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BUCKLING OF A RECTANGULAR PLATE ON AN ELASTIC FOUNDATION, COMPRESSED IN TWO DIRECTIONS

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Richard Charles Warren

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Robert W. Little
Major professor

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BUCKLING OF A RECTANGULAR PLATE ON AN ELASTIC FOUNDATION, COMPRESSED IN TWO DIRECTIONS

By

Richard Charles Warren

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ABSTRACT

BUCKLING OF A RECTANGULAR PLATE ON AN ELASTIC FOUNDATION, COMPRESSED IN TWO DIRECTIONS

By

Richard Charles Warren

The purpose of this research is to determine the first linear buckling mode of a rectangular elastic plate, resting on a Wrinkler foundation under various edge loading conditions. Two cases of boundary conditions are considered, simply supported all around and clamped all around. The solution for the simply supported case is found in closed form, but a numerical approximation is employed for the clamped case. The results are compared to solutions of the circular plate by Wolkowisky and to Hetenyi's solutions for beams on elastic foundations.

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D	$= \frac{Eh^2}{12(1-\nu^2)}$, the flexural rigidity of the plate
E	Young's modulus of plate material
ν	Poisson's ratio of plate material
h	plate thickness
k	foundation stiffness
U, V, W	deflection in the x,y and z direction respectively
φ	Airy stress function
ϵ	perturbation parameter
L	differential operator
λ	eigenvalue
c_1	$= \frac{\lambda h}{D} - \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}$
c_2	$= \frac{\lambda h}{D} + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}$
$\alpha_1, \alpha_2, \beta_1, \beta_2$	separation constants
a, b	dimensions of the plate
σ	edge stresses
σ_{cr}	critical edge stress
Λ	$= \frac{2\lambda h a^2}{D\pi^2}$
K	$= a^2\sqrt{\frac{k}{D}}$
p, q	real numbers
P	$= -\bar{c} + \frac{2\lambda h}{D} q$
Q	$= -\bar{c} + \frac{2\lambda h}{D} p$
$\bar{c}$	$= \pi^2 (\frac{n^2}{a^2} + \frac{m^2}{b^2})$

α_m, β_n	arguments of trigonometric functions
d_{mp}, e_{nq}	expansion coefficients of the eigenfunctions X and Y
c	c_1 or c_2

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CHAPTER I

INTRODUCTION

The use of plates in nautical and aeronautical construction spawned the need for buckling solutions of plates of various shapes under different edge loading conditions and has resulted in considerable examination of these problems as evidenced in the literature. The advent of solid propellent rocket motors, the need for a better understanding of rock mechanics in mining and drilling operations, and the increased demand on highways and airfields have recently precipitated the need for solutions to buckling of plates and shells on various types of foundations. One such foundation model is the Wrinkler foundation, composed of closely spaced linearly independent springs whose restoring force is linearly proportional to the deflection of the foundation surface with opposite sense. The present research provides the buckling solution for a rectangular plate, resting on a Wrinkler foundation and clamped or simply supported at its edges. The boundary conditions and foundation model were chosen for the purpose of comparing these results to Hetenyi's [1] solutions for beams on elastic foundations, and Wolkowisky's [2] solutions for the circular plate embedded in elastic springs.

Solutions to the buckling of plates on elastic foundations by edge thrusts in the plane of the plate have become numerous in the last fifteen years, with the solutions generally following previous approaches where the elastic foundations were not considered.

The mathematical solution for the non-linear boundary value problem of the buckled plate was first given by Friedrichs and Stoker [3]. A circular plate was considered, under uniform radial pressure and simply supported edge conditions. The non-linear von Karman plate equations [4] were utilized and simplified to a system of two ordinary non-linear differential equations where radial symmetry was taken into account. Buckling occurs when the edge load P_e reaches some critical load P_E. The authors sought to determine the stress state when the ratio P_e/P_E became greater than unity. A perturbation technique was used, but this technique was manageable only for the first few eigenvalues of P_e/P_E and a power series method was needed to obtain a higher range of values. The power series method was useful, provided solutions for the first few eigenvalues have been obtained but this method also became cumbersome as the ratio increased further and the authors were eventually forced into an asymptotic analysis as the ratio became larger than fifteen. Bodner [5] solved the clamped case of

the problem presented by Friedrichs and Stoker using their scheme.

Kline and Hancock [6] found the buckling solution of a circular plate on a Wrinkler type elastic foundation for the clamped and simply supported cases. They used a differential equation developed by Yi-Yuan Yu [7] for the deflection of a circular plate on an elastic foundation under the action of edge thrusts and lateral loads. This equation is based on the classical small deflection theory and hence, is a linear fourth order ordinary differential equation. By assuming no lateral load, they obtained a solution which gave the initial buckling load for any given circular plate in the linear range.

Wolkowisky [2] extended the work of Friedrichs and Stoker by placing the non-linear circular plate problem on a Wrinkler type elastic foundation. This added a term (the restoring force of the foundation) to the bending-stretching equation of the non-linear von Karman plate equations. Wolkowisky's work closely followed the method used by Friedrichs and Stoker, Except that, instead of the power series method, a numerical approach called the "shooting technique" was used after transforming the equations to a system of first order differential equations.

In their book, <u>Beams, Plates and Shells on Elastic Foundations</u>, Vlasov and Leont'ev [8] used series solutions to solve buckling problems of rectangular plates compressed by loads in one direction under various types of boundary conditions. Datta [9] found the thermal buckling solutions for triangular and elliptical plates on elastic foundations by conformally transforming the boundary onto the unit circle. He used the method of K. Munakata [10], who found the buckling solution of the rectangular plate by conformal mapping of the rectangle into a unit circle. However, the resulting differential equation is considerably more complicated than the original biharmonic equation, and hence, this method seems of little value here.

The buckling solutions for plates on elastic foundations, uniformly compressed in one direction are treated by: Datta [11], buckling of non-homogeneous rectangular plates; Ariman [12], buckling of thick plates, rectangular and infinite; and Sabir [13], a finite element solution for a rectangular plate.

An infinitely long elastic plate, simply supported at its long edges and compressed in the longitudinal direction was investigated by Seide [14]. The plate rested on but was not attached to a Wrinkler type foundation. The deflections were governed by two differential equations: one for the region in contact and the other for the region of separation. The boundary conditions took the form of continuity of deflection, slope, moment, and shear in the region of contact.

Jacquot [15] developed a method for the prediction of buckling of plates under the influence of elastic constraints, where each constraint was modeled in the form of a linear

spring. A buckling equation from Timoshenko and Woinosky-Krieger's book Theory of Plates and Shells [16], was used with a product of Dirac delta functions added to the right hand side to represent an elastic constraint. Each elastic constraint required a differential equation of this type. This technique was illustrated by application to the buckling of a square plate, simply supported at the edges, under uniform compression in two directions, with a single elastic constraint.

The first portion of the present research determines the first linear buckling mode of a rectangular elastic plate under uniform ($\sigma_{xx} = \sigma_{yy}$ = a constant along the edges) compression on a Wrinkler elastic foundation when the thickness and foundation stiffness are allowed to vary. This allows comparison to the circular case by Wolkowisky. This is followed by the case for non-uniform loading (the loading function is constant along each edge but $\sigma_{xx} \neq \sigma_{yy}$). The solution to the differential equation of the circular plate admits the simply supported and the clamped boundary conditions without difficulty, thus only the clamped case was carried out in detail by Wolkowisky. This is not the case for the rectangular plate so both cases are illustrated.

The mathematical formulation used in this problem is based on the non-linear Foppl-von Karman plate equations, which involve the plate dflection W and the Airy stress function φ as functions of the independent variables x and y. These equations are written in operator form and linearized using a perturbation technique, resulting in a system of linear partial differential equations. To obtain a buckling solution under a uniform load, it is necessary to make a choice for the Airy stress function satisfying $\sigma_{xx} = \sigma_{yy}$ = a constant on the boundary. For the non-uniform case φ must be chosen so that σ_{xx} and σ_{yy} are constants along the edge but can be independently varied according to some specified parameter. It is found that, when buckling initially occurs, no nodal lines are present provided that the foundation stiffness is relatively weak. However as the stiffness of the elastic foundation is increased, more and more nodal lines may appear for the first buckling mode.

A closed form solution for the buckling load of a rectangular plate can be found for the simply supported plate, but not for the clamped plate [17]. Some writers [18, 19] have given approximate solutions by the strain energy technique: Sezawa [19] found a closed form solution to the lateral vibration of a rectangular plate clamped at its four edges by assuming the plate clamped at the mid-points along the edges resulting in residual slopes along some parts of the boundary.

Exact solutions for buckling of rectangular clamped plates have been found by Taylor [17] and by Iyengar and Narasimhan [20]. Taylor's method satisfied one set of boundary conditions and the differential equation term by term, and approximated the other set of boundary conditions, while Iyengar and Narasimhan's approach satisfied both sets of boundary conditions term by term and approximated the differential equation. Both solutions involve the truncation of an infinite determinate. However, Taylor's method requires a considerable amount of numerical work compared to Iyengar and Narasimhan's method, and since both solutions coincide, the method by Iyengar and Narasimhan is used.

The present research can be extended to finding the buckling load beyond the first step of the perturbation technique. However, the power series method used in the solutions for circular plates has no analogue in the case for the rectangular plate [21] and "an exact solution for the rectangular plate valid for an unlimited range of the [buckling load] presents seemingly insurmountable difficulaties" [14].

CHAPTER II

MATHEMATICAL FORMULATION

An elastic rectangular plate of constant thickness is attached to an elastic foundation and is either simply supported or clamped all around. The foundation exerts a lateral force that is linearly proportional to the deflection of the plate and the sense of which is opposite to the deflection. If the behavior of the foundation in any particular region is independent of the behavior of the foundation in an adjacent region, the foundation is called a Wrinkler foundation. The plate is edge loaded in its plane and the following cases are considered: 1) uniform (the load is constant along each edge and $\sigma_{xx} = \sigma_{yy}$), 2) non-uniform (the load is constant along each edge and $\sigma_{xx} \neq \sigma_{yy}$).

The differential equation governing the bending of an elastic plate subjected to lateral loads can be expressed in terms of the biharmonic operator and the lateral load $p_z(x,y)$:

$$D\nabla^4 W = p_z(x,y), \qquad (2.01)$$

where W is the deflection of the middle surface. Equation (2.01) is based on the assumption that no external forces act parallel to the middle surface. The external lateral load of the plate is carried by the internal transverse shear s and by the internal bending moments m. The moments and shear forces are related by

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} = s_x \tag{2.02}$$

$$\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = s_y$$

Equilibrium requires that

$$\frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} = -P_z \tag{2.03}$$

Substitution of equations (2.02) into (2.03) yields

$$\frac{\partial^2 m_x}{\partial x^2} + 2\,\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = -P_z\,(x,y) \tag{2.04}$$

The bending moments are obtained by integration of the normal stress components:

$$m_x = \int_{-h/2}^{h/2} \sigma_x z dz, \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z dz$$

$$m_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \quad \text{and} \quad m_{yx} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \tag{2.05}$$

but since the stress tensor is symmetric $\sigma_{xy} = \sigma_{yx}$ and hence $m_{xy} = m_{yx}$. The stresses are related to the strains by the equations:

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \qquad (2.06)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\epsilon_{xy}$$

and the strains to the displacements by

$$\epsilon_x = -z\frac{\partial^2 W}{\partial x^2},\ \epsilon_y = -z\frac{\partial^2 W}{\partial y^2} \text{ and } \epsilon_{xy} = -z\frac{\partial^2 W}{\partial x \partial y} \qquad (2.07)$$

where $\frac{-\partial^2 W}{\partial x^2}$ and $\frac{-\partial^2 W}{\partial y^2}$ are the curvature change of the deflected middle surface and $-\frac{\partial^2 W}{\partial x \partial y}$ is the warping of the plate. Substitution of equations (2.07) into (2.06) and (2.06) into (2.05) and equations (2.05) into (2.04) yields equation (2.01). If edge loads are present, equation (2.01) must be modified to handle bending and stretching simultaneously. Since the foundation exerts a lateral force whose sense is opposite to the plate displacement, equation (1.01) is modified to include membrane forces and the effect of the foundation and becomes

$$\frac{D}{h}\nabla^4 W = \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y} - \frac{k}{h}W, \qquad (2.08)$$

where φ is the Airy stress function for plane stress which is related to the average stress across the thickness of the plate by the equations

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2},\ \sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2}, \text{ and } \sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y}. \qquad (2.09)$$

The applied bending load, $p_z(x,y)$ in equation (2.01) has been replaced by $-\frac{k}{h}W(x,y)$, the restoring force per unit area of the elastic foundation. The constant, k, is the foundation stiffness and h is the plate thickness. The constant, $D = \frac{Eh^3}{12(1-\nu^2)}$, is the flexural rigidity of the plate, where E is Young's modulus and ν is Poisson's ratio. Equation (2.08) involves two unknown functions φ and W, and so a second equation, called the compatibility equation, which relates φ and W is necessary. This equation can be developed from the strain displacement equations by using the assumption that the squares and products of the slopes of the deflection of the middle surface are of the same order of magnitude as the strains there. A nonlinear relationship is obtained between the strains and the variables U, V, and W, which are the deflections of the plate in the x, y, and z directions, respectively. A linear relationship is obtained between the strains and the stresses such that

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \epsilon_{yy} &= \frac{\partial V}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \qquad (2.10) \\ \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right) = \frac{(1+\nu)}{E}\sigma_{xy} \; . \end{aligned}$$

These expressions differ from the expressions for strain in the linear theory by the quadratic terms in W. The deflections U and V are eliminated through cross differentiation of equation (2.10), and then equations (2.09) are substituted for the stresses resulting in

$$\frac{1}{E}\nabla^4\varphi = \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} \; . \qquad (2.11)$$

The solution to the rectangular plate with simply supported boundary conditions (zero deflection and zero normal moment along the edge) is found under uniform edge loading and carried out in detail for the square plate. The buckling loads are expressed as a function of the foundation stiffness and plate thickness. The case for non-uniform loading follows with particular examples of $\sigma_{xx} = -\sigma_{yy}$ and $\sigma_{yy} = 0$, $\sigma_{xx} \neq 0$. The same loading conditions and examples are solved for the clamped boundary conditions (zero deflection and zero slope along the edge).

