

AN INVESTIGATION OF THE POWER FUNCTION
FOR THE TEST OF INDEPENDENCE
IN 2×2 CONTINGENCY TABLES

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
William Leonard Harkness
1959



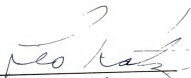
This is to certify that the
thesis entitled
An Investigation of the Power Function
for the Test of Independence
in 2×2 Contingency Tables

presented by

William Leonard Harkness

has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Statistics


Major professor

Date August 15, 1959

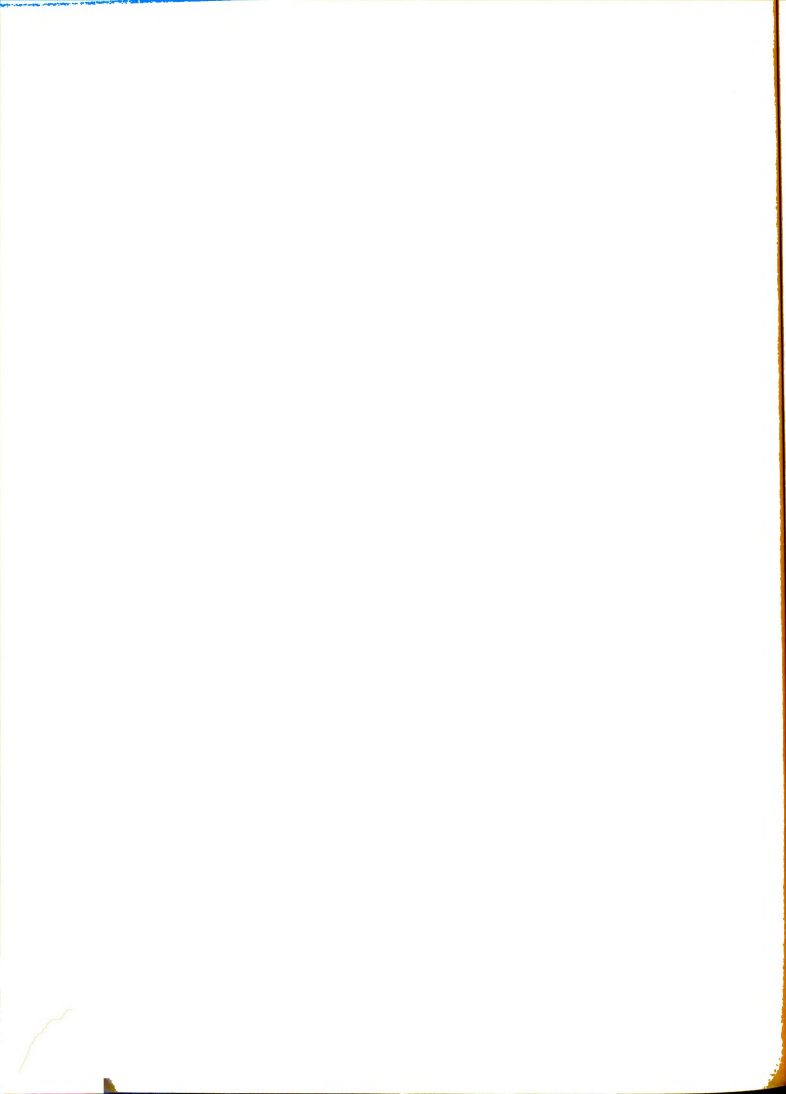
O-169





RETURNING MATERIALS:
Place in book drop to
remove this checkout from
your record. FINES will
be charged if book is
returned after the date
stamped below.

OCT 18 1999



AN INVESTIGATION OF THE POWER FUNCTION FOR THE TEST
OF INDEPENDENCE IN 2×2 CONTINGENCY TABLES

By

WILLIAM LEONARD HARKNESS

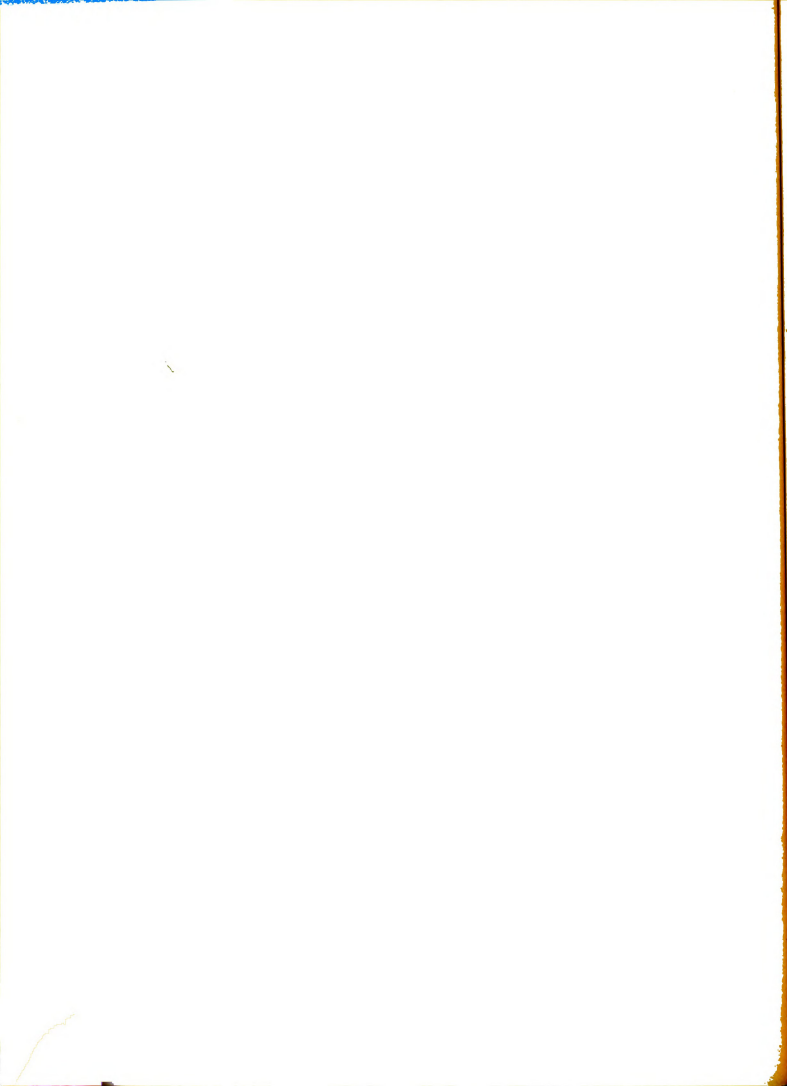
A THESIS

Submitted to the School of Graduate Studies of
Michigan State University of Agriculture and
Applied Science in partial fulfillment of
the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics

1959



014979
4/17/61

William Leonard Harkness
Candidate for the degree of
Doctor of Philosophy

Final examination, July 29, 1959, 9:00 A.M., Physics-Mathematics Building

Dissertation: An Investigation of the Power Function
for the Test of Independence in 2×2
Contingency Tables

Outline of Studies

Major subjects: Mathematical Statistics, Probability
Minor subjects: Algebra, Analysis

Biographical Items

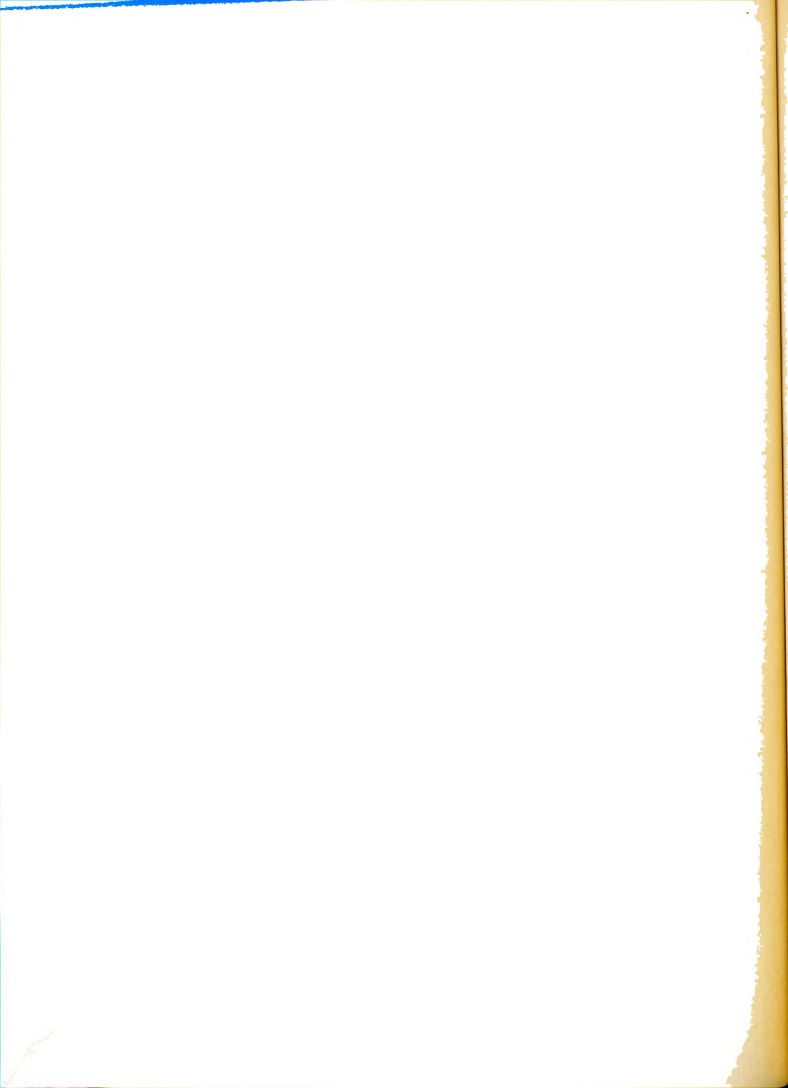
Born, June 25, 1934, Lansing, Michigan

Undergraduate Studies, Hillsdale College, 1951-52,
Michigan State College, 1952-55

Graduate Studies, Michigan State University 1955-56,
cont. 1957-59, University of Chicago, 1956-57.

Experience: Graduate Assistant, Michigan State College,
1955, Special Graduate Research Assistant
1955-56, cont. 1957-59, University Fellow,
University of Chicago, 1956-57, Mathematician,
Institute for Air Weapons Research 1957,
Temporary Instructor, Michigan State University,
1959

Member of Phi Kappa Phi, Pi Mu Epsilon, Sigma Xi,
Institute of Mathematical Statistics, American
Mathematical Society

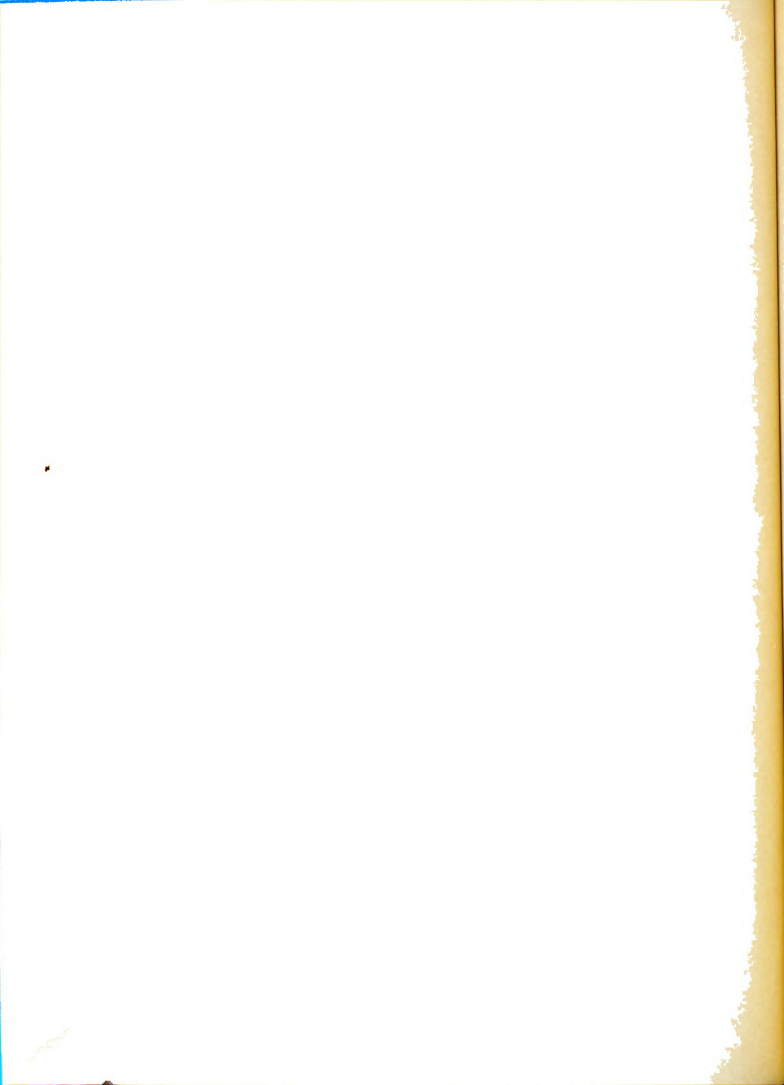


ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. Leo Katz, his major professor, for suggesting the problem and for his continuous support, endless patience, and encouragement during the completion of the problem.

The writer also deeply appreciates the financial support of the Office of Naval Research which made it possible for him to complete this investigation.

The author also is very grateful to Dr. J. F. Hannan for several invaluable suggestions, and to Mrs. Helen Spence, who set up the machine computation of the tables.



The author wishes to dedicate this thesis to his wife, Mary Lou, for her kind understanding and patience during the course of the writing of this manuscript.

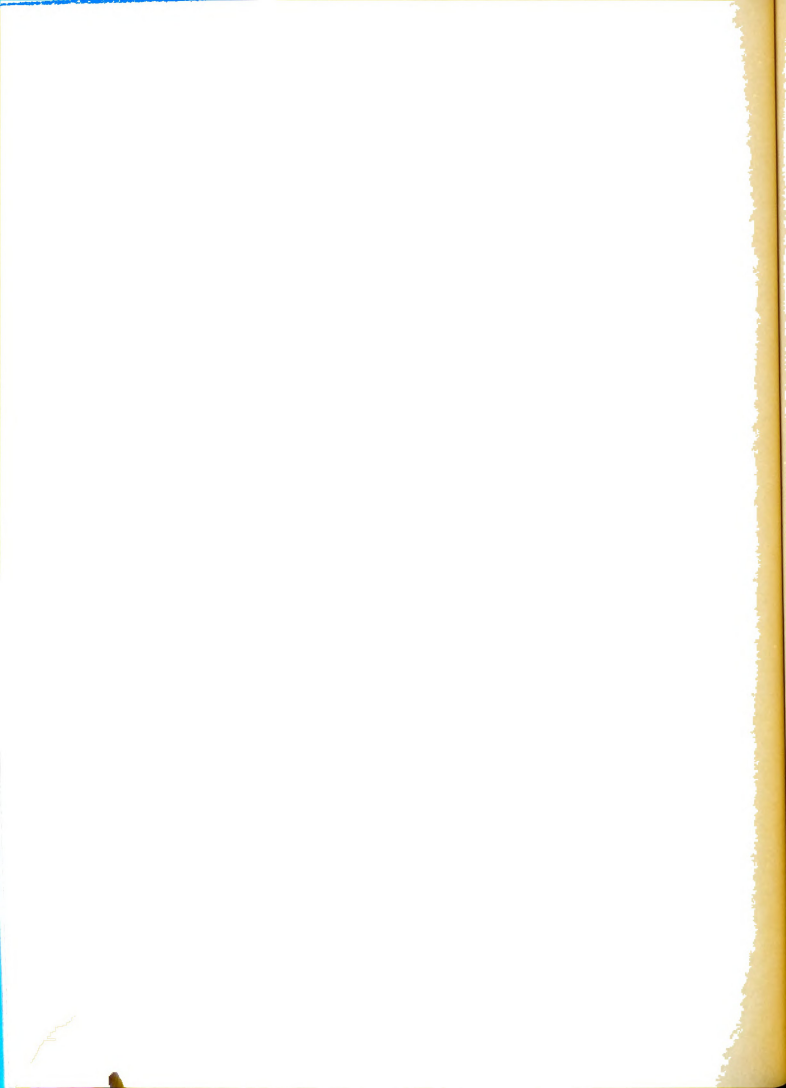
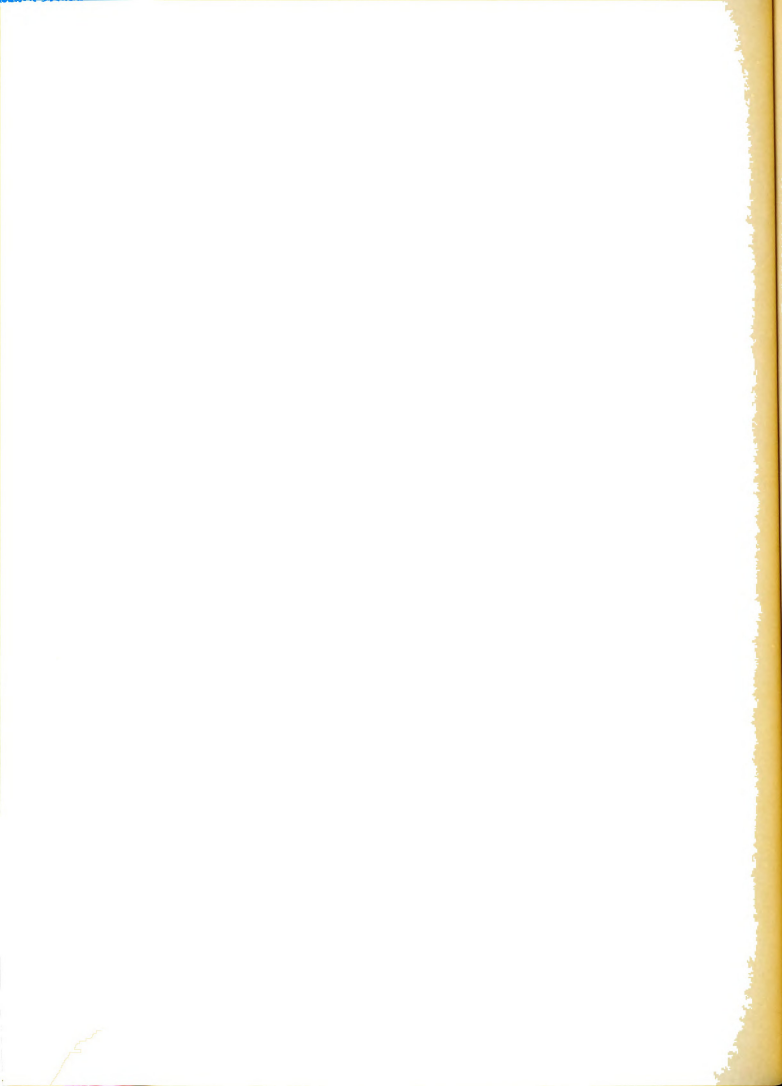
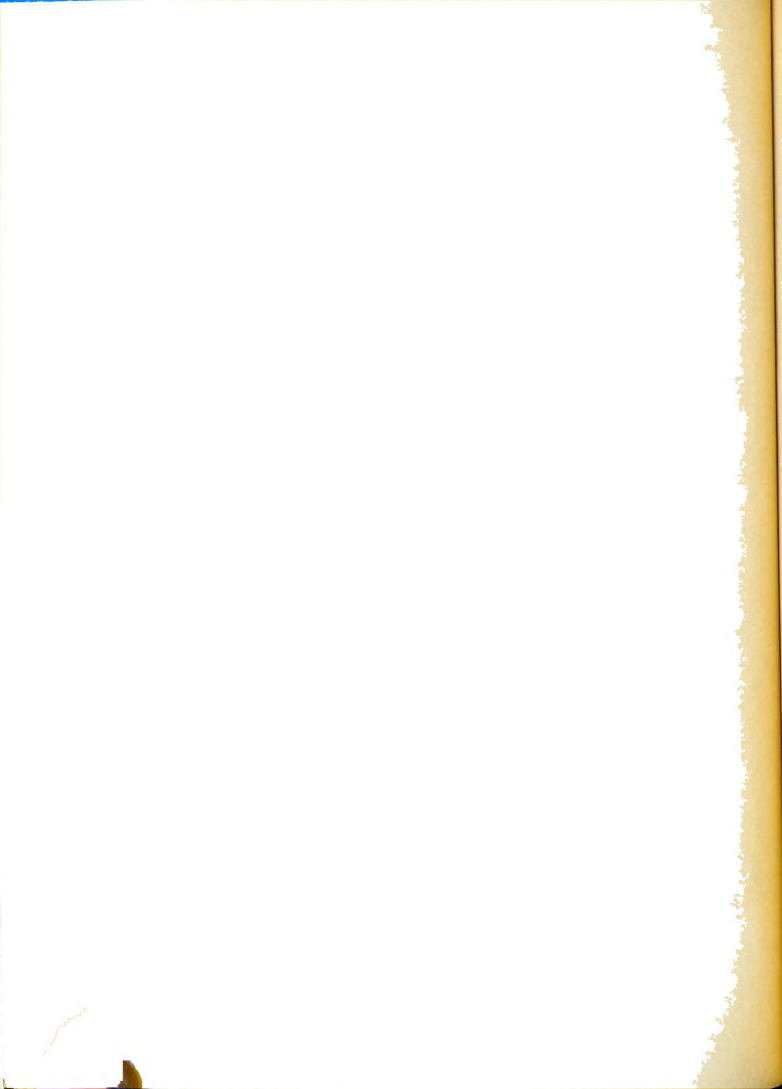


TABLE OF CONTENTS

CHAPTER 1	ANALYTICAL REVIEW OF PREVIOUS WORK	1
1.1	Abstract Models in 2 x 2 Contingency Tables	1
1.2	Probability Models	3
1.3	Statement of Hypotheses, and the Problem of Testing for Independence	6
1.4	Order of Presentation	10
1.5	The Uniformly Most Powerful Unbiased Test for One-Sided Alternatives and Two-Sided Alternatives	13
1.6	2 x 2 Independence Trial	22
1.7	2 x 2 Comparative Trial	27
1.8	The Double Dichotomy	37
1.9	Results	39
CHAPTER 2	AN INVESTIGATION OF A PROBABILITY FUNCTION ..	41
2.1	General Properties	41
2.2	Limiting Distributions and Approximations	54
CHAPTER 3	POWER FOR THE TEST OF INDEPENDENCE	71
3.1	2 x 2 Independence Trial	71
3.2	2 x 2 Comparative Trial	75
3.3	Double Dichotomy	79
3.4	Asymptotic Power	84
CHAPTER 4	COMPARISON OF POWER FUNCTIONS	92
4.1	Preliminaries	92
4.2	Comparison of Exact and Approximate Power	93
SUMMARY		97
APPENDIX A.	EXACT POWER FOR THE DOUBLE DICHOTOMY	100
A.1	Tables of $P_{10}(\lambda, P_A, P_B)$	103
A.2	Tables of $P_{20}(\lambda, P_A, P_B)$	108
A.3	Tables of $P_{30}(\lambda, P_A, P_B)$	113
APPENDIX B.	EXACT POWER FOR THE 2 x 2 COMPARATIVE TRIAL	116
B.1	Tables of $P_{10}(p_1, p_2, m_1)$	118
B.2	Tables of $P_{20}(p_1, p_2, m_1)$	122
B.3	Tables of $P_{30}(p_1, p_2, m_1)$	131



APPENDIX C. EXACT POWER IN THE 2 x 2 INDEPENDENCE TRIAL	145
C.1 Tables of $P_{10}(t \mid m_1, m_2)$	147
C.2 Tables of $P_{20}(t \mid m_1, m_2)$	150
C.3 Tables of $P_{30}(t \mid m_1, m_2)$	153
APPENDIX D. COMPARISON OF POWER FUNCTIONS	158
D.1 Tables of the Three Exact Power Functions and the Normal Approximation	159
APPENDIX E. Approximations to Power in the 2 x 2 Comparative Trial	171
E.1 Graphs	172
E.2 Tables of Sillitto's and Patnaik's Approximation, and Values of $P_n(p_1, p_2, m_1), P_n(t \mid m_1, m_2) \Big _{t = \frac{p_1 q_2}{p_2 q_1}}$	175
BIBLIOGRAPHY	177

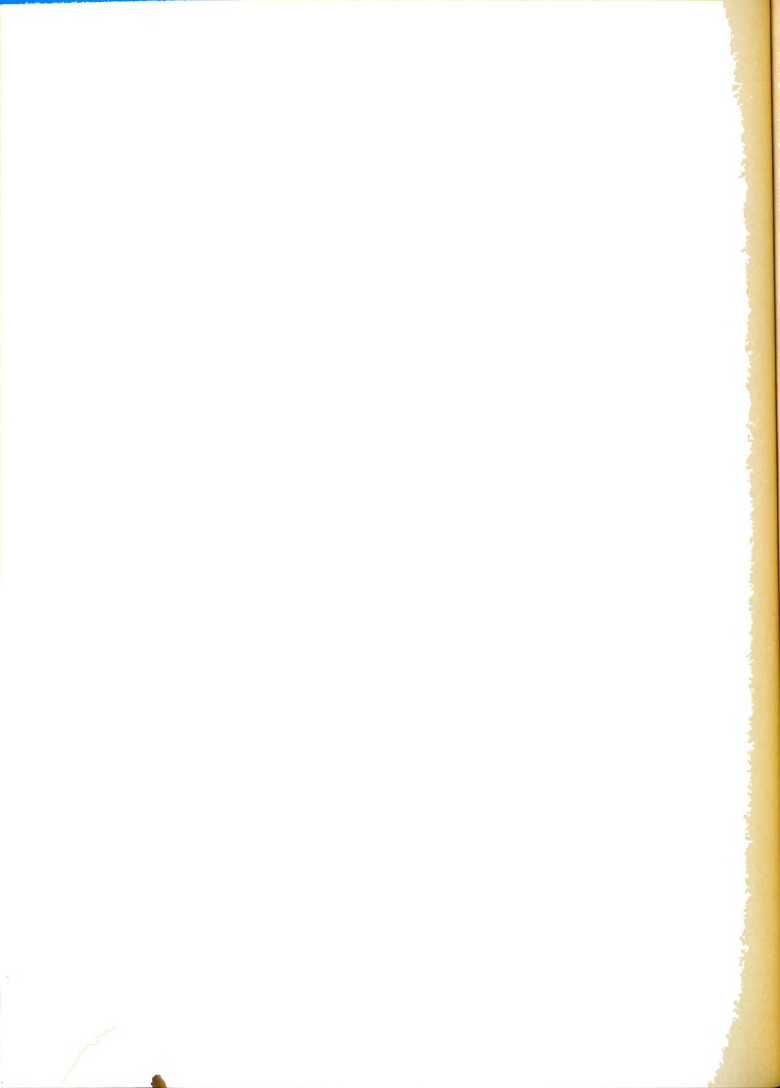


This thesis is concerned with an examination of the power function for the test of independence in 2×2 contingency tables. Three distinct types of experiments leading to the presentation of data in the form of a 2×2 table have been delineated, and several tests for independence for each have been proposed, but not much is known about the power functions of these tests.

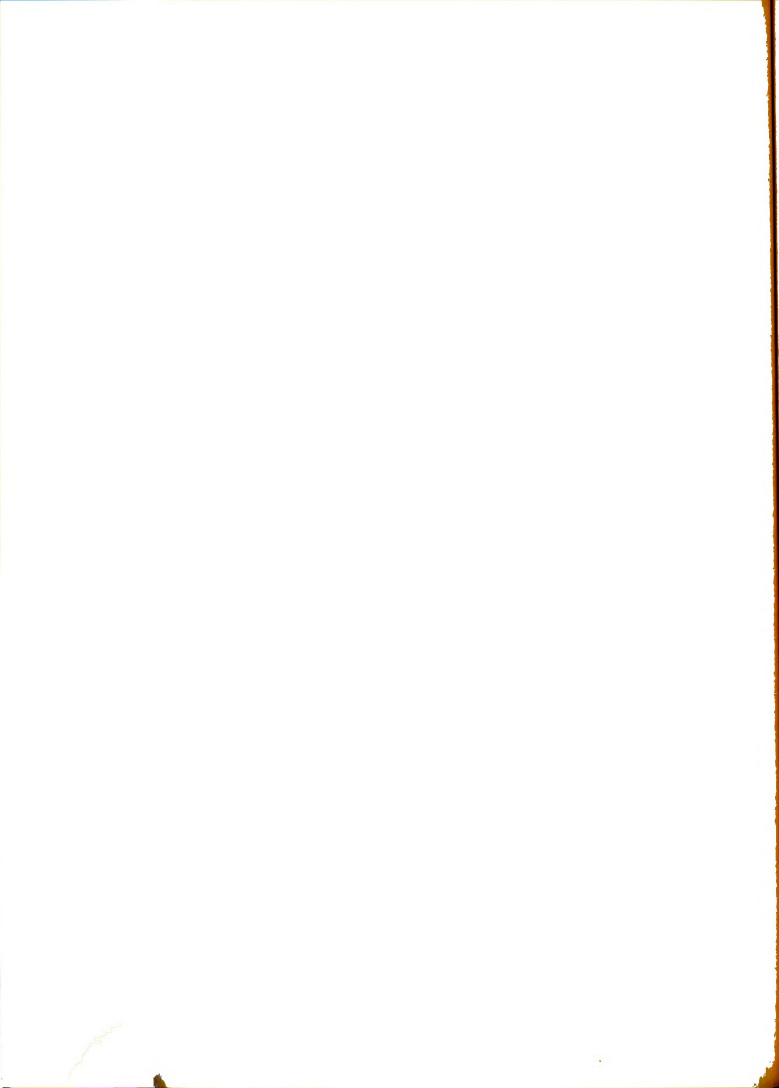
In Chapter I, which is a systematic review of previous work in 2×2 tables, the uniformly most powerful unbiased test for independence is discussed quite thoroughly. All the results and computations in this thesis are based on this test. The content of Chapter 2 is a study of the probability function $k(m_1, m_2, n; t) h(n_1 | m_1, m_2, n) t^{n_1}$, $0 < t < \infty$, where $h(n_1 | m_1, m_2, n)$ is the ordinary hypergeometric function, and $k(m_1, m_2, n; t)$ is the reciprocal of the sum of $h(n_1 | m_1, m_2, n) t^{n_1}$ over all possible values of n_1 ; some asymptotic properties are included in this study.

The exact power function for the test of independence in each of the three cases is given in Chapter 3. In this chapter, the three power functions are related to one another, and asymptotic power is investigated. The asymptotic power function for the χ^2 -test of independence is given here. Also, several approximations to power are proposed in Chapter 3.

Finally, Chapter 4 serves to unify the results obtained in previous chapters, and some of the approximations proposed



in Chapter 3 are compared with the exact power. Rather extensive tables of exact power for each of the three cases are in the appendices. These exact computations provide the means for evaluation the adequacy of the various approximations.

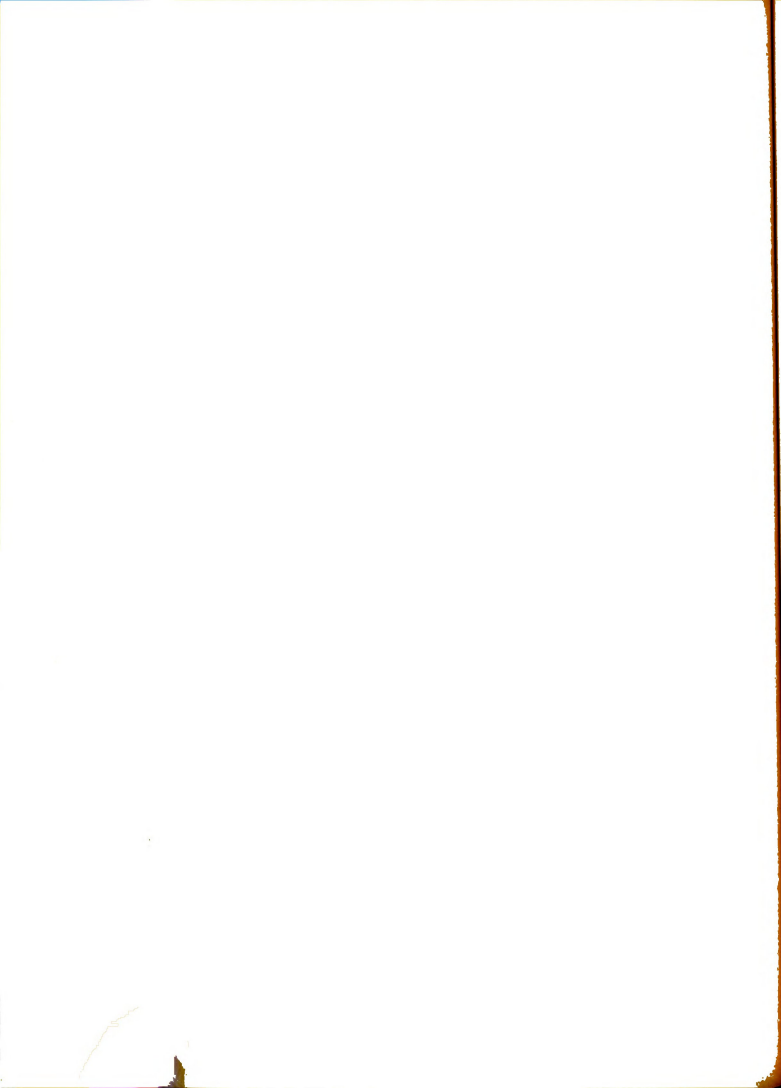


Chapter 1. Analytical Review of Previous Work

1.1 Abstract Models in 2 x 2 Contingency Tables.

One of the classical problems in mathematical statistics is that of testing for independence in 2 x 2 contingency tables. Karl Pearson's [17] 1900 paper in which he proposed the χ^2 test for independence, appears to be the first on the subject. Since that time, many other writers have devoted themselves to the same problem. For the most part, their attention has been directed at the case where the sample size is moderately large, and consequently, for this case, the problem can be said to be solved for all practical purposes. The problem of testing for independence in small samples, however, has led to considerable controversy. Several tests have been proposed, but basic disagreement remains. This is partially due to the fact that there are three distinct experimental situations which lead to the presentation of data in the form of 2 x 2 contingency tables. Abstractly, these three experiments may be described in the following manner:

I. A total of n similar balls, m_1 marked A_1 and $n - m_1$ marked A_2 , are placed in an urn, then withdrawn randomly in order. They are then placed in order in a row of n cells, of which m_2 have been labeled B_1 , $n - m_2$ labeled B_2 . The result of the



experiment is presented in Table I, where n_1 is the observed number of balls marked A_1 in receptables labeled B_1 . Both sets of marginal totals are fixed.

TABLE I

	B_1	B_2	
A_1	n_1	n_3	m_1
A_2	n_2	n_4	$n - m_1$
	m_2	$n - m_2$	n

II. From two urns, A_1 and A_2 , each containing a large number of balls marked B_1 and B_2 samples of m_1 and $n - m_1$ are taken. It is observed that m_2 of the balls are labeled B_1 , and $n - m_2$ are labeled B_2 . It is assumed that the proportion of balls marked B_1 in urn A_1 is p_1 , $i = 1, 2$. With this type of experiment, one set of marginal totals is fixed in Table I, namely, m_1 and $n - m_1$. Table II gives the relevant probabilities of occurrence of the balls with specified markings.

TABLE II

	B_1	B_2
A_1	p_1	$1 - p_1$
A_2	p_2	$1 - p_2$

III. A total of n similar balls is randomly selected from an urn containing a large number of balls, each ball labeled A_1 or A_2 and also labeled B_1 or B_2 . An observed result of the experiment is represented in the form of Table I, where none



of the marginal totals are fixed. It is assumed that the probabilities of occurrence of the various markings of the balls is given by Table III, together with the marginal sums.

TABLE III

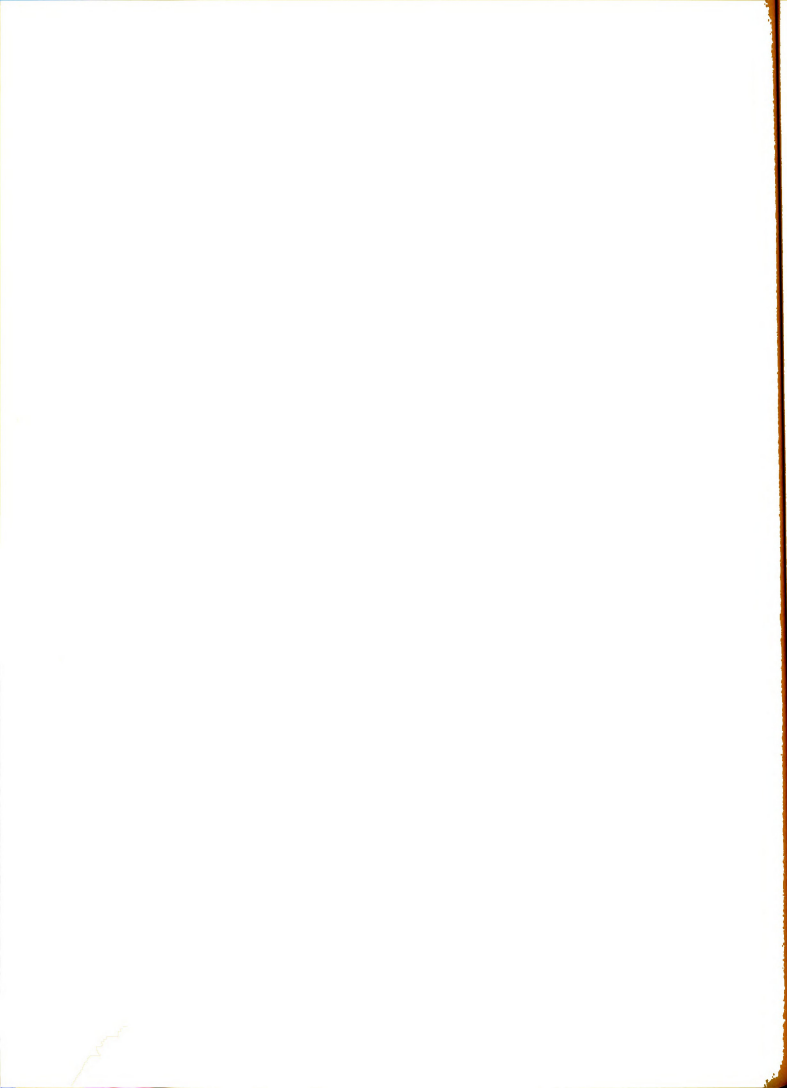
	B_1	B_2	
A_1	π_1	π_3	P_A
A_2	π_2	π_4	$1 - P_A$
	P_B	$1 - P_B$	1

Following G. A. Barnard's [3] nomenclature, we will call I the 2×2 "independence trial", II the 2×2 "comparative trial", and III the double dichotomy.

1.2. Probability Models.

Let us now consider the appropriate probability model for each of the three experiments described above. It will be seen that these probability models have a natural "hierarchical order" in that the probability model for the 2×2 independence trial and the 2×2 comparative trial are obtained as conditional probabilities of the probability distribution in the double dichotomy. Considering first, then, the double dichotomy, the probability of observing the sample point (n_1, n_2, n_3, n_4) is, by the multinomial probability law,

$$(1.2.1) \quad \Pr \{n_1, n_2, n_3, n_4\} = \frac{n!}{n_1! n_2! n_3! n_4!} \pi_1^{n_1} \pi_2^{n_2} \pi_3^{n_3} \pi_4^{n_4}$$



If we replace π_1 by $\lambda P_A P_B$, π_3 by $P_A(1 - \lambda P_B)$
 π_2 by $P_B(1 - \lambda P_A)$ π_4 by $1 - P_A - P_B + \lambda P_A P_B$

$$\text{where } \max \left[0, \frac{P_A + P_B - 1}{P_A P_B} \right] < \lambda < \min \left[\frac{1}{P_A}, \frac{1}{P_B} \right],$$

and replace n_2 by $(m_2 - n_1)$, n_3 by $(m_1 - n_1)$

and n_4 by $(n - m_1 - m_2 + n_1)$

then (1.2.1) may be rewritten as

$$(1.2.2) \quad \Pr \{n_1, m_1, m_2 \mid n, \lambda\} \\
= b(m_1; n, P_A) b(n_1; m_1, \lambda P_B) b\left(m_2 - n_1; n - m_1, \frac{P_B(1 - \lambda P_A)}{1 - P_A}\right)$$

where $b(x; n, p)$ is the usual binomial probability.

Thus, (1.2.2) gives the probability of the sample point (n_1, m_1, m_2) , given n, λ, P_A, P_B . Computing the conditional probability of (n_1, m_2) , given m_1, n, λ, P_A , and P_B , one obtains

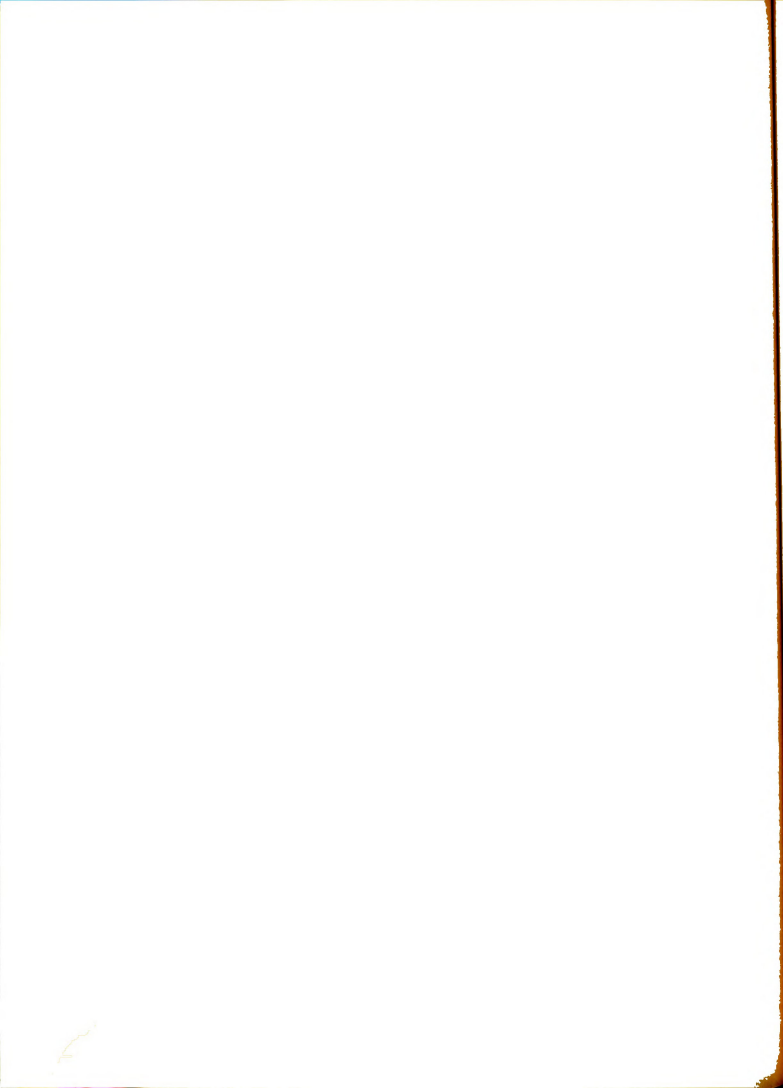
$$(1.2.3) \quad \Pr \{n_1, m_2 \mid m_1, n, \lambda\} \\
= b(n_1; m_1, \lambda P_B) \cdot b\left(m_2 - n_1; n - m_1, \frac{P_B(1 - \lambda P_A)}{1 - P_A}\right)$$

Letting $p_1 = \lambda P_B$

$$p_2 = \frac{P_B(1 - \lambda P_A)}{1 - P_A},$$

(1.2.3) yields the probability model for the 2×2 comparative trial.

When the 2×2 comparative trial is discussed without



reference to the double dichotomy, $\Pr \{n_1, m_2 \mid p_1, p_2, m_1, n\}$ will be used to denote the probability function rather than $\Pr \{n_1, m_2 \mid m_1, n, \lambda\}$.

