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FOR THE SOLUTION OF
ANISOTROPIC THIN PLATE BENDING PROBLEMS

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A NEW METHOD
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By

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ABSTRACT

A NEW METHOD
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By

Benjamin Chin-wen Wu

A numerical method for the solution of thin plate problems is presented. With the conventional assumptions for thin plates implied, plates of arbitrary plan form, subjected to arbitrary loading and boundary conditions, and made of anisotropic material are considered. This method is developed from the concept of the indirect boundary integral method.

The indirect boundary integral method uses the Green's function of a clamped circular plate of isotropic material. To solve an isotropic thin plate problem, the first step is to embed the real plate into the fictitious clamped circular plate for which the Green's function is known. Along the embedded contour, N points are prescribed, at which the boundary conditions for the original problem are specified. The numerical solution of the problem is then to find the magnitude of the set of N line forces and N ring moments imposed along the embedded contour such that the boundary conditions at the N boundary points are satisfied. With this method, problems with clamped and simply supported boundaries can be easily solved. For a free edge, however, due to the logarithmic nature of the Green's function and the fact that fourth order derivatives must be taken for the fictitious ring moments in the boundary condition equations, there are second order singularity difficulties during the numerical integration along the embedded contour.

In this thesis, three major modifications are introduced. These are (1)the set of fictitious moments are replaced by an additional set of fictitious forces, and the entire set of fictitious forces is located outside of the embedded contour, (2)the numerical integration is replaced by a simple summing process, and (3)the Green's function for a clamped circular plate is replaced by the Green's function of an infinite plate. With these modifications, significant improvements in solution accuracy and computing efficiency have been achieved. The second order singularity difficulties associated with free edges are avoided, and due to the simplicity of the new method, the computing costs are reduced by about sixty percent. Since the Green's functions for orthotropic and anisotropic infinite plates are also available, the new method is readily extended to orthotropic and anisotropic thin plate bending problems.

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INTRODUCTION

The thin plate bending problem is one of the most common problems in structural engineering. Design engineers encounter it daily. Plate theories and methods of solution can be traced back to the early eighteenth century. Famous names like Euler, Bernoulli, Lagrange, Navier, and Kirchhoff were all involved in the development of plate theories. In this century, Nadai, Love, Huber, Timoshenko, Lekhnitskii, von Karman, and Reissner, to name a few, are well-known for their work related to plate problems.

Mathematically, the thin plate bending problem is a typical boundary value problem. Since solving the problem reduces to finding a solution satisfying the governing fourth order partial differential equation and all the boundary condition equations, exact solutions are available only for special cases. In addition to many common methods for a wide range of problems shown in [1], the method of complex variables has been successfully applied [2,3,4,5,6] solving many additional problems. However, for a generalized problem, numerical techniques such as finite difference and finite element methods must be employed, [7,8].

In this dissertation, a different numerical method is introduced. Developed from the concept of an indirect boundary integration equation method [9], the new numerical method employs a known Green's function, the scheme of embedding the real plate in a fictitious plate for which the Green's function is known, and the imposition of fictitious forces so that all the boundary conditions are satisfied. This method is very effective because of the simplicity of its formulation. It can solve constant thickness plate problems with arbitrary plan form, arbitrary loading and boundary conditions, and anisotropic material properties. Since the fictitious forces are located far away from the plate boundary, this method gives more accurate results near the boundary than does the boundary integral method.

The procedure of this method can be summarized in three steps. The first is to find a Green's function of a certain type of plate problem. Many of them are available. Take isotropic problems for example. The Green's functions for a clamped circular plate [1,3,10], and for an infinite plate [1] are two possible choices. The second step is to embed the real plate in the aforementioned fictitious plate for which the Green's function is known, and prescribing a set of N boundary points. Since there are two boundary condition equations associated with each boundary point, a set of 2N fictitious forces are placed around the plate boundary. Solution of the problem, therefore, involves the determination of this set of 2N fictitious forces such that the 2N boundary condition equations are satisfied. The third step is to superimpose these 2N fictitious forces onto the actual loadings of the plate and compute the deflections and bending moments inside the plate.

There are four chapters in this dissertation. The governing fourth order partial differential equations for isotropic, orthotropic, and anisotropic plate problems and their associated boundary condition equations are reviewed in Chapter I. Chapter II introduces the indirect boundary integral method originally derived by Altiero and Sikarskie [9]. Their method is modified by moving the integration contour to the outside of the plate boundary. In so doing, the second order singularity difficulties for the free edge boundary conditions, encountered in their work, are avoided. In the meantime, a significant improvement in the solution accuracy is noticed. Chapter III illustrates the new point-force method for the solution of a generalized thin plate bending problem. The point-force method contains three major alterations over the boundary integral method, though the basic concept remains. The three changes are (1)the set of fictitious moments are replaced by a second set of fictitious forces, (2)the integration is replaced by an algebraic summing process, and (3)the Green's function for a clamped circular plate is replaced by the Green's function of an infinite plate. These changes have made the original method more effective. A saving of sixty percent for computing costs is realized. More importantly, it is due to the successful use of the Green's function of an isotropic infinite plate and, the Green's functions for orthotropic and anisotropic infinite plates are readily known [11,12,13], the new method can be extended to solve general orthotropic and anisotropic plate problems as well. Chapter IV presents several comments regarding to this new point-force method.

In order to verify the method, several test cases have been solved. The results are tabulated and graphed, and the computer programs are included in the Appendices. Though these computer programs are specifically designed for the example problems, they can be easily revised to accommodate general problems.

CHAPTER I

PRELIMINARIES ON PLATE THEORY

I.1 GOVERNING EQUATIONS

Following the assumptions involved in the well-known Kirchhoff-Love small deflection plate theory, all of the stress components within the plate can be expressed in terms of the vertical deflections, $w(x,y)$. Therefore, for static equilibrium, the governing differential equation can be derived in terms of the deflection function and the two independent coordinate variables x and y. For the three material types, namely isotropic, orthotropic, and anisotropic, the derivation of the governing differential equations can be found in most texts on plate theory [1,7,14].

Assuming constant thickness, for isotropic plates, the governing differential equation is

$$\frac{\partial^4 w(x,y)}{\partial x^4} + 2\,\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y} = \frac{q(x,y)}{D} \tag{1}$$

where $w(x,y)$ is the vertical deflection of the plate after bending, $q(x,y)$ is the load in the vertical direction, and D is the flexural rigidity of the plate defined by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

where E is the Young's modulus, ν is the Poisson's ratio, and h is the plate thickness.

For an orthotropic plate which has its geometric coordinates aligned with the principal material directions, the governing equation is

$$D_x \frac{\partial^4 w(x,y)}{\partial x^4} + 2H \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w(x,y)}{\partial y^4} = q(x,y), \qquad (2)$$

where,

$$D_x = \frac{E_x \cdot h^3}{12(1-\nu_x \nu_y)}; \quad D_y = \frac{E_y \cdot h^3}{12(1-\nu_x \nu_y)}; \text{ and } H = D_x \nu_y + \frac{Gh^3}{12} .$$

The subscripts x and y indicate the principal directions of the material constants, and G is the modulus of rigidity. Finally, the governing equation for an anisotropic plate, with one plane of material symmetry parallel to the middle surface of the plate is

$$D_{11} \frac{\partial^4 w(x,y)}{\partial x^4} + 4D_{16} \frac{\partial^4 w(x,y)}{\partial x^3 \partial y} + 2(D_{12}+2D_{66}) \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2}$$

$$+ 4D_{26} \frac{\partial^4 w(x,y)}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w(x,y)}{\partial y^4} = q(x,y), \qquad (3)$$

where D_{ij} are associated with the material constants, and are determined as follows:

$$D_{11} = \frac{h^3}{12} \frac{(a_{22}a_{66}-a_{26}^2)}{\text{det.}}; \quad D_{22} = \frac{h^3}{12} \frac{(a_{11}a_{66}-a_{16}^2)}{\text{det.}};$$

$$D_{66} = \frac{h^3}{12} \frac{(a_{11}a_{22}-a_{12}^2)}{\text{det.}}; \quad D_{12} = \frac{h^3}{12} \frac{(a_{16}a_{26}-a_{12}a_{66})}{\text{det.}};$$

$$D_{16} = \frac{h^3}{12} \frac{(a_{12}a_{26}-a_{22}a_{16})}{\text{det.}}; \quad D_{26} = \frac{h^3}{12} \frac{(a_{12}a_{16}-a_{11}a_{26})}{\text{det.}};$$

and,

$$\text{det.} = \begin{vmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{vmatrix} \quad ;$$

a_{ij} is the material constant matrix, i.e.,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{Bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} .$$

The coefficient, a_{ij}, are:

$$a_{11} = \frac{1}{E_x} \quad ; \quad a_{12} = -\frac{\nu_x}{E_x} \quad ; \quad a_{16} = \frac{\eta_{x,xy}}{G_{xy}} = \frac{\eta_{xy,x}}{E_x} \quad ;$$

$$a_{22} = \frac{1}{E_y} \quad ; \quad a_{66} = \frac{1}{G_{xy}} \quad ; \quad a_{26} = \frac{\eta_{y,xy}}{G_{xy}} = \frac{\eta_{xy,y}}{E_y} \quad ;$$

where $\eta_{xy,x}$, $\eta_{xy,y}$, and $\eta_{x,xy}$, $\eta_{y,xy}$ are called the coefficients of mutual influence of the first kind and the second kind, respectively, [15,16]. Physically, it is clear that they represent mutual influences between shear strains and normal stresses and between normal strains and shear stresses.

I.2 BOUNDARY CONDITIONS

Only the three major types of boundary conditions, namely, (1)clamped, (2)simply supported, and (3)free, are considered in this dissertation. Many others such as elastically-supported edges can also be handled with minor changes. The equations associated with these three major types of boundary conditions are

(1) Rigidly clamped edge (B_c):

$$w(x,y) = 0 \quad ; \quad \frac{\partial w(x,y)}{\partial n} = 0 \quad . \tag{4a}$$

(2) Simply supported edge (B_s):

$$w(x,y) = 0 \quad ; \quad M_n = 0 \quad . \tag{4b}$$

(3) Free edge (B_f):

$$M_n = 0 \quad ; \quad N_n + \frac{\partial H_{tn}}{\partial s} = 0 \quad , \tag{4c}$$

where $\frac{\partial}{\partial s}$ is the derivative along the contour arc, M_n is the unit edge bending moment, N_n is the unit edge shear force, H_{tn} is the unit edge twisting moment, and the subscript n means acting along the normal direction of the edge. These values can be written in terms of their components in the x and y directions as

$$M_n = M_x \cdot n_x^{\,2} + M_y \cdot n_y^{\,2} + 2H_{xy} \cdot n_x \cdot n_y \tag{5a}$$

$$H_{tn} = (M_y - M_x) \cdot n_x \cdot n_y + H_{xy} \cdot (n_x^{\,2} - n_y^{\,2}) \tag{5b}$$

$$N_n = N_x \cdot n_x + N_y \cdot n_y \tag{5c}$$

where n_x and n_y are direction cosines of an outward normal to the contour arc. For an anisotropic problem, the bending moment, twisting moment, and shear force in the x and y directions can be written in terms of the deflection function w(x,y) as

$$M_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{6a}$$

$$M_y = -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{6b}$$

$$H_{xy} = -\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{6c}$$

$$N_x = -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + 3D_{16}\frac{\partial^3 w}{\partial x^2 \partial y} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x \partial y^2} + D_{26}\frac{\partial^3 w}{\partial y^3}\right] \tag{6d}$$

$$N_y = -\left[D_{16}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x^2 \partial y} + 3D_{26}\frac{\partial^3 w}{\partial x \partial y^2} + D_{22}\frac{\partial^3 w}{\partial y^3}\right] \tag{6e}$$

Substituting Eqs. (5) and (6) into Eq. (4), the boundary condition equations for anisotropic plates can be written explicitly as

$$w(x,y) = 0 \qquad \text{on } B_c + B_s \tag{7a}$$

$$\frac{\partial w(x,y)}{\partial x}\cdot n_x + \frac{\partial w(x,y)}{\partial y}\cdot n_y = 0 \qquad \text{on } B_c \tag{7b}$$

$$\frac{\partial^2 w(x,y)}{\partial x^2}\left[D_{11}\cdot n_x^2 + 2D_{16}\cdot n_x\cdot n_y + D_{12}\cdot n_y^2\right]$$

$$+ \frac{\partial^2 w(x,y)}{\partial x \partial y}\left[2D_{16}\cdot n_x^2 + 4D_{66}\cdot n_x\cdot n_y + 2D_{26}\cdot n_y^2\right] +$$

$$+ \frac{\partial^2 w(x,y)}{\partial y^2} [D_{12} \cdot n_x{}^2 + 2D_{26} \cdot n_x \cdot n_y + D_{22} \cdot n_y{}^2] = 0 \text{ on } B_s + B_f \tag{7c}$$

$$\frac{\partial^3 w(x,y)}{\partial x^3} [D_{11} \cdot n_x (1+n_y{}^2) + 2D_{16} \cdot n_y{}^3 - D_{12} \cdot n_x \cdot n_y{}^2]$$

$$+ \frac{\partial^3 w(x,y)}{\partial x^2 \partial y} [4D_{16} \cdot n_x + D_{12} \cdot n_y (1+n_x{}^2) + 4D_{66} \cdot n_y{}^3 - D_{11} \cdot n_x{}^2 \cdot n_y$$

$$-2D_{26} \cdot n_x \cdot n_y{}^2] + \frac{\partial^3 w(x,y)}{\partial x \partial y^2} [4D_{26} \cdot n_y + D_{12} \cdot n_x (1+n_y{}^2) + 4D_{66} \cdot n_x{}^3$$

$$-D_{22} \cdot n_x \cdot n_y{}^2 - 2D_{16} \cdot n_x{}^2 \cdot n_y] + \frac{\partial^3 w(x,y)}{\partial y^3} [D_{22} \cdot n_y (1+n_x{}^2)$$

$$+2D_{26} \cdot n_x{}^3 - D_{12} \cdot n_x{}^2 \cdot n_y] = 0 \qquad \text{on } B_f \tag{7d}$$

B_c is the clamped portion of the boundary B, B_s is the simply supported portion of B, and B_f is the free portion of B. Clearly, $B=B_c+B_s+B_f$. For orthotropic plate problems, with D_{11}, D_{22}, and D_{66} being replaced by D_x, D_y, and D_k, respectively; D_{12} by $\nu_x D_y$ or $\nu_y D_x$; and $D_{16}=D_{26}=0$; Eqs.(6) and (7) can be reduced to

$$M_x = -D_x \left[\frac{\partial^2 w(x,y)}{\partial x^2} + \nu_y \frac{\partial^2 w(x,y)}{\partial y^2} \right] \tag{8a}$$

$$M_y = -D_y \left[\frac{\partial^2 w(x,y)}{\partial y^2} + \nu_x \frac{\partial^2 w(x,y)}{\partial x^2} \right] \tag{8b}$$

$$H_{xy} = -2D_k \frac{\partial^2 w(x,y)}{\partial x \partial y} \tag{8c}$$

$$N_x = -\frac{\partial}{\partial x}\left[D_x \frac{\partial^2 w(x,y)}{\partial x^2} + H\frac{\partial^2 w(x,y)}{\partial y^2}\right] \tag{8d}$$

$$N_y = -\frac{\partial}{\partial y}\left[H \frac{\partial^2 w(x,y)}{\partial x^2} + D_y\frac{\partial^2 w(x,y)}{\partial y^2}\right] \tag{8e}$$

where,

$$D_k = \frac{G \cdot h^3}{12}, \quad \text{and} \quad H = D_x\nu_y + 2D_k$$

$$w(x,y) = 0 \qquad \text{on } B_c + B_s \tag{9a}$$

$$\frac{\partial w(x,y)}{\partial x}\cdot n_x + \frac{\partial w(x,y)}{\partial y}\cdot n_y = 0 \qquad \text{on } B_c \tag{9b}$$

$$\frac{\partial^2 w(x,y)}{\partial x^2}(D_x\cdot n_x{}^2 + \nu_x D_y\cdot n_y{}^2) + \frac{\partial^2 w(x,y)}{\partial y^2}(D_y\cdot n_y{}^2 + \nu_y D_x\cdot n_x{}^2)$$

$$+ \frac{\partial^2 w(x,y)}{\partial x\partial y}(4D_k\cdot n_x\cdot n_y) = 0 \qquad \text{on } B_s + B_f \tag{9c}$$

$$\frac{\partial^3 w(x,y)}{\partial x^3}[D_x\cdot n_x{}^3 + D_x\cdot n_x\cdot n_y{}^2(2-\nu_y)]$$

$$+ \frac{\partial^3 w(x,y)}{\partial x^2\partial y}[\nu_x D_y\cdot n_y(1+n_x{}^2)+4D_k\cdot n_y{}^3-D_x\cdot n_x{}^2\cdot n_y]$$

$$+ \frac{\partial^3 w(x,y)}{\partial x\partial y^2}[\nu_y D_x\cdot n_x(1+n_y{}^2)+4D_k\cdot n_x{}^3-D_y\cdot n_x\cdot n_y{}^2]$$

$$+ \frac{\partial^3 w(x,y)}{\partial y^3}[D_y\cdot n_y{}^3+ D_y\cdot n_x{}^2\cdot n_y(2-\nu_x)] = 0 \qquad \text{on } B_f \tag{9d}$$

For isotropic plate problems, these two sets of equations can be further reduced by having $D_k = \frac{D(1-\nu)}{2}$, $\nu_x=\nu_y=\nu$, and $D_x=D_y=H=D$:

$$M_x = -D\left[\frac{\partial^2 w(x,y)}{\partial x^2} + \nu\frac{\partial^2 w(x,y)}{\partial y^2}\right] \tag{10a}$$

$$M_y = -D\left[\frac{\partial^2 w(x,y)}{\partial y^2} + \nu\frac{\partial^2 w(x,y)}{\partial x^2}\right] \tag{10b}$$

$$H_{xy} = -D(1-\nu)\frac{\partial^2 w(x,y)}{\partial x\partial y} \tag{10c}$$

$$N_x = -D\frac{\partial}{\partial x}\left[\nabla^2 w(x,y)\right] \tag{10d}$$

$$N_y = -D\frac{\partial}{\partial y}\left[\nabla^2 w(x,y)\right] \tag{10e}$$

where,

$$\nabla^2 w(x,y) = \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}$$

and;

$$w(x,y) = 0 \qquad \text{on } B_c + B_s \tag{11a}$$

$$\frac{\partial w(x,y)}{\partial x}\cdot n_x + \frac{\partial w(x,y)}{\partial y}\cdot n_y = 0 \qquad \text{on } B_c \tag{11b}$$

$$\frac{\partial^2 w(x,y)}{\partial x^2}(n_x{}^2+\nu n_y{}^2) + \frac{\partial^2 w(x,y)}{\partial y^2}(n_y{}^2+\nu n_x{}^2)$$

$$+ \frac{\partial^2 w(x,y)}{\partial x\partial y}\, 2(1-\nu)\cdot n_x\cdot n_y = 0 \qquad \text{on } B_s + B_f \tag{11c}$$

$$\frac{\partial^3 w(x,y)}{\partial x^3} n_x[1+n_y^2(1-\nu)] + \frac{\partial^3 w(x,y)}{\partial y^3} n_y[1+n_x^2(1-\nu)]$$

$$+ \frac{\partial^3 w(x,y)}{\partial x^2 \partial y} n_y[(2\nu-1)n_x^2 + (2-\nu)\cdot n_y^2]$$

$$+ \frac{\partial^3 w(x,y)}{\partial x \partial y^2} n_x[(2\nu-1)n_y^2 + (2-\nu)\cdot n_x^2] = 0 \qquad \text{on } B_f \qquad (11d)$$

I.3 THE GREEN'S FUNCTION METHOD

Solving a plate problem is, mathematically, to find the deflection function $w(x,y)$ such that the governing differential equation as well as the prescribed boundary conditions are all satisfied. For the special problem of a concentrated force applied at an arbitrary location, the solution is called the "Green's function" of the problem, and is often written as $G(x,y;\xi,\eta)$. That is, with the prescribed boundary conditions, a Green's function will provide the deflection at any point (x,y) when there is a concentrated force located at some point (ξ,η). The deflection function for a distributed load $q(x,y)$ over a region R inside the plate can be written, using superposition, as:

$$w(x,y) = \iint_R G(x,y;\xi,\eta)\cdot q(\xi,\eta)\,d\xi d\eta \qquad (12)$$

This is called the Green's function method or the influence function method, [1]. The Green's function can be either in closed form or in infinite series form, and varies with the problem. For isotropic problems, there are many Green's functions available. Some examples are given here.

The Green's function for a clamped circular plate is [1,3,10],

$$G(x,y;\xi,\eta) = \frac{1}{16\pi Da^2}\{(a^2-x^2-y^2)(a^2-\xi^2-\eta^2)$$

$$+[a^2(x-\xi)^2+a^2(y-\eta)^2]\,\ell n\frac{a^2(x-\xi)^2+a^2(y-\eta)^2}{(a^2-x^2-y^2)(a^2-\xi^2-\eta^2)+a^2(x-\xi)^2+a^2(y-\eta)^2}\} \tag{13}$$

where a is the plate radius.

There are two well known Green's functions for a simply supported rectangular plate, and both of them are of the infinite series type. The one with double trigonometric series is called the Navier's solution, while the single trigonometric series function is named after Levy, [1,7]

$$G(x,y;\xi,\eta)=\frac{4}{D\pi^4 ab}\sum_m\sum_n\frac{\sin\frac{m\pi x}{a}\,\sin\frac{m\pi y}{a}\,\sin\frac{m\pi\xi}{a}\,\sin\frac{m\pi\eta}{a}}{(\frac{m^2}{a^2}+\frac{n^2}{a^2})^2} \tag{14}$$

$m=1,2,3,4,\ldots\infty;\quad n=1,2,3,4,\ldots\infty$

and,

$$G(x,y;\xi,\eta)=\frac{a^2}{D\pi^3}\sum_m\left(1+\beta_m\coth\beta_m-\frac{\beta_m y}{b}\coth\frac{\beta_m y}{b}-\frac{\beta_m\eta}{b}\coth\frac{\beta_m\eta}{b}\right)$$

$$\times\ \frac{\sinh\frac{\beta_m\eta}{b}\,\sinh\frac{\beta_m y}{b}\,\sin\frac{m\pi x}{a}\,\sin\frac{m\pi\xi}{a}}{m^3\sinh\beta_m} \tag{15}$$

where, $\beta_m=\frac{m\pi b}{a}$, and $m=1,2,3,4,\ldots\infty$; if $y<\eta$, replace η by $b-\eta$; if $y\geq\eta$, replace y by b-y.

There is also a Green's function for an infinite plate, [1]

$$G(x,y;\xi,\eta)=\frac{1}{16\pi D}[(x-\xi)^2+(y-\eta)^2]\,\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2} \tag{16}$$

where a is an arbitrary reference radius.

There are also Green's functions for orthotropic and anisotropic plates [11,12,13,14], though not as many. They will be discussed later.

The major limitation to the Green's function method is that one must know the Green's function of a specific plate shape and boundary conditions before Eq. (12) can be employed.

CHAPTER II

INTEGRAL EQUATIONS APPROACHES

II.1 THE BOUNDARY INTEGRAL METHOD

Several investigators have applied boundary-integral techniques to thin plate problems. Jaswon, et al,[17,18,19], have developed a "direct" approach. Altiero and Sikarskie [9] have developed an "indirect" version of the boundary integral equation. The indirect approach is outlined here. The problem of interest is a thin plate of arbitrary plan form and arbitrary boundary conditions, subjected to an arbitrarily-distributed load, Figure 1. However, due to numerical difficulties in the evaluation of second order singularities for a free edge, plates with free edges were excluded in [9].

A plate problem is solved similar to an elasticity problem [20,21]. The plate of interest is embedded in a fictitious plate of the same material for which the Green's function is known. In order to satisfy the boundary conditions of the original problem, a set of fictitious line forces and a set of fictitious ring moments are introduced along the boundary of the embedded plate. The problem is therefore solved if the magnitude of these fictitious forces and moments can be determined such that the original boundary conditions are satisfied.

Knowing that the influence function for a point moment is simply the derivative of the Green's function for a point force with respect to the direction at which the moment is oriented, the influence function for the moment

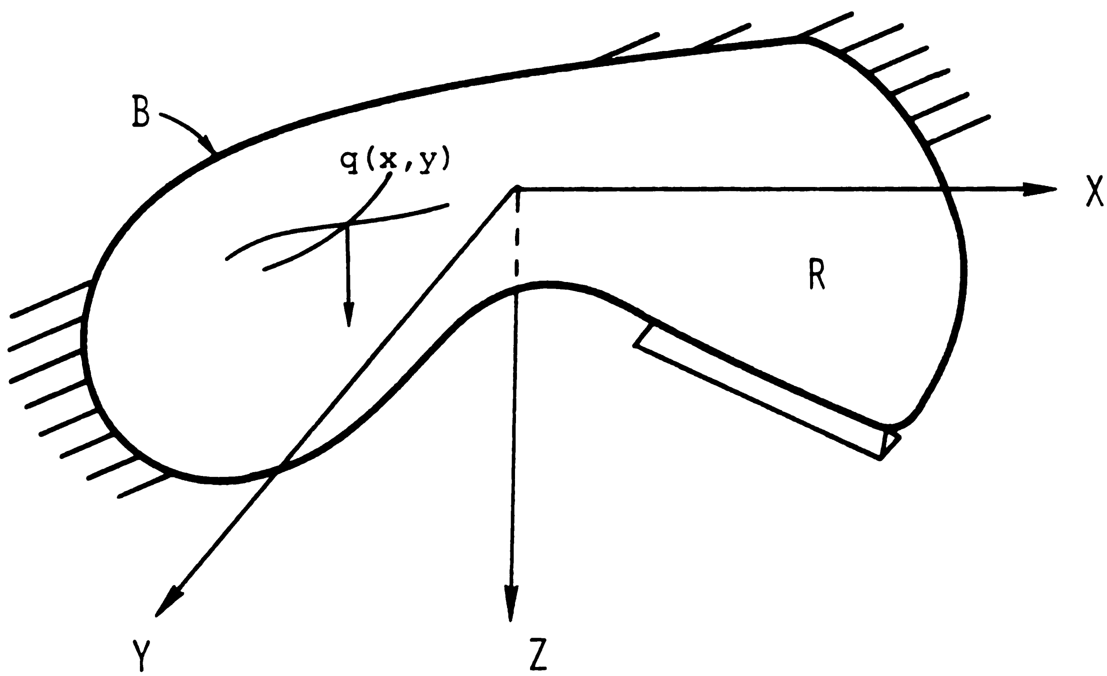

Figure 1. Problem of Interest

can be easily derived:

$$H(x,y;\xi,\eta) = -\frac{\partial G(x,y;\xi,\eta)}{\partial \bar{n}} = -\frac{\partial G}{\partial x}\cdot n_x - \frac{\partial G}{\partial y}\cdot n_y \tag{17}$$

where $G(x,y;\xi,\eta)$ is the known Green's function of a point force located at (ξ,η); $H(x,y;\xi,\eta)$ is the derived influence function of a point moment located at the same place (ξ,η) which is oriented along the direction $\bar{n}$, and n_x and n_y are the direction cosines of the outward normal vector $\bar{n}$ with respect to the x and y coordinates, respectively. The negative sign is for convention only.

Let P* represent a set of fictitious forces, M_n* represent a set of fictitious moments, and $q(x,y)$ be the given distributed lateral load. By superposition, the deflection at any point (x,y) can therefore be computed from the following deflection equation.

$$w(x,y)=\iint_R q(\xi,\eta)G(x,y;\xi,\eta)d\xi d\eta+\oint_{B^*} P^*(\xi,\eta)G(x,y;\xi,\eta)ds(\xi,\eta)$$
$$+\oint_{B^*} M_n^*(\xi,\eta)\left[-n_x(\xi,\eta)\frac{\partial G(x,y;\xi,\eta)}{\partial \xi}-n_y(\xi,\eta)\frac{\partial G(x,y;\xi,\eta)}{\partial \eta}\right]ds(\xi,\eta) \tag{18}$$

where R is the region over which the distributed load $q(x,y)$ is prescribed, B is the boundary of the embedded plate, and $n_x(\xi,\eta)$ and $n_y(\xi,\eta)$ are the direction cosines of the unit outward normal to the plate boundary at the loading point (ξ,η). Values of P* and M_n^* are to be determined from the boundary condition equations. The boundary condition equations are shown in Eq. (4), and they can be written in terms of the x and y coordinates as shown in Eq. (11).

The Green's function chosen in [9] is the Green's function for a clamped circular plate, Eq. (13). See Figure 2. Substituting this Green's function into the boundary condition equations, one therefore can determine the set of fictitious forces and moments from a set of two boundary conditions. However, since both plan form and boundary conditions are arbitrary, a numerical method must be employed. By modeling the plate with an N-sided polygon, and assuming the fictitious force and moment remain constant along each of the straight edges of this polygon, one can determine a set of N fictitious forces and N fictitious moments such that the boundary conditions at the N mid-points of the N-sided polygon are satisfied. Take the zero deflection boundary condition, for example, which occurs on both clamped and simply supported edges. One will have

$$\begin{aligned}
&\sum_{k=1}^{n} P_k^*(Q_k)\left[\int_{S_k} G(x,y;\xi,\eta)\,ds(\xi,\eta)\right] \\
&+\sum_{k=1}^{n} M_{n_k}^*(Q_k)\left[-n_x(Q_k)\int_{S_k} \frac{\partial G(x,y;\xi,\eta)}{\partial \xi}\,ds(\xi,\eta)\right. \\
&\left.-n_y(Q_k)\int_{S_k} \frac{\partial G(x,y;\xi,\eta)}{\partial \eta}\,ds(\xi,\eta)\right] = -\iint_R q(\xi,\eta)\,G(x,y;\xi,\eta)\,d\xi d\eta
\end{aligned} \tag{19}$$

where Q_k represents the centerpoint of the kth side of the polygon B, and S_k is a coordinate along the kth side, i.e., $0 \leq S_k \leq \Delta S_k$. Now, if the boundary conditions on B are forced to be satisfied at each of N locations (x_B, y_B), Eqs. (19) lead to a system of 2N linear algebraic equations for 2N unknowns P_k^* and $M_{n_k}^*$, k=1,2,3,...N. This system of equations can be expressed in matrix form by

$$\begin{bmatrix} RM \end{bmatrix} \begin{Bmatrix} P^* \\ \hline M_n^* \end{Bmatrix} = \begin{Bmatrix} RL \end{Bmatrix} \tag{20}$$

where RM is a 2N×2N matrix, the elements of which are the line integrals of Eq. (19), and RL is a 2N×1 column matrix consisting of the area integrals of Eq. (19). Once Eq. (20) is solved for the unknown fictitious forces and moments, P* and M_n^*, the displacements at any internal point can be found using

$$w(x,y)=\iint q(\xi,\eta)G(x,y;\xi,\eta)d\xi d\eta+\sum_{k=1}^{n} P_k^*(Q_k)\left[\int_{S_k} G(x,y;\xi,\eta)ds(\xi,\eta)\right]$$

$$+\sum_{k=1}^{n} Mn_k^*(Q_k)\left[-n_x(Q_k)\int_{S_k}\frac{\partial G(x,y;\xi,\eta)}{\partial\xi}ds(\xi,\eta)\right. \tag{21}$$

$$\left.-n_y(Q_k)\int_{S_k}\frac{\partial G(x,y;\xi,\eta)}{\partial\eta}ds(\xi,\eta)\right]$$

This method works satisfactorily if free boundary conditions are not included. For a free edge, the two boundary condition equations are as shown in Eq.(11). It can be seen that, associated with the fictitious moments, there are eight terms involving fourth order derivatives of the Green's function, Eq.(13). When integrating along the kth side of the polygon B, there will be difficulties in the evaluation of the second order singularities. That is, there will be terms such as

$$\int_{S_k}\frac{1}{(x-\xi)^2+(y-\eta)^2}ds_k(\xi,\eta)$$

which pose difficulties when $\xi\to x$ and $\eta\to y$. In addition, like other boundary integral methods, the errors in the region near the boundary can be substantial. Due to these two dificiencies, an "auxiliary boundary" method has been developed. This method is presented in the following section.

II.2 THE AUXILIARY BOUNDARY METHOD

The only difference between the current method and the boundary integral method discussed in the previous section is that an integration path, B*, is chosen different from the plate boundary, B; see Figure 3. In so

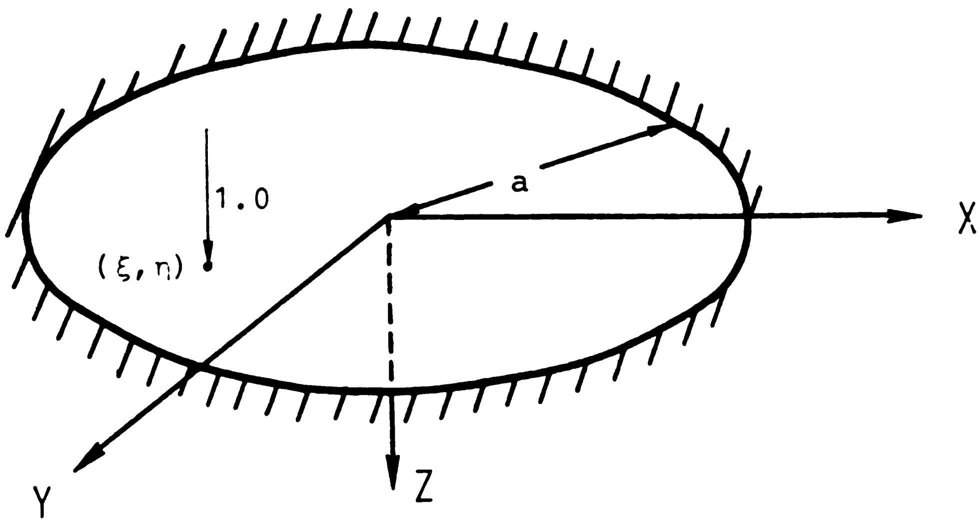

Figure 2. Problem for which Analytic Solution is Known.

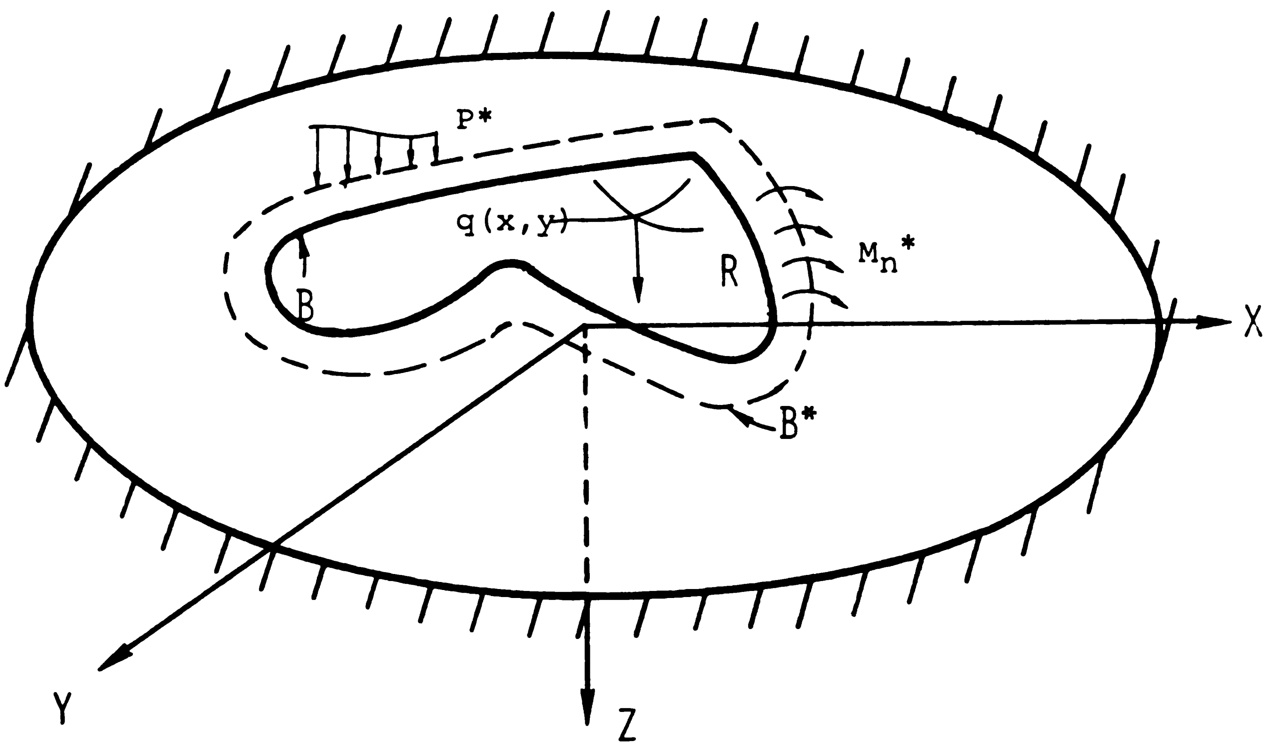

Figure 3. Fictitious Problem

doing, the singularities which arise in the integrand during numerical integration for the free boundary condition are avoided. Since integration is now carried out along the fictitious integration path, B*, there is no need to model the plate with an N-sided polygon. Instead, there are N boundary points prescribed on B where boundary conditions are to be satisfied. This, combined with the fact that the fictitious forces and moments are now located away from the boundary, provide significant improvements in the solution accuracy. For example, at the center point of a clamped rectangular plate, the errors for displacement and bending moments are reduced from 1.8 and 8.0 percent, shown in [9], to 0.04 and 1.6 percent, respectively.

