = \begin{bmatrix} I_m & P_{12}P_{22}^{-1} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} P_{11}-P_{12}P_{22}^{-1}P_{21} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ P_{22}^{-1}P_{21} & I_n \end{bmatrix}$$

which reduces to $\det(P) = \det(P_{22}) \det(P_{11} - P_{12}P_{22}^{-1}P_{21})$.

This completes the proof of Lemma C1. □

Lemma C2 : Let G and K be matrices of orders m x n and n x m respectively, then

$$\det(I_m + G\,K) = \det(I_n + K\,G)$$

where I_m and I_n are the Identity matrices of orders m x m and n x n.

Proof of Lemma C2 : See Plotkin [40].

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