Summing (1.2.3) over all possible values of n_1 gives the marginal probability for m_2 as

$$(1.2.4) \quad \Pr \{m_2 \mid m_1, n, \lambda\} \\ = \sum_{n_1} b(n_1; m_1, \lambda P_B) b\left(m_2 - n_1; n - m_1, \frac{P_B(1 - \lambda P_A)}{1 - P_A}\right).$$

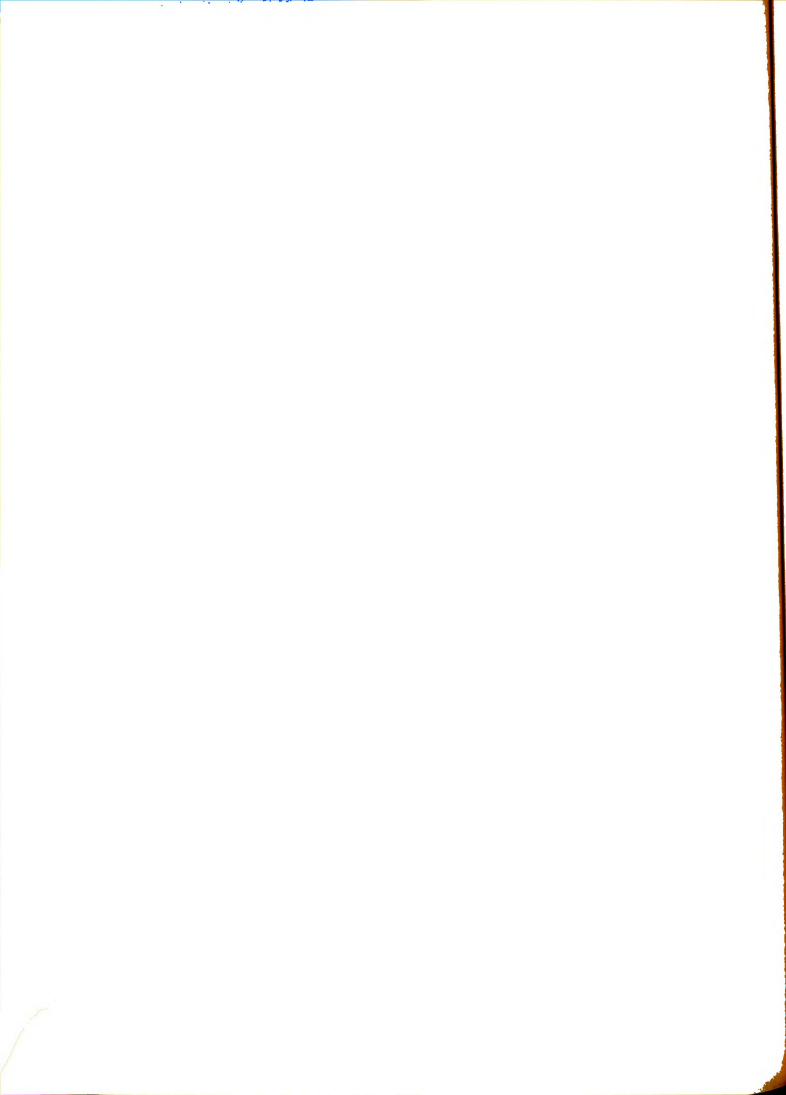
$\Pr \{m_2 \mid m_1, n, \lambda\}$ is seen to be the convolution of two binomial distributions. Hence, from (1.2.3) and (1.2.4), we obtain the conditional probability

$$(1.2.5) \quad \Pr \{n_1 \mid m_1, m_2, n, \lambda\} = \frac{\Pr \{n_1, m_2 \mid m_1, n, \lambda\}}{\Pr \{m_2 \mid m_1, n, \lambda\}} \\ = \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum_j \binom{m_1}{j} \binom{n - m_1}{m_2 - j} t^j}$$

$$\text{where } t = \frac{\lambda(1 - P_A - P_B) + \lambda P_A P_B}{(1 - \lambda P_A)(1 - \lambda P_B)} = \frac{p_1 q_2}{p_2 q_1},$$

$$q_i = 1 - p_i, \quad i = 1, 2.$$

$$0 < t < \infty.$$



It is thus natural to take (1.2.5) as the appropriate probability model for the 2×2 independence trial. Note that if $t = 1$, (1.2.5) will become

$$(1.2.6) \quad \Pr \{n_1 \mid m_1, m_2, n\} = \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} \div \binom{n}{m_2} .$$

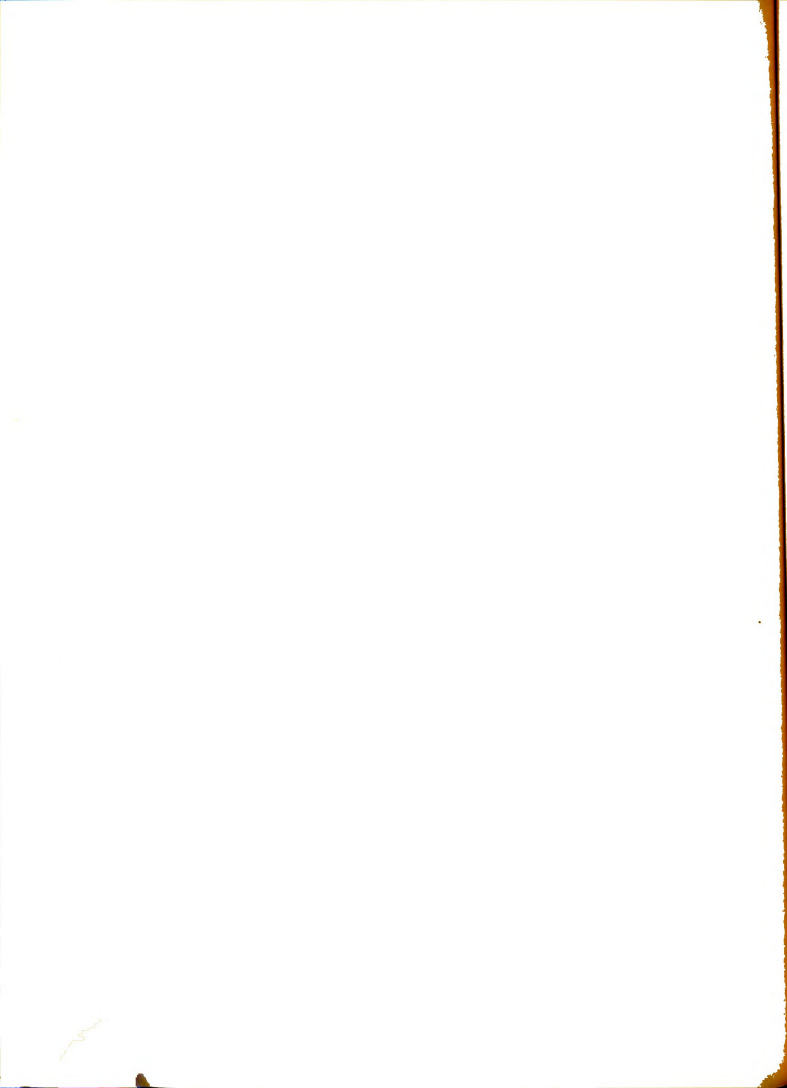
The distribution (1.2.5) is not new (see E. Sverdrup [24], E. Lehmann [11] and L. Katz [9]). However, no systematic study of its properties has been carried out. Because of its importance in investigating the power of the test of independence in 2×2 contingency tables, Chapter 2 has been devoted to a study of this distribution.

$$\text{The probability function} \quad \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}$$

was obtained here conditionally, but as a probability function it need not be regarded as depending on quantities λ, P_A, P_B , and when it is desired to indicate that it does not, the notation $\Pr \{n_1 \mid m_1, m_2, n; t\}$ will be used.

Section 1.3. Statement of Hypotheses, and the Problem of Testing for Independence.

In each of the three experimental situations described in 1.1, statisticians are interested in carrying out a test of a statistical hypothesis. More specifically, they are interested in testing for independence. For the double



dichotomy, the hypothesis of independence takes the form

$$H_0: \pi_1 = P_A P_B,$$

while for the 2×2 comparative trial, one has

$$H_0: p_1 = p_2.$$

In the 2×2 independence trial, the null hypothesis is that the markings A_1 or A_2 are independent of the labelings B_1 or B_2 . If the hypothesis is correct, the distribution of n_1 (i.e. the number marked A_1 and B_1) given m_1 and m_2 , is the hypergeometric distribution (1.2.6), corresponding to $t = 1$. Since $\pi_1 = P_A P_B \iff \lambda = 1$, $p_1 = p_2 \iff \lambda = 1$, and $t = 1 \iff \lambda = 1$, we may specify independence by $H_0: \lambda = 1$, for each case.

If $\lambda = 1$, (1.2.2) and (1.2.3) reduce to

$$\begin{aligned} (1.3.1) \quad \Pr \{n_1, m_1, m_2 \mid n\} \\ = b(m_1; n, P_A) b(m_2; n, P_B) h(n_1 \mid m_1, m_2, n) \end{aligned}$$

where $h(n_1 \mid m_1, m_2, n)$ is the hypergeometric probability function.

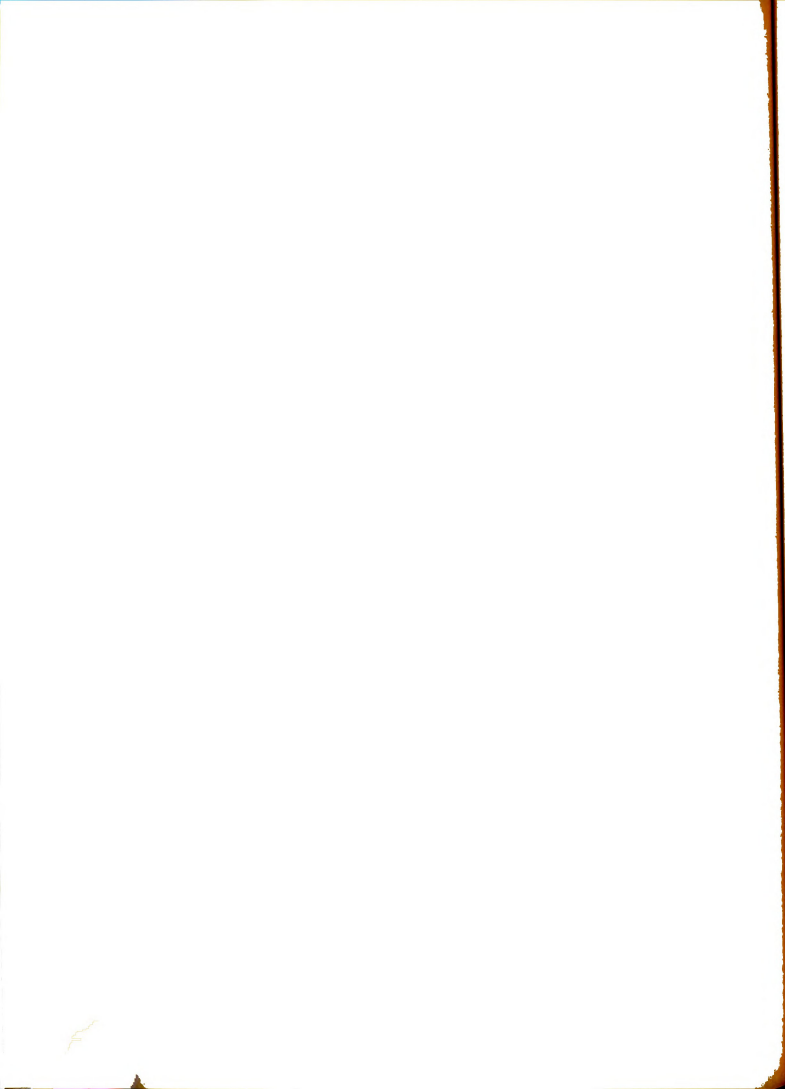
$$\begin{aligned} (1.3.2) \quad \Pr \{n_1, m_2 \mid m_1, n\} \\ = b(m_2; n, P_B) h(n_1 \mid m_1, m_2, n) \end{aligned}$$

and the conditional distribution of n_1 , given m_1, m_2, P_A, P_B , and $\lambda = 1$, given by (1.2.6), holds for all three cases.

Any alternative hypothesis may be expressed as

$$H_1: \lambda \neq 1,$$

for any of the three cases, so that H_1 is composite.



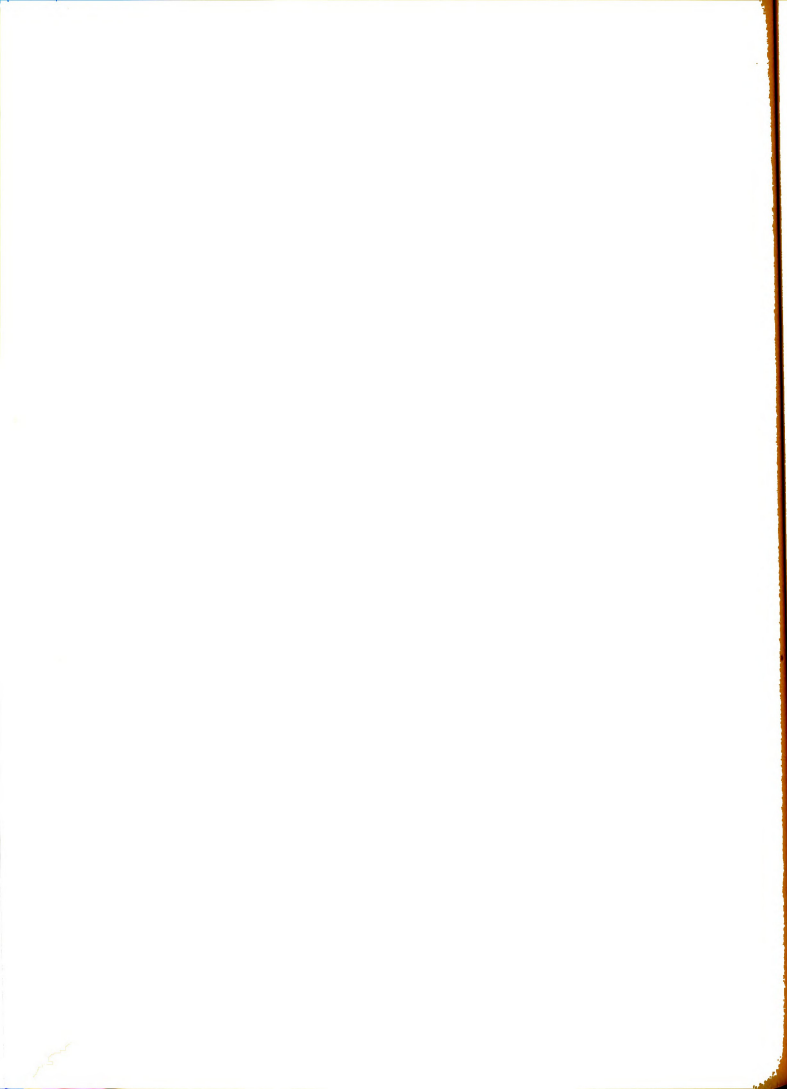
In terms of λ , H_0 is simple. The nuisance parameters P_A and P_B make it composite.

Ideally, one would hope to find a uniformly most powerful test for the class of alternative hypotheses. The case for one-tailed tests has been disposed of for all practical purposes by Tocher [25], Sverdrup [24], and Katz [9]. They showed that the same test procedure should be used in each of the three situations, and that the test, which is a slight modification of Fisher's classic test, is most powerful, in the sense of Neyman and Pearson. This test procedure will be completely described in Section 1.5.

For two-tailed tests, no uniformly most powerful test exists. Several tests have been proposed, for each of the three cases, with disagreements as to the appropriate one. The main point about which the controversy hinges is whether the marginal totals are intrinsic or nuisance parameters.

One school of thought, led by Fisher, maintains that the marginal totals per se furnish no relevant information as to the probabilities of the observed frequencies, and hence are nuisance parameters. Hence, Fisher would advocate a conditional test, given the marginal totals, so that each of the three cases would be handled in a similar way.

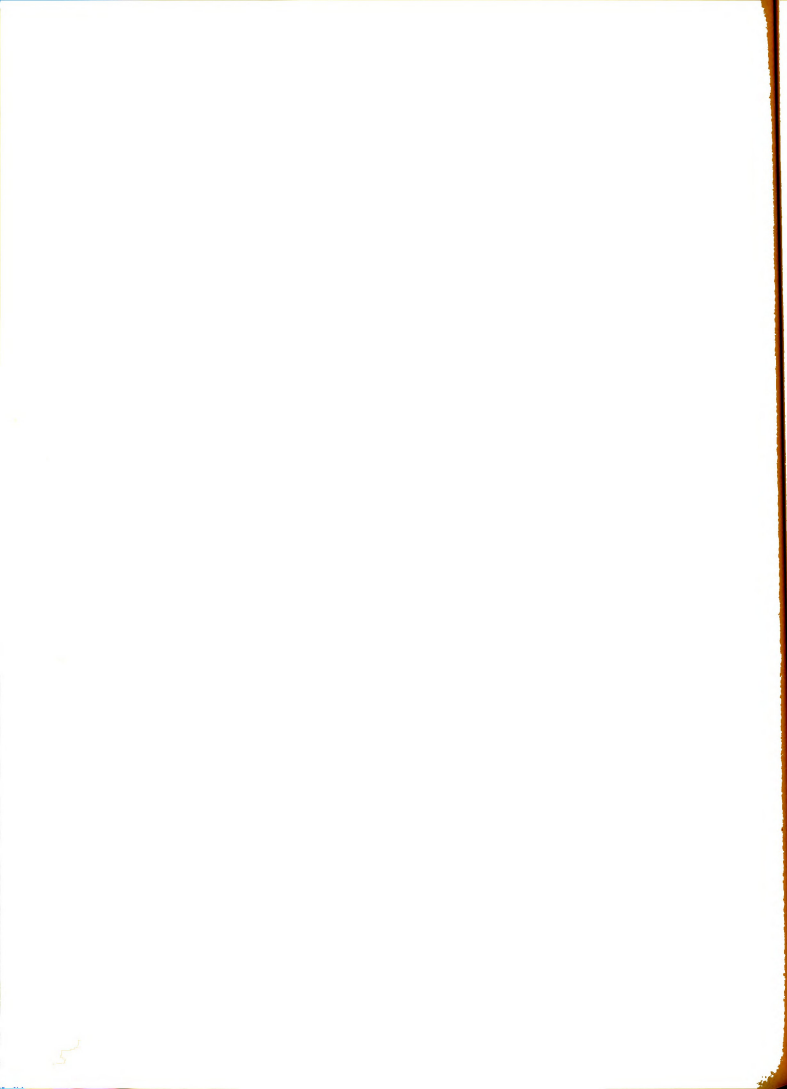
Another view is taken by E. S. Pearson [16], among others, who believes "it is an artificial procedure to restrict the experimental probability set to a linear set," as Fisher does. Barnard [3], 1947, page 136, expressed the opinion that "significance tests for the 2×2 independence trial will not



necessarily be appropriate for the 2×2 comparative trial," as well as the double dichotomy. He constructed an alternative test for the 2×2 comparative trial claiming this test had greater power than Fisher's "exact" test. However, Barnard [2], in 1949, wrote

"On the 2×2 table I arrived at a test, the CSM test, which seemed to be considerably more powerful than the "exact" test of Professor R. A. Fisher, by taking as a reference set a class of results different from that considered by Professor Fisher. This led to some controversy with Professor Fisher, in which he maintained that the Neyman-Pearson notion, that the reference set involved in a test of significance consists of the set of all results which could have arisen in the given circumstances, was ill-conceived. In private correspondence following on this controversy, Professor Fisher drew my attention to a particular case where there did seem to be some difficulty in using the Neyman-Pearson approach. I discussed this case in another paper (Barnard, 1947b), in which I attempted to show how the Neyman-Pearson approach could be extended to cover such a case. However, I was not myself satisfied with the position, and further meditation has led me to think that Professor Fisher was right after all."

When uniformly most powerful tests do not exist, various procedures are available. A very commonly used technique is

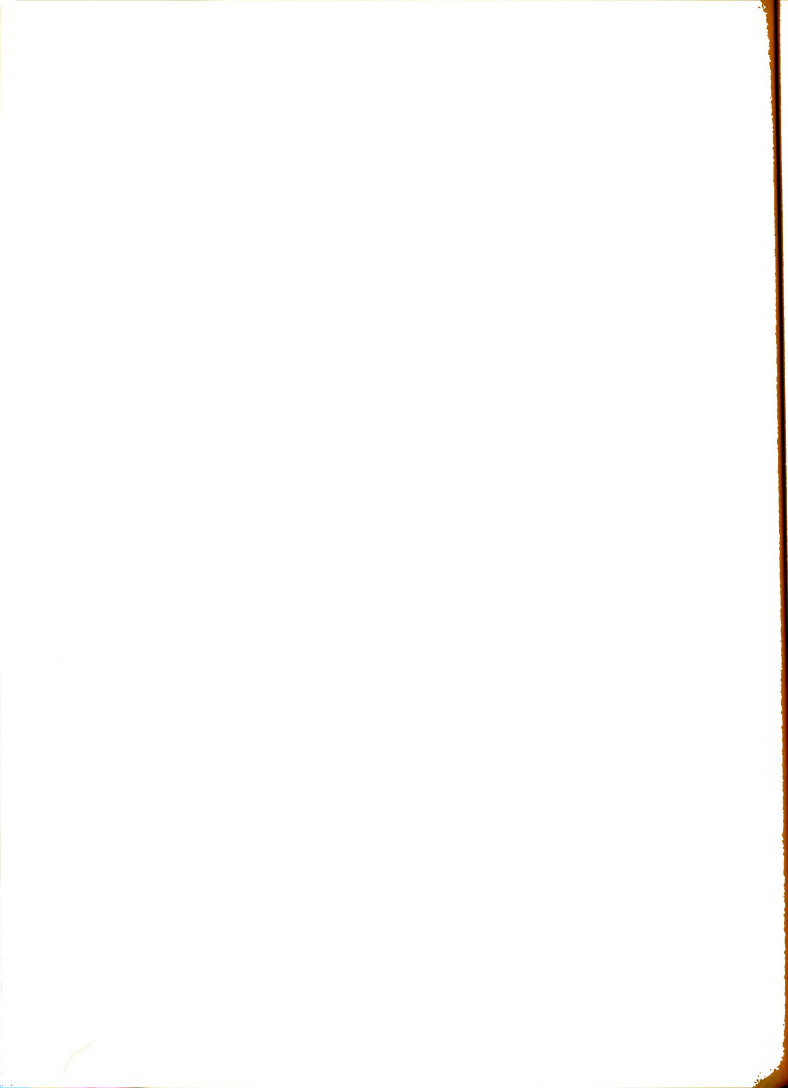


to restrict the class of possible tests to a smaller class of tests, with the hope of finding in this smaller class of tests one which is uniformly most powerful. For example, one might require that the test be a "similar" test, or that it be "unbiased", or that it be an "invariant" test. There are circumstances in which making such restrictions would be quite reasonable, and then again, some statisticians might feel there is good reason why such tests should not be made. In restricting ourselves to a similar test of size α we are requiring that the test make incorrect decisions at not more than the full allowable rate for all hypotheses under H_0 . The principle of unbiasedness seems to be a very reasonable one - requiring that a test should accept the alternative H_1 more frequently when to accept is the correct decision than when it is incorrect.

In the two-tailed tests which have been proposed for the test of independence, generally the class of tests has not been restricted, and more or less "subjective" criteria have been used to judge the efficacy of the test. Clearly, the merits of a test should be judged from its power function, and not by intricate intuitive processes.

1.4 Order of Presentation

While many people have written on 2×2 contingency tables, for the most part they have considered the three distinct types of experiments as separate problems, except for large sample sizes, where the χ^2 test becomes applicable in all three situations. As a result, there is an abundance



of papers on the subject, but no very unified treatment of the whole area. Thus, as a preliminary, in the remainder of Chapter I a comprehensive and systematic survey of past work is included. This survey reveals that there are still some unsolved problems left. It shows that whereas many tests for independence have been proposed, very little has been done on the power function. This is particularly true for the 2×2 independence trial and the double dichotomy. Our main concern, therefore, is an examination of the power function for the test of independence.

In section 1.5 of this chapter, the test on which we will be basing computations of power for all three cases is described. This test is the uniformly most powerful unbiased test, first proposed by Katz in 1942, and discussed in some detail by Sverdrup in 1953, and Tocher in 1950. Sections 1.6 and 1.7 and 1.8 are devoted specifically to past treatment of the 2×2 independence trial, 2×2 comparative trial and the double dichotomy.

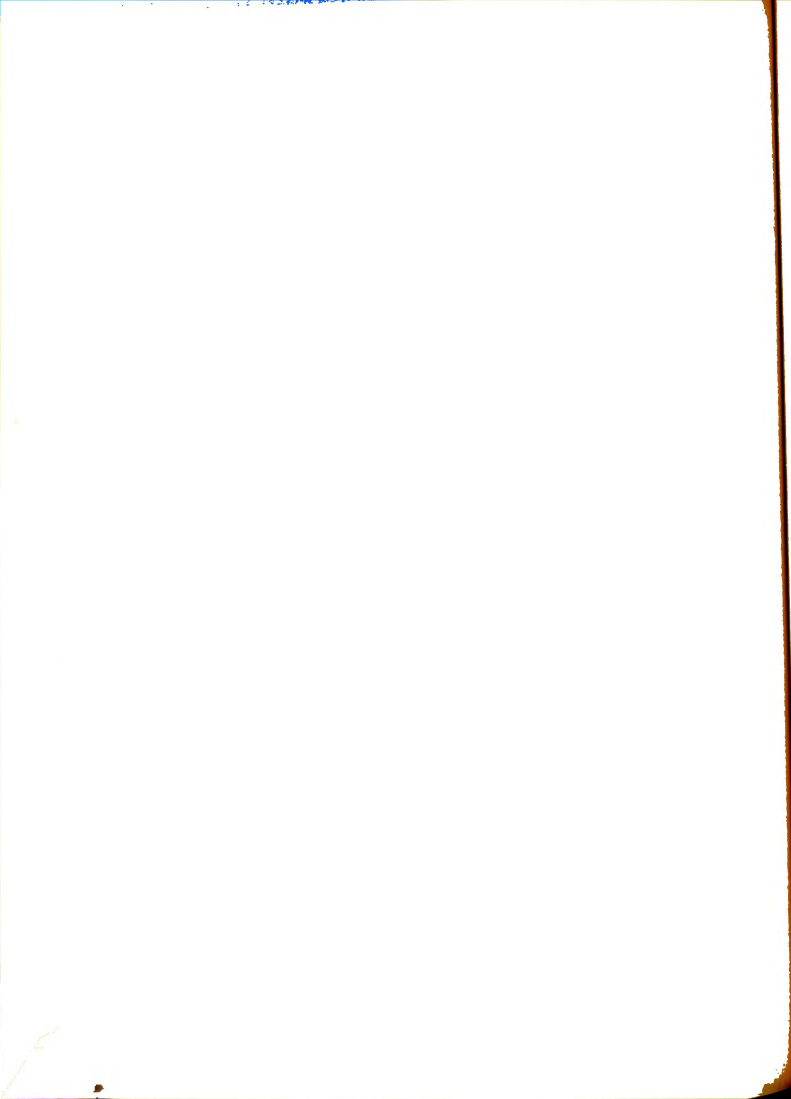
In order to evaluate power for the above mentioned unbiased test of independence, some properties of the conditional distribution given by (1.2.5) are described in Chapter 2, including moments, asymptotic distributions, and several approximations. It will be shown in section 2.2 that the asymptotic distribution of (1.2.5) is normal, and this result will be the foundation and key tool in the study of the power function for the test of independence. The derivation of the asymptotic distribution is patterned after



Feller's normal approximation for the binomial distribution, and the assumptions made are essentially the same as for the binomial case.

In Chapter 3, the exact power function for each of the Problems I, II, and III is given. The order of presentation begins in section 3.1 with the 2×2 independence trial, and proceeds naturally to the 2×2 comparative trial and the double dichotomy in sections 3.2, and 3.3 respectively. Several approximations to the power are given in these sections, using in the case of the 2×2 comparative trial an approximation given by Sillitto [21]. Theorem 3.4.B in Chapter 3 shows that, asymptotically, there is no difference in the power functions, for "corresponding" alternative hypotheses and suitable choice of marginal totals in the 2×2 independence trial and the 2×2 comparative trial. As a consequence of this theorem, an additional approximation to power for the 2×2 comparative trial and the double dichotomy is proposed. Also in 3.4. the asymptotic power function for the χ^2 -test of independence is studied.

Finally, Chapter 4 serves to unify the results obtained in Chapter 3, and some of the approximations proposed in Chapter 3 are compared with the exact power. Rather extensive tables of exact power for each of the three cases are in the appendix. These exact computations provide the means for evaluating the adequacy of the various approximations.



1.5 The Uniformly Most Powerful Unbiased Test for One-Sided Alternatives and Two-Sided Alternatives.

As noted on page 11, Katz [9], Tocher [25], and Sverdrup [24] are responsible for the development of the tests to be described here.

Katz, in 1942, assumed that any alternative distribution to the null distribution for the 2 x 2 independence trial could be taken as that given in (1.2.5), i.e.

$$(1.5.1) \Pr \{ n_1 \mid m_1, m_2, n; \lambda \} = \binom{m_1}{n_1} \binom{n-m_1}{m_2-n_1} t^{n_1} \div \sum_j \binom{m_1}{j} \binom{n-m_1}{m_2-j} t^j$$

$$\text{where } t = \frac{\lambda(1 - P_A - P_B + \lambda P_A P_B)}{(1 - \lambda P_A)(1 - \lambda P_B)}$$

Thus, the test for independence amounts to testing

$$H_0: t = 1$$

$$\text{vs. } H_1: t \neq 1$$

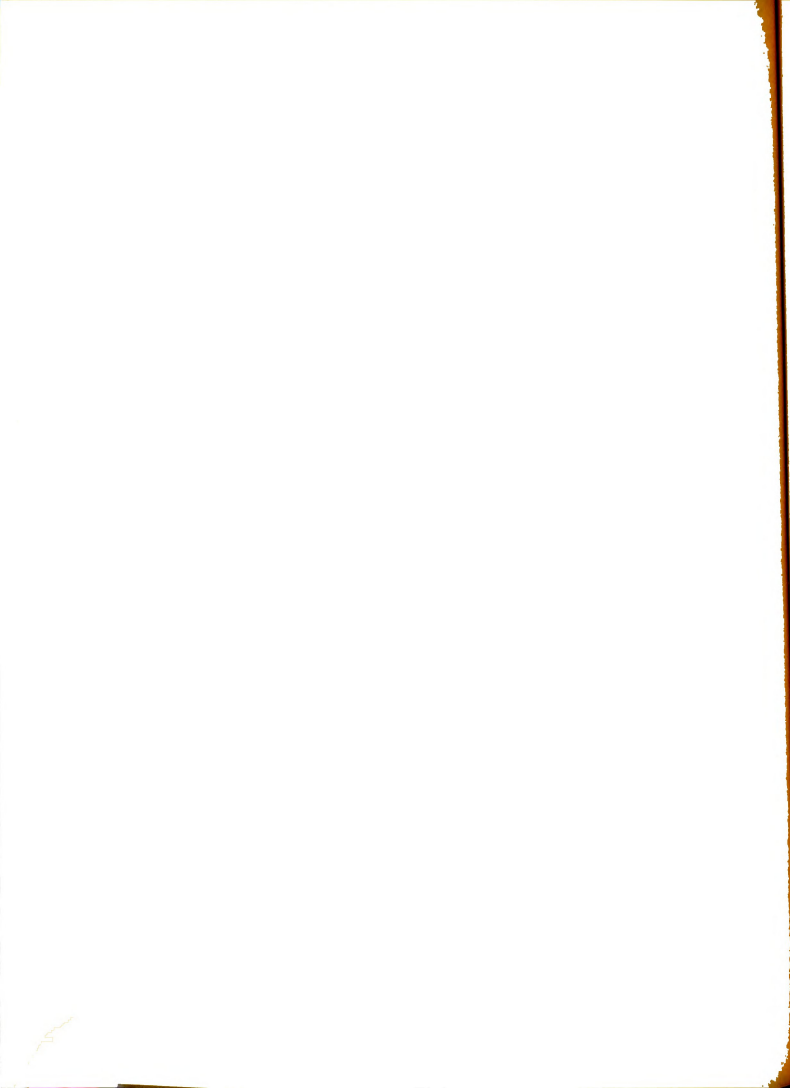
or, equivalently, to $H_0: \lambda = 1$ vs. $\lambda \neq 1$.

Considering first the special case

$$H_0: t = 1$$

$$\text{vs. } H_1: t = t_0 ; t_0 \neq 1$$

and applying the Neyman-Pearson lemma for testing a simple hypothesis against a simple alternative, Katz finds that H_0 should be rejected for all values of n_1 such that $t_0^{n_1} \geq k$, where k is some fixed constant chosen so that the test has size α , or



$$(1.5.2) \quad n_1 \log t_0 > \log k.$$

For all $t < 1$, the inequality (1.5.2) is satisfied for all $n_1 \leq a$; for all $t > 1$, by all $n_1 \geq b$. Thus, if we wish to test the hypothesis $t = 1$ only with respect to one-sided alternative hypotheses $t < 1$ (or $t > 1$) the uniformly most powerful critical region is a tail of the conditional distribution (1.2.6), $n_1 \leq a$ (or $n_1 \geq b$).

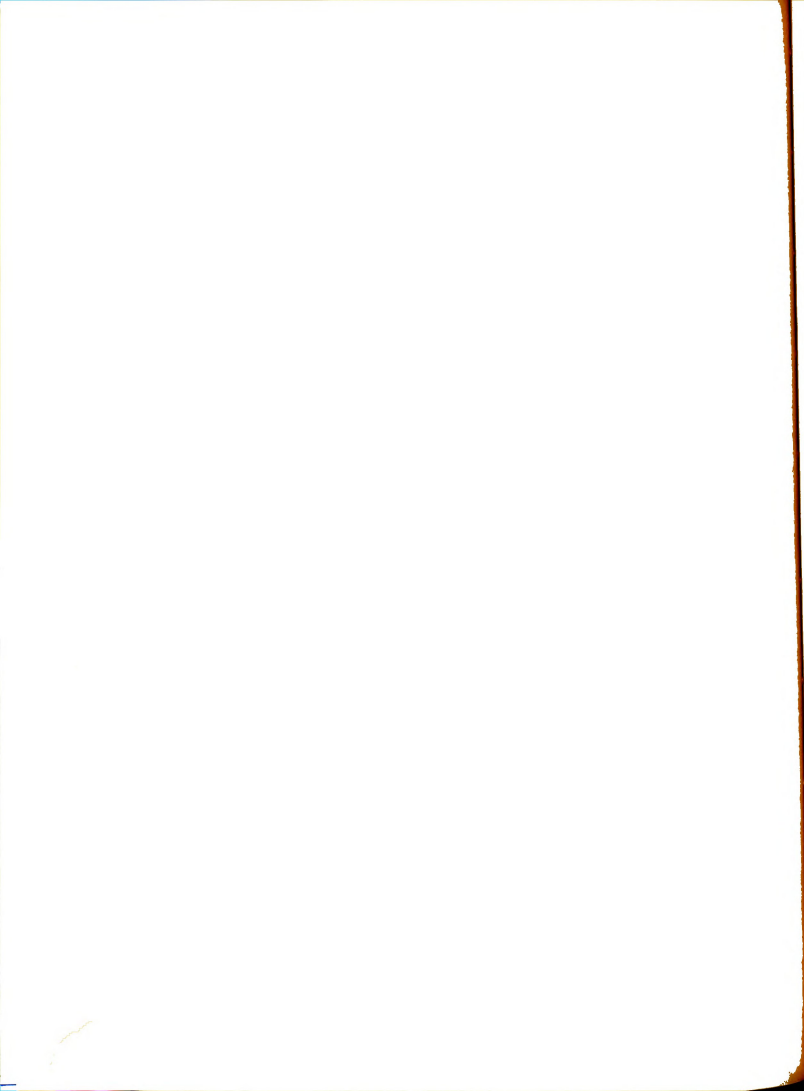
For testing alternatives $t \neq 1$, no uniformly most powerful test exists. Katz, therefore, restricts the class of tests to those which are unbiased, in the expectation that within this smaller class of tests a uniformly most powerful test exists. Katz notes that a necessary condition for unbiasedness is that the power function have a minimum at $t = 1$, or that

$$(1.5.3) \quad \frac{\partial}{\partial t} \left[\frac{\sum_{n_1 \in w(m_1, m_2)} \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum_j \binom{m_1}{j} \binom{n - m_1}{m_2 - j} t^j} \right]_{t=1} = 0$$

where $w(m_1, m_2)$ is the critical region for fixed m_1, m_2 , and n .

A little algebra reduces the necessary condition (1.5.3) for unbiasedness to the form

$$(1.5.4) \quad \left[\sum_j \Pr \{j | m_1, m_2, n\} \right] \left[\sum_{n_1 \in w(m_1, m_2)} n_1 \Pr \{n_1 | m_1, m_2, n\} \right] \\ = \left[\sum_j j \Pr \{j | m_1, m_2, n\} \right] \left[\sum_{n_1 \in w(m_1, m_2)} \Pr \{n_1 | m_1, m_2, n\} \right]$$



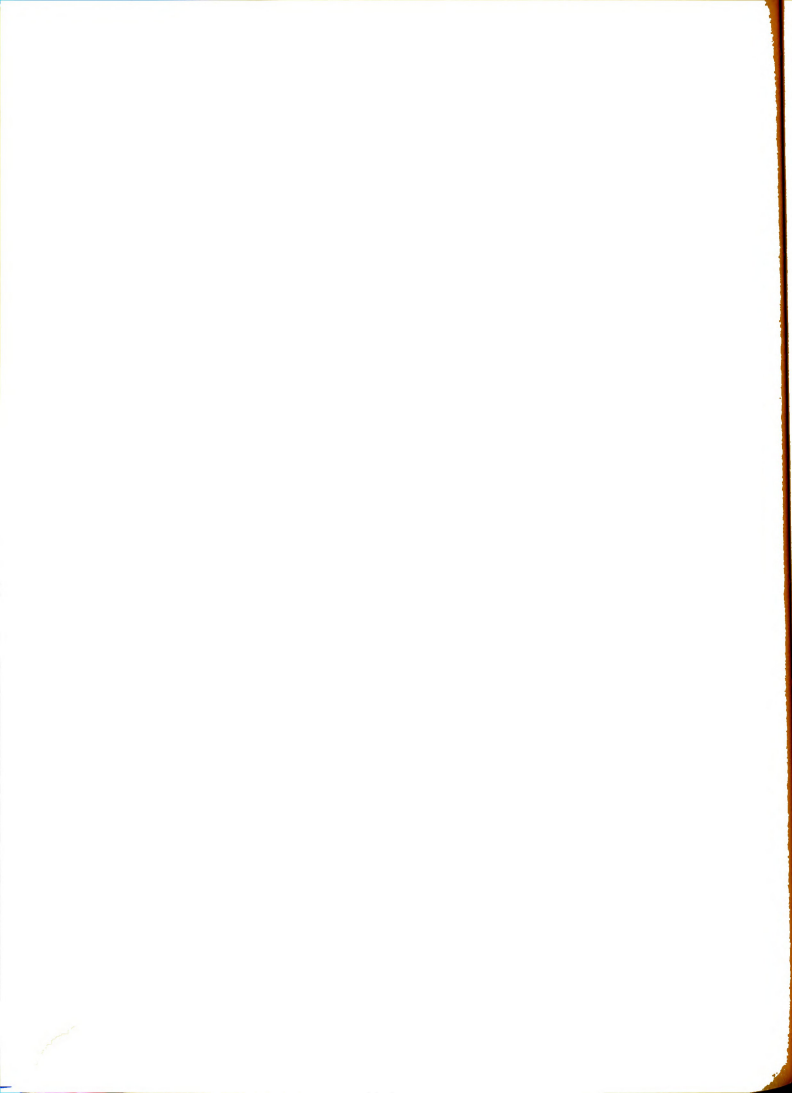
This requires that the mean value of n_1 in the critical region be the same as the mean value for the entire range of n_1 . Thus, the critical region $w(m_1, m_2)$ must contain values above the mean and below the mean, such that

$$(1.5.5) \quad \sum_{n_1 \in w(m_1, m_2)} n_1 \Pr \{n_1 | m_1, m_2, n\} = \frac{\alpha m_1 m_2}{n},$$

since the expected value of n_1 over the whole range of n_1 is $\frac{m_1 m_2}{n}$.

Because n_1 takes on only integer values, it is not always possible to obtain exact size α tests. In order to obtain such tests, a randomized decision rule must be employed. The final form for the unbiased test of independence as given by Katz is to reject H_0 against $H_1: t \neq 1$ for those n_1 such that

$$(1.5.6) \quad \sum_{n_1 < a} \Pr \{n_1 | m_1, m_2, n\} + \epsilon_1 \Pr \{a | m_1, m_2, n\} \\ + \epsilon_2 \Pr \{b | m_1, m_2, n\} + \sum_{n_1 > b} \Pr \{n_1 | m_1, m_2, n\} \\ = \alpha$$



and

$$\begin{aligned}
 (1.5.7) \quad & \sum_{n_1 < a} n_1 \Pr \{n_1 | m_1, m_2, n\} + a \varepsilon_1 \Pr \{a | m_1, m_2, n\} \\
 & + b \varepsilon_2 \Pr \{b | m_1, m_2, n\} + \sum_{n_1 > b} n_1 \Pr \{m_1, m_2, n\} \\
 & = \alpha \frac{m_1 m_2}{n},
 \end{aligned}$$

where a, b, ε_1 , and, ε_2 are determined by these equations and $0 \leq \varepsilon_1, \varepsilon_2 < 1$.

In the one-sided case (testing $t=1$ vs. $t<1$) we have that H_0 is to be rejected for values of n_1 such that

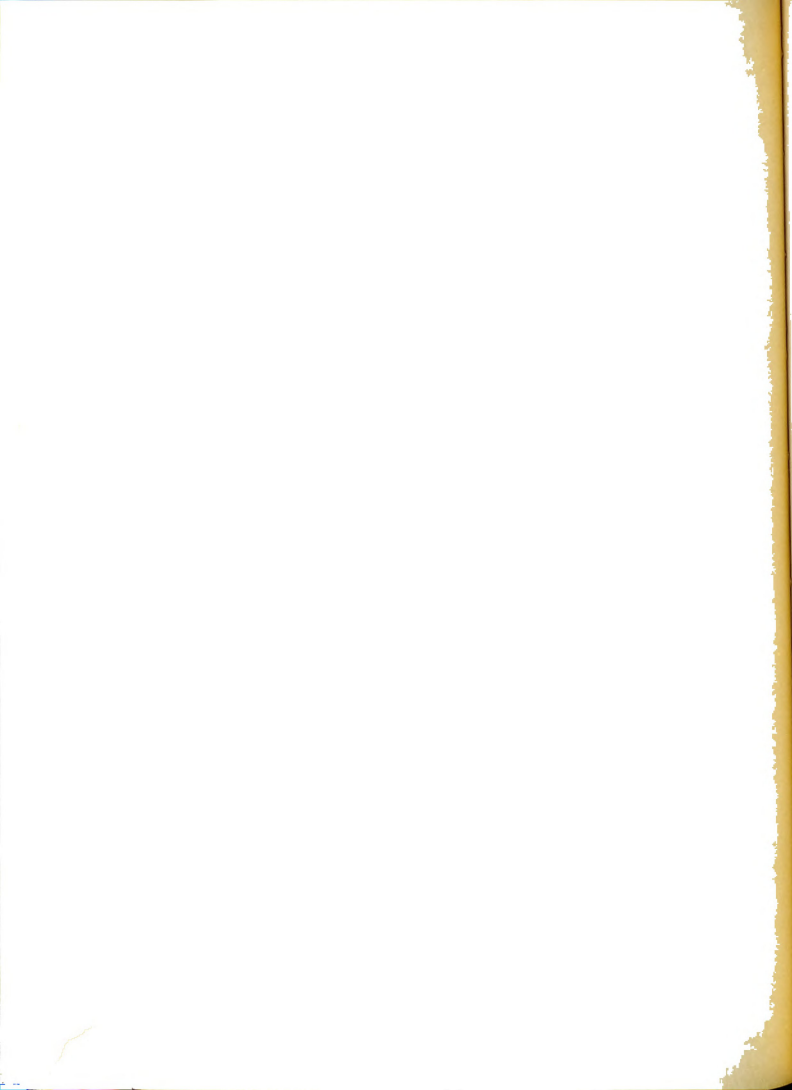
$$(1.5.8) \quad \sum_{n_1 < a} \Pr \{n_1 | m_1, m_2, n\} + \varepsilon \Pr \{a | m_1, m_2, n\} = \alpha,$$

where $0 \leq \varepsilon < 1$, and a and ε are determined such that the equality holds.

Tocher showed that the uniformly most powerful similar one-tailed test for any of the three problems is the conditional test given by Katz in (1.5.8), with, of course, the test in the 2×2 independence trial being uniformly most powerful among all tests. In order to establish this, Tocher proved the following lemma, which is a generalization of the Neyman-Pearson lemma to composite hypotheses.

Tocher's Lemma:

Let $p(j, \theta_0)$ be the probability of the j^{th} possible event in the countable set $I = \{1, 2, \dots\}$, under H_0 , and $p(j, \theta_1)$ be the probability under H_1 . Let $\phi(j, \theta_0) = \frac{\partial}{\partial \theta_0} \left\{ \log p(j, \theta_0) \right\}$.



For each value of θ_0 divide I into sets S_i with equal ϕ , and assume $p(j, \theta_0)$ satisfies conditions such that this division is invariant with respect to θ_0 . Thus $I = \bigcup_{i=1}^{\infty} S_i$. Denote the events in S_i for each i by $s_{i1}, s_{i2}, \dots$, and form the conditional probability under H_0 of s_{ij} given $s_{ij} \in S_i$, denoted by $\Pr(s_{ij} | S_i, \theta_0)$. Then

$$\Pr(s_{ij} | S_i, \theta) = \frac{p(j, \theta_0)}{\sum_{j=1}^{\infty} p(s_{ij}, \theta_0)} = \frac{p(j, \theta_0)}{p(S_i)} ,$$

$$\text{where } p(S_i) = \sum_{j=1}^{\infty} p(s_{ij}, \theta_0).$$

Choose any set of constants w_j , $0 \leq w_j \leq 1$, so that

$$\sum_{j=1}^{\infty} w_j \Pr(s_{ij} | S_i, \theta_0) = \alpha , \quad (i = 1, 2, \dots).$$

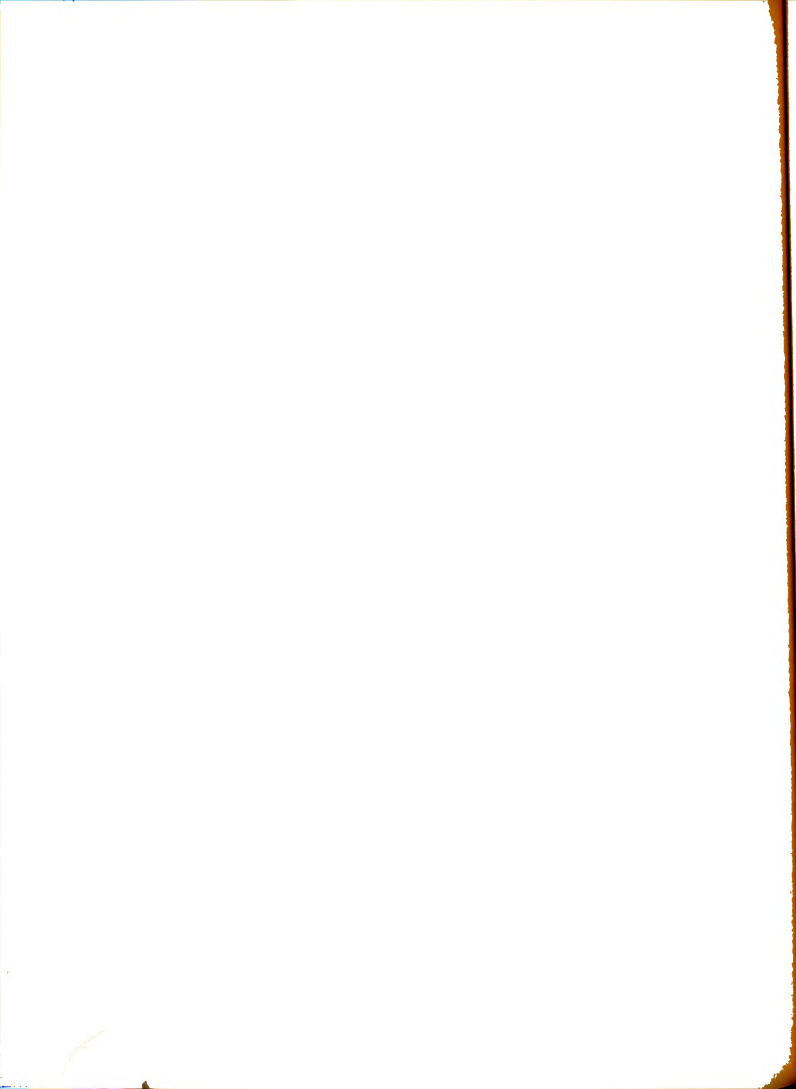
Let $\lambda_j = \frac{p(j, \theta_1)}{p(j, \theta_0)}$ be the likelihood ratio of the two

probability distributions under H_1 and H_0 respectively.