Following the aforementioned procedure but integrating along B* instead of B, plate problems with mixed boundary condition of all the three types can now be solved. Writing the boundary condition equations more explicitly, Eq.(11) become

$$\oint_{B^*} P^*(\xi,\eta)G(x,y;\xi,\eta)ds(\xi,\eta)$$
$$+\oint_{B^*} M_n^*(\xi,\eta)[-n_x(\xi,\eta)\frac{\partial G(x,y;\xi,\eta)}{\partial \xi}-n_y(\xi,\eta)\frac{\partial G(x,y;\xi,\eta)}{\partial \eta}]ds(\xi,\eta)$$
$$= -\iint_R q(\xi,\eta)G(x,y;\xi,\lambda)d\xi d\eta \tag{22a}$$

if (x,y) is on $B_c + B_s$;

$$\oint_{B^*} P^*(\xi,\eta)[n_x(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial x}+n_y(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial y}]ds(\xi,\eta)$$
$$+\oint_{B^*} M_n^*(\xi,\eta)[-n_x(\xi,\eta)n_x(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial \xi}$$
$$-n_x(\xi,\eta)n_y(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y\partial \xi}-n_y(\xi,\eta)n_x(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial \eta}$$
$$-n_y(\xi,\eta)n_y(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y\partial \eta}]ds(\xi,\eta) \tag{22b}$$

$$=-\iint q(\xi,\eta)\,[n_x(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial x}+n_y(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial y}]\,d\xi d\eta$$

if (x,y) on B_c;

$$\oint_{B^*} P^*(\xi,\eta)\,[n_x^2(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x^2}+2n_x(x,y)\,n_y(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial y}$$

$$+n_y^2(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y^2}]\,ds(\xi,\eta)$$

$$+\oint_{B^*} M_n^*(\xi,\eta)\,[-n_x(\xi,\eta)\,n_x^2(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial\xi}$$

$$-2n_x(\xi,\eta)\,n_x(x,y)\,n_y(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y\partial\xi}$$

$$-n_x(\xi,\eta)\,n_y^2(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^2\partial\xi}-n_y(\xi,\eta)\,n_x^2(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial\eta}$$

$$-2n_y(\xi,\eta)\,n_x(x,y)\,n_y(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y\partial\eta}$$

$$-n_y(\xi,\eta)\,n_y^2(x,y)\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^2\partial\eta}]\,ds(\xi,\eta)$$

$$=-\iint_R q(\xi,\eta)\,[n_x^2(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x^2}+2n_x(x,y)\,n_y(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial y}$$

$$+n_y^2(x,y)\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y^2}]\,d\xi d\eta \qquad (22c)$$

if (x,y) is on B_s;

$$\oint_{B^*} P^*(\xi,\eta)\{[n_x^2(x,y)+\nu n_y^2(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x^2}$$

$$+[2(1-\nu)n_x(x,y)n_y(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial y}$$

$$+[n_y^2(x,y)+\nu n_x^2(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y^2}\}ds(\xi,\eta)$$

$$+\oint_{B^*} M_n^*(\xi,\eta)\{-n_x(\xi,\eta)[n_x^2(x,y)+\nu n_y^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial\xi}$$

$$-n_x(\xi,\eta)[2(1-\nu)n_x(x,y)n_y(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y\partial\xi}$$

$$-n_x(\xi,\eta)[n_y^2(x,y)+\nu n_x^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^2\partial\xi}$$

$$-n_y(\xi,\eta)[n_x^2(x,y)+\nu n_y^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial\eta}$$

$$-n_y(\xi,\eta)[2(1-\nu)n_x(x,y)n_y(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y\partial\eta}$$

$$-n_y(\xi,\eta)[n_y^2(x,y)+\nu n_x^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^2\partial\eta}\}ds(\xi,\eta)$$

$$= -\iint_R q(\xi,\eta)\{[n_x^2(x,y)+\nu n_y^2(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x^2}$$

$$+[2(1-\nu)n_x(x,y)n_y(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x\partial y}$$

$$+[n_y^2(x,y)+\nu n_x^2(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial y^2}\}d\xi d\eta \qquad (22d)$$

if (x,y) is on B_f;

$$\oint_{B^*} P^*(\xi,\eta)\{[n_x^3(x,y)+(2-\nu)n_x(x,y)n_y^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^3}$$

$$+[(2\nu-1)n_x^2(x,y)n_y(x,y)+(2-\nu)n_y^3(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial y}$$

$$+[(2-\nu)n_x^3(x,y)+(2\nu-1)n_x(x,y)n_y^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y^2}$$

$$+[(2-\nu)n_x^2(x,y)n_y(x,y)+n_y^3(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^3}\}ds(\xi,\eta)$$

$$+\oint_{B^*} M_n^*(\xi,\eta)\{-n_x(\xi,\eta)[n_x^3(x,y)+(2-\nu)n_x(x,y)n_y^2(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x^3\partial y}$$

$$-n_x(\xi,\eta)\,[(2\nu-1)n_x^2(x,y)n_y(x,y)+(2-\nu)n_y^3(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x^2\partial y\partial\xi}$$

$$-n_x(\xi,\eta)\,[(2-\nu)n_x^3(x,y)+(2\nu-1)n_x(x,y)n_y^2(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x\partial y^2\partial\xi}$$

$$-n_x(\xi,\eta)\,[(2-\nu)n_x^2(x,y)n_y(x,y)+n_y^3(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial y^3\partial\xi}$$

$$-n_y(\xi,\eta)\,[n_x^3(x,y)+(2-\nu)n_x(x,y)n_y^2(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x^3\partial\eta}$$

$$-n_y(\xi,\eta)\,[(2\nu-1)n_x^2(x,y)n_y(x,y)+(2-\nu)n_y^3(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x^2\partial y\partial\eta}$$

$$-n_y(\xi,\eta)\,[(2-\nu)n_x^3(x,y)+(2\nu-1)n_x(x,y)n_y^2(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial x\partial y^2\partial\eta}$$

$$-n_y(\xi,\eta)\,[(2-\nu)n_x^2(x,y)n_y(x,y)+n_y^3(x,y)]\frac{\partial^4 G(x,y;\xi,\eta)}{\partial y^3\partial\eta}\}ds(\xi,\eta)$$

$$= -\iint_R q(\xi,\eta)\{[n_x^3(x,y)+(2-\nu)n_x(x,y)n_y^2(x,y)]\frac{\partial^2 G(x,y;\xi,\eta)}{\partial x^3}$$

$$+[(2\nu-1)n_x^2(x,y)n_y(x,y)+(2-\nu)n_y^3(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x^2\partial y}$$

$$+[(2-\nu)n_x^3(x,y)+(2\nu-1)n_x(x,y)n_y^2(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial x\partial y^2}$$

$$+[(2-\nu)n_x^2(x,y)n_y(x,y)+n_y^3(x,y)]\frac{\partial^3 G(x,y;\xi,\eta)}{\partial y^3}\}d\xi d\eta \quad (22e)$$

if (x,y) is on B_f. The derivatives of the Green's function are listed in Appendix A.

The solution is then obtained using the same numerical procedure shown in the previous section. However, the plate is no longer modeled with an N-sided polygon. Instead, N points are assigned along the boundary, and two of the above equations are satisfied at each of these N points. This is done by adjusting the magnitudes of the unknown fictitious forces P* and moments M_n* at the N meshes along the fictitious integration path B*. Thus, solving a set of 2N×2N linear algebraic equations for the set of fictitious forces and moments on B*, and substituting into the deflection equation Eq.(21), the problem of Figure 1 is solved.

A plate problem of arbitrary plan form, arbitrary lateral load, and arbitrary boundary conditions has now been solved. The complete computer program is shown in Appendix B.

Two example problems are illustrated here. In both cases uniformly-loaded square plates are considered. The dimensions of the plate and the fictitious contour B* are shown in Figure 4. The plate is embedded in a fictitious circular plate with a radius of 80m. Other

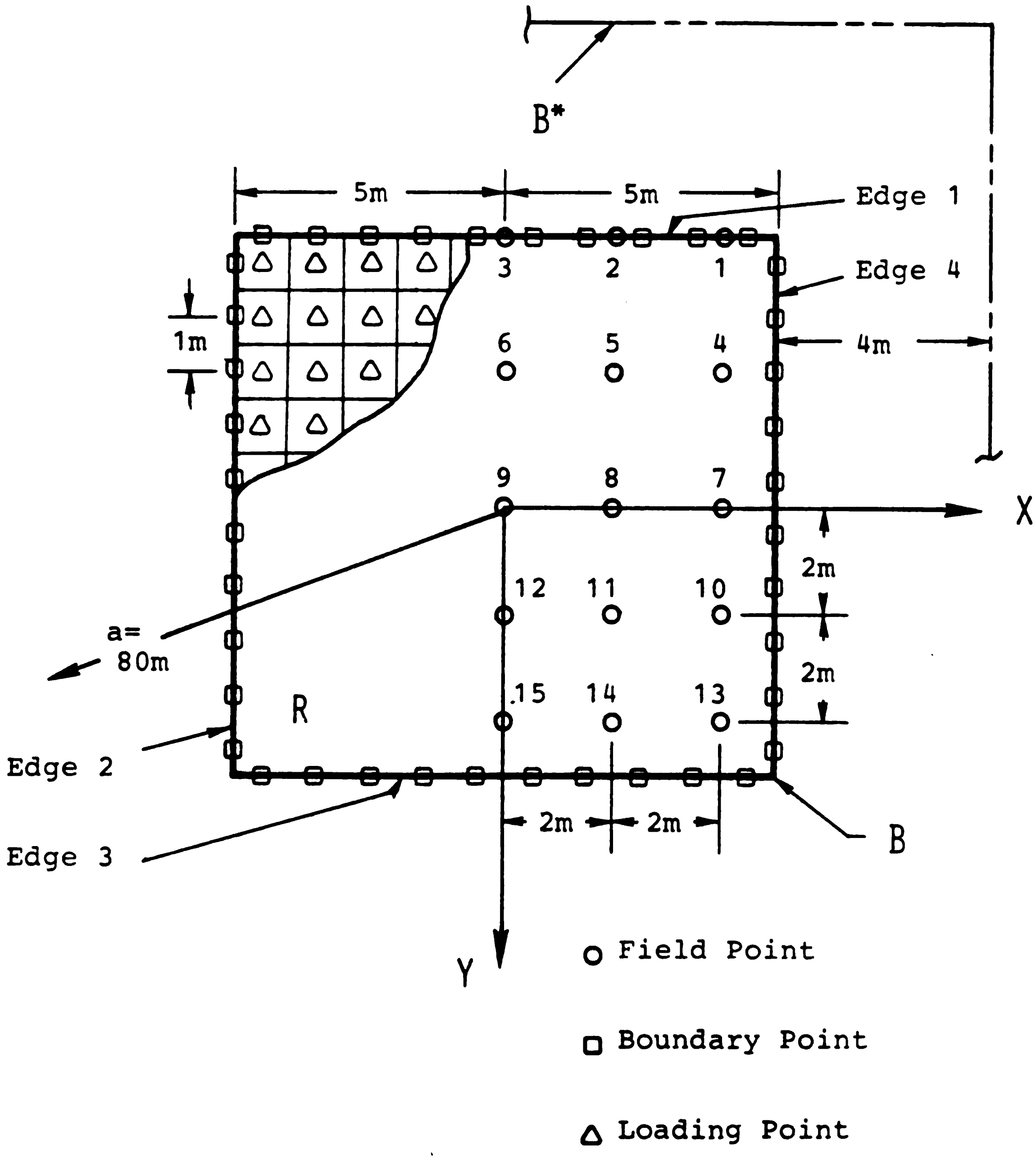

Figure 4. A Square Plate with an Auxiliary Integration Contour.

constants are $E=2.0684\times10^5$ MPa, $\nu=0.3$, $h=0.01$m, and $q=1$N/m^2. In the first example, edge one is free, edges two and four are simply supported, and edge three is clamped. In the second example, edges one and three are free while the other two are simply supported. The boundary conditions are satisfied at forty points spaced at a distance 1m from each other and .5m from the corner; see Figure 4. The contour B* is located 4m away from the plate boundary and is divided into forty intervals. Integration over an interval B* is done by simply multiplying the value of the integrand at the center of an interval by the interval length. Integration over R is accomplished by subdividing R into 100 equal divisions and multiplying the value of the integrand at the centerpoint of each division by the area of that division. This area subdivision is also shown in Figure 4.

Results of displacements and bending moments at each of the fifteen field points shown in Figure 4 are presented in Tables 1 and 2. When compared with the exact solutions from [1], the average errors are less than three percent for both deflections and bending moments. It may be noted that locations one, two, and three are on edge one of the boundary and between two adjacent boundary points where the boundary conditions are forced to be satisfied. Therefore, accuracy of results at these locations are expected to be the worst. Yet, the errors are less than four percent. Thus, the method using an auxiliary integration path is a great improvement over the previous indirect boundary integral method, especially when results near the boundary are needed.

Table 1

Comparison of Numerical and Exact Results for a Square Plate,
Edge One Free, Edges Two and Four Simply Supported, and Edge Three Clamped (Fig. 4)

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	1.856	1.917	3.281	3.687	3.845	4.271	0.000	0.000	0.000
2	4.790	4.966	3.672	8.266	8.521	3.091	0.000	0.000	0.000
3	5.881	6.065	3.125	9.760	9.989	2.346	0.000	0.000	0.000
4	1.405	1.440	2.542	3.142	3.183	1.308	1.020	1.040	1.961
5	3.618	3.710	2.554	6.933	7.049	1.673	2.289	2.308	0.830
6	4.438	4.547	2.449	8.096	8.236	1.729	2.692	2.711	0.706
7	0.949	0.967	1.937	2.360	2.378	0.763	1.068	1.035	-3.054
8	2.426	2.473	1.948	4.912	4.960	0.967	2.387	2.387	-1.801
9	2.966	3.024	1.930	5.618	5.677	1.038	2.794	2.747	-1.707
10	0.504	0.513	1.772	1.332	1.325	-0.5255	0.378	0.339	-10.275
11	1.271	1.291	1.570	2.364	2.360	-0.1861	0.555	0.492	-11.281
12	1.545	1.569	1.563	2.560	2.560	0.000	0.519	0.447	-13.900
13	0.087	0.088	1.628	-0.247	-0.287	16.937	-1.772	-1.837	3.668
14	0.211	0.214	1.366	-1.084	-1.360	4.797	-4.802	-4.918	2.416
15	0.254	0.257	1.345	-1.416	-1.469	3.743	-5.927	-6.058	2.210

Table 2

Comparison of Numerical and Exact Results for a Square Plate

Edges One and Three Free, Edges Two and Four Simply Supported (Fig. 4)

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	2.467	2.543	3.086	4.736	4.915	3.780	0.000	0.000	0.000
2	6.385	6.579	3.043	11.010	11.332	2.925	0.000	0.000	0.000
3	7.855	8.092	3.008	13.100	13.464	2.779	0.000	0.000	0.000
4	2.211	2.267	2.542	4.475	4.550	1.676	0.847	0.861	1.665
5	5.729	5.871	2.475	10.418	10.622	1.958	1.837	1.841	0.218
6	7.047	7.225	2.534	12.401	12.649	1.998	2.136	2.137	0.047
7	2.154	2.202	2.243	4.429	4.500	1.603	1.030	0.993	-3.621
8	5.571	5.703	2.356	10.292	10.478	1.807	2.305	2.249	-2.429
9	6.853	7.010	2.299	12.240	12.470	1.879	2.704	2.642	-2.293

CHAPTER III

A NEW METHOD

III.1 ISOTROPIC PLATE PROBLEMS

Though the use of the boundary integral method to solve isotropic plate problems has been proved successful, the boundary condition equations, Eqs.(22) and Appendix A, are quite lengthy, especially for a free edge. To simplify the formulation, it is reasonable to consider replacing the set of fictitious moments by a second set of fictitious forces. In so doing, there is no need to evaluate all the derivatives of the Green's function with respect to ξ and η, since they are associated with the fictitious moment M_n* only. With this simplification, the length of boundary condition equations, Eqs.(22), can be reduced by about fifty percent.

In practice, there are two simple ways that one can enter twice as many fictitious forces P*. One can either double the number of meshes along the integration contour, or define a second integration contour. Tests indicate that the latter provides somewhat better results. On the other hand, during numerical integration, the fictitious force is assumed constant along each mesh. It is therefore logical to replace this evenly distributed line force by a concentrated point force and place it at the center of the mesh. With this change, together with the elimination of the fictitious moments, the deflection function w(x,y) of Eq.(21) can be reduced to the following form.

$$w(x,y) = \iint_R G(x,y;\xi,\eta)\cdot q(\xi,\eta)\,d\xi d\eta + \sum_k^{2n} G(x,y;\xi,\eta)\cdot P_k^*(\xi,\eta) \qquad k=1,2,3,\ldots \tag{23}$$

Following a numerical procedure similar to that of section II.2, this simplified method was tested using the same square plate. The radius of the fictitious clamped circular plate was kept at 80m. The first set of fictitious forces were located at four meters away from the plate boundary, and the second set were located at two meters away from the first set, Figure 5. The results were almost identical to those obtained using both fictitious forces and moments.

Thus far the boundary integral method has been modified somewhat, in that the boundary integration has been replaced with an algebraic summing process. Though the elimination of the fictitious moments has been successful, the method is still tedious if free boundaries are involved. In order to further simplify the method, the well known Green's function for an infinite plate, [1], is introduced. It is

$$G(x,y;\xi,\eta) = \frac{1}{16\pi D}\left[(x-\xi)^2+(y-\eta)^2\right]\ln\frac{(x-\xi)^2+(y-\eta)^2}{a^2}\ . \tag{24}$$

Mathematically, this is the fundamental solution to the isotropic plate problem with a unit force at (ξ,η). The beauty of this Green's function is that the denomenator in the logarithmic term is a constant, a^2, where a is an arbitrary reference radius at which the deflection is zero. When derivatives are evaluated, this new Green's function gives a much shorter form than that obtained from the

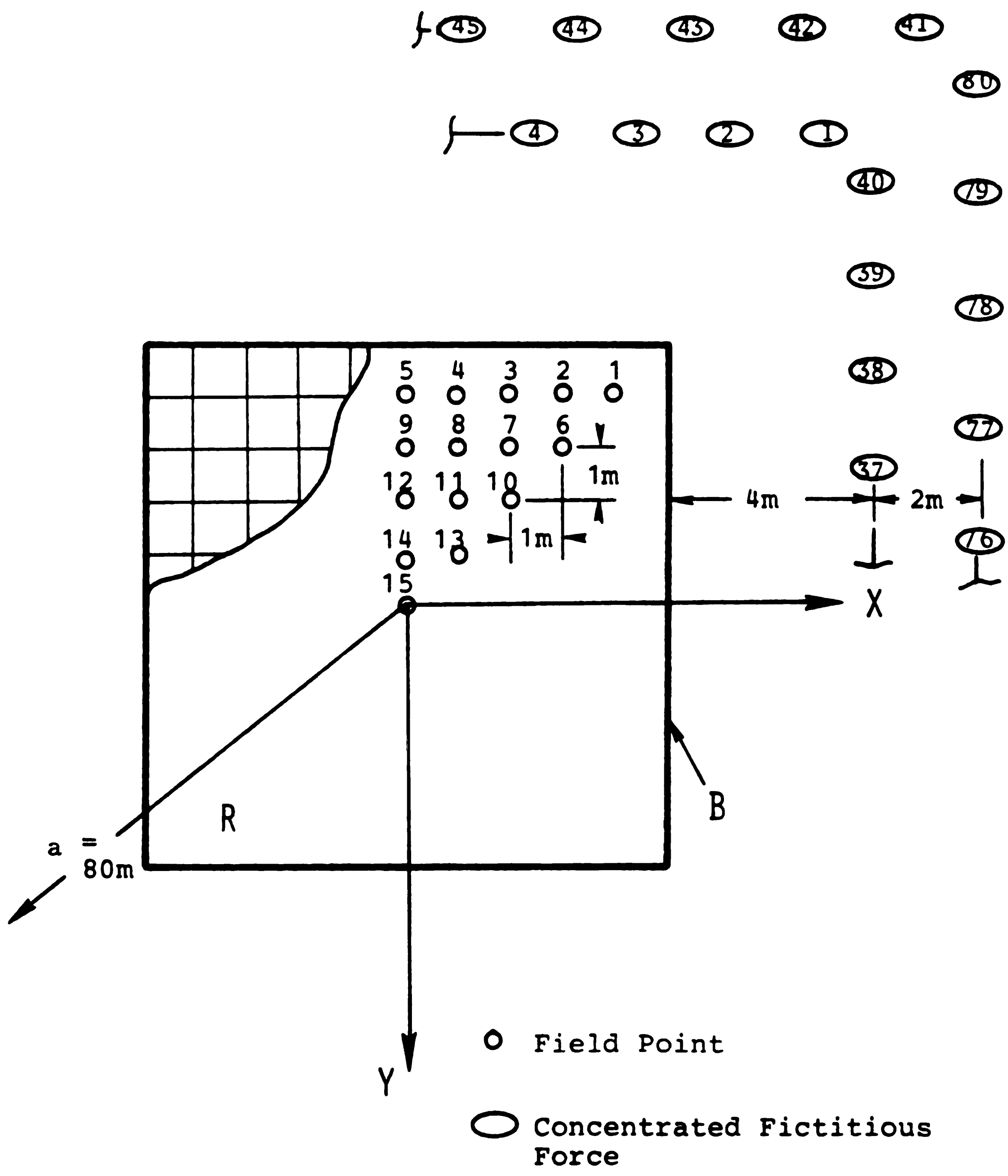

Figure 5. A Square Plate with Two Sets of
Fictitious Forces.

Green's function of a clamped circular plate. Consider $\frac{\partial^3 w}{\partial x^3}$ for example. The new Green's function gives

$$\frac{\partial^3 w}{\partial x^3} = \frac{1}{4\pi D}\left\{\iint_R \frac{(x-\xi)^3+3(y-\eta)^2(x-\xi)}{(x-\xi)^2+(y-\eta)^2}\, q(\xi,\eta)\, d\xi d\eta \right.$$

$$\left. + \sum_i^{2N} \frac{(x-\xi)^3+3(y-\eta)^2(x-\xi)}{(x-\xi)^2+(y-\eta)^2}\, P_i^*(\xi,\eta)\right\} \qquad (25)$$

while the old Green's function produces a very tedious expression.

In order to assess the advanteges of using this new Green's function, the previously-solved example problems, i.e., square plates, are repeated. The results are unchanged. However, the saving of computing time is large, approximately sixty percent, as shown in Table 3.

As an illustration of the capability for solving plate problems with odd plan forms, a simply supported and uniformly loaded equilateral triangular plate has also been treated; see Figure 6. For this triangular plate, each side is ten meters long and discretized into ten boundary points. There are, therefore, thirty boundary points in all. To simulate the evenly distributed loading condition of one Newton per square meter, one hundred 0.4333 Newton concentrated forces are placed at the centroids of the one hundred little equilateral triangles which form the plate. The sixty fictitious forces are equally spaced along two contours four and six meters away around the plate boundary. The results are compared with the exact solutions obtained from [1] shown in Table 4 and Figure 7. The errors are quite small.

With the new Green's function and all the other simplifications, the formulation of the new method becomes neat and simple. All the boundary condition equations can be written explicitly. Similar to Eqs.(22), we now have

Table 3

Comparison of Computing Costs

Isotropic Square Plate under Uniformly Distributed Load
40 Boundary Points, 100 Internal Forces, and 15 Field Points

Case No. 1: All the four edges are clamped.
Case No. 2: Edge one and three are free; and edges two and four are simply supported.

	Boundary Integral Method	The New Method	Savings %
Case No. 1	0.857*	0.361*	58
Case No. 2	0.949*	0.383*	60

*Execution CP seconds on a CDC Cyber-176 computer.

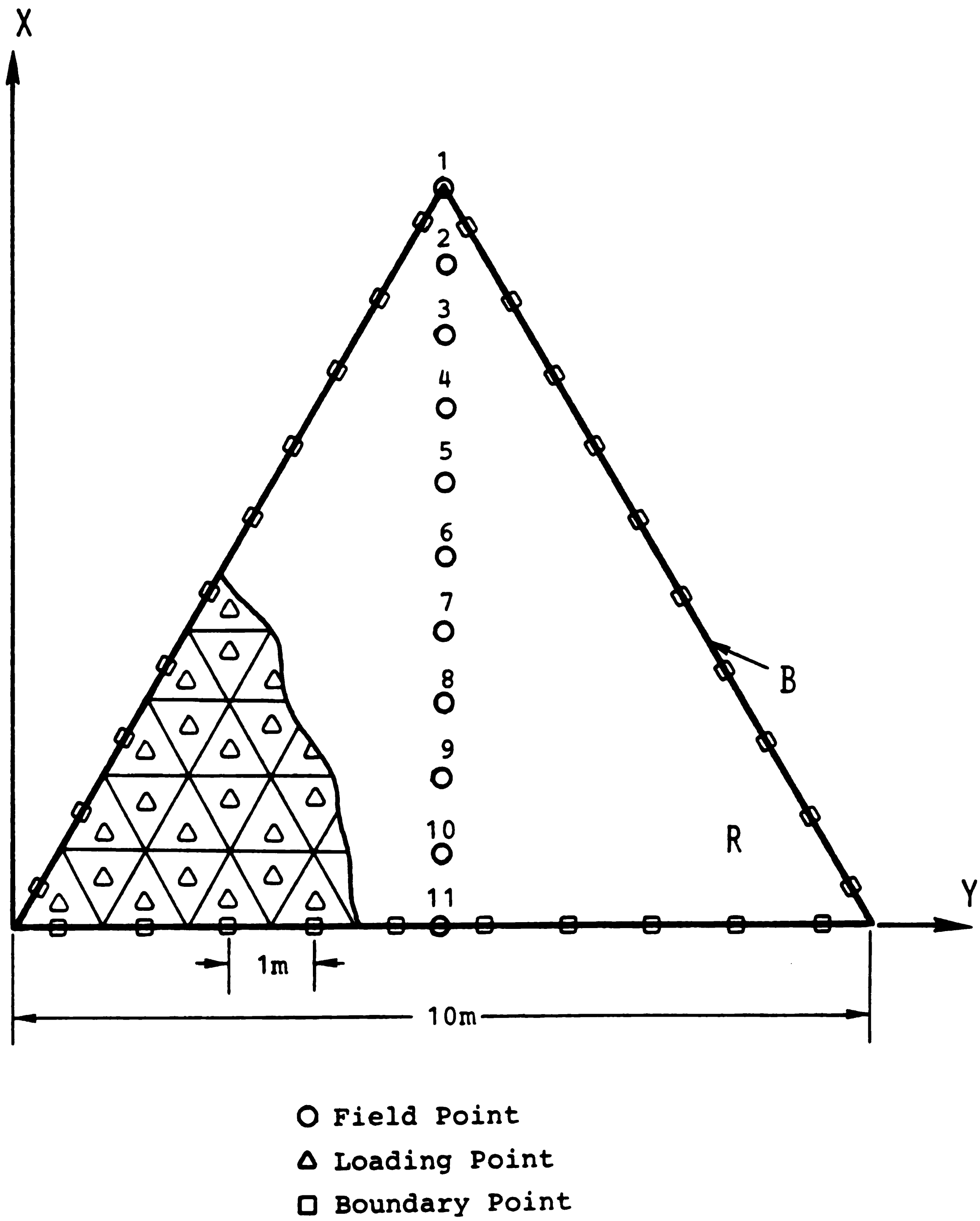

Figure 6. A Simply Support Equilateral Triangular Isotropic Plate.

Table 4

Comparison of Results of a Simply Supported Triangular Isotropic Plate
(Figure 6)

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.000	-.638E-4	*	-.219E-5	0.0188	*	.219E-5	0.0021	*
2	0.00515	0.00538	4.595	0.612	0.660	7.929	-0.392	-0.409	4.337
3	0.0337	0.0344	2.179	1.125	1.129	0.356	-0.345	-0.366	6.087
4	0.0906	0.0922	1.732	1.526	1.571	2.949	.00984	-.00356	*
5	0.166	0.169	1.395	1.800	1.799	-0.056	0.540	0.524	-2.963
6	0.242	0.245	1.306	1.934	1.974	2.068	1.111	1.111	-0.450
7	0.294	0.298	1.231	1.913	1.907	-0.314	1.597	1.588	-0.564
8	0.302	0.306	1.228	1.732	1.758	1.501	1.860	1.859	-0.054
9	0.253	0.257	1.282	1.350	1.349	-0.815	1.770	1.766	-0.226
10	0.147	0.149	2.628	0.781	0.811	3.908	1.194	1.197	0.251
11	0.000	-.298E-4	*	.299E-5	-0.0080	*	.516E-5	.00165	*

*Large percentage error, but small absolute error.

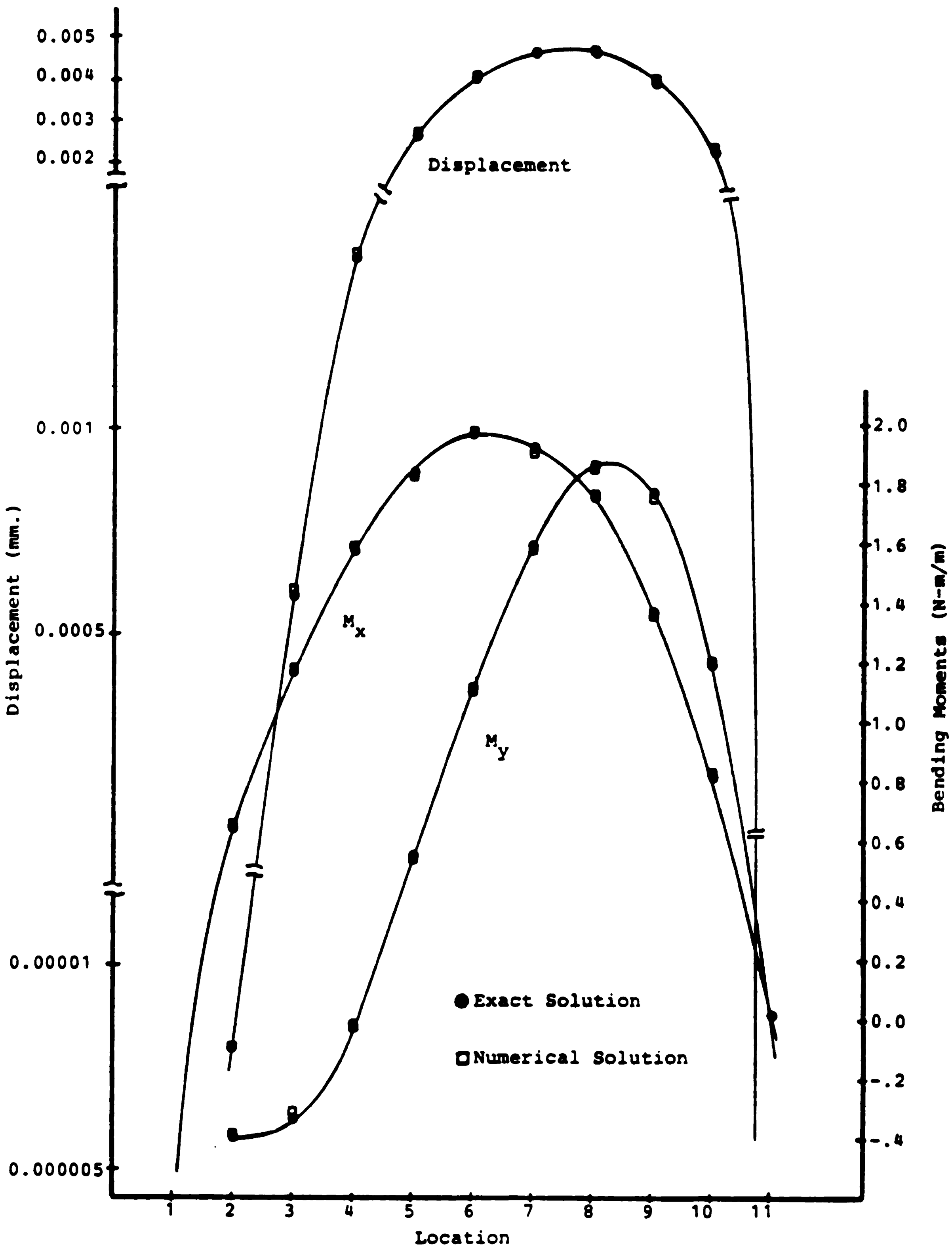

Figure 7. Comparison of Results of a Triangular Plate.

$$\sum_{i=1}^{2N} P_i^*(\xi,\eta)\,[(x-\xi)^2+(y-\eta)^2]\,\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}$$

$$= -\iint_R q(\xi,\eta)\,[(x-\xi)^2+(y-\eta)^2]\,\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}\,d\xi d\eta \qquad (26a)$$

if (x,y) is on $B_c + B_s$;

$$\sum_{i=1}^{2N} P_i^*(\xi,\eta)\,[1+\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\,[(x-\xi)n_x+(y-\eta)n_y]$$

(26b)

$$= -\iint_R q(\xi,\eta)\,[1+\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\,[(x-\xi)n_x+(y-\eta)n_y]$$

if (x,y) is on B_c;

$$\sum_{i=1}^{2N} P_i^*(\xi,\eta)\,\{n_x^2[\frac{2(x-\xi)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]$$

$$+ 2n_x n_y\frac{2(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)^2}$$

$$+ n_y^2[\frac{2(y-\eta)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\}$$

$$= -\iint q(\xi,\eta)\,\{n_x^2[\frac{2(x-\xi)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]$$

$$+ 2n_x n_y\frac{2(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)^2} \qquad (26c)$$

$$+ n_y^2[\frac{2(y-\eta)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\}\,d\xi d\eta$$

if (x,y) is on B_s;

$$\sum_{i=1}^{2N} P_i^*(\xi,\eta)\{[n_x^3+(2-\nu)n_x n_y^2]\frac{(x-\xi)[(x-\xi)^2+3(y-\eta)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}$$

$$+\ [(2\nu-1)n_x^2 n_y+(2-\nu)n_y^3]\frac{(y-\eta)[(y-\eta)^2-(x-\xi)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}$$

$$+\ [(2-\nu)n_x^3+(2\nu-1)n_x n_y^2]\frac{(x-\xi)[(x-\xi)^2-(y-\eta)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}$$

$$+\ [(2-\nu)n_x^2 n_y+n_y^3]\frac{(y-\eta)[(y-\eta)^2+3(x-\xi)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}\}$$

$$=\ -\iint_R q(\xi,\eta)\{[n_x^3+(2-\nu)n_x n_y^2]\frac{(x-\xi)[(x-\xi)^2+3(y-\eta)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}$$

$$+\ [(2\nu-1)n_x^2 n_y+(2-\nu)n_y^3]\frac{(y-\eta)[(y-\eta)^2-(x-\xi)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}$$

$$+\ [(2-\nu)n_x^3+(2\nu-1)n_x n_y^2]\frac{(x-\xi)[(x-\xi)^2-(y-\eta)^2]}{[(x-\xi)^2+(y-\eta)]^2}$$

$$+\ [(2-\nu)n_x^2 n_y+n_y^3]\frac{(y-\eta)[(y-\eta)^2+3(x-\xi)^2]}{[(x-\xi)^2+(y-\eta)^2]^2}\}d\xi d\eta \qquad (26d)$$

if (x,y) is on B_f; and

$$\sum_{i=1}^{2N} P_i^*(\xi,\eta)\{(n_x^2+\nu n_y^2)[\frac{2(x-\xi)^2}{(x-\xi)^2+(y-\eta)^2}+1+\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]$$

$$+\ \ 2(1-\nu)n_x n_y\ \frac{2(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)}$$

$$+\ (n_y^2+\nu n_x^2)[\frac{2(y-\eta)^2}{(x-\xi)^2+(y-\eta)^2}+1+\ell n\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\}\ =$$

$$= -\iint_R q(\xi,\eta)\{(n_x^2+\nu n_y^2)[\frac{2(x-\xi)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ln\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]$$

$$+ \quad 2(1-\nu)n_x n_y \frac{2(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)^2} \tag{26e}$$

$$+ (n_y^2+\nu n_x^2)[\frac{2(y-\eta)^2}{(x-\xi)^2+(y-\eta)^2} + 1 + \ln\frac{(x-\xi)^2+(y-\eta)^2}{a^2}]\}d\xi d\eta$$

if (x,y) is on B_f. Note that the common constants such as $\frac{1}{16\pi D}$, involved in both sides of the equations have been deleted.

With the new Green's function and boundary condition equations, and following the same numerical procedure shown previously to solve a set of 2N×2N linear algebraic equations for the unknowns of 2N fictitious forces, a general isotropic plate problem with arbitrary plan form, loading and boundary conditions can be solved. Although there are many improvements from the original boundary integral method [9], there are added numerical questions to be studied. In the original method, the radius of the fictitious plate involved in the Green's function is the only value to be chosen before analysis. An improper selection of this radius will result in poor solution accuracy. Fortunately, it has been found that good results can be obtained for a wide range of values of this radius. Take a 10m square plate for example. No change in solution has been noticed for values of this radius selected between 80m and 8000m.

For the new point-force method, on the other hand, in addition to the reference radius a in Eq.(24), the locations of the fictitious forces must also be determined. It has been observed from the numerical tests of this 10m plate problem that the fictitious forces must be placed within a narrow band 1m to 10m away from the plate boundary. Less accurate solutions will result if they are placed within 1m from the boundary, and no solution can be obtained if they are located farther than 10m away. It is conceivable that, like the boundary integral method, when the fictitious forces are too close to the boundary, it is impossible to get good results for those field points near the boundary. This is simply due to the fact that the boundary conditions are not satisfied everywhere along the boundary, but at those discretized boundary points only. On the other hand, when these fictitious forces are placed too far from the plate boundary, the influence due to each individual fictitious force is so weak that together with computing truncation errors, the RM matrix may become ill-conditioned.

Some results for a 10m clamped square plate are shown in Tables 5 through 8 to illustrate the change of solutions when fictitious forces are placed at different locations. Tables 5, 6, and 7 are for double-looped fictitious forces, and the double loops are at 4m and 6m, 1m and 3m, and 0.5m and 2.5m away from the plate boundary, respectively; see Figure 5 for reference. Table 8 is for a single-looped approach. That is, all the 2N fictitious forces are distributed along a single contour surrounding the plate. This contour is 4m away from the plate boundary; see Figure 4 for reference. The results are compared with the exact solution published in [1]. The five locations indicated in these tables correspond to locations 1,6,10,13 and 15 shown in Figure 5. The discrepencies between M_x and M_y of the exact solution were due to the fact that they were obtained from truncated infinite series solutions.