Equations (2.08) and (2.11) can be written in operator form by defining

$$L_1 = \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (2.12)$$

and

$$L_2 = \frac{\partial^2 W}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2 W}{\partial y^2}\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2 W}{\partial x \partial y}\frac{\partial^2}{\partial x \partial y}, \qquad (2.13)$$

where $L_1 = L_1(x,y)$ and $L_2 = L_2(W(x,y))$. Replacing equations (2.08) and (2.11) with their equivalent form in operator notation gives

$$(L_1 + \frac{k}{D})W = \frac{h}{D} L_2\varphi \qquad (2.14)$$

$$L_2 W = -\frac{2}{E} L_1\varphi . \qquad (2.15)$$

These equations can be linearized through a perturbation technique if the functions W and φ are expanded in terms of some unspecified parameter ϵ. If expansions are made having the form

$$W = \varepsilon W_1 + \varepsilon^2 W_2 + \ldots \tag{2.16}$$

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots , \tag{2.17}$$

and these values for W and φ are substituted into equation (2.14) and equation (2.15), this allows for a grouping of terms of comparable effects in the buckling problem:

$$\varepsilon^0 : L_1\varphi_0 = 0 \tag{2.18}$$

$$\varepsilon^1 : (L_1 + \frac{k}{D}) W_1 = \frac{h}{D} L_2\varphi_0 \tag{2.19}$$

$$L_1\varphi_1 = 0 \tag{2.20}$$

$$\varepsilon^2 : (L_1 + \frac{k}{D}) W_2 = \frac{h}{D}(L_2\varphi_1 + L_2\varphi_0) \tag{2.21}$$

$$L_2 W_1 = -\frac{2}{E} L_1\varphi_2 . \tag{2.22}$$

For the solution to equation (2.19), an Airy stress function φ_0, is needed that satisfies equation (2.18) and equations (2.09) such that $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy} =$ some constant on the boundary. Therefore, φ_0 must be an even function in both x and y, and may be written as

$$\varphi_0 = -\lambda (x^2 + y^2) \tag{2.23}$$

where the negative sign has been chosen to indicate compression. This gives

$$\sigma_{xx} = \sigma_{yy} = -2\lambda \tag{2.24}$$

If φ_0 is substituted into equation (2.07), and it is noted that L_2 is an operator dependent upon W_1 in equation (2.19), $L_2\varphi_0$ gives

$$L_2\varphi_0 = -2\lambda L_1^{1/2} W \tag{2.25}$$

Substituting this into (2.19) gives:

$$(L_1 + \frac{k}{D}) W_1 = - \frac{h}{D}\, 2\lambda L_1^{1/2} W_1 \tag{2.26}$$

or

$$(L_1 + 2\lambda\, \frac{h}{D}\, L_1^{1/2} + \frac{k}{D}) W_1 = 0 \; . \tag{2.27}$$

This can be factored into:

$$(L_1^{1/2} + c_1)\,(L_1^{1/2} + c_2) W_1 = 0 \tag{2.28}$$

where

$$c_1 = \frac{\lambda h}{D} - \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}} \tag{2.29}$$

and

$$c_2 = \frac{\lambda h}{D} + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}} \; . \tag{2.30}$$

Since the operators are commutative, this factorization is unique.

The complete solution [26] to equation (2.28) can be obtained by a linear combination of the solutions to the equations

$$(L_1^{1/2} + c_1) W_1^{(1)} = 0 \tag{2.31}$$

and

$$(L_1^{1/2} + c_2) W_1^{(2)} = 0 \; . \tag{2.32}$$

Using separation of variables, the solution for W_1 takes the form, $W_1 = X(x)Y(y)$, yielding:

$$\frac{X''}{X} + \frac{Y''}{Y} + c_1 = 0 \tag{2.33}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} + c_2 = 0 \tag{2.34}$$

or

$$\begin{aligned} X'' + \alpha_1^2 X &= 0 \\ Y'' + \beta_1^2 Y &= 0 \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} X'' + \alpha_2^2 X &= 0 \\ Y'' + \beta_2^2 Y &= 0 \end{aligned} \tag{2.36}$$

where

$$\alpha_1^2 + \beta_1^2 = c_1 \tag{2.37}$$

and

$$\alpha_2^2 + \beta_2^2 = c_2 \ . \tag{2.38}$$

The solution to W_1 is then:

$$\begin{aligned} W_1 = {} & A_1 \sin \alpha_1 x \sin \beta_1 y + B_1 \sin \alpha_1 x \cos \beta_1 y \\ & + C_1 \cos \alpha_1 x \sin \beta_1 y + D_1 \cos \alpha_1 x \cos \beta_1 y \\ & + A_2 \sin \alpha_2 x \sin \beta_2 y + B_2 \sin \alpha_2 x \cos \beta_2 y \\ & + C_2 \cos \alpha_2 x \sin \beta_2 y + D_2 \cos \alpha_2 x \cos \beta_2 y \ . \end{aligned} \tag{2.39}$$

A fourth order partial differential equation in two variables must be able to satisfy two boundary conditions along each edge. For most values of λ, the only solution is $W_1 \equiv 0$; the special values of λ for which non-trivial solutions exist are the eigenvalues and the corresponding solutions $W_1(x,y)$ are the eigenfunctions. The homogeneous differential equation, in conjunction with the homogeneous boundary conditions uniquely determines the shape of the buckled plate together with a set of eigenvalues leaving the amplitude arbitrary.

CHAPTER III

SIMPLY SUPPORTED CASE

III.1 Uniform Loading

Consider a uniform rectangular plate over a domain defined by $0 \leq x \leq a$ and $0 \leq y \leq b$. The boundaries of the domain are straight lines $x = 0, a$ and $y = 0, b$, and the origin is chosen at one of the corners of the plate as shown in Figure 3.1.

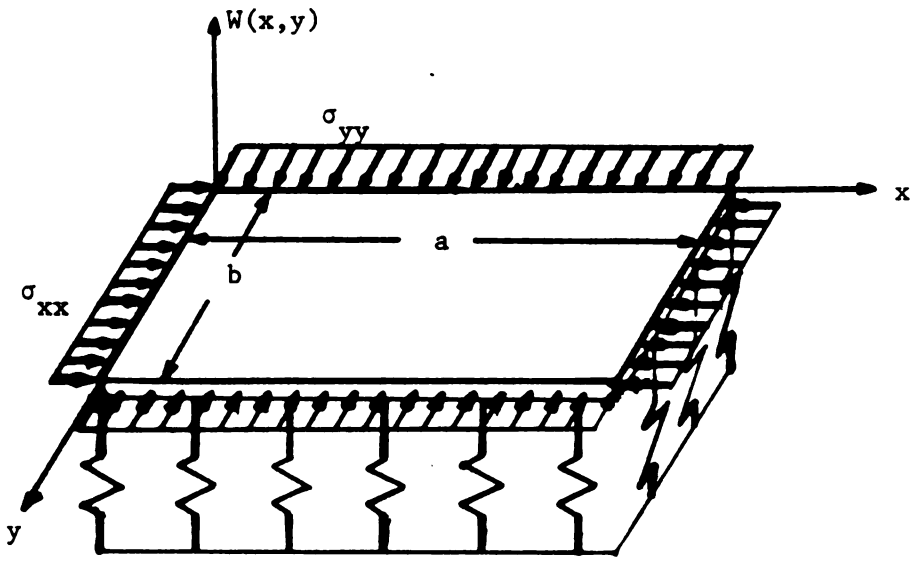

Figure 3.1 Rectangular Plate on an Elastic Foundation for Simply Supported Case

In the case of simply supported edge conditions all around, the requirement of no deflection and no normal moment on the edge take the form

$$w_1 = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along} \quad x = 0, a \tag{3.01}$$

$$w_1 = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along} \quad y = 0, b \ . \tag{3.02}$$

Because equation (2.33) is composed of products of trigonometric functions, the boundary conditions of no deflection or normal moment on the edge impose similar conditions on the trigonometric coefficients. When the boundary conditions $w_1(0,y) = w_1(x,0) = 0$ are applied to equation (2.33), the linear independence of the trigonometric functions gives

$$B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = 0$$

and equation (1.33) reduces to

$$w_1(x,y) = A_1 \sin \alpha_1 x \sin \beta_1 y + A_2 \sin \alpha_2 x \sin \beta_2 y \ . \tag{3.03}$$

When the boundary conditions $w_1(a,y) = w_2(x,b) = 0$ are used, equation (2.03) becomes

$$w_1(a,y) = A_1 \sin \alpha_1 a \sin \beta_1 y + A_2 \sin \alpha_2 a \sin \beta_2 y \tag{3.04}$$

$$w_1(x,b) = A_1 \sin \alpha_1 x \sin \beta_1 b + A_2 \sin \alpha_2 x \sin \beta_2 b \ . \tag{3.05}$$

As indicated by equation (2.31), α_1 and β_1 are dependent on c_1 and equation (1.33) denotes the dependency of α_2 and β_2 on c_2. Then α_1, α_2, c_1 and c_2 can be eliminated by using equations (2.29), (2.30), (2.37) and (2.38) so that equations (3.03) through (3.05) can be re-written as

$$W_1(x,y) = A_1 \sin \sqrt{\frac{\lambda h}{D} - \beta_1^2 - \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; x \sin \beta_1 y$$
$$+ A_2 \sin \sqrt{\frac{\lambda h}{D} - \beta_2^2 + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; x \sin \beta_2 y \quad (3.06)$$

or

$$W_1(a,y) = A_1 \sin \sqrt{\frac{\lambda h}{D} - \beta_1^2 - \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; a \sin \beta_1 y$$
$$+ A_2 \sin \sqrt{\frac{\lambda h}{D} - \beta_2^2 + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; a \sin \beta_2 y$$
$$= 0 \quad (3.07)$$
$$W_1(x,b) = A_1 \sin \sqrt{\frac{\lambda h}{D} - \beta_1^2 - \sqrt{(\frac{\lambda h}{d})^2 - \frac{k}{D}}}\; x \sin \beta_1 b$$
$$+ A_2 \sin \sqrt{\frac{\lambda h}{D} - \beta_2^2 + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; x \sin \beta_2 b$$
$$= 0 \; .$$

For a general nontrivial solution to equations (3.07) to exist, the arguments of the sine functions must be set equal to some integer multiple of π. That is:

$$\sqrt{\frac{\lambda h}{D} - \beta_1^2 - \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; a = n_1 \pi \quad (3.08)$$

$$\beta_1 b = m_1 \pi \quad (3.09)$$

$$\sqrt{\frac{\lambda h}{D} - \beta_2^2 + \sqrt{(\frac{\lambda h}{D})^2 - \frac{k}{D}}}\; a = n_2 \pi \quad (3.10)$$

$$\beta_2 b = m_2 \pi \quad (3.11)$$

where n_1, n_2, m_1 and m_2 are integers not equal to zero.