Order the points in S_i for each i by descending likelihood ratio, i.e., say $s_{ij} \leq s_{ik}$ if

$$\lambda_{ij} = \frac{p(s_{ij}, \theta_1)}{p(s_{ij}, \theta_0)} \geq \frac{p(s_{ik}, \theta_1)}{p(s_{ik}, \theta_0)} = \lambda_{ik} .$$

Relabel the points of S_i according to their likelihood ratio



ordering, say $s_{1j_1}, s_{1j_2}, s_{1j_3}, \dots$. Then the procedure

defined by

$$(1.5.9) \quad w_{1j_k} = 1, \quad j_k = 1, 2, \dots, a-1$$

$$w_{1a} = \frac{\sum_{j_k=1}^{a-1} \Pr(s_{1j_k} | s_1, \theta)}{\Pr(s_1 | s_1, \theta)} \quad j_k = a$$

$$w_{1j_k} = 0 \quad j_k = a+1, a+2, \dots$$

where a is determined by

$$\sum_{j_k=1}^{a-1} \Pr(s_{1j_k} | s_1, \theta) \leq \alpha < \sum_{j_k=1}^a \Pr(s_{1j_k} | s_1, \theta),$$

is the best similar procedure to test the composite hypothesis H_0 against the alternative H_1 .

Tocher then applied this lemma to finding the uniformly most powerful similar test (one-tailed) for the 2×2 comparative trial and the double dichotomy. We will now illustrate the application of this lemma in deriving his test for independence in the double dichotomy, noting beforehand that the test is the conditional test, for fixed marginal totals, with randomization on the boundaries. Let us recall (see page 7) that the hypothesis of independence in the double dichotomy takes the form



$$H_0: \pi_1 = P_A P_B$$

$$\text{vs. } H_1: \pi_1 = \lambda P_A P_B, \max \left[0, \frac{P_A - P_B - 1}{P_A P_B} \right] \leq \lambda \leq \min \left[\frac{1}{P_A}, \frac{1}{P_B} \right].$$

The probability of the sample point (n_1, m_1, m_2) under H_1 may be written as

$$(1.5.10) \quad \Pr \{n_1, m_1, m_2 \mid \lambda\} = b(m_1; n, P_A) b(m_2; n, P_B) h(n_1 \mid m_1, m_2, n).$$

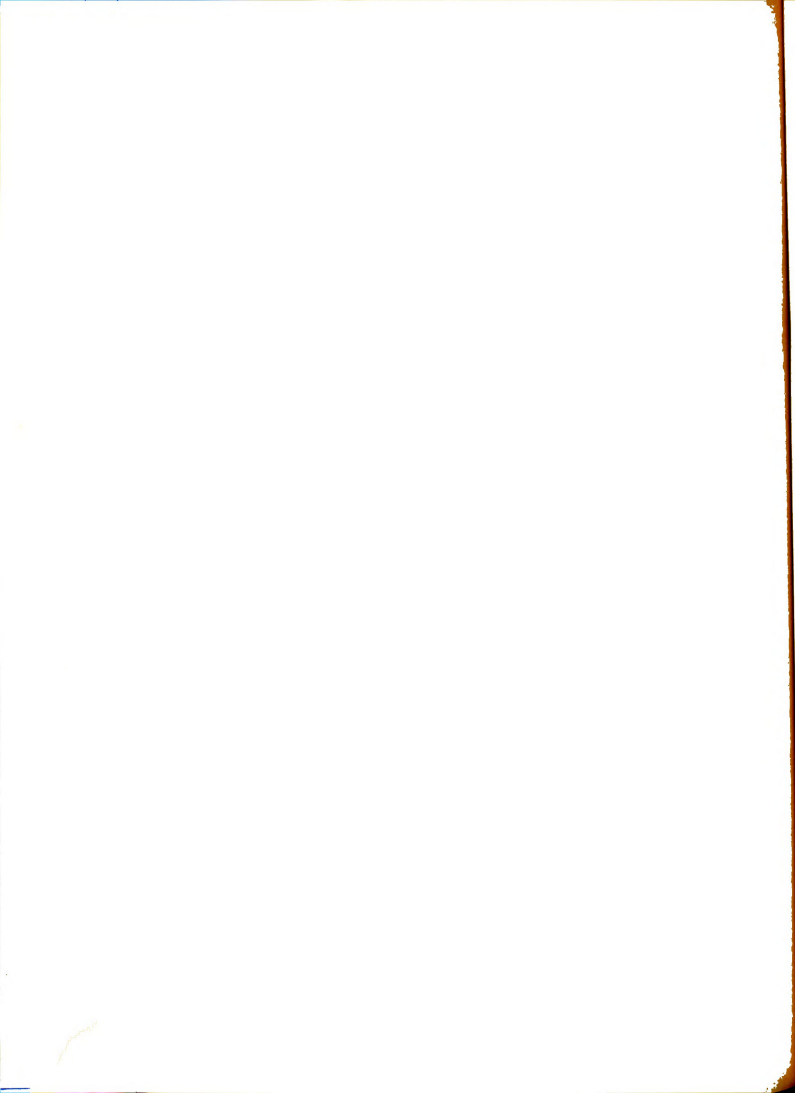
$$\left\{ \left[\frac{1 - \lambda P_A}{1 - P_A} \right]^{m_2} \left[\frac{1 - \lambda P_B}{1 - P_B} \right]^{m_1} \left[\frac{1 - P_A - P_B + \lambda P_A P_B}{(1 - \lambda P_A)(1 - \lambda P_B)} \right]^{n - m_1 - m_2} \cdot \left[\frac{1 - P_A - P_B + \lambda P_A P_B}{(1 - \lambda P_A)(1 - \lambda P_B)} \right]^{n_1} \right\}$$

When $\lambda = 1$, (1.5.10) clearly reduces to (1.3.1).

Hence,

$$(1.5.11) \quad \phi = \begin{bmatrix} \frac{\partial}{\partial P_A} \log \Pr(n_1, m_1, m_2) \\ \frac{\partial}{\partial P_B} \log \Pr(n_1, m_1, m_2) \end{bmatrix} = \begin{bmatrix} \frac{m_1 - n P_A}{P_A(1 - P_A)} \\ \frac{m_2 - n P_B}{P_B(1 - P_B)} \end{bmatrix}.$$

Thus the sets $S_{(m_1, m_2)}$ of equal ϕ have equal m_1 and m_2 , and the decomposition of the sample space into sets $S_{(m_1, m_2)}$ is invariant with respect to P_A and P_B . The conditional probability of n_1 within these sets under H_0 is given by (1.2.6), namely,



$$(1.5.12) \quad \Pr \{n_1 \mid m_1, m_2, n\} = \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1}}{\binom{n}{m_2}},$$

The likelihood ratio

$$(1.5.13) \quad L(\lambda) = \frac{\Pr \{(n_1, m_1, m_2) \mid n, \lambda\}}{\Pr \{n_1, m_1, m_2 \mid n\}} = \begin{array}{l} \text{the four factors} \\ \text{enclosed in brackets} \\ \text{in (1.5.10)} \end{array}$$

within each set $S_{(m_1, m_2)}$ is a monotone function of n_1 , increasing when $\lambda > 1$, decreasing when $\lambda < 1$. Thus, the common best similar procedure for the test against all alternatives $\lambda < 1$, say, is the conditional test, for fixed m_1 and m_2 , defined by

$$(1.5.14) \quad w(n_1, m_1, m_2) = 1, \quad n_1 < a(m_1, m_2)$$

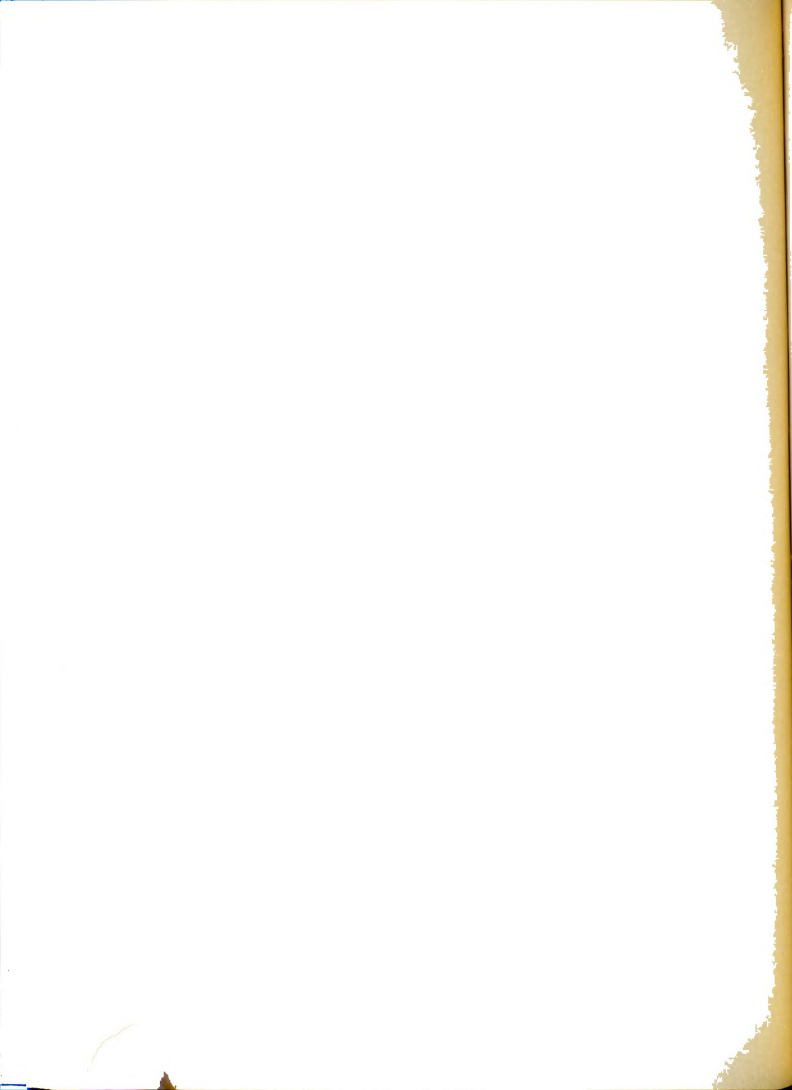
$$w(a[m_1, m_2], m_1, m_2) = \alpha - \frac{\sum_{n_1 \in W(m_1, m_2)} \Pr \{n_1 \mid m_1, m_2, n\}}{\Pr \{a(m_1, m_2) \mid m_1, m_2\}}$$

$$w(n_1, m_1, m_2) = 0, \quad n_1 > a(m_1, m_2)$$

where $a(m_1, m_2)$, for fixed m_1 and m_2 , is determined by

$$\sum_{n_1 \in W(m_1, m_2)} \Pr \{n_1 \mid m_1, m_2, n\} \leq \alpha < \sum_{n_1 \in W(m_1, m_2)} \Pr \{n_1 \mid m_1, m_2, n\}$$

For fixed m_1 and m_2 , this test says to reject H_0 if $n_1 < a(m_1, m_2)$ with probability 1, $n_1 = a(m_1, m_2)$ with probability ϵ and to accept H_0 if $n_1 > a(m_1, m_2)$ where



$a(m_1, m_2)$ and ϵ are determined such that

$$(1.5.15) \quad \sum_{n_1 < a} \Pr \{ n_1 \mid m_1, m_2, n \} + \epsilon \Pr \{ a \mid m_1, m_2, n \} = \alpha.$$

This is Katz' condition (1.5.8).

Before giving Sverdrup's results, let us summarize the various tests that may be of interest. There are six possible test situations for the three experimental models, with two distinct types for each model. That is, in all three problems, a test is to be made on $\lambda = 1$ against $\lambda < 1$ (or $\lambda > 1$) or $\lambda = 1$ against $\lambda \neq 1$. Recalling the discussion on page 7 in Section 1.3, we can summarize the hypotheses, null and alternative, in the following table.

Table IV

		I	II	III
Null Hypothesis	$\lambda=1$	Independent	$p_1 = p_2$	$\pi_1 = P_A P_B$
One-tailed alternative	$\lambda < 1$	"negative" dependence	$p_1 < p_2$	$\pi_1 < P_A P_B$
	$\lambda > 1$	"positive" dependence	$p_1 > p_2$	$\pi_1 > P_A P_B$
Two-tailed alternative	$\lambda \neq 1$	Dependence	$p_1 \neq p_2$	$\pi_1 \neq P_A P_B$

If we let T_1 be the one-tailed test defined by (1.5.8), and T_2 be the two-tailed test defined by (1.5.6) and (1.5.7), then Sverdrup has proven the following theorem:

Theorem:

For the one-tailed tests, test T_1 is the uniformly most powerful unbiased test at the level of significance α . For



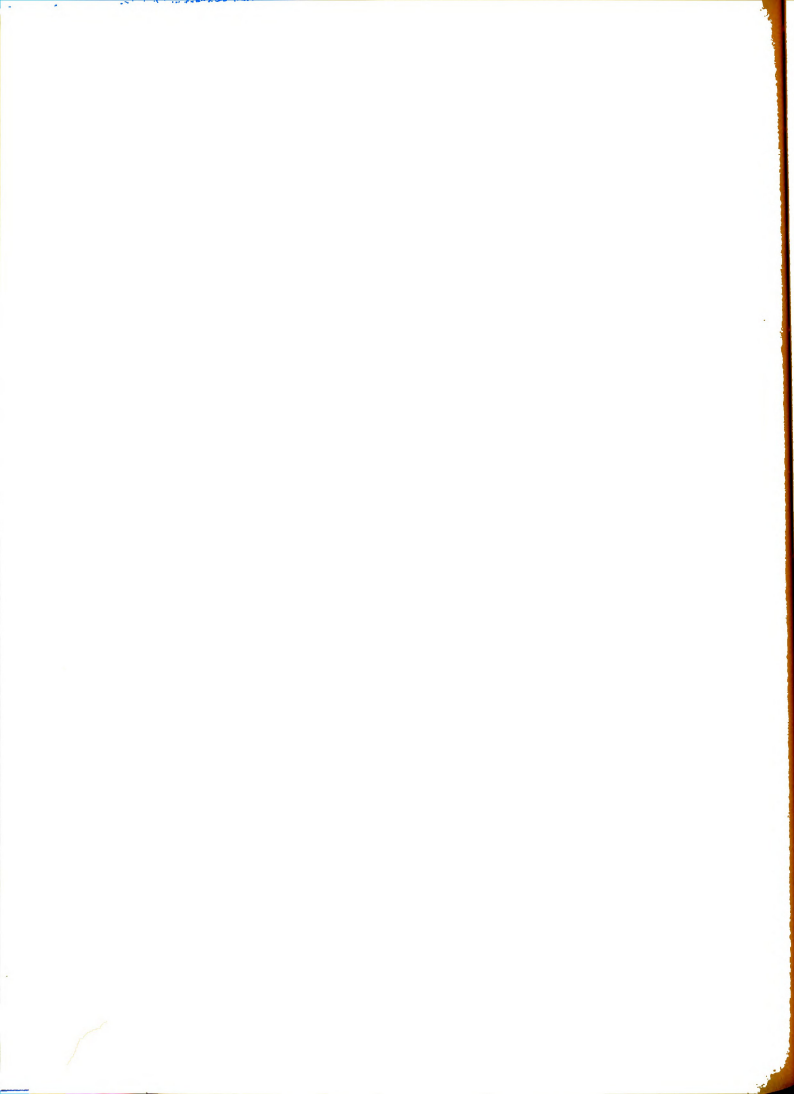
two-tailed tests, T_2 is the uniformly most powerful unbiased test at the level of significance α . Test T_1 is also uniformly most powerful among all tests in the wider class of all similar tests, and T_1 is uniformly most powerful among all tests for independence in the 2×2 independence trial.

Thus Tocher and Sverdrup have shown that properties of the test procedures given by Katz extend to the 2×2 comparative trial and the double dichotomy with each of Tocher and Sverdrup obtaining similar results for one-tailed tests.

In the remainder of this chapter, a description will be given of some other tests which have been proposed, for each of the three problems, beginning in the next section with the 2×2 independence trial, and in sections 1.7, and 1.8 discussing the 2×2 comparative trial and the double dichotomy.

1.6 2×2 Independence Trial

The situation in the 2×2 independence trial is best illustrated by Fisher's tea-tasting experiment. A lady is given n cups of tea, m_1 of which had milk added first, then the tea, and $n - m_1$ cups with tea put in first, followed by the milk. The cups are presented to her in random order. Her problem is to sort the cups. If she is told the number of cups with milk added first presumably she will guess that m_1 of the cups had milk added first, and the rest with tea added first. It is therefore natural to regard both sets of marginal totals as fixed, in repeated sampling.



In the 2×2 independence trial, the null hypothesis is that the markings A_1 or A_2 are independent of the labelings B_1 or B_2 . If the hypothesis is correct, the distribution of n_1 , the number marked A_1 and B_1 , given m_1 and m_2 is the hypergeometric distribution(1.2.6), i.e.

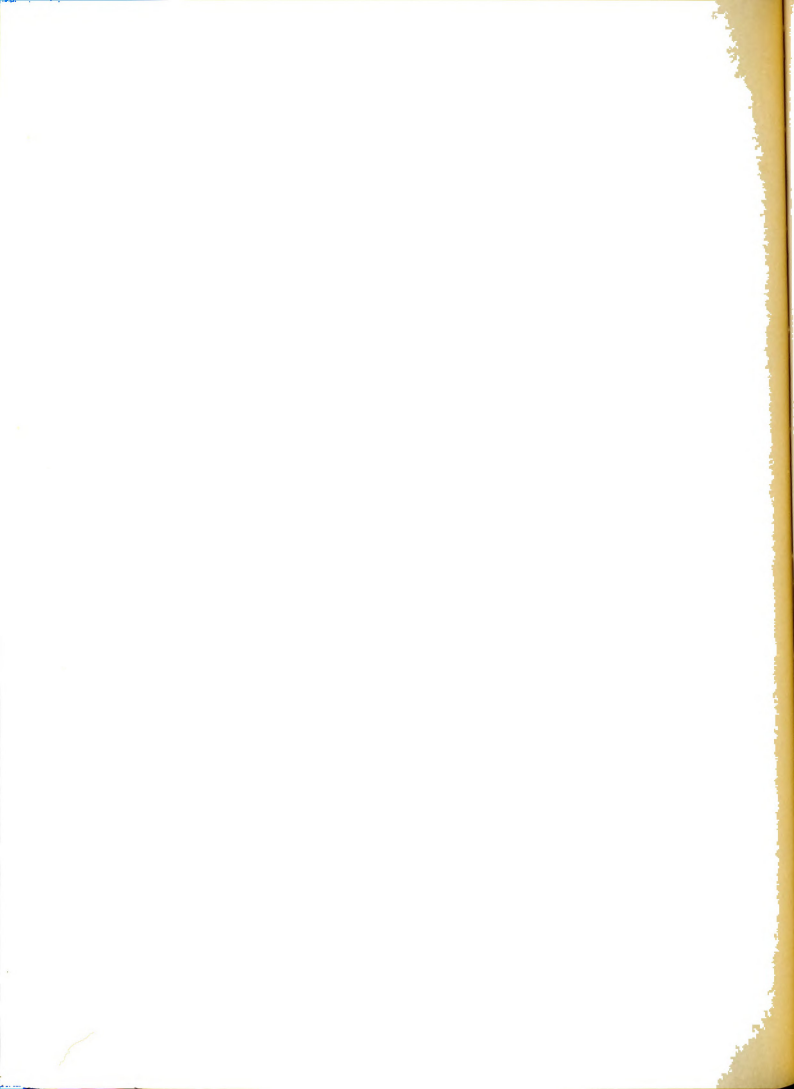
$$(1.6.1) \Pr \{n_1 | m_1, m_2, n\} = \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1}}{\binom{n}{m_2}}$$

where the range of n_1 is given by

$$\max[0, m_1 + m_2 - n] \leq n_1 \leq \min[m_1, m_2]$$

Equation(1.6.1), for the null hypothesis, has been described by Yates as Fisher's "exact" distribution based on the hypergeometric probability law. It is a completely known one-variate discrete probability distribution. Except for specifying non-independence, the only clearly defined alternatives to independence are those proposed by Katz.

For one-tailed tests, the region of rejection of H_0 usually consists of extreme values in one of the tails, rejecting all those values in the tail whose probabilities sum to a number equal to or less than α , given in advance. By employing a randomized decision rule, we can make the test of exact size α . However, for a two-tailed test, several alternative regions are possible, since no uniformly most powerful test exists. Some criterion must therefore be adopted in order to determine which of the several possible critical

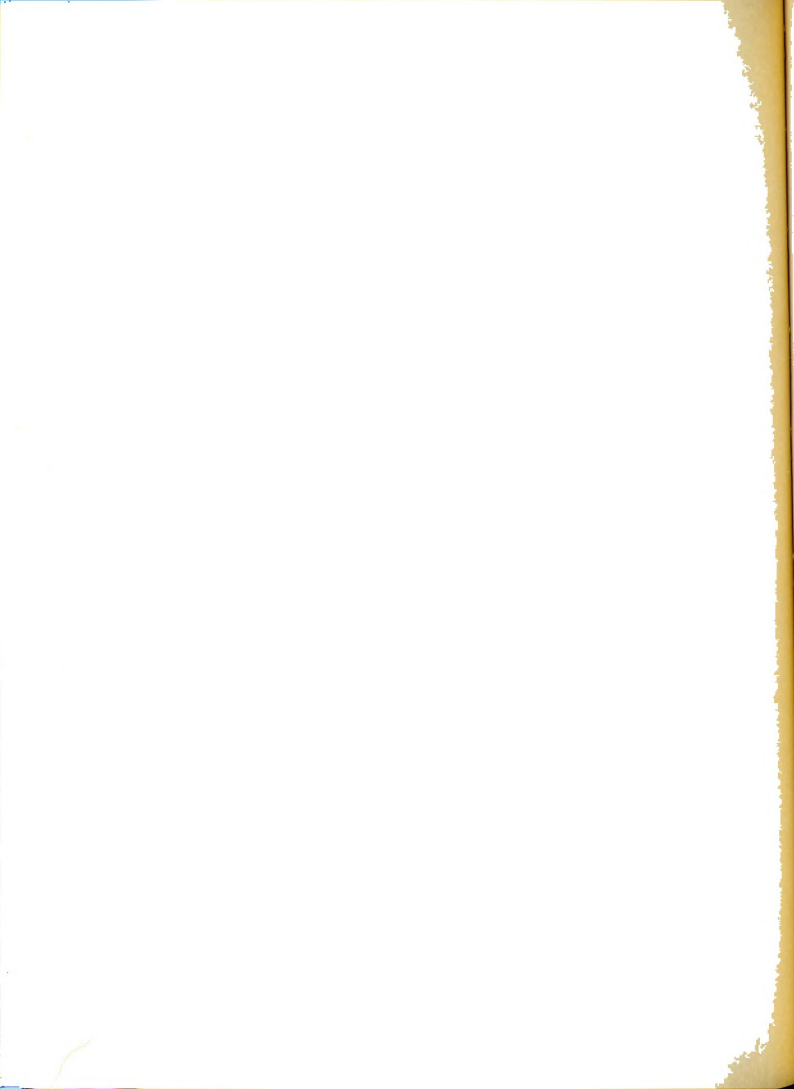


regions should be taken.

The usual convention is to make the probability associated with each tail region sum to $\alpha/2$ or less. Because n_1 assumes only discrete values it is not always possible to obtain critical regions of size exactly α in terms of integer values of n_1 . In order to obtain exact size α - tests, we must adopt a procedure of randomization for a decision. In this way, we can put probability $\alpha/2$ in each tail, under H_0 .

P. Armsen [1] suggests two other possible rules for the selection of points in the critical region, which are denoted by D_2 and D_3 in his paper.

- D_2 . "Arrange the possible events in ascending order of the size of their probabilities under the null hypothesis; include in the $100\alpha\%$, two-tailed rejection region those events for which the cumulative sum of these ordered probabilities is smaller than or equal to α ."
- D_3 . "Define $F(E)$, or the 'first tail' probability, as the cumulative sum of the probabilities under the null hypothesis of all possible events which are more extreme, in the same direction, than given event E , including the probability of E itself. Define $S(E)$, or the 'second tail' probability, as the cumulative sum of probabilities, starting with that of the event most extreme in the opposite direction as compared with E , and cumulating up to but not exceeding the value of $F(E)$. If and only if, $F(E) + S(E) \leq \alpha$, include E in the rejection region for the two-tailed $100\alpha\%$ level of significance."



One might also restrict the tests to a class of "nice" tests, and then look for one which is uniformly most powerful within this class as was done in 1.5. In any case, a class of alternative distributions for the alternative hypotheses must be specified. It is clear that in order to decide which test is "best", one must examine the power function of the test. Katz, Tocher and Sverdrup proposed their test with this consideration in mind.

Finally, as Armsen has pointed out, there are certain peculiarities in any of the first three definitions of the critical regions. It is possible, in certain cases, to construct, on the basis of his definitions, critical regions in which all points of the region come from one tail of the distribution.

The classic test of the null hypothesis is that given by Fisher [7]. It consists in computing (1.6.1) for the probability of the observed value and all values less likely. If the sum of these probabilities is equal to or less than α , then H_0 is rejected. Extensive tables have been prepared by Finney [6], Latscha [10], and others, for small n , indicating the critical points for which the hypothesis of independence is to be rejected. For small values of n , exact computation of the tail terms is practical, but for large n , it is quite tedious. One may, in this case, approximate the exact hypergeometric distribution by a normal distribution with a mean and variance corresponding to the mean and variance of (1.6.1).

$$(1.6.2) \quad E(n_1 | m_1, m_2, n) = \frac{m_1 m_2}{n} = \bar{n}_1$$



$$(1.6.3) \quad V(n_1 | m_1, m_2, n) = \frac{m_1(n - m_1)m_2(n - m_2)}{n^2(n - 1)} = h^2$$

Let

$$(1.6.4) \quad u = \frac{n_1 - \frac{m_1 m_2}{n}}{\sqrt{\frac{m_1(n - m_1)m_2(n - m_2)}{n^2(n - 1)}}} = \frac{n_1 - \bar{n}_1}{h},$$

and write u_α for the 100α -percent point of the standard normal distribution, i.e., u_α is defined by the equation

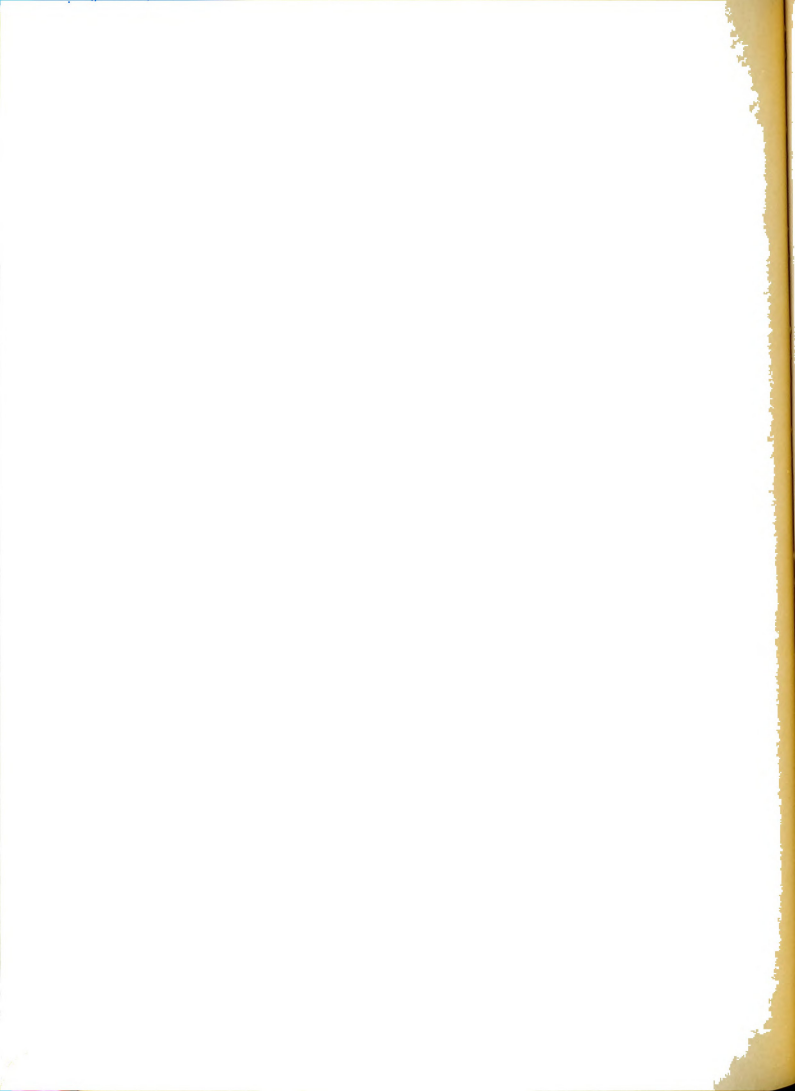
$$(1.6.5) \quad \int_{u_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \alpha.$$

Then, for a one-tailed test, reject H_0 if $u > u_\alpha$ or $u < u_{1-\alpha}$, depending on the appropriate case. For two-tailed tests, reject if $|u| > u_{\frac{\alpha}{2}}$. Very often a correction

factor for continuity is included in u , i.e., the absolute value of the numerator of u is reduced by $\frac{1}{2}$. If this is done, then for very large n , we get the usual χ^2 test in a 2×2 table, with one degree of freedom, if we square the corrected u , i.e.,

$$(1.6.6) \quad \chi^2 = \frac{n(|n_1 n - m_1 m_2| - \frac{n}{2})^2}{m_1(n - m_1)m_2(n - m_2)}$$

We reject if $\chi^2 \geq \chi_\alpha^2$, where χ_α^2 is the α th percent point of χ^2 with one degree of freedom.



1.7 2 x 2 Comparative Trial

To fix the notion of a 2 x 2 comparative trial, we give an example. Groups of m_1 men and $n - m_1$ women are randomly selected. They are then examined to see whether they are smokers or non-smokers. It is assumed that the proportion of men smokers is p_1 , and the proportion of women smokers is p_2 . In repeated sampling, it is supposed that we always take the same number of men in each sample, and the same number of women, but that the number of smokers in each sub-sample is free to vary.

In the 2 x 2 comparative trial, the hypothesis to be tested is the composite hypothesis $p_1 = p_2 = p$, against alternatives $p_1 \neq p_2$.

The probability of observing (n_1, n_2) is, in general,

$$\begin{aligned} (1.7.1) \quad \Pr \{ (n_1, n_2) \mid p_1, p_2, m_1, n \} &= b(n_1; m_1, p_1) b(n_2; n - m_1, p_2) \\ &= b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2) \end{aligned}$$

In contrast to the 2 x 2 independence trial, the probability distribution (1.7.1) is over a two-dimensional lattice of points. This distinguishes the problem from the one discussed previously. It is necessary for testing purposes to decide whether m_2 is an intrinsic parameter or a nuisance parameter. Several tests for the null hypothesis have been proposed.

Barnard's test [3] may be described as follows. "Taking rectangular axes in a plane, one can represent an observed result (as in Table I) as the point whose coordinates are



(n_1, n_2) , where $0 \leq n_1 \leq m_1$, $0 \leq n_2 \leq n - m_1$ (n_1, n_2 integral).

Call the totality of possible points a lattice diagram. The points in this lattice diagram will be given a total ordering.

First, the same rank should be given to the point

$(m_1 - n_1, n - m_1 - n_2)$ as to the point (n_1, n_2) . This is called the symmetry condition, or condition S. Secondly, two

points which, respectively, have the same abscissa or the same ordinate as (n_1, n_2) , and which lie farther from the

diagonal line joining $(0,0)$ and $(m_1, n - m_1)$ should be considered as indicating a wider difference than (n_1, n_2) .

This is called the 'convexity' condition, or condition C.

Conditions S and C generate a partial ordering. To make it a total ordering, one further condition, called the maximum

condition, or condition M, is imposed. Conditions C and S require that the points $(m_1, 0)$ and $(0, n - m_1)$ be given the

lowest rank. Associate a function P with these two points

$$(1.7.2) \quad P(0, n - m_1, p) = p^{m_1} (1-p)^{n-m_1} + p^{n-m_1} (1-p)^{m_1},$$

where $p_1 = p_2 = p$ in (1.7.1)

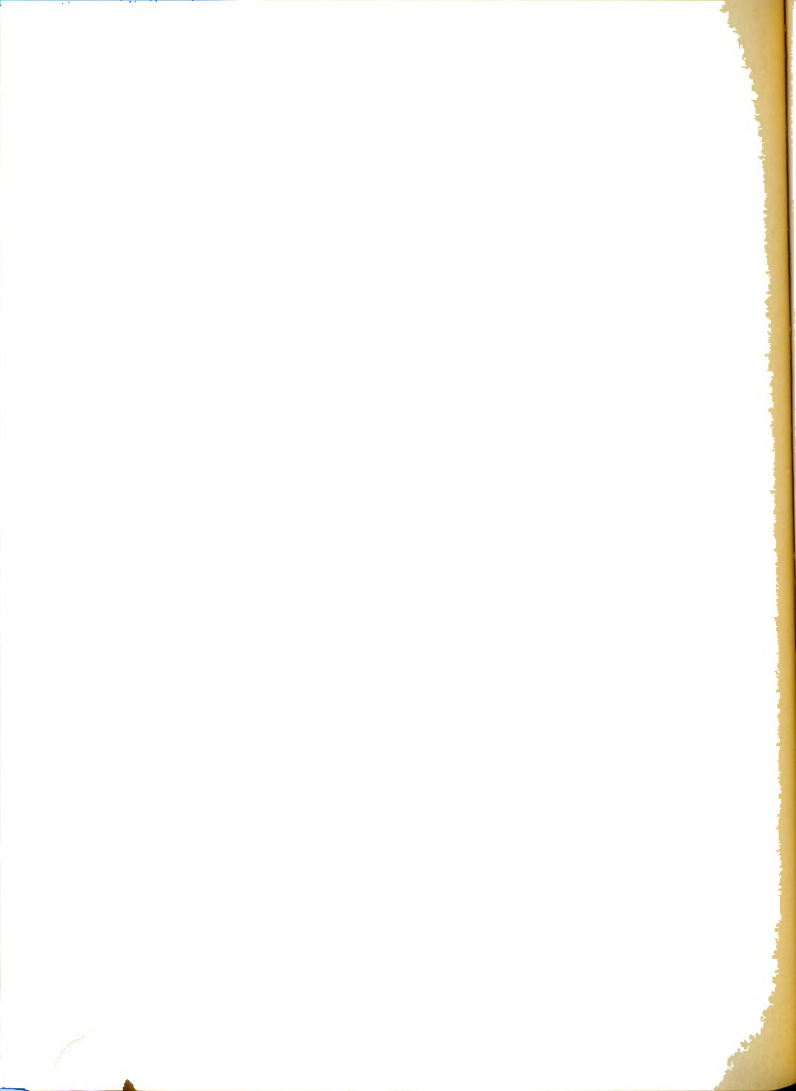
Let

$$(1.7.3) \quad P_m(0, n - m_1) = \max_{0 < p < 1} P(0, n - m_1, p).$$

Considering only points for which $\frac{n_1}{m_1} < \frac{n_2}{n - m_1}$

(by symmetry), if the first $(K - 1)$ points (a_1, b_1) ,

$(a_2, b_2), \dots, (a_{K-1}, b_{K-1})$ in order of increasing rank have been chosen, and (a_{K-1}, b_{K-1}) is associated with



the function

$$(1.7.4) \quad P(a_{K-1}, b_{K-1}, p) = P(a_{K-2}, b_{K-2}, p) \\ + \binom{m_1}{n_1} \binom{n-m_1}{n_2} \left[p^{m_2(1-p)^{n-m_2+p}} p^{n-m_2(1-p)^{m_2}} \right],$$

then the K^{th} point, (a_K, b_K) , is that point, of all points (n_1, n_2) permitted by the C condition, for which

$$(1.7.5) \quad P_m(n_1, n_2) = \max_{0 < p < 1} P(a_K, b_K, p)$$

is least; (a_K, b_K) is then associated with the function

$$(1.7.6) \quad P(a_K, b_K, p) = P(a_{K-1}, b_{K-1}, p) \\ + \binom{m_1}{n_1} \binom{n-m_1}{n_2} \left[p^{m_2(1-p)^{n-m_2}} + p^{n-m_2(1-p)^{m_2}} \right].$$

If r points give the same value of $P_m(n_1, n_2)$ and this value is less than that associated with any other permissible point, assign the same rank to each and replace the second term in $P(a_K, b_K, p)$ by the corresponding sum over all these points.

The next point after them will be denoted as the $(K + r)^{\text{th}}$ point in the ordering. Barnard then associates the

$$P_m(a_K, b_K) = \max_{0 < p < 1} P(a_K, b_K, p) \text{ as the significance level of}$$

the point (a_K, b_K) . This test is called the C S M test, for obvious reasons, by Barnard. Barnard's procedure leads to the assignment of significance levels to points generally quite a bit smaller than those assigned by Fisher, in his Conditional test.



If the marginal totals are large, Barnard's test is difficult to apply, since the computation becomes prohibitive. When this is the case, a normal approximation may be used.

If the null hypothesis is true, then, from (1.7.1)

$$(1.7.7) \Pr \{n_1, m_2 \mid p_1 m_1, n\} = h \{n_1 \mid m_1, m_2, n\} b \{m_2 \mid n, p\}$$

where $p_1 = p_2 = p$.

Using the normal approximation to the hypergeometric distribution given by

$$(1.7.8) \Pr \{n_1 \mid m_1, m_2, n\} \simeq \int_{n_1 - \frac{1}{2}}^{n_1 + \frac{1}{2}} \frac{1}{\sqrt{2\pi} h} e^{-\frac{1}{2} \left(\frac{x - \bar{n}_1}{h} \right)^2} dx,$$

a critical region is constructed containing all those points $(n_1, m_2 - n_1)$ such that

$$(1.7.9) \frac{n_1 + \frac{1}{2} - \bar{n}_1}{h} \leq -u_{\frac{\alpha}{2}} \quad \text{and} \quad \frac{n_1 - \frac{1}{2} - \bar{n}_1}{h} \geq u_{\frac{\alpha}{2}}.$$

$$\text{Let } L_1 = \bigcup_{m_2=0}^n \left\{ (n_1, m_2 - n_1) : \frac{n_1 + \frac{1}{2} - \bar{n}_1}{h} \leq -u_{\frac{\alpha}{2}} \right\}$$

$$\text{and } L_2 = \bigcup_{m_2=0}^n \left\{ (n_1, m_2 - n_1) : \frac{n_1 - \frac{1}{2} - \bar{n}_1}{h} \geq u_{\frac{\alpha}{2}} \right\}$$

$$\text{and put } L = L_1 \cup L_2.$$

Then, if a one-tailed test is performed, H_0 is rejected at level $\frac{\alpha}{2}$ if the observed sample point falls in the



appropriate component of the critical region, either L_1 or L_2 , while for two-tailed tests, H_0 is rejected at level α if the sample point is in L .

Pearson notes that since the probability density is discrete, the "true" probability of rejection will quite generally be much smaller than α . For each value of m_2

let $\alpha_{m_2} = \sum_{n_1 \in L} \Pr \{n_1 | m_1, m_2, n\}$. Then $\alpha_{m_2} \leq \alpha$,

for each $m_2 = 0, 1, \dots, n$, and hence $\sum_{m_2=0}^n \Pr \{m_2 | p, n\} \alpha_{m_2} \leq \alpha$.

He then remarks that whereas the $\frac{1}{2}$ - correction for continuity is appropriate for the Problem I, "it is not helpful for the Problem II where we are concerned with a two-dimensional experimental probability set." He proposes, therefore, that the correction for continuity be omitted in (1.7.8). If this is done, then new critical regions L_1' and L_2' are obtained, where

$$L_1' = \bigcup_{m_2=0}^n \left\{ (n_1, m_2 - n_1) : \frac{n_1 - \bar{n}_1}{h} \leq -u_{\frac{\alpha}{2}} \right\} \text{ and}$$

$$L_2' = \bigcup_{m_2=0}^n \left\{ (n_1, m_2 - n_1) : \frac{n_1 - \bar{n}_1}{h} \geq u_{\frac{\alpha}{2}} \right\}.$$

Corresponding to α_{m_2} , one has

$$\alpha_{m_2}' = \sum_{n \in L'} \Pr \{n_1 | m_1, m_2, n\}, \quad L' = L_1' \cup L_2'.$$



With this modification, α'_{m_2} will be sometimes less and sometimes greater than α , and, hence, "in the balance, it seems likely that the chance of the point $(n_1, m_2 - n_1)$ falling in L' will lie closer to α than when the correction is used." We note, however, that one can obtain regions similar to the sample space by use of a randomized test.

If u is defined as in (1.6.4), then $E(u) = 0$, $V(u) = 1$, independently of p . The distribution of u depends on p , but Pearson feels "we may not in the long run do too badly by assuming u to be normal."

It is Pearson's opinion that the test of H_0 against H_1 should be performed as follows:

- (a) If any of the marginal totals are "small", apply Barnard's test.
- (b) If the marginal totals are all large, use the normal approximation, with the $\frac{1}{2}$ - factor for continuity omitted.

Patnaik [14] has given two approximations to the power of the test described by Pearson. The exact power function for this test is

$$(1.7.10) \quad \sum_{m_2=0}^n \sum_{n_1} \Pr \{ (n_1, m_2) \mid p_1, p_2, m_1, n \}$$

where the inner sum is over $n_1 \in L'_1$ or L'_2 as the case may be for one-tailed tests, and over $n_1 \in L'$ for two-tailed tests. His approximation is derived thru the normal approximations of the type involving hypergeometric and binomial



distributions. In the process of deriving his approximation to the power function, Patnaik also gives an approximation to the conditional distribution of n_1 given m_1 and m_2 under H_1 , which is quite interesting, but unfortunately may give absurd results, and which casts serious doubts on the validity of his approximation to the power. Patnaik's procedure in deriving the approximation follows below.