The computer program for isotropic plate problems with arbitrary plan form, loading, and boundary conditions using this simple point-force method is shown in Appendix C. It is believed that for a plate of any size and shape, good solution accuracy can be achieved if the locations of fictitious forces are selected properly. The determination of

Table 5

Comparison of Results of a Clamped Square Plate
Double-Looped Fictitious Forces at 4m and 6m Away from the Plate Boundary

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.016	0.016	0.000	-0.337	-0.371	9.740	-0.336	-0.371	10.221
2	0.137	0.137	-0.150	0.181	0.145	-19.500	0.181	0.145	-19.900
3	0.361	0.361	-0.073	1.151	1.115	-3.070	1.151	1.115	-3.127
4	0.576	0.576	0.000	1.974	1.939	-1.790	1.975	1.939	-1.823
5	0.664	0.664	0.000	2.290	2.254	-1.550	2.291	2.254	-1.572

Table 6

Comparison of Results of a Clamped Square Plate
Double-Looped Fictitious Forces at 1m and 3m Away from the Plate Boundary

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.016	0.016	0.000	-0.337	-0.373	10.514	-0.336	-0.373	11.296
2	0.137	0.137	-0.08	0.181	0.149	-17.562	0.181	0.149	-17.970
3	0.361	0.360	-0.146	1.151	1.116	-3.056	1.151	1.116	-3.113
4	0.576	0.576	-0.091	1.974	1.938	-1.860	1.975	1.938	-1.890
5	0.664	0.664	-0.870	2.290	2.253	-1.61	2.291	2.253	-1.635

Table 7

Comparison of Results of a Clamped Square Plate
Double-Looped Fictitious Forces at 0.5m and 2.5m Away from the Plate Boundary

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.016	0.016	-2.300	-0.337	-0.367	8.630	-0.336	-0.367	9.104
2	0.137	0.135	-1.188	0.181	0.149	-17.414	0.181	0.149	-17.823
3	0.361	0.360	-1.091	1.151	1.108	-3.750	1.151	1.108	-3.808
4	0.576	0.570	-0.912	1.974	1.928	-2.368	1.975	1.928	-2.397
5	0.664	0.658	-0.870	2.290	2.243	-2.054	2.291	2.243	-2.078

Table 8

Comparison of Results of a Clamped Square Plate

Single-Looped Fictitious Forces at 4m Away from the Plate Boundary

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.016	0.015	-6.868	-0.337	-0.398	18.005	-0.336	-0.398	18.523
2	0.137	0.137	-0.192	0.181	0.146	-19.190	0.181	0.146	-19.598
3	0.361	0.361	-0.058	1.151	1.116	-3.030	1.151	1.116	-3.088
4	0.576	0.576	0.000	1.974	1.940	-1.776	1.975	1.940	-1.806
5	0.664	0.664	0.000	2.290	2.255	-1.533	2.291	2.255	-1.558

of these locations may not be an easy task, and this numerical question requires further study.

On the other hand, for multiply connected plates, seemingly there will be difficulties if the "holes" are small and many boundary points are prescribed at the holes, since placing many fictitious forces in a small area inside a hole will certainly lead to numerical problems. Further study is needed in the search for optimum locations for fictitious forces.

III.2 ANISOTROPIC PLATE PROBLEMS

Due to the increased use of composite and multilayered plates for strength and weight reduction, anisotropic plate problems are becoming more and more important. With mid-plane symmetry of the material properties, the governing equation is Eq.(3), and the boundary conditions are given by Eqs.(5) and (7). They are far more complicated than when the plate is made from an isotropic material. Finite difference and finite element methods are generally used to obtain a solution. In this dissertation, a new numerical method is introduced. Using the Green's function for an anisotropic infinite plate and the same point-force technique shown previously, solution of an anisotropic plate problem with arbitrary plan form, loading, and boundary conditions is obtainable.

Since general anisotropic problems are very difficult to solve, they are often reduced to orthotropic problems through coordinate transformation or approximation, whenever possible. Therefore, orthotropic problems will be discussed first. For an orthotropic material, there are three mutually perpendicular planes of symmetry with respect to the elastic properties of the material, and the problems are greatly simplified compared with general anisotropic problems. In practice, it appears that orthotropic problems are more common than the general anisotropic plate problems. Reinforced decks in civil, marine, and aerospace engineering, and plates made of layered composite materials are typical examples.

III.2.1 ORTHOTROPIC PROBLEMS

For an orthotropic problem, the governing differential equation, Eq.(2), is a special form of Eq.(3), the equation for anisotropic problems, with $D_{16}=D_{26}=0$. The boundary condition equations are also simpler than those for their anisotropic counterparts; see Eqs.(4) and (8). It is due to these simplifications that solutions are obtainable for many problems. Bares and Massonet [22] used a beam and grid analogy, Vinson and Brull [23] used a power series expansion, and Rajappa [24] tried a Maclaurin's series. In addition, the application of finite difference method is clearly presented by Szilard [7], and the theory of finite element method is explicitly shown in Zienkiewicz's text [8]. For classic approaches, texts [14,26] of Lekhnitskii and Huber, respectively, are probably the most important.

For an orthotropic material, if the geometric coordinates are aligned with the principal material directions, the governing differential equation for equilibrium can be shown in Eq.(2). There are four material constants, namely E_x, E_y, ν_x, and G_{xy}, where E_x and E_y are the two Young's moduli evaluated along the x and y directions, respectively; ν_x is the Poisson's ratio in the x direction due to normal stress in the y direction; G_{xy} is the shear modulus. The other Poisson's ratio ν_y is related to ν_x by Betti's reciprocal theorem

$$\nu_y = \frac{E_y \nu_x}{E_x}$$

and therefore is not an independent material constant.

In order to apply the new point-force method, the first requirement is to find the Green's function of some appropriate problem. There are two Green's functions readily available for a simply supported rectangular plate, namely Navier's double series solution and Levy's single series solution:

$$G(x,y;\xi,\eta)=\frac{4b^3}{\pi^4 a}\sum_m^\infty\sum_n^\infty \frac{\sin\frac{m\pi\xi}{a}\ \sin\frac{m\pi x}{a}\ \sin\frac{m\pi\eta}{b}\ \sin\frac{m\pi y}{b}}{D_x(\frac{bm}{a})^4+2H(n\frac{bm}{a})^2+D_y n^4} \tag{27}$$

$$m=1,2,3,\ldots \quad , \ n=1,2,3,\ldots$$

and

$$G(x,y;\xi,\eta)=\frac{2b^2}{\pi^3 D_x D_y}\cdot\frac{1}{(\beta^2-\lambda^2)}\sum_n^\infty \frac{\sin\frac{n\pi\eta}{b}\ \sin\frac{n\pi y}{b}}{n^3}\ \times$$

$$\times\left[\frac{\beta\sinh\frac{n\pi\lambda(a-\xi)}{b}\sinh\frac{n\pi\lambda x}{b}}{\sinh\frac{n\pi\lambda a}{b}}-\frac{\lambda\sinh\frac{n\pi\eta(a-\xi)}{b}\ \sinh\frac{n\pi\beta x}{b}}{\sinh\frac{n\pi\beta a}{b}}\right] \tag{28}$$

$$n=1,2,3,\ldots$$

for $0\leq x<\xi$; and substitute x by (a-x) and (a-ξ) by ξ for $\xi\leq x\leq a$, where β and λ are the roots of the characteristic equation (which will be discussed later), a and b are the dimensions of the rectangular plate, and ξ and η are the location of the point force. It is known that the Navier's double series solution converges slowly. However, due to its simplicity in higher order derivatives, it was also tested along with Levy's single series solution. Before the full development for orthotropic problems, these two Green's function were evaluated for their efficiency in isotropic problems. For a square plate under uniformly distributed load, the results were disappointing for both approaches. For a solution accuracy greater than ninety percent, more than one hundred terms of Levy's series were needed, and the number is even higher for Navier's series. The computing costs were formidable. Therefore, the idea of using either approach was abandoned.

Since the fast converging Levy's series failed to yield satisfactory results in the application of the point-force method, it was clear that the Green's function to be adopted for the method must be in closed form. One of the currently existing Green's function in closed form is given in [12] for an infinite plate. Depending on inter-relationships among the material constants, this Green's function contains a group of three independent equations. These equations are derived in terms of several new material parameters. Therefore, it is necessary to introduce these new material parameters prior to the presentation of the governing equations. Let

$$\rho = \frac{H}{\sqrt{D_x D_y}} \quad , \quad \text{and} \quad \varepsilon^4 = \frac{D_x}{D_y} \tag{29}$$

then, the governing partial differential equation, Eq.(2), can be re-written in the form

$$\frac{\partial^4 w(x,y)}{\partial^4 y} + 2\rho\varepsilon^2 \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \varepsilon^4 \frac{\partial^4 w(x,y)}{\partial x^4} = \frac{q(x,y)}{D_y} \, . \tag{30}$$

Since the Green's function is the solution for the Dirac delta loading function of this equation, and this equation can be integrated in its homogeneous form, i.e., $q(x,y)=0$ for $(x,y)\neq(\xi,\eta)$, we can write the Green's function symbolically as

$$D_1 D_2 D_3 D_4 G(x,y;\xi,\eta)=0, \tag{31}$$

where the D's are linear differential operators, in the form of

$$D_i = \frac{\partial}{\partial y} - r_i \cdot \frac{\partial}{\partial x} \tag{32}$$

and r_i are determined as the roots of the characteristic equation

$$r^4 + 2\rho\varepsilon^2 r^2 + \varepsilon^4 = 0 . \tag{33}$$

These roots are either complex or pure-imaginary as shown by Lekhnitskii [27]. That is, the roots are in the form of

$$r_{1,2} = \pm i\beta \quad ; \quad r_{3,4} = \pm i\lambda \ . \tag{34}$$

Depending on the value of ρ, either greater than, equal to, or less than unity, the values β and λ can be easily determined by using either one of the following three equations,

$$\beta = \varepsilon\sqrt{\rho+\sqrt{\rho^2-1}} \ , \quad \lambda = \varepsilon\sqrt{\rho-\sqrt{\rho^2-1}} \tag{35a}$$

for $\rho>1$;

$$\beta = \varepsilon \qquad , \quad \lambda = \varepsilon \tag{35b}$$

for $\rho=1$; and

$$\beta = \mu_1 + i\mu_2 \quad , \quad \lambda = \mu_1 - i\mu_2 \tag{35c}$$

for $\rho<1$, where

$$\mu_{1,2} = \varepsilon\sqrt{\frac{1\pm\rho}{2}} \ .$$

It is clear that, depending on the material constants, each of the three criteria must be considered. For an infinite plate, Mossakowski [12] has derived three different Green's functions for these three different material types. With

$$D_o = \sqrt{D_x D_y} = \varepsilon^2 D_y,$$

we have

$$G(x,y;\xi,\eta) = \frac{1}{8\pi D_o(\beta^2-\lambda^2)}\{\beta[(x-\xi)^2-\lambda^2(y-\eta)^2]\,\ell n\frac{(x-\xi)^2+\lambda^2(y-\eta)^2}{a^2}$$

$$-4\lambda\beta(x-\xi)(y-\eta)[\text{arc tg}\frac{\lambda(y-\eta)}{(x-\xi)} - \text{arc tg}\frac{\beta(y-\eta)}{(x-\xi)}]$$

$$-\lambda[(x-\xi)^2-\beta^2(y-\eta)^2]\,\ell n\frac{(x-\xi)^2+\beta^2(y-\eta)^2}{a^2}$$

$$-3(\beta-\lambda)[(x-\xi)^2+\lambda\beta(y-\eta)^2]\} \tag{36a}$$

for $\rho>1$;

$$G(x,y;\xi,\eta) = \frac{1}{32\pi D_o}\{\frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{\mu_1}\ell n\frac{(x-\xi)^4+2\rho\varepsilon^2(x-\xi)^2(y-\eta)^2+\varepsilon^4(y-\eta)^4}{a^4}$$

$$-\frac{2[(x-\xi)^2-\varepsilon^2(y-\eta)^2]}{\mu_2} + \text{arc tg}\frac{2\mu_1\mu_2(y-\eta)^2}{(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2}$$

$$-\frac{2\varepsilon^2(x-\xi)(y-\eta)}{\mu_1\mu_2}\ell n\frac{\mu_1^2(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]^2}{\mu_1^2(y-\eta)^2+[(x-\xi)+\mu_2(y-\nu)]^2}$$

$$-\frac{6[(x-\xi)^2+\varepsilon^2(y-\eta)^2]}{\mu_1}\} \tag{36b}$$

for $\rho<1$; and

$$G(x,y;\xi,\eta)=\frac{1}{16\pi\varepsilon D_o}\{[(x-\xi)^2+\varepsilon^2(y-\eta)^2]\ell n\frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{a^2}$$

$$-[3(x-\xi)^2+\varepsilon^2(y-\eta)^2]\} \qquad (36c)$$

for $\rho=1$.

The second order derivatives of these equations are also given in [12]. Since they are needed not only in the boundary condition equations for a simply supported or free edge, but also in the determination of bending moments after fictitious point forces are computed, they are worth including in the following. Other derivatives are listed in Appendix D. There are three sets of equations, one set for each Green's function.

$$\frac{\partial^2 G}{\partial x^2}=\frac{1}{4\pi D_o(\beta^2-\lambda^2)}[\beta\ell n\frac{(x-\xi)^2+\lambda^2(y-\eta)^2}{a^2}-\lambda\ell n\frac{(x-\xi)^2+\beta^2(y-\eta)^2}{a^2}]$$

$$\frac{\partial^2 G}{\partial y^2}=\frac{\varepsilon^2}{4\pi D_o(\beta^2-\lambda^2)}[\beta\ell n\frac{(x-\xi)^2+\beta^2(y-\eta)^2}{a^2}-\lambda\ell n\frac{(x-\xi)^2+\lambda^2(y-\eta)^2}{a^2}]$$

$$\frac{\partial^2 G}{\partial x\partial y}=\frac{\varepsilon^2}{2\pi D_o(\beta^2-\lambda^2)}[\text{arc tg}\frac{\beta(y-\eta)}{(x-\xi)}-\text{arc tg}\frac{\lambda(y-\eta)}{(x-\xi)}] \qquad (37a)$$

for $\rho>1$;

$$\frac{\partial^2 G}{\partial x^2}=\frac{1}{16\pi D_o}[\frac{1}{\mu_1}\ell n\frac{(x-\xi)^4+2\rho\varepsilon^2(x-\xi)^2(y-\eta)^2+\varepsilon^4(y-\eta)^4}{a^4}$$

$$-\frac{2}{\mu_2}\text{arc tg}\frac{2\mu_1\mu_2(y-\eta)^2}{(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2}]$$

$$\frac{\partial^2 G}{\partial y^2}=\frac{\varepsilon^2}{16\pi D_o}[\frac{1}{\mu_1}\ell n\frac{(x-\xi)^4+2\rho\varepsilon^2(x-\xi)^2(y-\eta)^2+\varepsilon^4(y-\eta)^4}{a^4}+$$

$$+\ \frac{2}{\mu_2}\ \text{arc tg}\frac{2\mu_1\mu_2(y-\eta)^2}{(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2}$$

$$\frac{\partial^2 G}{\partial x\partial y} = \frac{-\varepsilon^2}{16\pi D_o\mu_1\mu_2}\ell n\frac{\mu_1{}^2(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]^2}{\mu_1{}^2(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]^2} \tag{37b}$$

for $\rho<1$; and

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{8\pi\varepsilon D_o}[\ell n\frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{a^2} - \frac{2\varepsilon^2(y-\eta)^2}{(x-\xi)^2+\varepsilon^2(y-\eta)}]$$

$$\frac{\partial^2 G}{\partial y^2} = \frac{\varepsilon}{8\pi D_o}[\ell n\frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{a^2} + \frac{2\varepsilon^2(y-\eta)^2}{(x-\xi)^2+\varepsilon^2(y-\eta)}]$$

$$\frac{\partial^2 G}{\partial x\partial y} = \frac{1}{4\pi D_o}[\frac{\varepsilon(x-\xi)(y-\eta)}{(x-\xi)^2+\varepsilon^2(y-\eta)^2}] \tag{37c}$$

for $\rho=1$.

Following the same numerical procedure as in isotropic problems, orthotropic plate problems of arbitrary plan form, loading, and boundary conditions can now be solved. The added complexity is that depending on $\rho \gtreqless 1$, there are three sets of equations to be concerned with. In order to verify the results for all the three possibilities, three sample cases have been solved. Consider a simply supported 10m square plate, and let $E_x=2.068\times10^5$ MPa, $E_y=E_x/15$, $\nu_x=0.3$, and h=0.01m. Varying ρ from 0.1 to 1.0 and 10.0, the accuracy of all theses three sets of results of deflections and bending moments are excellent when compared with the double series solution shown in [14], taking 400 terms. It is due to the fact that the changes are minimal when more than 100 terms are taken in the double series solution, the 400-term double series solution is believed very close to the exact one. The comparisons are tabulated in Tables 9 through 11. The nine locations of field points are shown in Figure 8.

The large errors for the bending moments in the y direction of Table 11 are somewhat misleading, because their magnitudes are small in comparison with the bending moments in the x direction. The computer program for these examples is shown in Appendix E.

III.2.2 ANISOTROPIC PROBLEMS

An anisotropic thin plate is considered as a plate made with a material which has the mid-plane of the plate as the only plane of material symmetry. It is due to the complexity of its governing and boundary condition equations, Eqs.(3) and (7), that efforts are always made to reduce anisotropic problems to orthotropic problems. That is, methods such as coordinate transformation are often tried to eliminate the two material constants a_{16} and a_{26} in Eq.(3). This is, however, not always possible. In general, neglecting these constants often times will lead to large errors, [28]. Therefore, though techniques to the solution of orthotropic problems are more important than that of general anisotropic problems, methods for the latter must also be developed.

Since a large number of anisotropic problems are related to man-made layered composite materials, it is wise to review a few references that will provide a better understanding of composite anisotropic materials: [28] presents basic concepts, fundamental equations, and many interesting illustrations; [29] introduces many exotic materials, their mechanical properties and applications; [30] illustrates many matrix systems and their characters; and [31] gives many studies of the applications and their significant contributions to the aerospace industry.

The solution of anisotropic plate problems is again

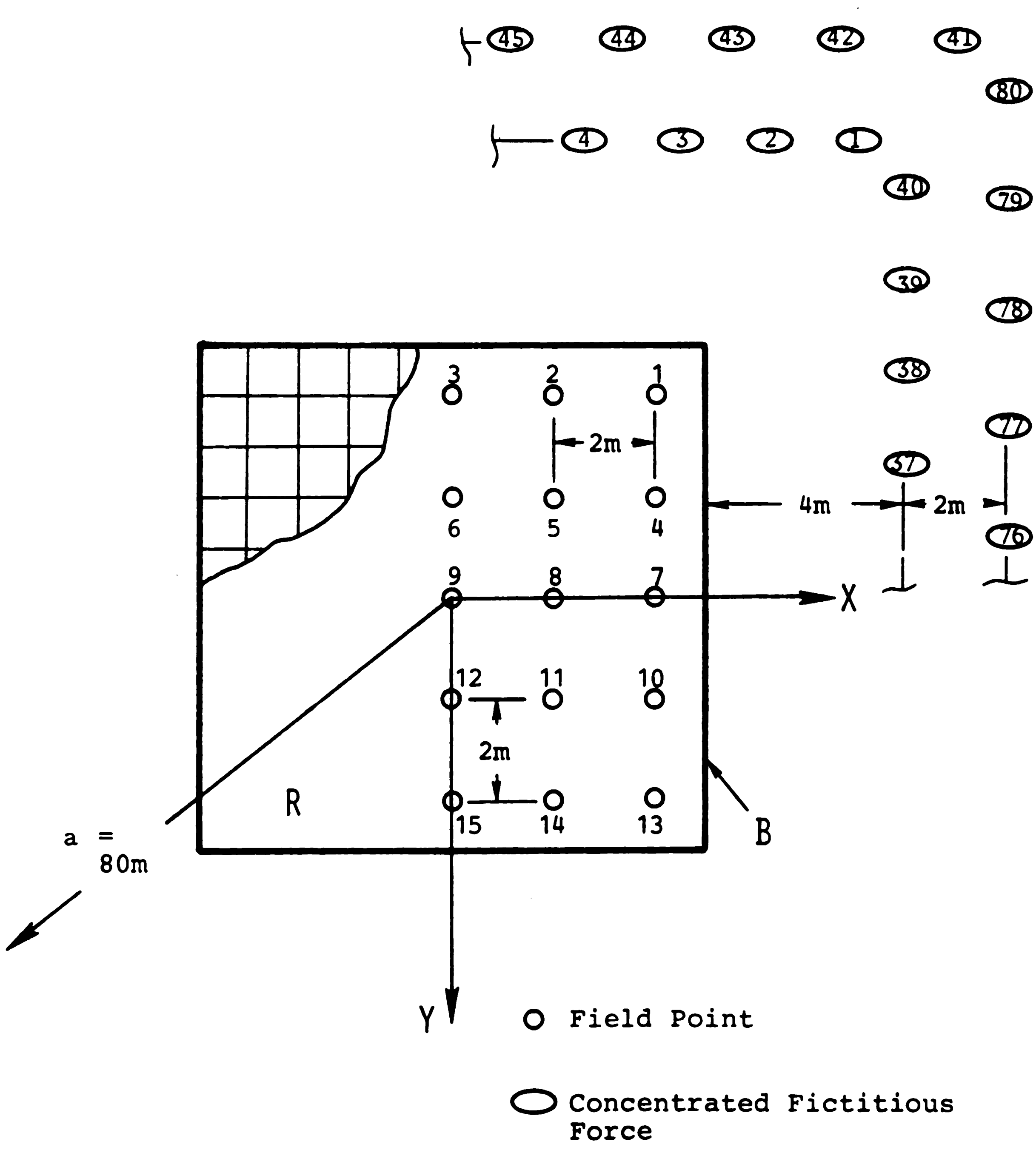

Figure 8. A Simply Supported Orthotropic Plate.

Table 9

Comparison of Results of a Simply Supported Orthotropic Square Plate

$E_x=2.068\times10^5$ MPa, $E_y=E_x/15$, $\nu_x=0.3$, $h=0.01$m

$\rho=0.1$

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.976	0.996	1.996	2.192	2.296	4.716	0.329	0.300	-8.967
2	2.497	2.543	1.821	4.515	4.596	1.794	0.775	0.763	-1.252
3	3.053	3.107	1.763	5.167	5.236	1.335	0.915	0.907	-0.831
4	2.205	2.235	1.347	4.323	4.418	2.198	0.343	0.303	-11.749
5	5.707	5.780	1.268	10.026	10.140	1.137	0.875	0.842	-3.762
6	7.016	7.103	1.243	11.910	12.030	1.008	1.073	1.048	-2.330
7	2.546	2.571	0.948	4.879	4.974	1.947	0.280	0.237	-15.297
8	6.601	6.674	1.111	11.496	11.613	1.018	0.707	0.666	-5.799
9	8.120	8.210	1.109	13.732	13.858	0.918	0.864	0.824	-4.664

Table 10

Comparison of Results of a Simply Supported Orthotropic Square Plate

$E_x=2.068\times10^5$ MPa, $E_y=E_x/15$, $\nu_x=0.3$, $h=0.01$m

$\rho=1.0$

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.684	0.695	1.670	1.578	1.614	2.307	0.225	0.195	-13.233
2	1.742	1.770	1.598	3.146	3.182	1.145	0.531	0.510	-4.049
3	2.127	2.160	1.527	3.578	3.610	0.889	0.625	0.611	-2.208
4	1.561	1.578	1.070	3.186	3.232	1.444	0.241	0.205	-14.938
5	4.021	4.063	1.039	7.072	7.128	0.791	0.603	0.567	-6.001
6	4.766	4.983	4.556	8.272	8.329	0.689	0.735	0.699	-4.833
7	1.778	1.837	3.340	3.624	3.671	1.297	0.216	0.179	-17.123
8	4.699	4.743	0.928	8.209	8.266	0.694	0.538	0.499	-7.246
9	5.771	5.823	0.900	9.669	9.728	0.610	0.656	0.617	-5.949

Table 11

Comparison of Results of a Simply Supported Orthotropic Square Plate

$E_x=2.068\times10^5$ MPa, $E_y=E_x/15$, $\nu_x=0.3$, h=0.01m

$\rho=10.0$

Loct.	W(true) mm.	W(num.) mm.	%Error	MX(true) N-m/m	MX(num.) N-m/m	%Error	MY(true) N-m/m	MY(num.) N-m/m	%Error
1	0.176	0.176	0.158	0.489	0.460	-5.914	0.058	0.015	-73.766
2	0.437	0.441	0.869	0.790	0.754	-4.547	0.132	0.089	-32.780
3	0.529	0.534	0.912	0.845	0.808	-4.333	0.157	0.114	-27.331
4	0.407	0.408	0.365	1.044	1.022	-2.107	0.066	0.022	-65.801
5	1.020	1.025	0.546	1.798	1.764	-1.891	0.148	0.102	-30.951
6	1.234	1.244	0.752	1.938	1.902	-1.858	0.175	0.129	-26.367
7	0.481	0.484	0.443	1.211	1.190	-1.734	0.668	0.025	-96.285
8	1.209	1.216	0.537	2.127	2.093	-1.599	0.151	0.105	-30.464
9	1.467	1.475	0.506	2.299	2.263	-1.566	0.178	0.1323	-25.591

obtained using a known Green's function for an infinite anisotropic plate [11,13] and applying a set of fictitious forces surrounding the plate boundary such that all the boundary conditions are satisfied. Similar to the previous problems, the numerical procedure is to solve the 2N×2N algebraic boundary condition equations for the unknown magnitude of fictitious forces. The only added work is in the determination of the complex roots of the characteristic polynomial equation. IMSL computer subroutine ZPOLR has been conveniently employed for this purpose.

The characteristic equation for the homogeneous solution of Eq.(3) is [13,14],

$$r^4 + 4\frac{D_{26}}{D_{22}} r^3 + 2\frac{D_{12}+2D_{66}}{D_{22}} r^2 + 4\frac{D_{16}}{D_{22}} r + \frac{D_{11}}{D_{22}} = 0 \qquad (38)$$

where the roots r_i are involved in the four linear differential operators $\frac{\partial}{\partial y} - r_i \cdot \frac{\partial}{\partial x}$, the same as in the orthotropic formulation. Solving this fourth order algebraic equation, the roots can be determined in the form of

$$r_{1,2} = \alpha \pm i\beta \; ; \; r_{3,4} = \gamma \pm i\lambda$$

They are all complex values as proved in [27].

For an infinite plate, the Green's function shown in [13] is

$$G(x,y;\xi,\eta) = \frac{1}{8\pi D_{22}\phi_1\phi_2}\{\frac{(\alpha-\gamma)^2-(\beta^2-\lambda^2)}{\beta}R_1(x,y;\xi,\eta)$$

$$+ \frac{(\alpha-\gamma)^2+(\beta^2-\lambda^2)}{\lambda}R_3(x,y;\xi,\eta)$$

$$+ 4(\alpha-\gamma)[S_1(x,y;\xi,\eta)-S_3(x,y;\xi,\eta)]\} \qquad (39)$$

where,

$$\phi_1 = (\alpha-\gamma)^2+(\beta-\lambda)^2 \ ; \quad \phi_2 = (\alpha-\gamma)^2+(\beta+\lambda)^2 \ ;$$

$$R_1(x,y;\xi,\eta)=\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}$$

$$\times\ \{\ell n\frac{[(x-\xi)+\alpha(y-\eta)]^2+\beta^2(y-\eta)^2}{a^2} - 3\}$$

$$- 4\beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]\text{arc tg}\frac{\beta(y-\eta)}{(x-\xi)+\alpha(y-\eta)} \ ;$$

$$S_1(x,y;\xi,\eta)= \beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]$$

$$\times\ \{\ell n\frac{[(x-\xi)+\alpha(y-\eta)]^2+\beta^2(y-\eta)^2}{a^2} - 3\}$$

$$+\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}\text{arc tg}\frac{\beta(y-\eta)}{(x-\xi)+\alpha(y-\eta)};$$

and $R_3(x,y;\xi,\eta)$ and $S_3(x,y;\xi,\eta)$ are obtained by replacing α and β by γ and λ, respectively.

As with the orthotropic Green's function, the first order derivatives are quite lengthy. The second order derivatives, however, can be reduced to very compact forms. Since they are the most important derivatives, they are listed here. Others are shown in Appendix F.

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\{\frac{(\alpha-\gamma)^2-(\beta^2-\lambda^2)}{\beta}L_1(x,y;\xi,\eta)$$

$$+ \frac{(\alpha-\gamma)^2+(\beta^2-\lambda^2)}{\lambda}L_3(x,y;\xi,\eta)+4(\alpha-\gamma)[N_1(x,y;\xi,\eta)-N_3(x,y;\xi,\eta)]\} \tag{40a}$$

$$\frac{\partial^2 G}{\partial y^2} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\{\frac{(\alpha^2+\beta^2-2\alpha\gamma)(\alpha^2+\beta^2)+(\alpha^2-\beta^2)(\gamma^2+\lambda^2)}{\beta}L_1(x,y;\xi,\eta)$$

$$+ \frac{(\gamma^2+\lambda^2-2\alpha\gamma)(\gamma^2+\lambda^2)+(\gamma^2-\lambda^2)(\alpha^2+\beta^2)}{\lambda}L_3(x,y;\xi,\eta)$$

$$- 4[\alpha(\gamma^2+\lambda^2)-\gamma(\alpha^2+\beta^2)][N_1(x,y;\xi,\eta)-N_3(x,y;\xi,\eta)]\} \quad (40b)$$

$$\frac{\partial^2 G}{\partial x\partial y} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\{\frac{(\alpha-2\gamma)(\alpha^2-\beta^2)+\alpha(\gamma^2+\lambda^2)}{\beta}L_1(x,y;\xi,\eta)$$

$$+ \frac{(\gamma-2\alpha)(\gamma^2+\lambda^2)+\gamma(\alpha^2+\beta^2)}{\lambda}L_3(x,y;\xi,\eta)$$

$$+ 2(\alpha^2+\beta^2-\gamma^2-\lambda^2)[N_1(x,y;\xi,\eta)-N_3(x,y;\xi,\eta)]\} \quad (40c)$$

where,

$$L_1(x,y;\xi,\eta) = \ell n\frac{[(x-\xi)+\alpha(y-\eta)]^2+\beta^2(y-\eta)^2}{a^2}$$

$$N_1(x,y;\xi,\eta) = \text{arc tg}\frac{\beta(y-\eta)}{(x-\xi)+\alpha(y-\eta)}$$

and $L_3(x,y;\xi,\eta)$ and $N_3(x,y;\xi,\eta)$ are obtained by replacing α and β by γ and λ, respectively. It is worth noting that Eqs.(39) and (40) can be easily reduced to Eqs.(36) and (37) by making $\alpha=\gamma=0$ for $\rho>1.0$; $\alpha=\mu_2$, $\gamma=-\mu_2$, $\beta=\lambda=\mu_1$ for $\rho<1.0$; and $\alpha=\gamma\to 0$, $\beta=\lambda\to\varepsilon$ for $\rho=1.0$.

Following the same numerical procedure shown in the previous two sections, using the new point-force method, the solution of an anisotropic thin plate problem with arbitrary plan form, loading, and boundary conditions can be obtained. For the verification of results, however, due to lack of exact solutions available for anisotropic problems to be compared with, a different approach must be taken. An orthotropic plate problem will become apparently anisotropic if the geometric coordinates are made different from the principal material directions. Therefore, solutions of orthotropic plate problems can be used to validate the equations for general anisotropic problems. This approach can be summarized in four steps as shown in the following. First, an angle of rotation for the geometric coordinates is chosen arbitrarily, and corresponding to the new coordinate system, locations of the boundary points, the fictitious forces, the field points, and the unit outward normals of the boundary points are determined. The second step is to compute the six flexural rigidity constants D_{ij} used in Eq.(3), [14]. The next step is to employ the Green's function of the infinite anisotropic plate to solve the pseudo-anisotropic problem. The final step is to determine the displacements and bending moments at the prescribed field points in the original coordinate system using coordinate transformation, and then make comparison with the orthotropic solutions.

With this validation method, orthotropic plate example problems shown earlier with all the three types of ρ, i.e., greater than, equal to, and less than unity have been tested against four coordinate rotation angles, namely 15, 30, 45, and 60 degrees. The discrepencies of results were within one percent and were believed to be due to truncation errors during the added numerical processes. The computer program for this validation is shown in Appendix H, while the program for a general anisotropic plate problem is shown in Appendix G.

It must be noted that the two flexural rigidity constants D_{16} and D_{26} are based on the two material constants a_{16} and a_{26} of Eq.(3). In order to investigate the influence due to these two material constants, several sample problems using a simply supported square plate have been tested. Since the

the material constant matrix shown in Eq.(3) must be positive definite, a_{16} and a_{26} were selected to be less than a_{11} and a_{22}, respectively. Under this condition, take four typical cases with $a_{16}=0.1/E_x$, $a_{26}=0.1/E_y$; $a_{16}=0.9/E_x$, $a_{26}=0.9/E_y$; $a_{16}=0.1/E_x$, $a_{26}=0.9/E_y$; and $a_{16}=0.9/E_x$, $a_{26}=0.1/E_y$ for example, it has been found that the differences in displacements and bending moments were smaller than one percent. However, when the shear modulus G_{xy} is small in comparison with E_x and E_y, a_{16} and a_{26} can be made greater than a_{11} and a_{22}; their contribution to the solution may become significant.

CHAPTER IV

CLOSURE

Starting from a boundary integral equation method for an isotropic thin plate problem with boundary clamped or/and simply supported, a very efficient numerical solution to problems with arbitrary plan form, arbitrary loading and boundary conditions, and anisotropic material, has been developed. The method uses the known Green's functions of isotropic, orthotropic, and anisotropic infinite plates. The problem is solved after the real plate is embedded in the fictitious infinite plate, and the boundary conditions at the N prescribed boundary points are forced to be satisfied with an imposed set of 2N calculated fictitious forces located somewhere outside the plate boundary. Though no efforts have been made to compare with the two leading numerical methods, the finite element and the finite difference methods, it is believed that the new method has the following two advantages: (1)since the Green's function is the exact solution to a point force problem, and there are no assumed polynomials for results, high solution accuracy is expected; (2)due to the fact that the equations are simple, and the modeling is for the plate boundary only, the current method is easier to use.

Large percentage errors indicated in all tables are somewhat misleading. Take the simply supported triangular plate problem for example. Percentage errors shown in Table 4 are huge at certain locations. However, the real errors are small as shown in Figure 7.

During the development of the current method, it was found that though series type Green's functions were easy for formulation, they were not suitable for the current method. This was due to the fact that large number of terms of the series were needed to provide acceptable solution accuracy, and this would lead to formidable computing costs.

Though the current method is efficient, numerical questions remain. An improper choice of the locations for the fictitious forces may result in poor solution accuracy or no solution at all. Therefore, in order to take full advantage of this method, some further studies should be made so that the locations of fictitious forces chosen will bring optimum results. In the meantime, due to the involvement of many "looped" summing processes, Eqs. (19), (21), (22), and (26), it is also necessary to do sensitivity studies to minimize the numbers of boundary points, fictitious forces, and internal forces for least computing cost.

All the five computer programs developed for this thesis research are shown in Appendices B, C, E, G, and H. The first is for the boundary integral method. It uses the Green's function of a clamped circular plate. The second is for the new point-force method for isotropic problems. The third and the fourth are for orthotropic and anisotropic problems, respectively. The fifth is a method employed to validate the equations for general anisotropic problems, using exact solutions for orthotropic plate problems. All these computer programs are coded in FORTRAN, and their flow chart is shown on the next page, Figure 9.

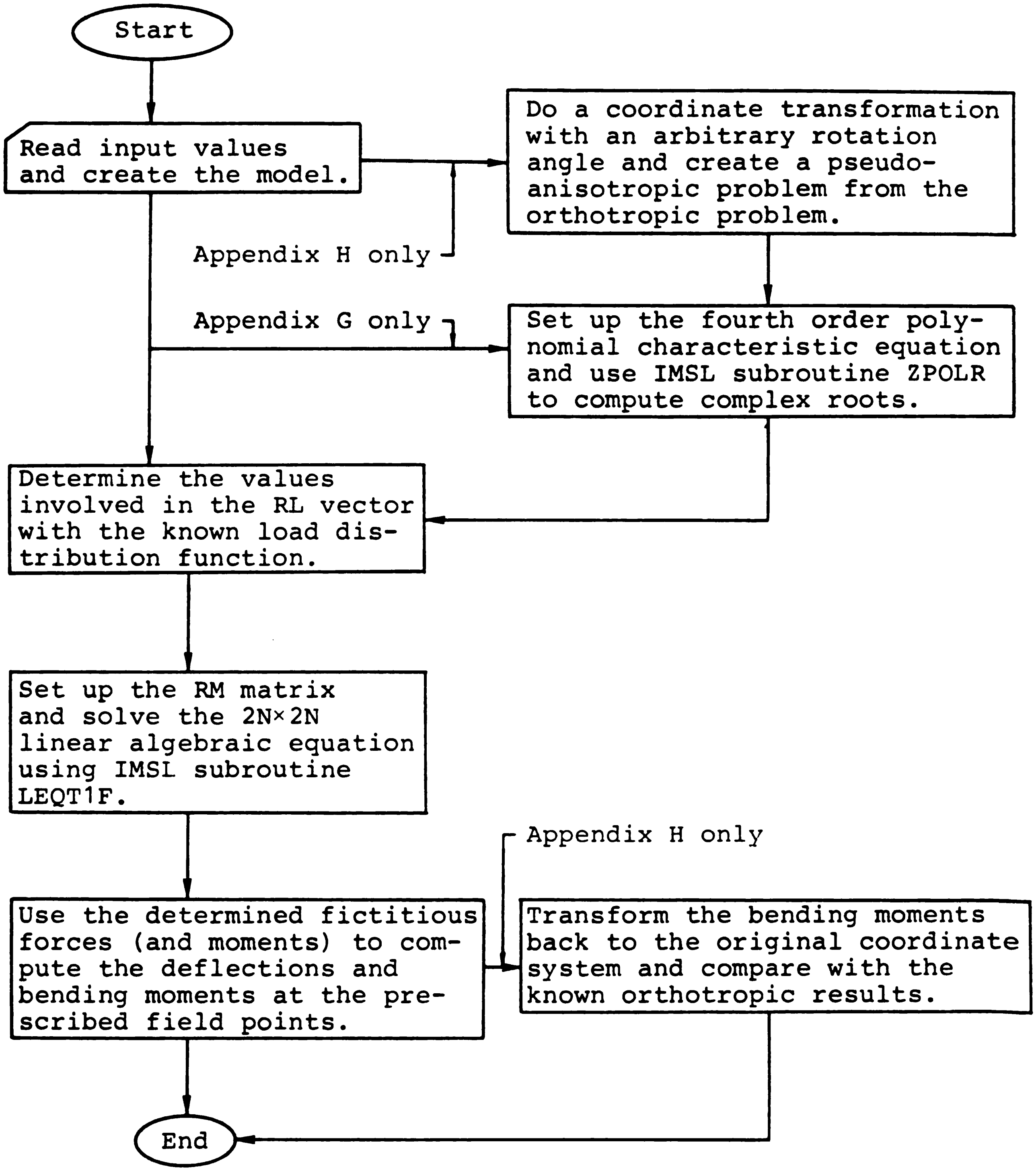

Figure 9

The flow chart for the programs
shown in Appendices B, C, E, G, and F.