Substituting (3.09) into (3.08) and (3.11) into (3.10), and setting $a = b$ for a square plate, and choosing the integers n_1, m_1, n_2, m_2 equal to 1 for the first buckling node, equation (3.08) can be written as

$$\Lambda - \frac{2}{\pi^2}\sqrt{\frac{\pi^4}{4}\Lambda^2 - K^2} = 4 \qquad (3.12)$$

and equation (3.10) becomes

$$\Lambda + \frac{2}{\pi^2}\sqrt{\frac{\pi^4}{4}\Lambda^2 - K^2} = 4 \qquad (3.13)$$

where the substitutions

$$\frac{\lambda h}{D} = \frac{\Lambda\pi^2}{2a^2} \qquad (3.14)$$

and

$$\frac{k}{D} = \frac{K^2}{a^4} \qquad (3.15)$$

have been made. When $K = 0$, the case in which the elastic foundation is absent, equation (3.12) has no solution implying that $A_1 = 0$. For this case, equation (3.13) has the solution $\Lambda = 2$ and A_2 is arbitrary. The solution to (3.14) becomes

$$\lambda = \frac{D\pi^2}{a^2 h} \qquad (3.16)$$

or

$$\sigma_{cr} = -2\lambda = \frac{-2D\pi^2}{a^2 h} \qquad (3.17)$$

which agrees with Timoshenko's solution [17] for the buckling of a simply supported rectangular plate.

As K increases, $A_1 = 0$ remains valid, Λ increases A_2 remains arbitrary and the radical in equation (3.15) approaches zero. At the point at which the radical in equation (3.15) equals zero, the radical in equation (3.14) also equals zero. Hence, $\Lambda = 4$, $K = 2\pi^2$ and A_1 and A_2 are both arbitrary. For $K > 2\pi^2$, equation (3.15) has no solution and, hence, $A_2 = 0$. Equation (3.14), however, has the solution $\Lambda = 4$ at $K = 2\pi^2$ and Λ increases as K increases beyond $2\pi^2$; and A_1 is now arbitrary. This process can be continued for different values of n and m as K and Λ increase.

A more general approach can be taken by substituting equation (3.06), with the substitutions of (3.08) through (3.11) into the differential equation (2.20),

$$(L_1 + \frac{2\lambda h}{D} L_1^{1/2} + \frac{k}{D}) W_1 = 0 \ .$$

This gives:

$$\{[(\frac{n_1\pi}{a})^2 + (\frac{m_1\pi}{b})^2]^2 - \frac{2\lambda h}{D}[(\frac{n_1\pi}{a})^2 + (\frac{m_1\pi}{b})^2] + \frac{k}{D}\}A_1 \sin\alpha_1 x \sin\beta_1 y$$

$$+ \{[(\frac{n_2\pi}{a})^2 + (\frac{m_2\pi}{b})^2]^2 - \frac{2\lambda h}{D}[(\frac{n_2\pi}{a})^2 + (\frac{m_2\pi}{b})^2]$$

$$+ \frac{k}{D}\}A_2 \sin\alpha_2 x \sin\beta_2 y = 0 \qquad (3.18)$$

which implies that

$$\{[(\frac{n_1\pi}{a})^2 + (\frac{m_1\pi}{b})^2]^2 - \frac{2\lambda h}{D}[(\frac{n_1\pi}{a})^2 + (\frac{m_1\pi}{b})^2]$$

$$+ \frac{k}{D}\}A_1 = 0 \qquad (3.19)$$

and

$$\{[(\frac{n_2\pi}{a})^2 + (\frac{m_2\pi}{b})^2]^2 - \frac{2\lambda h}{D}[(\frac{n_2\pi}{a})^2 + (\frac{m_2\pi}{b})^2]$$

$$+ \frac{k}{D}\}A_2 = 0 \ . \qquad (3.20)$$

Multiplying through by a^4 and writing a general equation for (3.19) or (3.20), depending on whether n_1 and m_1 or n_2 and m_2 are used, gives

$$(n^2 + \frac{a^2}{b^2} m^2)^2 \pi^4 - \frac{2\lambda h a^2}{D} (n^2 + \frac{a^2}{b^2} m^2) \pi^2 + \frac{ka^4}{D} = 0 . \quad (3.21)$$

Using equations (3.12) and (3.13) and examining the case a = b, for a square plate, yields

$$(n^2 + m^2)^2 - (n^2 + m^2)\Lambda + \frac{K^2}{\pi^4} = 0 \quad (3.22)$$

This gives a series of quadratic curves in K, each of which is tangent to the line $K = \frac{\pi^2}{2} \Lambda$. To see this, one can solve for K in (3.22) with the assumption that there exists a line, $K = r\Lambda$, through the origin which is tangent to

$$K^2 = (n^2 + m^2)\pi^4 \Lambda - (n^2 + m^2)^2 \pi^4 . \quad (3.23)$$

If $K = r\Lambda$, the tangency requirements in the solution of the quadratic for Λ in equation (3.23) dictate that the discriminate vanishes, or $r = \frac{\pi^2}{2}$, for which

$$K = \frac{\pi^2}{2} \Lambda \quad (3.24)$$

with the points of tangency

$$\Lambda = 2(n^2 + m^2) \quad (3.25)$$

$$K = \pi^2 (n^2 + m^2) \quad (3.26)$$

To the left of these points of tangency $A_1 = 0$ and A_2 is arbitrary and to the right of these points A_1 is arbitrary and $A_2 = 0$.

Equation (3.22) is plotted in Figure 3.2.

The solid curve in Figure 2 is the line along which buckling will occur. The numbers in parentheses represent values for n and m: (n_2, m_2) to the left of the points of tangency (equation (3.20)) and (n_1, m_1) to the right of the points (equation (3.19)).

Letting (n_a, m_a) represent the node numbers for one curve and (n_b, m_b) the node numbers of the next consecutive curve, and letting $A = n_a^2 + m_a^2$ and letting $B = n_b^2 + m_b^2$, then the intersection of two consecutive curves from two equations of the form (3.22), occurs at the point

$$\Lambda = \frac{(n_a^2 + m_a^2)^2 - (n_b^2 + m_b^2)^2}{(n_a^2 + m_a^2) - (n_b^2 + m_b^2)} = \frac{A^2 - B^2}{A - B} = A + B \qquad (3.27)$$

and

$$K = \sqrt{A\pi^4 (\Lambda - A)} = \pi^2 \sqrt{AB} \, . \qquad (3.28)$$

For the intersection of the first two curves marked (1,1) and (1,2), $n_a = m_a = 1$, and $n_b = m_b = 2$; $A = 2$ and $B = 5$ and therefore

$$\Lambda = 7 \qquad (3.29)$$

$$K = \sqrt{10}\,\pi^2 . \qquad (3.30)$$

For $0 \leq K \leq \sqrt{10}\,\pi^2$, the buckling is symmetric in two directions. For the next intersection, $n_a = 1$, $m_a = 2$, $n_b = m_b = 2$ and $\Lambda = 13$ and $K = 2\pi^2\sqrt{10}$, $\sqrt{10}\,\pi^2 \leq K \leq 2\sqrt{10}\,\pi^2$, and the buckling is symmetric in one direction and antisymmetric in the other direction. Continuing on, for $2\sqrt{10}\,\pi^2 < K < 4\sqrt{5}\,\pi^2$, buckling is antisymmetric in two directions.

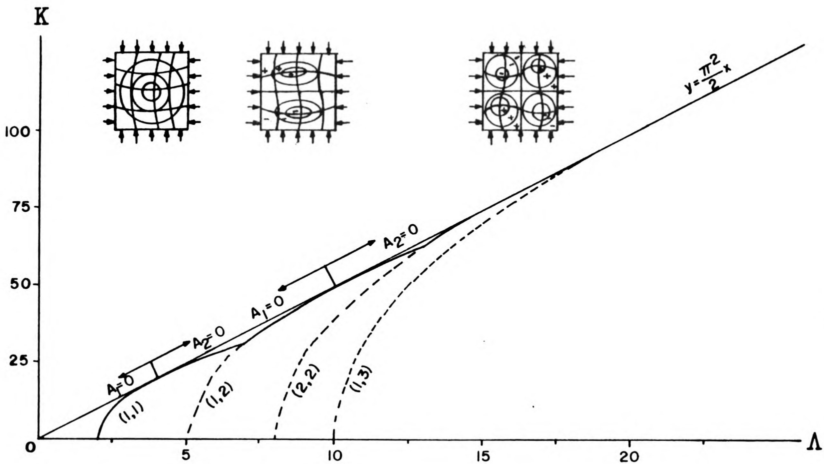

Figure 3.2 Buckling of a simply supported rectangular plate on a Wrinkler foundation

The critical load when $K = 0$, occurs at:

$$\sigma_{cr} = -2\lambda = -\frac{D\pi^2}{a^2 h}(n^2 + m^2) \qquad (3.31)$$

or for a nonsquare plate

$$\sigma_{cr} = -2\lambda = -\frac{D\pi^2}{a^2 h}(n^2 + m^2 \frac{a^2}{b^2}) \qquad (3.32)$$

which corresponds to Timoshenko's solution [15].

These results compare favorably with Heteyni's [1] results for a simply supported beam on an elastic foundation as illustrated in Figure 3.3 where k is the foundation stiffness, l the length and EI is the flexural rigidity of the beam. These values are plotted against N_{cr}/N_e which is the ratio of the critical buckling load to the Euler load for a hinged end bar of length l and flexural rigidity EI.

III.2 The Case of Nonuniform Loading

From equation (2.13):

$$(L_1 + \frac{k}{D}) W_1 = \frac{h}{D} L_2 \varphi_0 \qquad (3.33)$$

choose

$$\varphi_0 = -\lambda(px^2 + qy^2) \quad \text{for the nonuniform case,} \qquad (3.34)$$

where p and q are values to be chosen depending on the desired loading conditions.

From equation (3.34), the edge stresses are

$$\sigma_{xx} = -2\lambda q, \qquad (3.35)$$

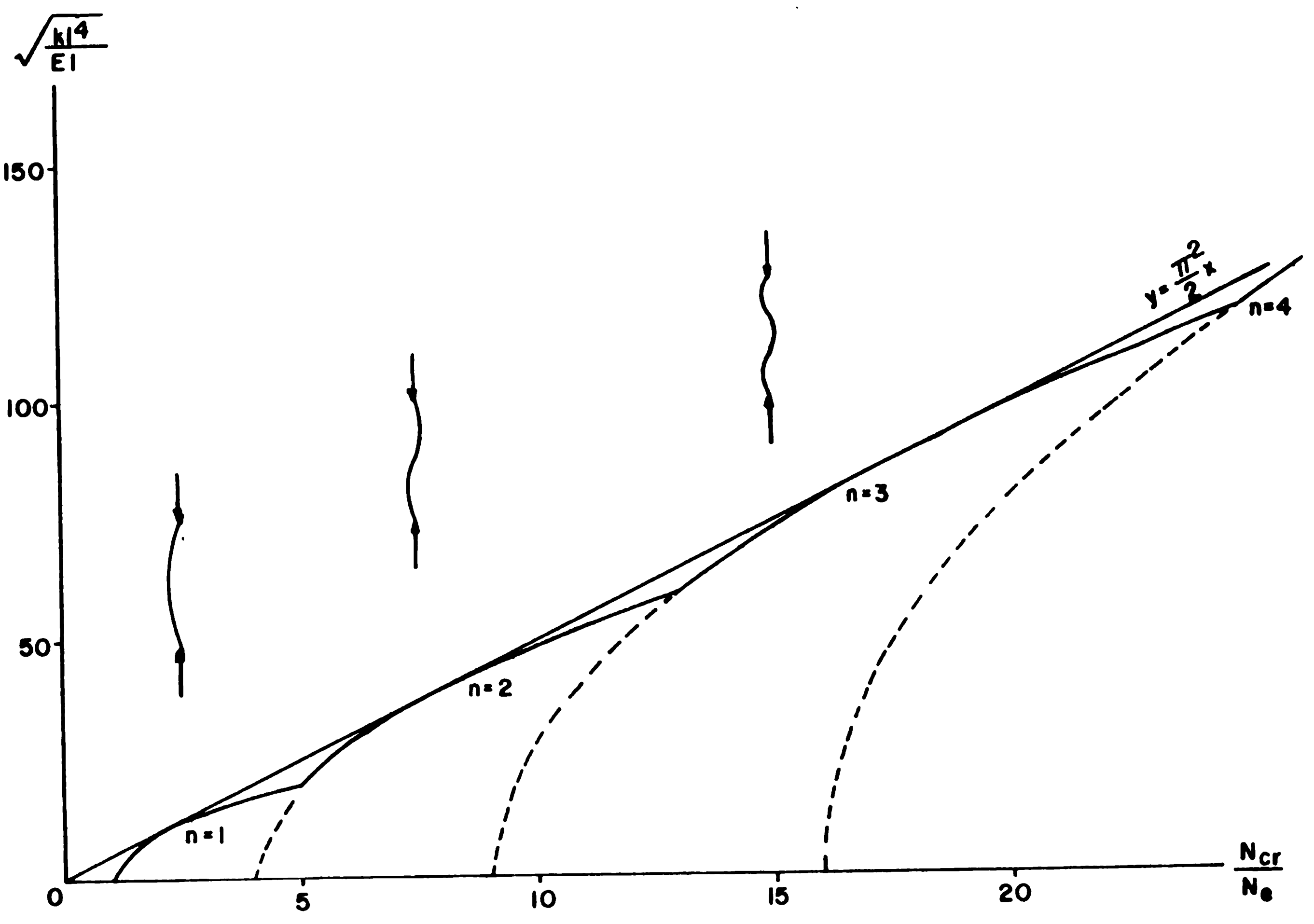

Figure 3.3 Buckling of a beam on a Wrinkler foundation

$$\sigma_{yy} = -2\lambda p \tag{3.36}$$

and

$$\sigma_{xy} = 0 \ . \tag{3.37}$$

From the right hand side of equation (3.33),

$$L_2(W_1)\varphi_0 = -2\lambda \left(q \frac{\partial^2 W_1}{\partial x^2} + p \frac{\partial^2 W_1}{\partial y^2}\right) \ . \tag{3.38}$$