Rewriting (1.7.1), one has

$$(1.7.11) \quad \Pr \{n_1, m_2 - n_1 \mid p_1, p_2, m_1, n\} \\ = \binom{n}{m_2} q_1^{m_1} p_2^{m_2} q_2^{n-m_1-m_2} \frac{\binom{m_1}{n_1} \binom{n-m_1}{m_2-n_1}}{\binom{n}{m_2}} t^{n_1}$$

$$\text{where } t = \frac{p_1 q_2}{p_2 q_1}, \quad q_i = 1 - p_i, \quad i = 1, 2.$$

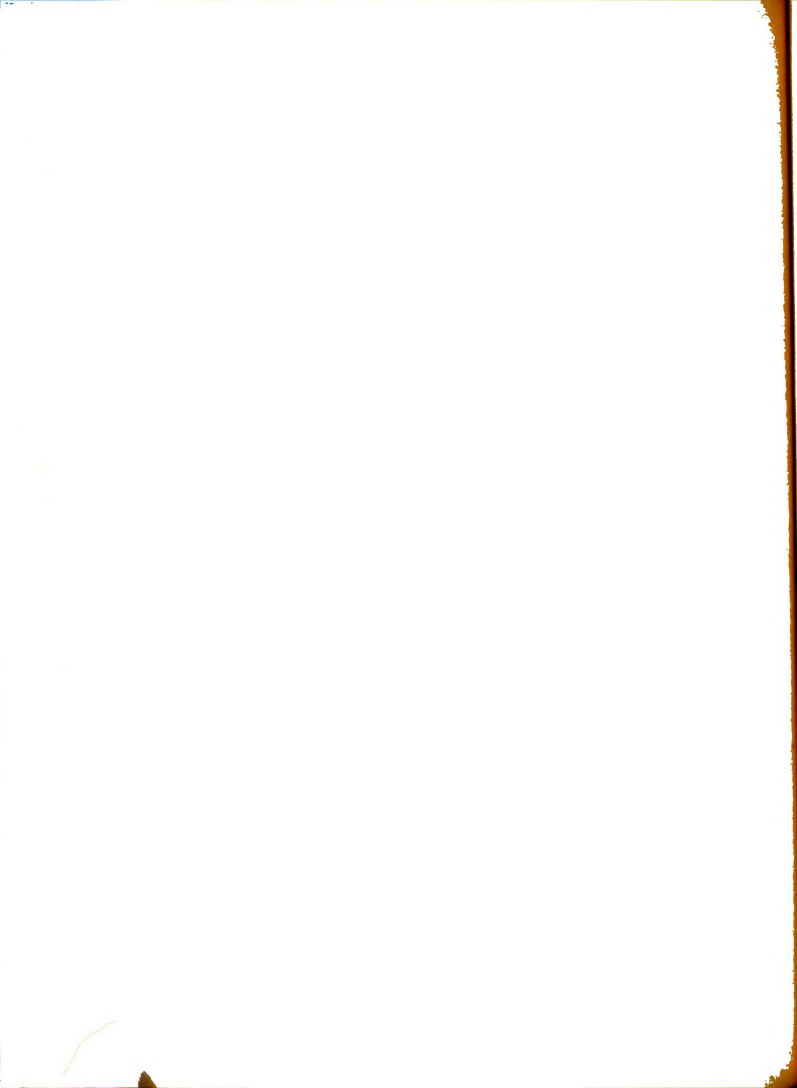
For m_1, m_2 , and n large, the hypergeometric probability term in (1.7.11) is approximately equal to the ordinate of a normal distribution with mean $\frac{m_1 m_2}{n}$ and variance

$$h^2 = \frac{m_1 m_2 (n-m_1)(n-m_2)}{n^2 (n-1)} \quad (\text{see Feller [5] page 180) and so}$$

Patnaik replaces this term by the ordinate of a normal distribution with this mean and variance, obtaining as an approximation for (1.7.11)

$$(1.7.12) \quad \binom{n}{m_2} q_1^{m_1} p_2^{m_2} q_2^{n-m_1-m_2} \frac{1}{\sqrt{2\pi h}} e^{-\frac{1}{2h^2}(n_1 - \frac{m_1 m_2}{n})^2} t^{n_1}$$

Writing t^{n_1} as $e^{n_1 \log t}$ and then completing the square



in n_1 , Patnaik finds that (1.7.12) may be written as

$$(1.7.13) \left[\binom{n}{m_2} q_1^{m_1} p_2^{m_2} q_2^{n-m_1-m_2} t^{\frac{m_1 m_2}{n} + h^2 \log t} \right] \\ \cdot \left[\frac{1}{\sqrt{2\pi} h} e^{-\frac{\frac{1}{2}(n_1 - \frac{m_1 m_2}{n} - h^2 \log t)^2}{h^2}} \right]$$

and notes that the first factor in brackets is independent of n_1 , so that the last factor is approximately the conditional distribution of n_1 given m_1, m_2, n , and t , i.e.,

a normal distribution with mean $\frac{m_1 m_2}{n} + h^2 \log t$ and variance h^2 . With this result, he then assumes that the conditional power function of the Pearson test, for fixed m_2 , is approximately given by

$$(1.7.14) \quad \beta(m_2) = 1 - \int_{-\frac{u_\alpha}{2} - h(m_2)}^{\frac{u_\alpha}{2} - h(m_2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

where, following his notation, $\beta(m_2)$ denotes the conditional

power function, and $h(m_2) = \sqrt{\frac{m_1(n-m_1)}{n^2(n-1)} \frac{m_2(n-m_2)}{n^2(n-1)}} \log t$. Next

he approximates the distribution of m_2 by a normal distribution with mean $\gamma = m_1 p_1 + (n-m_1) p_2$,

and variance

$$\sigma^2 = m_1 p_1 (1-p_1) + (n-m_1) p_2 (1-p_2) = \sigma_1^2 + \sigma_2^2;$$

γ and σ^2 are the exact mean and variance of m_2 . This



leads to the approximation

$$(1.7.15) \quad \beta \sim \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left(\frac{m_2 - \gamma}{\sigma} \right)^2} \beta(m_2) dm_2,$$

where β denotes the unrestricted power function. Since this still represents no essential simplification, he considers two approximations to (1.7.15). The first consists in expanding $\beta(m_2)$ in a Taylor's Series (about γ) to obtain

$$(1.7.16) \quad \beta \sim \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left(\frac{m_2 - \gamma}{\sigma} \right)^2} \left[\sum_{j=0}^{\infty} \frac{\beta^{(j)}(\gamma)}{j!} (m_2 - \gamma)^j \right]$$

$$= \beta(\gamma) + \frac{\sigma^2 \beta''(\gamma)}{2!} + \frac{3\sigma^4 \beta^{(4)}(\gamma)}{4!} + \dots$$

and then taking $\beta(\gamma)$ as a first term approximation. Thus, his first approximation to power is given by

$$(1.7.17) \quad \beta \sim 1 - \int_{-u\sigma/2 - h(\gamma)}^{u\sigma/2 - h(\gamma)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

The second approximation is derived thru using the method of approximate product-integration developed by R. E. Beard. This leads to his second approximation, given by

$$(1.7.18) \quad \beta \sim 1/6 \beta(\gamma - \sqrt{3} \sigma) + 2/3 \beta(\gamma) + 1/6 \beta(\gamma + \sqrt{3} \sigma).$$

Patnaik claims that (1.7.18) gives a better approximation than by using the first three or four terms of the above-



mentioned Taylor's series. Thus, Patnaik chooses (1.7.18) as his approximation to the power function for the test of independence in the 2×2 comparative trial. It should be noted, however, that the assumption that m_2 is normally distributed is irrelevant as far as the first two terms of (1.7.16) are concerned, since

$$\begin{aligned}
 (1.7.19) \quad & \sum_{m_2=0}^n \Pr \{m_2 \mid p_1, p_2, m_1, n\} \sum_{j=0}^{\infty} \beta^{(j)}(\gamma) \frac{(m_2 - \gamma)^j}{j!} \\
 &= \beta(\gamma) + \frac{\sigma^2 \beta''(\gamma)}{2!} + \left[(q_1 - p_1) \sigma_1^2 + (q_2 - p_2) \sigma_2^2 \right] \\
 &\quad \cdot \frac{\beta'''(\gamma)}{3!} + \dots
 \end{aligned}$$

Both of Patnaik's approximations rely heavily on the approximation to the conditional distribution of n_1 given m_1, m_2, n , and t . In section 2.2, it will be shown that this approximation may lead to rather non-sensical results, and hence that both of Patnaik's approximations may be unreliable. The implication of the results in 2.2 on Patnaik's approximations will be discussed in section 3.2.

Sillitto [21] also obtained an approximation to power for the same test. The approximation is based on the arc sine transformation for a binomial variable. Let

$$(1.7.20) \quad c(p_1, p_2, m_1) = \frac{\arcsin \sqrt{p_2} - \arcsin \sqrt{p_1}}{\sigma}$$



where $\sigma = \frac{1}{2} \sqrt{\frac{1}{m_1} + \frac{1}{n-m_1}} = \frac{1}{2} \sqrt{\frac{n}{m_1(n-m_1)}}$, and the angle is measured in radians. Then Sillitto gives as an approximation to power

$$(1.7.21) \quad \beta \sim 1 - \left\{ \phi \left[u_{\frac{\alpha}{2}} - c(p_1, p_2, m_1) \right] - \phi \left[-u_{\frac{\alpha}{2}} - c(p_1, p_2, m_1) \right] \right\}$$

for two-tailed tests, and

$$(1.7.22) \quad \beta \sim 1 - \phi [u_{\alpha} - c(p_1, p_2, m_1)]$$

for one-tailed tests.

For comparisons of Patnaik's and Sillitto's approximations, see table E.2.1 and E.2.2 in appendix E. It appears from the available data that Sillitto's approximation is much more satisfactory than Patnaik's.

One could also make a conditional test, for fixed m_2 and put probability $\alpha/2$ or less in each tail, and then compute the power function. C. C. Sekar[20] and others, have given tables of the power function for the special case where $m_1 = n - m_1 = \frac{n}{2}$, and the critical region for each m_2 is symmetric, i.e., if $(n_1, m_2 - n_1)$ is in the critical region, then so is $(m_2 - n_1, n_1)$.

1.8 The Double Dichotomy

As in the previous two sections, an example of an experiment illustrating the situation in the double dichotomy is given.



A group of n college students, selected at random from the totality of all college students, are classified according to sex and according to whether or not they are drinkers or non-drinkers. The result of the classification is represented in the form of Table V, along with the hypothetical proportions of all students having the double classification:

TABLE V

	Male	Female	
Drinker	n_1	n_3	m_1
Non-drinker	n_2	n_4	$n - m_1$
	m_2	$n - m_2$	n

TABLE VI

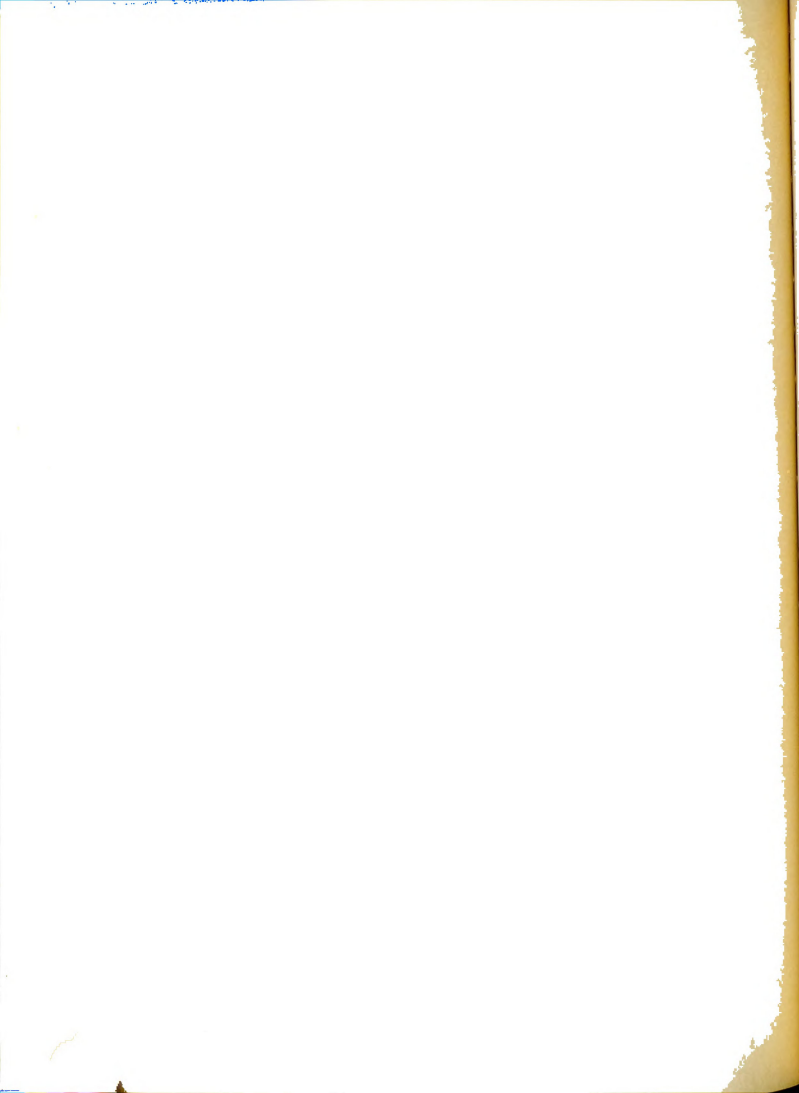
	Male	Female	
Drinker	π_1	π_2	P_A
Non-Drinker	π_3	π_4	$1 - P_A$
	P_B	$1 - P_B$	1

It is natural to regard both sets of marginal totals as free to vary in repeated experimentation.

The hypothesis of independence takes the form

$$H_0: \pi_1 = P_A P_B.$$

If H_0 is true, the two classifications are independent. As previously noted (page 7) any alternative hypothesis may be



expressed as

$$H_1: \pi_1 = \lambda P_A P_B,$$

where λ satisfies the inequalities

$$\frac{P_A + P_B - 1}{P_A P_B} \leq \lambda \leq \frac{1}{\max[P_A, P_B]}$$

The double dichotomy appears to have received the least attention of the three cases. For small samples, Fisher, as in the previous cases, asserts the test should be performed regarding the marginal totals as fixed by the observed sample totals; in other words, the conditional test for fixed m_1 and m_2 is the appropriate test. Pearson [16] suggests that Barnard's method be extended, for small n , to this double dichotomy, and for large n suggests the use of the normal approximation in an analogous fashion to the 2×2 comparative trial.

There is no available literature on the power of any test in the double dichotomy.

1.9 Results

In concluding this survey of past work, one sees that while much has been done with 2×2 contingency tables, there are some aspects which have not been investigated. Thus, in the remainder of this thesis, the author proposes to do the following:

1. Investigate the properties of the conditional distribution $\Pr \{n_1 \mid m_1, m_2, n; t\}$ given by equation (1.2.5),



including its asymptotic distribution and several approximations to it. As a part of this investigation, it is shown by Theorem (2.2.F) that there are at least two published results in the literature which may be invalid, namely, the results of Patnaik [14] on the power function for the test of independence in the 2×2 comparative trial, and Moore's [13] power function for a test for randomness in a sequence of two alternatives involving a 2×2 table.

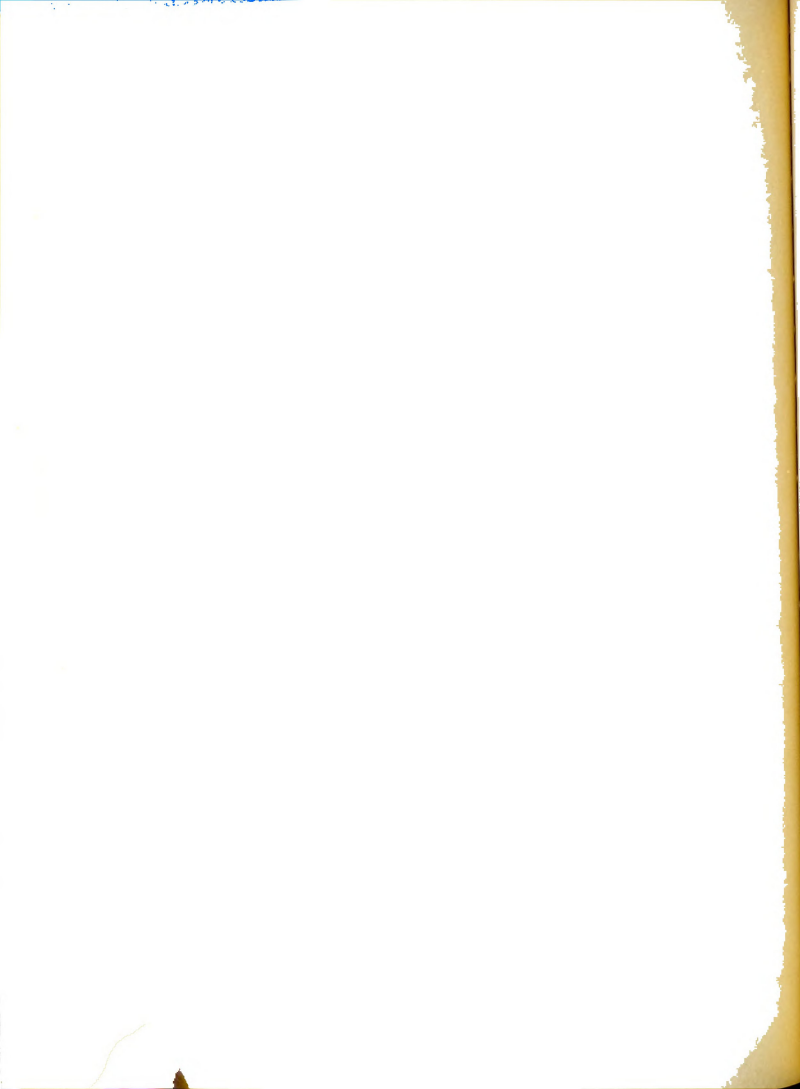
2. Give the exact power functions for the test of independence based on the uniformly most powerful unbiased test described in Section 1.5, for each of the 2×2 independence trial, 2×2 comparative trial, and the double dichotomy, and some approximations to these exact power functions.

3. Investigate some relations between these exact power functions, and prove that for large sample sizes there is little difference in power between the three cases for appropriately chosen alternative hypotheses and marginal totals.

Theorem (3.4.A) in Chapter 3 gives the limiting power function in the 2×2 independence trial, with modest conditions, and Theorem (3.4.B) shows that asymptotically the power functions for the test of independence in the 2×2 independence trial, 2×2 comparative trial, and double dichotomy, are equal.

4. Study the asymptotic power function for the χ^2 -test.

5. Investigate the adequacy of various approximations by making comparisons between the various approximations.



2.1 General Properties

The conditional probability of n_1 , given the two marginal totals m_1 and m_2 , and the dependence parameter t , $0 < t < \infty$, obtained in section 1.2 and given by (1.25) takes the form

$$(2.1.1) \Pr \{n_1 | m_1, m_2, n; t\} = \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum_i \binom{m_1}{i} \binom{n - m_1}{m_2 - i} t^i} = \frac{\Pr \{n_1 | m_1, m_2, n\} t^{n_1}}{\sum_i \Pr \{i | m_1, m_2, n\} t^i}$$

where $\max [0, m_1 + m_2 - n] \leq n_1 \leq \min [m_1, m_2]$, $0 \leq m_1, m_2 \leq n$.

In order to evaluate power for the various models, some of the properties of this distribution will be useful. Let

$$a = -m_1; \quad b = -m_2; \quad c = n - m_1 - m_2 + 1.$$

Then

$$\begin{aligned} \sum_{j=0}^{m_1} \binom{m_1}{j} \binom{n - m_1}{m_2 - j} t^j &= \sum_{j=0}^{m_1} \frac{\Gamma(1-a) \Gamma(c-b) t^j}{\Gamma(j+1) \Gamma(c+j) \Gamma(1-a-j) \Gamma(1-b-j)} \\ (2.1.2) \quad &= \frac{\Gamma(c-b)}{\Gamma(1-b) \Gamma(c)} \Gamma(1-a) \Gamma(c) \Gamma(1-b) \\ &\quad \cdot \sum_{j=0}^{m_1} \frac{t^j}{\Gamma(j+1) \Gamma(c+j) \Gamma(1-a-j) \Gamma(1-b-j)} \\ &= \frac{\Gamma(c-b)}{\Gamma(1-b) \Gamma(c)} F(a, b; c; t) \quad \text{where } F(a, b; c; t) = {}_2F_1(a, b; c; t) \end{aligned}$$

is the well-known hypergeometric function (see [22], page 1).



In this notation, the probability generating function defined by

$$(2.1.3) \quad P(\theta) \equiv \sum \Pr \{n_1 | m_1, m_2, n; t\} \theta^{n_1}$$

may be expressed as the ratio of two hypergeometric functions

$$(2.1.4) \quad P(\theta) = \frac{F(a, b; c; t\theta)}{F(a, b; c; t)} = \frac{\sum \Pr \{n_1 | m_1, m_2, n\} (\theta t)^{n_1}}{\sum_{n_1} \Pr \{n_1 | m_1, m_2, n\} (t)^{n_1}}.$$

Differentiating k times with respect to θ , one has, upon setting $\theta = 1$,

$$(2.1.5) \quad P^{(k)}(1) = \frac{(a)_k (b)_k}{(c)_k} \frac{F(a+k, b+k; c+k; t)}{F(a, b; c; t)}$$

where $(d)_k = d(d-1)\cdots(d-k+1)$.

In particular, denoting the k^{th} moments about the origin and the mean by μ_k' and μ_k respectively,

$$\mu_1' = P'(1) = \frac{ab}{c} t \frac{F(a+1, b+1; c+1; t)}{F(a, b; c; t)}$$

$$(2.1.6) \quad \mu_2' = P''(1) + P'(1) = \frac{a(a+1) b(b+1)}{c(c+1)} t^2 \frac{F(a+2, b+2; c+2; t)}{F(a, b; c; t)} \\ + \frac{ab}{c} t \frac{F(a+1, b+1; c+1; t)}{F(a, b; c; t)}$$

$$\mu_2 = P''(1) + P'(1) - (P'(1))^2 = \mu_2' - (\mu_1')^2$$

$$\mu_3' = P'''(1) + 3P''(1) + P'(1)$$



and in general, using the relation

$$(2.1.7) \quad t^k = \sum_{j=0}^k S(k, j) (t)_j$$

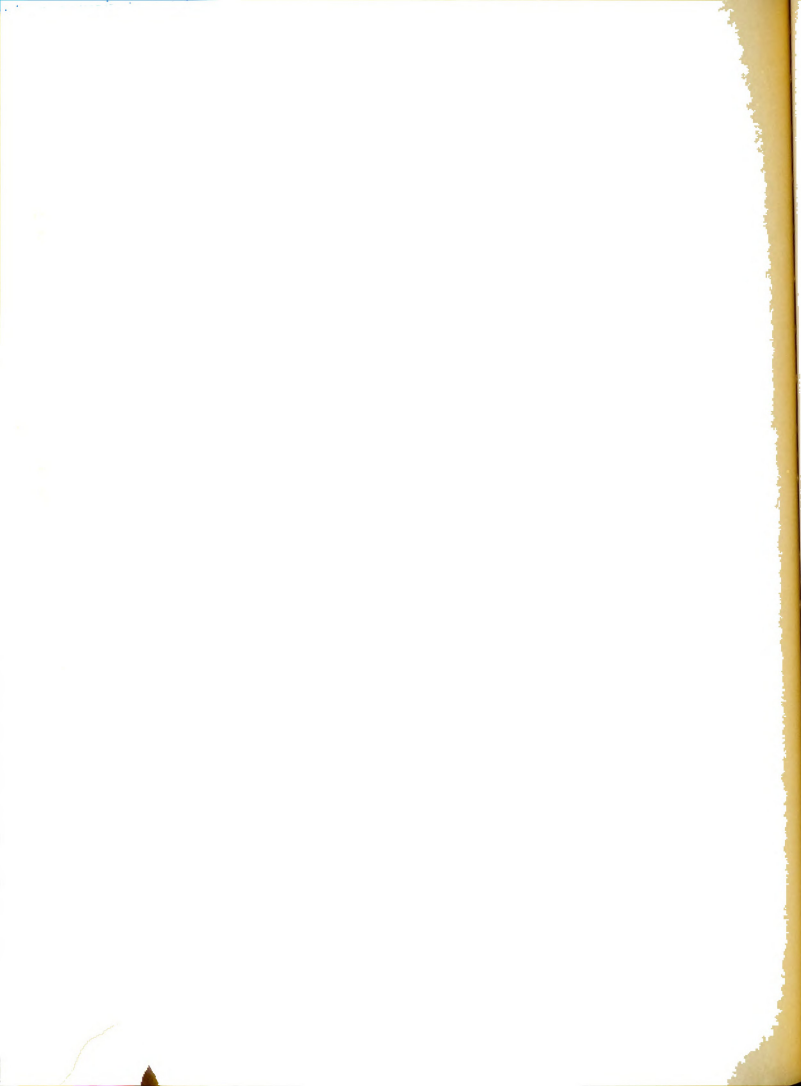
where $S(k, j)$ is the Sterling number of the 2nd kind, defined by (2.1.7) (see Riordan [18], pages 32-33), one has

$$\begin{aligned} (2.1.8) \quad \mu_k' &= E(n_1^k) = E \sum_{j=0}^k S(k, j) (n_1)_j = \sum_{j=0}^k S(k, j) E(n_1)_j \\ &= \sum_{j=0}^k S(k, j) P^{(j)}(1) \\ &= \sum_{j=0}^k S(k, j) \frac{(a)_j (b)_j}{(c)_j} t^j \frac{F(a+j, b+j; c+j; t)}{F(a, b; c; t)}. \end{aligned}$$

The central moments are found from the well-known relations

$$(2.1.9) \quad \mu_k = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \mu_i (\mu_1')^{k-i}, \quad k = 2, 3, 4, \dots$$

Computationally, we would probably prefer to use



$$\begin{aligned}
 (2.1.10) \quad \mu_k' &= \frac{\sum_j j^k \binom{m_1}{j} \binom{n-m_1}{m_2-j} t^j}{\sum_j \binom{m_1}{j} \binom{n-m_1}{m_2-j} t^j} \\
 &= \frac{\sum_j j^k \Pr \{j | m_1, m_2, n\} t^j}{\sum_j \Pr \{j | m_1, m_2, n\} t^j}
 \end{aligned}$$

in calculating moments about the origin.

The moment μ_k' is a function of t . Differentiating μ_k' , as given by (2.1.10), with respect to t , one gets

$$(2.1.11) \quad \frac{d}{dt} \mu_k' = \frac{\mu_{k+1}' - \mu_k' \mu_1'}{t}, \quad k = 1, 2, 3, \dots$$

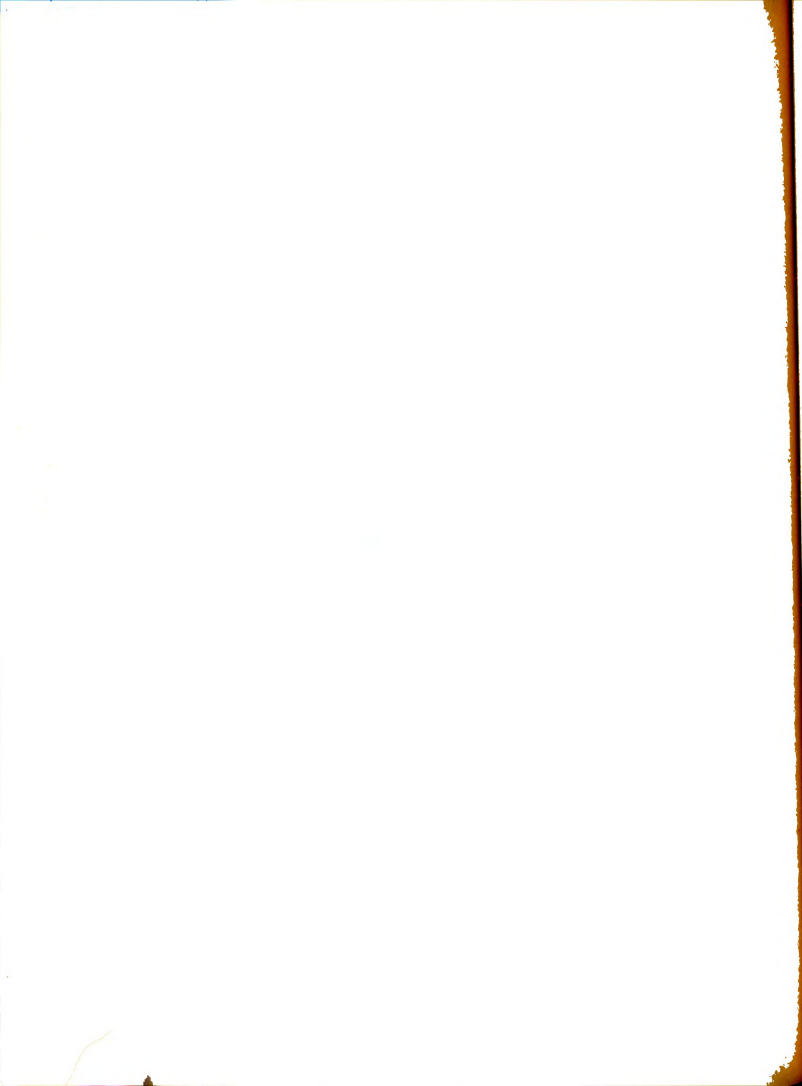
Setting $k = 1$, we find in particular that

$$(2.1.12) \quad \frac{d}{dt} \mu_1' = \frac{\mu_2' - (\mu_1')^2}{t} = \frac{\mu_2}{t} > 0$$

since $t > 0$, which confirms the obvious fact that μ_1' is a monotone increasing function of t . Putting $k = 2$ in (2.1.11), obtaining

$$\frac{d}{dt} \mu_2' = \frac{\mu_3' - \mu_2' \mu_1'}{t}$$

noting that $\mu_2 = \mu_2' - (\mu_1')^2$, and using (2.1.9) and (2.1.12), it is easily seen that



$$(2.1.13) \text{ (a)} \quad \frac{d}{dt} \mu_2 = \frac{\mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3}{t} = \frac{\mu_3}{t}$$

$$(2.1.13) \text{ (b)} \quad \frac{d^2}{dt^2} \mu_1' = \frac{\mu_3 - \mu_2}{t^2}.$$

Using well-known linear relations between hypergeometric functions (see Snow [22], pages 31-32), one derives some recurrence relations between moments. For example, using the linear relation

$$(2.1.14) \quad t(1-t) (a+1) (b+1) F(a+2, b+2; c+2; t) + (c+1)[c-(a+b+1)t]$$

$$\cdot F(a+1, b+1; c+1; t) = c(c+1) F(a, b; c; t)$$

and (2.1.5) and (2.1.6), it follows, after some simplification, that

$$(2.1.15) \quad \mu_2' = P''(1) + P'(1) = \frac{ab}{1-t} t - \frac{[c-(a+b+1)t]\mu_1'}{1-t} + \mu_1'$$

$$= \frac{m_1 m_2 t - [n - (m_1 + m_2)(1-t)] \mu_1'}{1-t}.$$

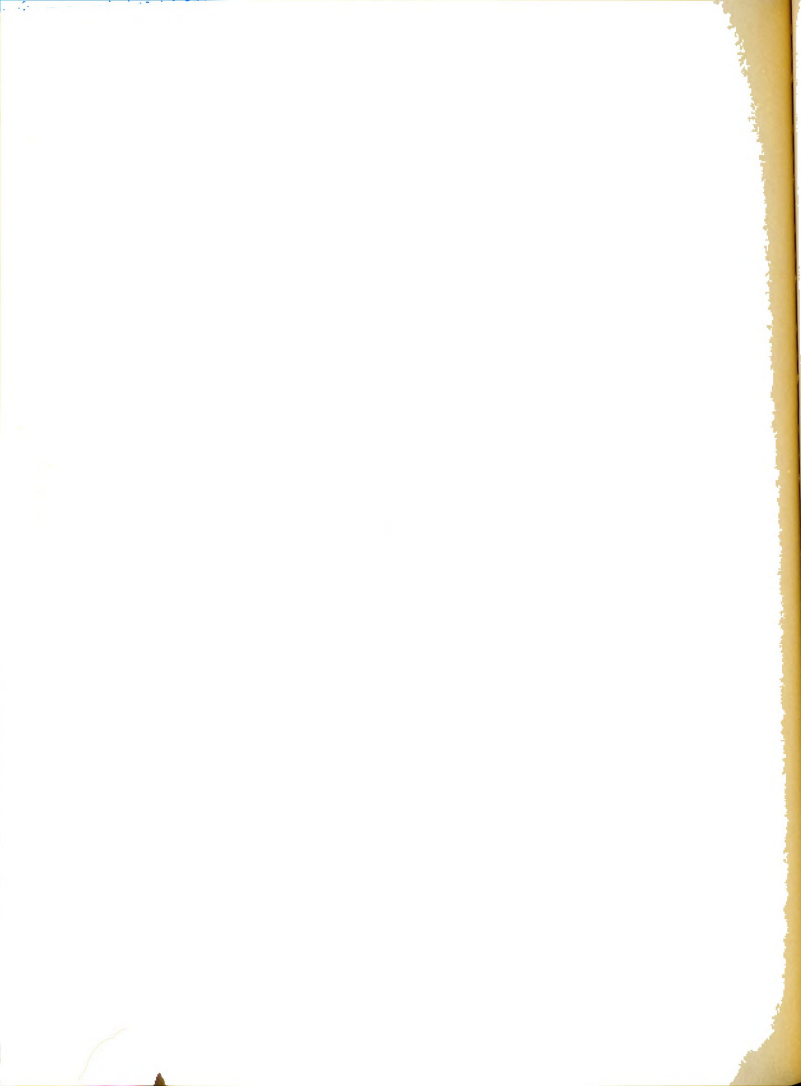
Using this form for μ_2' , one then obtains,

$$(2.1.16) \quad \mu_2 = \frac{m_1 m_2 t - [n - (m_1 + m_2)(1-t) + (1-t)\mu_1'] \mu_1'}{1-t}.$$

Using the same relation (2.1.14), of contiguity above, after some algebra, one finds that

$$(2.1.17) \quad \mu_3' = \frac{[m_1 m_2 t + n - m_1 - m_2] \mu_1' - [(n-1) - (m_1 + m_2)(1-t)] \mu_2'}{1-t}$$

and



$$(2.1.18) \quad \mu_4' = \left\{ [m_1 m_2 t + 2(n-m_1-m_2)] \mu_1' \right. \\ \left. + [(m_1-2)(m_2-2)t + 3(n-m_1-m_2+3) + 3t(m_1+m_2-5) - 11(1-t)] \mu_2' \right. \\ \left. - [(n-m_1-m_2-3) + (m_1+m_2+1)t] \mu_3' \right\} (1-t)^{-1}$$

Other recurrence forms may also be derived, using linear relations between hypergeometric functions. Using (2.1.14) again, one finds the relation

(2.1.19)

$$E \{n_1 | m_1, m_2, n; t\} = \frac{m_1 m_2 t}{(1-t)[E(n_1 | m_1-1, m_2-1, n-1; t) - (m_1+m_2-1)] + n},$$

from which one can compute $E \{n_1 | m_1, m_2, n; t\}$ from smaller values of m_1, m_2 , and n .

As soon as μ_1' is computed, we can obtain μ_2 , μ_3' , and μ_4' by substituting this value into the recurrence relations given by (2.1.16), (2.1.17), and (2.1.18) respectively, and once these moments are known, μ_3 and μ_4 can be calculated using (2.1.9). The variance μ_2 could also be obtained using (2.1.12), which asserts that

$$\mu_2 = t \frac{d}{dt} \mu_1'.$$

An example illustrating the computation of the first four moments about the origin, and the central moments μ_2, μ_3 , and μ_4 , is given at the end of this section.

The procedure just outlined for computed moments is still quite tedious, even for small values of m_1, m_2 , and n . Therefore, we now consider approximations for the moments.



Solving for μ_1' in (2.1.16), the relation between μ_2 and μ_1' takes the form

$$(2.1.20) \quad \mu_1' = \left[\frac{n-(m_1+m_2)(1-t)}{2(t-1)} \right] \pm \sqrt{\left[\frac{n-(m_1+m_2)(1-t)}{2(1-t)} \right]^2 + \frac{m_1 m_2 t}{1-t}} - \mu_2$$

where the negative square root is taken if $t > 1$ and the positive root if $t < 1$. If $t = 1$, then we have

$$\mu_1' = \frac{m_1 m_2}{n}.$$

Defining $\lambda_{(n)}^+$ by

$$(2.1.21) \quad \lambda_{(n)}^+ = \frac{n}{m_1 m_2} \left\{ \left[\frac{n-(m_1+m_2)(1-t)}{2(t-1)} \right] + \sqrt{\left[\frac{n-(m_1+m_2)(1-t)}{2(1-t)} \right]^2 + \frac{m_1 m_2 t}{1-t}} \right\}$$

and $\lambda_{(n)}^-$ in the same manner, except the negative square root replaces the positive square root, we obtain the inequalities

$$(2.1.22)(a) \quad \mu_1' \leq \lambda_{(n)}^+ \frac{m_1 m_2}{n} \quad \text{if } t \leq 1,$$

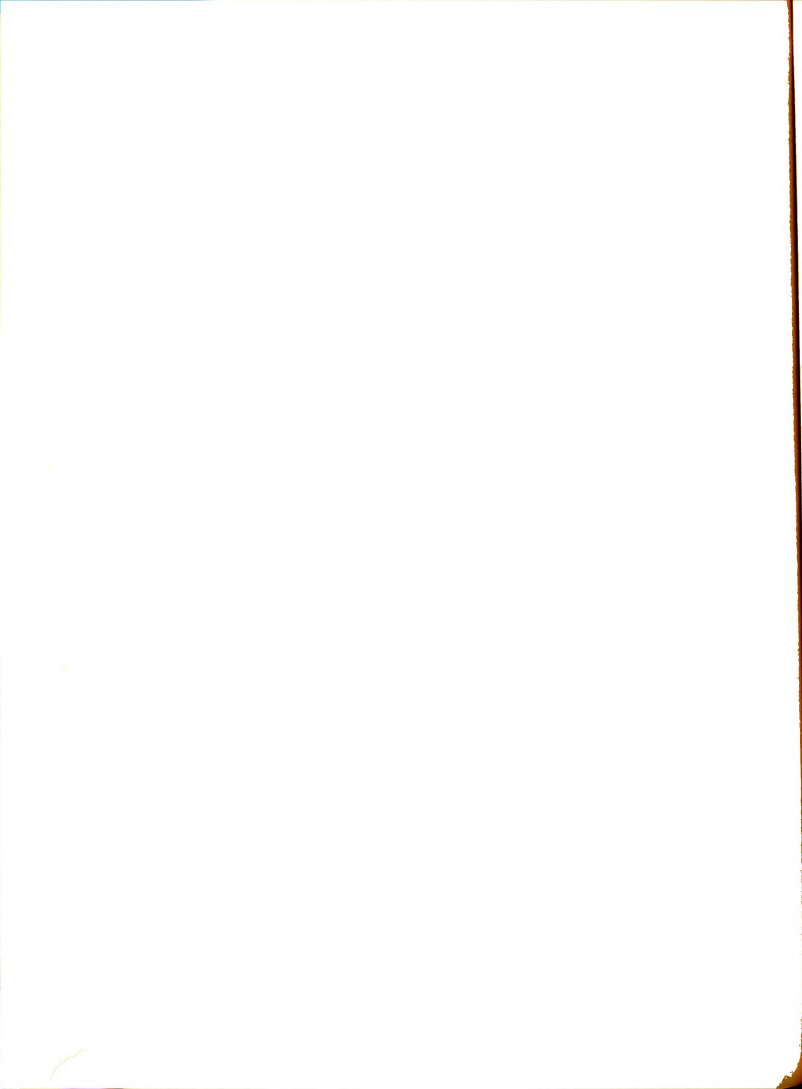
and

$$(2.1.22)(b) \quad \mu_1' \geq \lambda_{(n)}^- \frac{m_1 m_2}{n} \quad \text{if } t \geq 1,$$

using the fact that $\mu_2 \geq 0$. It is easily verified that

$\lambda_{(n)}^+$ and $\lambda_{(n)}^-$ satisfy the quadratic equation in $\lambda_{(n)}$ (so that $\lambda_{(n)}^+$ and $\lambda_{(n)}^-$ are the roots of the equation)

$$\frac{\lambda_{(n)} \left(1 - \frac{m_1}{n} - \frac{m_2}{n} + \lambda_{(n)} \frac{m_1 m_2}{n^2} \right)}{(1 - \lambda_{(n)} \frac{m_1}{n})(1 - \lambda_{(n)} \frac{m_2}{n})} = t, \quad t \text{ fixed.}$$



If, now, we suppose that $\frac{m_1}{n} \rightarrow P_A$ and $\frac{m_2}{n} \rightarrow P_B$
as $n \rightarrow \infty$, then

(2.1.23)

$$\lambda_{(n)}^+ \xrightarrow{n \rightarrow \infty} \frac{1}{P_A P_B} \left\{ \left[\frac{1 - (P_A + P_B)(1-t)}{2(t-1)} \right] + \sqrt{\left[\frac{1 - (P_A + P_B)(1-t)}{2(t-1)} \right]^2 + \frac{t P_A P_B}{1-t}} \right\} = \lambda^+.$$

$$\lambda_{(n)}^- \xrightarrow{n \rightarrow \infty} \frac{1}{P_A P_B} \left\{ \left[\frac{1 - (P_A + P_B)(1-t)}{2(t-1)} \right] - \sqrt{\left[\frac{1 - (P_A + P_B)(1-t)}{2(t-1)} \right]^2 + \frac{t P_A P_B}{1-t}} \right\} = \lambda^-$$

and λ^+ and λ^- are roots of the equation

$$(2.1.24) \quad \frac{\lambda (1 - P_A - P_B + \lambda P_A P_B)}{\lambda (1 - \lambda P_A)(1 - \lambda P_B)} = t.$$

We recall from section 1.2 that this constant arose in deriving the conditional probability of n_1 , given $m_1, m_2, n, \lambda, P_A$, and P_B . The implication of (2.1.22)(a), (2.1.22)(b), (2.1.23) is that

$$(2.1.25) \quad \lim_{n \rightarrow \infty} \frac{\mu_1'}{n} \leq \lambda^+ P_A P_B \quad \text{if } t \leq 1$$

$$\lim_{n \rightarrow \infty} \frac{\mu_1'}{n} \geq \lambda^- P_A P_B \quad \text{if } t \geq 1.$$



In the next section, it will be shown that the equality sign holds in (2.1.25). Hence, for m_1, m_2 , and n moderately large, μ_1' can be approximated very well by

$$(2.1.26) \quad \theta_{(n)}^+ = \lambda_{(n)}^+ \frac{m_1 m_2}{n} \quad \text{if } t \leq 1,$$

$$\theta_{(n)}^- = \lambda_{(n)}^- \frac{m_1 m_2}{n} \quad \text{if } t \geq 1.$$

Furthermore, it will be proven that, under the same hypotheses on the order of magnitude of m_1 and m_2 ,

$$(2.1.27) \quad \lim_{n \rightarrow \infty} \frac{\mu_2}{n} = \left[\sum_{i=1}^4 \frac{1}{\pi_i} \right]^{-1}$$

where

$$\pi_1 = \lambda P_A P_B$$

$$\pi_2 = P_B - \lambda P_A P_B$$

$$\pi_3 = P_A - \lambda P_A P_B$$

$$\pi_4 = 1 - P_A - P_B + \lambda P_A P_B.$$

Hence, as an approximation to μ_2 we have

$$(2.1.28) \quad \hat{\mu}_2 = n \left[\sum_{i=1}^4 \frac{1}{\pi_i'} \right]^{-1}$$

where π_i' is obtained from π_i by replacing P_A and P_B by

$\frac{m_1}{n}$ and $\frac{m_2}{n}$ respectively, and λ by $\lambda_{(n)}^+$ if $\lambda \leq 1$, and by

$\lambda_{(n)}^-$ if $\lambda \geq 1$.



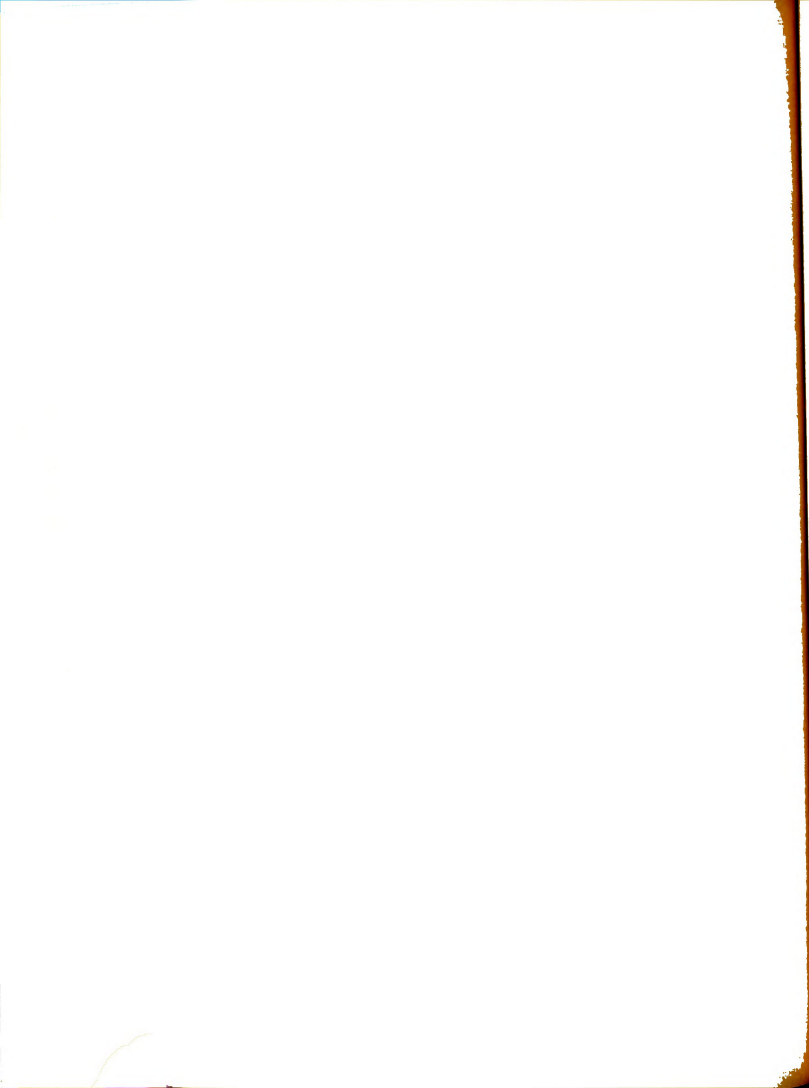
For small values of m_1, m_2 , and n , the approximation (2.1.26) can be improved if we replace μ_2 by $\hat{\mu}_2$ in (2.1.20), i.e., we approximate μ_1' by (2.1.29)

$$\hat{\mu}_1' = \left[\frac{n - (m_1 + m_2)(1-t)}{2(t-1)} + \sqrt{\frac{n - (m_1 + m_2)(1-t)}{2(t-1)} + \frac{m_1 m_2 t}{1-t} - \hat{\mu}_2} \right]$$

where the positive square root Δ is taken if $t < 1$, and the negative square root if $t > 1$.

If it is desired to approximate μ_3' , μ_4' , μ_3 , and μ_4 , then the recurrence relations given in (2.1.17) and (2.1.18), together with (2.1.9), may be used with μ_1' replaced by $\hat{\mu}_1'$, being sure to take the correct root. However, these approximations, especially for μ_4' and μ_4 , will not be as close to the true values as $\hat{\mu}_1'$ is to μ_1' , since a very small error in approximating μ_1' can build up to a sizable error when μ_4' is approximated.

For small values of m_1, m_2 , and n , (2.1.1), (2.1.10), and (2.1.16) may be used to compute exact probabilities, moments about the origin, and the variance, using tables of the hypergeometric probability distribution. Roksar [19] has four-place tables of the hypergeometric distribution for selected values of m_1, m_2 , and n , ranging to $n = 100$. To approximate the moments, (2.1.28) and (2.1.29) may be used. We now give two examples illustrating some of the recurrence relations, approximations and other properties of the probability function (2.1.1).



Example 1. We take $m_1 = 6$, $m_2 = 8$, $n = 20$, $t = 2$. The individual probabilities, accurate to nine decimal places, are given in Table VII

Table VII

n_1	$\Pr \{n_1 6, 8, 20; 2\}$
0	.003 489 268
1	.047 852 816
2	.209 356 071
3	.372 188 572
4	.279 141 429
5	.081 204 779
6	.006 767 065

Exact computation of the moments about the origin and the mean value μ_1' , using (2.1.10) and (2.1.9), yield

$$\begin{aligned}
 \mu_1' &= 3.14632267 & \mu_1 &= 0 \\
 \mu_2' &= 10.974970922 & \mu_2 &= 1.07562455 \\
 \mu_3' &= 41.24912768 & \mu_3 &= -.05019623 \\
 \mu_4' &= 164.52813316 & \mu_4 &= 3.27492641
 \end{aligned}$$

Since in the next section we will show that n_1 is asymptotically normally distributed, as an indication of the rate of convergence to the normal distribution, we compute

$$\text{also } \beta_1 = \frac{\mu_3'^2}{\mu_2'^3} = .002025 \quad \text{and} \quad \beta_2 = \frac{\mu_4'}{\mu_2'^2} = 2.83061.$$



These values compare very well with the values 0 and 3 for a normally distributed random variable. The two constants β_1 and β_2 are measures of the skewness and peakedness of a distribution.