APPENDICES

APPENDIX A

DERIVATIVES OF THE GREEN'S FUNCTION FOR A CLAMPED CIRCULAR PLATE

APPENDIX A

DERIVATIVES OF THE GREEN'S FUNCTION FOR A CLAMPED CIRCULAR PLATE

The Green's function shown in Eq.(10) can be written as

$$G(x,y;\xi,\eta) = \frac{a}{16\pi D}\{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2 \ln \frac{r_{12}{}^2}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}\}$$

where, $r_1^2 = \frac{x^2+y^2}{a^2}$, $r_2^2 = \frac{\xi^2+\eta^2}{a^2}$, and $r_{12} = \frac{(x-\xi)^2+(y-\eta)^2}{a^2}$.

For simplicity, from now on the variables x, y, ξ, and η are all made non-dimensional. That is, the variables x, y, ξ, and η shown in the following equations are actually the ratios of $\frac{x}{a}$, $\frac{y}{a}$, $\frac{\xi}{a}$, and $\frac{\eta}{a}$, respectively.

$$\frac{\partial G}{\partial x} = \frac{a^2}{8\pi D}\{x \cdot r_2{}^2 - \xi - \frac{r_{12}{}^2(x \cdot r_2{}^2-\xi)}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$+(x-\xi)\ln \frac{r_{12}{}^2}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}\}$$

$$\frac{\partial G}{\partial y} = \frac{a^2}{8\pi D}\{y \cdot r_2{}^2 - \eta - \frac{r_{12}{}^2(y \cdot r_2{}^2-\eta)}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$+(y-\eta)\,\ell n\,\frac{r_{12}^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\;\}$$

$$\frac{\partial G}{\partial \xi} = -\frac{a^2}{8\pi D}\{\xi\cdot r_1 - x - \frac{r_{12}^{2}(\xi\cdot r_1^{2}-x)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$+(\xi-x)\,\ell n\,\frac{r_{12}^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\;\}$$

$$\frac{\partial G}{\partial \eta} = -\frac{a^2}{8\pi D}\{\eta\cdot r_2 - y - \frac{r_{12}^{2}(\eta\cdot r_2^{2}-y)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$+(\eta-y)\,\ell n\,\frac{r_{12}^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\;\}$$

$$\frac{\partial^2 G}{\partial^2 x} = \frac{a^2}{8\pi D}\{r_2^{2} + \frac{2(x-\xi)^2}{r_{12}^{2}} + \frac{4(x-\xi)(\xi-x\cdot r_2^{2})-r_{12}^{2}\cdot r_2^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$+\frac{2r_{12}^{2}(\xi-x\cdot r_2^{2})^2}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^2} + \ell n\frac{r_{12}^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\;\}$$

$$\frac{\partial^2 G}{\partial x\,\partial y} = \frac{a^2}{8\pi D}\{\frac{2(x-\xi)(y-\eta)}{r_{12}^{2}} + \frac{2(y-\eta)(\xi-x\cdot r_2^{2})+(x-\xi)(\eta-y\cdot r_2^{2})}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$+\frac{2r_{12}^{2}(\xi-x\cdot r_2^{2})(\eta-y\cdot r_2^{2})}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\;\}$$

$$\frac{\partial^2 G}{\partial y^2} = \frac{a^2}{8\pi D}\{r_2^{2} + \frac{2(y-\eta)^2}{r_{12}} + \frac{4(y-\eta)(\eta-y\cdot r_2^{2})-r_{12}^{2}r_2^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$+\frac{2r_{12}^{2}(\eta-y\cdot r_2^{2})^{2}}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^{2}}+\ln\frac{r_{12}^{2}}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\}$$

$$\frac{\partial^2 G}{\partial x\partial\xi}=-\frac{a^2}{8\pi D}\{\frac{2(x-\xi)^2}{r_{12}^{2}}+\frac{2r_{12}^{2}(x-\xi\cdot r_1^{2})(x\cdot r_2^{2}-\xi)}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^{2}}+\ln r_{12}^{2}$$

$$+\frac{(1-2x\xi)(1-r_1^{2})(1-r_2^{2})-2(x-\xi)(x-\xi\cdot r_1^{2})-2(x-\xi)(x\cdot r_2^{2}-\xi)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$-\ln[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]\}$$

$$\frac{\partial^2 G}{\partial x\partial\eta}=-\frac{a^2}{8\pi D}\{\frac{2(y-\xi)(x-\eta)}{r_{12}^{2}}+\frac{2r_{12}^{2}(y-\eta\cdot r_1^{2})(x\cdot r_2^{2}-\xi)}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^{2}}$$

$$+\frac{(-2x\eta)(1-r_1^{2})(1-r_2^{2})-2(x-\xi)(y-\eta\cdot r_1^{2})-2(y-\eta)(x\cdot r_2^{2}-\xi)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\}$$

$$\frac{\partial^2 G}{\partial y\partial\xi}=-\frac{a^2}{8\pi D}\{\frac{2(x-\xi)(y-\eta)}{r_{12}^{2}}+\frac{2r_{12}^{2}(x-\xi\cdot r_1^{2})-(y\cdot r_2^{2}-\eta)}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^{2}}$$

$$+\frac{(-2y\xi)(1-r_1^{2})(1-r_2^{2})-2(y-\eta)(x-\xi\cdot r_1^{2})-2(x-\xi)(y\cdot r_2^{2}-\eta)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}\}$$

$$\frac{\partial^2 G}{\partial y\partial\eta}=-\frac{a^2}{8\pi D}\{\frac{2(y-\eta)^2}{r_{12}^{2}}+\frac{2r_{12}^{2}(y-\eta\cdot r_1^{2})(y\cdot r_2^{2}-\eta)}{[(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}]^{2}}+\ln r_{12}^{2}$$

$$+\frac{(1-2y\eta)(1-r_1^{2})(1-r_2^{2})-2(y-\eta)(y-\eta\cdot r_1^{2})-2(y-\eta)(y\cdot r_2^{2}-\eta)}{(1-r_1^{2})(1-r_2^{2})+r_{12}^{2}}$$

$$- \ell n\, [(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]\}$$

$$\frac{\partial^3 G}{\partial x^2 \partial \xi} = -\frac{a^2}{4\pi D}\{ \frac{3(x-\xi)}{r_{12}{}^2} - \frac{2(x-\xi)^3}{r_{12}{}^2} - \frac{x(1-2x\cdot\xi)(1-r_2{}^2)+\xi(1-r_1{}^2)(1-r_2{}^2)}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$- \frac{(x-\xi)(1-2x\xi)-2(\xi-x\cdot r_2{}^2)+(x-\xi)r_2{}^2}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2} - \frac{4r_{12}{}^2(x-\xi r_1{}^2)(\xi-x\cdot r_2{}^2)^2}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^3}$$

$$+ \frac{(1-2x\xi)(\xi-x\cdot r_2{}^2)(1-r_1{}^2)(1-r_2{}^2)-4(x-\xi)(x-\xi r_1{}^2)(\xi-x\cdot r_2{}^2)}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}$$

$$+ \frac{2(x-\xi)(\xi-x\cdot r_2{}^2)^2+r_{12}{}^2 r_2{}^2(x-\xi r_1{}^2)-r_{12}{}^2(1-2x\xi)(\xi-x\cdot r_2{}^2)}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}\}$$

$$\frac{\partial^3 G}{\partial x^2 \partial \eta} = -\frac{a^2}{4\pi D}\{ \frac{(y-\eta)}{r_{12}{}^2} - \frac{2(x-\xi)^2(y-\eta)}{r_{12}{}^4} - \frac{4r_{12}{}^2(y-\eta r_1{}^2)(\xi-x\cdot r_2{}^2)^2}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^3}$$

$$- \frac{x(-2x\eta)(1-r_2{}^2)+\eta(1-r_1{}^2)(1-r_2{}^2)+(y-\eta r_1{}^2)+(x-\xi)(-2x\eta)+(y-\eta)r_2{}^2}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$+ \frac{(-2x\eta)(\xi-x\cdot r_2{}^2)(1-r_1{}^2)(1-r_2{}^2)-4(x-\xi)(y-\eta r_1{}^2)(\xi-x\cdot r_2{}^2)+2(y-\eta)(\xi-x\cdot r_2{}^2)^2}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}$$

$$\frac{+r_{12}{}^2 r_2{}^2(y-\eta r_1{}^2)-r_{12}{}^2(-2x\eta)(\xi-x\cdot r_2{}^2)}{}\}$$

$$\frac{\partial^3 G}{\partial y^2 \partial \xi} = -\frac{a^2}{4\pi D}\left\{\frac{(x-\xi)}{r_{12}^{\,2}} - \frac{2(y-\eta)^2(x-\xi)}{r_{12}^{\,4}} - \frac{4r_{12}^{\,2}(x-\xi r_1^{\,2})(\eta-y\cdot r_2^{\,2})^2}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}\right.$$

$$-\frac{y(-2y\xi)(1-r_2^{\,2})+\xi(1-r_1^{\,2})(1-r_2^{\,2})+(x-\xi r_1^{\,2})+(y-\eta)(-2y\xi)+(x-\xi)r_2^{\,2}}{(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}}$$

$$+\frac{(-2y\xi)(\eta-y\cdot r_2^{\,2})(1-r_1^{\,2})(1-r_2^{\,2})-4(y-\eta)(x-\xi r_1^{\,2})(\eta-y\cdot r_2^{\,2})+2(x-\xi)(\eta-y\cdot r_2^{\,2})^2}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^2}$$

$$\left.\underline{+r_{12}^{\,2}r_2^{\,2}(x-\xi r_1^{\,2})+r_{12}^{\,2}(-2y\xi)(\eta-y\cdot r_2^{\,2})}\right\}$$

$$\frac{\partial^3 G}{\partial y^2 \partial \eta} = -\frac{a^2}{4\pi D}\left\{\frac{3(y-\eta)}{r_{12}^{\,2}} - \frac{2(y-\eta)^3}{r_{12}^{\,4}} - \frac{4r_{12}^{\,2}(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}\right.$$

$$-\frac{y(1-2y\eta)(1-r_2^{\,2})+\eta(1-r_1^{\,2})(1-r_2^{\,2})+(y-\eta r_1^{\,2})+(y-\eta)(1-2y\eta)-2(\eta-y\cdot r_2^{\,2})+(y-\eta)r_2^{\,2}}{(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}}$$

$$+\frac{(1-2y\eta)(\eta-y\cdot r_2^{\,2})(1-r_1^{\,2})(1-r_2^{\,2})-4(y-\eta)(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})+2(y-\eta)(\eta-y\cdot r_2^{\,2})^2}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^2}$$

$$\left.\underline{+r_{12}^{\,2}r_2^{\,2}(y-\eta r_1^{\,2})-r_{12}^{\,2}(1-2y\eta)(\eta-y\cdot r_2^{\,2})}\right\}$$

$$\frac{\partial^3 G}{\partial x \partial y \partial \xi} = -\frac{a^2}{4\pi D}\left\{\frac{y-\eta}{r_{12}^{\,2}} - \frac{2(x-\xi)^2(y-\eta)}{r_{12}^{\,4}} - \frac{4r_{12}^{\,2}(x-\xi r_1^{\,2})(\xi-x\cdot r_2^{\,2})(\eta-y\cdot r_2^{\,2})}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}\right.$$

$$+\frac{(\xi-x\cdot r_2^2)(1-2y\eta)(1-r_1^2)(1-r_2^2)-2(\xi-x\cdot r_2^2)(y-\eta)(y-\eta r_1^2)+2(\xi-x\cdot r_2^2)\cdot}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}$$

$$\frac{\cdot(y-\eta)(\eta-y\cdot r_2^2)-2(x-\xi)(y-\eta r_1^2)(\eta-y\cdot r_2^2)+2(\eta-y\cdot r_2^2)\eta x\cdot r_{12}^2}{}$$

$$+\frac{x(r_2^2-1)(1-2y\eta)+2x\eta(y-\eta)+(\xi-x\cdot r_2^2)}{(1-r_1^2)(1-r_2^2)+r_{12}^2}\ \}$$

$$\frac{\partial^3 G}{\partial x^3}=\frac{a^2}{4\pi D}\{\frac{3(x-\xi)}{r_{12}^2}-\frac{2(x-\xi)^2}{r_{12}^4}+\frac{3(\xi-x\cdot r_2^2)-3r_2^2(x-\xi)}{(1-r_1^2)(1-r_2^2)+r_{12}^2}+\frac{4r_{12}^2(\xi-x\cdot r_2^2)^3}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^3}$$

$$+\frac{6(\xi-x\cdot r_2^2)^2(x-\xi)-3r_{12}^2r_2^2(\xi-x\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}\ \}$$

$$\frac{\partial^3 G}{\partial y^3}=\frac{a^2}{4\pi D}\{\frac{3(y-\eta)}{r_{12}^2}-\frac{2(y-\eta)^2}{r_{12}^4}+\frac{3(\eta-y\cdot r_2^2)-3r_2^2(y-\eta)}{(1-r_1^2)(1-r_2^2)+r_{12}^2}+\frac{4r_{12}^2(\eta-y\cdot r_2^2)^3}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^3}$$

$$+\frac{6(y-\eta)(\eta-y\cdot r_2^2)^2-3r_{12}^2r_2^2(\eta-y\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}\ \}$$

$$\frac{\partial^3 G}{\partial x^2\partial y}=\frac{a^2}{4\pi D}\{\frac{y-\eta}{r_{12}^2}-\frac{2(x-\xi)^2(y-\eta)}{r_{12}^4}+\frac{\eta-y\cdot r_2^2-r_2^2(y-\eta)}{(1-r_1^2)(1-r_2^2)+r_{12}^2}$$

$$+\frac{4(x-\xi)(\xi-x\cdot r_2^2)(\eta-y\cdot r_2^2)-r_{12}^2r_2^2(\eta-y\cdot r_2^2)+2(y-\eta)(\xi-x\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}$$

$$+\frac{4r_{12}^2(\xi-x\cdot r_2^2)^2(\eta-y\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^3}\ \}$$

$$\frac{\partial^3 G}{\partial x \partial y^2} = \frac{a^2}{4\pi D}\left\{\frac{x-\xi}{r_{12}^{\;2}} - \frac{2(y-\eta)^2(x-\xi)}{r_{12}^{\;4}} + \frac{\xi - x\cdot r_2^{\;2} - r_2^{\;2}(x-\xi)}{(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}} + \frac{4r_{12}^{\;2}(\eta - y\cdot r_2^{\;2})^2(\xi - x\cdot r_2^{\;2})}{[(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}]^3}\right.$$

$$\left. + \frac{4(y-\eta)(\eta-y\cdot r_2^{\;2})(\xi-x\cdot r_2^{\;2}) - r_{12}^{\;2}r_2^{\;2}(\xi - x\cdot r_2^{\;2}) + 2(x-\xi)(\eta - y\cdot r_2^{\;2})^2}{[(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}]^2}\right\}$$

$$\frac{\partial^4 G}{\partial x^3 \partial \xi} = -\frac{a^2}{4\pi D}\left\{\frac{3}{r_{12}^{\;2}} + \frac{8(x-\xi)^4}{r_{12}^{\;6}} - \frac{12(x-\xi)^2}{r_{12}^{\;4}} - \frac{24r_{12}^{\;2}(x-\xi r_1^{\;2})(\xi - x\cdot r_2^{\;2})}{[(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}]^4}\right.$$

$$+ \frac{-(1-r_2^{\;2})(1-2x\xi) + 4x\xi(1-r_2^{\;2}) - 2(1-2x\xi) + 2\xi(x-\xi) - 3r_2^{\;2}}{(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}}$$

$$+ \frac{-4x(1-r_2^{\;2})(\xi-x\cdot r_2^{\;2})(1-2x\xi) \;\; -4\xi(1-r_1^{\;2})(1-r_2^{\;2})(\xi-x\cdot r_2^{\;2}) - 6(\xi - x\cdot r_2^{\;2})\cdot}{}$$

$$\frac{\cdot(x-\xi r_1^{\;2}) - 8(\xi-x\cdot r_2^{\;2})(x-\xi)(1-2x\xi) + 6(\xi-x\cdot r_2^{\;2}) - 12r_2^{\;2}(x-\xi)(\xi-x\cdot r_2^{\;2})}{[(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}]^2}$$

$$\frac{-r_2^{\;2}(1-r_1^{\;2})(1-r_2^{\;2})(1-2x\xi) + 6r_2^{\;2}(x-\xi)(x-\xi r_1^{\;2}) + 2r_{12}^{\;2}r_2^{\;2}(1-2x\xi) + 2r_{12}^{\;2}\xi(\xi-x\cdot r_2^{\;2})}{}$$

$$+ \frac{4(1-2x\xi)(\xi-x\cdot r_2^{\;2})^2(1-r_1^{\;2})(1-r_2^{\;2}) - 24(x-\xi)(x-\xi r_1^{\;2})(\xi-x\cdot r_2^{\;2}) \; +}{[(1-r_1^{\;2})(1-r_2^{\;2})+r_{12}^{\;2}]^3}$$

$$\left.\frac{+8(x-\xi)(\xi-x\cdot r_2^{\;2})^3 + 12r_{12}^{\;2}r_2^{\;2}(x-\xi r_1^{\;2})(\xi-x\cdot r_2^{\;2}) - 8r_{12}^{\;2}(1-2x\xi)(\xi-x\cdot r_2^{\;2})^2}{}\right\}$$

$$\frac{\partial^4 G}{\partial x^3 \partial\eta} = -\frac{a^2}{4\pi D}\{\frac{-6(x-\xi)(y-\eta)}{r_{12}{}^4} + \frac{8(x-\xi)^3(y-\eta)}{r_{12}{}^6} - \frac{24r_2{}^2(y-r_1{}^2\eta)(\xi-x\cdot r_2{}^2)}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^4}$$

$$+ \frac{8x^2\eta(1-r_2{}^2)(\xi-x\cdot r_2{}^2)-4\eta(1-r_1{}^2)(1-r_2{}^2)(\eta-x\cdot r_2{}^2)-6(y-\eta r_1)(\xi-x\cdot r_2{}^2)+}{}$$

$$\frac{+12x\eta(x-\xi)(\xi-x\cdot r_2{}^2)+2r_2{}^2x\eta(1-r_1{}^2)(1-r_2{}^2)+6r_2{}^2(x-\xi)(y-\eta r_1{}^2)-6r_2{}^2(y-\eta)\cdot}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}$$

$$\frac{\cdot(\xi-x\cdot r_2{}^2)-4x\eta r_{12}{}^2r_2{}^2+4x\eta(x-\xi)(\xi-x\cdot r_2{}^2)+2\eta r_{12}{}^2(\xi-x\cdot r_2{}^2)}{}$$

$$+ \frac{-8x\eta(\xi-x\cdot r_2{}^2)^2(1-r_1{}^2)(1-r_2{}^2)-24(x-\xi)(y-\eta r_1{}^2)(\xi-x\cdot r_2{}^2) +}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^3}$$

$$\frac{+8(y-\eta)(\xi-x\cdot r_2{}^2)^3+12r_{12}{}^2r_2{}^2(y-\eta r_1{}^2)(\xi-x\cdot r_2{}^2)+16r_{12}{}^2x\eta(\xi-x\cdot r_2{}^2)}{}\}$$

$$\frac{\partial^4 G}{\partial y^3 \partial\xi} = -\frac{a^2}{4\pi D}\{\frac{-6(x-\xi)(y-\eta)}{r_{12}{}^4} + \frac{8(x-\xi)(y-\eta)^3}{r_{12}{}^6} + \frac{6y\xi(1-r_2{}^2)+4y\xi+2\xi(y-\eta)}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$+ \frac{8y^2\xi(1-r_2{}^2)(\eta-y\cdot r_2{}^2)-4\xi(1-r_1{}^2)(1-r_2{}^2)(\eta-y\cdot r_2{}^2)-6(x-\xi r_1{}^2)(\eta-y\cdot r_2{}^2)+}{}$$

$$\frac{+12y\xi(y-\eta)(\eta-y\cdot r_2{}^2)+2y\xi r_2{}^2(1-r_1{}^2)(1-r_2{}^2)+6r_2{}^2(y-\eta)(x-\xi r_1{}^2) -}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}$$

$$\frac{-6r_2{}^2(x-\xi)(\eta-y\cdot r_2{}^2)-4y\xi r_{12}{}^2r_2{}^2+4y\xi(y-\eta)(\eta-y\cdot r_2{}^2)+2\xi r_{12}{}^2(\eta-y\cdot r_2{}^2)}{}$$

$$+\frac{-8y\xi(1-r_1^2)(1-r_2^2)(\eta-y\cdot r_2^2)^2-24(y-\eta)(x-\xi r_1^2)(\eta-y\cdot r_2^2)\ +}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^3}$$

$$\frac{+8(x-\xi)(\eta-y\cdot r_2^2)^3+12r_{12}^2r_2^2(x-\xi r_1^2)(\eta-y\cdot r_2^2)+16r_{12}^2y\xi(\eta-y\cdot r_2^2)}{}$$

$$-\frac{24r_{12}^2(x-\xi r_1^2)(\eta-y\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^4}\ \}$$

$$\frac{\partial^4 G}{\partial x^2\partial y\partial\xi}=-\frac{a^2}{4\pi D}\{-\frac{6(x-\xi)(y-\eta)}{r_{12}^4}+\frac{8(x-\xi)^3(y-\eta)}{r_{12}^6}+\frac{2y\xi(1-r_2^2)+2y\xi}{(1-r_1^2)(1-r_2^2)+r_{12}^2}$$

$$+\frac{-2x(1-2x\xi)(1-r_2^2)(\eta-y\cdot r_2^2)-2\xi(1-r_1^2)(1-r_2^2)(\eta-y\cdot r_2^2)\ -}{}$$

$$\frac{-2(x-\xi r_1^2)(\eta-y\cdot r_2^2)-2(x-\xi)(1-2x\xi)(\eta-y\cdot r_2^2)+4(\xi-x\cdot r_1^2)(\eta-y\cdot r_2^2)\ -}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}$$

$$\frac{-2(x-\xi)r_2^2(\eta-y\cdot r_2^2)-2y(1-2x\xi)(\xi-x\cdot r_2^2)(1-r_2^2)+8y\xi(x-\xi)(\xi-x\cdot r_2^2)+}{}$$

$$\frac{+2(y-\eta)r_2^2(x-\xi r_1^2)-2r_{12}^2r_2^2y\xi-2(y-\eta)(1-2x\xi)(\xi-x\cdot r_2^2)}{}$$

$$+\frac{4(1-2x\xi)(\xi-x\cdot r_2^2)(1-r_1^2)(1-r_2^2)(\eta-y\cdot r_2^2)-16(x-\xi)(x-\xi r_1^2)(\xi-x\cdot r_2^2)\cdot}{}$$

$$\frac{\cdot(\eta-y\cdot r_2^2)+8(x-\xi)(\xi-x\cdot r_2^2)^2(\eta-y\cdot r_2^2)+4r_{12}^2r_2^2(x-\xi r_1^2)(\eta-y\cdot r_2^2)-}{}$$

$$\frac{-4r_{12}^{\,2}(1-2x\xi)(\xi-x\cdot r_2^{\,2})(\eta-y\cdot r_2^{\,2})-8(y-\eta)(x-\xi\ r_1^{\,2})(\xi-x\cdot r_2^{\,2})\ +}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}$$

$$\frac{+8r_{12}^{\,2}y\xi(\xi-x\cdot r_2^{\,2})}{} - \frac{24r_{12}^{\,2}(x-\xi r_1^{\,2})(\xi-x\cdot r_2^{\,2})^2(\eta-y\cdot r_2^{\,2})}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^4}$$

$$\frac{\partial^4 G}{\partial y^3\partial\eta} = -\frac{a^2}{4\pi D}\left\{\frac{3}{r_{12}^{\,2}} - \frac{12(y-\eta)^2}{r_{12}^{\,4}} + \frac{8(y-\eta)^4}{r_{12}^{\,6}} - \frac{24r_{12}^{\,2}(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})^3}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^4}\right.$$

$$+\frac{-(1-r_2^{\,2})(1-2y\eta)+4y\eta(1-r_2^{\,2})-2(1-2y\eta)+2\eta(y-\eta)-3r_2^{\,2}}{(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}}$$

$$+\frac{-4y(1-2y\eta)(1-r_2^{\,2})(\eta-y\cdot r_2^{\,2})-4\eta(1-r_1^{\,2})(1-r_2^{\,2})(\eta-y\cdot r_2^{\,2})-6(\eta-y\cdot r_2^{\,2})\cdot}{}$$

$$\frac{\cdot(y-\eta r_1^{\,2})-8(y-\eta)(1-2y\eta)(\eta-y\cdot r_2^{\,2})+6(\eta-y\cdot r_2^{\,2})-6r_2^{\,2}(y-\eta)(\eta-y\cdot r_2^{\,2})-}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^2}$$

$$\frac{-r_2^{\,2}(1-2y\eta)(1-r_1^{\,2})(1-r_2^{\,2})+6r_2^{\,2}(y-\eta)(y-\eta r_1^{\,2})+2r_{12}^{\,2}r_2^{\,2}(1-2y\eta)+2r_{12}^{\,2}(\eta-y\cdot r_2^{\,2})}{}$$

$$+\frac{4(1-2y\eta)(1-r_1^{\,2})(1-r_2^{\,2})(\eta-y\cdot r_2^{\,2})^2-24(y-\eta)(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})^2+8(y-\eta)\cdot}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}$$

$$\left.\frac{\cdot(\eta-y\cdot r_2^{\,2})^3+12r_{12}^{\,2}r_2^{\,2}(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})-8r_{12}^{\,2}(1-2y\eta)(\eta-y\cdot r_2^{\,2})}{}\right\}$$

$$\frac{\partial^4 G}{\partial x^2 \partial y \partial \eta} = -\frac{a^2}{4\pi D}\{\frac{-1}{r_{12}^{\,2}} + \frac{8(x-\xi)^2(y-\eta)^2}{r_{12}^{\,6}} + \frac{2y\eta(1-r_2^{\,2})-(1-2y\eta)-r_2^{\,2}}{(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}}$$

$$+\frac{4x^2\eta(1-r_2^{\,2})(\eta-y\cdot r_2^{\,2})-2\eta(1-r_1^{\,2})(1-r_2^{\,2})(\eta-y\cdot r_2^{\,2})-2(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})+}{}$$

$$\frac{+4x\eta(x-\xi)(\eta-y\cdot r_2^{\,2})-2r_2^{\,2}(y-\eta)(\eta-y\cdot r_2^{\,2})+4xy\eta(\xi-x\cdot r_2^{\,2})(1-r_2^{\,2})-}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^2}$$

$$\frac{-4(x-\xi)(\xi-x\cdot r_2^{\,2})(1-2y\eta)+2(\xi-x\cdot r_2^{\,2})^2+2(y-\eta)r_2^{\,2}(y-\eta r_1^{\,2})+}{}$$

$$\frac{+r_{12}^{\,2}r_2^{\,2}(1-2y\eta)+4x\eta(y-\eta)(\xi-x\cdot r_2^{\,2})}{} + \frac{-8x\eta(\xi-x\cdot r_2^{\,2})(1-r_1^{\,2})(1-r_2^{\,2})\cdot}{}$$

$$\frac{\cdot(\eta-y\cdot r_2^{\,2})-16(x-\xi)(y-\eta r_1^{\,2})(\xi-x\cdot r_2^{\,2})(\eta-y\cdot r_2^{\,2})+8(y-\eta)(\xi-x\cdot r_2^{\,2})^2\cdot}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^3}$$

$$\frac{\cdot(\eta-y\cdot r_2^{\,2})+4r_{12}^{\,2}r_2^{\,2}(y-\eta r_1^{\,2})(\eta-y\cdot r_2^{\,2})+8r_{12}^{\,2}x\eta(\xi-x\cdot r_2^{\,2})(\eta-y\cdot r_2^{\,2})\ -}{}$$

$$\frac{-8(y-\eta)(y-\eta r_1^{\,2})(\xi-x\cdot r_2^{\,2})^2-4r_{12}^{\,2}(1-2y\eta)(\xi-x\cdot r_2^{\,2})^2}{}$$

$$-\frac{24r_{12}^{\,2}(y-\eta r_1^{\,2})(\xi-x\cdot r_2^{\,2})^2(\eta-y\cdot r_2^{\,2})}{[(1-r_1^{\,2})(1-r_2^{\,2})+r_{12}^{\,2}]^4}\}$$

$$\frac{\partial^4 G}{\partial x \partial y^2 \partial \xi} = -\frac{a^2}{4\pi D}\{-\frac{1}{r_{12}{}^2} + \frac{8(x-\xi)^2(y-\eta)^2}{r_{12}{}^2} + \frac{2x\xi(1-r_2{}^2)-(1-2x\xi)-r_2{}^2}{(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2}$$

$$+ \frac{4y^2\xi(1-r_2{}^2)(\xi-x\cdot r_2{}^2)-2\xi(1-r_1{}^2)(1-r_2{}^2)(\xi-x\cdot r_2{}^2)-2(x-\xi r_1{}^2)(\xi-x\cdot r_2{}^2)+}{}$$

$$\frac{+\,4y\xi(y-\eta)(\xi-x\cdot r_2{}^2)-2r_2{}^2(x-\xi)(\xi-x\cdot r_2{}^2)+4x\cdot y\xi(1-r_2{}^2)(\xi-y\cdot r_2{}^2)-}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^2}$$

$$\frac{-\,4(y-\eta)(1-2\xi x)(\eta-y\cdot r_2{}^2)+2(\eta-y\cdot r_2{}^2)^2+2(x-\xi)r_2{}^2(x-\xi r_1{}^2)+}{}$$

$$\frac{+\,r_{12}{}^2r_2{}^2(1-2x\xi)+4y\xi(x-\xi)(\eta-y\cdot r_2{}^2)}{} + \frac{-8y\xi(\eta-y\cdot r_2{}^2)(1-r_1{}^2)(1-r_2{}^2)\cdot}{}$$

$$\frac{(\xi-x\cdot r_2{}^2)-16(y-\eta)(x-\xi r_1{}^2)(\eta-y\cdot r_2{}^2)(\xi-x\cdot r_2{}^2)+8(x-\xi)(\eta-y\cdot r_2{}^2)^2\cdot}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^3}$$

$$\frac{(\xi-x\cdot r_2{}^2)+4r_{12}{}^2r_2{}^2(x-\xi r_1{}^2)(\xi-x\cdot r_2{}^2)+8y\xi r_{12}{}^2(\eta-y\cdot r_2{}^2)(\xi-x\cdot r_2{}^2)-}{}$$

$$\frac{-\,8(x-\xi)(x-\xi r_1{}^2)(\eta-y\cdot r_2{}^2)^2-4r_{12}{}^2(1-2x\xi)(\eta-y\cdot r_2{}^2)^2}{}$$

$$-\frac{24r_{12}{}^2(x-\xi r_1{}^2)(\eta-y\cdot r_2{}^2)^2(\xi-x\cdot r_2{}^2)}{[(1-r_1{}^2)(1-r_2{}^2)+r_{12}{}^2]^4}\}$$

$$\frac{\partial^4 G}{\partial x \partial y^2 \partial \eta} = -\frac{a^2}{4\pi D}\{-\frac{6(x-\xi)(y-\eta)}{r_{12}^4} + \frac{8(y-\eta)^3(x-\xi)}{r_{12}^6} + \frac{4x\eta - 2x\eta r_2^2}{(1-r_1^2)(1-r_2^2)+r_{12}^2}$$

$$+ \frac{-2y(1-2y\eta)(1-r_2^2)(\xi-x\cdot r_2^2)-2\eta(1-r_1^2)(1-r_2^2)(\xi-x\cdot r_2^2)-}{}$$

$$\frac{-2(y-\eta r_1^2)(\xi-x\cdot r_2^2)-2(y-\eta)(1-2y\eta)(\xi-x\cdot r_2^2)+4(\eta-y\cdot r_2^2)(\xi-x\cdot r_2^2)-}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^2}$$

$$\frac{-2r_2^2(y- \text{[illegible]} (\xi-x\cdot r_2^2)-2x(1-2y\eta)(\eta-y\cdot r_2^2)(1-r_2^2)+8(y-\eta)x\eta(\eta-y\cdot r_2^2)+}{\cdot}$$

$$\frac{+2r_2^2(x-\xi)(y-\eta r_1^2)-2x\eta r_{12}^2 r_2^2-2(x-\xi)(1-2y\eta)(\eta-y\cdot r_2^2)}{}$$

$$+ \frac{4(1-2y\eta)(\eta-y\cdot r_2^2)(1-r_1^2)(1-r_2^2)(\xi-x\cdot r_2^2)-16(y-\eta)(y-\eta r_1^2)\cdot}{}$$

$$\frac{\cdot(\eta-y\cdot r_2^2)(\xi-x\cdot r_2^2)+8(y-\eta)(\eta-y\cdot r_2^2)^2(\xi-x\cdot r_2^2)+4r_{12}^2 r_2^2(y-\eta r_1^2)\cdot}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^3}$$

$$\frac{\cdot(\xi-x\cdot r_2^2)-4r_{12}^2(1-2y\eta)(\eta-y\cdot r_2^2)(\xi-x\cdot r_2^2)-8(x-\xi)(y-\eta r_1^2)\cdot}{}$$

$$\frac{\cdot(\eta-y\cdot r_2^2)^2+8r_{12}^2 x\eta(\eta-y\cdot r_2^2)}{} - \frac{24r_{12}^2(y-\eta r_1^2)(\eta-y\cdot r_2^2)^2(\xi-x\cdot r_2^2)}{[(1-r_1^2)(1-r_2^2)+r_{12}^2]^4}\}$$