Equation (3.33) can then be written as

$$(L_1 + \frac{k}{D}) W_1 = - \frac{2\lambda h}{D} \left(q \frac{\partial^2 W_1}{\partial x^2} + p \frac{\partial^2 W_1}{\partial y^2}\right) \tag{3.39}$$

or

$$L_1 W_1 + \frac{2\lambda h}{D} \left(q \frac{\partial^2}{\partial x^2} + p \frac{\partial^2}{\partial y^2}\right) W_1 + \frac{k}{D} W_1 = 0 \tag{3.40}$$

This can be rewritten as

$$\left[(L_1^{1/2} + \frac{2\lambda h}{D} q) \frac{\partial^2}{\partial x^2} + (L_1^{1/2} + \frac{2\lambda h}{D} p) \frac{\partial^2}{\partial y^2} + \frac{k}{D} \right] W_1 = 0 \tag{3.41}$$

Again as in equations (2.25) and (2.26), if solutions are sought in the form

$$L_1^{1/2} W = -\bar{c} W, \tag{3.42}$$

then substitution into equation (3.41) yields:

$$\left[(-\bar{c} + \frac{2\lambda h}{D} q) \frac{\partial^2}{\partial x^2} + (-\bar{c} + \frac{2\lambda h}{D} p) \frac{\partial^2}{\partial y^2} + \frac{k}{D} \right] W_1 = 0 \tag{3.43}$$

or

$$\left(P \frac{\partial^2}{\partial x^2} + Q \frac{\partial^2}{\partial y^2} + \frac{k}{D}\right) W_1 = 0, \tag{3.44}$$

where

$$P = -\bar{c} + \frac{2\lambda h}{D} q \tag{3.45}$$

$$Q = -\bar{c} + \frac{2\lambda h}{D} p \tag{3.46}$$

when $p = q$, $P = Q$ and equation (3.43) becomes, using equation (3.42), a quadratic equation in c. The solution is the superposition of the solutions to equation (3.42) for the two different values of $\bar{c}$. This is the case for uniform loading previously developed. When $p \neq q$, the solution to equation (3.44) involves only one value of $\bar{c}$.

Separation of variables in equation (3.44) leads to

$$PX''Y + QXY'' + \frac{k}{D} XY = 0, \qquad (3.47)$$

or

$$PX'' + \alpha_3^2 X = 0 \qquad (3.48)$$

$$QY'' + \beta_3^2 Y = 0, \qquad (3.49)$$

where

$$\alpha_3^2 + \beta_3^2 = \frac{k}{D} . \qquad (3.50)$$

The solutions to equations (3.48) and (3.49) are

$$X = A' \sin \frac{\alpha_3}{\sqrt{P}} x + C' \cos \frac{\alpha_3}{\sqrt{P}} x, \quad P \neq 0 \qquad (3.51)$$

and

$$Y = B' \sin \frac{\beta_3}{\sqrt{Q}} y + D' \cos \frac{\beta_3}{\sqrt{Q}} y, \quad Q \neq 0 \qquad (3.52)$$

(When $p = q$, equation (3.41) reduces to equation (2.21). For P or Q equal to zero, this reduction is not possible.)

From equation (3.51) and (3.52),

$$W = A \sin \frac{\alpha_3}{\sqrt{P}} x \sin \frac{\beta_3}{\sqrt{Q}} y + B \sin \frac{\alpha_3}{\sqrt{P}} x \cos \frac{\beta_3}{\sqrt{Q}} y$$
$$+ C \cos \frac{\alpha_3}{\sqrt{P}} x \sin \frac{\beta_3}{\sqrt{Q}} y + D \cos \frac{\alpha_3}{\sqrt{P}} x \cos \frac{\beta_3}{\sqrt{Q}} y. \qquad (3.53)$$

Applying the boundary conditions for the simply supported case $X(0) = Y(0) = 0$ and $X''(0) = Y''(0) = 0$ yields

$$B = C = D = 0, \tag{3.54}$$

so that

$$W = A \sin \frac{\alpha_3}{\sqrt{P}} x \sin \frac{\beta_3}{\sqrt{Q}} y \; . \tag{3.55}$$

Applying the boundary conditions for the other two edges: $X(a) = Y(b) = 0$ and $X''(a) = Y''(b) = 0$ yield

$$A \sin \frac{\alpha_3}{\sqrt{P}} a = 0 \tag{3.56}$$

and

$$A \sin \frac{\beta_3}{\sqrt{Q}} b = 0 \; . \tag{3.57}$$

For a nontrivial solution to exist (for $A \neq 0$), the arguments of the trigonometric functions must be integer multiples of π or

$$\frac{\alpha_3}{\sqrt{P}} a = n\pi \tag{3.58}$$

and

$$\frac{\beta_3}{\sqrt{Q}} b = m\pi, \tag{3.59}$$

where m and n are integers not equal to zero.

Using equations (3.45), (3.46) and (3.50) in (3.58) and (3.59) gives

$$\sqrt{\frac{\frac{k}{D} - \beta_3^2}{\frac{2\lambda h}{D} q - \bar{c}}} = \frac{n\pi}{a} \tag{3.60}$$

and

$$\sqrt{\frac{\beta_3^2}{\frac{2\lambda h}{D}p - \bar{c}}} = \frac{m\pi}{b} \qquad (3.61)$$

Solving for β_3 in (2.61) and substituting this into (3.60) yields

$$\sqrt{\frac{\frac{k}{D} - \frac{m^2\pi^2}{b^2}\left(\frac{2\lambda h}{D} p - \bar{c}\right)}{\frac{2\lambda h}{D}q - \bar{c}}} = \frac{n\pi}{a} \qquad (3.62)$$

where $\bar{c}$ can be determined by substituting (3.55) into (3.42) and using (3.58) and (3.59):

$$\bar{c} = \frac{\alpha_3^2}{P} + \frac{\beta_3^2}{Q} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) . \qquad (3.63)$$

Substituting (3.63) into (3.62) and solving for λ yields:

$$\lambda = \frac{k}{2h\left(\frac{pm^2}{b^2} + \frac{qn^2}{a^2}\right)\pi^2} + \frac{\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^2 D\pi^2}{2h\left(\frac{pm^2}{b^2} + \frac{qn^2}{a^2}\right)} . \qquad (3.64)$$

For $p = q = 1$, equation (3.64) reduces to equation (3.21) for the case of uniform loading. Furthermore, for $k = 0$, $p = 0$, $q = 1$, and for $k = 0$, $p = q = 1$, equation (3.64) reduces to Timoshenko's solutions for the buckling of a rectangular plate compressed in one direction and uniformly compressed in two directions respectively.

The Case For $p = -q$, $\sigma_{xx} = -\sigma_{yy}$

Let $p = -q = 1$, $a = b$, then

$$\lambda = \frac{ka^2}{2h(m^2 - n^2)\pi^2} + \frac{(n^2 + m^2)^2 D\pi^2}{2a^2h(m^2 - n^2)} \qquad (3.65)$$

$m \neq n$ and the first buckling node is then $m = 2$, $n = 1$, or

$$\lambda = \frac{ka^2}{6h\pi^2} + \frac{25D\pi^2}{6a^2h} \qquad (3.66)$$

or

$$\sigma_{cr} = -2\lambda = -\frac{ka^2}{3h\pi^2} - \frac{25D\pi^2}{3a^2h} \, . \qquad (3.67)$$

For $k = 0$,

$$\sigma_{cr} = -8.33 \, \frac{D\pi^2}{a^2h} \, , \qquad (3.68)$$

which is the value given by Brush and Almroth [21].

CHAPTER IV

CLAMPED CASE

IV.1 Uniform Loading

The buckling solution for the clamped rectangular plate is not as straightforward as the solution for the simply supported case. A function satisfying the boundary conditions and the differential equation has not been found ([22], [16]) or cannot be found ([23]). Various approximate methods have been used, and the one found to be the most useful (based on rapidity of convergence and amount of numerical work) is the method by Iyengar and Narasimhan.

The displacement function $W(x,y)$ is expanded in a double orthogonal series composed of hyperbolic and trigonometric functions. The boundary conditions lead to two transcendental equations which can be satisfied by choosing the arguments of the functions. The satisfaction of the differential equation involves an infinite determinant which is truncated for an approximate solution.

From equation (1.20)

$$(L_1 + \frac{2\lambda h}{D} L_1^{1/2} + \frac{k}{D}) W_1 = 0 \ . \qquad (4.01)$$

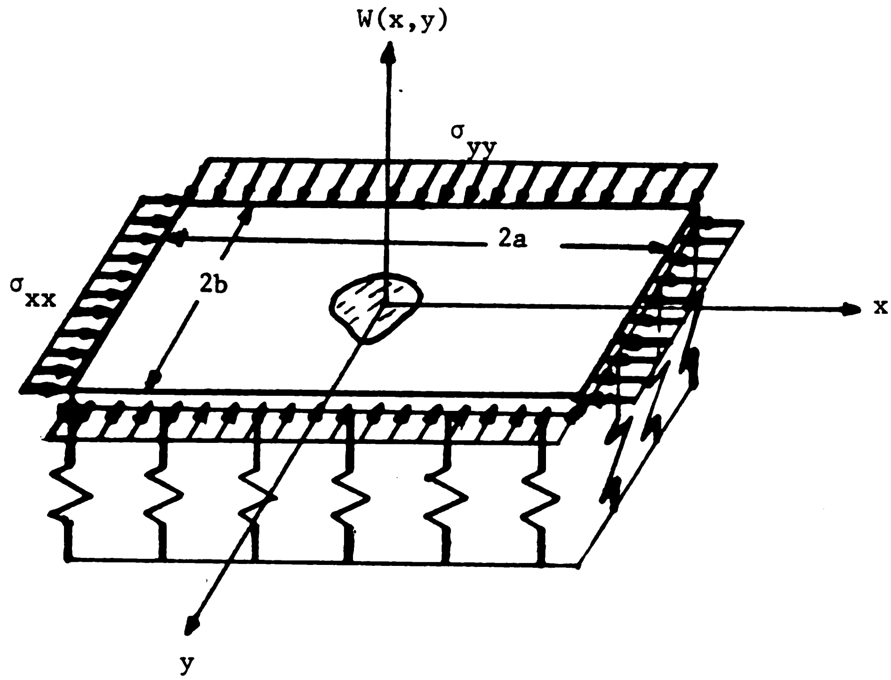

Figure 4.1 Rectangular Plate on an Elastic Foundation for Clamped Case

The symmetric form of buckling can be described by expanding the function W_1 in the following form, with the origin of the coordinates chosen as in Figure 4.1,

$$W_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n$$

$$= \sum_m \sum_n A_{mn} \left(\frac{\cos \alpha_m x/a}{\cos \alpha_m} - \frac{\text{ch } \alpha_m x/a}{\text{ch } \alpha_m}\right) \left(\frac{\cos \beta_n y/b}{\cos \beta_n} - \frac{\text{ch } \beta_n y/b}{\text{ch } \beta_n}\right) . \tag{4.02}$$