Using (2.1.21) we find that $\lambda_{(20)} = .1553778$, and so as a first approximation to μ_1' we have $\theta_{(20)} = 3.107556$. From (2.1.28), we find that $\hat{\mu}_2 = 1.018620$, so that the improved approximate value of μ_1' employing (2.1.29) is given by

$\hat{\mu}_1' = 3.144265$. The higher moments about the origin are approximated by using the recursion relations between moments; thus, from (2.1.15), (2.1.17), and (2.1.18), we find successively that

$$\hat{\mu}_2' = 10.9050$$

$$\hat{\mu}_3' = 39.1503$$

$$\hat{\mu}_4' = 104.8391 .$$

Except for μ_4' , the approximations seem to be adequate.

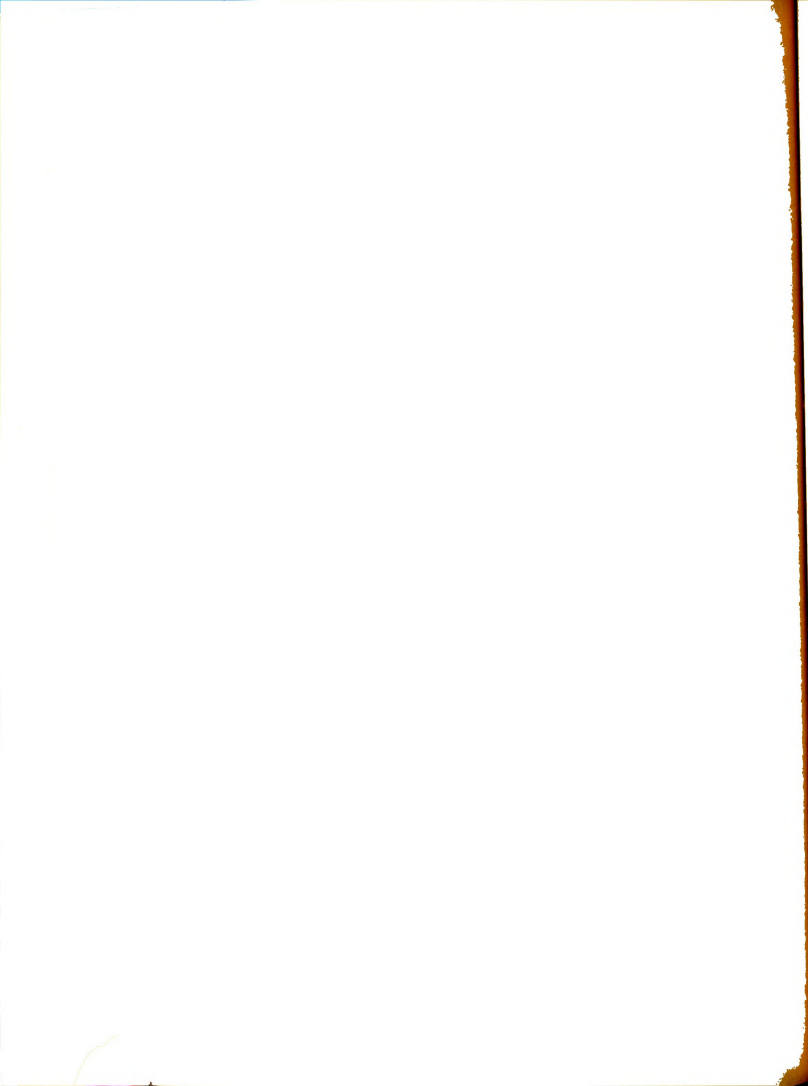
Example 2. Let $m_1 = 12$, $m_2 = 15$, $n = 30$, $t = 6$. The exact probabilities $\Pr\{n_1 | 12, 15, 30; t\}$ are given in Table VIII. The procedure for computing the exact and approximate values was given in Example 1, so only the numerical data is presented here.



Table VIII

n_1	$\Pr \{n_1 12, 15, 30; 6\}$	n_1	$\Pr \{n_1 12, 15, 30; 6\}$
0	.0000 0000	7	.0752 0799
1	.0000 0002	8	.2051 1270
2	.0000 0016	9	.3190 6421
3	.0000 0684	10	.2650 6873
4	.0001 5827	11	.1032 7353
5	.0020 8911	12	.0137 6980
6	.0162 4864		

$$\begin{aligned}
 \mu_1' &= 9.09946 & \hat{\mu}_1' &= 9.09660 & \theta_{(30)}^- &= 9.00000 \\
 \mu_2' &= 84.28220 & \hat{\mu}_2' &= 82.44000 & \mu_2 &= 1.48202 \\
 \mu_3' &= 793.51303 & \hat{\mu}_3' &= 791.03628 & \mu_3 &= -.38066 \\
 \mu_4' &= 7,584.74691 & \hat{\mu}_4' &= 7,940.10768 & \mu_4 &= 6.46250 \\
 \beta_1 &= .04452 & \beta_2 &= 2.94234 & \lambda_{(30)}^- &= 1.5
 \end{aligned}$$



2.2 Limiting Distributions and Approximations

The asymptotic properties of the probability function (2.1.1) will be of great importance in finding approximations for both the individual probabilities and the corresponding distribution function. In this section, three limit theorems, (2.2.D), (2.2.E) and (2.2.F), are given for (2.1.1).

Since (2.1.1) is of the form

$$(2.2.1) \quad \Pr \{n_1 | m_1, m_2, n; t\} = k(m_1, m_2, n; t) \Pr \{n_1 | m_1, m_2, n\} t^{n_1},$$

where $\Pr(n_1 | m_1, m_2, n)$ is the ordinary hypergeometric function as given by (1.2.6), and $k(m_1, m_2, n; t)$ is the reciprocal of the sum of $\Pr \{n_1 | m_1, m_2, n\} t^{n_1}$ over all possible values of n_1 , it would seem plausible that limit theorems for (2.1.1) would follow readily from corresponding limit theorems for $\Pr \{n_1 | m_1, m_2, n\}$. Three such limit theorems have been established by Feller for the hypergeometric function.

Th. 2.2.A. (Normal Convergence: Feller [5], page 180).

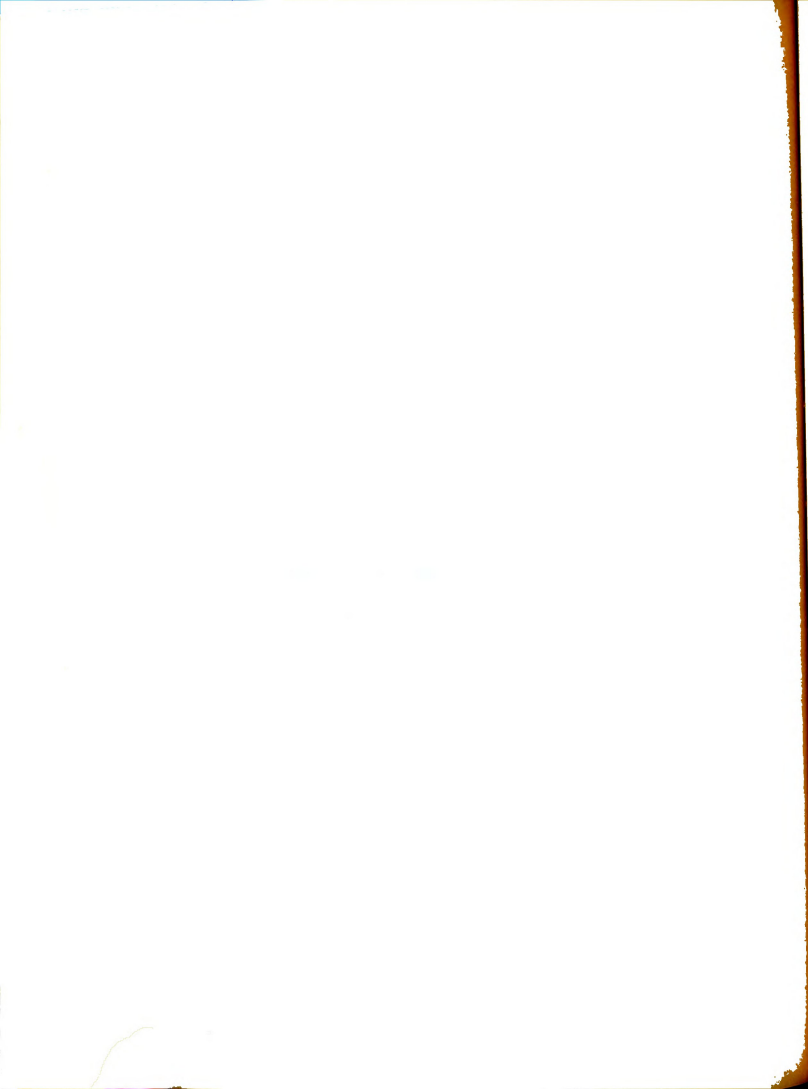
If $\frac{m_1}{n} \longrightarrow r, \frac{m_2}{n} \longrightarrow s, (0 < r, s < 1),$ and

$$\frac{n_1 - m_1 m_2 n^{-1}}{h} = x_{n_1} \longrightarrow x, \text{ where } h = \sqrt{\frac{m_1 m_2 (n-m_1) (n-m_2)}{n^2 (n-1)}},$$

then

$$(2.2.2) \quad \Pr \{n_1 | m_1, m_2, n\} \sim h^{-1} \phi(x_{n_1})$$

where the symbol " $\sim$ " means that the ratio of the two sides



tends to unity, and ϕ is the normal density $(2\pi)^{-\frac{1}{2}} \exp \left[\frac{-y^2}{2} \right]$.

Th. 2.2.B (Binomial Convergence: Feller [5], page 180).

If $\frac{m_1}{n} \longrightarrow r > 0$, $\frac{n-m_1}{n} \longrightarrow 1-r > 0$, then for each fixed m_2

$$(2.2.3) \quad \lim_n \Pr \{n_1 | m_1, m_2, n\} = \binom{m_2}{n_1} r^{n_1} (1-r)^{m_2-n_1}.$$

Th. 2.2.C (Poisson Convergence: Feller [5], page 180).

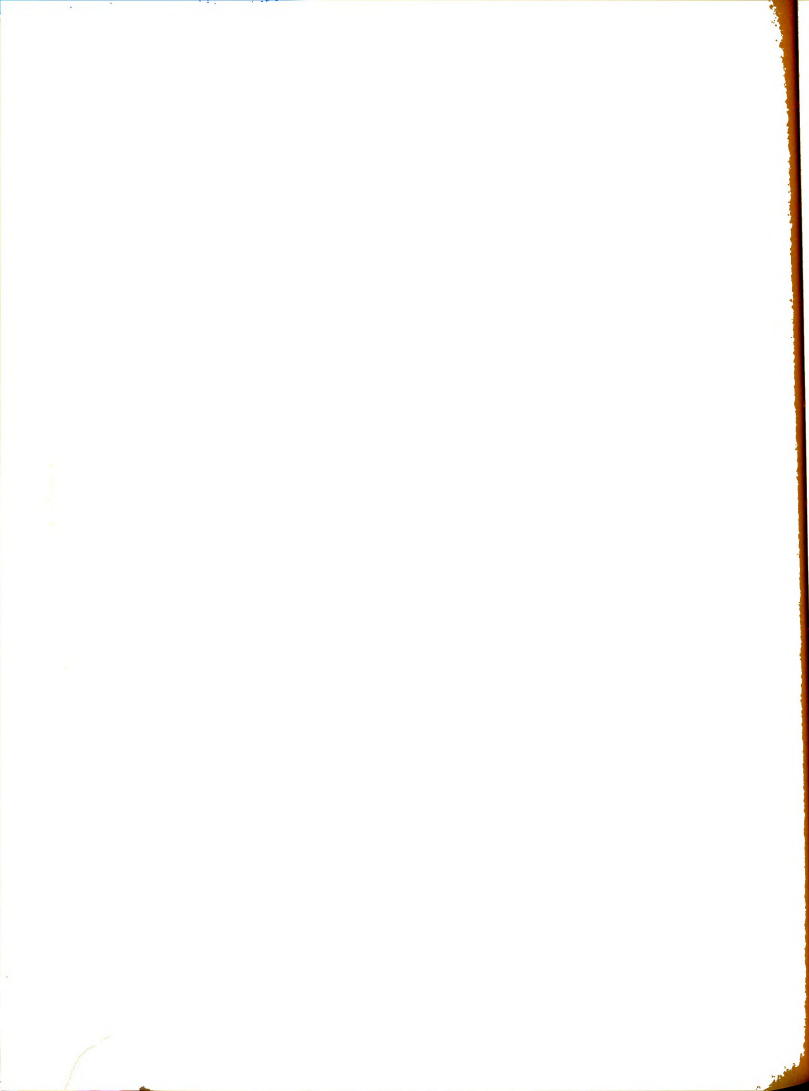
If $\frac{m_1 m_2}{n} \longrightarrow v$, $0 < v < \infty$, as $m_1, m_2, n \longrightarrow \infty$, then

$$(2.2.4) \quad \lim_n \Pr \{n_1 | m_1, m_2, n\} = \frac{e^{-v} v^{n_1}}{n!}.$$

In the next few pages, similar theorems on convergence to the normal, binomial, and Poisson laws will be proven for (2.1.1), with convergence to the binomial and Poisson Laws following almost trivially from Theorems 2.2.B and 2.2.C., whereas convergence to the normal density will be proven by suitably modifying Feller's proof of the convergence of the binomial distribution to the normal distribution.

Theorems 2.2.D and 2.2.E below are easy consequences of (2.2.3) and (2.2.4), and are therefore given first.

Th. 2.2.D If $\frac{m_1}{n} \longrightarrow r > 0$, $\frac{n-m_1}{n} \longrightarrow 1-r > 0$, then for



each fixed m_2 ,

$$\lim_{n \rightarrow \infty} \Pr \{n_1 | m_1, m_2, n; t\} = \binom{m_2}{n_1} \left(\frac{tr}{tr + 1 - r} \right)^{n_1} \left(\frac{1}{tr + 1 - r} \right)^{m_2 - n_1}.$$

Proof: In proving Th. 2.2.B, Feller has shown that

$$(2.2.5) \quad \binom{m_2}{n_1} \left(\frac{m_1}{n} - \frac{n_1}{n} \right)^{n_1} \left(\frac{n - m_1}{n} - \frac{m_2 - n_1}{n} \right)^{m_2 - n_1} \\ < \Pr \{n_1 | m_1, m_2, n\} < \binom{m_2}{n_1} \left(\frac{m_1}{n} \right)^{n_1} \left(\frac{n - m_1}{n} \right)^{m_2 - n_1} \left(1 - \frac{m_2}{n} \right)^{-m_2}$$

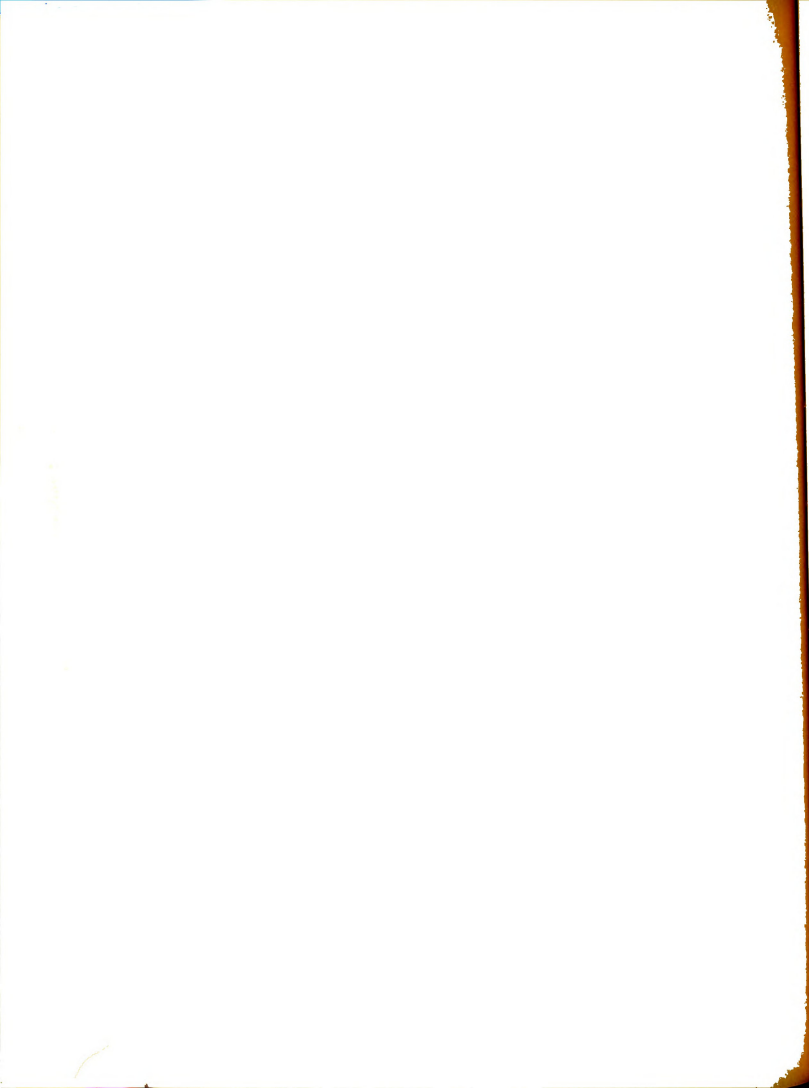
$$\text{Since } \Pr \{n_1 | m_1, m_2, n; t\} = \frac{\Pr \{n_1 | m_1, m_2, n\} t^{n_1}}{\sum_j \Pr \{j | m_1, m_2, n\} t^j},$$

the following two inequalities, using (2.2.5), are obtained:

$$(2.2.6)(a) \quad \Pr \{n_1 | m_1, m_2, n; t\} > \frac{\binom{m_2}{n_1} \left(\frac{m_1 t}{n} - \frac{n_1 t}{n} \right)^{n_1} \left(\frac{n - m_1}{n} - \frac{m_2 - n_1}{n} \right)^{m_2 - n_1}}{\sum_j \binom{m_2}{j} \left(\frac{m_1 t}{n} \right)^j \left(\frac{n - m_1}{n} \right)^{m_2 - j} \left(1 - \frac{m_2}{n} \right)^{-m_2}};$$

$$(2.2.6)(b) \quad \Pr \{n_1 | m_1, m_2, n; t\} < \frac{\binom{m_2}{n_1} \left(\frac{m_1 t}{n} \right)^{n_1} \left(\frac{n - m_1}{n} \right)^{m_2 - n_1} \left(1 - \frac{m_2}{n} \right)^{-m_2}}{\sum_j \binom{m_2}{j} \left(\frac{m_1 t}{n} - \frac{jt}{n} \right)^j \left(\frac{n - m_1}{n} - \frac{m_2 - j}{n} \right)^{m_2 - j}}.$$

For each fixed m_2 both of the sums in the denominators in



(2.2.6)(a) and (2.2.6)(b) are finite and positive, so that

$$(2.2.7)(a) \quad \lim_{n \rightarrow \infty} \Pr \{n_1 | m_1, m_2, n; t\} \geq \frac{\binom{m_2}{n_1} (rt)^{n_1} (1-r)^{m_2-n_1}}{\sum_j \binom{m_2}{j} (rt)^j (1-r)^{m_2-j}}$$

$$= \left[\binom{m_2}{n_1} \left(\frac{rt}{rt+1-r} \right)^{n_1} \left(\frac{1-r}{rt+1-r} \right)^{m_2-n_1} \right]$$

Similarly, it is easily shown that

$$(2.2.7)(b) \quad \lim_{n \rightarrow \infty} \Pr \{n_1 | m_1, m_2, n; t\} \leq \binom{m_2}{n_1} \left(\frac{rt}{rt+1-r} \right)^{n_1} \left(\frac{1-r}{rt+1-r} \right)^{m_2-n_1}$$

Thus, (2.2.7)(a) and (2.2.7)(b) imply the desired result.

Th.2.2.E If $\frac{m_1 m_2 t}{m_1 t + n - m_1} = v_n \longrightarrow v$, ($0 < v < \infty$), as

$$m_1, m_2, n \longrightarrow \infty,$$

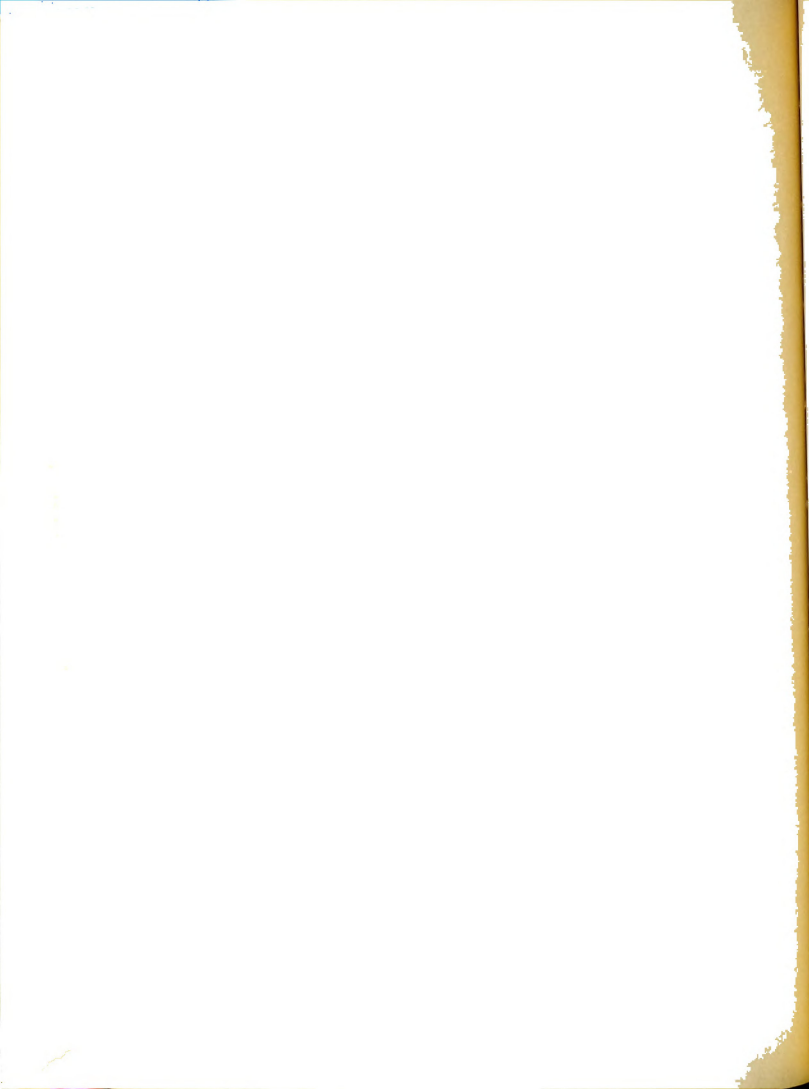
then

$$\lim_{n \rightarrow \infty} \Pr \{n_1 | m_1, m_2, n; t\} = \frac{e^{-v} v^{n_1}}{n_1!}.$$

Proof: Let $p_n = \frac{m_1 t}{m_1 t + n - m_1}$. Then the hypotheses imply

$p_n \longrightarrow 0$ in such a way that $m_2 p_n \longrightarrow v$, and the Poisson limit theorem for the binomial distribution implies the conclusion of the theorem.

One would probably suspect that convergence of (2.1.1),



suitably normalized, to the normal density would follow directly from Theorem 2.2.A, according to the heuristic argument below.

Write (2.1.1) in the form (2.2.1), and replace $\Pr\{n_1|m_1, m_2, n\}$ by its asymptotic equivalent normal form, i.e., replace $\Pr\{n_1|m_1, m_2, n\}$ by the ordinate of a normal distribution with mean $\frac{m_1 m_2}{n}$ and variance

$$h^2 = \frac{m_1 m_2 (n-m_1) (n-m_2)}{n^2(n-1)}, \text{ obtaining}$$

$$(2.2.8) \quad \Pr\{n_1|m_1, m_2, n; t\} \sim k(m_1, m_2, n; t)$$

$$\cdot \frac{1}{\sqrt{2\pi} h} e^{-\frac{1}{2h^2}(n_1 - \frac{m_1 m_2}{n})^2} e^{n_1 \log t}$$

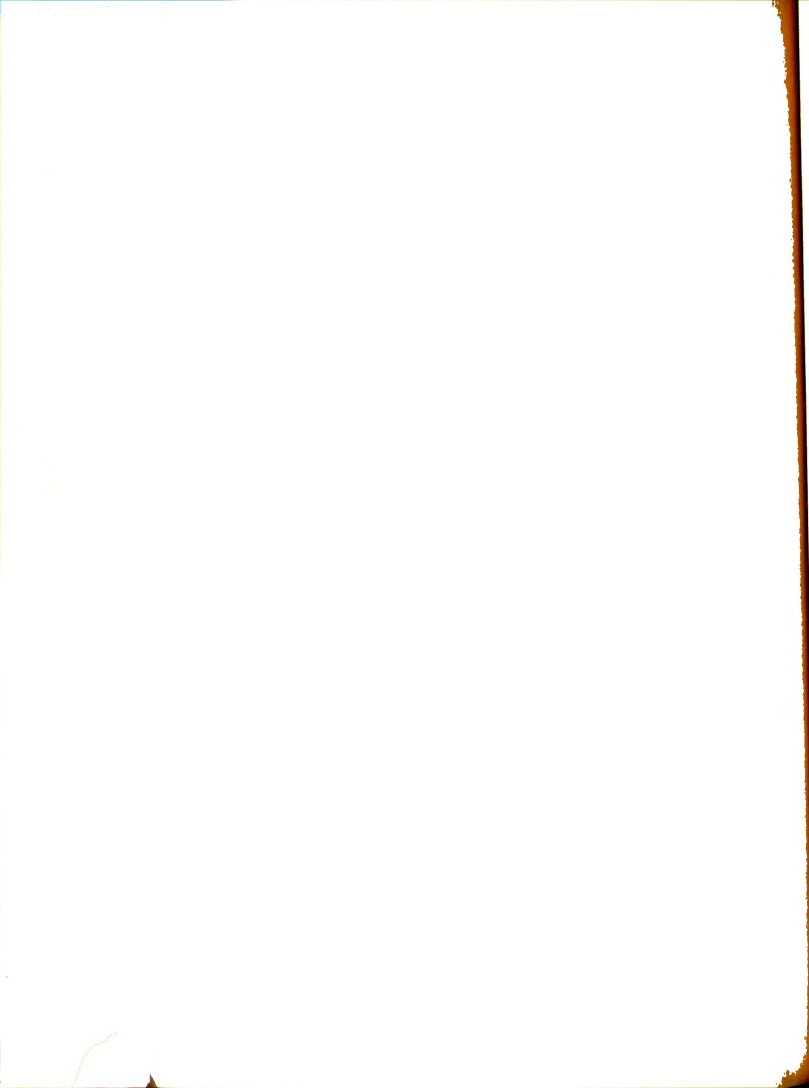
Completing the square in n_1 in the exponent and integrating to determine the constant $k(m_1, m_2, n; t)$, one would obtain as an approximation,

$$(2.2.9) \quad \Pr\{n_1|m_1, m_2, n; t\} \sim \frac{1}{\sqrt{2\pi} h} e^{-\frac{1}{2}(\frac{n_1 - \chi_1}{h})^2}$$

where $\chi_1 = \frac{m_1 m_2}{n} + h^2 \log t$, so that for large values of

m_1, m_2 and n , n_1 would be approximately normal, with mean χ_1

and variance h^2 .



This is purely an heuristic argument, but at first sight it seems plausible that a limit theorem could be obtained using the above as a rough outline. That this is not the case, however, is evident upon further consideration. If the above procedure were valid, then the same procedure applied to a binomial random variable should also be valid. However, if the binomial probability law is modified in the same manner that the hypergeometric law was modified, one would have

$$(2.2.10) \quad \Pr \{k; n, p; t\} = \frac{\binom{n}{k} p^k q^{n-k} t^k}{\sum_{j=0}^n \binom{n}{j} p^j q^{n-j} t^j} .$$

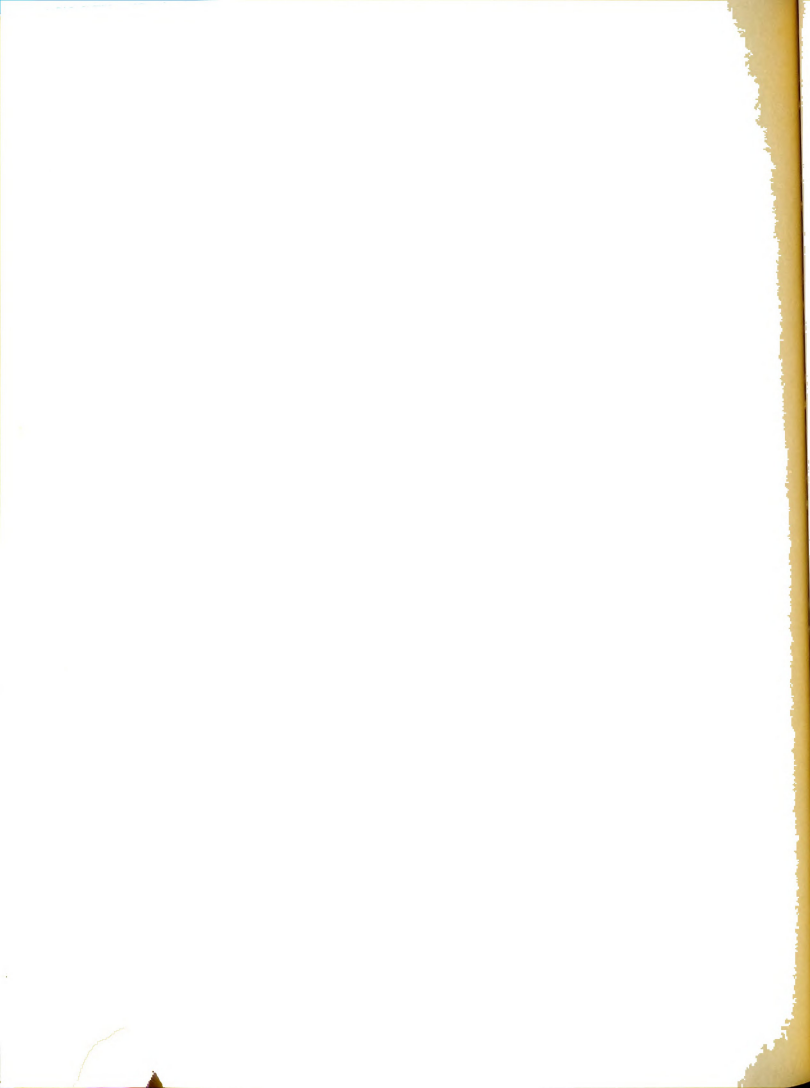
$$\text{Since } \binom{n}{k} p^k q^{n-k} \sim \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \frac{(k-np)^2}{\sigma^2}}, \quad \sigma^2 = npq,$$

one would find, following the procedure outlined above, that

$$(2.2.11) \quad \Pr \{k; n, p; t\} \sim \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \frac{(k-np-\sigma^2 \log t)^2}{\sigma^2}}$$

i.e., k would be approximately normal, with mean and variance

$$\bar{y} = np + \sigma^2 \log t, \quad \sigma^2 = npq$$



But, (2.2.10) can be written as

$$(2.2.12) \Pr \{k; n, p; t\} = \binom{n}{k} \left(\frac{pt}{pt+q} \right)^k \left(\frac{q}{pt+q} \right)^{n-k},$$

and it follows from the well-known limit theorem for the binomial distribution that k is asymptotically normal with mean

$$\gamma_2 = \frac{npt}{pt+q} \quad \text{and variance} \quad \sigma_2^2 = \frac{npqt}{(pt+q)^2}.$$

Comparing the two means γ and γ_2 and the two variances σ^2 and σ_2^2 , one has

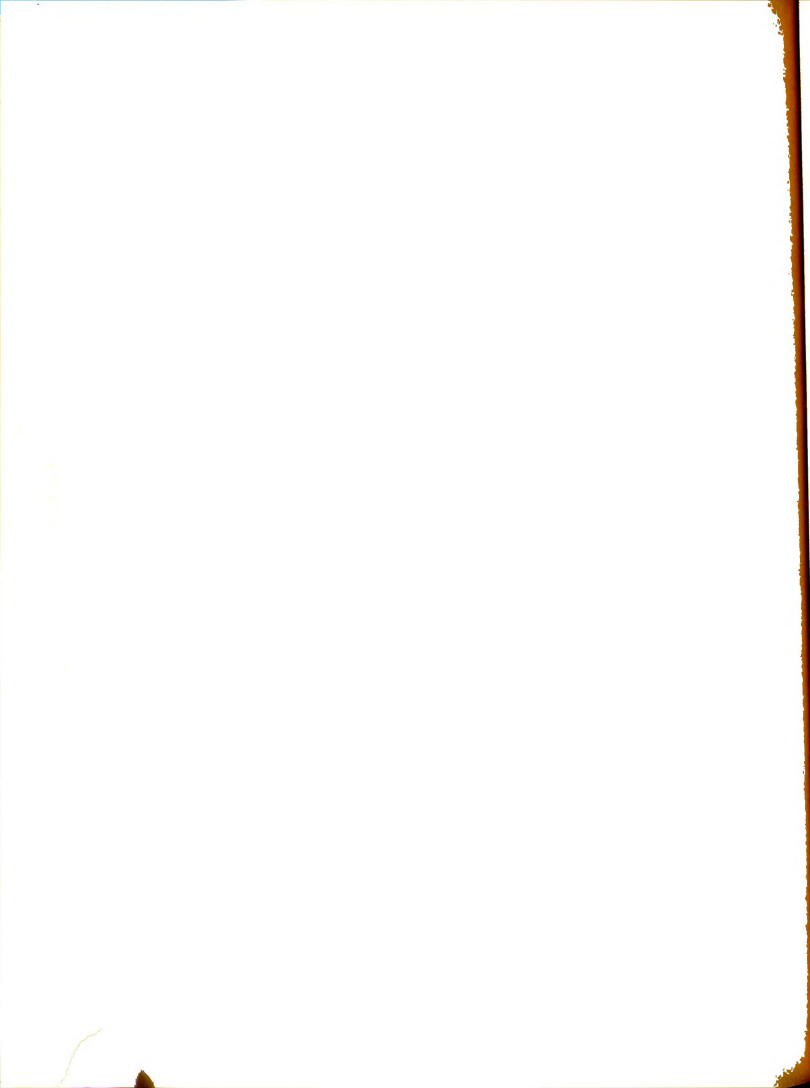
$$\frac{\gamma}{\gamma_2} = \frac{(pt+q)[1 + q \log t]}{t}, \quad \frac{\sigma^2}{\sigma_2^2} = \frac{(pt+q)^2}{t}$$

so that γ and σ^2 will depart monotonically away from the true mean and variance as t varies away from 1. In the neighborhood of $t = 1$, however, the results are not too dissimilar. It will be seen later that a similar result is obtained for the modified hypergeometric probability law. For very small values of t , the heuristic approximation (2.2.9) gives absurd results. For example, suppose $\log t < -4$,

and $m_1 = m_2 = \frac{n}{2}$. Then

$$\gamma = \frac{n}{4} + \frac{n}{16} \log t = n[.25 + .0625 \log t] < n(.25 - .25) = 0,$$

so that an everywhere positive random variable would have a negative expectation, which is impossible. Nevertheless, the approximation (2.2.9) has been used on at least two occasions

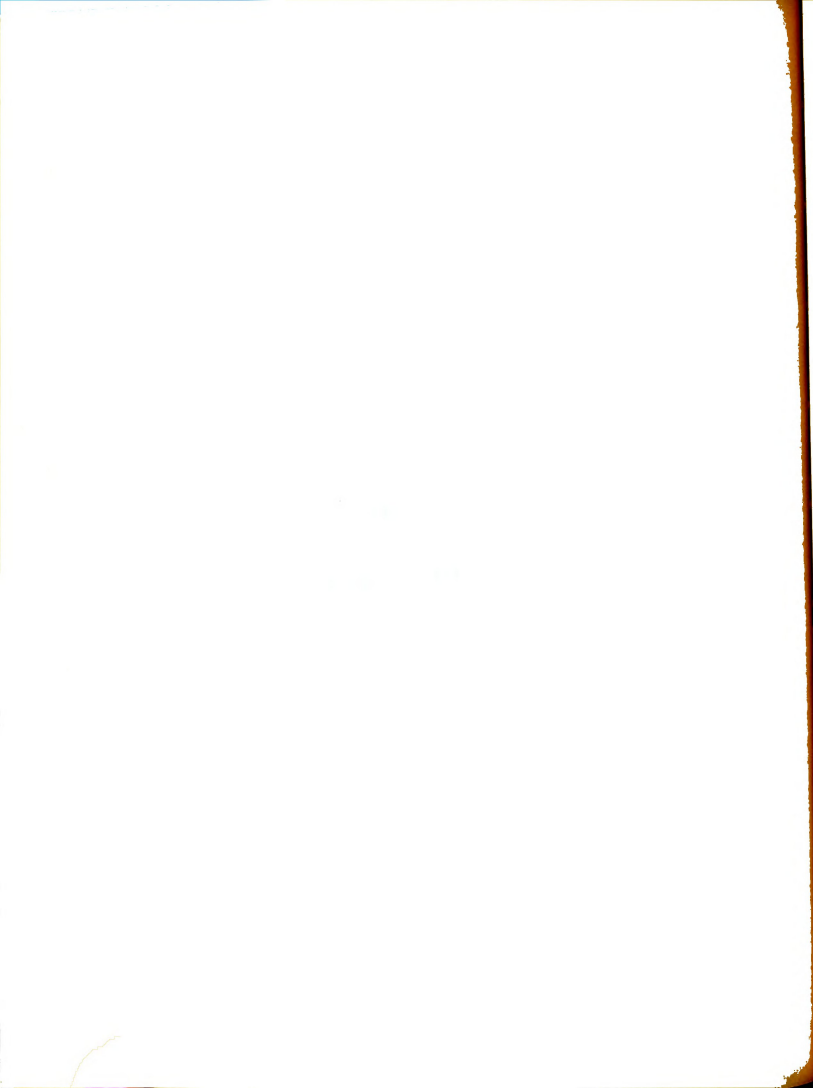


in approximating a power function; once by Moore [13] and again by Patnaik [14] in deriving an approximation to the power for the test of independence in the 2×2 comparative trial. Patnaik's approximation will be discussed later in section 3.2.

The crux of the matters discussed above is that an alternative approach towards finding the correct limit theorem corresponding to Th. 2.2.A must be adopted. This will be accomplished by following Feller's argument in his normal approximation to the binomial.

The asymptotic distribution could also be obtained by using a theorem due to Steck [23, Th. 2.2, page 248], but the following argument is less difficult and more direct; in addition, it gives an asymptotic formula for the individual terms. Before proceeding to the derivation, however, it will first be necessary to observe that Feller's binomial limit theorem can be stated in a stronger form than is usually given.

Putting $h = (npq)^{-\frac{1}{2}}$ and $x_k = (k-np)h$, Feller's binomial-approximation theorem asserts that $b(k;n,p) \sim h\phi(x_k)$ if $n \rightarrow \infty$, and $k \rightarrow \infty$ in such a way that $hx_k^3 \rightarrow 0$, and where p is held fixed. However, if p is free to vary but still $hx_k^3 \rightarrow 0$ as $n \rightarrow \infty$, $k \rightarrow \infty$ and $n-k \rightarrow \infty$, Feller's theorem still holds since the proof remains the same, except p is to be replaced by p_n , h by $h_n = (np_nq_n)^{-\frac{1}{2}}$, and x_k by $(k-np_n)h_n$. In the derivation of the asymptotic



formula for the individual probabilities $\Pr \{n_1 | m_1, m_2, n; t\}$ below, the quantities p_1 and p_2 to be introduced are not fixed, depending on n , but the notation will not show this.

The quantity t in $\Pr \{n_1 | m_1, m_2, n; t\}$ is a positive constant and independent of m_1, m_2 , and n . Write $\Pr \{n_1 | m_1, m_2, n; t\}$ in the form

$$\begin{aligned}
 (2.2.13) \quad \Pr \{n_1 | m_1, m_2, n; t\} &= \frac{b(n_1; m_1, p_1) \, b(m_2 - n_1; n - m_1, p_2)}{\sum_{n_1} b(n_1; m_1, p_1) \, b(m_2 - n_1; n - m_1, p_2)} \\
 &= \frac{\binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum_{n_1} \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}
 \end{aligned}$$

where p_1 and p_2 are chosen so that $\frac{p_1 q_2}{p_2 q_1} = t$ and

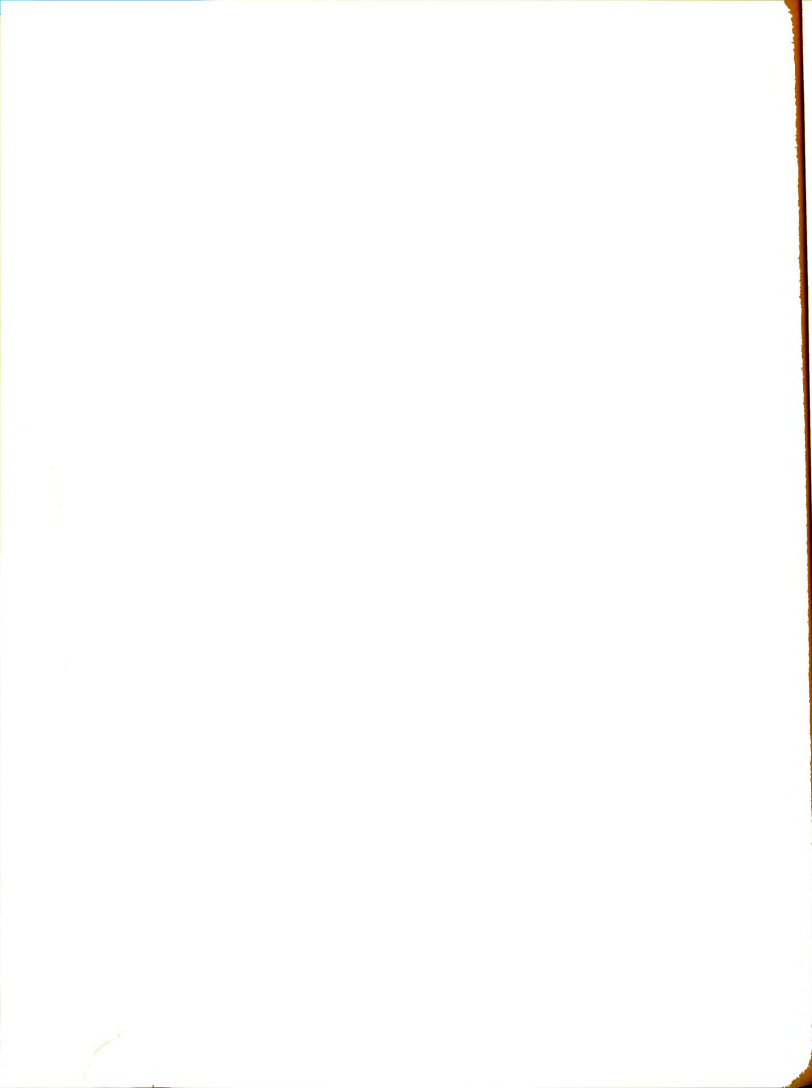
$$m_2 = m_1 p_1 + (n - m_1) p_2.$$

Letting $h_1 = (m_1 p_1 q_1)^{-\frac{1}{2}}$, $h_2 = [(n - m_1) p_2 q_2]^{-\frac{1}{2}}$, $h^2 = h_1^2 + h_2^2$,

$$(2.2.14) \quad h_1 = (m_1 p_1 q_1)^{-\frac{1}{2}}, \quad h_2 = [(n - m_1) p_2 q_2]^{-\frac{1}{2}}, \quad h^2 = h_1^2 + h_2^2,$$

$$x_{1, n_1} = h_1 (n_1 - m_1 p_1), \quad x_{2, n_1} = h_2 [n_1 - (n - m_1) p_2],$$

$$x_{n_1} = h (n_1 - m_1 p_1)$$



and noting that since $m_2 - (n-m_1)p_2 = m_1p_1$,

$$(2.2.15) \quad x_{2,m_2-n_1} = h_2[(m_2-n_1) - (n-m_1)p_2] = -h_2(n_1-m_1p_1),$$

two applications of the extended Feller-binomial-approximation yield

$$(2.2.16) \quad b(n_1; m_1, p_1) b(m_2-n_1; n-m_1, p_2) \sim h_1 \phi(x_{1,n_1}) h_2 \phi(x_{2,m_2-n_1})$$

as $h \rightarrow 0$, $hx_{n_1}^3 \rightarrow 0$. This is true since $h_1 \leq h$, so that

$$hx_{n_1}^3 \rightarrow 0 \text{ obviously implies } h_1 x_{1,n_1}^3 \rightarrow 0 \text{ and } hx_{2,m_2-n_1}^3 \rightarrow 0.$$

Conversely, if $h_1 x_{1,n_1}^3 \rightarrow 0$, and $h_2 x_{2,m_2-n_1}^3 \rightarrow 0$, then

$$hx_{n_1}^3 \rightarrow 0 \text{ follows from } h_1 x_{1,n_1}^3 = h_1 \left(\frac{h_1}{h}\right)^3 x_{n_1}^3$$

$$\text{and } h_2 x_{2,m_2-n_1}^3 = -h_2 \left(\frac{h_2}{h}\right)^3 x_{n_1}^3, \text{ and the fact that}$$

$h \rightarrow 0 \iff h_1 \text{ and } h_2 \rightarrow 0$. Using the definition of

h_1, x_{1,n_1} , and (2.2.15), (2.2.16) is easily transformed to

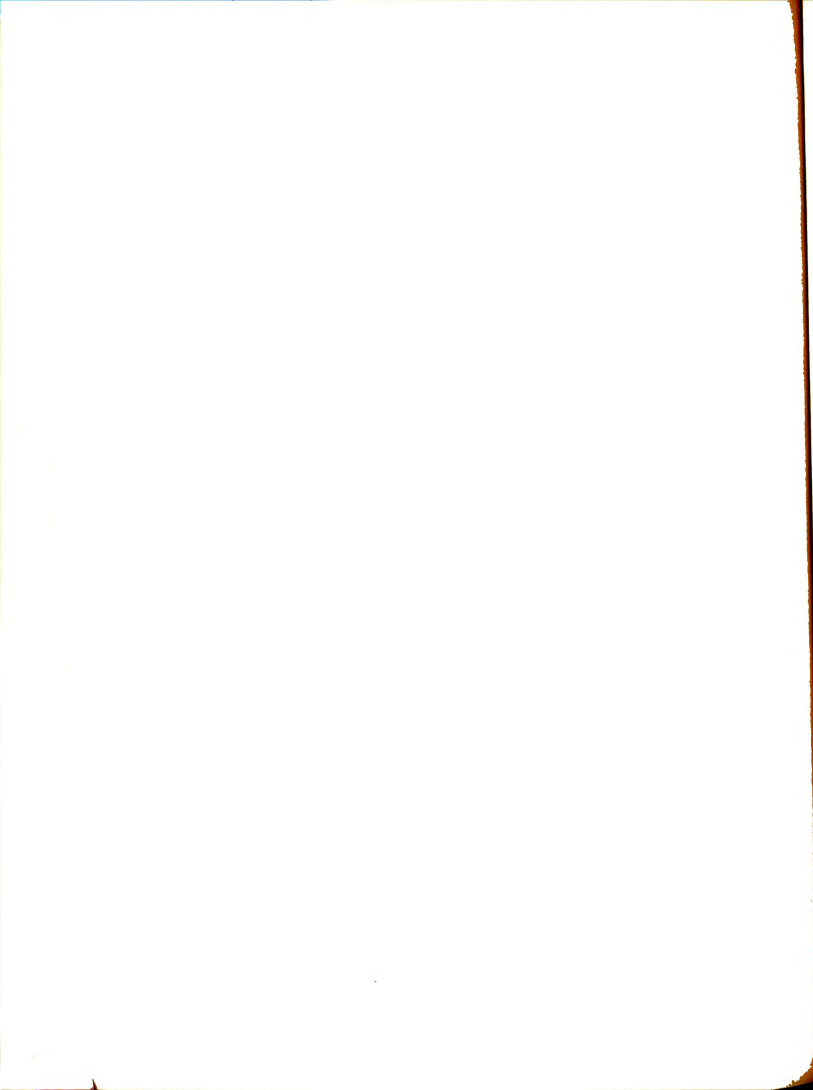
$$(2.2.17) \quad b(n_1; m_1, p_1) b(m_2-n_1; n-m_1, p_2) \sim (2\pi)^{-\frac{1}{2}} h_1 h_2 \phi(x_{n_1})$$

and hence

$$(2.2.18) \quad (2\pi)^{-\frac{1}{2}} \frac{h}{h_1 h_2} b(n_1; m_1, p_1) b(m_2-n_1; n-m_1, p_2) \sim h \phi(x_{n_1})$$

as $h, hx_{n_1}^3 \rightarrow 0$. Just as in Feller's treatment of the

binomial case, it follows that



$$(2.2.19) \quad \sum_{n_1=\alpha}^{\beta} (2\pi)^{\frac{1}{2}} \frac{h}{h_1 h_2} b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2)$$

$$\sim \sum_{\alpha}^{\beta} h \phi(x_{n_1}) \sim \Phi(x_{\beta + \frac{1}{2}}) - \Phi(x_{\alpha - \frac{1}{2}})$$

as $h, hx_{\alpha}^3, hx_{\beta}^3 \rightarrow 0$.