APPENDIX B

COMPUTER PROGRAM FOR THE BOUNDARY INTEGRAL METHOD

APPENDIX B

COMPUTER PROGRAM FOR THE BOUNDARY INTEGRAL METHOD

```
      PROGRAM PLCLOLD(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C
C  *********************************************************************
C                                                                      *
C  BOUNDARY INTEGRAL EQUATION METHOD                                   *
C                                                                      *
C                                                                      *
C  ISOTROPIC MATERIAL                                                  *
C  LINEAR PLATE THEORY                                                 *
C                                                                      *
C  ARBITRARY PLAN FORM                                                 *
C  ARBITRARY TRANSVERSE LOAD                                           *
C  ARBITRARY BOUNDARY CONDITIONS                                       *
C                                                                      *
C  THE FOLLOWING INPUT VALUES CAN BE ENTERED WITH FREE FORMAT......    *
C                                                                      *
C          NIP      =NUMBER OF INTERNAL LOAD POINTS                    *
C          NML      =NUMBER OF BOUNDARY MESH LENGTHES                  *
C          NFP      =NUMBER OF FIELD POINTS                            *
C          XB,YB    =POINTS ON B AT WHICH B.C. ARE SATISFIED           *
C          BANX,BANY=COMPONENTS OF UNIT NORMAL TO B AT XB,YB           *
C          XXB,YYB  =END POINTS OF MESHES ALONG THE PATH OF INTEGRATION *
C          XF,YF    =FIELD POINTS                                      *
C          XI,YI    =INTERNAL LOAD POINTS                              *
C          L        =LOCATION OF INTERNAL LOAD POINTS                  *
C                    (1=CORNER,2=EDGE,4=INTERNAL)                      *
C          Q        =VALUE OF TRANSVERSE LOAD AT INTERNAL LOAD POINT   *
C          DPSI     =FIELD SUBDIVISION SPACING ( X-DIRECTION)          *
C          DETA     =FIELD SUBDIVISION SPACING ( Y-DIRECTION)          *
C          EVALUE   =YOUNGS MODULUS OF THE MATERIAL                    *
C          HVALUE   =THICKNESS OF THE PLATE                            *
C          RADIUS   =RADIUS OF THE CIRCULAR CLAMPED PLATE              *
C          PR       =POISSON RATIO                                     *
C          NBTYPE   =BOUNDARY CONDITION TYPE FOR EACH MESH LENGTH      *
C                    NBTYPE=1 -- CLAMPED                               *
C                    NBTYPE=2 -- SIMPLY SUPPORTED                      *
C                    NBTYPE=3 -- FREE                                  *
C                                                                      *
C  *********************************************************************
C
      DIMENSION XB(36),YB(36),XXB(37),YYB(37),NBTYPE(37)
      DIMENSION XI(25),YI(25),L(25),Q(25)
      DIMENSION RLX(72),RB(72),RM(72,72),PS(72),WKAREA(72),RL(72)
      DIMENSION XF(17),YF(17),W(17),BMX(17),BMY(17)
      DIMENSION BANX(36),BANY(36)
C
C
C  FORMAT STATEMENTS
C
  100 FORMAT("OLOCT",6X,"XB",8X,"YB",7X,"ANX",8X,"ANY",6X,"XXB",7X,"YYB"
     1,4X,"NBTYPE",/"0"/(I5,6F10.4,I5))
  200     FORMAT ("OLOCT",16X,"XI",17X,"YI",14X,"L",23X,"Q"/"0"/
     1    (I4,3X,E20.8,1X,E20.8,5X,I4,10X,E20.8))
  300     FORMAT ("OLOCT",19X,"RLD",34X,"RLS"/"0"/(I4,8X,E20.8,16X,
     1    E20.8))
  400     FORMAT ("1LOCT",19X,"PSP",34X,"PSM"/"0"/(I4,8X,E20.8,16X,
     1    E20.8))
  500     FORMAT ("OLOCT",19X,"XF",35X,"YF"/"0"/(I4,8X,E20.8,16X,E20.8))
  600 FORMAT(1H1,8X,"NODE",12X"XF",16X,"YF",19X,"W",16X,"BMX",
     116X,"BMY"/(1H0,I10,2F20.10,3E20.12))
  700 FORMAT(1H1,"INPUT VALUES .....",//1X,"NML = ",I3,"  NFP =",I3,
     1"   NIP = ",I3,"    PR = ",F5.3,"   DPSI = ",F5.3,"  DETA = ",F5.3,
     1/1H0,"YOUNGS MODULUS =   ",E14.8,"     RADIUS OF THE PLATE = ",
     1F7.3,"    THICKNESS OF THE PLATE = ", F6.3)
  800 FORMAT(I5,40X,2F10.4,I5)
  900 FORMAT(////,1X,"SUM OF THE LATERAL LOADS = ",E20.12)
 1103 FORMAT(I10,2E20.12)
 1014 FORMAT(1H1,"THE FOLLOWING IS A LIST OF DOUBLE CHECKING OF B.C.S",
     1//1X,"NOTE .... IF ALL THE VALUES LISTED BELOW ARE IN THE ORDER OF
     1 E-9 OR LESS, RESULTS WILL BE IN GOOD SHAPE.",
     1//,4X"NML"10X,"B.C.  1",13X,"B.C.  2",//)
C
C  INPUT VALUES .........
C
      READ(5,*)NML,NIP,NFP,PR,DPSI,DETA,EVALUE,RADIUS,HVALUE
      WRITE(6,700)NML,NFP,NIP,PR,DPSI,DETA,EVALUE,RADIUS,HVALUE
      READ(5,*)(XB(I),I=1,NML)
      READ(5,*)(YB(I),I=1,NML)
      READ(5,*)(NBTYPE(I),I=1,NML)
      READ(5,*)(BANX(I),I=1,NML)
```

```
      READ(5,*)(BANY(I),I=1,NML)
      READ(5,*)(XXB(I),I=1,NML)
      READ(5,*)(YYB(I),I=1,NML)
      WRITE(6,100)(I,XB(I),YB(I),BANX(I),BANY(I),XXB(I),YYB(I),NBTYPE(I)
     1,I=1,NML)
      XXB(NML+1)=XXB(1)
      YYB(NML+1)=YYB(1)
      NBTYPE(NML+1)=NBTYPE(1)
      NMLP1=NML+1
      WRITE(6,800)NMLP1,XXB(NMLP1),YYB(NMLP1),NBTYPE(NMLP1)
      READ(5,*)(XF(I),I=1,NFP)
      READ(5,*)(YF(I),I=1,NFP)
      WRITE(6,500)(I,XF(I),YF(I),I=1,NFP)
      READ(5,*)(XI(I),I=1,NIP)
      READ(5,*)(YI(I),I=1,NIP)
      READ(5,*)(L(I),I=1,NIP)
      READ(5,*)(Q(I),I=1,NIP)
      WRITE(6,200)(I,XI(I),YI(I),L(I),Q(I),I=1,NIP)
C
C ******************************************************************
C        THE PLATE PROBLEM IS TO SOLVE THE FOLLOWING SET OF EQUATIONS  *
C ******************************************************************
C
C      .*                                   *.   .*  *.   .*       *.
C     *                                       * *      * *           *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     *   RM(I,J)             *   RM(I,J+NML) * *  F*  * *   RL(I)   *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     * *************         * ************* * * *** * = * ********* *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     * RM(I+NML,J)           * RM(I+NML,J+NML) * * M* * * RL(I+NML) *
C     *                       *               * *      * *           *
C     *                       *               * *      * *           *
C     *                                       * *      * *           *
C      *.                                    .*  *.  .*  *.         .*
C        *                                  *     *  *     *       *
C
C
C
C
C ******************************************************************
C        THE FIRST STEP IS TO DETERMINE THE RL VECTOR.                 *
C        THESE ARE THE VALUES OF AREA INTEGRALS OF BOUNDARY CONDI-     *
C        TION EQUATIONS.                                               *
C ******************************************************************
C
C THE FOLLOWING IS FOR TRANSFERING TO NON-DIMENSIONAL
C
      NBP=NML+1
      NML2=2*NML
      DPLATE=EVALUE*HVALUE**3/(12.*(1.-PR**2))
      PI=4.*ATAN(1.)
      COEF1=-1./(16.*PI)
      COEF2=2.0*COEF1
      DO 80 I=1,NML
      XB(I)=XB(I)/RADIUS
      YB(I)=YB(I)/RADIUS
      XXB(I)=XXB(I)/RADIUS
   80 YYB(I)=YYB(I)/RADIUS
      XXB(NMLP1)=XXB(1)
      YYB(NMLP1)=YYB(1)
      DO 81 I=1,NFP
      XF(I)=XF(I)/RADIUS
   81 YF(I)=YF(I)/RADIUS
      DO 82 I=1,NIP
      XI(I)=XI(I)/RADIUS
   82 YI(I)=YI(I)/RADIUS
C
C ******************************************************************
C        NCHECK IS AN ARGUMENT NUMBER FOR DOUBLE CHECKING B.C. VALUES. *
C        AFTER THE FICTITIOUS LOADS P* AND M* HAVE BEEN COMPUTED,      *
C        THESE VALUES ARE SUBSTITUTED BACK TO THE ORIGINAL BOUNDARY    *
C        CONDITION EQUATIONS TO DOUBLE CHECK WHETHER F(W)=ZERO ARE     *
C        SATISFIED. ACCORDING TO EXPERIMENTS, ERRORS .LE. E-10 INDICATE*
C        THE RM MATRIX IS NOT POORLY ILLCONDITIONED, AND BOTH          *
C        TRUNCATION AND ROUND-OFF ERRORS ARE NOT GREAT.                *
C ******************************************************************
```

```
C
      NCHECK=0
 1019 CONTINUE
C
C THE FOLLOWING EQUATIONS ARE FOR ALL THE THREE B.C. TYPES
C
          DO 5 I=1,NML
          RL(I)=0.0
          RL(I+NML)=0.0
      R1X=XB(I)
      R1Y=YB(I)
          R1S=R1X**2+R1Y**2
          Z1=1.0-R1S
          S=((XB(I+1)-XB(I))**2+(YB(I+1)-YB(I))**2)**0.5
      ANX=BANX(I)
      ANY=BANY(I)
          DO 4 J=1,NIP
          R2X=XI(J)
          R2Y=YI(J)
          R2S=R2X**2+R2Y**2
          Z2=1.0-R2S
          R12S=(R1X-R2X)**2+(R1Y-R2Y)**2
          Z3=Z1*Z2
          Z6=R1X*R2S-R2X
          Z7=R1X-R2X
          Z8=R1Y*R2S-R2Y
          Z9=R1Y-R2Y
          Z4=R12S/(Z3+R12S)
      IF(Z4.LE.1.E-12)GOTO 17
          Z5=ALOG(Z4)
          GO TO 18
   17     Z5=0.0
   18     Z10=(Z1*Z2+R12S*Z5)*Q(J)
C
      IF(NBTYPE(I).EQ.1)GOTO 30
C
      ANX2=ANX**2
      ANY2=ANY**2
      Z3R12=Z3+R12S
      Z3R12S=Z3R12**2
      Z3R12C=Z3R12*Z3R12S
      Z3R12Q=Z3R12S**2
C
      IF(NBTYPE(I).EQ.3)GOTO 34
C
      ZZ3=ANX2*(R2S+2.*Z7**2/R12S-(4.*Z7*Z6+R12S*R2S)/Z3R12
     1+2.*R12S*Z6**2/Z3R12S*Z5)
      ZZ4=4.*ANX*ANY*(Z7*Z9/R12S-(Z9*Z6+Z7*Z8)/Z3R12+R12S*Z6*Z8/Z3R12S)
      ZZ5=ANY2*(R2S+2.*Z9**2/R12S-(4.*Z9*Z8+R12S*R2S)/Z3R12
     1+2.*R12S*Z8**2/Z3R12S*Z5)
      Z11=(ZZ3+ZZ4+ZZ5)*Q(J)
      GOTO 34
   30 CONTINUE
          Z11=(ANX*(Z6-Z6*Z4+Z7*Z5)+ANY*(Z8-Z8*Z4+Z9*Z5))*Q(J)
   34 CONTINUE
          IF (L(J).GT.1) GO TO 1
          Z12=COEF1*DPSI*DETA/4.0
          Z13=COEF2*DPSI*DETA/4.0
          GO TO 3
    1     IF (L(J).GT.2) GO TO 2
          Z12=COEF1*DPSI*DETA/2.0
          Z13=COEF2*DPSI*DETA/2.0
          GO TO 3
    2     Z12=COEF1*DPSI*DETA
          Z13=COEF2*DPSI*DETA
    3 CONTINUE
C
      IF(NBTYPE(I).EQ.3)GOTO 50
C
          RL(I)=RL(I)+Z12*Z10
          RL(I+NML)=RL(I+NML)+Z13*Z11
      GOTO 4
C
C THE FOLLOWING IS FOR FREE EDGES
C
   50 CONTINUE
      CONST1=ANX*(ANX2+(2.-PR)*ANY2)
      CONST2=ANY*((2.*PR-1.)*ANX2+(2.-PR)*ANY2)
      CONST3=ANX*((2.*PR-1.)*ANY2+(2.-PR)*ANX2)
      CONST4=ANY*(ANY2+(2.-PR)*ANX2)
      CONST5=ANX2+PR*ANY2
```

```
      CONST6=2.*(1.-PR)*ANX*ANY
      CONST7=ANY2+PR*ANX2
      ZF1=1.-2.*R1X*R2X
      ZF2=1.-2.*R1Y*R2Y
      ZF3=R1X-R2X*R1S
      ZF4=R1Y-R2Y*R1S
      AA3=6.*Z7/R12S-4.*Z7**3/R12S**2-(6.*Z6+6.*R2S*Z7)/Z3R12
     1+(12.*Z6**2*Z7+6.*R12S*R2S*Z6)/Z3R12S-8.*R12S*Z6**3/Z3R12C
      DD3=6.*Z9/R12S-4.*Z9**3/R12S**2-(6.*R2S*Z9+6.*Z8)/Z3R12
     1+(12.*Z9*Z8**2+6.*R12S*R2S*Z8)/Z3R12S-8.*R12S*Z8**3/Z3R12C
      AA2=2.*Z9/R12S-4.*Z7**2*Z9/R12S**2+(-2.*R2S*Z9-2.*Z8)/Z3R12
     1+(8.*Z7*Z6*Z8+2.*R12S*R2S*Z8+4.*Z9*Z6**2)/Z3R12S
     1-8.*R12S*Z6**2*Z8/Z3R12C
      DD2=2.*Z7/R12S-4.*Z9**2*Z7/R12S**2-(2.*R2S*Z7+2.*Z6)/Z3R12
     1+(8.*Z9*Z6*Z8+2.*R12S*Z6*R2S+4.*Z7*Z8**2)/Z3R12S
     1-8.*R12S*Z8**2*Z6/Z3R12C
      AAA=R2S+2.*Z7**2/R12S-(4.*Z7*Z6+R12S*R2S)/Z3R12
     1+2.*R12S*Z6**2/Z3R12S+Z5
      DDD=R2S+2.*Z9**2/R12S-(4.*Z9*Z8+R12S*R2S)/Z3R12
     1+2.*R12S*Z8**2/Z3R12S+Z5
      GGG=2.*Z7*Z9/R12S-2.*(Z9*Z6+Z7*Z8)/Z3R12
     1+2.*R12S*Z6*Z8/Z3R12S
      TOTAL1= (CONST1*AA3+CONST2*AA2+CONST3*DD2+CONST4*DD3)*Z13*Q(J)
      RL(I)=RL(I)+TOTAL1
      TOTAL2= (CONST5*AAA+CONST6*GGG+CONST7*DDD)*Z13*Q(J)
      RL(I+NML)=RL(I+NML)+TOTAL2
    4    CONTINUE
    5    CONTINUE
         WRITE (6,300) (I,RL(I),RL(I+NML),I=1,NML)
C
C
C ******************************************************************
C        DETERMINATION OF COEFFICIENT MATRIX .....                *
C        RM(I,J), RM(I,J+NML), RM(I+NML,J), AND RM(I+NML,J+NML)   *
C ******************************************************************
C
         COEF1=-COEF1
         COEF2=-COEF2
         DO 8 I=1,NML
      R1X=XB(I)
      R1Y=YB(I)
         R1S=R1X**2+R1Y**2
         Z1=1.0-R1S
         S1=((XB(I+1)-XB(I))**2+(YB(I+1)-YB(I))**2)**0.5
      ANX1=BANX(I)
      ANY1=BANY(I)
      DO 7 J=1,NML
      R2X=(XXB(J+1)+XXB(J))/2.
      R2Y=(YYB(J+1)+YYB(J))/2.
         R2S=R2X**2+R2Y**2
         Z2=1.0-R2S
         S2=((XXB(J+1)-XXB(J))**2+(YYB(J+1)-YYB(J))**2)**0.5
         ANX2=(YYB(J)-YYB(J+1))/S2
         ANY2=(XXB(J+1)-XXB(J))/S2
         Z3=Z1*Z2
         R12S=(R1X-R2X)**2+(R1Y-R2Y)**2
      Z3R12=Z3+R12S
      Z3R12S=Z3R12**2
      Z3R12C=Z3R12*Z3R12S
      Z3R12Q=Z3R12S**2
      Z4=R12S/Z3R12
         Z5=ALOG(Z4)
C
      IF(NBTYPE(I).EQ.3)GOTO 51
C
         Z6=R1X-R2X*R1S
         Z7=R1X-R2X
         Z8=R1Y-R2Y*R1S
         Z9=R1Y-R2Y
         Z10=R1X*R2S-R2X
         Z11=R1Y*R2S-R2Y
         Z12=1.0-2.0*R1X*R2X
         Z13=-2.0*R1X*R2Y
         Z14=-2.0*R1Y*R2X
         Z15=1.0-2.0*R1Y*R2Y
         RM(I,J)=COEF1*(Z3+R12S*Z5)*S2
         RM(I,J+NML)=COEF2*(ANX2*(Z6-Z6*Z4+Z7*Z5)+ANY2*(Z8-Z8*Z4+Z9*Z5
     1   ))*S2
C
      IF(NBTYPE(I).EQ.2)GOTO 40
C
```

```
      RM(I+NML,J)=COEF2*(ANX1*(Z10-Z10*Z4+Z7*Z5)+ANY1*(Z11-Z11*Z4+Z9
     1   *Z5))*S2
      Z16=ANX1*ANX2*(2.0*(Z7**2)/R12S+(Z12*Z3-2.0*Z7*Z6-2.0*Z7*Z10)/
     1   Z3R12+(2.0*R12S*Z6*Z10)/Z3R12S+Z5)
      Z17=ANX1*ANY2*(2.0*Z9*Z7/R12S+(Z13*Z3-2.0*Z7*Z8-2.0*Z9*Z10)/
     1   Z3R12+(2.0*R12S*Z8*Z10)/Z3R12S)
      Z18=ANY1*ANX2*(2.0*Z7*Z9/R12S+(Z14*Z3-2.0*Z9*Z6-2.0*Z7*Z11)
     1   /Z3R12+(2.0*R12S*Z6*Z11)/Z3R12S)
      Z19=ANY1*ANY2*(2.0*(Z9**2)/R12S+(Z15*Z3-2.0*Z9*Z8-2.0*Z9*Z11)/
     1   Z3R12+(2.0*R12S*Z8*Z11)/Z3R12S+Z5)
      RM(I+NML,J+NML)=COEF2*(Z16+Z17+Z18+Z19)*S2
      GOTO 7
C
C     FOR SIMPLY SUPPORTED BOUNDARIES
C
    4 ZX1=-Z10
      ZY1=-Z11
      ZX1S=ZX1**2
      ZR1=R12S*R2S
      ZT1=(R2S+Z7**2*2./R12S+(4.*Z7*ZX1-ZR1)/Z3R12
     1+2.*R12S*ZX1**2/Z3R12S+Z5)*ANX1**2
      ZT2=4.*ANX1*ANY1*(Z7*Z9/R12S+(Z9*ZX1+Z7*ZY1)/Z3R12
     1+R12S*ZX1*ZY1/Z3R12S)
      ZT3=ANY1**2*(R2S+2.*Z9**2/R12S+(4.*Z9*ZY1-ZR1)/Z3R12
     1+2.*R12S*ZY1**2/Z3R12S+Z5)
      RM(I+NML,J)=COEF2*(ZT1+ZT2+ZT3)*S2
      ZT4=ANX1**2*ANX2*(6.*Z7/R12S-4.*Z7**3/R12S**2
     1-2.*(R1X*Z12*Z2+R2X*Z3+Z6+Z7*Z12-2.*ZX1+Z7*R2S)/Z3R12
     1+2.*(Z12*ZX1*Z3-4.*Z7*Z6*ZX1+2.*Z7*ZX1S+ZR1*Z6-R12S*Z12*ZX1)
     1/Z3R12S-8.*R12S*Z6*ZX1**2/Z3R12C)
      ZT5=ANX1**2*ANY2*(2.*Z9/R12S-4.*Z7**2*Z9/R12S**2
     1-2.*(R1X*Z13*Z2+R2Y*Z3+Z8+Z7*Z13+Z9*R2S)/Z3R12
     1+2.*(Z13*ZX1*Z3-4.*Z7*Z8*ZX1+2.*Z9*ZX1S+ZR1*Z8-R12S*Z13*ZX1)
     1/Z3R12S-8.*R12S*Z8*ZX1S/Z3R12C)
      ZT6=4.*ANX1*ANY1*ANX2*(Z9/R12S-2.*Z7**2*Z9/R12S**2
     1+(-R1Y*Z2*Z12-Z14*Z7+ZY1)/Z3R12
     1+(ZY1*Z12*Z3-2.*ZY1*Z7*Z6+2.*Z7*ZY1*ZX1-2.*Z9*Z6*ZX1
     1+2.*R2X*R1Y*R12S*ZX1)/Z3R12S
     1-4.*R12S*Z6*ZX1*ZY1/Z3R12C)
      ZT7=4.*ANX1*ANY1*ANY2*(Z7/R12S-2.*Z9**2*Z7/R12S**2
     1+(-R1X*Z2*Z15-Z13*Z9+ZX1)/Z3R12
     1+(ZX1*Z15*Z3-2.*ZX1*Z9*Z8+2.*Z9*ZX1*ZY1
     1-2.*Z7*Z8*ZY1+2.*R2Y*R1X*R12S*ZY1)/Z3R12S
     1-4.*R12S*Z8*ZX1*ZY1/Z3R12C)
      ZT8=ANY1**2*ANY2*(2.*Z7/R12S-4.*Z7*Z9**2/R12S**2
     1-2.*(R1Y*Z14*Z2+R2X*Z3+Z6+Z9*Z14+Z7*R2S)/Z3R12
     1+2.*(Z14*ZY1*Z3-4.*Z9*Z6*ZY1+2.*Z7*ZY1**2
     1+ZR1*Z6-R12S*Z14*ZY1)/Z3R12S
     1-8.*R12S*Z6*ZY1**2/Z3R12C)
      ZT9=ANY1**2*ANY2*(6.*Z9/R12S-4.*Z9**3/R12S**2
     1-2.*(R1Y*Z15*Z2+R2Y*Z3+Z8+Z9*Z15-2.*ZY1+Z9*R2S)/Z3R12
     1+2.*(Z15*ZY1*Z3-4.*Z9*Z8*ZY1+2.*Z9*ZY1**2
     1+ZR1*Z8-R12S*Z15*ZY1)/Z3R12S
     1-8.*R12S*Z8*ZY1**2/Z3R12C)
      RM(I+NML,J+NML)=COEF2*(ZT4+ZT5+ZT6+ZT7+ZT8+ZT9)*S2
      GOTO 7
C
C        FOR FREE EDGES
C
   51 CONTINUE
      Z6=R1X*R2S-R2X
      Z7=R1X-R2X
      Z9=R1Y-R2Y
      Z8=R1Y*R2S-R2Y
      CONST1=ANX1*(ANX1**2+(2.-PR)*ANY1**2)
      CONST2=ANY1*((2.*PR-1.)*ANX1**2+(2.-PR)*ANY1**2)
      CONST3=ANX1*((2.*PR-1.)*ANY1**2+(2.-PR)*ANX1**2)
      CONST4=ANY1*(ANY1**2+(2.-PR)*ANX1**2)
      CONST5=ANX1**2+PR*ANY1**2
      CONST6=2.*(1.-PR)*ANX1*ANY1
      CONST7=ANY1**2+PR*ANX1**2
      ZF1=1.-2.*R1X*R2X
      ZF2=1.-2.*R1Y*R2Y
      ZF3=R1X-R2X*R1S
      ZF4=R1Y-R2Y*R1S
      AA3=6.*Z7/R12S-4.*Z7**3/R12S**2-(6.*Z6+6.*R2S*Z7)/Z3R12
     1+(12.*Z6**2*Z7+6.*R12S*R2S*Z6)/Z3R12S-8.*R12S*Z6**3/Z3R12C
      BB3=6./R12S+16.*Z7**4/R12S**3-24.*Z7**2/R12S**2
     1+(-2.*Z2*ZF1+8.*R1X*R2X*Z2-4.*ZF1+4.*R2X*Z7-6.*R2S)/Z3R12
     1+(8.*R1X*Z2*Z6*ZF1+8.*R2X*Z3*Z6+12.*Z6*ZF3+16.*Z6*Z7*ZF1
     1+12.*Z6**2+12.*R2S*Z7*Z6-2.*R2S*Z3*ZF1+12.*R2S*Z7*ZF3
```

```
1+4.*R12S*R2S*ZF1-4.*R12S*R2X*Z6)/Z3R12S
1+(8.*ZF1*Z6**2*Z3-48.*Z7*ZF3*Z6**2-16.*Z7*Z6**3
1-24.*R12S*R2S*ZF3*Z6-16.*R12S*ZF1*Z6**2)/Z3R12C
1+48.*R12S*ZF3*Z6**3/Z3R12Q
 CC3=-12.*Z7*Z9/R12S**2+16.*Z7**3*Z9/R12S**3
1+(12.*R1X*R2Y*Z2+8.*R1X*R2Y+4.*R2Y*Z7)/Z3R12
1+(-16.*R1X**2*R2Y*Z2*Z6+8.*R2Y*Z3*Z6+12.*ZF4*Z6-24.*R1X*R2Y*Z6*Z7
1+4.*R2S*R1X*R2Y*Z3+12.*R2S*Z7*ZF4+12.*R2S*Z9*Z6
1-8.*R1X*R2Y*R12S*R2S-8.*R1X*R2Y*Z6*Z7-4.*R2Y*R12S*Z6)/Z3R12S
1+(-16.*R1X*R2Y*Z6**2*Z3-48.*Z7*ZF4*Z6**2-16.*Z9*Z6**3
1-24.*R12S*R2S*ZF4*Z6+32.*R12S*R1X*R2Y*Z6**2)/Z3R12C
1+48.*R12S*ZF4*Z6**3/Z3R12Q
 DD3=6.*Z9/R12S-4.*Z9**3/R12S**2-(6.*R2S*Z9+6.*Z8)/Z3R12
1+(12.*Z9*Z8**2+6.*R12S*R2S*Z8)/Z3R12S-8.*R12S*Z8**3/Z3R12C
 EE3=-12.*Z7*Z9/R12S**2+16.*Z7*Z9**3/R12S**3
1+(12.*R1Y*R2X*Z2+8.*R1Y*R2X+4.*R2X*Z9)/Z3R12
1+(-16.*R1Y**2*R2X*Z2*Z8+8.*R2X*Z3*Z8+12.*ZF3*Z8
1-24.*R1Y*R2X*Z9*Z8+4.*R1Y*R2X*R2S*Z3+12.*R2S*Z9*ZF3
1+12.*R2S*Z7*Z8-8.*R1Y*R2X*R12S*R2S-8.*R1Y*R2X*Z9*Z8
1-4.*R2X*R12S*Z8)/Z3R12S
1+(-16.*R1Y*R2X*Z3*Z8**2-48.*Z9*ZF3*Z8**2-16.*Z7*Z8**3
1-24.*R12S*R2S*ZF3*Z8+32.*R12S*R1Y*R2X*Z8**2)/Z3R12C
1+48.*R12S*ZF3*Z8**3/Z3R12Q
 FF3=6./R12S-24.*Z9**2/R12S**2+16.*Z9**4/R12S**3
1+(-2.*Z2*ZF2+8.*R1Y*R2Y*Z2-4.*ZF2+4.*R2Y*Z9-6.*R2S)/Z3R12
1+(8.*R1Y*ZF2*Z2*Z8+8.*R2Y*Z3*Z8+12.*Z8*ZF4+16.*Z9*ZF2*Z8
1+12.*Z8**2+12.*R2S*Z9*Z8-2.*R2S*ZF2*Z3+12.*R2S*Z9*ZF4
1+4.*R12S*R2S*ZF2-4.*R12S*R2Y*Z8)/Z3R12S
1+(8.*ZF2*Z3*Z8**2-48.*Z9*ZF4*Z8**2-16.*Z9*Z8**3
1-24.*R12S*R2S*ZF4*Z8-16.*R12S*ZF2*Z8**2)/Z3R12C
1+48.*R12S*ZF4*Z8**3/Z3R12Q
 AA2=2.*Z9/R12S-4.*Z7**2*Z9/R12S**2+(-2.*R2S*Z9-2.*Z8)/Z3R12
1+(8.*Z7*Z6*Z8+2.*R12S*R2S*Z8+4.*Z9*Z6**2)/Z3R12S
1-8.*R12S*Z6**2*Z8/Z3R12C
 BB2=-12.*Z7*Z9/R12S**2+16.*Z7**3*Z9/R12S**3
1+(4.*R1Y*R2X*Z2+4.*R1Y*R2X)/Z3R12
1+(4.*R1X*ZF1*Z2*Z8+4.*R2X*Z3*Z8+4.*ZF3*Z8+4.*Z7*ZF1*Z8
1+8.*Z6*Z8+4.*Z7*R2S*Z8+4.*R1Y*ZF1*Z6*Z2-16.*R1Y*R2X*Z7*Z6
1+4.*Z9*R2S*ZF3-4.*R12S*R2S*R1Y*R2X+4.*Z9*ZF1*Z6)/Z3R12S
1+(8.*ZF1*Z6*Z3*Z8-32.*Z7*ZF3*Z6*Z8-16.*Z7*Z6**2*Z8
1-8.*R12S*R2S*ZF3*Z8-8.*R12S*ZF1*Z6*Z8-16.*Z9*ZF3*Z6**2
1+16.*R12S*R1Y*R2X*Z6**2)/Z3R12C
1+48.*R12S*ZF3*Z6**2*Z8/Z3R12Q
 CC2=2./R12S-(4.*Z7**2+4.*Z9**2)/R12S**2+16.*Z7**2*Z9**2/R12S**3
1+(4.*R1Y*R2Y*Z2-2.*ZF2-2.*R2S)/Z3R12
1+(-8.*R1X**2*R2Y*Z2*Z8+4.*R2Y*Z3*Z8+4.*ZF4*Z8-8.*R1X*R2Y*Z7*Z8
1+4.*R2S*Z9*Z8-8.*R1X*R1Y*R2Y*Z6*Z2+8.*Z7*ZF2*Z6+4.*Z6**2
1+4.*Z9*R2S*ZF4+2.*R12S*R2S*ZF2-8.*R1X*R2Y*Z9*Z6)/Z3R12S
1+(-16.*R1X*R2Y*Z3*Z6*Z8-32.*Z7*ZF4*Z6*Z8-16.*Z9*Z6**2*Z8
1-8.*R12S*R2S*ZF4*Z8+16.*R12S*R1X*R2Y*Z6*Z8-16.*Z9*ZF4*Z6**2
1-8.*R12S*ZF2*Z6**2)/Z3R12C
1+48.*R12S*ZF4*Z6**2*Z8/Z3R12Q
 DD2=2.*Z7/R12S-4.*Z9**2*Z7/R12S**2-(2.*R2S*Z7+2.*Z6)/Z3R12
1+(8.*Z9*Z6*Z8+2.*R12S*Z6*R2S+4.*Z7*Z8**2)/Z3R12S
1-8.*R12S*Z8**2*Z6/Z3R12C
 EE2=2./R12S-4.*(Z7**2+Z9**2)/R12S**2+16.*Z7**2*Z9**2/R12S**3
1+(4.*R1X*R2X*Z2-2.*ZF1-2.*R2S)/Z3R12
1+(-8.*R1Y**2*R2X*Z2*Z6+4.*R2X*Z3*Z6+4.*ZF3*Z6-8.*R1Y*R2X*Z9*Z6
1+4.*R2S*Z7*Z6-8.*R1X*R1Y*R2X*Z2*Z8+8.*Z9*ZF1*Z8+4.*Z8**2
1+4.*Z7*R2S*ZF3+2.*R12S*R2S*ZF1-8.*R1Y*R2X*Z7*Z8)/Z3R12S
1+(-16.*R1Y*R2X*Z3*Z6*Z8-32.*Z9*ZF3*Z6*Z8-16.*Z7*Z8**2*Z6
1-8.*R12S*R2S*ZF3*Z6+16.*R1Y*R2X*R12S*Z6*Z8-16.*Z7*ZF3*Z8**2
1-8.*R12S*ZF1*Z8**2)/Z3R12C
1+48.*R12S*ZF3*Z8**2*Z6/Z3R12Q
 FF2=-12.*Z7*Z9/R12S**2+16.*Z9**3*Z7/R12S**3
1+4.*(R1X*R2Y*Z2+R1X*R2Y)/Z3R12
1+(4.*R1Y*ZF2*Z2*Z6+4.*R2Y*Z3*Z6+4.*ZF4*Z6+4.*Z9*ZF2*Z6+8.*Z6*Z8
1+4.*R2S*Z9*Z6+4.*R1X*ZF2*Z8*Z2-16.*Z9*R1X*R2Y*Z8+4.*R2S*Z7*ZF4
1-4.*R1X*R2Y*R12S*R2S+4.*Z7*ZF2*Z8)/Z3R12S
1+(8.*ZF2*Z6*Z8*Z3-32.*Z9*ZF4*Z6*Z8-16.*Z9*Z8**2*Z6
1-8.*R12S*R2S*ZF4*Z6-8.*R12S*ZF2*Z6*Z8-16.*Z7*ZF4*Z8**2
1+16.*R12S*R1X*R2Y*Z8**2)/Z3R12C
1+48.*R12S*ZF4*Z8**2*Z6/Z3R12Q
 AAA=R2S+2.*Z7**2/R12S-(4.*Z7*Z6+R12S*R2S)/Z3R12
1+2.*R12S*Z6**2/Z3R12S+Z5
 BBB=6.*Z7/R12S-4.*Z7**3/R12S**2-2.*(R1X*ZF1*Z2+R2X*Z3+ZF3
1+Z7*ZF1+2.*Z6+Z7*R2S)/Z3R12+2.*(-ZF1*Z6*Z3+4.*Z7*ZF3*Z6
1+2.*Z7*Z6**2+R12S*R2S*ZF3+R12S*ZF1*Z6)/Z3R12S
1-8.*R12S*ZF3*Z6**2/Z3R12C
 CCC=2.*Z9/R12S-4.*Z7**2*Z9/R12S**2-2.*(R1X*(-2.*R1X*R2Y)*Z2
1+R2Y*Z3+ZF4+Z7*(-2.*R1X*R2Y)+Z9*R2S)/Z3R12
```

```
     1+2.*(2.*R1X*R2Y*Z6*Z3+4.*Z7*ZF4*Z6+2.*Z9*Z6**2+R12S*R2S*ZF4
     1-2.*R12S*R1X*R2Y*Z6)/Z3R12S-8.*R12S*ZF4*Z6**2/Z3R12C
      DDD=R2S+2.*Z9**2/R12S-(4.*Z9*Z8+R12S*R2S)/Z3R12
     1+2.*R12S*Z8**2/Z3R12S+Z5
      EEE=2.*Z7/R12S-4.*Z7*Z9**2/R12S**2-2.*(R1Y**2*(-2.)*R2X*Z2
     1+R2X*Z3+ZF3-Z9*2.*R1Y*R2X+Z7*R2S)/Z3R12
     1+2.*(2.*R1Y*R2X*Z8*Z3+4.*Z9*ZF3*Z8+2.*Z7*Z8**2
     1+R12S*R2S*ZF3-R12S*2.*R1Y*R2X*Z8)/Z3R12S-8.*R12S*ZF3*Z8**2/Z3R12C
      FFF=6.*Z9/R12S-4.*Z9**3/R12S**2
     1-2.*(R1Y*ZF2*Z2+R2Y*Z3+ZF4+Z9*ZF2+2.*Z8+Z9*R2S)/Z3R12
     1+2.*(-ZF2*Z8*Z3+4.*Z9*ZF4*Z8+2.*Z9*Z8**2+R12S*R2S*ZF4
     1+R12S*ZF2*Z8)/Z3R12S-8.*R12S*ZF4*Z8**2/Z3R12C
      GGG=2.*Z7*Z9/R12S-2.*(Z9*Z6+Z7*Z8)/Z3R12
     1+2.*R12S*Z6*Z8/Z3R12S
      HHH=2.*Z9/R12S-4.*Z7**2*Z9/R12S**2+(-2.*R1Y*ZF1*Z2
     1+4.*R1Y*R2X*Z7-2.*Z8)/Z3R12
     1+(-2.*Z8*ZF1*Z3+4.*Z8*Z7*ZF3+4.*Z7*Z6*Z8+4.*Z9*ZF3*Z6
     1-4.*R2X*R1Y*R12S*Z6)/Z3R12S
     1-8.*R12S*ZF3*Z6*Z8/Z3R12C
      OOO=2.*Z7/R12S-4.*Z9**2*Z7/R12S**2+(-2.*R1X*Z2*ZF2
     1+4.*R1X*R2Y*Z9-2.*Z6)/Z3R12
     1+(-2.*Z6*ZF2*Z3+4.*Z6*Z9*ZF4+4.*Z6*Z9*Z8+4.*Z7*ZF4*Z8
     1-4.*R2Y*R1X*R12S*Z8)/Z3R12S-8.*R12S*ZF4*Z6*Z8/Z3R12C
      TOTAL3=(CONST1*AA3+CONST2*AA2+CONST3*DD2+CONST4*DD3)*S2
      RM(I,J)=TOTAL3*COEF2
      TOTAL4=((CONST1*BB3+CONST2*BB2+CONST3*EE2+CONST4*EE3)*ANX2
     1+(CONST1*CC3+CONST2*CC2+CONST3*FF2+CONST4*FF3)*ANY2)*S2
      RM(I,J+NML)=TOTAL4*COEF2
      TOTAL5=(CONST5*AAA+CONST6*GGG+CONST7*DDD)*S2
      RM(I+NML,J)=TOTAL5*COEF2
      TOTAL6=((CONST5*BBB+CONST6*HHH+CONST7*EEE)*ANX2
     1+(CONST5*CCC+CONST6*OOO+CONST7*FFF)*ANY2)*S2
      RM(I+NML,J+NML)=TOTAL6*COEF2
    7     CONTINUE
    8     CONTINUE
      DO 1015 I=1,NML2
 1015 RLX(I)=RL(I)
      IF(NCHECK.NE.0)GOTO 1100
C
C
C ***********************************************************************
C          SO FAR WE HAVE FORMED BOTH RM MATRIX AND RL VECTOR.          *
C          AN ISML SUBROUTINE LEQT1F IS THEN CALLED TO SLOVE THIS SET OF *
C          SIMULTANEOUS EQUATIONS FOR VALUES OF FICTITIOUS FORCES      *
C          AND BENDING MOMENTS ALONG THE THE PATH OF INTEGRATION.       *
C ***********************************************************************
C
C
C
      CALL LEQT1F(RM,1,NML2,NML2,RL,0,WKAREA,IER)
      DO 29 I=1,NML2
   29 PS(I)=RL(I)
      WRITE (6,400) (I,PS(I),PS(I+NML),I=1,NML)
      NCHECK=1
      GO TO 1019
 1100 CONTINUE
      WRITE(6,1014)
      DO 1101 I=1,NML2
      SUM=0.
      DO 1102 J=1,NML2
      SUM=SUM+RM(I,J)*PS(J)
 1102 CONTINUE
 1101 RB(I)=SUM-RLX(I)
      WRITE(6,1103)(I,RB(I),RB(I+NML),I=1,NML)
      DO 83 I=1,NML2
   83 PS(I)=-PS(I)/RADIUS
C
C ***********************************************************************
C          FROM HERE DOWN, DISPLACEMENTS AND BENDING MOMENTS AT ALL THE  *
C          FIELD POINTS WILL BE COMPUTED.                               *
C ***********************************************************************
C
      RSQD=RADIUS**2/DPLATE
      RQD=-RSQD*RADIUS
      COEF1=1./(16.*PI)
      COEF2=COEF1*2.
         DO 9 I=1,NFP
         W(I)=0.0
         BMX(I)=0.0
         BMY(I)=0.0
    9    CONTINUE
         PLOAD=0.
```

```
      DO 14 I=1,NFF
      R1X=XF(I)
      R1Y=YF(I)
      R1S=R1X**2+R1Y**2
      Z1=1.0-R1S
      DO 13 J=1,NIF
      R2X=XI(J)
      R2Y=YI(J)
      R2S=R2X**2+R2Y**2
      Z2=1.0-R2S
      R12S=(R1X-R2X)**2+(R1Y-R2Y)**2
      Z3=Z1*Z2
      Z4=Z3+R12S
      Z5=R12S/Z4
      Z6=R1X-R2X
      Z7=R1Y-R2Y
      Z8=R2X-R1X*R2S
      Z9=R2Y-R1Y*R2S
      Z10=(4.0*Z6*Z8-R12S*R2S)/Z4
      Z11=(4.0*Z7*Z9-R12S*R2S)/Z4
      Z12=(2.0*R12S*(Z8**2))/(Z4**2)
      Z13=(2.0*R12S*(Z9**2))/(Z4**2)
   IF(R12S.LE.1.E-12)GOTO 19
      Z14=(2.0*(Z6**2))/R12S
      Z15=(2.0*(Z7**2))/R12S
      Z16=ALOG(Z5)
      GO TO 20
19    Z16=0.0
C
C     IF SOME FIELD POINTS ARE ASSIGNED AT THE LOAD POINTS BY        *
C     MISTAKE, BOTH BMX AND BMY WILL BE AUTOMATICALLY ASSIGNED       *
C     TO BE 10000000000.                                             *
C
      BMX(I)=10000000000.
      BMY(I)=10000000000.
20    Z17=(Z3+R12S*Z16)*Q(J)
      Z18=(R2S+Z14+Z10+Z12+Z16)*Q(J)
      Z19=(R2S+Z15+Z11+Z13+Z16)*Q(J)
      IF (L(J).GT.1) GO TO 10
      Z20=COEF1*DPSI*DETA/4.0
      Z21=COEF2*DPSI*DETA/4.0
      GO TO 12
10    IF (L(J).GT.2) GO TO 11
      Z20=COEF1*DPSI*DETA/2.0
      Z21=COEF2*DPSI*DETA/2.0
      GO TO 12
11    Z20=COEF1*DPSI*DETA
      Z21=COEF2*DPSI*DETA
12    W(I)=W(I)+Z20*Z17*FSQD
      BMX(I)=BMX(I)+(-Z21*Z18-PR*Z21*Z19)
      BMY(I)=BMY(I)+(-PR*Z21*Z18-Z21*Z19)
   IF(I.GE.2)GOTO 13
   FLOAD=FLOAD+Q(J)*Z20/COEF1
13    CONTINUE
14    CONTINUE
      DO 16 I=1,NFP
      R1X=XF(I)
      R1Y=YF(I)
      R1S=R1X**2+R1Y**2
      Z1=1.0-R1S
      DO 15 J=1,NML
      R2X=(XXB(J+1)+XXB(J))/2.0
      R2Y=(YYB(J+1)+YYB(J))/2.0
      R2S=R2X**2+R2Y**2
      Z2=1.0-R2S
      S2=((XXB(J+1)-XXB(J))**2+(YYB(J+1)-YYB(J))**2)**0.5
      ANX2=(YYB(J)-YYB(J+1))/S2
      ANY2=(XXB(J+1)-XXB(J))/S2
      Z3=Z1*Z2
      R12S=(R1X-R2X)**2+(R1Y-R2Y)**2
      Z4=Z3+R12S
      Z5=R12S/Z4
      Z6=R1X-R2X
      Z7=R1Y-R2Y
      Z8=R2X-R1X*R2S
      Z9=R2Y-R1Y*R2S
      Z10=(4.0*Z6*Z8-R12S*R2S)/Z4
      Z11=(4.0*Z7*Z9-R12S*R2S)/Z4
      Z12=(2.0*R12S*(Z8**2))/(Z4**2)
      Z13=(2.0*R12S*(Z9**2))/(Z4**2)
      Z14=(2.0*(Z6**2))/R12S
```

```
      Z15=(2.0*(Z7**2))/R12S
      Z16=ALOG(Z5)
      Z17=R1X-R2X*R1S
      Z18=R1Y-R2Y*R1S
      Z19=1.0-2.0*R1X*R2X
      Z20=-2.0*R1X*R2Y
      Z21=-2.0*R1Y*R2X
      Z22=1.0-2.0*R1Y*R2Y
      Z23=6.0*Z6/R12S
      Z24=2.0*Z7/R12S
      Z25=2.0*Z6/R12S
      Z26=6.0*Z7/R12S
      Z27=-4.0*(Z6**3)/(R12S**2)
      Z28=-4.0*(Z6**2)*Z7/(R12S**2)
      Z29=-4.0*Z6*(Z7**2)/(R12S**2)
      Z30=-4.0*(Z7**3)/(R12S**2)
      Z31=-2.0*(R1X*Z19*Z2+R2X*Z3+Z17+Z6*Z19-2.0*Z8+Z6*R2S)/Z4
      Z32=-2.0*(R1X*Z20*Z2+R2Y*Z3+Z18+Z6*Z20+Z7*R2S)/Z4
      Z33=-2.0*(R1Y*Z21*Z2+R2X*Z3+Z17+Z7*Z21+Z6*R2S)/Z4
      Z34=-2.0*(R1Y*Z22*Z2+R2Y*Z3+Z18+Z7*Z22-2.0*Z9+Z7*R2S)/Z4
      Z35=2.0*(Z19*Z8*Z3-4.0*Z6*Z17*Z8+2.0*Z6*(Z8**2)+R12S*R2S*Z17-
     1 R12S*Z19*Z8)/(Z4**2)
      Z36=2.0*(Z20*Z8*Z3-4.0*Z6*Z18*Z8+2.0*Z7*(Z8**2)+R12S*R2S*Z18-
     1 R12S*Z20*Z8)/(Z4**2)
      Z37=2.0*(Z21*Z9*Z3-4.0*Z7*Z17*Z9+2.0*Z6*(Z9**2)+R12S*R2S*Z17-
     1 R12S*Z21*Z9)/(Z4**2)
      Z38=2.0*(Z22*Z9*Z3-4.0*Z7*Z18*Z9+2.0*Z7*(Z9**2)+R12S*R2S*Z18-
     1 R12S*Z22*Z9)/(Z4**2)
      Z39=-8.0*(R12S*Z17*(Z8**2))/(Z4**3)
      Z40=-8.0*(R12S*Z18*(Z8**2))/(Z4**3)
      Z41=-8.0*(R12S*Z17*(Z9**2))/(Z4**3)
      Z42=-8.0*(R12S*Z18*(Z9**2))/(Z4**3)
      Z43=COEF1*(Z3+R12S*Z16)*S2
      Z44=COEF2*(ANX2*(Z17-Z17*Z5+Z6*Z16)+ANY2*(Z18-Z18*Z5+Z7*Z16))*
     1 S2
      W(I)=W(I)+(Z43*PS(J)+Z44*PS(J+NML))*ROD
      Z45=COEF2*(R2S+Z14+Z10+Z12+Z16)*S2
      Z46=COEF2*(R2S+Z15+Z11+Z13+Z16)*S2
      Z47=COEF2*(ANX2*(Z23+Z27+Z31+Z35+Z39)+ANY2*(Z24+Z28+Z32+Z36+
     1 Z40))*S2
      Z48=COEF2*(ANX2*(Z25+Z29+Z33+Z37+Z41)+ANY2*(Z26+Z30+Z34+Z38+
     1 Z42))*S2
      BMX(I)=BMX(I)+(-Z45*PS(J)-Z47*PS(J+NML)-PR*Z46*PS(J)-PR*Z48*
     1 PS(J+NML))*(-RADIUS)
      BMY(I)=BMY(I)+(-PR*Z45*PS(J)-PR*Z47*PS(J+NML)-Z46*PS(J)-Z48*
     1 PS(J+NML))*(-RADIUS)
   15 CONTINUE
   16 CONTINUE
      DO 84 I=1,NFP
      XF(I)=XF(I)*RADIUS
   84 YF(I)=YF(I)*RADIUS
      WRITE(6,600)(I,XF(I),YF(I),W(I),BMX(I),BMY(I),I=1,NFP)
      WRITE(6,900)PLOAD
C     PLOAD IS THE SUM OF THE DISTRIBUTED LATERAL LOADS APPLIED.
      STOP
      END
60,220,128,0.3,1.0,1.0,30.E6,80.,0.4
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,20*-5.,
-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,20*5.,
10*-10.,-9.5,-8.5,-7.5,-6.5,-5.5,-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.
5.5,6.5,7.5,8.5,9.5,10*10.,9.5,8.5,7.5,6.5,5.5,4.5,3.5,2.5,
1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,-5.5,-6.5,-7.5,-8.5,-9.5
60*1
10*0.,20*-1.,10*0.,20*1.
10*-1.,20*0.,10*1.,20*0.
10.,8.,6.,4.,2.,0.,-2.,-4.,-6.,-8.,-10.
20*-10.,-8.,-6.,-4.,-2.,0.,2.,4.,6.,8.,10.,19*10.
11*-15.,-13.5,-12.,-10.5,-9.,-7.5,-6.,-4.5,-3.,-1.5,0.,1.5,3.,4.5,6.,7.5
10.5,12.,13.5,15.,10*15.,13.5,12.,10.5,9.,7.5,6.,4.5,3.,1.5,0.,
-1.5,-3.,-4.5,-6.,-7.5,-9.,-10.5,-12.,-13.5
4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,
4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,
4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,
4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,
4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,1.5,0.5,4.5,3.5,2.5,
4.5,3.5,2.5,1.5,0.5,23*0.
5*-10.,5*-9.,5*-8.,5*-7.,5*-6.,5*-5.,5*-4.,5*-3.,5*-2.,5*-1.,5*0.,
5*1.,5*2.,5*3.,5*4.,5*5.,5*6.,5*7.,5*8.,5*9.,5*10.,
-9.5,-8.5,-7.5,-6.5,-5.5,-4.5,-3.5,-2.5,-1.5,-0.5
0.5,1.5,2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5,-10.,0.,10.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
```

```
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
10*9.5,10*8.5,10*7.5,10*6.5,10*5.5,10*4.5,10*3.5,10*2.5,10*1.5,10*0.5,
10*-0.5,10*-1.5,10*-2.5,10*-3.5,10*-4.5,10*-5.5,10*-6.5,10*-7.5,10*-8.5,
20*4
20*1.
```