The boundary conditions for the clamped case take the form

$$W_1 = 0, \; \frac{\partial W_1}{\partial x} = 0 \quad \text{at} \quad x = \pm a \tag{4.03}$$

$$W_1 = 0, \; \frac{\partial W_1}{\partial y} = 0 \quad \text{at} \quad y = \pm b \; . \tag{4.04}$$

$W_1 = 0$ is automatically satisfied at $x = \pm a$, $y = \pm b$. The conditions on $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$ can be met at the boundaries provided the following equations are satisfied

$$\tan \alpha_m + \tanh \alpha_m = 0 \tag{4.05}$$

$$\tan \beta_n + \tanh \beta_n = 0 \; . \tag{4.06}$$

From this

$$\alpha_m = \beta_m ; \; \alpha_1 = 2.3650, \; \alpha_2 = 5.4978.. \; . \tag{4.07}$$

Substituting (4.02) into (4.01) yields

$$\sum_m \sum_n A_{mn} \Big[\Big(\alpha_m^4 + \frac{a^4}{b^4} \alpha_n^4 + \frac{ka^4}{D}\Big) X_m Y_n + \frac{2a^2}{b^2} \alpha_m^2 \alpha_n^2 X'' Y''$$

$$- \frac{2\lambda h}{D} a^2 \Big(\alpha_m^2 X_m'' Y_n + \frac{a^2}{b^2} \alpha_n^2 X_m Y_n''\Big) \Big] = 0 \tag{4.08}$$

where

$$X_m'' = \frac{\cos \alpha_m x/a}{\cos \alpha_m} + \frac{\operatorname{ch} \alpha_m x/a}{\operatorname{ch} \alpha_m} \tag{4.09}$$

and

$$Y_n'' = \frac{\cos \beta_n y/b}{\cos \beta_n} + \frac{\operatorname{ch} \beta_n y/b}{\operatorname{ch} \beta_n} \; . \tag{4.10}$$

The functions considered here form complete sets in their intervals[1] so that X_m'' can be expressed as an expansion of X_m, and likewise Y_n'' can be expressed as an expansion of Y_n:

$$X_m'' = \sum_{p=1}^{\infty} d_{mp} X_p \tag{4.11}$$

---

1 These functions satisfy the normal modes of vibration of a beam and hence satisfy a self-adjoint differential equation.

$$Y_n'' = \sum_{q=1}^{\infty} e_{nq} Y_q \ . \tag{4.12}$$

Because of orthogonality, the coefficients d_{mp}, e_{nq} can be determined by multiplying (4.11) through X_p and (4.12) through by Y_q and integrating from $-a$ to a. This yields:

$$\int_{-a}^{a} X_m'' X_p dx = d_{mp} \int_{-a}^{a} X_p^2 dx \tag{4.13}$$

or

$$d_{mp} = \frac{\int_{-a}^{a} X_m'' X_p dx}{\int_{-a}^{a} X_p^2 dx} \tag{4.14}$$

where

$$\int_{-a}^{a} X_p^2 dx = 2a \tag{4.15}$$

$$\int_{-a}^{a} X_m'' X_p dx = a\left[\frac{-2 \tanh \alpha_m}{\alpha_m} - \frac{1}{ch^2 \alpha_m} + \frac{1}{\cos^2 \alpha_m}\right] \qquad p = m \tag{4.16}$$

$$\int_{-a}^{a} X_m'' X_p dx = \frac{8\alpha_p^2 a}{\alpha_p^4 - \alpha_m^4}(\alpha_m \tanh \alpha_m - \alpha_p \tanh \alpha_p) \qquad p \neq m \ . \tag{4.17}$$

The same procedure is carried out for Y_n and Y_n'' and it is found that

$$d_{nm} = e_{nm},\ d_{11} = .5499,\ d_{12} = -.4356,\ d_{21} = -.0805,$$
$$d_{22} = .8181. \tag{4.18}$$

For a square plate, $a = b$ and equation (4.08) becomes

$$\sum_m \sum_n A_{mn} [(\alpha_m^4 + \alpha_n^4 + \frac{k}{D} a^4) X_m Y_n + 2\alpha_m \alpha_n X_m'' Y_n''$$

$$- \frac{2\lambda h a^2}{D} (\alpha_m^2 X_m'' Y_n + \alpha_n^2 X_m Y_n'')] = 0 \tag{4.19}$$

or

$$\sum_m \sum_n A_{mn} [(\alpha_m^4 + \alpha_n^4 + \frac{k}{D} a^4) X_m Y_n + 2\alpha_m^2 \alpha_n^2 \sum_p d_{mp} X_p \sum_q d_{nq} Y_q$$

$$- \frac{2\lambda h a^2}{D} (\alpha_m^2 Y_n \sum_p d_{mp} X_p + \alpha_n^2 X_m \sum_q d_{nq} Y_q)] = 0 . \tag{4.20}$$

Using only the first term of the series as a first approximation, $(m = n = 1)$ gives

$$A_{11} [2\alpha_1^4 (1 + d_{11}^2) + \frac{ka^4}{D} - \frac{4\lambda h a^2}{D} \alpha_1^2 d_{11}] X_1(x) Y_1(y)$$

$$= 0 \tag{4.21}$$

or for $A_{11} \neq 0$,

$$2\alpha_1^4 (1 + d_{11}^2) + \frac{ka^4}{D} - \frac{4\lambda h a^2}{D} \alpha_1^2 d_{11} = 0 \tag{4.22}$$

or solving for λ:

$$\lambda = \frac{6.63D}{a^2 h} + \frac{ka^2}{12.3h} , \tag{4.23}$$

then

$$\sigma_{cr} = -2\lambda = \frac{-13.25D}{a^2 h} + \frac{-ka^2}{6.15h} . \tag{4.24}$$

For $k = 0$ this compares with Timoshenko's solution of $\frac{13.15D}{a^2 h}$. For $k \neq 0$, let

$$\Lambda = \frac{2\lambda h a^2}{D\pi^2} \tag{4.25}$$

$$K = a^2 \sqrt{\frac{k}{D}} , \tag{4.26}$$

then equation (4.24) can be written as

$$\Lambda - 1.647(10^{-2})K - 1.343 = 0 \, . \tag{4.27}$$

This is plotted as the "symmetric" curve in Figure 4.2.

For antisymmetric buckling, sines and hyperbolic sines are used instead of cosines and hyperbolic cosines. This time let

$$W_1 = \sum_m \sum_n X_m Y_n = \sum_m \sum_n A_{mn} \left(\frac{\sin \alpha_m x/a}{\sin \alpha_m} - \frac{\text{sh } \alpha_m x/a}{\text{sh } \alpha_m}\right) \left(\frac{\sin \beta_n y/b}{\sin \beta_n} - \frac{\text{sh } \beta_n y/b}{\text{sh } \beta_n}\right) \tag{4.28}$$

The boundary conditions give

$$\cot \alpha_m - \coth \alpha_m = 0 \tag{4.29}$$

$$\cot \beta_n - \coth \beta_n = 0 \tag{4.30}$$

which implies that

$$\alpha_n = \beta_n, \ \alpha_1 = 3.9266, \ \alpha_2 = 7.0686\ldots \, . \tag{4.31}$$

Substituting (4.28) into (4.1) gives a similar equation as to (4.08), where

$$X_m'' = \frac{\sin \alpha_m x/a}{\sin \alpha_m} + \frac{\text{sh } \alpha_m x/a}{\text{sh } \alpha_m} \tag{4.32}$$

$$Y_n'' = \frac{\sin \alpha_n y/b}{\sin \alpha_n} + \frac{\text{sh } \alpha_n y/b}{\text{sh } \alpha_n} \, . \tag{4.33}$$

Using (4.11) and (4.12), the equivalent equations for (4.16) and (4.17) are

$$\int_{-a}^{a} X_m'' X_p \, dx = a\left[\frac{-2 \coth \alpha_p}{\alpha_p} + \frac{1}{\sin^2 \alpha_p} + \frac{1}{\text{sh}^2 \alpha_p}\right], p = m \tag{4.34}$$

$$\int_{-a}^{a} X_m'' X_p \, dx = \frac{8\alpha_p^2 a}{\alpha_p^4 - \alpha_m^4}[\alpha_m \coth \alpha_m - \alpha_p \coth \alpha_p] p \neq m \tag{4.35}$$

Again, using the same procedure for Y_n and Y_n'', yields

$$d_{nm} = e_{nm},\ d_{11} = .7467,\ d_{12} = -.2777,\ d_{21} = -.0857,$$
$$d_{22} = .8585. \tag{4.36}$$

Using only the first term of the series leads to

$$2\alpha_1^4(1 + d_{11}^2) + \frac{ka^4}{D} - \frac{4\lambda ha^2}{D}\alpha_{11}^2 d_{11} = 0 \tag{4.37}$$

or

$$\lambda = \frac{16.9D}{a^2 h} + \frac{ka^2}{46.02h} . \tag{4.38}$$

This can be written as

$$\Lambda - 4.403\,(10^{-3})K - 3.261 = 0 . \tag{4.39}$$

This equation is plotted as the "antisymmetric" curve in Figure 4.2.

For the second order approximation, $m = 1,\ n = 1,2.$ Using equation (4.20) for the symmetric case,

$$\begin{aligned} A_{11}\{[2\alpha_1^4(1 + d_{11}^2) &+ \frac{ka^4}{D}]X_1Y_1 + 2\alpha_1^4 d_{11}d_{12}X_1Y_2 \\ &- \frac{2\lambda ha^2}{D}\alpha_1^2(2d_{11}X_1Y_1 + d_{12}X_1Y_2)\} \\ + A_{12}\{2\alpha_1^2\alpha_2^2(d_{11}d_{21}X_1Y_1 &+ d_{11}d_{22}X_1Y_2) + (\alpha_1^4 + \alpha_2^4 + \frac{ka^4}{D})X_1Y_2 \\ &- \frac{2\lambda ha^2}{D}[\alpha_2^2 d_{21}X_1Y_1 + (\alpha_1^2 d_{11} + \alpha_2^2 d_{22})X_1Y_2]\} \\ = 0 \end{aligned} \tag{4.40}$$

The linear independence of the functions X_1Y_1 and X_1Y_2 implies that

$$\begin{aligned} \{A_{11}[2\alpha_1^4(1 + d_{11}^2) &+ \frac{ka^4}{D} - \frac{4\lambda ha^2}{D}\alpha_1^2 d_{11}] \\ &+ A_{12}[2\alpha_1^2\alpha_2^2 d_{11}d_{21} - \frac{2\lambda ha^2}{D}\alpha_2^2 d_{21}]\}X_1Y_1 = 0 \end{aligned} \tag{4.41}$$

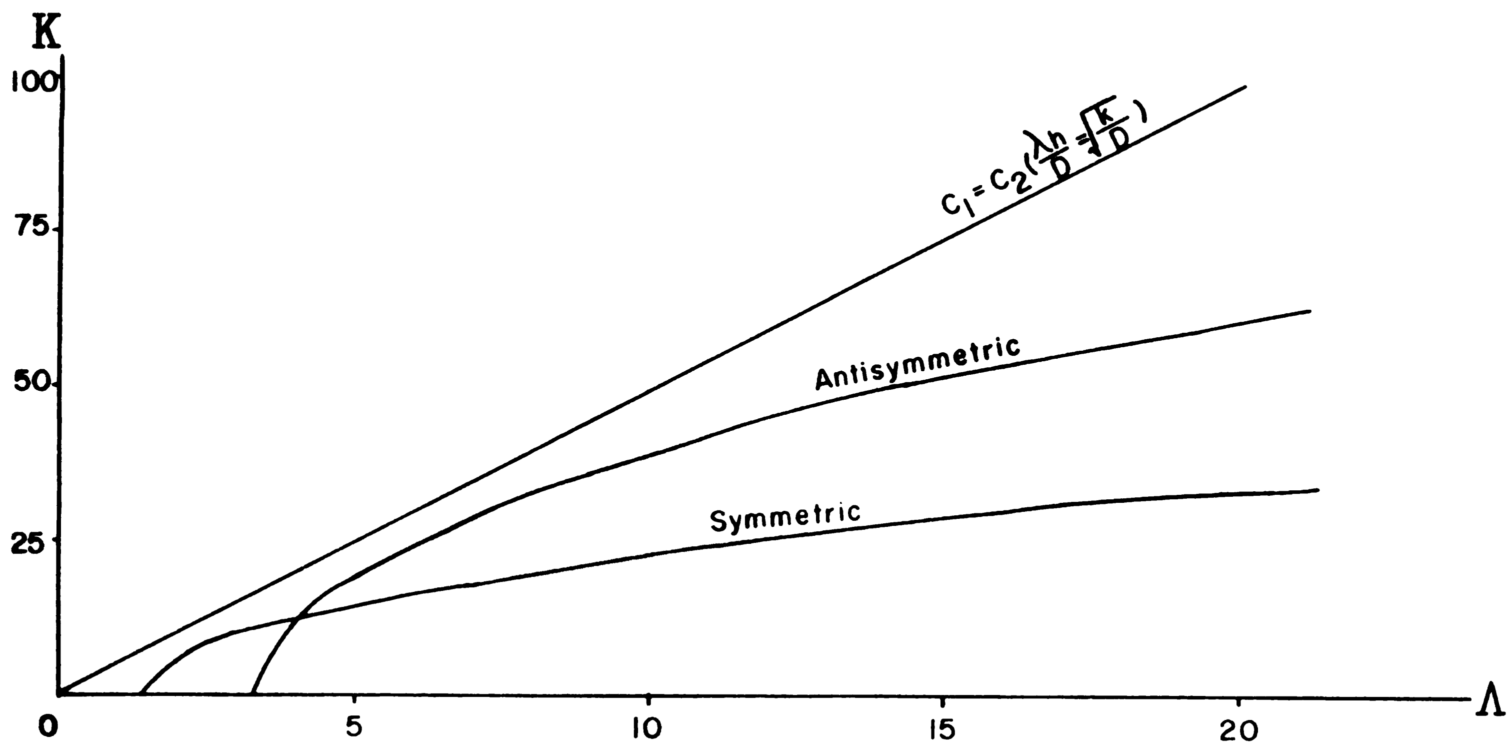

Figure 4.2 Buckling of a clamped rectangular plate on a Wrinkler foundation. First approximation