Thus, if there exists tails $n_1 < \alpha$ and $n_1 > \beta$, with $x_{\alpha} \rightarrow -\infty$, $x_{\beta} \rightarrow +\infty$ while still hx_{α}^3 and $hx_{\beta}^3 \rightarrow 0$, such that the sum of the left hand side of (2.2.17) over these tail values converges to zero, then

$$\sum_{n_1} (2\pi)^{\frac{1}{2}} \frac{h}{h_1 h_2} b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2) \rightarrow 1, \text{ and so}$$

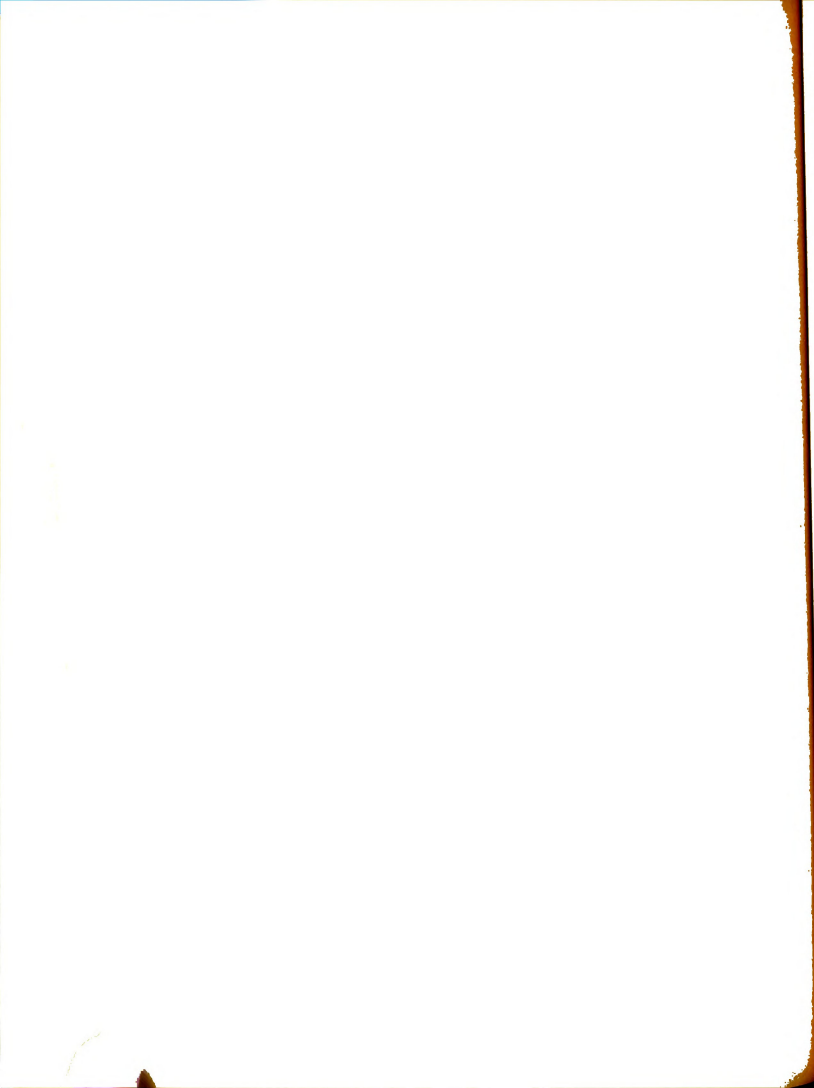
$$(2.2.20) \quad \Pr\{n_1 | m_1, m_2, n; t\} \sim h \phi(x_{n_1})$$

as $h, hx_{n_1}^3 \rightarrow 0$.

But if $\beta > m_1 p_1$ and is such that $x_{\beta} \rightarrow \infty$ while

$hx_{\beta}^3 \rightarrow 0$, then

$$\begin{aligned} \sum_{n_1 > \beta} b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2) &< b(\beta; m_1, p_1) \sum_{n_1 > \beta} b(m_2 - n_1; n - m_1, p_2) \\ &< b(\beta; m_1, p_1) \sim h_1 \phi(x_{1, \beta}), \text{ and therefore} \end{aligned}$$



$$(2.2.21) \quad \sum_{n_1 > \beta} (2\pi)^{\frac{1}{2}} \frac{h}{h_1 h_2} b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2)$$

$$< (2\pi)^{\frac{1}{2}} \frac{h}{h_2} \delta(x_{1,\beta}) \leq (2\pi)^{\frac{1}{2}} (1 + \frac{h_1}{h_2}) \delta(x_{1,\beta}) \longrightarrow 0,$$

and similarly for $\sum_{n_1 < \alpha} (2\pi)^{\frac{1}{2}} \frac{h}{h_1 h_2} b(n_1; m_1, p_1) b(m_2 - n_1; n - m_1, p_2).$

Therefore (2.2.19) is established.

The remaining question, since (2.2.19) is expressed in terms of m_1, m_2, n , and t only thru the parameters p_1 and p_2 , is to try to interpret the conditions $h, h x_{n_1}^3 \longrightarrow 0$ in terms of m_1, m_2 , and n for t fixed in $(0, \infty)$.

The condition $h \longrightarrow 0$ clearly implies that h_1 and $h_2 \longrightarrow 0$, or $m_1 p_1 q_1 \longrightarrow \infty$ and $(n - m_1) p_2 q_2 \longrightarrow \infty$, and therefore $m_1 p_1, m_1 q_1, (n - m_1) p_2, (n - m_1) q_2 \longrightarrow \infty$. But then

$$m_2 = m_1 p_1 + (n - m_1) p_2 \longrightarrow \infty$$

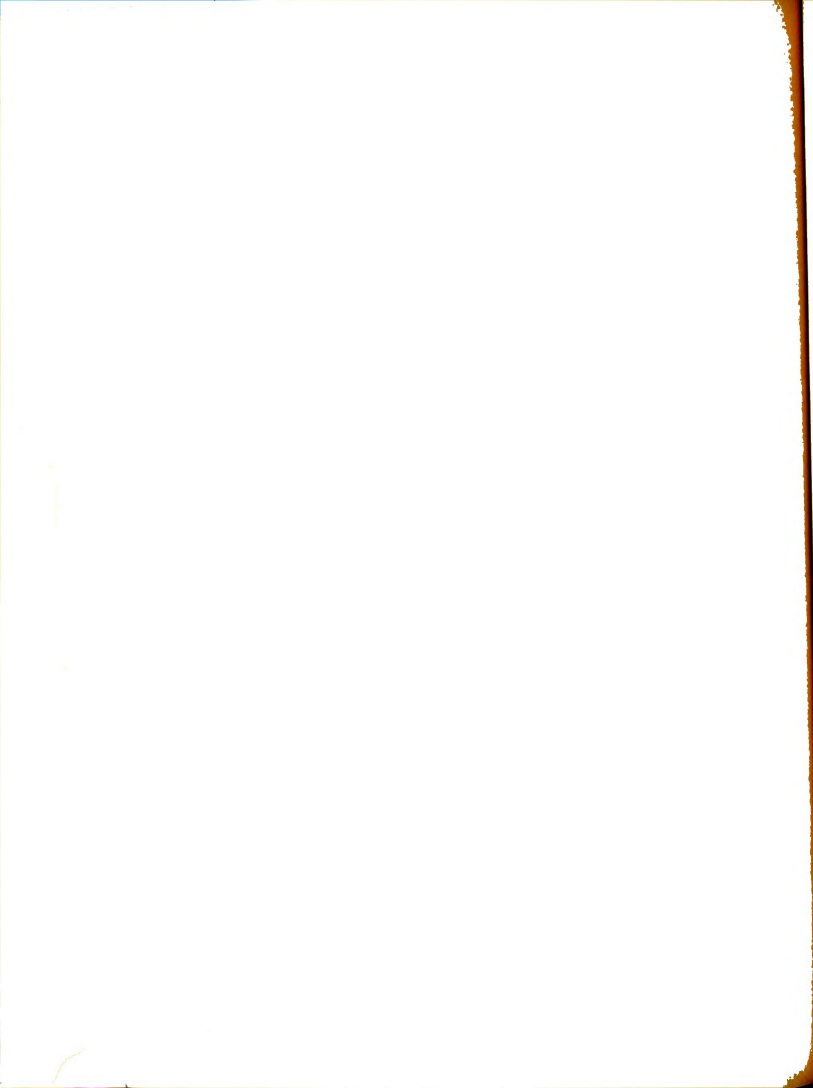
and

$$(n - m_2 = m_1 q_1 + (n - m_1) q_2 \longrightarrow \infty$$

Conversely

$$m_2 \longrightarrow \infty \iff \max. [m_1 p_1, (n - m_1) p_2] \longrightarrow \infty$$

$$n - m_2 \longrightarrow \infty \iff \max. [m_1 q_1, (n - m_1) q_2] \longrightarrow \infty$$

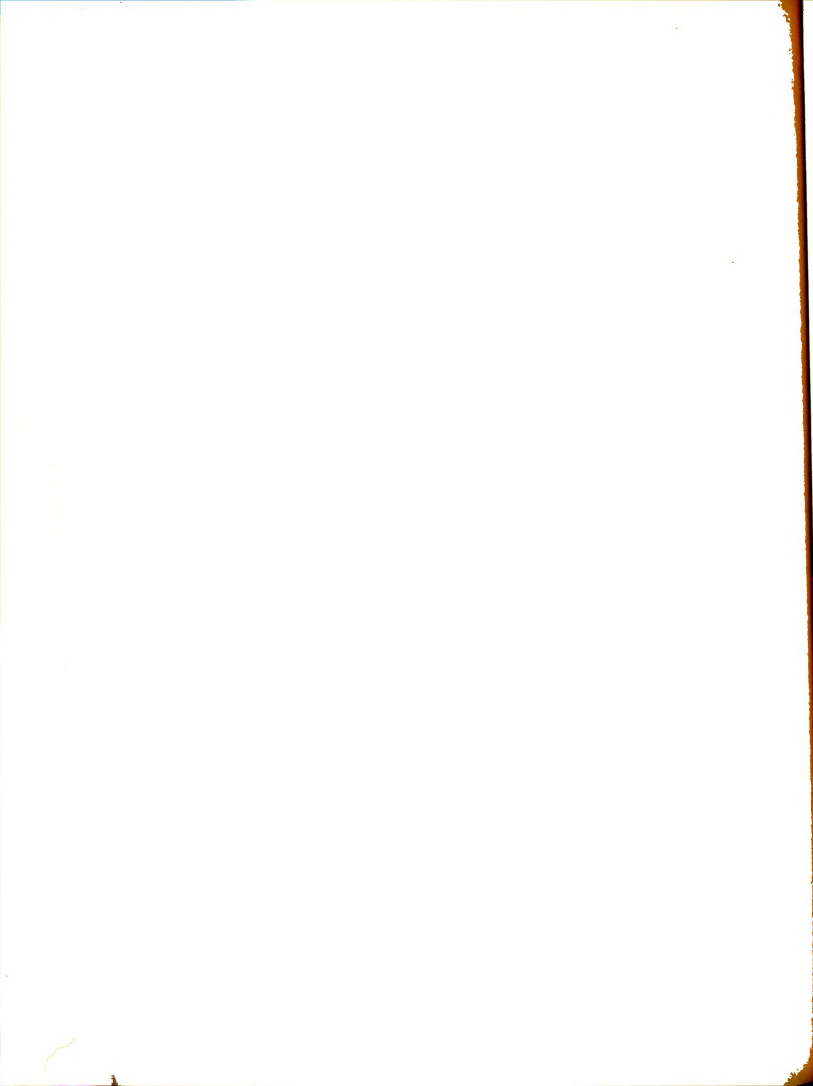


and with t held constant and equal to $\frac{p_1 q_2}{p_2 q_1}$, if m_2 and $n - m_2 \rightarrow \infty$ then either $m_1 p_1 q_1$ or $(n - m_1) p_2 q_2 \rightarrow \infty$. Finally, if both m_1 and $n - m_1 \rightarrow \infty$, then obviously $m_1 p_1 q_1$ and $(n - m_1) p_2 q_2 \rightarrow \infty$. Hence, it remains only to determine the values of n_1 such that $h x_{n_1}^3 \rightarrow 0$, or equivalently, those n_1 such that both $h_1 x_{1, n_1}^3$ and $h_2 x_{2, m_2 - n_1}^3 \rightarrow 0$.

In general it may be rather difficult to determine the values of n_1 for which $h x_{n_1}^3 \rightarrow 0$, but for the important special case when m_1 and m_2 are both of the same order of magnitude as n , it is very easy. Thus, suppose $\frac{m_1}{n} \rightarrow P_A$, $\frac{m_2}{n} \rightarrow P_B$. Then the two conditions, $t = \frac{p_1 q_2}{p_2 q_1}$ ($0 < t < \infty$) and $m_2 = m_1 p_1 + (n - m_1) p_2$ imply $nh_1^2 = \frac{n}{m_1 p_1 q_1} \rightarrow c_1$, $nh_2^2 = \frac{n}{(n - m_1) p_2 q_2} \rightarrow c_2$, where c_1 and c_2 are finite positive constants and consequently

$$h x_{n_1}^3 = h^4 (n_1 - m_1 p_1)^3 = \left[\left(\frac{m_1}{n} p_1 q_1 \right)^{-1} + \left[\left(\frac{n - m_1}{n} \right) p_2 q_2 \right]^{-1} \right]^2 \frac{(n_1 - m_1 p_1)^3}{n^2},$$

and hence $h x_{n_1}^3 \rightarrow 0$ if and only if $\frac{(n_1 - m_1 p_1)^3}{n^2} \rightarrow 0$,



which is the same condition as in Feller's binomial case.

The above results are collected in the following theorem.

Th.2.2.F. If α_1 and β_1 vary so that $hx_{\alpha_1}^3 \rightarrow 0$ and $hx_{\beta_1}^3 \rightarrow 0$, then

$$\Pr \{n_1 | m_1, m_2, n; t\} \sim h \phi(x_{n_1}).$$

uniformly for all $\alpha_1 < n_1 < \beta_1$, and

$$(2.2.22) \quad \sum_{n_1=\alpha_1}^{\beta_1} \Pr \{n_1 | m_1, m_2, n; t\} \sim \phi(x_{\beta_1 + \frac{1}{2}}) - \phi(x_{\alpha_1 - \frac{1}{2}}).$$

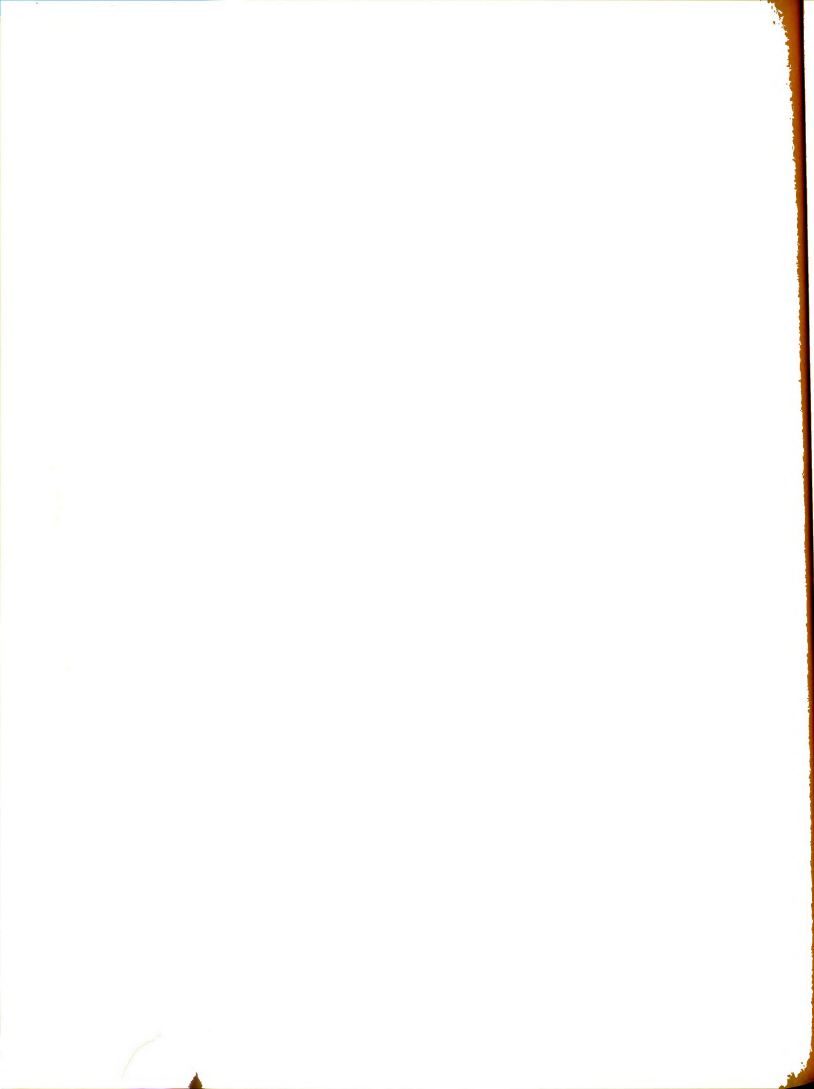
The last assertion of Th.2.2.F follows exactly as in Feller. Finally, it follows, again as in Feller, that for every fixed $a < b$,

$$(2.2.23) \quad \Pr \{a \leq h(n_1 - m_1 p_1) \leq b\} \rightarrow \bar{\Phi}(b) - \bar{\Phi}(a).$$

The quantities p_1 and p_2 have not yet been expressed in terms of m_1, m_2, n , and t . Solving first for p_2 in terms of p_1 and t , one has $p_2 = p_1(p_1 + q_1 t)^{-1}$, and then solving for p_1 in the resulting quadratic equation

$m_2 = m_1 p_1 + (n - m_1) p_1 (p_1 + q_1 t)^{-1}$, it is found that

$$(2.2.24) \quad p_1 = \frac{\lambda(n)^+ m_2}{n}, \quad p_2 = \left(\frac{m_2}{n}\right) \left(1 - \frac{\lambda(n)^+ m_1}{n}\right) \left(1 - \frac{m_1}{n}\right)^{-1}$$



if $t < 1$, [if $t > 1$, $\lambda_{(n)}^+$ is to be replaced by $\lambda_{(n)}^-$] where $\lambda_{(n)}^+$

and $\lambda_{(n)}^-$ are given by (2.1.21), and satisfy the equation

$$t = \frac{\lambda_{(n)} \left(1 - \frac{m_1}{n} - \frac{m_2}{n} + \frac{\lambda_{(n)} m_1 m_2}{n^2} \right)}{\left(1 - \frac{\lambda_{(n)} m_1}{n} \right) \left(1 - \frac{\lambda_{(n)} m_2}{n} \right)}. \quad \text{Routine algebra}$$

shows that, putting $\sigma^2 = \frac{1}{n^2}$, omitting the + and - signs,

$$(2.2.25) \quad \sigma^2 = n \left[\sum_{i=1}^4 \frac{1}{\pi_i'} \right]^{-1}$$

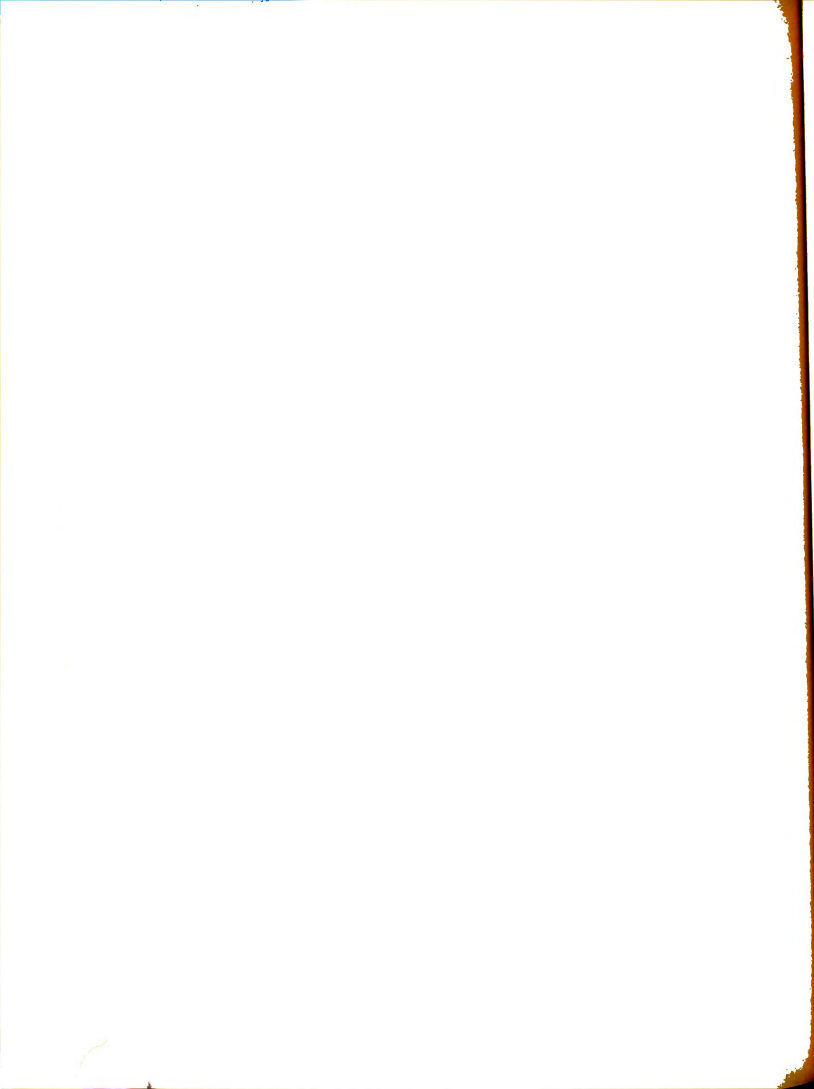
where $\pi_1' = \lambda_{(n)} \left(\frac{m_1}{n} \right) \left(\frac{m_2}{n} \right)$, $\pi_2' = \left(\frac{m_1}{n} \right) \left(1 - \lambda_{(n)} \frac{m_2}{n} \right)$,

$\pi_3' = \left(\frac{m_2}{n} \right) \left(1 - \lambda_{(n)} \frac{m_1}{n} \right)$, and $\pi_4' = 1 - \frac{m_1}{n} - \frac{m_2}{n} + \lambda_{(n)} \left(\frac{m_1}{n} \right) \left(\frac{m_2}{n} \right)$,

so that the asymptotic mean and variance, expressed as functions of m_1, m_2, n , and t , are given by

$$(2.2.26) \quad \theta_{(n)} = m_1 p_1 = \frac{\lambda_{(n)} m_1 m_2}{n} \quad \text{and} \quad \sigma^2 = n \left[\sum_{i=1}^4 \frac{1}{\pi_i'} \right]^{-1}.$$

Therefore, if $\frac{m_1}{n} \longrightarrow P_A$, $\frac{m_2}{n} \longrightarrow P_B$,



$$(2.2.27) \quad \lim_{n \rightarrow \infty} \frac{\mu_1}{n} = \lambda P_A P_B, \quad \lim_{n \rightarrow \infty} \frac{\mu_2}{n} = \left(\sum_{i=1}^4 \frac{1}{\pi_i} \right)^{-1},$$

where $\lambda = \lim_{n \rightarrow \infty} \lambda_{(n)}$ verifying the assertions made in

section 2.1 in equations numbered (2.1.25) and (2.1.27), with π_1 as defined in (2.1.27).

Finally, in concluding this chapter, several approximations to the probability function (2.1.1), based on the limit theorems established above, are given below. If n is large, and $\frac{m_1}{n} = r$, with m_2 small as compared to n , then the binomial approximation

$$(2.2.28) \quad \Pr \{n_1 | m_1, m_2, n; t\} \sim \binom{m_2}{n_1} \left(\frac{rt}{rt+1-r} \right)^{n_1} \left(\frac{1-r}{rt+1-r} \right)^{m_2-n_1}$$

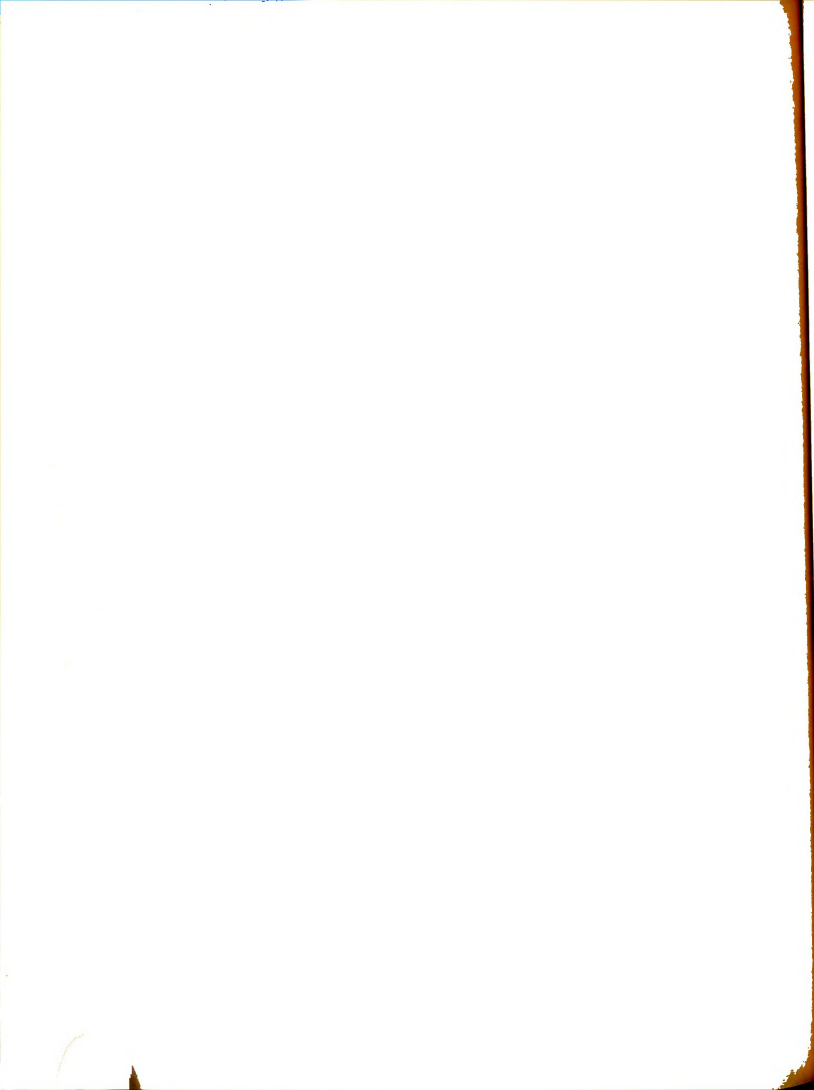
should be suitable. If $\frac{m_1}{n}$ is small and $v_n = \frac{m_1 m_2 t}{m_1 t + n - m_1}$ is

of moderate magnitude, then the Poisson approximation

$$(2.2.29) \quad \Pr \{n_1 | m_1, m_2, n; t\} \sim \frac{e^{-v_n} v_n^{n_1}}{n_1!}$$

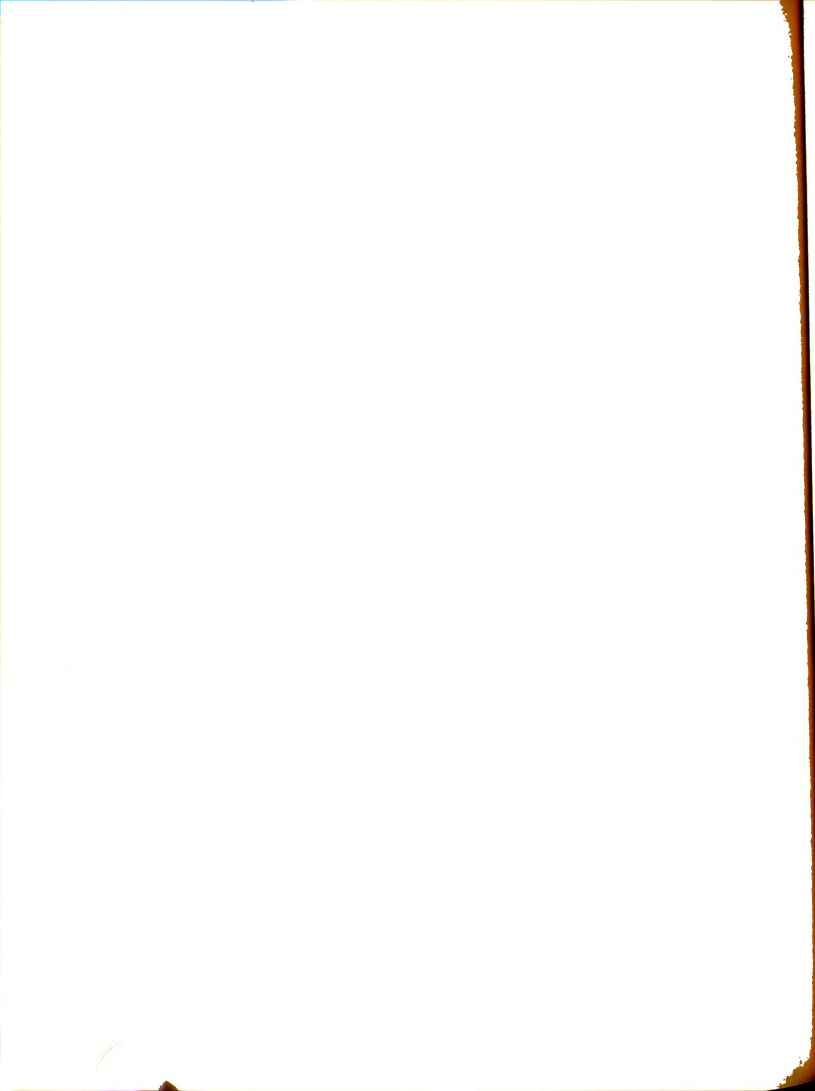
may be used. Finally, if $m_1, m_2, n-m_1$, and $n-m_2$ are all moderately large, then the normal approximation

$$(2.2.30) \quad \Pr \{n_1 | m_1, m_2, n; t\} \sim \frac{1}{\sqrt{2\pi} \sigma} \exp \left[-\frac{1}{2} \left(\frac{n_1 - \theta(n)}{\sigma} \right)^2 \right]$$



will give satisfactory results.

It is recommended that Patnaik's approximation not be used.



Chapter 3. Power for the Test of Independence

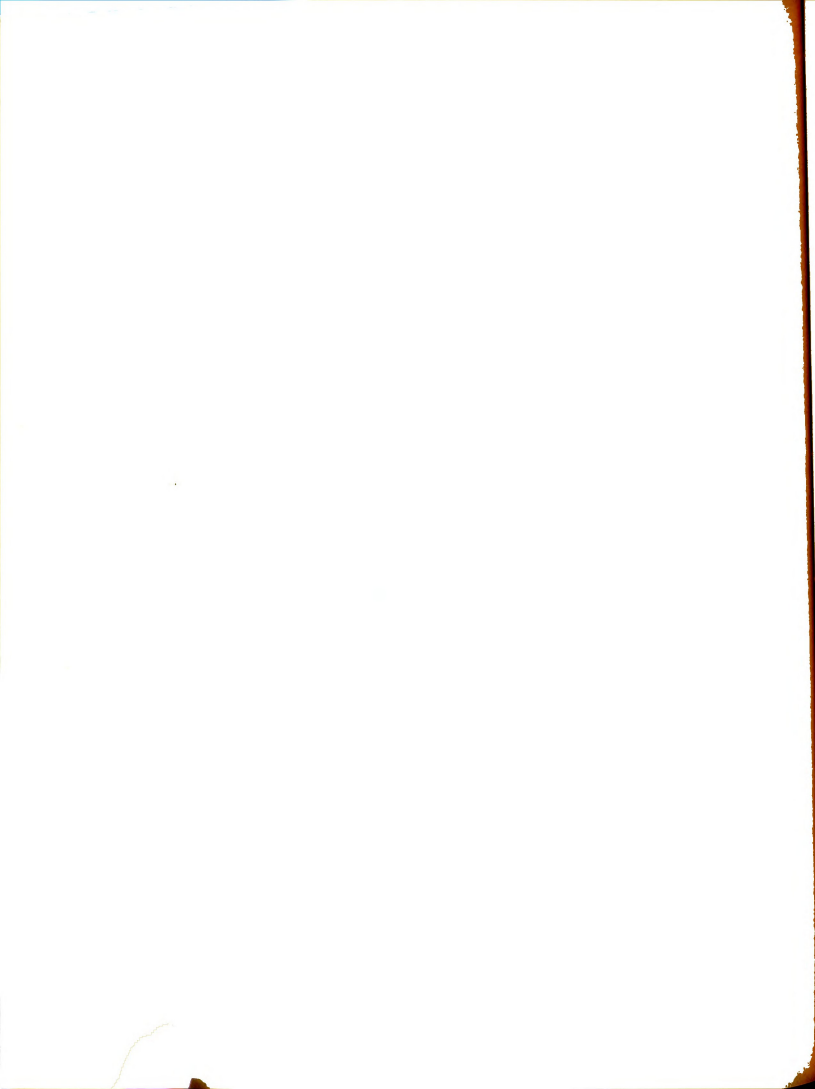
3.1 2 x 2 Independence Trial

In all subsequent discussion of power functions, the power will be evaluated for the unbiased test described in section 1.5.

For the 2 x 2 independence trial, the exact power of this test is given by

$$\begin{aligned}
 (3.1.1) \quad P_n(\lambda, P_A, P_B, m_1, m_2) &= \frac{\sum_{n_1} \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}}{\sum_{n_1} \binom{m_1}{n_1} \binom{n - m_1}{m_2 - n_1} t^{n_1}} \\
 &= \frac{\sum_{n_1} \Pr \{n_1 | m_1, m_2, n\} t^{n_1}}{\sum_{n_1} \Pr \{n_1 | m_1, m_2, n\} t^{n_1}} \\
 &= \frac{n_{EW}(m_1, m_2)}{\sum_{n_1} \Pr \{n_1 | m_1, m_2, n\} t^{n_1}}
 \end{aligned}$$

where $t = \frac{\lambda(1 - P_A - P_B + \lambda P_A P_B)}{(1 - \lambda P_A)(1 - \lambda P_B)}$, and $w(m_1, m_2)$ is the region of rejection of H_0 , for fixed m_1 and m_2 , defined by equations (1.5.6) and (1.5.7). The notation $P_n(t | m_1, m_2)$



will also be used to denote the power function in the 2×2 independence trial, since it is much easier to tabulate power as a function of t .

Using tables of the hypergeometric distribution, (3.1.1) is easily evaluated for small values of the parameters.

Notice that

$$\begin{aligned} (3.1.2) \quad P_n(\lambda, P_A, P_B, m_1, m_2) &= P_n(\lambda, P_A, P_B, m_2, m_1) \\ &= P_n(\lambda, P_A, P_B, n-m_1, n-m_2) \end{aligned}$$

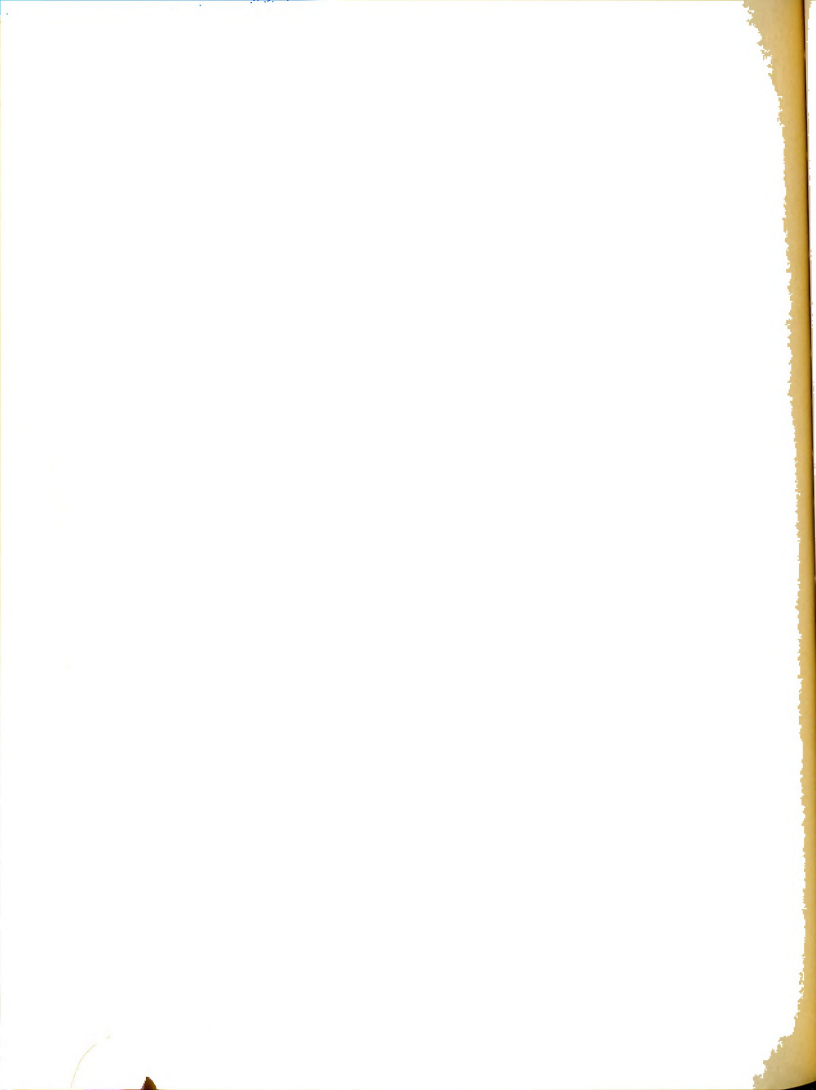
using the symmetry of $\Pr \{n_1 | m_1, m_2, n\}$.

When n is large, and m_1 and m_2 are both fairly large, the computation of (3.1.1) becomes laborious. Therefore, there is a need of approximating this expression. Fortunately, such approximations may be immediately obtained from the corresponding approximations to $\Pr \{n_1 | m_1, m_2, n, \lambda\}$.

If all of $m_1, m_2, n-m_1, n-m_2$ are large enough to require approximations, but are of moderate magnitude, then the normal approximation (2.2.30) may be used. Employing a $\frac{1}{2}$ factor for continuity, this approximation becomes

(3.1.3)

$$\begin{aligned} P_n(\lambda, P_A, P_B, m_1, m_2) &\sim \phi \left(\frac{a - \frac{1}{2} - \theta(n)}{\sigma} \right) \\ &+ \varepsilon_1 \left[\Phi \left(\frac{a + \frac{1}{2} - \theta(n)}{\sigma} \right) - \Phi \left(\frac{a - \frac{1}{2} - \theta(n)}{\sigma} \right) \right] \\ &+ \varepsilon_2 \left[\Phi \left(\frac{b + \frac{1}{2} - \theta(n)}{\sigma} \right) - \Phi \left(\frac{b - \frac{1}{2} - \theta(n)}{\sigma} \right) \right] + \left[1 - \Phi \left(\frac{b + \frac{1}{2} - \theta(n)}{\sigma} \right) \right] \end{aligned}$$



$$\begin{aligned}
&= (1 - \varepsilon_1) \phi\left(\frac{a - \frac{1}{2} - \theta(n)}{\sigma}\right) + \varepsilon_1 \phi\left(\frac{a + \frac{1}{2} - \theta(n)}{\sigma}\right) - (1 - \varepsilon_2) \phi\left(\frac{b + \frac{1}{2} - \theta(n)}{\sigma}\right) + 1 \\
&\quad - \varepsilon_2 \phi\left(\frac{b - \frac{1}{2} - \theta(n)}{\sigma}\right)
\end{aligned}$$

where a , b , ε_1 , and ε_2 are determined by equations (1.5.6) and (1.5.7), and $\theta(n)$ and σ^2 are given by (2.2.26).

If m_2 is small compared to n , the binomial approximation (2.2.28) can be used, while if both m_1 and m_2 are small relative to n , the Poisson approximation (2.2.29) is suitable. The summations may be performed with the aid of tables of the incomplete beta function or the incomplete gamma function. No computations of the power function are included for the Poisson and binomial approximations.