APPENDIX C

COMPUTER PROGRAM FOR THE POINT-FORCE METHOD

APPENDIX C

COMPUTER PROGRAM FOR THE POINT-FORCE METHOD

```
      PROGRAM NEWPLCL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C ***********************************************************************
C
C POINT FORCE METHOD FOR ISOTROPIC PLATE BENDING PROBLEMS *****
C
C ARBITRARY PLAN FORM, TRANSVERSE LOAD, AND BOUNDARY CONDITIONS
C
C
C REQUIRED INPUT VALUES ---
C
C        NBP       =NUMBER OF BOUNDARY POINTS
C        NIP       =NUMBER OF INTERNAL LOAD POINTS
C        NFP       =NUMBER OF FIELD POINTS
C        XB,YB     =POINTS ON B AT WHICH B.C. ARE SATISFIED.
C        BANX,BANY =COMPONENTS OF UNIT NORMAL TO B AT XB,YB.
C        XXB,YYB   =END POINTS OF MESHES AROUND B WHERE FICTITIOUS
C                   FORCES ARE ASSIGNED.
C        XF,YF     =FIELD POINTS
C        XI,YI     =INTERNAL LOAD POINTS
C        PR        =POISSON*S RATIO
C        EVALUE    =YOUNG*S MODULUS
C        HVALUE    =PLATE THICKNESS
C        RADIUS    =RADIUS OF THE FICTITIOUS CIRCULAR PLATE OF WHICH
C                   THE DISPLACEMENT AT THE CIRCUMFERENTIAL BOUNDARY
C                   IS SET TO ZERO.
C        NBTYPE    =BOUNDARY CONDITION TYPE AT EACH BOUNDARY POINTS
C                   NBTYPE = 1 --- CLAMPED
C                   NBTYPE = 2 --- SIMPLY SUPPORTED
C                   NBTYPE = 3 --- FREE
C
C ***********************************************************************
C
      DIMENSION XB(40),YB(40),XXB(81),YYB(81),NBTYPE(40)
      DIMENSION XI(100),YI(100),DEL(2)
      DIMENSION RB(80),RM(80,80),PS(80),WKAREA(80),RL(80)
      DIMENSION XF(181),YF(181),W(181),BMX(181),BMY(181)
      DIMENSION BANX(40),BANY(40)
C
  100 FORMAT(*0LOCT*,6X,*XB*,8X,*YB*,7X,*ANX*,8X,*ANY*,4X,*NBTYPE*,
     +/*0*/(I5,4F10.4,I5))
  110 FORMAT(*0LOCT*,11X,*XXB*,14X,*YYB*/*0*/(I4,8X,F9.2,8X,F9.2))
  200 FORMAT (*0LOCT*,11X,*XI*,17X,*YI*,14X,/*0*/
     +(I4,11X,F6.2,11X,F6.3))
  300 FORMAT (*0LOCT*,19X,*RLD*,34X,*RLS*/*0*/(I4,8X,E20.8,16X,
     1E20.8))
  400 FORMAT (*1LOCT*,19X,*PSP*/*0*/(I4,8X,E20.8))
  500 FORMAT(*0LOCT*,11X,*XF*,17X,*YF*/*0*/(I4,11X,F6.3,11X,F6.3))
  600 FORMAT(1H1,8X,*NODE*,12X*XF*,16X,*YF*,19X,*W*,16X,*BMX*,
     116X,*BMY*/(1H0,I10,2F20.10,3E20.12))
  700 FORMAT(1H1,*INPUT VALUES .....*,//1X,*NBP = *,I3,*  NFP =*,I3,
     +*  NIP = *,I3,*  PR = *,F5.3,
     1/1H0,*YOUNGS MODULUS =  *,E14.8,*    RADIUS OF THE PLATE =  *,
     +F7.1,*  THICKNESS OF THE PLATE =*,F6.3)
C
C INPUT VALUES .........
C
      READ(5,*)NBP,NIP,NFP,PR,EVALUE,RADIUS,HVALUE
      WRITE(6,700)NBP,NFP,NIP,PR,EVALUE,RADIUS,HVALUE
```

```
      READ(5,*)(XB(I),I=1,NBP)
      READ(5,*)(YB(I),I=1,NBP)
      READ(5,*)(NBTYPE(I),I=1,NBP)
      READ(5,*)(BANX(I),I=1,NBP)
      READ(5,*)(BANY(I),I=1,NBP)
      WRITE(6,100)(I,XB(I),YB(I),BANX(I),BANY(I),NBTYPE(I)
     1,I=1,NBP)
C
C ASSIGN LOCATIONS OF FIELD POINTS, (XF,YF).
C
      X0=5.0$Y0=5.0
      XF(1)=10.-X0$YF(1)=-10.+Y0
      DO 41 I=2,11
      XF(I)=XF(I-1)-0.5
   41 YF(I)=YF(I-1)
      DO 42 J=1,10
      K=J*17
      DO 42 I=1,11
      XF(I+K)=XF(I)
   42 YF(I+K)=YF(I+K-17)+1.0
      XF(12)=10.-X0$YF(12)=-9.5+Y0
      DO 43 I=13,17
      XF(I)=XF(I-1)-1.0
   43 YF(I)=YF(I-1)
      DO 44 J=1,9
      K=J*17
      DO 44 I=12,17
      XF(I+K)=XF(I)
   44 YF(I+K)=YF(I+K-17)+1.0
      READ(5,*)(XI(I),I=1,NIP)
      READ(5,*)(YI(I),I=1,NIP)
C
C ASSIGN LOCATIONS OF FICTITIOUS FORCES, (XXB,YYB).
C
      NBP2=NBP*2$NBP2P1=NBP2+1
      NBPP1=NBP+1
      DIST1=4.0 $ DIST2=2.0
      DEL(1)=(10.+2.*DIST1)/10.
      DEL(2)=(10.+2.*DIST1+2.*DIST2)/10.
      XXB(1)=5.+DIST1-DEL(1)$YYB(1)=-5.-DIST1
      XXB(41)=5.+DIST1+DIST2-DEL(2)$YYB(41)=-5.-DIST1-DIST2
      DO 28 J=1,2
      DELT=DEL(1)$IF(J.EQ.2)DELT=DEL(2)
      DO 25 I=2,10
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)-DELT
   25 YYB(K)=YYB(K-1)
      DO 26 I=11,20
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   26 YYB(K)=YYB(K-1)+DELT
      DO 27 I=21,30
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)+DELT
   27 YYB(K)=YYB(K-1)
      DO 28 I=31,40
      K=I
```

```
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   28 YYB(K)=YYB(K-1)-DELT
      XXB(81)=XXB(41) $ YYB(81)=YYB(41)
      WRITE(6,110)(I,XXB(I),YYB(I),I=1,NBF2P1)
      WRITE(6,500)(I,XF(I),YF(I),I=1,NFP)
      WRITE(6,200)(I,XI(I),YI(I),I=1,NIP)
      DPLATE=EVALUE*HVALUE**3/(12.*(1.-PR**2))
      PI=4.*ATAN(1.)
      COEF1=1./(16.*PI*DPLATE)
      COEF2=COEF1*2 $ COEF4=COEF2**2 $ QLOAD=1.0 $ R2=RADIUS**2
C
C SET UP THE RL VECTOR AS SHOWN IN APPENDIX A.
C
      DO 5 I=1,NBP
      RL(I)=0.0
      RL(I+NBP)=0.0
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 6 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
      Z1=R1X-R2X$Z2=R1Y-R2Y$R12S=Z1**2+Z2**2
      Z5=ALOG(R12S/R2)
      IF(NBTYPE(I).EQ.3)GOTO 18
      RL(I)=RL(I)-QLOAD*(R12S*Z5)
      IF(NBTYPE(I).EQ.1)GOTO 17
C
C FOR SIMPLY SUPPORTED EDGES ONLY ---
C
      Z1S=Z1**2 $ Z2S=Z2**2 $ ANXS=ANX**2 $ ANYS=ANY**2
      RL(I+NBP)=RL(I+NBP)-QLOAD*((2.*Z1S/R12S+1.+Z5)*ANXS
     ++4.*Z1*Z2/R12S*ANX*ANY+(2.*Z2S/R12S+1.+Z5)*ANYS)
      GOTO 6
C
C FOR CLAMPED EDGES ONLY ---
C
   17 RL(I+NBP)=RL(I+NBP)-QLOAD*(Z1*ANX+Z2*ANY)*(1.+Z5)
      GOTO 6
C
C FOR FREE EDGES ONLY ---
C
   18 Z1S=Z1**2 $ Z2S=Z2**2 $ ANXS=ANX**2 $ ANYS=ANY**2
      R12SS=R12S**2
      RL(I)=RL(I)-QLOAD*((ANXS+PR*ANYS)*(2.*Z1S/R12S+1.+Z5)
     ++(4.*(1.-PR)*ANX*ANY)*Z1*Z2/R12S
     ++(ANYS+PR*ANXS)*(2.*Z2S/R12S+1.+Z5))
      C1=ANX*(1.+ANYS*(1.-PR))
      C2=((2.*PR-1.)*ANXS+(2.-PR)*ANYS)*ANY
      C3=((2.*PR-1.)*ANYS+(2.-PR)*ANXS)*ANX
      C4=ANY*(1.+ANXS*(1.-PR))
      RL(I+NBP)=RL(I+NBP)-QLOAD*(C1*Z1*(Z1S+3.*Z2S)/R12SS
     ++C2*Z2*(Z2S-Z1S)/R12SS+C3*Z1*(Z1S-Z2S)/R12SS
     ++C4*Z2*(Z2S+Z1S*3.)/R12SS)
    6 CONTINUE
    5 CONTINUE
C
C SET UP THE RM MATRIX AS SHOWN IN APPENDIX B, AND NOTE THAT
```

```
C FICTITIOUS FORCES ARE LOCATED AT THE CENTER OF THE ASSIGNED MESHES.
C
      DO 8 I=1,NBP
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 7 J=1,NBP2
      R2X=(XXB(J+1)+XXB(J))/2.
      R2Y=(YYB(J+1)+YYB(J))/2.
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 33
      R2X=(XXB(1)+XXB(NBP))/2.
      R2Y=(YYB(1)+YYB(NBP))/2.
   33 CONTINUE
      Z1=R1X-R2X$Z2=R1Y-R2Y$R12S=Z1**2+Z2**2
      Z5=ALOG(R12S/R2)
      IF(NBTYPE(I).EQ.3)GOTO 19
      RM(I,J)=R12S*Z5
      IF(NBTYPE(I).EQ.1)GOTO 20
      Z1S=Z1**2 $ Z2S=Z2**2 $ ANXS=ANX**2 $ ANYS=ANY**2
      RM(I+NBP,J)=((2.*Z1S/R12S+1.+Z5)*ANXS
     ++4.*Z1*Z2/R12S*ANX*ANY+(2.*Z2S/R12S+1.+Z5)*ANYS)
      GOTO 7
   20 RM(I+NBP,J)=(Z1*ANX+Z2*ANY)*(1.+Z5)
      GOTO 7
   19 Z1S=Z1**2 $ Z2S=Z2**2 $ ANXS=ANX**2 $ ANYS=ANY**2
      R12SS=R12S**2
      RM(I,J)=((ANXS+PR*ANYS)*(2.*Z1S/R12S+1.+Z5)
     ++(4.*(1.-PR)*ANX*ANY)*Z1*Z2/R12S
     ++(ANYS+PR*ANXS)*(2.*Z2S/R12S+1.+Z5))
      C1=ANX*(1.+ANYS*(1.-PR))
      C2=((2.*PR-1.)*ANXS+(2.-PR)*ANYS)*ANY
      C3=((2.-PR-1.)*ANYS+(2.-PR)*ANXS)*ANX
      C4=ANY*(1.+ANXS*(1.-PR))
      RM(I+NBP,J)=(C1*Z1*(Z1S+3.*Z2S)/R12SS
     ++C2*Z2*(Z2S-Z1S)/R12SS+C3*Z1*(Z1S-Z2S)/R12SS
     ++C4*Z2*(Z2S+Z1S*3.)/R12SS)
    7 CONTINUE
    8 CONTINUE
C
C SOLVE THE SET OF LINEAR EQUATIONS TO DETERMINE THE FICTITIOUS FORCES.
C
      CALL LEQT1F(RM,1,NBP2,NBP2,RL,0,WKAREA,IER)
      DO 29 I=1,NBP2
   29 PS(I)=RL(I)
      WRITE (6,400) (I,PS(I),I=1,NBP2)
      DO 9 I=1,15
      W(I)=0.0
      BMX(I)=0.0
      BMY(I)=0.0
    9 CONTINUE
C
C COMPUTE DISPLACEMENTS AND BENDING MOMENTS AT THE PRESCRIBED FIELD POIN
C
      DO 14 I=1,15
      R1X=XF(I)
      R1Y=YF(I)
      DO 13 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
```

```
      Z1=R1X-R2X$Z2=R1Y-R2Y$R12S=Z1**2+Z2**2
      Z1S=Z1**2 $ Z2S=Z2**2
      Z5=ALOG(R12S/R2)
      W(I)=W(I)+COEF1*(R12S*Z5)*QLOAD
      Z6=1.+2.*Z1S/R12S+Z5 $ Z7=1.+2.*Z2S/R12S+Z5
      Z8=DPLATE*COEF2*QLOAD
      BMX(I)=BMX(I)-Z8*(Z6+PR*Z7) $ BMY(I)=BMY(I)-Z8*(Z7+PR*Z6)
      IF(I.GE.2)GOTO 13
   13 CONTINUE
   14 CONTINUE
      DO 16 I=1,15
      R1X=XF(I)
      R1Y=YF(I)
      DO 15 J=1,NBP2
      R2X=(XXB(J+1)+XXB(J))/2.0
      R2Y=(YYB(J+1)+YYB(J))/2.0
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 11
      R2X=(XXB(1)+XXB(NBP))/2.
      R2Y=(YYB(1)+YYB(NBP))/2.
   11 CONTINUE
      Z1=R1X-R2X$Z2=R1Y-R2Y$R12S=Z1**2+Z2**2
      Z5=ALOG(R12S/R2)
      Z1S=Z1**2 $ Z2S=Z2**2
      W(I)=W(I)+FS(J)*COEF1*(R12S*Z5)
      Z6=1.+2.*Z1S/R12S+Z5 $ Z8=1.+2.*Z2S/R12S+Z5
      Z11=PS(J)*COEF2*Z6 $ Z12=PS(J)*COEF2*Z8
      BMX(I)=BMX(I)-DPLATE*(Z11+PR*Z12)
      BMY(I)=BMY(I)-DPLATE*(Z12+PR*Z11)
   15 CONTINUE
   16 CONTINUE
      WRITE(6,600)(I,XF(I),YF(I),W(I),BMX(I),BMY(I),I=1,15)
       END
```

```
40,100,181,0.3,30.E6,80.,0.4
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,10*-5.,
-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,10*5.
10*-5.,-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,
10*5.,4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5
40*2
10*0.,10*-1.,10*0.,10*1.
10*-1.,10*0.,10*1.,10*0.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
10*-4.5,10*-3.5,10*-2.5,10*-1.5,10*-0.5,
10*0.5,10*1.5,10*2.5,10*3.5,10*4.5
```

APPENDIX D

DERIVATIVES OF THE GREEN'S FUNCTION OF AN INFINITE ORTHOTROPIC PLATE

APPENDIX D

DERIVATIVES OF THE GREEN'S FUNCTION OF AN INFINITE ORTHOTROPIC PLATE

For $\varepsilon > 1.0$:

$$\frac{\partial G}{\partial x} = \frac{1}{4\pi D_o(\beta^2-\lambda^2)} \{ (x-\xi) \; [\beta \ln \frac{(x-\xi)^2+\lambda^2(y-\eta)^2}{a^2} - \lambda \ln \frac{(x-\xi)^2+\beta^2(y-\eta)^2}{a^2}$$

$$+ \frac{\beta\{(x-\xi)^2+\lambda^2(y-\eta)^2\}}{(x-\xi)^2+\lambda^2(y-\eta)^2} - \frac{\lambda\{(x-\xi)^2+\beta^2(y-\eta)^2\}}{(x-\xi)^2+\beta^2(y-\eta)^2}] + 2\beta\lambda(y-\eta)[\text{ arc tg } \frac{\beta(y-\eta)}{x-\xi}$$

$$- \text{ arc tg } \frac{\lambda(y-\eta)}{x-\xi}] - 3(x-\xi)(\beta-\lambda) \}$$

$$\frac{\partial G}{\partial y} = \frac{\varepsilon^2}{4\pi D_o(\beta^2-\lambda^2)} \{ (y-\eta)[\beta \ln \frac{(x-\xi)^2+\beta^2(y-\eta)^2}{a^2} - \lambda \ln \frac{(x-\xi)^2+\lambda^2(y-\eta)^2}{a^2}$$

$$- \frac{\lambda\{(x-\xi)^2+\lambda^2(y-\eta)^2\}}{(x-\xi)^2+\lambda^2(y-\eta)^2} + \frac{\beta\{(x-\xi)^2+\beta^2(y-\eta)^2\}}{(x-\xi)^2+\beta^2(y-\eta)}] + 2(x-\xi)[\text{arc tg} \frac{\beta(y-\eta)}{x-\xi}$$

$$- \text{ arc tg } \frac{\lambda(y-\eta)}{x-\xi}] - 3(y-\eta)(\beta-\lambda) \}$$

$$\frac{\partial^3 G}{\partial x^3} = \frac{x-\xi}{2\pi D_o(\beta^2-\lambda^2)}\left[\frac{\beta}{(x-\xi)^2+\lambda^2(y-\eta)^2} - \frac{\lambda}{(x-\xi)^2+\beta^2(y-\eta)^2}\right]$$

$$\frac{\partial^3 G}{\partial y^3} = \frac{-\varepsilon^2(y-\eta)}{2\pi D_o(\beta^2-\lambda^2)}\left[\frac{\lambda^3}{(x-\xi)^2+\lambda^2(y-\eta)^2} - \frac{\beta^3}{(x-\xi)^2+\beta^2(y-\eta)^2}\right]$$

$$\frac{\partial^3 G}{\partial x^2 \partial y} = \frac{-\varepsilon^2(y-\eta)}{2\pi D_o(\beta^2-\lambda^2)}\left[\frac{\beta}{(x-\xi)^2+\beta^2(y-\eta)^2} - \frac{\lambda}{(x-\xi)^2+\lambda^2(y-\eta)^2}\right]$$

$$\frac{\partial^3 G}{\partial x \partial y^2} = \frac{-\varepsilon^2(x-\xi)}{2\pi D_o(\beta^2-\lambda^2)}\left[\frac{\lambda}{(x-\xi)^2+\lambda^2(y-\eta)^2} - \frac{\beta}{(x-\eta)^2+\beta^2(y-\eta)}\right]$$

For $\rho < 1.0$:

$$\frac{\partial G}{\partial x} = \frac{1}{16\pi D_o}\Big\{\frac{x-\xi}{\mu_1}\Big[\ln \frac{(x-\xi)^4+2\rho\varepsilon^2(x-\xi)^2(y-\eta)^2+\varepsilon^4(y-\eta)^4}{a^4} - 6\Big]$$

$$+ \frac{2\,[(x-\xi)^2+\varepsilon^2(y-\eta)^2]\,[(x-\xi)^3+\rho\varepsilon^2(x-\xi)(y-\eta)^2]}{\mu_1\,[(x-\xi)^4+2\rho\varepsilon^2(x-\xi)^2(y-\eta)^2+\varepsilon^4(y-\eta)^4]} - \frac{2(x-\xi)}{\mu_2}\,\mathrm{arc\ tg}\,\frac{2\mu_1\mu_2(y-\eta)^2}{(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2}$$

$$+ \frac{(x-\xi)^2-\varepsilon^2(y-\eta)^2}{\mu_2}\cdot\frac{4\mu_1\mu_2(x-\xi)(y-\eta)^2}{[(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2]^2+[2\mu_1\mu_2(y-\eta)^2]^2}$$

$$- \frac{\varepsilon^2(y-\eta)}{\mu_1\mu_2}\ \ln\frac{\mu_1^{\,2}(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]^2}{\mu_1^{\,2}(y-\eta)^2+[(x-\xi)+\mu_2(y-\eta)]^2} -$$

$$- \frac{\varepsilon^2 (x-\xi)(y-\eta)}{\mu_1 \mu_2} \left[\frac{2\{(x-\xi)-\mu_2(y-\eta)\}}{\mu_1{}^2 (y-\eta)^2 + \{(x-\xi)-\mu_2(y-\eta)\}^2} - \frac{2\{(x-\xi)+\mu_2(y-\eta)\}}{\mu_1{}^2(y-\eta)^2+\{(x-\xi)+\mu_2(y-\eta)\}^2} \right]\}$$

$$\frac{\partial G}{\partial y} = \frac{1}{16\pi D_o} \{ \frac{\varepsilon^2 (y-\eta)}{\mu_1} [\ell n \frac{(x-\xi)^4 + 2\rho\varepsilon^2 (x-\xi)^2 (y-\eta)^2 + \varepsilon^4 (y-\eta)^4}{a^4} - 6]$$

$$+ \frac{2\{(x-\xi)^2 + \varepsilon^2 (y-\eta)^2\}\{\rho\varepsilon^2 (x-\xi)^2 (y-\eta) + \varepsilon^4 (y-\eta)^3\}}{\mu_1 \{(x-\xi)^4 + 2\rho\varepsilon^2 (x-\xi)^2 (y-\eta)^2 + \varepsilon^4 (y-\eta)^4\}} + \frac{2\varepsilon^2 (y-\eta)}{\mu_2} \cdot$$

$$\cdot \text{arc tg} \frac{2\mu_1\mu_2 (y-\eta)^2}{(x-\xi)^2 + \rho\varepsilon^2 (y-\eta)^2} - \frac{(x-\xi)^2 - \varepsilon^2 (y-\eta)^2}{\mu_2} \cdot \frac{4\mu_1\mu_2 (y-\eta)(x-\xi)^2}{[(x-\xi)^2 + \rho\varepsilon^2 (y-\eta)^2]^2 + [2\mu_1\mu_2 (y-\eta)^2]^2}$$

$$- \frac{\varepsilon^2 (x-\xi)}{\mu_1 \mu_2} \ell n \frac{\mu_1{}^2 (y-\eta)^2 + [(x-\xi)-\mu_2 (y-\eta)]^2}{\mu_1{}^2 (y-\eta)^2 + [(x-\xi)+\mu_2 (y-\eta)]^2} - \frac{\varepsilon^2 (x-\xi)(y-\eta)}{\mu_1 \mu_2} \cdot$$

$$\cdot \left[\frac{2\mu_1{}^2 (y-\eta) - 2\mu_2 \{(x-\xi)-\mu_2 (y-\eta)\}}{\mu_1{}^2 (y-\eta)^2 + \{(x-\xi)-\mu_2 (y-\eta)\}^2} - \frac{2\mu_1{}^2 (y-\eta) + 2\mu_2 \{(x-\xi)+\mu_2 (y-\eta)\}}{\mu_1{}^2 (y-\eta)^2 + \{(x-\xi)+\mu_2 (y-\eta)\}^2} \right]\}$$

$$\frac{\partial^3 G}{\partial x^3} = \frac{1}{4\pi D_o} \{ \frac{1}{\mu_1} \frac{(x-\xi)^3 + \rho\varepsilon^2 (x-\xi)(y-\eta)^2}{(x-\xi)^4 + 2\rho\varepsilon^2 (x-\xi)^2 (y-\eta)^2 + \varepsilon^4 (y-\eta)^4}$$

$$+ \frac{2\mu_1 (x-\xi)(y-\eta)^2}{[(x-\xi)^2 + \rho\varepsilon^2 (y-\eta)^2]^2 + [2\mu_1\mu_2 (y-\eta)^2]^2} \}$$

$$\frac{\partial^3 G}{\partial y^3} = \frac{1}{4\pi D_o} \{ \frac{1}{\mu_1} \frac{\rho\varepsilon^2 (x-\xi)^2 (y-\eta) + \varepsilon^4 (y-\eta)^3}{(x-\xi)^4 + 2\rho\varepsilon^2 (x-\xi)^2 (y-\eta)^2 + \varepsilon^4 (y-\eta)^4} +$$

$$+ \frac{2\mu_1(y-\eta)(x-\xi)^2}{[(x-\xi)^2+\rho\varepsilon^2(y-\eta)^2]^2+[2\mu_1\mu_2(y-\eta)^2]^2}\}$$

$$\frac{\partial^3 G}{\partial x^2 \partial y} = \frac{-\varepsilon^2}{8\pi D_o \mu_1 \mu_2}\{\frac{(x-\xi)-\mu_2(y-\eta)}{\mu_1{}^2(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]^2}$$

$$- \frac{(x-\xi)+\mu_2(y-\eta)}{\mu_1{}^2(y-\eta)^2+[(x-\xi)+\mu_2(y-\eta)]^2}\}$$

$$\frac{\partial^3 G}{\partial x \partial y^2} = \frac{-\varepsilon^2}{8\pi D_o \mu_1 \mu_2}\{\frac{\varepsilon^2(y-\eta)-\mu_2(x-\xi)}{\mu_1{}^2(y-\eta)^2+[(x-\xi)-\mu_2(y-\eta)]}$$

$$- \frac{\varepsilon^2(y-\eta)+\mu_2(x-\xi)}{\mu_1{}^2(y-\eta)^2+[(x-\xi)+\mu_2(y-\eta)]}\}$$

For $\rho = 1.0$:

$$\frac{\partial G}{\partial x} = \frac{(x-\xi)}{8\pi\varepsilon D_o}\{\ell n \frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{a^2} - 2\}$$

$$\frac{\partial G}{\partial y} = \frac{\varepsilon(y-\eta)}{8\pi D_o}\ \ell n\ \frac{(x-\xi)^2+\varepsilon^2(y-\eta)^2}{a^2}$$

$$\frac{\partial^3 G}{\partial x^3} = \frac{1}{4\pi\varepsilon D_o}\ \frac{(x-\xi)^3-\varepsilon^2(y-\eta)(x-\xi)}{\{(x-\xi)^2+\varepsilon^2(y-\eta)\}^2}$$

$$\frac{\partial^3 G}{\partial y^3} = \frac{\varepsilon(y-\eta)}{4\pi D_o} \frac{3(x-\xi)^2+\varepsilon^2(y-\eta)^2}{(x-\xi)^2+\varepsilon^2(y-\eta)^2}$$

$$\frac{\partial^3 G}{\partial x^2 \partial y} = \frac{1}{4\pi D_o} \frac{\{\varepsilon^2(y-\eta)^2-(x-\xi)^2\}\varepsilon(y-\eta)^2}{\{(x-\xi)^2+\varepsilon^2(y-\eta)^2\}^2}$$

$$\frac{\partial^3 G}{\partial x \partial y^2} = \frac{1}{4\pi D_o} \frac{(x-\xi)^2-\varepsilon^2(y-\eta)^2\varepsilon(x-\xi)}{\{(x-\xi)^2+\varepsilon^2(y-\eta)^2\}^2}$$