and

$$\{A_{11}[2\alpha_1^4 d_{11}d_{12} - \frac{2\lambda ha^2}{D}\alpha_1^2 d_{12}] + A_{12}[2\alpha_1^2\alpha_2^2 d_{11}d_{22} + \alpha_1^4 + \alpha_2^4 + \frac{ka^4}{D} - \frac{2\lambda ha^2}{D}(\alpha_1^2 d_{11} + \alpha_2^2 d_{22})]\}X_1Y_x = 0 \qquad (4.42)$$

To satisfy these equations for all values of x and y, the coefficients of X_1Y_1 and X_1Y_2 must vanish, or

$$\begin{bmatrix} 2\alpha_1^4(1+d_{11}^2) + \frac{ka^4}{D} - \frac{4\lambda ha^2}{D}\alpha_{11}^2 d_{11} & 2\alpha_1^2\alpha_2^2 d_{11}d_{21} - \frac{2\lambda ha^2}{D}\alpha_2^2 d_{21} \\ 2\alpha_1^4 d_{11}d_{12} - \frac{2\lambda ha^2}{D}\alpha_1^2 d_{12} & \alpha_1^4 + \alpha_2^4 + \frac{ka^4}{D} + 2\alpha_1^2\alpha_2^2 d_{11}d_{22} - \frac{2\lambda ha^2}{D}(\alpha_1^2 d_{11} + \alpha_2^2 d_{22}) \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix} = 0 \qquad (4.43)$$

For the nontrivial solution to exist, the coefficients of the determinant must vanish, or

$$(2\alpha_1^4(1+d_{11}^2) + \frac{ka^4}{D} - \frac{4\lambda ha^2}{D}\alpha_1^2 d_{11})[\alpha_1^4 + \alpha_2^4 + \frac{ka^4}{D} + 2\alpha_1^2\alpha_2^2 d_{11}d_{22} - \frac{2\lambda ha^2}{D}(\alpha_1^2 d_{11} + \alpha_2^2 d_{22})] - (2\alpha_1^2\alpha_2^2 d_{11}d_{21} - \frac{2\lambda ha^2}{D}\alpha_2^2 d_{21})(2\alpha_1^4 d_{11}d_{12} - \frac{2\lambda ha^2}{D}\alpha_1^2 d_{12}) = 0 \qquad (4.44)$$

This may be written

$$660.28\left(\frac{\lambda ha^2}{D}\right)^2 - \left(17880.82 + 67.90\frac{ka^4}{D}\right)\frac{\lambda ha^2}{D} + \left(\frac{ka^4}{D}\right)^2 + 1178.65\frac{ka^4}{D} + 89.93.63 = 0 \qquad (4.45)$$

which can be expressed as

$$\Lambda^2 - (5.49 + .0208K^2)\Lambda + 6.219(10^{-5})K^4 + .0733K^2 + 5.547 = 0 . \quad (4.46)$$

Equation (4.46) is plotted as the symmetric curve in Figure 4.3.

For the antisymmetric case, the same procedure is carried out using equations (4.28) instead of (4.02):

$$4936.28\left(\frac{\lambda ha^2}{D}\right)^2 - \left(250213 + 154.88\,\frac{ka^4}{D}\right)\frac{\lambda ha^2}{D} + \left(\frac{ka^4}{D}\right)^2 + 4462.87\,\frac{ka^4}{D} + 2746189.04 = 0 \quad (4.47)$$

or

$$\Lambda^2 - (10.27 + .0064K^2)\Lambda + 8.32(10^{-6})K^4 + .037K^2 + 22.84 = 0 . \quad (4.48)$$

Equation (4.48) is plotted as the antisymmetric curve in Figure 4.3.

For the third order approximation, $m = 1, 2$, $n = 1, 2$, and equation (4.20) becomes

$$\begin{aligned} &A_{11}\{[2\alpha_1^4(1 + d_1^2) + \tfrac{k}{D}a^4]X_1Y_1 + 2\alpha_1^4(d_{11}^2X_1Y_1 + d_{11}d_{12}X_1Y_2 \\ &\quad + d_{11}d_{12}X_2Y_1) - 2\,\tfrac{\lambda h}{D}\,\alpha^2\alpha_1^2(2d_{11}X_1Y_1 + d_{12}X_1Y_2 + d_{12}X_2Y_1)\} \\ &\quad + A_{12}\{(\alpha_1^4 + \alpha_2^4 + \tfrac{k}{D}a^4)X_1Y_2 + 2\alpha_1^2\alpha_2^2(d_{11}d_{21}X_1Y_1 \\ &\quad + d_{12}d_{22}X_1Y_2 + d_{12}d_{21}X_2Y_1) - 2\,\tfrac{\lambda h}{D}\,a^2(\alpha_1^2d_{11}X_1Y_2 \\ &\quad + \alpha_2^2d_{21}X_1Y_1 + \alpha_2^2d_{22}X_1Y_2)\} \\ &\quad + A_{21}\{(\alpha_2^4 + \alpha_1^4 + \tfrac{k}{D}a^4)X_2Y_1 + 2\alpha_2^2\alpha_1^2(d_{11}d_{21}X_1Y_1 \\ &\quad + d_{21}d_{12}X_1Y_2 + d_{11}d_{22}X_2Y_1) - 2\,\tfrac{\lambda h}{D}\,a^2[\alpha_2^2(d_{21}X_1Y_1 \\ &\quad + d_{22}X_2Y_1) + \alpha_1^2d_{11}X_2Y_1]\} = 0 \end{aligned} \quad (4.49)$$

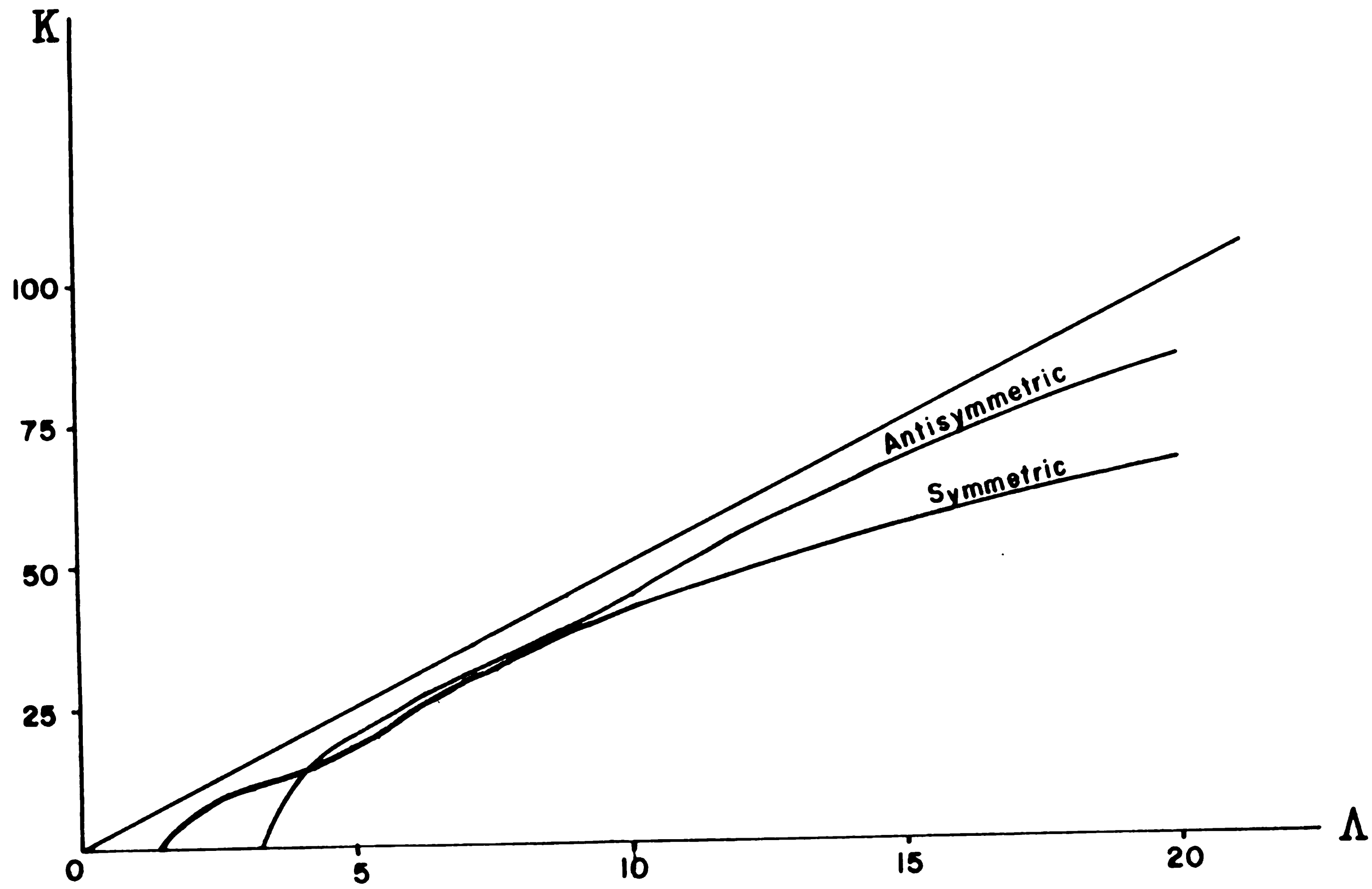

Figure 4.3 Buckling of a clamped rectangular plate on a Wrinkler foundation. Second approximation

which leads to the determinant

$$\begin{vmatrix} 2\alpha_1^4(1+d_{11}^2) + \frac{k}{D}a^4 - 4\frac{\lambda h}{D}a^2\alpha_1^2 d_{11} & 2\alpha_1^2\alpha_2^2 d_{11}d_{21} - 2\frac{\lambda h}{D}a^2\alpha_2^2 d_{21} & 2\alpha_2^2\alpha_1^2 d_{11}d_{21} - 2\frac{\lambda h}{D}a^2\alpha_2^2 d_{21} \\ 2\alpha_1^4 d_{11}d_{12} - 2\frac{\lambda h}{D}a^2\alpha_1^2 d_{12} & \alpha_1^4 + \alpha_2^4 + \frac{k}{D}a^4 + 2\alpha_1^2\alpha_2^2 d_{11}d_{22} - 2\frac{\lambda h}{D}a^2(\alpha_1^2 d_{11} + \alpha_2^2 d_{22}) & 2\alpha_2^2\alpha_1^2 d_{21}d_{12} \\ 2\alpha_1^4 d_{11}d_{12} - 2\frac{\lambda h}{D}a^2\alpha_1^2 d_{12} & 2\alpha_1^2\alpha_2^2 d_{12}d_{21} & \alpha_2^4 + \alpha_1^4 + \frac{k}{D}a^4 + 2\alpha_1^2\alpha_2^2 d_{11}d_{22} - 2\frac{\lambda h}{D}a^2(\alpha_2^2 d_{22} + \alpha_1^2 d_{11}) \end{vmatrix} = 0 \qquad (4.50)$$

The symmetric case may be written

$$\Lambda^3 - (2.526 \times 10^{-2}K^2 + 9.646)\Lambda^2 + (1.433 \times 10^{-4}K^4 + 1.83 \times 10^{-1}K^2 + 2.831 \times 10^1)\Lambda - 2.351 \times 10^{-7}K^6 - 5.349 \times 10^{-4}K^4 - 3.248 \times 10^{-1}K^2 - 2.293 \times 10^1 = 0, \qquad (4.51)$$

which is plotted as the symmetric curve in Figure 4.4, and the antisymmetric case may be written

$$\Lambda^3 - (8.315 \times 10^{-3}K^2 + 1.729 \times 10^1)\Lambda^2 + (2.046 \times 10^{-5}K^4 + 1.017 \times 10^{-1}K^2 + 9.488 \times 10^1)\Lambda - 1.572 \times 10^{-8}K^6 - 1.287 \times 10^{-4}K^4 - 3.041 \times 10^{-1}K^2 - 1.601 \times 10^2 = 0, \qquad (4.52)$$

which is plotted as the antisymmetric curve in Figure 4.4.