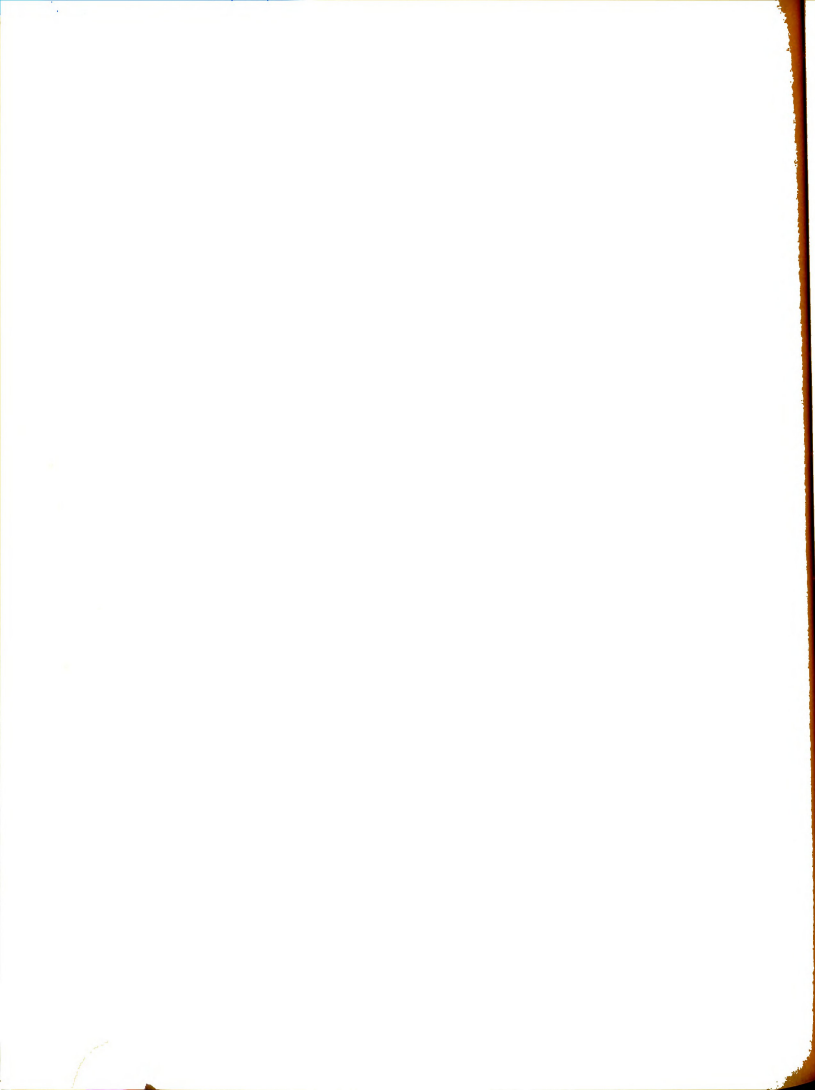
One can also approximate the power for the test of independence by using the test procedure described in section 1.6, based on assuming a normal approximation under the null hypothesis. The test consists of rejecting H_0 when $|u| > u_{\frac{\alpha}{2}}$

where u and $u_{\frac{\alpha}{2}}$ are defined by (1.6.4) and (1.6.5)

respectively. Evaluating the power of this test assuming the normal approximation (2.2.30) for (2.1.1), yields

$$(3.1.4) \quad P_n(\lambda, P_A, P_B, m_1, m_2) \sim$$

$$1 - \left[\phi\left(\frac{\frac{hu_{\frac{\alpha}{2}}}{2} + (1 - \lambda(n)) \frac{m_1 m_2}{n}}{\sigma}\right) - \phi\left(\frac{-\frac{hu_{\frac{\alpha}{2}}}{2} + (1 - \lambda(n)) \frac{m_1 m_2}{n}}{\sigma}\right) \right]$$

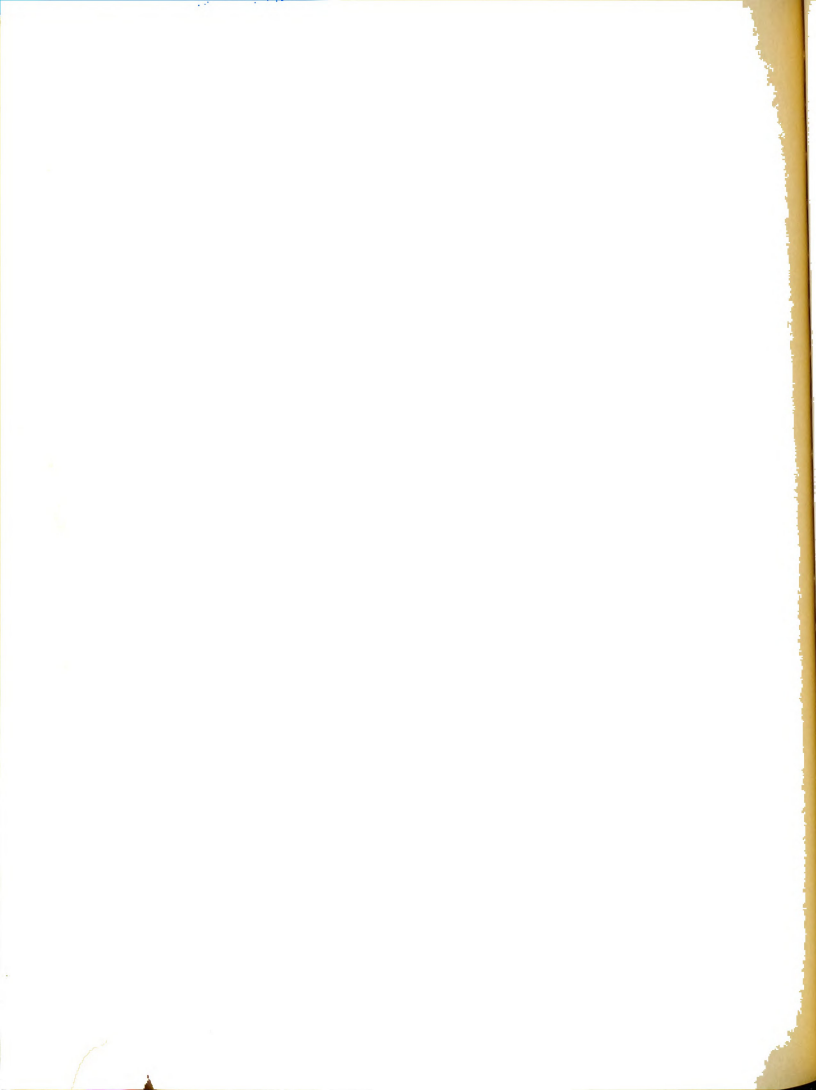


where $h^2 = \frac{m_1(n-m_1)m_2(n-m_2)}{n^2(n-1)}$, $\lambda_{(n)}$ is given by (2.1.21),

and σ^2 by (2.2.26).

Exact values have been computed from (3.1.1) for various values of m_1, m_2, n , and t , and compared with approximate values from (3.1.3) and (3.1.4). These exact and approximate values may be found in the appendix in tables D.3.1, D.3.2, and D.3.3.

In section 3.4 there is further discussion of the power function (3.1.1) including the relation between the power function for the 2×2 independence trial and the 2×2 comparative trial and double dichotomy. In Chapter 4, the adequacy of the approximations (3.1.3) and (3.1.4) is discussed.



3.2 2 x 2 Comparative Trial

The probability distribution of the sample point $(n_1, m_2 - n_1)$, is given by

(3.2.1)

$$\Pr \{n_1, m_2 | m_1, n, \lambda\} = \binom{m_1}{n_1} p_1^{n_1} (1-p_1)^{m_1-n_1} \binom{n-m_1}{m_2-n_1} p_2^{m_2-n_1} \cdot (1-p_2)^{n-m_1-m_2+n_1}$$

where $p_1 = \lambda P_B$, and $p_2 = \frac{P_B(1-\lambda P_A)}{1-P_A}$.

Using the unbiased test procedure discussed in section 1.5, the exact power function for

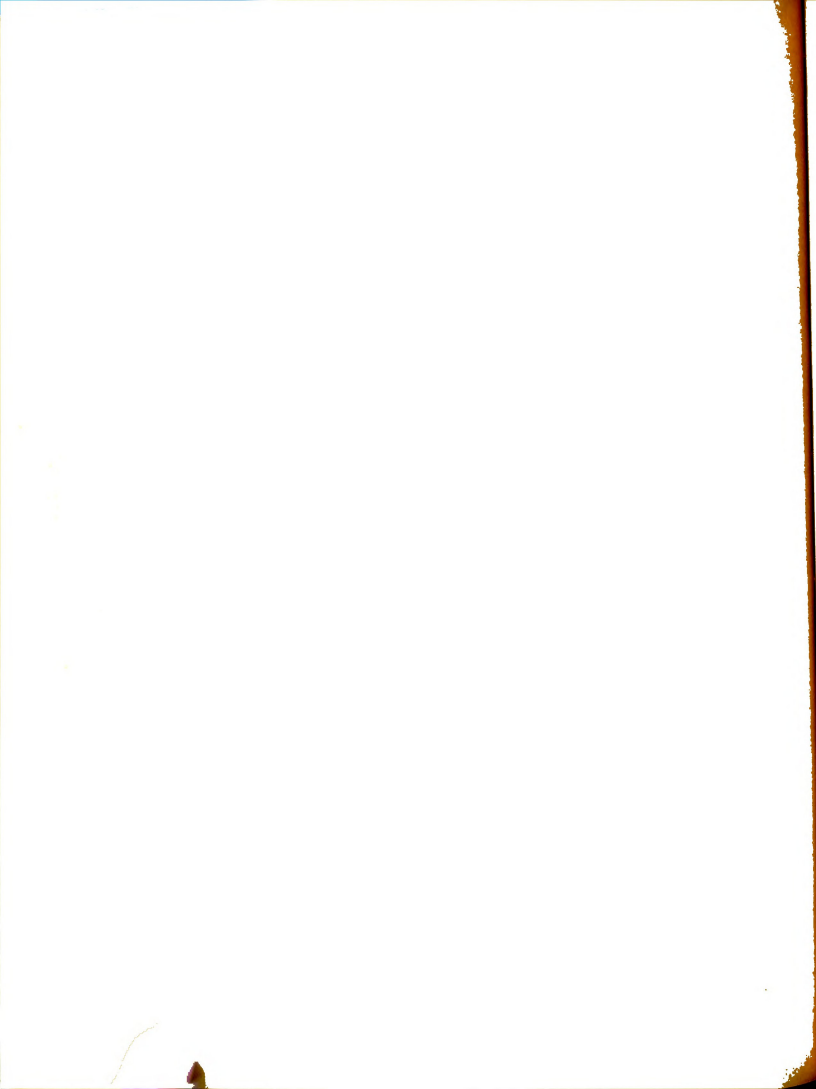
$$H_0: p_1 = p_2$$

$$H_1: p_1 \neq p_2, \text{ or, since } t = \frac{p_1(1-p_2)}{p_2(1-p_1)} = \frac{\lambda(1-P_A-P_B + \lambda P_A P_B)}{(1-\lambda P_A)(1-\lambda P_B)}, t \neq 1$$

is

$$(3.2.2) \quad P_n(\lambda, P_A, P_B, m_1) = \sum_{m_2=0}^n \sum_{n_1 \in w_{m_1}(m_2)} \Pr \{n_1, m_2 | m_1, n, \lambda\},$$

where $w_{m_1}(m_2)$ is the critical region with m_1 fixed. It is convenient to use the notation $P_n(\lambda, P_A, P_B, m_1)$ to denote the power function, but the notation $P_n(p_1, p_2, m_1)$ will also be used to indicate that the power function is a function of p_1 and p_2 when it is desired to think in terms of the 2 x 2 comparative trial by itself.



Since

$$(3.2.3) \quad \Pr \{m_2 | m_1, n, \lambda\} = \sum_{n_1} \Pr \{n_1, m_2 | m_1, n, \lambda\}$$

$$= (1-p_1)^{m_1} p_2^{m_2} (1-p_2)^{n-m_1-m_2} \sum_{n_1} \binom{m_1}{n_1} \binom{n-m_1}{m_2-n_1} t^{n_1}$$

and

$$(3.2.4) \quad \Pr \{n_1, m_2 | m_1, n, \lambda\} = \Pr \{n_1 | m_1, m_2, n, \lambda\} \Pr \{m_2 | m_1, n, \lambda\}$$

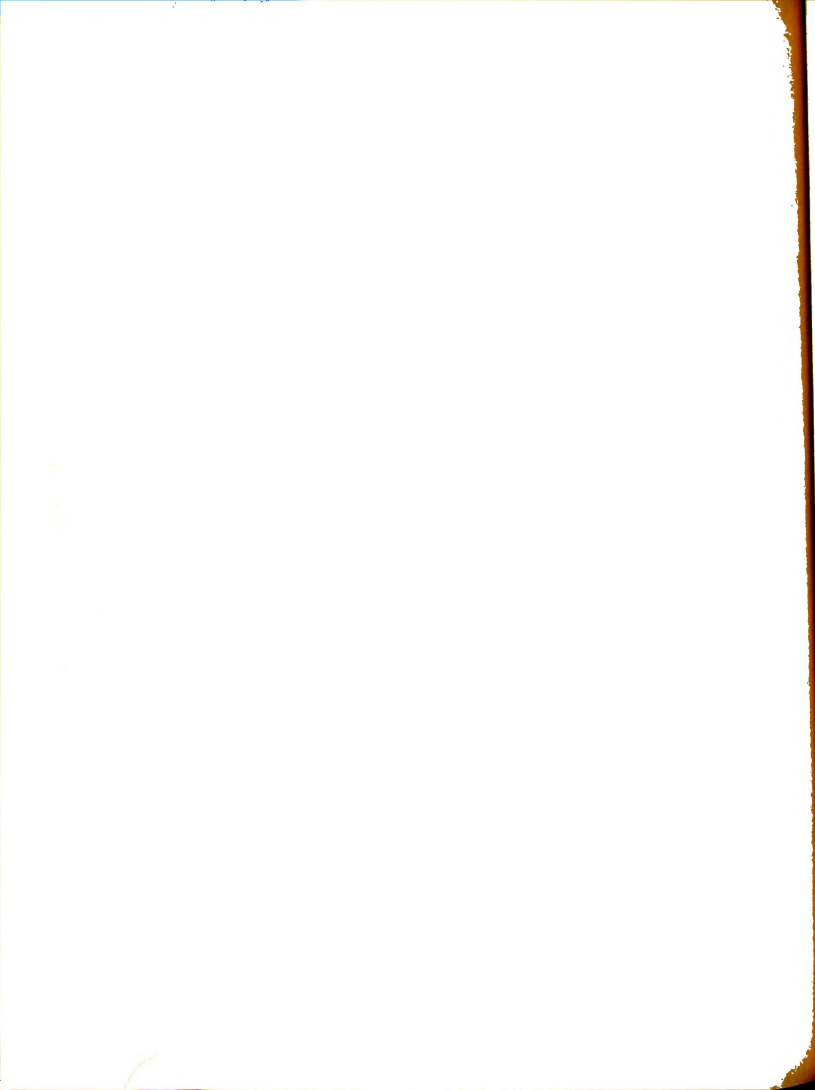
where $\Pr \{n_1 | m_1, m_2, n, \lambda\}$ is given by (1.2.5), one sees that the expression in (3.2.2) for the exact power function may be rewritten as

(3.2.5)

$$\begin{aligned} P_n(\lambda, P_A, P_B, m_1) &= \sum_{m_2=0}^n \Pr \{m_2 | m_1, n, \lambda\} \sum_{n_1 \in W_{m_1}(m_2)} \Pr \{n_1 | m_1, m_2, n, \lambda\} \\ &= \sum_{m_2=0}^n \Pr \{m_2 | m_1, n, \lambda\} P_n(\lambda, P_A, P_B, m_1, m_2) \end{aligned}$$

where $P_n(\lambda, P_A, P_B, m_1, m_2)$ is the power function in the 2×2 independence trial.

This last form of the power function is interesting but not very useful for computing. In disguised form, (3.2.3) is the convolution of two independent random variables X and Y with X and Y distributed binomially $b(n_1; m_1, p_1)$ and $b(m_2 - n_1; n - m_1, p_2)$ respectively.



It is thus seen that either one must evaluate the exact power function by forming sums of products of binomial probabilities, or by finding suitable approximations. The tabled exact values in the appendix of the power function (3.2.1) were computed by forming such sums, and three approximations are suggested below; the first has already appeared in the literature, whereas the last two have not.

First, there is Sillitto's approximation (1.7.21) which is repeated for the sake of completeness. Letting

$$c(p_1, p_2, m_1) = \frac{\sin^{-1} \sqrt{p_2} - \sin^{-1} \sqrt{p_1}}{\frac{1}{2} \sqrt{\frac{1}{m_1} + \frac{1}{n-m_1}}}$$

Sillitto's approximation is

$$(3.2.6) \quad P_n(p_1, p_2, m_1) \sim 1 - \int_{-\frac{u}{2}}^{\frac{u}{2} - c(p_1, p_2, m_1)} \frac{1}{\sqrt{\frac{1}{2\pi}}} e^{-\frac{y^2}{2}} dy,$$

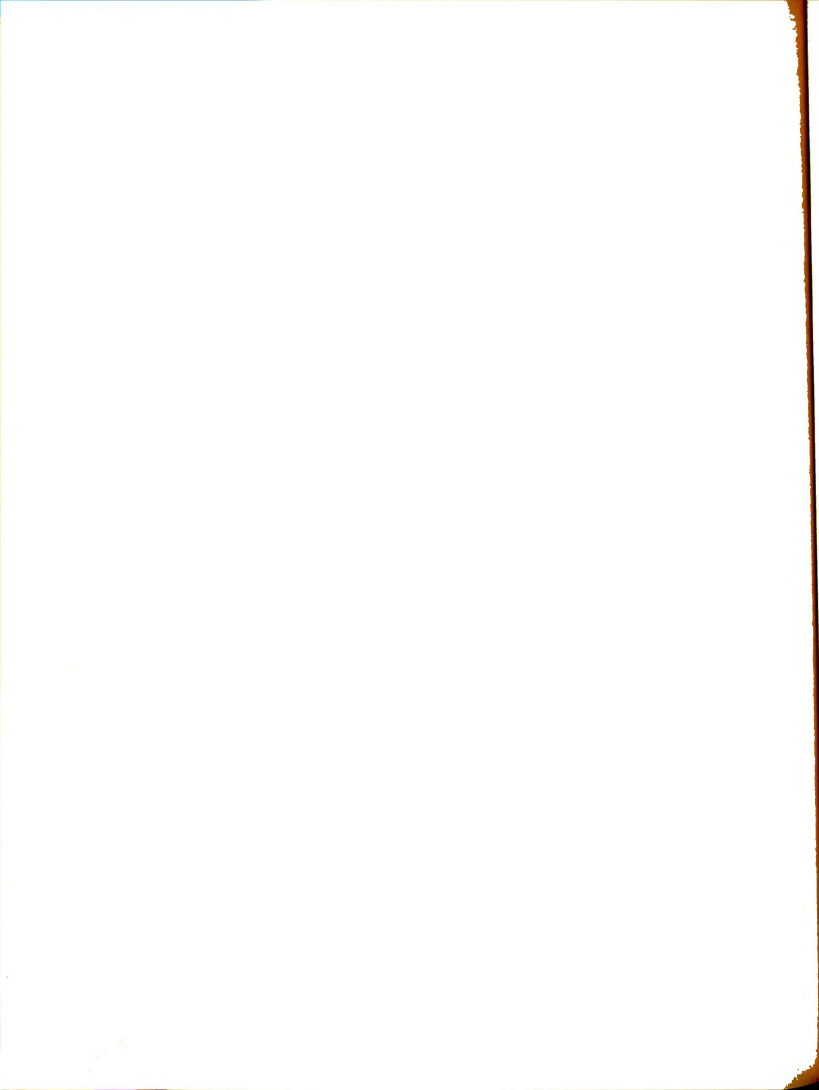
where the angles are measured in radians.

A second approximation which is considerably easier to evaluate than Sillitto's but perhaps not as good may be obtained as follows.

Let $u = \frac{x_1 - x_2}{\sigma_{x_1 - x_2}}$, where $x_1 = \frac{n_1}{m_1}$, $x_2 = \frac{m_2 - n_1}{n - m_1}$, and

$$\sigma_{x_1 - x_2}^2 = \frac{p_1 q_1}{m_1} + \frac{p_2 q_2}{n - m_1}. \text{ Then } E(u) = \frac{p_1 - p_2}{\sigma_{x_1 - x_2}}, V(u) = 1$$

If H_0 is true, $E(u) = 0$. Assume that u is normally



distributed. If H_0 is rejected whenever $|u| > \frac{u_{\alpha}}{2}$, then an approximation to power is given by

$$(3.2.7) \quad P_n(p_1, p_2, m_1) \sim 1 - \int_{-\frac{u_{\alpha}}{2}}^{\frac{u_{\alpha}}{2}} e^{-\frac{t^2}{2}} dt$$

To obtain the third approximation, we assume the normal approximation (3.1.4) for the conditional power function $P_n(\lambda, p_A, p_B, m_1, m_2)$ and expand it as a function of m_2 in a Taylor Series about the mean γ of m_2 , where

$\gamma = m_1 p_1 + (n - m_1) p_2$. Then from (3.2.5), one has

$$(3.2.8) \quad P_n(\lambda, p_A, p_B, m_1) \sim \sum_{m_2=0}^n \Pr \{m_2, m_1, n, \lambda\} \left[P_n(\lambda, p_A, p_B, m_1, \gamma) + P_n'(\lambda, p_A, p_B, m_1, \gamma)(m_2 - \gamma) + \frac{P_n''(\lambda, p_A, p_B, m_1, \gamma)(m_2 - \gamma)^2}{2!} + \dots \right]$$

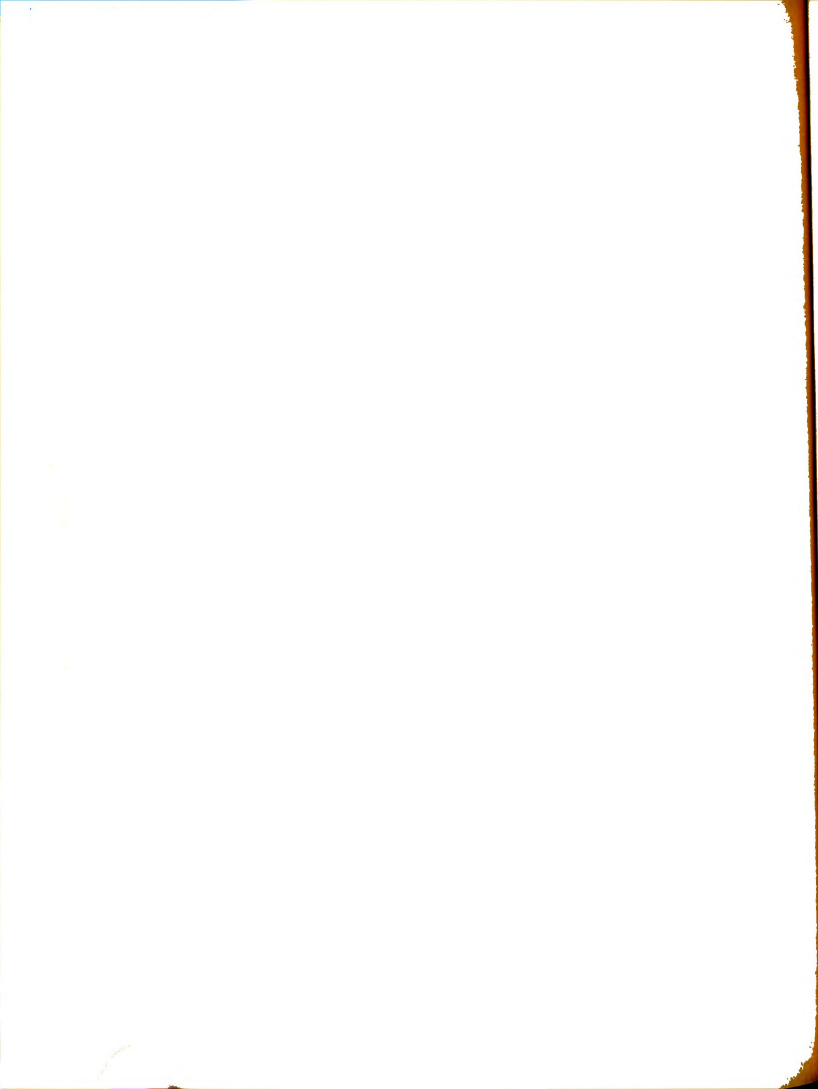
$$= P_n(\lambda, p_A, p_B, m_1, \gamma) + \frac{P_n''(\lambda, p_A, p_B, m_1, \gamma) \sigma^2}{2!} + \dots,$$

where $\sigma^2 = m_1 p_1 q_1 + (n - m_1) p_2 q_2$, and $P_n^{(j)}(\lambda, p_A, p_B, m, \gamma)$ is the j^{th} derivative of $P_n(\lambda, p_A, p_B, m_1, m_2)$ evaluated at $m_2 = \gamma$.

Then as a first term approximation, one has

$$(3.2.9) \quad P_n(\lambda, p_A, p_B, m_1) \sim P_n(\lambda, p_A, p_B, m_1, \gamma).$$

It will be recalled from section 1.7 that Patnaik also gave an approximation to the power function. In view of Theorem 2.2.F and the remarks preceding and following the proof of the theorem, it is seen that there is considerable



doubt as to the validity of the approximation. Perhaps the approximation can be justified on the grounds that as the sample size gets larger one is interested in only those values of t near one (since for any fixed value of t the power of the test for independence tends to 1 as the integers $m_1 \rightarrow \infty, n \rightarrow \infty$, which means that the test is consistent), in which case the mean and variance of Patnaik's approximation to the conditional distribution of n_1 , given m_1, m_2, n, t , may not differ significantly from the true mean and variance. Nevertheless, there is nothing in the way of simplicity in his approximation which recommends its use over (3.2.9), which is based on sound considerations.

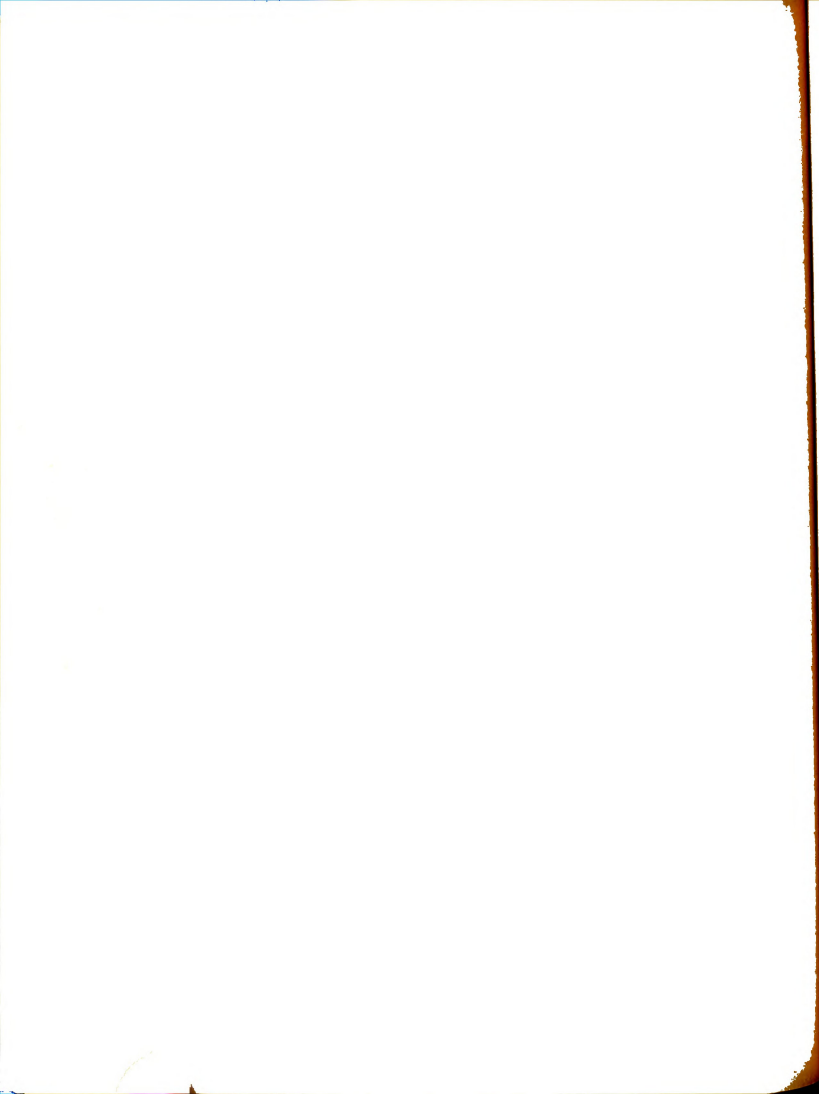
3.3 Double Dichotomy

For the double dichotomy classification, the probability of observing the sample point (n_1, m_1, m_2) , given λ, P_A, P_B , and n , is

$$(3.3.1) \quad \Pr \{n_1, m_1, m_2 \mid n, \lambda\} \\ = b(m; n, P_A) b(n_1; m_1, \lambda P_B) b\left(m_2 - n_1; n - m_1, \frac{P_B(1 - \lambda P_A)}{(1 - P_A)}\right)$$

and the exact power function for the test of independence, given in section 1.5., is

$$(3.3.2) \quad P_n(\lambda, P_A, P_B) = \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{n_1 \in w_{m_1, m_2}} \Pr \{n_1, m_1, m_2 \mid n, \lambda\}$$



where w_{m_1, m_2} is the set sum $\bigcup_{m_1, m_2} w(m_1, m_2)$ with $w(m_1, m_2)$

the critical region for fixed m_1 and m_2 as determined by (1.5.6) and (1.5.7). Since

$$(3.3.3) \quad \Pr \{n_1, m_1, m_2 \mid n, \lambda\} = \Pr \{m_1 \mid P_A, n\} \Pr \{n_1, m_2 \mid m_1, n, \lambda\}$$

where $\Pr \{m_1 \mid P_A, n\} = \binom{n}{m_1} P_A^{m_1} (1-P_A)^{n-m_1}$ and $\Pr \{n_1, m_2 \mid m_1, n, \lambda\}$

is given by (3.2.1), with $p_1 = \lambda P_B$ and $p_2 = \frac{P_B(1-\lambda P_A)}{1-P_A}$,

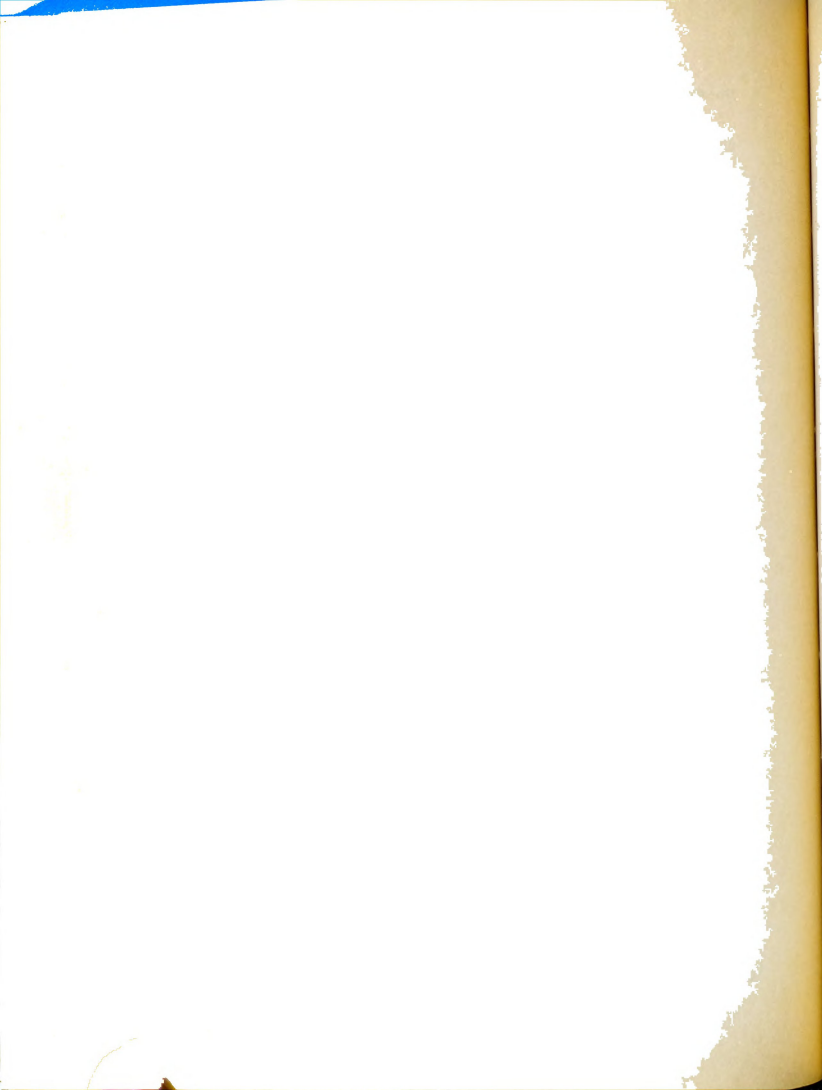
it follows that (3.3.2) can be rewritten as

$$(3.3.4) \quad P_n(\lambda, P_A, P_B) = \sum_{m_1=0}^n \Pr \{m_1 \mid n, P_A\} P_n(\lambda, P_A, P_B, m_1)$$

with $P_n(\lambda, P_A, P_B, m_1)$ given by (3.2.2).

Thus, the power function can be evaluated by weighting the power function in the 2×2 comparative trial with binomial probabilities. However, even for n quite small, exact evaluation of (3.3.2) is very tedious, and for n moderately large the computation of (3.3.2) becomes prohibitive.

With the aid of the Michigan State University electronic computer, exact values were computed for $n = 10, 20$, and 30 , with P_A and P_B assuming the values $.1, .2, .3, .4, .5$, $P_B \leq P_A$, and with $\lambda = .1, .2, .3, \dots, \frac{1}{P_A}$, with the exception that for $n = 30$ exact values are not available for $P_A = .2$, $P_B = .2$; $P_A = .2$, $P_B = .1$; and $P_A = .1$, $P_B = .1$. These exact values



are given in Appendix A.

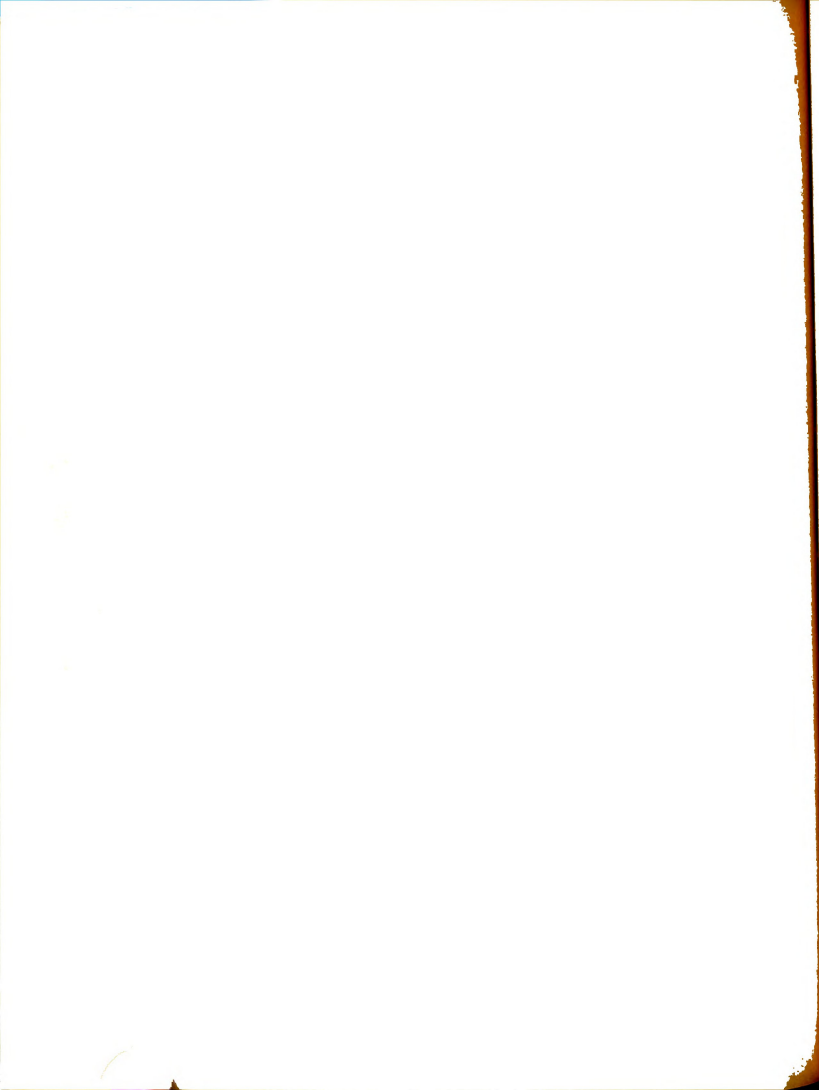
In section 3.4, it is shown that the ratio of $P_n(\lambda, P_A, P_B, m_1)$ and $P_n(\lambda, P_A, P_B, m_1, m_2)$ $\left[\begin{matrix} m_1 = nP_A \\ m_2 = nP_B \end{matrix} \right]$ converges to 1 as $n \rightarrow \infty$, and also the ratio of $P_n(\lambda, P_A, P_B)$ and $P_n(\lambda, P_A, P_B, m_1, m_2)$ $\left[\begin{matrix} m_1 = nP_A \\ m_2 = nP_B \end{matrix} \right]$ goes to 1 as $n \rightarrow \infty$. This

suggests several approximations for $P_n(\lambda, P_A, P_B)$, namely, the exact power function $P_n(\lambda, P_A, P_B, m_1)$ $\left[\begin{matrix} m_1 = nP_A \end{matrix} \right]$ in the 2×2 comparative trial and any of the three approximations in 3.2 for $P_n(\lambda, P_A, P_B, m_1)$ $\left[\begin{matrix} m_1 = nP_A \end{matrix} \right]$. Also, one may use the exact power function $P_n(\lambda, P_A, P_B, m_1, m_2)$ $\left[\begin{matrix} m_1 = nP_A \\ m_2 = nP_B \end{matrix} \right]$ for the test

of independence in the 2×2 independence trial, evaluated for marginal totals equal to the expected values of m_1 and m_2 , as an approximation.

In appendix D.1, $P_n(\lambda, P_A, P_B)$; $P_n(\lambda, P_A, P_B, m_1)$ $\left[\begin{matrix} m_1 = nP_A \end{matrix} \right]$ and $P_n(\lambda, P_A, P_B, m_1, m_2)$ $\left[\begin{matrix} m_1 = nP_A \\ m_2 = nP_B \end{matrix} \right]$ are compared for several

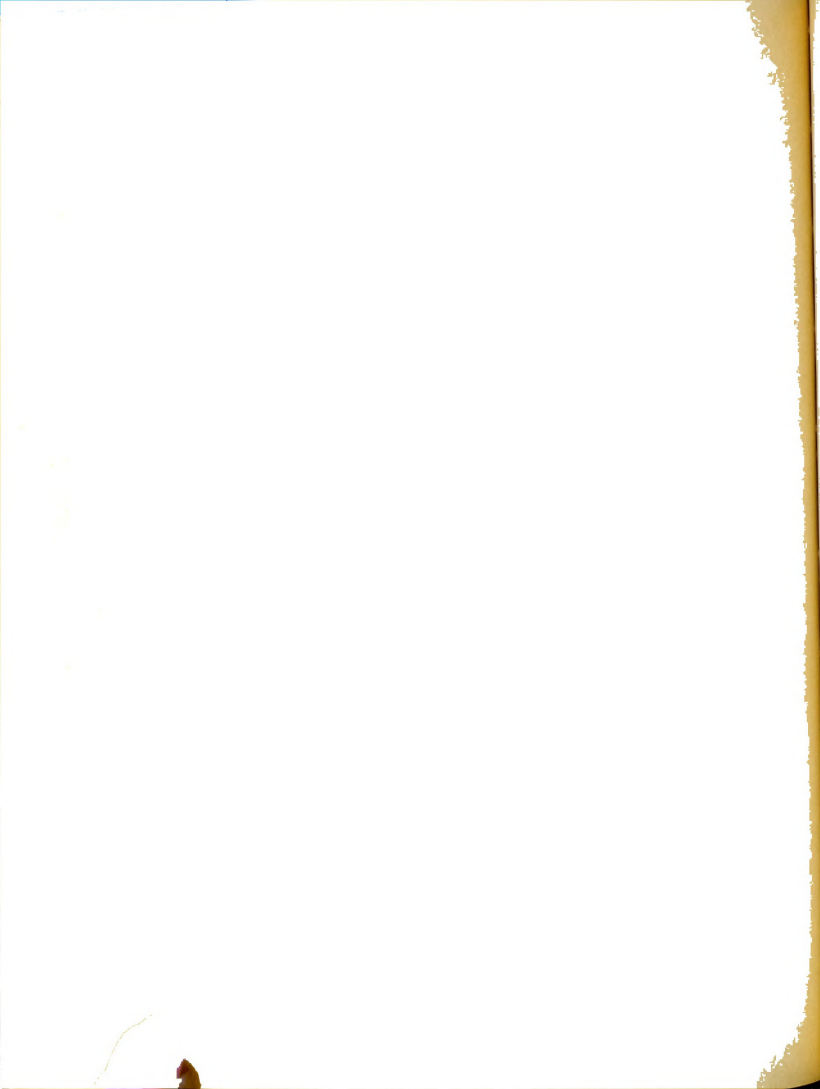
values of n, P_A, P_B , and λ . Finally, one can use the normal approximation 3.1.4, with $\lambda_{(n)}$ replaced by λ . Thus, as a very easily computed approximation to $P_n(\lambda, P_A, P_B)$, we have



(3.3.5)

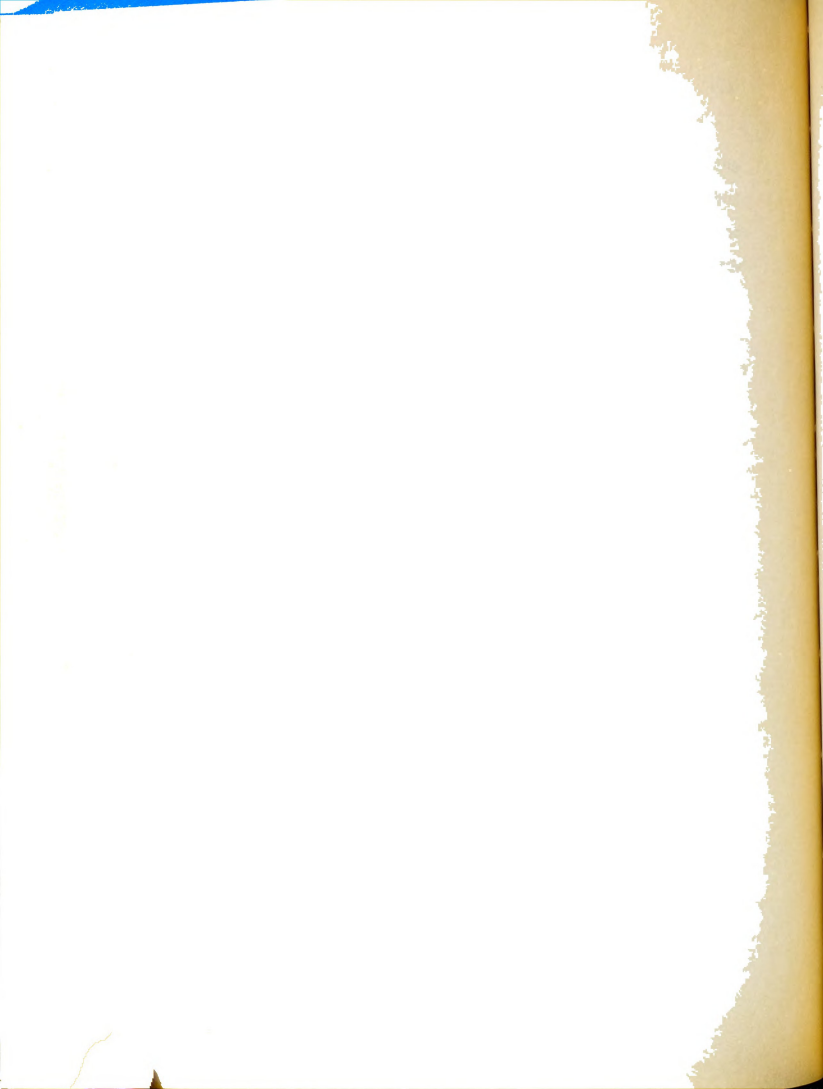
$$P_n(\lambda, P_A, P_B) \sim \phi\left(\frac{-u \frac{\alpha}{2} h + (1-\lambda) n P_A P_B}{\sigma}\right) + 1 - \phi\left(\frac{u \frac{\alpha}{2} h + (1-\lambda) n P_A P_B}{\sigma}\right)$$

The approximate values given by (3.3.5) are compared with the exact values as computed from (3.3.2) for a large number of cases, and judging from the simplicity of the approximation and the accuracy, (3.3.5) should be quite adequate. These comparisons may be found in appendix D.



3.4 Asymptotic Power

We shall be concerned in this section with an investigation of the asymptotic power function for the test of independence in the 2×2 independence trial, the 2×2 comparative trial, and the double dichotomy. The results which are to be presented in this section are not surprising, but rather they confirm what statisticians have believed for some time about the nature of the power function for the test of independence in 2×2 contingency tables for large sample sizes, i.e., that there is probably very little difference in power for the test of independence in the 2×2 independence trial, the 2×2 comparative trial, and the double dichotomy. Thus, our main contribution to the theory of the 2×2 contingency table given in this section is the limiting power function for the test of independence in the 2×2 independence trial. It follows almost trivially from this result that, asymptotically, power for the test of independence is the same for each of the three possible experimental situations leading to the presentation of data in the form of a 2×2 contingency table. In turn, the limiting power function in the 2×2 independence trial is almost a trivial consequence of Th.2.2.F on the asymptotic distribution of the conditional distribution of n_1 , given m_1, m_2, n , and t . The main results of this section are contained in Theorems 3.4.A and 3.4.B.



We recall that the hypothesis of independence takes the form $H_0: t = 1$, while any alternative to independence is given by $H_1: t \neq 1$; H_0 and H_1 may be expressed also by $H_0: \lambda = 1$ vs. $H_1: \lambda \neq 1$.

Under the null hypothesis (by Th.2.2.F) the conditional distribution of n_1 , given m_1, m_2 , and n , is asymptotically normal with mean $\frac{m_1 m_2}{n}$ and variance $h^2 = \frac{m_1(n-m_1)m_2(n-m_2)}{n^2(n-1)}$

if the conditions of Th.2.2.F, with $t=1$, are satisfied. This implies that, asymptotically, the unbiased test for independence described in 1.5 consists of rejecting H_0 at level α if

$$u = \left| \frac{n_1 - \frac{m_1 m_2}{n}}{h} \right| \geq u_{\frac{\alpha}{2}}.$$

The main tool in studying power for large samples will be Th. 2.2.F on the asymptotic distribution of n_1 under the alternative hypothesis that $t \neq 1$. If $t > 0$ is arbitrary but fixed, and the conditions of Th. 2.2.F are satisfied, then it follows that

$$(3.4.1) \quad P_n(t|m_1, m_2) \sim \phi \left(\frac{-u_{\frac{\alpha}{2}} h + \frac{m_1 m_2}{n} - m_1 p_1}{\sigma} \right) + 1 - \phi \left(\frac{u_{\frac{\alpha}{2}} h + \frac{m_1 m_2}{n} - m_1 p_1}{\sigma} \right),$$

where $\sigma^2 = \left[\frac{1}{m_1 p_1 q_1} + \frac{1}{(n-m_1) p_2 q_2} \right]^{-1}$, and p_1 and p_2 are



determined such that $t = \frac{p_1(1-p_2)}{p_2(1-p_1)}$, $m_2 = m_1 p_1 + (n-m_1)p_2$.