APPENDIX E

COMPUTER PROGRAM FOR AN ORTHOTROPIC PROBLEM

APPENDIX E

COMPUTER PROGRAM FOR AN ORTHOTROPIC PROBLEM

```
      PROGRAM ORTPLCL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C *********************************************************************
C
C POINT FORCE METHOD FOR ORTHOTROPIC PLATE BENDING PROBLEMS,
C ARBITRARY PLAN FORM, TRANSVERSE LOAD, AND BOUNDARY CONDITIONS ***
C
C SHOWN HERE IS AN EXAMPLE FOR A SIMPLY SUPPORTED SQUARE PLATE.
C
C REQUIRED INPUT VALUES ---
C
C        NBP        =NUMBER OF BOUNDARY POINTS
C        NIP        =NUMBER OF INTERNAL LOAD POINTS
C        NFP        =NUMBER OF FIELD POINTS
C        XB,YB      =POINTS ON B AT WHICH B.C. ARE SATISFIED.
C        BANX,BANY  =COMPONENTS OF UNIT NORMAL TO B AT XB,YB.
C        XXB,YYB    =END POINTS OF MESHES AROUND B WHERE FICTITIOUS
C                    FORCES ARE ASSIGNED.
C        XF,YF      =FIELD POINTS
C        XI,YI      =INTERNAL LOAD POINTS
C        VX         =POISSON≠S RATIO IN X DIRECTION,
C                    DUE TO STRESS IN Y DIRECTION
C        EX,EY      =YOUNG≠S MODULI IN X AND Y DIRECTIONS,
C                    RESPECTIVELY
C        HVALUE     =PLATE THICKNESS
C        RADIUS     =RADIUS OF THE FICTITIOUS CIRCULAR PLATE OF WHICH
C                    THE DISPLACEMENT AT THE CIRCUMFERENTIAL BOUNDARY
C                    IS SET TO ZERO.
C
C *********************************************************************
C
      DIMENSION XB(40),YB(40),XXB(81),YYB(81),NBTYPE(81)
      DIMENSION XI(100),YI(100),DEL(2)
      DIMENSION RLX(80),RB(80),RM(80,80),PS(80),WKAREA(80),RL(80)
      DIMENSION XF(181),YF(181),W(181),BMX(181),BMY(181)
      DIMENSION BANX(40),BANY(40),RMX(80,80)
      REAL LUMDA,LUMDA2,MU1,MU2
C
  100 FORMAT(≠0LOCT≠,6X,≠XB≠,8X,≠YB≠,7X,≠ANX≠,8X,≠ANY≠,4X,≠NBTYPE≠,
     +/≠0≠/(I5,4F10.4,I5))
  110 FORMAT(≠0LOCT≠,11X,≠XXB≠,14X,≠YYB≠/≠0≠/(I4,8X,F9.2,8X,F9.2))
  200 FORMAT (≠0LOCT≠,11X,≠XI≠,17X,≠YI≠,14X,/≠0≠/
     +(I4,11X,F6.2,11X,F6.3))
  400 FORMAT (≠0LOCT≠,19X,≠PSP≠/≠0≠/(I4,8X,E20.8))
  500 FORMAT(≠0LOCT≠,11X,≠XF≠,17X,≠YF≠/≠0≠/(I4,11X,F6.3,11X,F6.3))
  600 FORMAT(1H1,8X,≠NODE≠,12X≠XF≠,16X,≠YF≠,19X,≠W≠,16X,≠BMX≠,
     116X,≠BMY≠/(1H0,I10,2F20.10,3E20.12))
  700 FORMAT(1H1,≠INPUT VALUES .....≠,//1X,≠NML = ≠,I3,≠  NIP =≠,I3,
     1≠  NFP = ≠,I3,≠    PR = ≠,F5.3,≠  DPSI = ≠,F5.3,≠  DETA = ≠,F5.3,
     +/1H0,≠ EX = ≠,E10.4,≠ EY = ≠,E10.4,≠ VX = ≠,F5.2,≠ GXY = ≠,E10.4
     +/1H0,≠ RADIUS OF THE PLATE = ≠,
     +F7.1,≠  THICKNESS OF THE PLATE =≠,F6.3,≠   DIST= ≠,F5.1)
 1103 FORMAT(I5,5X,E20.12,5X,E20.12)
 1014 FORMAT(1H1,≠THE FOLLOWING IS A LIST OF DOUBLE CHECKING OF B.C.S≠,
     1//1X,≠NOTE .... IF ALL THE VALUES LISTED BELOW ARE IN THE ORDER OF
     1 E-9 OR LESS, RESULTS WILL BE IN GOOD SHAPE.≠,
     1//,4X≠NML≠10X,≠B.C.  1≠,13X,≠B.C.  2≠,//)
  701 FORMAT(1H1,≠DX = ≠,E15.6,≠ DY = ≠,E15.6,≠ H = ≠,E15.6,≠ G = ≠,
     +E15.6,≠ VY = ≠,F5.2,≠ RHO = ≠,F6.2)
```

```
  702 FORMAT(1H0,*EPSLON = *,E15.7,*  BETA = *,E15.7,
     +*    LUMDA = *,E15.7)
C
C INPUT VALUES .........
C
      READ(5,*)NML,NIP,NFP,PR,DPSI,DETA,EX,EY,VX,GXY,RADIUS,HVALUE,DIST
      WRITE(6,700)NML,NIP,NFP,PR,DPSI,DETA,EX,EY,VX,GXY,
     +RADIUS,HVALUE,DIST
      READ(5,*)(XB(I),I=1,NML)
      READ(5,*)(YB(I),I=1,NML)
      READ(5,*)(NBTYPE(I),I=1,NML)
      READ(5,*)(BANX(I),I=1,NML)
      READ(5,*)(BANY(I),I=1,NML)
      READ(5,*)(XI(I),I=1,NIP)
      READ(5,*)(YI(I),I=1,NIP)
      WRITE(6,100)(I,XB(I),YB(I),BANX(I),BANY(I),NBTYPE(I)
     1,I=1,NML)
C
C
      X0=5.0$Y0=5.0
      XF(1)=10.-X0$YF(1)=-10.+Y0
      DO 41 I=2,11
      XF(I)=XF(I-1)-0.5
41    YF(I)=YF(I-1)
      DO 42 J=1,10
      K=J*17
      DO 42 I=1,11
      XF(I+K)=XF(I)
42    YF(I+K)=YF(I+K-17)+1.0
      XF(12)=10.-X0$YF(12)=-9.5+Y0
      DO 43 I=13,17
      XF(I)=XF(I-1)-1.0
43    YF(I)=YF(I-1)
      DO 44 J=1,9
      K=J*17
      DO 44 I=12,17
      XF(I+K)=XF(I)
44    YF(I+K)=YF(I+K-17)+1.0
      NML2=NML*2$NML2P1=NML2+1
      NMLP1=NML+1
      NOPTION=1
      DIST1=4.0 $ DIST2=2.0
      DEL(1)=(10.+2.*DIST1)/10.
      DEL(2)=(10.+2.*DIST1+2.*DIST2)/10.
      XXB(1)=5.+DIST1-DEL(1)$YYB(1)=-5.-DIST1
      XXB(41)=5.+DIST1+DIST2-DEL(2)$YYB(41)=-5.-DIST1-DIST2
      DO 28 J=1,2
      DELT=DEL(1)$IF(J.EQ.2)DELT=DEL(2)
      DO 25 I=2,10
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)-DELT
 25   YYB(K)=YYB(K-1)
      DO 26 I=11,20
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
 26   YYB(K)=YYB(K-1)+DELT
      DO 27 I=21,30
      K=I
```

```
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)+DELT
   27 YYB(K)=YYB(K-1)
      DO 28 I=31,40
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   28 YYB(K)=YYB(K-1)-DELT
      XXB(81)=XXB(41) $ YYB(81)=YYB(41)
      PI=4.*ATAN(1.)  $ Q=1.0
      VY=EY*VX/EX $ DX=EX*HVALUE**3/(12.*(1.-VX*VY))
      DY=EY*HVALUE**3/(12.*(1.-VX*VY)) $ D0=SQRT(DX*DY)
      NCOUNT=1
   38 CONTINUE
      DK=GXY*HVALUE**3/12.
      H=DX*VY+2.*DK $ RHO=H/D0
      WRITE(6,110)(I,XXB(I),YYB(I),I=1,NML2P1)
      WRITE(6,500)(I,XF(I),YF(I),I=1,NFP)
      WRITE(6,200)(I,XI(I),YI(I),I=1,NIP)
      E4=DX/DY $ E2=SQRT(E4) $ EPSLON=SQRT(E2)
      FACTOR=1.E-6 $ QLOAD=Q*DETA*DPSI $ R2=RADIUS**2
      WRITE(6,701)DX,DY,H,GXY,VY,RHO
      IF(RHO-1.0)2,1,3
C     FOR RHO .EQ. 1.0 **********
    1 COEF1=1./(16.*PI*EPSLON*D0) $ COEF2=2.*COEF1
      NTYPE=1
      GOTO 4
C     FOR RHO .LT. 1.0 **********
    2 COEF1=1./(32.*PI*D0) $ COEF2=2.*COEF1
      MU1=EPSLON*SQRT((1.+RHO)/2.)
      MU2=EPSLON*SQRT((1.-RHO)/2.)
      NTYPE=2
      GOTO 4
C     FOR RHO .GT. 1.0 **********
    3 BETA=EPSLON*SQRT(RHO+SQRT(RHO**2-1.)) $ BETA2=BETA**2
      LUMDA=EPSLON*SQRT(RHO-SQRT(RHO**2-1.)) $ LUMDA2=LUMDA**2
      WRITE(6,702)EPSLON,BETA,LUMDA
      COEF1=1./(8.*PI*D0*(BETA2-LUMDA2)) $ COEF2=COEF1*2.
      NTYPE=3
    4 CONTINUE
      DO 5 I=1,NML
      RL(I)=0.0
      RL(I+NML)=0.0
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 6 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2
      AA=DX*ANXS+DY*ANYS*VX
      BB=DY*ANYS+DX*ANXS*VY
      CC=2.*ANX*ANY*DK
C
      GOTO(17,18,19),NTYPE
   17 R12S=Z1S+Z2S*E2 $ Z5=ALOG(R12S/R2)
      C1=Z5-2.*E2*Z2S/R12S
      C2=Z5+2.*E2*Z2S/R12S
```

```
      C3=EPSLON*Z1*Z2/R12S
      RL(I)=RL(I)-QLOAD*(R12S*Z5-(3.*Z1S+E2*Z2S))
      RL(I+NML)=RL(I+NML)-QLOAD*(AA*C1+BB*E2*C2+2.*EPSLON*CC*C3)*FACTOR
      GOTO 6
   18 Z7=ALOG((Z1S**2+2.*RHO*E2*Z1S*Z2S+E4*Z2S**2)/R2**2)
      Z8=ATAN(2.*MU1*MU2*Z2S/(Z1S+RHO*E2*Z2S))
      IF(Z8.LT.0.)Z8=Z8+PI
      Z9=ALOG(((MU1*Z2)**2+(Z1-MU2*Z2)**2)/((MU1*Z2)**2+(Z1+MU2*Z2)**2))
      RL(I)=RL(I)-QLOAD*((Z1S+E2*Z2S)/MU1*Z7-2.*(Z1S-E2*Z2S)/MU2*Z8
     +-2.*E2*Z1*Z2/(MU1*MU2)*Z9-6.*(Z1S+E2*Z2S)/MU1)
      C1=Z7/MU1-2.*Z8/MU2 $ C2=Z7/MU1+2.*Z8/MU2 $ C3=-Z9/(MU1*MU2)
      RL(I+NML)=RL(I+NML)-QLOAD*(AA*C1+BB*E2*C2+CC*E2*C3)*FACTOR
      GOTO 6
   19 ZLOG1=ALOG((Z1S+LUMDA2*Z2S)/R2) $ ZLOG2=ALOG((Z1S+BETA2*Z2S)/R2)
      IF(ABS(Z1).LE.1.E-6)GOTO 12
      Z7=ATAN(LUMDA*Z2/Z1) $ Z8=ATAN(BETA*Z2/Z1)
      GOTO 45
   12 Z7=PI/2. $ Z8=Z7
   45 CONTINUE
      IF(Z7.LT.0.)Z7=Z7+PI
      IF(Z8.LT.0.)Z8=Z8+PI
      C1=BETA*ZLOG1-LUMDA*ZLOG2 $ C2=-LUMDA*ZLOG1+BETA*ZLOG2
      C3=-Z7+Z8
      RL(I)=RL(I)-QLOAD*(BETA*(Z1S-LUMDA2*Z2S)*ZLOG1-4.*LUMDA*BETA*Z1*Z2
     +*(Z7-Z8)-LUMDA*(Z1S-BETA2*Z2S)*ZLOG2-3.*(BETA-LUMDA)
     +*(Z1S+LUMDA*BETA*Z2S))
      RL(I+NML)=RL(I+NML)-QLOAD*(AA*C1+BB*E2*C2+2.*E2*C3)*FACTOR
    6 CONTINUE
    5 CONTINUE
C
C
      DO 8 I=1,NML
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 7 J=1,NML2
      R2X=(XXB(J+1)+XXB(J))/2.
      R2Y=(YYB(J+1)+YYB(J))/2.
      IF(NOPTION.EQ.2.OR.J.NE.NML)GOTO 33
      R2X=(XXB(1)+XXB(NML))/2.
      R2Y=(YYB(1)+YYB(NML))/2.
   33 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2
      AA=DX*ANXS+DY*ANYS*VX
      BB=DY*ANYS+DX*ANXS*VY
      CC=2.*ANX*ANY*DK
C
      GOTO (20,21,22),NTYPE
   20 R12S=Z1S+Z2S*E2 $ Z5=ALOG(R12S/R2)
      C1=Z5-2.*E2*Z2S/R12S
      C2=Z5+2.*E2*Z2S/R12S
      C3=EPSLON*Z1*Z2/R12S
      RM(I,J)=R12S*Z5-(3.*Z1S+E2*Z2S)
      RM(I+NML,J)=(AA*C1+BB*E2*C2+2.*EPSLON*CC*C3)*FACTOR
      GOTO 7
   21 Z7=ALOG((Z1S**2+2.*RHO*E2*Z1S*Z2S+E4*Z2S**2)/R2**2)
      Z8=ATAN(2.*MU1*MU2*Z2S/(Z1S+RHO*E2*Z2S))
      IF(Z8.LT.0.)Z8=Z8+PI
```

```
      Z9=ALOG(((MU1*Z2)**2+(Z1-MU2*Z2)**2)/((MU1*Z2)**2+(Z1+MU2*Z2)**2))
      RM(I,J)=((Z1S+E2*Z2S)/MU1*Z7-2.*(Z1S-E2*Z2S)/MU2*Z8
     +-2.*E2*Z1*Z2/(MU1*MU2)*Z9-6.*(Z1S+E2*Z2S)/MU1)
      C1=Z7/MU1-2.*Z8/MU2 $ C2=Z7/MU1+2.*Z8/MU2 $ C3=-Z9/(MU1*MU2)
      RM(I+NML,J)=(AA*C1+BB*E2*C2+CC*E2*C3)*FACTOR
      GOTO 7
   22 ZLOG1=ALOG((Z1S+LUMDA2*Z2S)/R2) $ ZLOG2=ALOG((Z1S+BETA2*Z2S)/R2)
      IF(ABS(Z1).LE.1.E-6)GOTO 23
      Z7=ATAN(LUMDA*Z2/Z1) $ Z8=ATAN(BETA*Z2/Z1)
      GOTO 46
   23 Z7=PI/2. $ Z8=Z7
   46 CONTINUE
      IF(Z7.LT.0.)Z7=Z7+PI
      IF(Z8.LT.0.)Z8=Z8+PI
      C1=BETA*ZLOG1-LUMDA*ZLOG2 $ C2=-LUMDA*ZLOG1+BETA*ZLOG2
      C3=-Z7+Z8
      RM(I,J)=(BETA*(Z1S-LUMDA2*Z2S)*ZLOG1-4.*LUMDA*BETA*Z1*Z2
     +*(Z7-Z8)-LUMDA*(Z1S-BETA2*Z2S)*ZLOG2-3.*(BETA-LUMDA)
     +*(Z1S+LUMDA*BETA*Z2S))
      RM(I+NML,J)=(AA*C1+BB*E2*C2+2.*E2*C3)*FACTOR
    7 CONTINUE
    8 CONTINUE
      DO 10 I=1,NML
      DO 10 J=1,NML2
      RMX(I,J)=RM(I,J)
   10 RMX(I+NML,J)=RM(I+NML,J)
      DO 1015 I=1,NML2
 1015 RLX(I)=RL(I)
C
C
      CALL LEQT1F(RM,1,NML2,NML2,RL,0,WKAREA,IER)
      DO 29 I=1,NML2
   29 PS(I)=RL(I)
      WRITE (6,400) (I,PS(I),I=1,NML2)
      WRITE(6,1014)
      DO 1101 I=1,NML2
      SUM=0.
      DO 1102 J=1,NML2
      SUM=SUM+RMX(I,J)*PS(J)
 1102 CONTINUE
 1101 RB(I)=SUM-RLX(I)
      WRITE(6,1103)(I,RB(I),RB(I+NML),I=1,NML)
      DO 9 I=1,NFP
      W(I)=0.0
      BMX(I)=0.0
      BMY(I)=0.0
    9 CONTINUE
      DO 14 I=1,NFP
      R1X=XF(I)
      R1Y=YF(I)
      DO 13 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
C
      GOTO (30,31,32),NTYPE
   30 R12S=Z1S+Z2S*E2 $ Z5=ALOG(R12S/R2)
      Z6=2.*E2*Z2S/R12S
      W(I)=W(I)+QLOAD*(R12S*Z5-(3.*Z1S+E2*Z2S))*COEF1
      BMX(I)=BMX(I)-QLOAD*COEF2*DX*((Z5-Z6)+VY*E2*(Z5+Z6))
```

```
      BMY(I)=BMY(I)-QLOAD*COEF2*DY* (E2*(Z5+Z6)+VX*(Z5-Z6))
      GOTO 13
   31 Z7=ALOG((Z1S**2+2.*RHO*E2*Z1S*Z2S+E4*Z2S**2)/R2**2)
      Z8=ATAN(2.*MU1*MU2*Z2S/(Z1S+RHO*E2*Z2S))
      IF(Z8.LT.0.)Z8=Z8+PI
      Z9=ALOG(((MU1*Z2)**2+(Z1-MU2*Z2)**2)/((MU1*Z2)**2+(Z1+MU2*Z2)**2))
      W(I)=W(I)+QLOAD*((Z1S+E2*Z2S)/MU1*Z7-2.*(Z1S-E2*Z2S)/MU2*Z8
     +-2.*E2*Z1*Z2/(MU1*MU2)*Z9-6.*(Z1S+E2*Z2S)/MU1)*COEF1
      W2X=Z7/MU1-Z8*2./MU2 $ W2Y=Z7/MU1+2.*Z8/MU2
      BMX(I)=BMX(I)-DX*QLOAD*COEF2*(W2X+E2*VY*W2Y)
      BMY(I)=BMY(I)-DY*QLOAD*COEF2*(W2Y*E2+VX*W2X)
      GOTO 13
   32 ZLOG1=ALOG((Z1S+LUMDA2*Z2S)/R2) $ ZLOG2=ALOG((Z1S+BETA2*Z2S)/R2)
      IF(ABS(Z1).LE.1.E-6)GOTO 24
      Z7=ATAN(LUMDA*Z2/Z1) $ Z8=ATAN(BETA*Z2/Z1)
      GOTO 47
   24 Z7=PI/2. $ Z8=Z7
   47 CONTINUE
      IF(Z7.LT.0.)Z7=Z7+PI
      IF(Z8.LT.0.)Z8=Z8+PI
      W(I)=W(I)+QLOAD*(BETA*(Z1S-LUMDA2*Z2S)*ZLOG1-4.*LUMDA*BETA*Z1*Z2
     +*(Z7-Z8)-LUMDA*(Z1S-BETA2*Z2S)*ZLOG2-3.*(BETA-LUMDA)
     +*(Z1S+LUMDA*BETA*Z2S))*COEF1
      W2X=BETA*ZLOG1-LUMDA*ZLOG2 $ W2Y=BETA*ZLOG2-LUMDA*ZLOG1
      BMX(I)=BMX(I)-DX*QLOAD*COEF2*(W2X+E2*VY*W2Y)
      BMY(I)=BMY(I)-DY*QLOAD*COEF2*(W2Y*E2+VX*W2X)
   13 CONTINUE
   14 CONTINUE
      DO 16 I=1,NFP
      R1X=XF(I)
      R1Y=YF(I)
      DO 15 J=1,NML2
      R2X=(XXB(J+1)+XXB(J))/2.0
      R2Y=(YYB(J+1)+YYB(J))/2.0
      IF(NOPTION.EQ.2.OR.J.NE.NML)GOTO 11
      R2X=(XXB(1)+XXB(NML))/2.
      R2Y=(YYB(1)+YYB(NML))/2.
   11 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
C
      GOTO (34,35,36),NTYPE
   34 R12S=Z1S+Z2S*E2 $ Z5=ALOG(R12S/R2)
      Z6=2.*E2*Z2S/R12S
      W(I)=W(I)+PS(J)*COEF1*(R12S*Z5-(3.*Z1S+E2*Z2S))
      BMX(I)=BMX(I)-PS(J)*COEF2*DX*((Z5-Z6)+VY*E2*(Z5+Z6))
      BMY(I)=BMY(I)-PS(J)*COEF2*DY*(E2*(Z5+Z6)+VX*(Z5-Z6))
      GOTO 15
   35 Z7=ALOG((Z1S**2+2.*RHO*E2*Z1S*Z2S+E4*Z2S**2)/R2**2)
      Z8=ATAN(2.*MU1*MU2*Z2S/(Z1S+RHO*E2*Z2S))
      IF(Z8.LT.0.)Z8=Z8+PI
      Z9=ALOG(((MU1*Z2)**2+(Z1-MU2*Z2)**2)/((MU1*Z2)**2+(Z1+MU2*Z2)**2))
      W(I)=W(I)+PS(J)*((Z1S+E2*Z2S)/MU1*Z7-2.*(Z1S-E2*Z2S)/MU2*Z8
     +-2.*E2*Z1*Z2/(MU1*MU2)*Z9-6.*(Z1S+E2*Z2S)/MU1)*COEF1
      W2X=Z7/MU1-Z8*2./MU2 $ W2Y=Z7/MU1+2.*Z8/MU2
      BMX(I)=BMX(I)-DX*PS(J)*COEF2*(W2X+E2*VY*W2Y)
      BMY(I)=BMY(I)-DY*PS(J)*COEF2*(W2Y*E2+VX*W2X)
      GOTO 15
   36 ZLOG1=ALOG((Z1S+LUMDA2*Z2S)/R2) $ ZLOG2=ALOG((Z1S+BETA2*Z2S)/R2)
      IF(ABS(Z1).LE.1.E-6)GOTO 37
      Z7=ATAN(LUMDA*Z2/Z1) $ Z8=ATAN(BETA*Z2/Z1)
```

```
      GOTO 48
   37 Z7=PI/2. $ Z8=Z7
   48 CONTINUE
      IF(Z7.LT.0.)Z7=Z7+PI
      IF(Z8.LT.0.)Z8=Z8+PI
      W(I)=W(I)+PS(J)*(BETA*(Z1S-LUMDA2*Z2S)*ZLOG1-4.*LUMDA*BETA*Z1*Z2
     +*(Z7-Z8)-LUMDA*(Z1S-BETA2*Z2S)*ZLOG2-3.*(BETA-LUMDA)
     +*(Z1S+LUMDA*BETA*Z2S))*COEF1
      W2X=BETA*ZLOG1-LUMDA*ZLOG2 $ W2Y=BETA*ZLOG2-LUMDA*ZLOG1
      BMX(I)=BMX(I)-DX*PS(J)*COEF2*(W2X+E2*VY*W2Y)
      BMY(I)=BMY(I)-DY*PS(J)*COEF2*(W2Y*E2+VX*W2X)
   15 CONTINUE
   16 CONTINUE
      WRITE(6,600)(I,XF(I),YF(I),W(I),BMX(I),BMY(I),I=1,NFP)
      GOTO(53,53,51,52,53),NCOUNT
   49 RHO=0.5 $ NCOUNT=2
      GOTO 38
   50 RHO=2.0 $ NCOUNT=3
      GOTO 38
   51 RHO=0.1 $ NCOUNT=4
      GOTO 38
   52 RHO=10. $ NCOUNT=5
      GOTO 38
   53 STOP
       END
```

```
40,100,181,0.3,1.0,1.0,30.E6,2.0E6,0.3,7.5E6,80.,0.4,4.0
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,10*-5.,
-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,10*5.
10*-5.,-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,
10*5.,4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5
40*2
10*0.,10*-1.,10*[illegible].,10*1.
10*-1.,10*0.,10*1.,10*0.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
10*-4.5,10*-3.5,10*-2.5,10*-1.5,10*-0.5,
10*0.5,10*1.5,10*2.5,10*3.5,10*4.5
```

APPENDIX F

DERIVATIVES OF THE GREEN'S FUNCTION OF AN INFINITE ANISOTROPIC PLATE

APPENDIX F

DERIVATIVES OF THE GREEN'S FUNCTION OF AN INFINITE ANISOTROPIC PLATE

For simplicity, the derivatives are written in terms of the four constants ϕ_1, ϕ_2, ϕ_3, and ϕ_4, and the eight functions L_1, L_3, R_1, R_3, N_1, N_3, S_1, and S_3 as shown in Eq. (38).

$$\frac{\partial G}{\partial x} = \frac{1}{8\pi D_{22}\phi_1\phi_2}\left[\phi_3\frac{\partial R_1}{\partial x} + \phi_4\frac{\partial R_3}{\partial x} + 4(\alpha-\gamma)\left(\frac{\partial S_1}{\partial x} - \frac{\partial S_3}{\partial x}\right)\right]$$

$$\frac{\partial G}{\partial y} = \frac{1}{8\pi D_{22}\phi_1\phi_2}\left[\phi_3\frac{\partial R_1}{\partial y} + \phi_4\frac{\partial R_3}{\partial y} + 4(\alpha-\gamma)\left(\frac{\partial S_1}{\partial y} - \frac{\partial S_3}{\partial y}\right)\right]$$

$$\frac{\partial^3 G}{\partial x^3} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\left[\phi_3\frac{\partial L_1}{\partial x} + \phi_4\frac{\partial L_3}{\partial x} + 4(\alpha-\gamma)\left(\frac{\partial N_1}{\partial x} - \frac{\partial N_3}{\partial x}\right)\right]$$

$$\frac{\partial^3 G}{\partial y^3} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\left\{\frac{(\alpha^2+\beta^2-2\alpha\gamma)(\alpha^2+\beta^2)+(\alpha^2-\beta^2)(\gamma^2+\lambda^2)}{\beta}\frac{\partial L_1}{\partial y}\right.$$

$$+\frac{(\gamma^2+\lambda^2-2\alpha\gamma)(\gamma^2+\lambda^2)+(\gamma^2-\lambda^2)(\alpha^2+\beta^2)}{\lambda}\frac{\partial L_3}{\partial y}$$

$$\left.-4[\alpha(\gamma^2+\lambda^2)-\gamma(\alpha^2+\beta^2)]\left(\frac{\partial N_1}{\partial y} - \frac{\partial N_3}{\partial y}\right)\right\}$$

$$\frac{\partial^3 G}{\partial x^2 \partial y} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\left[\phi_3\frac{\partial L_1}{\partial y}+\phi_4\frac{\partial L_3}{\partial y} + 4(\alpha-\gamma)\left(\frac{\partial N_1}{\partial y} - \frac{\partial N_3}{\partial y}\right)\right]$$

$$\frac{\partial^3 G}{\partial x \partial y^2} = \frac{1}{4\pi D_{22}\phi_1\phi_2}\left\{\frac{(\alpha^2+\beta^2-2\alpha\gamma)(\alpha^2+\beta^2)+(\alpha^2-\beta^2)(\gamma^2+\lambda^2)}{\beta}\frac{\partial L_1}{\partial x}\right.$$

$$+\frac{(\gamma^2+\lambda^2-2\alpha\gamma)(\gamma^2+\lambda^2)+(\gamma^2-\lambda^2)(\alpha^2+\beta^2)}{\lambda}\frac{\partial L_3}{\partial x}$$

$$\left.- 4[\alpha(\gamma^2+\lambda^2)-\gamma(\alpha^2+\beta^2)]\left(\frac{\partial N_1}{\partial x} - \frac{\partial N_3}{\partial x}\right)\right\}$$

where,

$$\phi_1 = (\alpha-\gamma)^2+(\beta-\lambda)^2 \quad ; \quad \phi_2 = (\alpha-\gamma)^2+(\beta+\lambda)^2$$

$$\phi_3 = \frac{(\alpha-\gamma)^2-(\beta^2-\lambda^2)}{\beta}; \quad \phi_4 = \frac{(\alpha-\gamma)^2+(\beta^2-\lambda^2)}{\lambda}$$

$$L_1 = \ln\frac{[(x-\xi)+\alpha(y-\eta)]^2+\beta^2(y-\eta)^2}{a^2}$$

$$L_3 = \ln\frac{[(x-\xi)+\gamma(y-\eta)]^2+\lambda^2(y-\eta)^2}{a^2}$$

$$R_1 = \{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}\cdot(L_1-3)$$

$$R_3 = \{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}\cdot(L_3-3)$$

$$N_1 = \text{arc tg}\,\frac{\beta(y-\eta)}{(x-\xi)+\alpha(y-\eta)} \quad ; \quad N_3 = \text{arc tg}\,\frac{\lambda(y-\eta)}{(x-\xi)+\gamma(y-\eta)}$$

$$S_1 = \beta(y-\eta)[(x-\xi)+\alpha(y-\eta)](L_1-3)+\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}N_1$$

$$S_3 = \lambda(y-\eta)[(x-\xi)+\gamma(y-\eta)](L_3-3)+\{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}N_3$$

$$\frac{\partial L_1}{\partial x} = \frac{2\{(x-\xi)+\alpha(y-\eta)\}}{\{(x-\xi)+\alpha(y-\eta)\}^2+\beta^2(y-\eta)^2}$$

$$\frac{\partial L_3}{\partial x} = \frac{2\{(x-\xi)+\gamma(y-\eta)\}}{\{(x-\xi)+\gamma(y-\eta)\}^2+\lambda^2(y-\eta)^2}$$

$$\frac{\partial L_1}{\partial y} = \frac{2\alpha(x-\xi)+2(\alpha^2+\beta^2)(y-\eta)}{\{(x-\xi)+\alpha(y-\eta)\}^2+\beta^2(y-\eta)^2}$$

$$\frac{\partial L_3}{\partial y} = \frac{2\gamma(x-\xi)+2(\gamma^2+\lambda^2)(y-\eta)}{\{(x-\xi)+\gamma(y-\eta)\}^2+\lambda^2(y-\eta)^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{-\beta(y-\eta)}{\{(x-\xi)+\alpha(y-\eta)\}^2+\beta^2(y-\eta)^2}$$

$$\frac{\partial N_3}{\partial x} = \frac{-\gamma(y-\eta)}{\{(x-\xi)+\gamma(y-\eta)\}^2+\lambda^2(y-\eta)^2}$$

$$\frac{\partial N_1}{\partial y} = \frac{\beta(x-\xi)}{\{(x-\xi)+\alpha(y-\eta)\}^2+\beta^2(y-\eta)^2}$$

$$\frac{\partial N_3}{\partial y} = \frac{\lambda(x-\xi)}{\{(x-\xi)+\gamma(y-\eta)\}^2+\lambda^2(y-\eta)^2}$$

$$\frac{\partial R_1}{\partial x} = 2[(x-\xi)+\alpha(y-\eta)](L_1-3)+\frac{\partial L_1}{\partial x}\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}$$

$$-4\beta(y-\eta)N_1-4\beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]\frac{\partial N_1}{\partial x}$$

$$\frac{\partial R_3}{\partial x} = 2[(x-\xi)+\gamma(y-\eta)](L_3-3)+\frac{\partial L_3}{\partial x}\{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}$$

$$-4\lambda(y-\eta)N_3-4\lambda(y-\eta)[(x-\xi)+\gamma(y-\eta)]\frac{\partial N_3}{\partial x}$$

$$\frac{\partial R_1}{\partial y} = 2[\alpha(x-\xi)+(\alpha^2-\beta^2)(y-\eta)](L_1-3)+\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}\frac{\partial L_1}{\partial y}$$

$$-4\beta[(x-\xi)+2\alpha(y-\eta)]N_1-4\beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]\frac{\partial N_1}{\partial y}$$

$$\frac{\partial R_3}{\partial y} = 2[\gamma(x-\xi)+(\gamma^2-\lambda^2)(y-\eta)](L_3-3)+\{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}\frac{L}{y}$$

$$-4\lambda[(x-\xi)+2\gamma(y-\eta)]N_3-4\lambda(y-\eta)[(x-\xi)+\gamma(y-\eta)]\frac{\partial N_3}{\partial y}$$

$$\frac{\partial S_1}{\partial x} = \beta(y-\eta)(L_1-3)+\beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]\frac{\partial L_1}{\partial x} + 2[(x-\xi)+\alpha(y-\eta)]\cdot N_1$$

$$+\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}\frac{\partial N_1}{\partial x}$$

$$\frac{\partial S_3}{\partial x} = \lambda(y-\eta)(L_3-3)+\lambda(y-\eta)[(x-\xi)+\gamma(y-\eta)]\frac{\partial L_3}{\partial x} + 2[(x-\xi)+\gamma(y-\eta)]\cdot N_3$$

$$+\{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}\frac{\partial N_3}{\partial x}$$

$$\frac{\partial S_1}{\partial y} = \beta[(x-\xi)+2\alpha(y-\eta)](L_1-3)+\beta(y-\eta)[(x-\xi)+\alpha(y-\eta)]\frac{\partial L_1}{\partial y}$$

$$+ 2[\alpha(x-\xi)+(\alpha^2-\beta^2)(y-\eta)]\cdot N_1+\{[(x-\xi)+\alpha(y-\eta)]^2-\beta^2(y-\eta)^2\}\frac{\partial N_1}{\partial y}$$

$$\frac{\partial S_3}{\partial y} = \lambda[(x-\xi)+2\gamma(y-\eta)](L_3-3)+\lambda(y-\eta)[(x-\xi)+\gamma(y-\eta)]\frac{\partial L_3}{\partial y}$$

$$+ 2[\gamma(x-\xi)+(\gamma^2-\lambda^2)(y-\eta)]\cdot N_3+\{[(x-\xi)+\gamma(y-\eta)]^2-\lambda^2(y-\eta)^2\}\frac{\partial N_3}{\partial y}$$