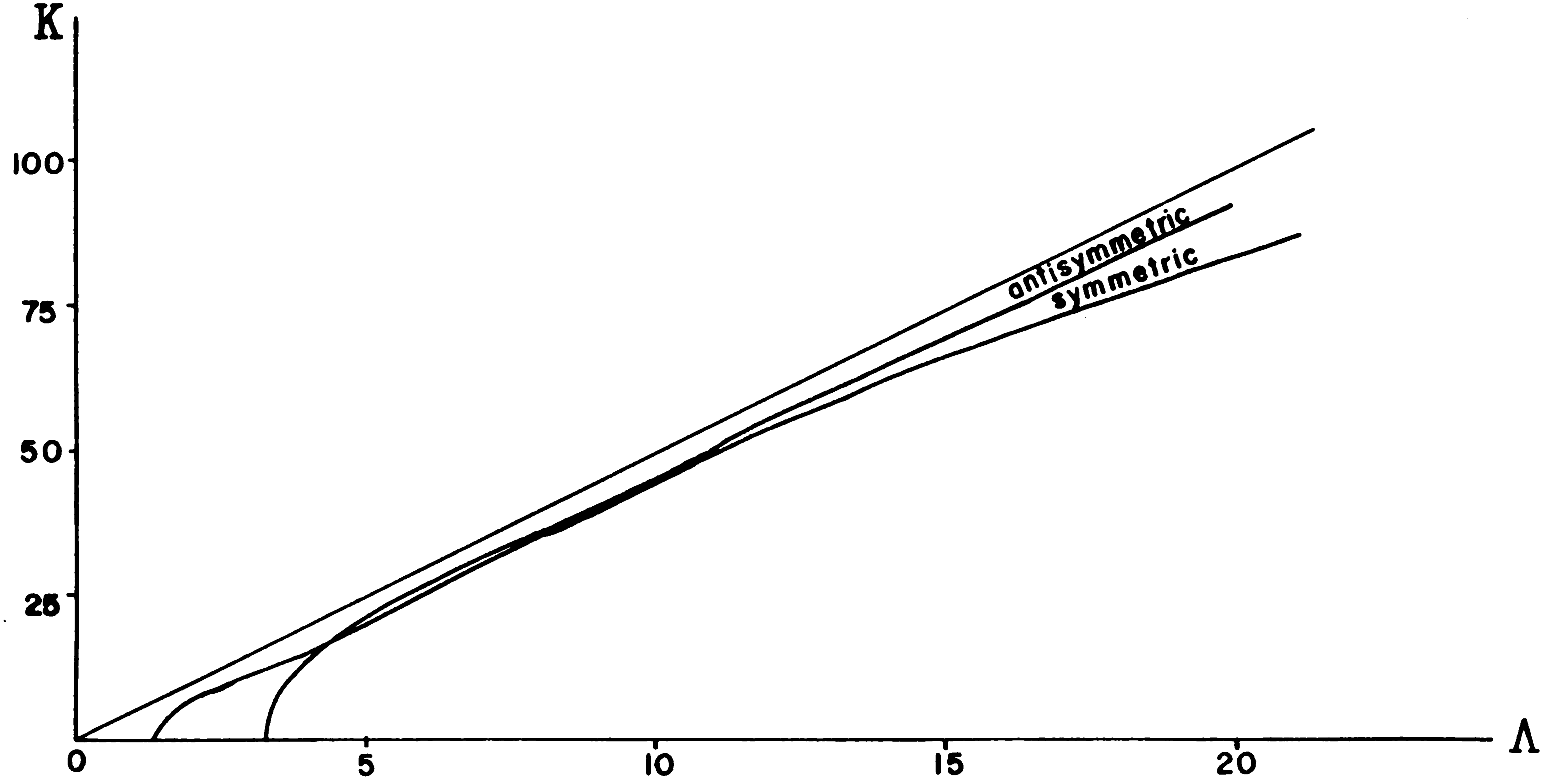

Figure 4.4 Buckling of a clamped rectangular plate on a Wrinkler foundation. Third approximation

The convergence for values of Λ near $K = 0$ is good for lower order approximations. Results for larger values of K or Λ requires the evaluation of higher order approximations. A ninth order determinate is therefore evaluated and the results are plotted in Figure 4.5. This is followed by a plot of Hetenyi's solution for buckling of a clamped beam on an elastic foundation in Figure 4.6.

The convergence for several orders of Λ vs. K for the symmetric and the antisymmetric case is plotted in Figures 4.7 and 4.8.

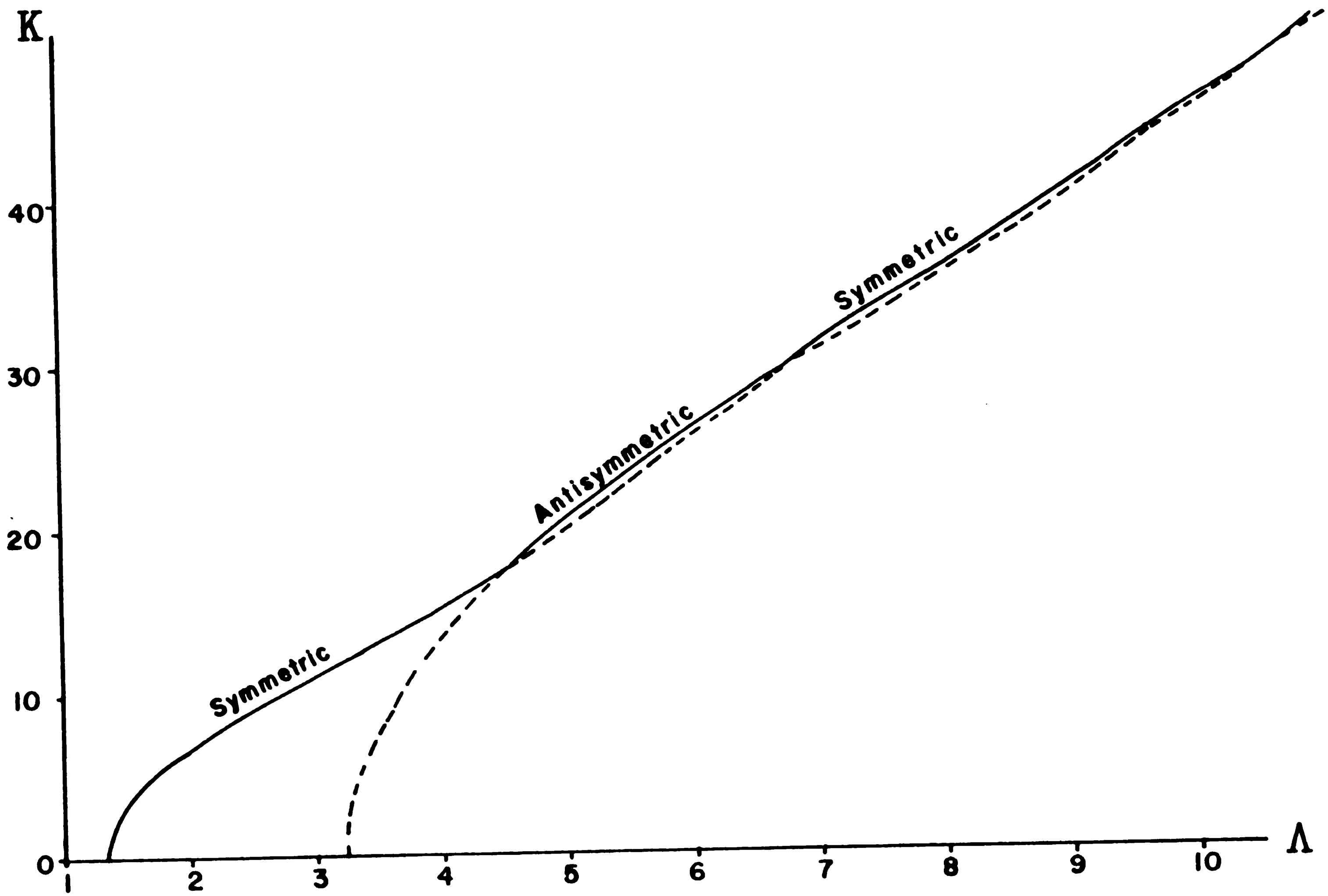

Figure 4.5 Buckling of a clamped rectangular plate on a Wrinkler foundation. Ninth approximation

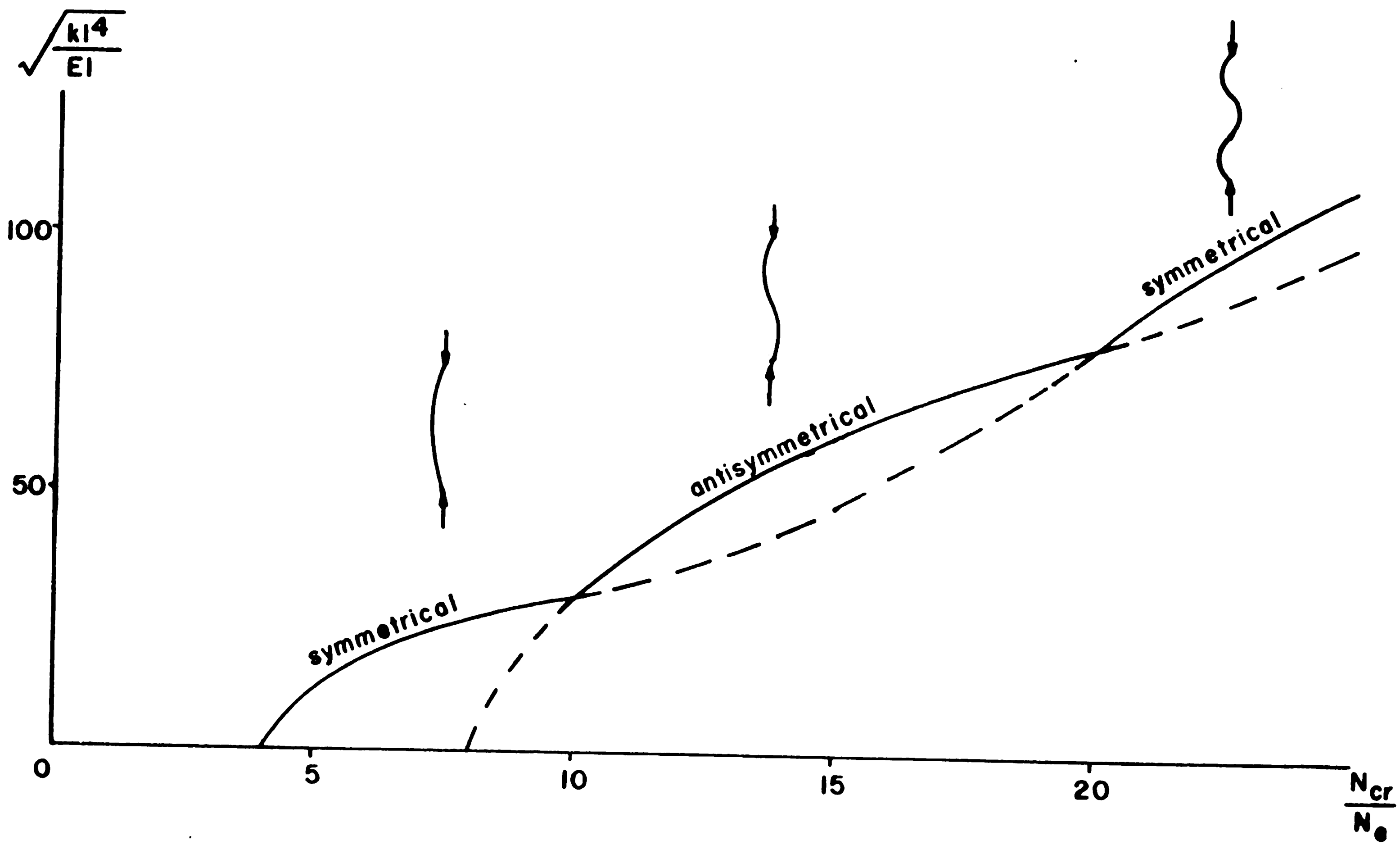

Figure 4.6 Buckling of a beam on a Wrinkler foundation. Clamped Case

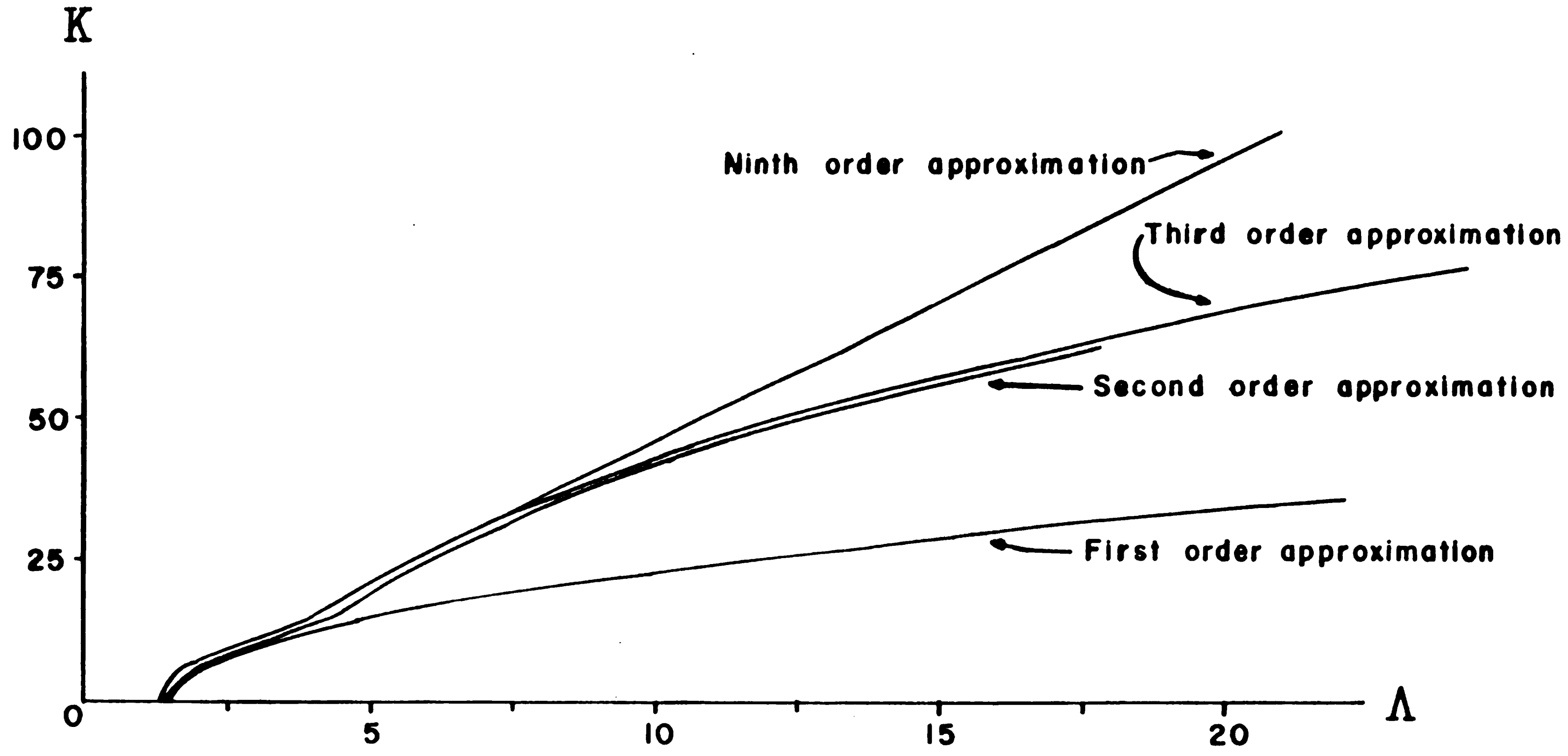

Figure 4.7 Convergence of the eigenvalues Λ, for each value of K for the symmetric case.

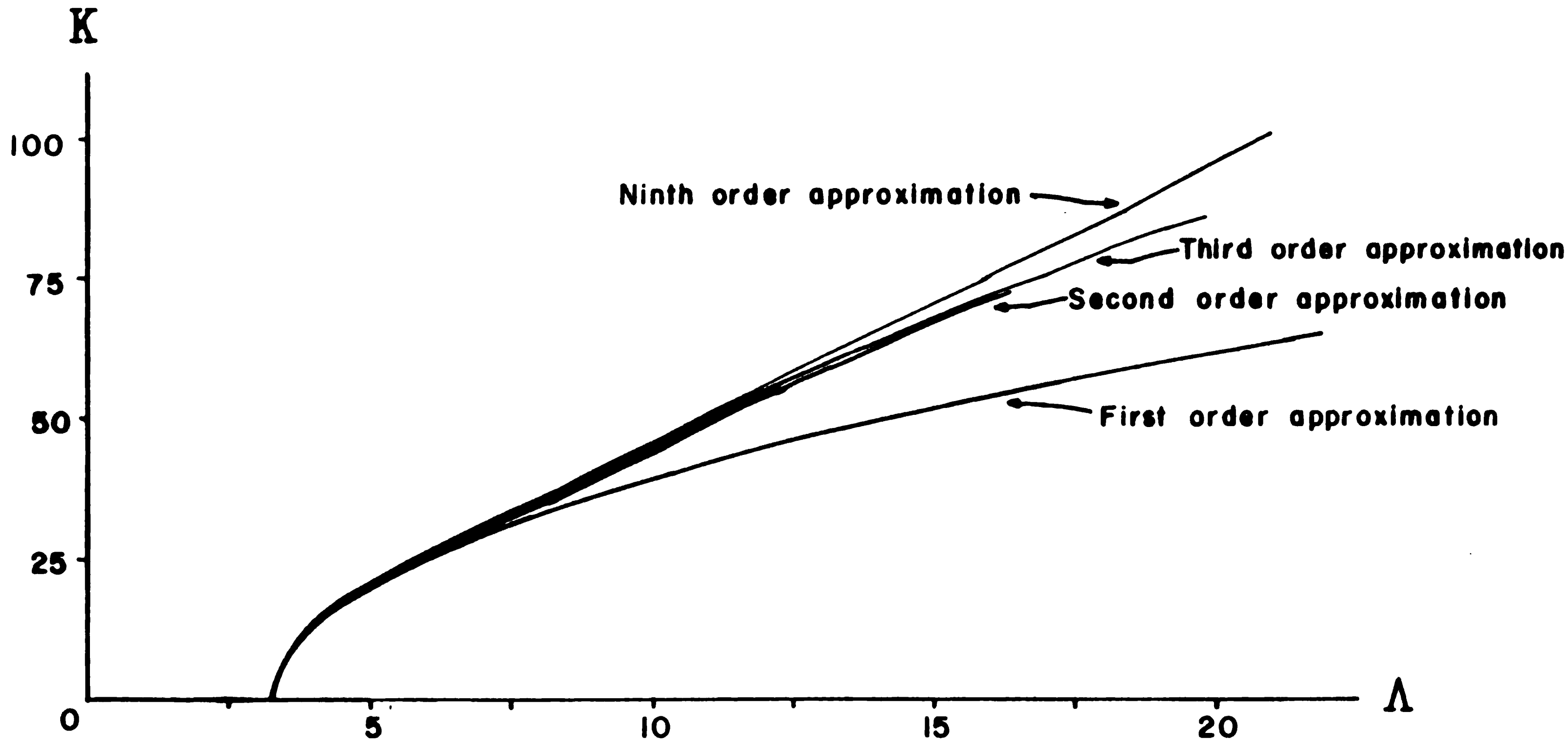

Figure 4.8 Convergence of the eigenvalues Λ, for each value of K for the antisymmetric case.

The Case of Non-Uniform Loading IV.2

The case for non-uniform loading leads to equation (3.40):

$$L_1 W_1 + 2\frac{\lambda h}{D}\left(q\frac{\partial^2}{\partial x^2} + p\frac{\partial^2}{\partial y^2}\right)W_1 + \frac{k}{D}W_1 = 0 \qquad (4.53)$$

Using equations (4.02) through (4.13), equation (4.21) is replaced by

$$W = \sum_m \sum_n A_{mn}\Big[(\alpha_m^4 + \alpha_n^4 + \frac{k}{D}a^4)X_m Y_n + 2\alpha_m^2\alpha_n^2 \sum_s d_{ms}X_s \sum_r d_{nr}Y_r - 2\frac{\lambda h}{D}a^2(\alpha_m^2 q Y_n \sum_s d_{ms}X_s + \alpha_n^2 p X_m \sum_r d_{nr}Y_r)\Big] = 0. \qquad (4.54)$$

The first term approximation $(n = m = 1)$ yields

$$\Big[A_{11}(2\alpha_1^4(1 + d_{11}^2) + \frac{k}{D}a^4 - 2\frac{\lambda h}{D}a^2\alpha_1^2 d_{11}(p+q)\Big]X_1Y_1 = 0 \qquad (4.55)$$

or

$$2\alpha_1^4(1 + d_{11}^2) + \frac{k}{D}a^4 - 2\frac{\lambda h}{D}a^2\alpha_1^2 d_{11}(p+q) = 0 \qquad (4.56)$$

which, for the symmetric case becomes

$$81.5 + \frac{k}{D}a^4 - 6.15\frac{\lambda h a^2}{D}(p+q) = 0 \qquad (4.57)$$

Solving for λ:

$$\lambda = \frac{13.25D}{a^2(p+q)h} + \frac{ka^2}{(6.15)(p+q)h}, \qquad (4.58)$$

which implies

$$\sigma_{cr} = -2\lambda = -\frac{26.5D}{a^2h(p+q)} - \frac{ka^2}{3.08h(p+q)}. \qquad (4.58)$$

For $p = 0$,

$$\sigma_{cr} = - \frac{10.74\pi^2 D}{a^2 h}$$

which compares with

$$\sigma_{cr} = \frac{10.07\pi^2 D}{a^2 h}$$

from Timoshenko and Gere [17].

CHAPTER V

CONCLUSIONS

The non-linear Föppl-von Karman plate equations are written in operator form and linearized using a perturbation technique. A choice of the Airy stress function ϕ leads to a linear homogeneous partial differential equation with homogeneous boundary conditions. The problem becomes one of determining the parameter λ for which non-trivial solutions exist.

The buckling solution to the rectangular plate on an elastic foundation shows a similarity to the solution for the circular plate on an elastic foundation. In both cases the elastic foundation causes the plates to assume a shape dependent on the stiffness of the foundation. Also, if the foundation is absent or if the foundation stiffness is very weak, the plates buckle into the first mode with no nodal lines. For stiffer foundations, the first buckling mode can assume shapes with many nodes.

An interesting phenomenon of the circular plate in the linearized case was an even distribution of the "ridges" and "valleys" for a given foundation stiffness. As the edge load was increased, the solution of the resulting non-linear boundary value problem yielded a boundary layer effect or a migration of the ridges and valleys to the edge of the plate. It would be of interest to solve the non-linear boundary value problem of the rectangular plate to detect the presence of a boundary layer effect.

The perturbation technique used in the present research provides a means of going beyond the linearized case, although this presents difficulty owing the nature of a two dimensional operator. Solutions in the circular case were easier to handle because the axisymmetric loading led to an ordinary differential equation. The additional sequence of differential equations beyond the first linearized equation may provide an advantage over or be an aid to other methods of solution in post buckling behavior as in the case treated by Friedrichs and Stoker. This, of course, can only be made apparent by further investigation.

A critical load vs. foundation stiffness plot was made for both the simply supported and the clamped cases and compared to plots of beams under the same boundary conditions. This became an aid in determining the progression of buckling from symmetric to antisymmetric and back to symmetric buckling as the foundation stiffness increased.

A closed form solution for the buckling of a plate clamped on four sides does not exist, consequently a numerical method was used. The technique chosen was singled out for its lack of numerical computation and for its rapidity of convergence. When $K = 0$, good results are obtained using only a second order determinant. However, for large values of K it becomes necessary to evaluate determinants of much higher order.

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