APPENDIX G

COMPUTER PROGRAM FOR AN ANISOTROPIC PROBLEM

APPENDIX G

COMPUTER PROGRAM FOR AN ANISOTROPIC PROBLEM

```
      PROGRAM ANIPLCL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C ***********************************************************************
C
C POINT FORCE METHOD FOR ANISOTROPIC PLATE BENDING PROBLEMS,
C ARBITRARY PLAN FORM, TRANSVERSE LOAD, AND BOUNDARY CONDITIONS ***
C
C SHOWN HERE IS AN EXAMPLE FOR A SIMPLY SUPPORTED SQUARE PLATE.
C
C REQUIRED INPUT VALUES ---
C
C           NBP       =NUMBER OF BOUNDARY POINTS
C           NIP       =NUMBER OF INTERNAL LOAD POINTS
C           NFP       =NUMBER OF FIELD POINTS
C           XB,YB     =POINTS ON B AT WHICH B.C. ARE SATISFIED.
C           BANX,BANY =COMPONENTS OF UNIT NORMAL TO B AT XB,YB.
C           XXB,YYB   =END POINTS OF MESHES AROUND B WHERE FICTITIOUS
C                      FORCES ARE ASSIGNED.
C           XF,YF     =FIELD POINTS
C           XI,YI     =INTERNAL LOAD POINTS
C           VX        =POISSON*S RATIO IN X DIRECTION
C                      DUE TO STRESS IN Y DIRECTION
C           EX,EY     =YOUNG*S MODULI IN X AND Y DIRECTIONS,
C                      RESPECTIVELY
C           GXY       =SHEAR MODULUS
C           HVALUE    =PLATE THICKNESS
C           RADIUS    =RADIUS OF THE FICTITIOUS CIRCULAR PLATE OF WHICH
C                      THE DISPLACEMENT AT THE CIRCUMFERENTIAL BOUNDARY
C                      IS SET TO ZERO.
C
C ***********************************************************************
C
      DIMENSION XB(40),YB(40),XXB(81),YYB(81)
      DIMENSION XI(100),YI(100),DEL(2)
      DIMENSION RLX(80),RB(80),RM(80,80),PS(80),WKAREA(80),RL(80)
      DIMENSION XF(181),YF(181),W(181),BMX(181),BMY(181)
      DIMENSION BANX(40),BANY(40),RMX(80,80)
      REAL AVECTOR(5),MX,MY,MXY
      COMPLEX ERROR,ROOT(4)
C
  100 FORMAT(≠0LOCT≠,6X,≠XB≠,8X,≠YB≠,7X,≠ANX≠,8X,≠ANY≠,4X,
     +/≠0≠/(I5,4F10.4))
  110 FORMAT(≠0LOCT≠,11X,≠XXB≠,14X,≠YYB≠/≠0≠/(I4,8X,F9.2,8X,F9.2))
  200 FORMAT (≠0LOCT≠,11X,≠XI≠,17X,≠YI≠,14X,/≠0≠/
     +(I4,11X,F6.3,11X,F6.3))
  400 FORMAT (≠1LOCT≠,19X,≠PSP≠/≠0≠/(I4,8X,E20.8))
  500 FORMAT(≠0LOCT≠,11X,≠XF≠,17X,≠YF≠/≠0≠/(I4,11X,F6.3,11X,F6.3))
  600 FORMAT(1H1,8X,≠NODE≠,12X≠XF≠,16X,≠YF≠,19X,≠W≠,16X,≠BMX≠,
     116X,≠BMY≠/(1H0,I10,2F20.10,3E20.12))
  700 FORMAT(1H1,≠INPUT VALUES .....≠,//1X,≠NBP = ≠,I3,≠  NIP =≠,I3,
     +≠  NFP = ≠,I3,
     +/1H0,≠ EX = ≠,E10.4,≠ EY = ≠,E10.4,≠ VX = ≠,F5.2,≠ GXY = ≠,E10.4,
     +/1H0,≠ RADIUS OF THE PLATE = ≠,
     +F7.1,≠  THICKNESS OF THE PLATE =≠,F6.3)
  703 FORMAT(///,1X,≠THE COEFFICIENTS OF THE CHARACTERISTIC ≠,
     +≠POLYNOMIAL ARE -----≠,/1X,5E12.5)
  704 FORMAT(1H0,≠THE FOUR ROOTS OF THE CHARACTERISTIC EQUATION:≠,
     +(/1X,2E12.5))
  705 FORMAT(1H0,≠ ERROR FOR ROOT NO. ≠,I2,2X,2E13.4,≠ IS ≠,2E10.4)
```

```
  707 FORMAT(1H0,≠THE FOUR CONSTANTS ARE :≠,
     +/1X,≠ALPHA = ≠,E10.4,≠     BETA = ≠,E10.4,
     +≠      GAMMA = ≠,E10.4,≠     LUMBDA = ≠,E10.4)
 1103 FORMAT(I5,5X,E20.12,5X,E20.12)
 1014 FORMAT(1H1,≠THE FOLLOWING IS A LIST OF DOUBLE CHECKING OF B.C.S≠,
     1//,4X≠NBP≠10X,≠B.C.  1≠,13X,≠B.C.  2≠,//)
  708 FORMAT(1H0,≠RADIUS = ≠,E10.3,≠    DIST1 = ≠,F6.1,
     +≠      DIST2 = ≠,F6.1)
  801 FORMAT(1H1,≠THE BENDING RIGIDITIES ARE -----≠,
     +/1X,≠D11 = ≠,E10.4,≠   D12 = ≠,E10.4,≠   D22 = ≠,E10.4,
     +/1X,≠D66 = ≠,E10.4,≠   D16 = ≠,E10.4,≠   D26 = ≠,E10.4)
  802 FORMAT(1X,≠SOMETHING IS WRONG WITH THE INPUT MATERIAL CONSTANTS≠,
     +≠; THE DETERMINANT IS EITHER NEGATIVE OR ZERO≠,/1X,
     +≠COMPUTATION IS TERMINATED, DET = ≠,E15.7)
  805 FORMAT(1H1,≠THE AIJ ARE ---≠,/1X,≠A11= ≠,E15.7,≠  A12= ≠,
     +E15.7,≠  A22= ≠,E15.7,≠  A66= ≠,E15.7,≠  A16= ≠,E15.7,
     +≠  A26= ≠,E15.7)
C
C INPUT VALUES .........
C
      READ(5,*)NBP,NIP,NFP,EX,EY,VX,GXY,RADIUS,HVALUE
      READ(5,*)A16,A26
      READ(5,*)(XB(I),I=1,NBP)
      READ(5,*)(YB(I),I=1,NBP)
      READ(5,*)(BANX(I),I=1,NBP)
      READ(5,*)(BANY(I),I=1,NBP)
      READ(5,*)(XI(I),I=1,NIP)
      READ(5,*)(YI(I),I=1,NIP)
      NBP2=NBP*2$NBP2P1=NBP2+1
      NOPTION=1
      PI=4.*ATAN(1.)
C
C
      A11=1./EX $ A12=-VX/EX $ A22=1./EY $ A66=1./GXY
      WRITE(6,805)A11,A12,A22,A66,A16,A26
      DET=(A11*A22-A12**2)*A66+2.*A12*A16*A26-A11*A26**2-A22*A16**2
      IF(DET.GT.0.)GOTO 40
      WRITE(6,802)DET
      STOP
   40 CONTINUE
      ZZ=HVALUE**3/(12.*DET)
      D11=(A22*A66-A26**2)*ZZ $ D22=(A11*A66-A16**2)*ZZ
      D12=(A16*A26-A12*A66)*ZZ $ D66=(A11*A22-A12**2)*ZZ
      D16=(A12*A26-A22*A16)*ZZ $ D26=(A12*A16-A11*A26)*ZZ
      RADIUS=80. $ DIST1=2.0 $ DIST2=2.
      WRITE(6,700)NBP,NIP,NFP,EX,EY,VX,GXY,
     +RADIUS,HVALUE
      WRITE(6,708)RADIUS,DIST1,DIST2
      X0=5.0$Y0=5.0
      XF(1)=10.-X0$YF(1)=-10.+Y0
      DO 41 I=2,11
      XF(I)=XF(I-1)-0.5
   41 YF(I)=YF(I-1)
      DO 42 J=1,10
      K=J*17
      DO 42 I=1,11
      XF(I+K)=XF(I)
   42 YF(I+K)=YF(I+K-17)+1.0
      XF(12)=10.-X0$YF(12)=-9.5+Y0
      DO 43 I=13,17
```

```
      XF(I)=XF(I-1)-1.0
   43 YF(I)=YF(I-1)
      DO 44 J=1,9
      K=J*17
      DO 44 I=12,17
      XF(I+K)=XF(I)
   44 YF(I+K)=YF(I+K-17)+1.0
      DEL(1)=(10.+2.*DIST1)/10.
      DEL(2)=(10.+2.*DIST1+2.*DIST2)/10.
      XXB(1)=5.+DIST1-DEL(1)$YYB(1)=-5.-DIST1
      XXB(41)=5.+DIST1+DIST2-DEL(2)$YYB(41)=-5.-DIST1-DIST2
      DO 28 J=1,2
      DELT=DEL(1)$IF(J.EQ.2)DELT=DEL(2)
      DO 25 I=2,10
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)-DELT
   25 YYB(K)=YYB(K-1)
      DO 26 I=11,20
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   26 YYB(K)=YYB(K-1)+DELT
      DO 27 I=21,30
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)+DELT
   27 YYB(K)=YYB(K-1)
      DO 28 I=31,40
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   28 YYB(K)=YYB(K-1)-DELT
      XXB(81)=XXB(41) $ YYB(81)=YYB(41)
      WRITE(6,100)(I,XB(I),YB(I),BANX(I),BANY(I),I=1,NBP)
      WRITE(6,110)(I,XXB(I),YYB(I),I=1,NBP2P1)
      WRITE(6,500)(I,XF(I),YF(I),I=1,NFP)
      WRITE(6,200)(I,XI(I),YI(I),I=1,NIP)
      WRITE(6,801)D11,D12,D22,D66,D16,D26
C
C SET UP AND SOLVE THE FOURTH DEGREE CHARACTERISTEC POLYNOMIAL.
C
      AVECTOR(1)=1.0 $ AVECTOR(2)=4.*D26/D22
      AVECTOR(3)=2.*(D12+2.*D66)/D22
      AVECTOR(4)=4.*D16/D22 $ AVECTOR(5)=D11/D22
      WRITE(6,703)(AVECTOR(I),I=1,5)
      CALL ZPOLR(AVECTOR,4,ROOT,IER)
      WRITE(6,704)(ROOT(I),I=1,4)
      DO 706 I=1,4
      ERROR=AVECTOR(1)*ROOT(I)**4+AVECTOR(2)*ROOT(I)**3+AVECTOR(3)*
     +ROOT(I)**2+AVECTOR(4)*ROOT(I)+AVECTOR(5)
  706 WRITE(6,705)I,ROOT(I),ERROR
      R1=REAL(ROOT(1)) $ R2=AIMAG(ROOT(1))
      R3=REAL(ROOT(3)) $ R4=AIMAG(ROOT(3))
      WRITE(6,707)R1,R2,R3,R4
      R1S=R1**2 $ R2S=R2**2 $ R3S=R3**2 $ R4S=R4**2
C
      CONSTG=(R1-R3)**2+(R2+R4)**2 $ CONSTH=(R1-R3)**2+(R2-R4)**2
      COEF1=1./(8.*PI*D22*CONSTG*CONSTH) $ COEF2=2.*COEF1
      CONST1=((R1-R3)**2-(R2S-R4S))/R2
```

```
      CONST2=((R1-R3)**2+(R2S-R4S))/R4
      CONST3=4.*(R1-R3)
      CONST5=((R1-2.*R3)*(R1S+R2S)+R1*(R3S+R4S))/R2
      CONST6=((R3-2.*R1)*(R3S+R4S)+R3*(R1S+R2S))/R4
      CONST7=2.*(R1S+R2S-R3S-R4S)
      CONST8=((R1S+R2S-2.*R1*R3)*(R1S+R2S)+(R1S-R2S)*(R3S+R4S))/R2
      CONST9=((R3S+R4S-2.*R1*R3)*(R3S+R4S)+(R3S-R4S)*(R1S+R2S))/R4
      CONST10=4.*(R1*(R3S+R4S)-R3*(R1S+R2S))
      QLOAD=1.0 $ A2=RADIUS**2
      DO 5 I=1,NBP
      RL(I)=0.0
      RL(I+NBP)=0.0
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 6 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2 $ ANXY=ANX*ANY
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 60
      FUNCN1=PI/2.
      GOTO 61
   60 FUNCN1=ATAN(R2*Z2/Z3)
   61 IF(ABS(Z4).GT.1.E-6)GOTO 62
      FUNCN3=PI/2.
      GOTO 63
   62 FUNCN3=ATAN(R4*Z2/Z4)
   63 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      AA=D11*ANXS+D12*ANYS+2.*D16*ANXY
      BB=ANXS*D12+D22*ANYS+2.*ANXY*D26
      CC=2.*D16*ANXS+2.*D26*ANYS+4.*D66*ANXY
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      RL(I)=RL(I)-QLOAD*(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*
     +(FUNCS1-FUNCS3))
      RL(I+NBP)=RL(I+NBP)-QLOAD*(AA*WXX+BB*WYY+CC*WXY)
    6 CONTINUE
    5 CONTINUE
C
C
      DO 8 I=1,NBP
      R1X=XB(I)
      R1Y=YB(I)
      ANX=BANX(I)
      ANY=BANY(I)
      DO 7 J=1,NBP2
```

```
      R2X=(XXB(J+1)+XXB(J))/2.
      R2Y=(YYB(J+1)+YYB(J))/2.
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 33
      R2X=(XXB(1)+XXB(NBP))/2.
      R2Y=(YYB(1)+YYB(NBP))/2.
   33 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2 $ ANXY=ANX*ANY
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 70
      FUNCN1=PI/2.
      GOTO 71
   70 FUNCN1=ATAN(R2*Z2/Z3)
   71 IF(ABS(Z4).GT.1.E-6)GOTO 72
      FUNCN3=PI/2.
      GOTO 73
   72 FUNCN3=ATAN(R4*Z2/Z4)
   73 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      AA=D11*ANXS+D12*ANYS+2.*D16*ANXY
      BB=ANXS*D12+D22*ANYS+2.*ANXY*D26
      CC=2.*D16*ANXS+2.*D26*ANYS+4.*D66*ANXY
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      RM(I,J)=(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
      RM(I+NBP,J)=(AA*WXX+BB*WYY+CC*WXY)
    7 CONTINUE
    8 CONTINUE
      DO 10 I=1,NBP
      DO 10 J=1,NBP2
      RMX(I,J)=RM(I,J)
   10 RMX(I+NBP,J)=RM(I+NBP,J)
      DO 1015 I=1,NBP2
 1015 RLX(I)=RL(I)
C
C
      CALL LEQT1F(RM,1,NBP2,NBP2,RL,0,WKAREA,IER)
      DO 29 I=1,NBP2
   29 PS(I)=RL(I)
      WRITE (6,400) (I,PS(I),I=1,NBP2)
      WRITE(6,1014)
      DO 1101 I=1,NBP2
      SUM=0.
      DO 1102 J=1,NBP2
      SUM=SUM+RMX(I,J)*PS(J)
 1102 CONTINUE
 1101 RB(I)=SUM-RLX(I)
      WRITE(6,1103)(I,RB(I),RB(I+NBP),I=1,NBP)
      DO 9 I=1,NFP
      W(I)=0.0
```

```
      BMX(I)=0.0
      BMY(I)=0.0
    9 CONTINUE
      DO 14 I=1,NFP
      R1X=XF(I)
      R1Y=YF(I)
      DO 13 J=1,NIP
      R2X=XI(J)
      R2Y=YI(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+P1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 80
      FUNCN1=PI/2.
      GOTO 81
   80 FUNCN1=ATAN(R2*Z2/Z3)
   81 IF(ABS(Z4).GT.1.E-6)GOTO 82
      FUNCN3=PI/2.
      GOTO 83
   82 FUNCN3=ATAN(R4*Z2/Z4)
   83 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      FUNCP1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      W(I)=W(I)+(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
     +*COEF1*QLOAD
      MX=-COEF2*(D11*WXX+D12*WYY+2.*D16*WXY)*QLOAD
      MY=-COEF2*(D12*WXX+D22*WYY+2.*D26*WXY)*QLOAD
      MXY=-COEF2*(D16*WXX+D26*WYY+2.*D66*WXY)*QLOAD
      BMX(I)=BMX(I)+MX
      BMY(I)=BMY(I)+MY
   13 CONTINUE
   14 CONTINUE
      DO 16 I=1,NFP
      R1X=XF(I)
      R1Y=YF(I)
      DO 15 J=1,NBP2
      R2X=(XXB(J+1)+XXB(J))/2.0
      R2Y=(YYB(J+1)+YYB(J))/2.0
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 11
      R2X=(XXB(1)+XXB(NBP))/2.
      R2Y=(YYB(1)+YYB(NBP))/2.
   11 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+P1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 90
      FUNCN1=PI/2.
      GOTO 91
   90 FUNCN1=ATAN(R2*Z2/Z3)
```

```
   91 IF(ABS(Z4).GT.1.E-6)GOTO 92
      FUNCN3=PI/2.
      GOTO 93
   92 FUNCN3=ATAN(R4*Z2/Z4)
   93 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      W(I)=W(I)+(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
     +*COEF1*PS(J)
      MX=-COEF2*(D11*WXX+D12*WYY+2.*D16*WXY)*PS(J)
      MY=-COEF2*(D12*WXX+D22*WYY+2.*D26*WXY)*PS(J)
      MXY=-COEF2*(D16*WXX+D26*WYY+2.*D66*WXY)*PS(J)
      BMX(I)=BMX(I)+MX
      BMY(I)=BMY(I)+MY
   15 CONTINUE
   16 CONTINUE
      WRITE(6,600)(I,XF(I),YF(I),W(I),BMX(I),BMY(I),I=1,NFP)
          END
```

```
40,100,181,30.E6,2.0E6,0.3,8.8E4,80.,0.4
0.,0.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,10*-5.,
-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,10*5.
10*-5.,-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,
10*5.,4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5
10*0.,10*-1.,10*0.,10*1.
10*-1.,10*0.,10*1.,10*0.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
10*-4.5,10*-3.5,10*-2.5,10*-1.5,10*-0.5,
10*0.5,10*1.5,10*2.5,10*3.5,10*4.5
      IF(DET.GT.0.)GOTO 40
```

APPENDIX H

COMPUTER PROGRAM FOR THE VERIFICATION OF EQUATIONS USED FOR ANISOTROPIC PROBLEMS

APPENDIX H

COMPUTER PROGRAM FOR THE VERIFICATION OF EQUATIONS USED FOR ANISOTROPIC PROBLEMS

```
      PROGRAM PLCLCHK(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C ********************************************************************
C
C THIS IS TO DOUBLE CHECK THE ANISOTROPIC PLATE PROGRAM,
C USING A SIMPLY SUPPORTED SQUARE PLATE.
C
C REQUIRED INPUT VALUES ---
C
C        NBP        =NUMBER OF BOUNDARY POINTS
C        NIP        =NUMBER OF INTERNAL LOAD POINTS
C        NFP        =NUMBER OF FIELD POINTS
C        XB,YB      =POINTS ON B AT WHICH B.C. ARE SATISFIED.
C        BANX,BANY  =COMPONENTS OF UNIT NORMAL TO B AT XB,YB.
C        XXB,YYB    =END POINTS OF MESHES AROUND B WHERE FICTITIOUS
C                    FORCES ARE ASSIGNED.
C        XF,YF      =FIELD POINTS
C        XI,YI      =INTERNAL LOAD POINTS
C        VX         =POISSON*S RATIO IN X DIRECTION
C                    DUE TO STRESS IN Y DIRECTION
C        EX,EY      =YOUNG*S MODULI IN X AND Y DIRECTIONS,
C                    RESPECTIVELY
C        GXY        =SHEAR MODULUS
C        HVALUE     =PLATE THICKNESS
C        RADIUS     =RADIUS OF THE FICTITIOUS CIRCULAR PLATE OF WHICH
C                    THE DISPLACEMENT AT THE CIRCUMFERENTIAL BOUNDARY
C                    IS SET TO ZERO.
C
C ********************************************************************
C
      DIMENSION XB(40),YB(40),XXB(81),YYB(81)
      DIMENSION XI(100),YI(100),DEL(2)
      DIMENSION RLX(80),RB(80),RM(80,80),PS(80),WKAREA(80),RL(80)
      DIMENSION XF(181),YF(181),W(181),BMX(181),BMY(181)
      DIMENSION BANX(40),BANY(40),RMX(80,80)
      DIMENSION BMXX(181),BMYY(181),BXY(181)
      DIMENSION XB1(40),YB1(40),XXB1(81),YYB1(81),XI1(100),YI1(100)
      DIMENSION XF1(181),YF1(181),BANX1(40),BANY1(40)
      REAL AVECTOR(5),MX,MY,MXY
      COMPLEX ERROR,ROOT(4)
C
  100 FORMAT(*0LOCT*,6X,*XB*,8X,*YB*,7X,*ANX*,8X,*ANY*,4X,*NBTYPE*,
     +/*0*/(I5,4F10.4,I5))
  110 FORMAT(*0LOCT*,11X,*XXB*,14X,*YYB*/*0*/(I4,8X,F9.2,8X,F9.2))
  200 FORMAT (*0LOCT*,11X,*XI*,17X,*YI*,14X,/*0*/
     +(I4,11X,F6.3,11X,F6.3))
  300 FORMAT (*0LOCT*,19X,*RLD*,34X,*RLS*/*0*/(I4,8X,E20.8,16X,
     1E20.8))
  400 FORMAT (*1LOCT*,19X,*PSP*/*0*/(I4,8X,E20.8))
  500 FORMAT(*0LOCT*,11X,*XF*,17X,*YF*/*0*/(I4,11X,F6.3,11X,F6.3))
  600 FORMAT(1H1,8X,*NODE*,12X*XF*,16X,*YF*,19X,*W*,16X,*BMX*,
     116X,*BMY*/(1H0,I10,2F20.10,3E20.12))
  700 FORMAT(1H1,*INPUT VALUES .....*,//1X,*NBP = *,I3,*  NIP =*,I3,
     +*  NFP = *,I3,
     +/1H0,* EX = *,E10.4,* EY = *,E10.4,* VX = *,F5.2,* GXY = *,E10.4,
     +/1H0,* RADIUS OF THE PLATE = *,
     +F7.1,*  THICKNESS OF THE PLATE =*,F6.3)
  703 FORMAT(///,1X,*THE COEFFICIENTS OF THE CHARACTERISTIC *,
     +*POLYNOMIAL ARE -----*,/1X,5E12.5)
```

```
  704 FORMAT(1H0,≠THE FOUR ROOTS OF THE CHARACTERISTIC EQUATION:≠,
     +(/1X,2E12.5))
  705 FORMAT(1H0,≠ ERROR FOR ROOT NO. ≠,I2,2X,2E13.4,≠ IS ≠,2E10.4)
  707 FORMAT(1H0,≠THE FOUR CONSTANTS ARE :≠,
     +/1X,≠ALPHA = ≠,E10.4,≠      BETA = ≠,E10.4,
     +≠      GAMMA = ≠,E10.4,≠      LUMBDA = ≠,E10.4)
 1103 FORMAT(I5,5X,E20.12,5X,E20.12)
 1014 FORMAT(1H1,≠THE FOLLOWING IS A LIST OF DOUBLE CHECKING OF B.C.S≠,
     1//,4X≠NBP≠10X,≠B.C.  1≠,13X,≠B.C.  2≠,//)
  708 FORMAT(1H0,≠RADIUS = ≠,E10.3,≠      DIST1 = ≠,F6.1,
     +≠      DIST2 = ≠,F6.1)
  801 FORMAT(1H1,≠THE BENDING RIGIDITIES ARE -----≠,
     +/1X,≠D11 = ≠,E10.4,≠    D12 = ≠,E10.4,≠    D22 = ≠,E10.4,
     +/1X,≠D66 = ≠,E10.4,≠    D16 = ≠,E10.4,≠    D26 = ≠,E10.4)
  805 FORMAT(1H1,≠THE AIJ ARE ---≠,/1X,≠A11= ≠,E15.7,≠  A12= ≠,
     +E15.7,≠  A22= ≠,E15.7,≠  A66= ≠,E15.7,≠  A16= ≠,E15.7,
     +≠  A26= ≠,E15.7)
C
C INPUT VALUES .........
C
      READ(5,*)NBP,NIP,NFP,EX,EY,VX,GXY,RADIUS,HVALUE
      READ(5,*)(XB(I),I=1,NBP)
      READ(5,*)(YB(I),I=1,NBP)
      READ(5,*)(BANX(I),I=1,NBP)
      READ(5,*)(BANY(I),I=1,NBP)
      READ(5,*)(XI(I),I=1,NIP)
      READ(5,*)(YI(I),I=1,NIP)
      NBP2=NBP*2$NBP2P1=NBP2+1
      NOPTION=1
      PI=4.*ATAN(1.)
C
C
      NCOUNT=1
      A16=0. $ A26=0.
      A11=1./EX $ A12=-VX/EX $ A22=1./EY $ A66=1./GXY
      WRITE(6,805)A11,A12,A22,A66,A16,A26
      DET=(A11*A22-A12**2)*A66+2.*A12*A16*A26-A11*A26**2-A22*A16**2
      ZZ=HVALUE**3/(12.*DET)
      D11=(A22*A66-A26**2)*ZZ $ D22=(A11*A66-A16**2)*ZZ
      D12=(A16*A26-A12*A66)*ZZ $ D66=(A11*A22-A12**2)*ZZ
      D16=(A12*A26-A22*A16)*ZZ $ D26=(A12*A16-A11*A26)*ZZ
      RADIUS=80. $ DIST1=2.0 $ DIST2=2.
      DK=GXY*HVALUE**3/12. $ VY=EY*VX/EX
      DX=D11 $ DY=D22 $ D3=DX*VX+2.*DK
      H=D3 $ RHO=H/SQRT(DX*DY)
      WRITE(6,700)NBP,NIP,NFP,EX,EY,VX,GXY,
     +RADIUS,HVALUE
      WRITE(6,708)RADIUS,DIST1,DIST2
      WRITE(6,709)DX,DY,RHO
  709 FORMAT(///1X,≠DX = ≠,E15.7,5X,≠DY =≠,E15.7,5X,≠RHO =≠,F10.5)
      X0=5.0$Y0=5.0
      XF(1)=10.-X0$YF(1)=-10.+Y0
      DO 41 I=2,11
      XF(I)=XF(I-1)-0.5
   41 YF(I)=YF(I-1)
      DO 42 J=1,10
      K=J*17
      DO 42 I=1,11
      XF(I+K)=XF(I)
   42 YF(I+K)=YF(I+K-17)+1.0
```

```
      XF(12)=10.-X0$YF(12)=-9.5+Y0
      DO 43 I=13.17
      XF(I)=XF(I-1)-1.0
   43 YF(I)=YF(I-1)
      DO 44 J=1,9
      K=J*17
      DO 44 I=12,17
      XF(I+K)=XF(I)
   44 YF(I+K)=YF(I+K-17)+1.0
      DEL(1)=(10.+2.*DIST1)/10.
      DEL(2)=(10.+2.*DIST1+2.*DIST2)/10.
      XXB(1)=5.+DIST1-DEL(1)$YYB(1)=-5.-DIST1
      XXB(41)=5.+DIST1+DIST2-DEL(2)$YYB(41)=-5.-DIST1-DIST2
      DO 28 J=1,2
      DELT=DEL(1)$IF(J.EQ.2)DELT=DEL(2)
      DO 25 I=2,10
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)-DELT
   25 YYB(K)=YYB(K-1)
      DO 26 I=11.20
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   26 YYB(K)=YYB(K-1)+DELT
      DO 27 I=21,30
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)+DELT
   27 YYB(K)=YYB(K-1)
      DO 28 I=31,40
      K=I
      IF(J.EQ.2)K=K+40
      XXB(K)=XXB(K-1)
   28 YYB(K)=YYB(K-1)-DELT
      XXB(81)=XXB(41) $ YYB(81)=YYB(41)
      PHI=0.
  804 WRITE(6,807)PHI
  807 FORMAT(1H0,*** WHEN PHI IS *,F7.3,5X,*DEGREES --*)
      PHI=PHI*PI/180.
      SINP=SIN(PHI) $ COSP=COS(PHI)
      DO 301 I=1,181
      XF1(I)=XF(I)*COSP+YF(I)*SINP
  301 YF1(I)=-XF(I)*SINP+YF(I)*COSP
      DO 302 I=1,40
      XB1(I)=XB(I)*COSP+YB(I)*SINP
      YB1(I)=-XB(I)*SINP+YB(I)*COSP
      BANX1(I)=BANX(I)*COSP+BANY(I)*SINP
  302 BANY1(I)=-BANX(I)*SINP+BANY(I)*COSP
      DO 303  I=1,81
      XXB1(I)=XXB(I)*COSP+YYB(I)*SINP
  303 YYB1(I)=-XXB(I)*SINP+YYB(I)*COSP
      DO 304 I=1,100
      XI1(I)=XI(I)*COSP+YI(I)*SINP
  304 YI1(I)=-XI(I)*SINP+YI(I)*COSP
      WRITE(6,100)(I,XB1(I),YB1(I),BANX1(I),BANY1(I),I=1,NBP)
      WRITE(6,110)(I,XXB1(I),YYB1(I),I=1,NBP2P1)
      WRITE(6,500)(I,XF1(I),YF1(I),I=1,NFP)
      WRITE(6,200)(I,XI1(I),YI1(I),I=1,NIP)
      COSP2=COSP**2 $ COSP4=COSP2**2
```

```
      SINP2=SINP**2 $ SINP4=SINP2**2
      COS2P=COS(2.*PHI) $ SIN2P=SIN(2.*PHI)
      D11=DX*COSP4+2.*D3*SINP2*COSP2+DY*SINP4
      D22=DX*SINP4+2.*D3*SINP2*COSP2+DY*COSP4
      D66=DK+(DX+DY-2.*D3)*SINP2*COSP2
      D12=DY*VX+(DX+DY-2.*D3)*SINP2*COSP2
      D16=0.5*(DY*SINP2-DX*COSP2+D3*COS2P)*SIN2P
      D26=0.5*(DY*COSP2-DX*SINP2-D3*COS2P)*SIN2P
      WRITE(6,801)D11,D12,D22,D66,D16,D26
      AVECTOR(1)=1.0 $ AVECTOR(2)=4.*D26/D22
      AVECTOR(3)=2.*(D12+2.*D66)/D22
      AVECTOR(4)=4.*D16/D22 $ AVECTOR(5)=D11/D22
      WRITE(6,703)(AVECTOR(I),I=1,5)
      CALL ZPOLR(AVECTOR,4,ROOT,IER)
      WRITE(6,704)(ROOT(I),I=1,4)
      DO 706 I=1,4
      ERROR=AVECTOR(1)*ROOT(I)**4+AVECTOR(2)*ROOT(I)**3+AVECTOR(3)*
     +ROOT(I)**2+AVECTOR(4)*ROOT(I)+AVECTOR(5)
  706 WRITE(6,705)I,ROOT(I),ERROR
      R1=REAL(ROOT(1)) $ R2=AIMAG(ROOT(1))
      R3=REAL(ROOT(3)) $ R4=AIMAG(ROOT(3))
      WRITE(6,707)R1,R2,R3,R4
      R1S=R1**2 $ R2S=R2**2 $ R3S=R3**2 $ R4S=R4**2
      CONSTG=(R1-R3)**2+(R2+R4)**2 $ CONSTH=(R1-R3)**2+(R2-R4)**2
      COEF1=1./(8.*PI*D22*CONSTG*CONSTH) $ COEF2=2.*COEF1
      CONST1=((R1-R3)**2-(R2S-R4S))/R2
      CONST2=((R1-R3)**2+(R2S-R4S))/R4
      CONST3=4.*(R1-R3)
      CONST5=((R1-2.*R3)*(R1S+R2S)+R1*(R3S+R4S))/R2
      CONST6=((R3-2.*R1)*(R3S+R4S)+R3*(R1S+R2S))/R4
      CONST7=2.*(R1S+R2S-R3S-R4S)
      CONST8=((R1S+R2S-2.*R1*R3)*(R1S+R2S)+(R1S-R2S)*(R3S+R4S))/R2
      CONST9=((R3S+R4S-2.*R1*R3)*(R3S+R4S)+(R3S-R4S)*(R1S+R2S))/R4
      CONST10=4.*(R1*(R3S+R4S)-R3*(R1S+R2S))
      QLOAD=1.0 $ A2=RADIUS**2
      DO 5 I=1,NBP
      RL(I)=0.0
      RL(I+NBP)=0.0
      R1X=XB1(I)
      R1Y=YB1(I)
      ANX=BANX1(I)
      ANY=BANY1(I)
      DO 6 J=1,NIP
      R2X=XI1(J)
      R2Y=YI1(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2 $ ANXY=ANX*ANY
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 60
      FUNCN1=PI/2.
      GOTO 61
   60 FUNCN1=ATAN(R2*Z2/Z3)
   61 IF(ABS(Z4).GT.1.E-6)GOTO 62
      FUNCN3=PI/2.
      GOTO 63
   62 FUNCN3=ATAN(R4*Z2/Z4)
   63 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
```

```
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      AA=D11*ANXS+D12*ANYS+2.*D16*ANXY
      BB=ANXS*D12+D22*ANYS+2.*ANXY*D26
      CC=2.*D16*ANXS+2.*D26*ANYS+4.*D66*ANXY
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      RL(I)=RL(I)-QLOAD*(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*
     +(FUNCS1-FUNCS3))
      RL(I+NBP)=RL(I+NBP)-QLOAD*(AA*WXX+BB*WYY+CC*WXY)
    6 CONTINUE
    5 CONTINUE
C
C
      DO 8 I=1,NBP
      R1X=XB1(I)
      R1Y=YB1(I)
      ANX=BANX1(I)
      ANY=BANY1(I)
      DO 7 J=1,NBP2
      R2X=(XXB1(J+1)+XXB1(J))/2.
      R2Y=(YYB1(J+1)+YYB1(J))/2.
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 33
      R2X=(XXB(1)+XXB(NBP))/2.
      R2Y=(YYB(1)+YYB(NBP))/2.
   33 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      ANXS=ANX**2 $ ANYS=ANY**2 $ ANXY=ANX*ANY
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 70
      FUNCN1=PI/2.
      GOTO 71
   70 FUNCN1=ATAN(R2*Z2/Z3)
   71 IF(ABS(Z4).GT.1.E-6)GOTO 72
      FUNCN3=PI/2.
      GOTO 73
   72 FUNCN3=ATAN(R4*Z2/Z4)
   73 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      AA=D11*ANXS+D12*ANYS+2.*D16*ANXY
      BB=ANXS*D12+D22*ANYS+2.*ANXY*D26
      CC=2.*D16*ANXS+2.*D26*ANYS+4.*D66*ANXY
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
```

```
      RM(I,J)=(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
      RM(I+NBP,J)=(AA*WXX+BB*WYY+CC*WXY)
    7 CONTINUE
    8 CONTINUE
      DO 10 I=1,NBP
      DO 10 J=1,NBP2
      RMX(I,J)=RM(I,J)
   10 RMX(I+NBP,J)=RM(I+NBP,J)
      DO 1015 I=1,NBP2
 1015 RLX(I)=RL(I)
C
C
      CALL LEQT1F(RM,1,NBP2,NBP2,RL,0,WKAREA,IER)
      DO 29 I=1,NBP2
   29 PS(I)=PL(I)
      WRITE (6,400) (I,PS(I),I=1,NBP2)
      WRITE(6.1014)
      DO 1101 I=1,NBP2
      SUM=0.
      DO 1102 J=1,NBP2
      SUM=SUM+RMX(I,J)*PS(J)
 1102 CONTINUE
 1101 RB(I)=SUM-RLX(I)
      WRITE(6,1103)(I,RB(I),RB(I+NBP),I=1,NBP)
      DO 9 I=1,NFP
      W(I)=0.0
      BMX(I)=0.0
      BMY(I)=0.0
      BXY(I)=0.
      BMXX(I)=0.
      BMYY(I)=0.
    9 CONTINUE
      DO 14 I=1,NFP
      R1X=XF1(I)
      R1Y=YF1(I)
      DO 13 J=1,NIP
      R2X=XI1(J)
      R2Y=YI1(J)
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 80
      FUNCN1=PI/2.
      GOTO 81
   80 FUNCN1=ATAN(R2*Z2/Z3)
   81 IF(ABS(Z4).GT.1.E-6)GOTO 82
      FUNCN3=PI/2.
      GOTO 83
   82 FUNCN3=ATAN(R4*Z2/Z4)
   83 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
```

```
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      W(I)=W(I)+(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
     +*COEF1*QLOAD
      MX=-COEF2*(D11*WXX+D12*WYY+2.*D16*WXY)*QLOAD
      MY=-COEF2*(D12*WXX+D22*WYY+2.*D26*WXY)*QLOAD
      MXY=-COEF2*(D16*WXX+D26*WYY+2.*D66*WXY)*QLOAD
      BMX(I)=BMX(I)+MX
      BMY(I)=BMY(I)+MY
      BXY(I)=BXY(I)+MXY
      BMXX(I)=BMX(I)*COSP2+BMY(I)*SINP2-2.*BXY(I)*COSP*SINP
      BMYY(I)=BMX(I)*SINP2+BMY(I)*COSP2-2.*BXY(I)*SINP*COSP
C     THE NEGATIVE SIGN IS FOR A REVERSED ANGLE.
   13 CONTINUE
   14 CONTINUE
      DO 16 I=1,NFP
      R1X=XF1(I)
      R1Y=YF1(I)
      DO 15 J=1,NBP2
      R2X=(XXB1(J+1)+XXB1(J))/2.0
      R2Y=(YYB1(J+1)+YYB1(J))/2.0
      IF(NOPTION.EQ.2.OR.J.NE.NBP)GOTO 11
      R2X=(XXB1(1)+XXB1(NBP))/2.
      R2Y=(YYB1(1)+YYB1(NBP))/2.
   11 CONTINUE
      Z1=R1X-R2X $ Z2=R1Y-R2Y $ Z1S=Z1**2 $ Z2S=Z2**2
      FUNCL1=ALOG(((Z1+R1*Z2)**2+R2S*Z2S)/A2)
      FUNCL3=ALOG(((Z1+R3*Z2)**2+R4S*Z2S)/A2)
      Z3=Z1+R1*Z2 $ Z4=Z1+R3*Z2
      IF(ABS(Z3).GT.1.E-6)GOTO 90
      FUNCN1=PI/2.
      GOTO 91
   90 FUNCN1=ATAN(R2*Z2/Z3)
   91 IF(ABS(Z4).GT.1.E-6)GOTO 92
      FUNCN3=PI/2.
      GOTO 93
   92 FUNCN3=ATAN(R4*Z2/Z4)
   93 CONTINUE
      IF(FUNCN1.LT.0.)FUNCN1=FUNCN1+PI
      IF(FUNCN3.LT.0.)FUNCN3=FUNCN3+PI
      FUNCR1=((Z1+R1*Z2)**2-R2S*Z2S)*(FUNCL1-3.)-4.*R2*Z2*(Z1+R1*Z2)
     +*FUNCN1
      FUNCR3=((Z1+R3*Z2)**2-R4S*Z2S)*(FUNCL3-3.)-4.*R4*Z2*(Z1+R3*Z2)
     +*FUNCN3
      FUNCS1=R2*Z2*(Z1+R1*Z2)*(FUNCL1-3.)+((Z1+R1*Z2)**2-R2S*Z2S)*FUNCN1
      FUNCS3=R4*Z2*(Z1+R3*Z2)*(FUNCL3-3.)+((Z1+R3*Z2)**2-R4S*Z2S)*FUNCN3
      WXX=CONST1*FUNCL1+CONST2*FUNCL3+CONST3*(FUNCN1-FUNCN3)
      WXY=CONST5*FUNCL1+CONST6*FUNCL3+CONST7*(FUNCN1-FUNCN3)
      WYY=CONST8*FUNCL1+CONST9*FUNCL3-CONST10*(FUNCN1-FUNCN3)
      W(I)=W(I)+(CONST1*FUNCR1+CONST2*FUNCR3+CONST3*(FUNCS1-FUNCS3))
     +*COEF1*PS(J)
      MX=-COEF2*(D11*WXX+D12*WYY+2.*D16*WXY)*PS(J)
      MY=-COEF2*(D12*WXX+D22*WYY+2.*D26*WXY)*PS(J)
      MXY=-COEF2*(D16*WXX+D26*WYY+2.*D66*WXY)*PS(J)
      BMX(I)=BMX(I)+MX
      BMY(I)=BMY(I)+MY
      BXY(I)=BXY(I)+MXY
      BMXX(I)=BMX(I)*COSP2+BMY(I)*SINP2-2.*BXY(I)*COSP*SINP
      BMYY(I)=BMX(I)*SINP2+BMY(I)*COSP2-2.*BXY(I)*SINP*COSP
   15 CONTINUE
```

```
   16 CONTINUE
      WRITE(6,600)(I,XF(I),YF(I),W(I),BMXX(I),BMYY(I),I=1,NFP)
      GOTO(49,50,51,52,53)NCOUNT
   49 PHI=15. $ NCOUNT=2
      GOTO 804
   50 PHI=30. $ NCOUNT=3
      GOTO 804
   51 PHI=45. $ NCOUNT=4
      GOTO 804
   52 PHI=60. $ NCOUNT=5
      GOTO 804
   53 STOP
          END
```

```
40,100,181,30.E6,2.0E6,0.3,8.8E4,80.,0.4
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,10*-5.,
-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,10*5.
10*-5.,-4.5,-3.5,-2.5,-1.5,-0.5,0.5,1.5,2.5,3.5,4.5,
10*5.,4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5
10*0.,10*-1.,10*0.,10*1.
10*-1.,10*0.,10*1.,10*0.
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
4.5,3.5,2.5,1.5,0.5,-0.5,-1.5,-2.5,-3.5,-4.5,
10*-4.5,10*-3.5,10*-2.5,10*-1.5,10*-0.5,
10*0.5,10*1.5,10*2.5,10*3.5,10*4.5
```

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