Here the assumptions of Th. 2.2.F require that

$$(3.4.2) \quad \frac{\left(u_{\frac{\infty}{2}} h + \frac{m_1 m_2}{n} - m_1 p_1\right)^3}{\sigma^4} \longrightarrow 0$$

as $\sigma^{-1} \longrightarrow 0$. We shall limit ourselves to values of m_1, m_2, n , and t for which (3.4.2) is satisfied, since this will cover most cases of interest.

If t is kept fixed as $m_1, m_2, n \longrightarrow \infty$, then for fixed level of significance α it turns out in the cases we shall examine that the power of the test for independence tends to 1. In order to examine the situation in which the power is not close to 1 in large samples, we must either let the significance probability decrease to 0 as m_1, m_2 , and n increase, or consider a sequence of alternative hypotheses converging to the null hypothesis. We shall discuss the second case. In the 2×2 independence trial, we shall let $t \longrightarrow 1$ in such a way that $\sqrt{n}(1-t) \longrightarrow c$, where c is any arbitrary but fixed constant, and for the 2×2 comparative trial and double dichotomy, we shall let $\lambda \longrightarrow 1$ such that $\sqrt{n}(1-\lambda)$ converges to a constant.

Before proceeding to the main results of this section on asymptotic power, it will be necessary to first note that the assumption that t is fixed in Th. 2.2.F is superfluous, as long as $\sigma^{-1} \longrightarrow 0$ and $\sigma^{-1} x_{n_1}^3 \longrightarrow 0$. In all that follows it is assumed that t , P_A, P_B , and λ are positive; also, the quantities p_1 and p_2 used in the proof of Th. 3.4.A depend on



m_1, m_2, n , and t but this dependence will not be indicated by the notation.

Th. 3.4.A. For all real numbers $a, b (0 < a, b < 1)$ such that

$$\left| \frac{m_1 - na}{\sqrt{na(1-a)}} \right| \leq f_1(n) \quad \text{and} \quad \left| \frac{m_2 - nb}{\sqrt{nb(1-b)}} \right| \leq f_2(n), \quad \text{where } f_1$$

and f_2 are such that $n^{-\frac{1}{6}} f_1 \rightarrow 0, n^{-\frac{1}{6}} f_2 \rightarrow 0$, we have

$$(3.4.3) \quad P_n(t | m_1, m_2) \Big|_{t=1-\frac{c}{\sqrt{n}}} \xrightarrow{n \rightarrow \infty} \Phi(-u_{\frac{\alpha}{2}} + \delta) + 1 - \Phi(u_{\frac{\alpha}{2}} + \delta).$$

where $\delta = \sqrt{a(1-a)b(1-b)} \quad c$.

Proof: First we note that the hypotheses of the lemma imply

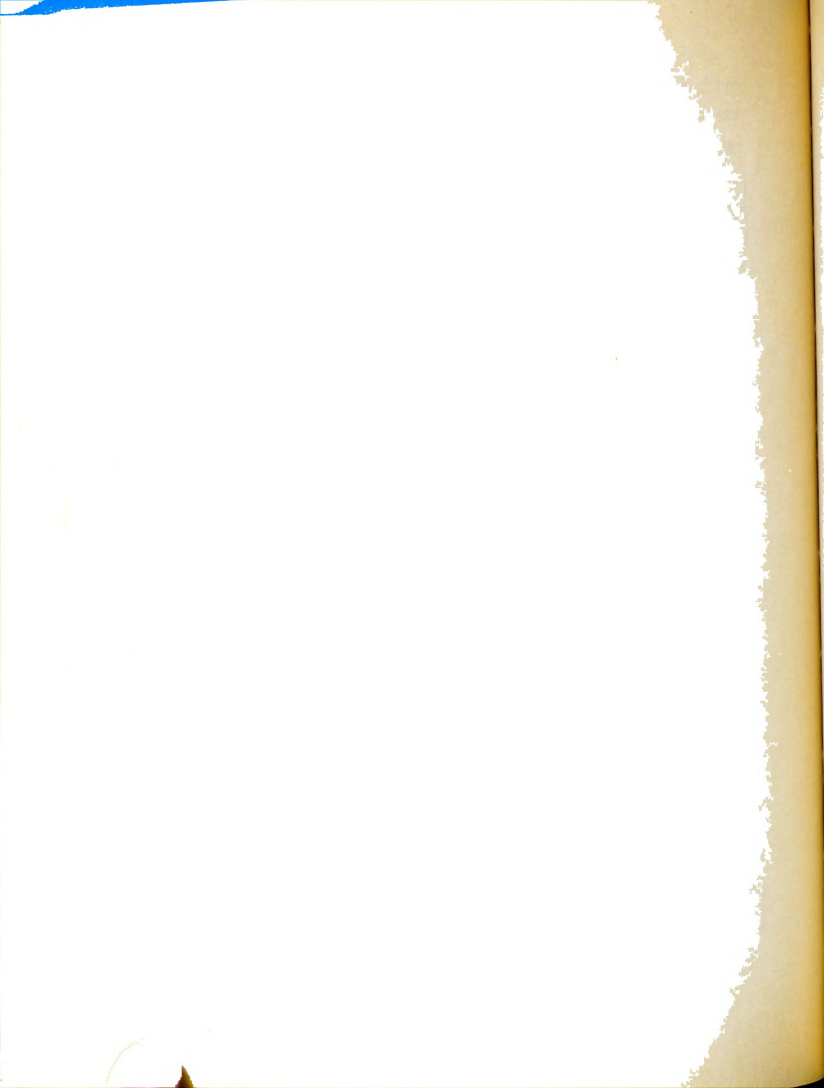
$\frac{m_1}{n} \rightarrow a, \frac{m_2}{n} \rightarrow b$. We will show that

$$(3.4.4) \quad \frac{u_{\frac{\alpha}{2}} h + \frac{m_1 m_2}{n} - m_1 p_1}{\sigma} \rightarrow u_{\frac{\alpha}{2}} + \delta,$$

where p_1 and σ are as given in (3.4.1). Since t is assumed to be positive, the choice of p_1 and p_2 such that

$$t = \frac{p_1 q_2}{p_2 q_1} = 1 - \frac{c}{\sqrt{n}} \quad \text{and} \quad m_2 = m_1 p_1 + (n - m_1) p_2$$

implies p_1, p_2, q_1 and q_2 are positive so that $\sigma^2 \rightarrow \infty$. It will be shown below that (3.4.2) is satisfied, so that (3.4.1) holds. To establish the theorem it is only necessary to show that (3.4.4) holds.



Putting $t = 1 - \frac{c}{\sqrt{n}} = \frac{p_1 q_2}{p_2 q_1}$, it is clear that as $n \rightarrow \infty$,

$|p_1 - p_2| \rightarrow 0$, so that using the fact that $m_2 = m_1 p_1 + (n - m_1) p_2$

and $\frac{m_1}{n} \rightarrow a$, $\frac{m_2}{n} \rightarrow b$, it follows that $p_1, p_2 \rightarrow b$,

and therefore, since

$$p_2 = p_1 \left(1 - \frac{c}{\sqrt{n}}\right)^{-1},$$

$$\sqrt{n} (p_2 - p_1) = \frac{\sqrt{n} p_1 q_1 c}{\sqrt{n} - c} \rightarrow b(1-b)c.$$

Also, from the above, it follows that

$$\frac{\sigma^2}{n} = \left[\frac{n}{m_1 p_1 q_1} + \frac{n}{(n - m_1) p_2 q_2} \right]^{-1} \rightarrow a(1-a)b(1-b).$$

Therefore

$$\frac{u_{\frac{\alpha}{2}} h}{\sigma} = \frac{u_{\frac{\alpha}{2}}}{2} \left[\frac{m_1 (n - m_1) m_2 (n - m_2) \sigma^2}{n^3 (n - 1)} \right]^{\frac{1}{2}} \rightarrow \frac{u_{\frac{\alpha}{2}}}{2},$$

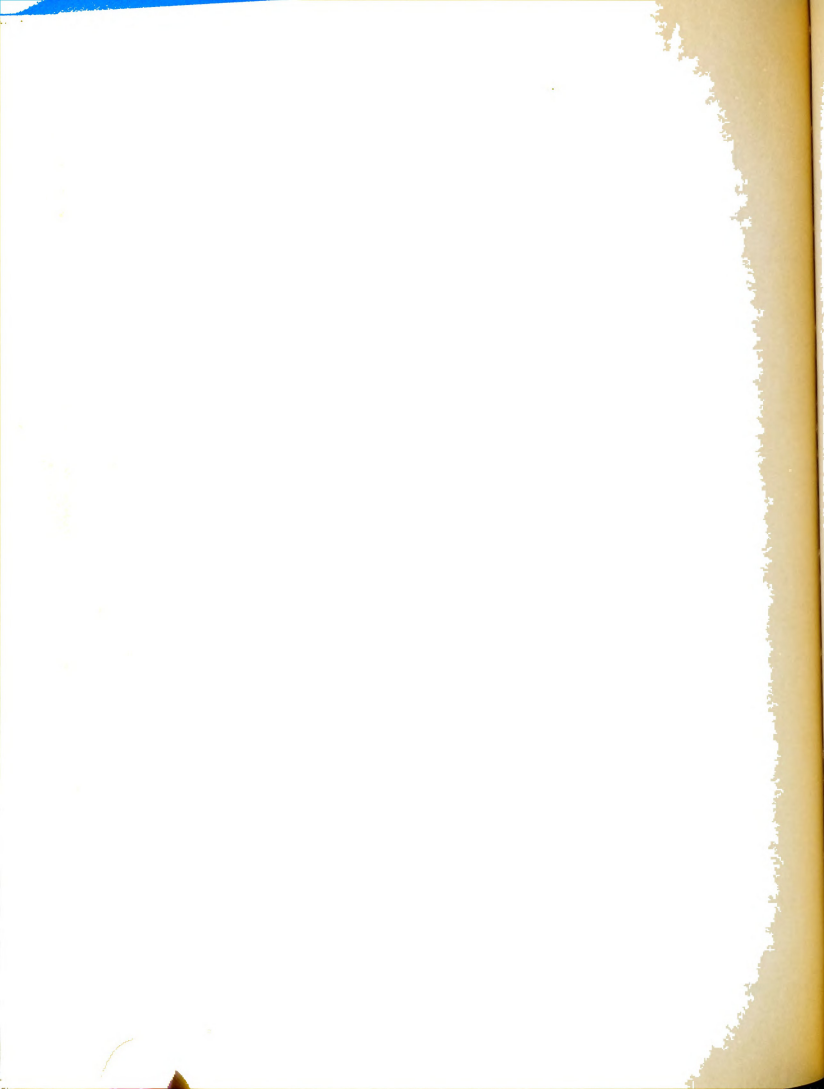
and

$$\left[\frac{m_1 m_2}{n} - m_1 p_1 \right] \sigma^{-1} = \frac{m_1}{n} \left[m_1 p_1 + (n - m_1) p_2 - n p_1 \right] \sigma^{-1}$$

$$= \left(\frac{m_1}{n} \right) \left(\frac{n - m_1}{n} \right) \left(\frac{\sigma^2}{n} \right)^{-\frac{1}{2}} \sqrt{n} (p_2 - p_1) \rightarrow [a(1-a)b(1-b)]^{\frac{1}{2}}.$$

We have thus established (3.4.4). It follows immediately that (3.4.2) holds, so that the theorem is proven.

We now consider a sequence of 2×2 tables with fixed marginal probabilities P_A and P_B . Applying Th. 3.4.A. with



$$a = P_A, b = P_B, \text{ and } t = \frac{\lambda(1-P_A-P_B+\lambda P_A P_B)}{(1-\lambda P_A)(1-\lambda P_B)}, \lambda = 1 - \frac{d}{\sqrt{n}},$$

we have that, upon rewriting the expression for t with $\lambda = 1 - \frac{d}{\sqrt{n}}$,

$$t = \frac{(1 - \frac{d}{\sqrt{n}})(Q_A Q_B - \frac{d}{\sqrt{n}})}{(Q_A + \frac{d}{\sqrt{n}} P_A)(Q_B + \frac{d}{\sqrt{n}} P_B)} = 1 - \frac{\frac{d}{\sqrt{n}}}{(Q_A + \frac{d}{\sqrt{n}} P_A)(Q_B + \frac{d}{\sqrt{n}} P_B)}$$

$$\text{and } \sqrt{n}(1-t) \rightarrow \frac{d}{Q_A Q_B},$$

$$(3.4.5) \quad P_n(\lambda, P_A, P_B, m_1, m_2) \Big|_{\lambda=1-\frac{d}{\sqrt{n}}} \longrightarrow \Phi \left(-u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right) + 1 - \Phi \left(u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right).$$

We immediately infer from Th. 3.4.A the following two corollaries:

Corollary 1. For all values of m_1 such that $\left| \frac{m_1 - n P_A}{\sqrt{n P_A Q_A}} \right| \leq f_1(n)$

where $n^{-\frac{1}{6}} f_1(n) \rightarrow 0$,

$$(3.4.6) \quad P_n(\lambda, P_A, P_B, m_1) \Big|_{\lambda=1-\frac{d}{\sqrt{n}}} \longrightarrow \Phi \left(-u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right) + 1 - \Phi \left(u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right).$$



Corollary 2.

$$(3.4.7) \quad P_n(\lambda, P_A, P_B) \Big]_{=1 - \frac{d}{\sqrt{n}}} \longrightarrow \phi \left(-u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right) + 1 - \phi \left(u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right).$$

Th. 3.4.A also shows that the limiting power function for the χ^2 -test for independence is the non-central χ^2 -distribution with non-centrality parameter $\frac{P_A P_B}{Q_A Q_B} d^2$, assuming that the conditions of the normal-approximation Theorem 2.2.F are satisfied, and $\lambda = 1 - \frac{d}{\sqrt{n}}$.

As a particular case of (3.4.1), we put $m_1 = nP_A, m_2 = nP_B$ and obtain, after some algebra,

$$(3.4.8) \quad P_n(\lambda, P_A, P_B, m_1, m_2) \sim \phi \left(\frac{-u_{\frac{\alpha}{2}} h' + n(1-\lambda) P_A P_B}{\sigma_{\pi}} \right) + 1 - \phi \left(\frac{u_{\frac{\alpha}{2}} h' + n(1-\lambda) P_A P_B}{\sigma_{\pi}} \right)$$

where $m_1 = nP_A, m_2 = nP_B, h' = \sqrt{n P_A P_B Q_A Q_B}$, $\sigma_{\pi}^2 = n \left[\sum_{i=1}^4 \frac{1}{\pi_i} \right]^{-1}$,

and $\pi_i, i = 1, 2, 3, 4$, is given by (2.1.27). Setting $\lambda = 1 - \frac{d}{\sqrt{n}}$

in (3.4.8), it is easily seen that

$$(3.4.9) \quad P_n(\lambda, P_A, P_B, m_1, m_2) \sim \phi \left(-u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right) + 1 - \phi \left(u_{\frac{\alpha}{2}} + \sqrt{\frac{P_A P_B}{Q_A Q_B}} d \right)$$



where $\lambda = 1 - \frac{d}{\sqrt{n}}$, $m_1 = nP_A$, $m_2 = nP_B$.

Therefore, from (3.4.5), (3.4.6), (3.4.7), and (3.4.9), we obtain the following three asymptotic relations, which we group together in Th.3.4.B.

Th. 3.4.B For all m_1, m_2 such that

$$\left| \frac{m_1 - nP_A}{\sqrt{nP_A Q_A}} \right| \leq f_1(n), \quad \left| \frac{m_2 - nP_B}{\sqrt{nP_B Q_B}} \right| \leq f_2(n)$$

where $n^{-\frac{1}{6}} f_1(n) \rightarrow 0$, $i = 1, 2$,

$$(3.4.10) \quad P_n(\lambda, P_A, P_B, m_1, m_2) \Big|_{\lambda=1-\frac{d}{\sqrt{n}}} \sim P_n(\lambda, P_A, P_B, m_1, m_2)$$

$$(3.4.11) \quad P_n(\lambda, P_A, P_B, m_1) \Big|_{\lambda=1-\frac{d}{\sqrt{n}}} \sim P_n(\lambda, P_A, P_B, m_1, m_2)$$

$$(3.4.12) \quad P_n(\lambda, P_A, P_B) \Big|_{\lambda=1-\frac{d}{\sqrt{n}}} \sim P_n(\lambda, P_A, P_B, m_1, m_2)$$

where $P_n(\lambda, P_A, P_B, m_1, m_2)$ on the right-hand-side of

(3.4.10), (3.4.11), and (3.4.12) is evaluated at $\lambda = 1 - \frac{d}{\sqrt{n}}$

$m_1 = nP_A$, and $m_2 = nP_B$ in each case.

(3.4.12) implies that for n moderately large we can approximate the power function for the test of independence in the double dichotomy quite well by evaluating the power



function in the 2×2 independence trial with marginal totals at their expectations, i.e., with $m_1 = nP_A$, $m_2 = nP_B$. The implication of (3.4.11) is that the power function in the 2×2 comparative trial can similarly be approximated by the power function in the 2×2 independence trial, this time with marginal totals m_1 and $m_2 = m_1 p_1 + (n - m_1) p_2$, where

$$p_1 = \lambda P_B \quad \text{and} \quad p_2 = \frac{P_B(1 - \lambda P_A)}{1 - P_A}. \quad \text{Also, the power function}$$

$$P_n(\lambda, P_A, P_B, m_1) \Big|_{m_1 = nP_A}, \quad \text{may also be used to approximate}$$

$$P_n(\lambda, P_A, P_B).$$

For numerical illustrations of these last approximations, see appendix D.



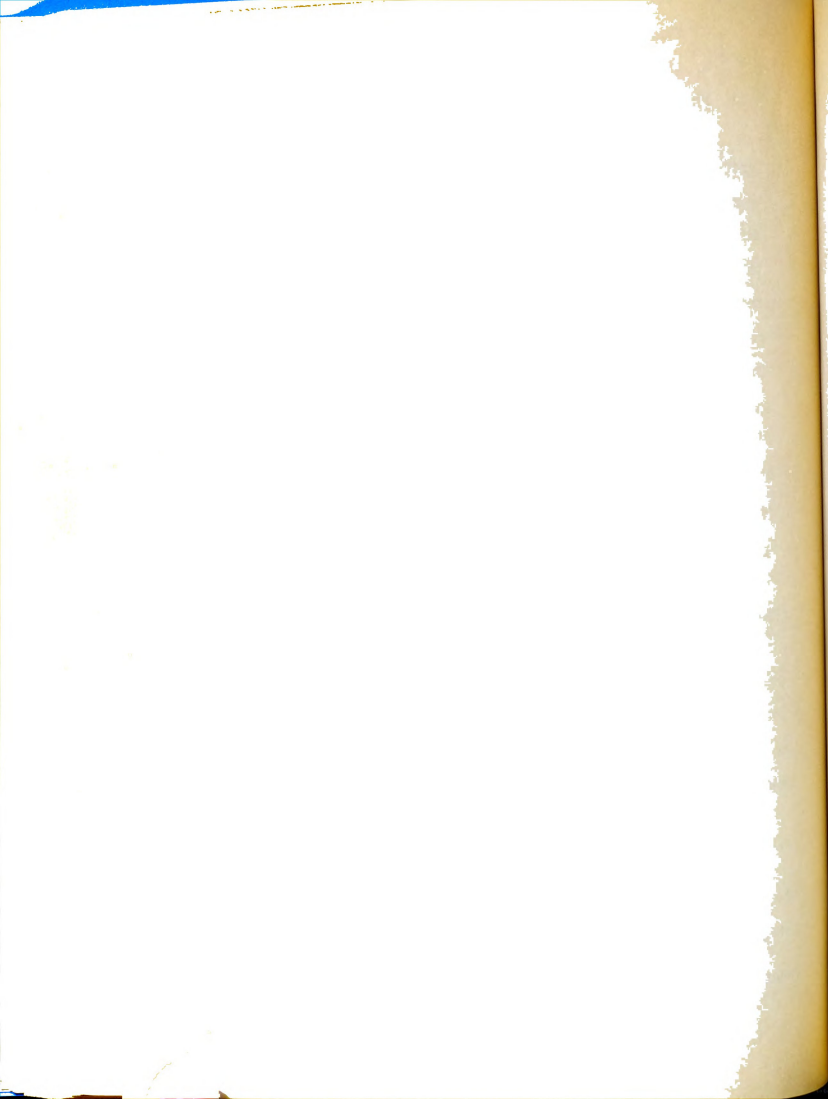
4.1 Preliminaries

In Chapter 3 we gave the exact power function for the uniformly most powerful unbiased test of independence in the 2×2 independence trial, 2×2 comparative trial, and the double dichotomy. We proposed several approximations for these exact power functions, including some based on asymptotic properties as described in section 3.4. We also suggested that the exact power functions for the 2×2 independence trial and 2×2 comparative trial might be used as approximations for the power function in the double dichotomy. In this chapter we shall examine the adequacy of some of these approximations. We shall also see how the various exact power functions compare for small values of m_1, m_2 , and n , knowing from the results in section 3.4 that for large values of m_1, m_2 , and n there is little difference between them.

The notation $P_n(\lambda, P_A, P_B)$ was used to denote the exact power function in the double dichotomy. The conditional power function, with one set of marginal totals fixed, say m_1 and $n - m_1$, was denoted by $P_n(\lambda, P_A, P_B, m_1)$, and is the exact power function in the 2×2 comparative trial, with $p_1 = \lambda P_B$,

and $p_2 = \frac{P_B(1 - \lambda P_A)}{1 - P_A}$. Going one step further, we denoted

the conditional power function with both sets of marginal totals fixed by $P_n(\lambda, P_A, P_B, m_1, m_2)$, which is the exact power function in the 2×2 independence trial. It is hoped that



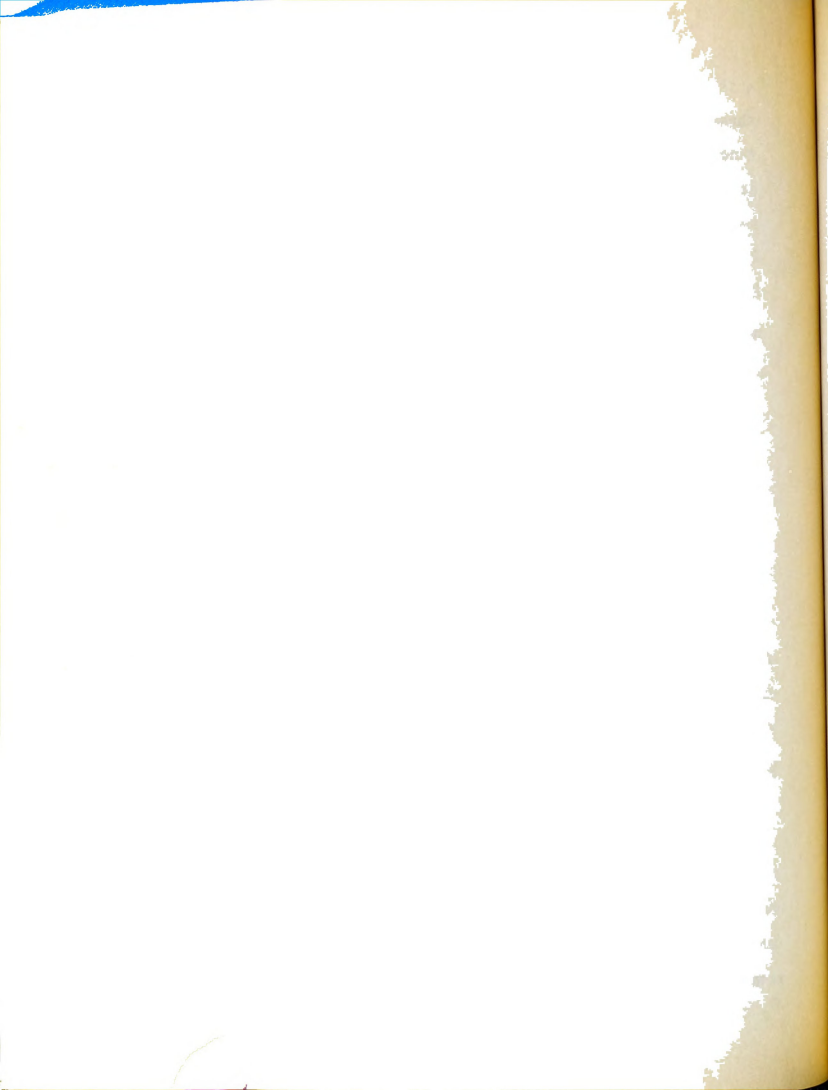
this choice of notation emphasizes the relation between the three exact power functions, i.e., that the exact power functions in the 2×2 independence trial and 2×2 comparative trial are conditional power functions with respect to the exact power function in the double dichotomy. In the 2×2 comparative trial, we will use the alternative notation $P_n(p_1, p_2, m_1)$ to denote the power function when p_1 and p_2 are not related to λ , P_A , and P_B .

4.2 Comparison of Exact and Approximate Power Functions.

Probably the most interesting tables of power which have been computed are given in Appendix D, pages 159-170. In these tables, values of the three exact power functions have been tabulated for certain combinations of n , P_A , and P_B together with a few approximate values using the normal approximations (3.1.4) and (3.35). The power function in the 2×2 independence trial is evaluated at $m_1 = nP_A$, and $m_2 = nP_B$, and the power function in the 2×2 comparative trial is evaluated at $m_1 = nP_A$. There are two important observations to be made from the data given there. The first is that these cases in which

$$P_n(\lambda, P_A, P_B) < P_n(\lambda, P_A, P_B, m_1) \Big]_{m_1=nP_A} < P_n(\lambda, P_A, P_B, m_1, m_2) \Big]_{\substack{m_1=nP_A \\ m_2=nP_B}},$$

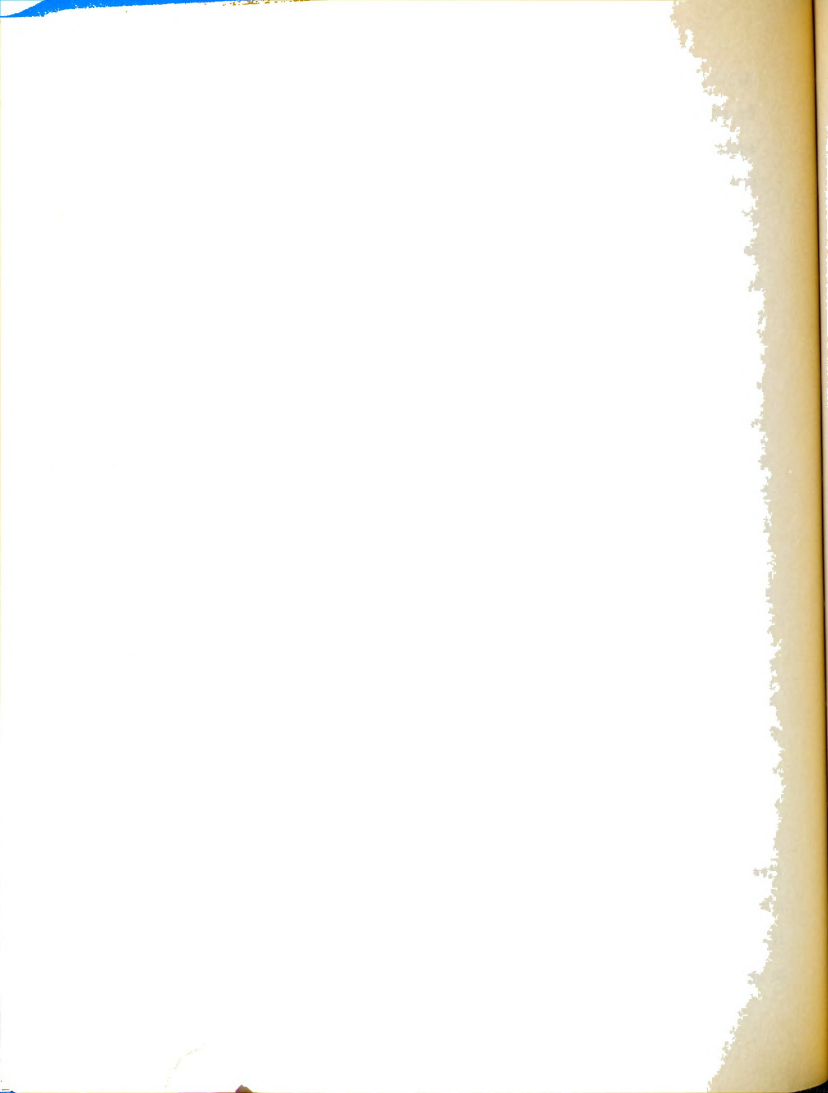
and also cases for which there are inequalities in the opposite direction, with the exception that there is no case in the tables for which power in the double dichotomy is greater than that in the 2×2 comparative trial. However, if $nP_A = 1$, then $P_n(\lambda, P_A, P_B, m_1) \Big]_{m_1=nP_A} \equiv \alpha$ for all λ . From table



A.1.1, $P_{10}(\lambda, .1, .1) > .05$ for all $\lambda \neq 1$, so that no theorem is possible concerning the order of the power functions. The second observation is that the functional values of the three power functions draw together as n increases; for $n = 10$, there are wide differences; at $n = 20$, the differences decrease, still being fairly large, and for $n = 30$, the differences are smaller, but there are cases where the differences are large enough to be considered seriously.

It is also rather interesting to examine the pairwise differences between the power functions. It appears that the differences between power in the 2×2 independence trial and the 2×2 comparative trial tend to be somewhat than the differences between power in the 2×2 comparative trial and the double dichotomy.

The adequacy of the normal approximation (3.3.5) to $P_n(\lambda, P_A, P_B)$ is reflected in the tables in Appendix D, section D.1. Values computed from (3.3.5) tend to overestimate $P_n(\lambda, P_A, P_B)$ for small and large values of λ , and underestimate for values of λ in the neighborhood of 1. Overall, the approximation seems to be quite adequate. Values of power in the 2×2 comparative trial, computed using Sillitto's and Patnaik's approximations, are given in E.2, together with the exact values $P_n(p_1, p_2, m_1)$. The tables given are for the special cases $m_1 = 10$, $n = 20$ and $m_1 = 15$, $n = 30$, since for these cases the region for rejecting H_0 is about the same for the unbiased test, and the Pearson test, on which Sillitto and Patnaik based their computations of



power. Pearson's test is described in section 1.7. It is very apparent that Patnaik's approximation is uniformly worse than Sillitto's approximation, which tends to overestimate the correct values. The exact power function $P_n(t|m_1, m_2)$ in the 2×2 independence trial may also be used to approximate

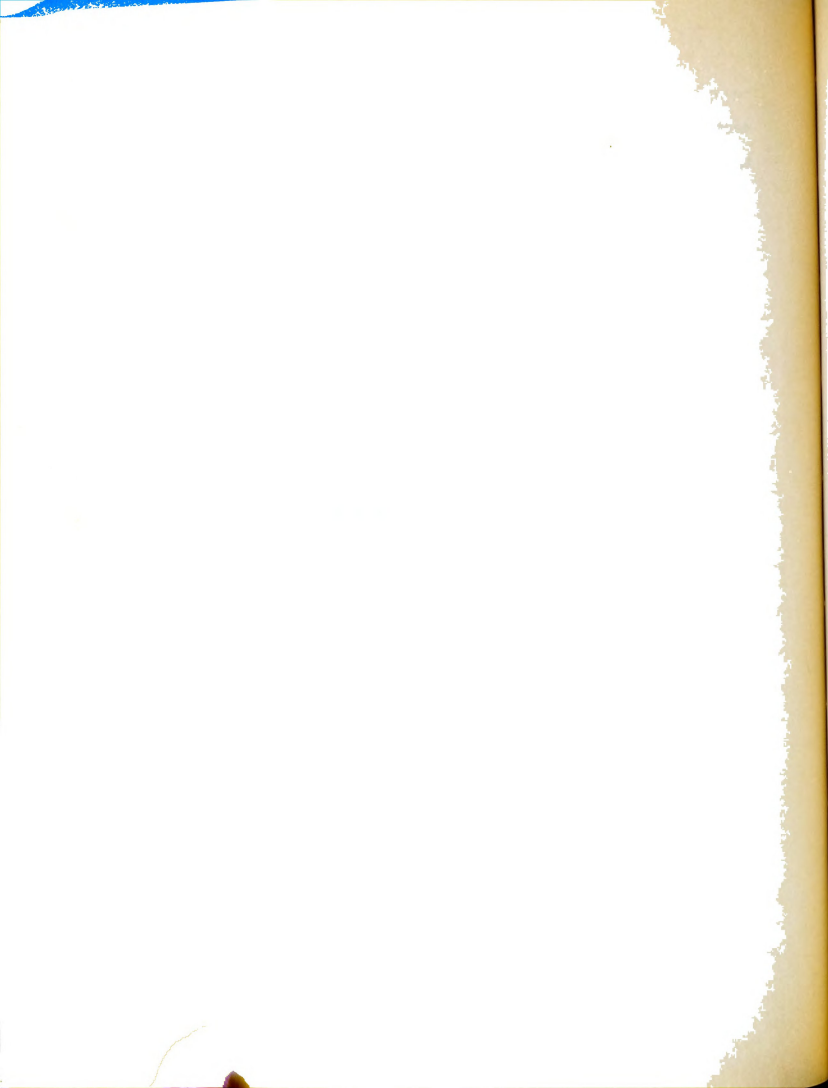
$P_n(p_1, p_2, m_1)$ by putting $t = \frac{p_1 q_2}{p_2 q_1}$ and $m_2 = m_1 p_1 + (n - m_1) p_2 = \gamma$.

γ is the expected value of m_2 . A few values were computed from $P_n(t|m_1, \gamma)$, and the values compared with those obtained from Sillitto's approximation. These values were sometimes closer to the correct values than Sillitto's, but there is not enough data to draw any conclusions. These values may also be found in E.2. In all, Sillitto's approximation appears to be quite accurate, and isn't difficult to compute, so for small integral values of m_1 and n it is probably to be preferred as an approximation. In the 2×2 independence trial, the normal approximation (3.1.4) should be satisfactory, as indicated by the data in D.1.

In view of Th.3.4.B we have that

$$P_n(t | m_1, \gamma) \sim P_n(p_1, p_2, m_1)$$

for a wide range of values of m_1 and with $t = \frac{p_1 q_2}{p_2 q_1}$. This implies that power in the 2×2 comparative trial should be nearly constant for all combinations of p_1 and p_2 for which t is constant.

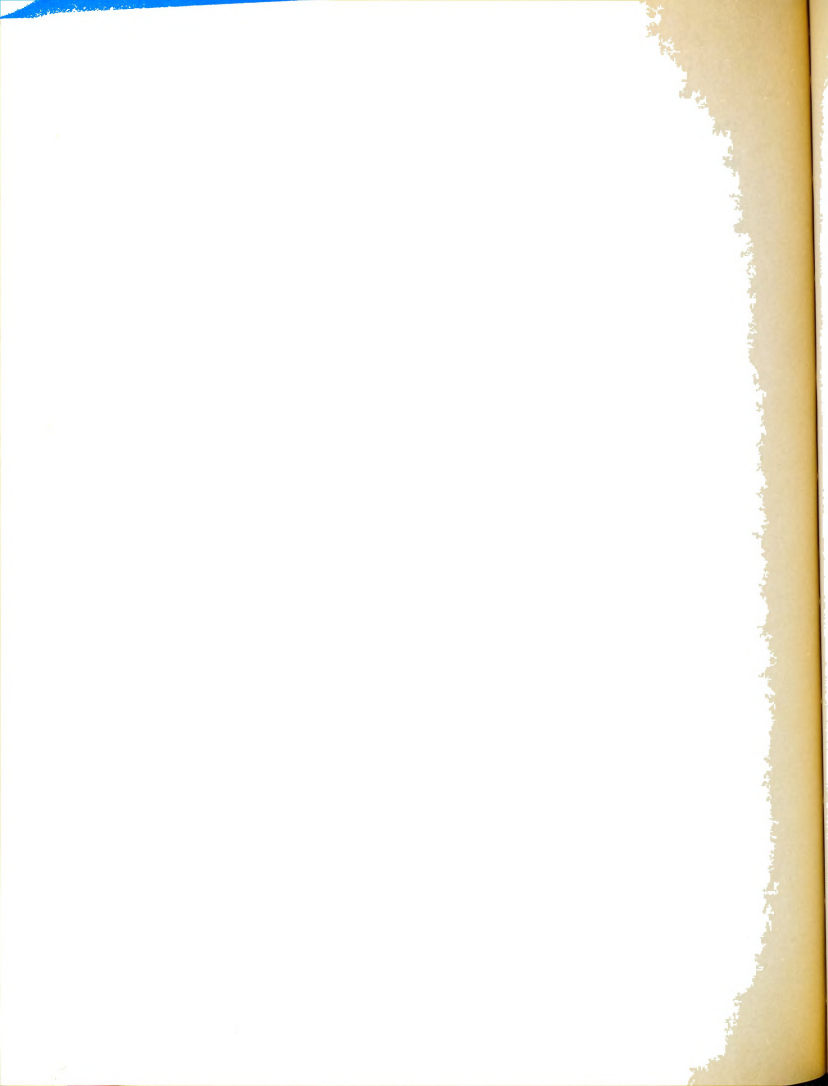


On the basis of the data in the appendices, we suggest that the following approximations to the power functions be used, considering both ease of computation and accuracy:

I. 2 x 2 Independence Trial — use the normal approximation given by (3.1.4).

II. 2 x 2 Comparative Trial — use either Sillitto's approximation given by (3.2.6), or the normal approximation (3.1.4), with m_2 evaluated at its expected value, given by
$$\bar{Y} = m_1 p_1 + (n - m_1) p_2.$$

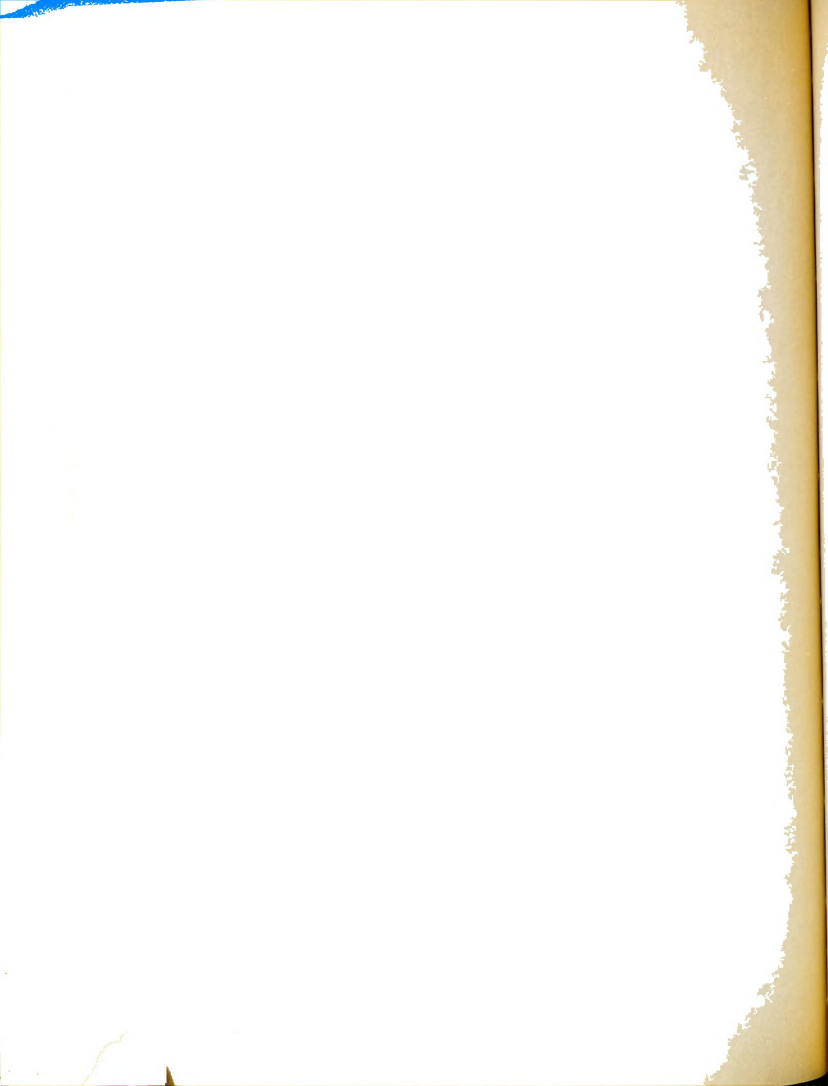
III. Double Dichotomy — use the normal approximation given in (3.3.5).



Summary

We have re-examined one of the oldest problems in mathematical statistics, a problem which has become classical within its field -- that of testing for independence in 2×2 tables. At various times in the past few years, it has been thought that the theory of 2×2 contingency tables was completely known, that it was dead as a subject of research. In view of the controversy that has stirred the attention of such noted statisticians as R. A. Fisher, E. S. Pearson, and G. A. Barnard, it is difficult to understand how the problem can be considered as solved in all its many facets. This is particularly true when we realize that very little was previously known about the power for the test of independence in small samples. This appears to be very surprising, since practicing statisticians and research workers have used the results of tests for independence in 2×2 tables for many years as a basis for making decision.

We have undertaken the task of remedying this situation. Throughout this thesis, our one underlying objective was to thoroughly examine the power for the test of independence in small samples, even though we have investigated power for large samples also. In preparation for this examination, we first tried to determine precisely what had been accomplished previously in 2×2 tables. It was discovered that only in one of three possible cases had much been done in the way of studying power for a test of independence, this case being the 2×2 comparative trial. Therefore, we chose a particular



test for independence, the uniformly most powerful unbiased test first proposed by Katz [9] in 1942, and were able to use Katz' formulation of alternative hypotheses as a basis for investigating power for this test of independence. The unique feature of Katz' formulation was that it provided a logical and consistent method for examining and comparing power in each of the three cases corresponding to both sets of marginal totals fixed, one set fixed, and neither fixed. It was necessary first to study some properties of the conditional distribution for fixed marginal totals under the class of alternative hypotheses. Once these properties were known, it was fairly easy to examine systematically power for the test of independence, and that is what we hope we have done in this paper. Briefly, we will now summarize what we have done.

1. In section 2.1 of Chapter 2, we gave expressions for computing moments of the modified hypergeometric probability function $\Pr \{n_1 | m_1, m_2, n; t\}$, including also several recursion relations between the moments of the distribution, and were able to indicate the form of the asymptotic mean and variance of this distribution.

2. In section 2.2 of the same chapter, it was shown that the limiting distribution of the conditional distribution of n_1 for fixed marginal totals was binomial, Poisson, or normal, depending on the mode in which the marginal totals m_1 and m_2 increased. As a result of Theorem 2.2.F, it was suggested that there are at least two published results in



the literature which may be invalid.

3. We gave the exact power function for the test of independence in the 2×2 independence trial, 2×2 comparative trial, and the double dichotomy in the first three sections of Chapter 3, and suggested several approximations for these power functions. We investigated the adequacy of these approximations in Chapter 4.

4. We examined asymptotic power for the test of independence in 3.4, and confirmed what many statisticians have believed for sometime - that for very large sample sizes, the difference in power between the three cases corresponding to the nature of the marginal totals is negligible.

5. We provided extensive tables of power for each of the three cases and used the values of exact power to evaluate the adequacy of the various approximations proposed in Chapter 3. The tables of exact power are in appendices A, B, C, and D. On the basis of these exact values, we suggested three specific approximations for the exact power functions.

We hope that we have succeeded in thoroughly examining the power for the tests of independence, and that the results and computations, which we have presented in this thesis, will be useful for applied statisticians and research workers.



Appendix A. Exact Power for the Double Dichotomy.

We consider a given population in which the members of the population are classified by two attributes, each having two categories, so that there are four distinct classes of members. We denote these classes by A_1B_1, A_2B_1, A_1B_2 , and A_2B_2 . The proportions of members in these four classes are π_1, π_2, π_3 , and π_4 respectively. Putting $P_A = \pi_1 + \pi_3$ and $P_B = \pi_1 + \pi_2$, we propose to test the null hypothesis H_0 of no association in the occurrence of one attribute and the occurrence of the second, i.e., that $\pi_1 = P_A P_B$, by selecting a sample of size n , and making use of the observed numbers in the four classes. Any alternative hypothesis may be expressed in the form $\pi_1 = \lambda P_A P_B$. We use the uniformly most powerful unbiased test, given in section 1.5, of Chapter 1, to test H_0 . The tables in this appendix give the exact values of the power function for this test. This power function is denoted by $P_n(\lambda, P_A, P_B)$, and its functional form is given by (3.3.2) on page 79. The significance level α is .05 in all the tables.

The values of $P_n(\lambda, P_A, P_B)$ were obtained by first computing the conditional power function $P_n(\lambda, P_A, P_B, m_1)$ for the 2×2 comparative trial, and then weighting these values with binomial probabilities, as suggested in 3.3, equation (3.3.4). All the computations were performed on the Michigan State University digital computer, MISTIC.



The tables are divided into three sections, A.1, A.2, and A.3, corresponding to $n = 10, 20$, and 30 respectively. In A.1 and A.2, tables of $P_{10}(\lambda, P_A, P_B)$ and $P_{20}(\lambda, P_A, P_B)$ are given for all 15 combinations of P_A and P_B with $P_B \leq P_A$, $P_A, P_B = .1, .2, .3, .4, .5$, and in A.3 tables of $P_{30}(\lambda, P_A, P_B)$ are given for the same combinations of P_A and P_B , except the three cases $P_A = P_B = .2$, $P_A = .2, P_B = .1$, and $P_A = P_B = .1$ are not available. For the combinations of P_A and P_B with $P_A = .3, .4$, and $.5$, $P_n(\lambda, P_A, P_B)$ was computed for $\lambda = .1, .2, .3, \dots, \left[\frac{1}{P_A} \right]$. For $P_A = .2$, the range on λ was from $.2$

to 4.8 with an increment of $.2$; for $P_A = .1$ and $n = 10$, the range of λ is $.4$ to 9.6 , the increment being $.4$, while for $n = 20$, the range on λ is $.3$ to 9.9 , increasing by $.3$.

To check the accuracy of the computations, we used the fact that the test of independence is a conditional test. Thus, since

$$\begin{aligned} P_n(\lambda, P_A, P_B) &= \sum_{m_1=0}^n \sum_{m_2=0}^n \Pr \{m_1, m_2 | n, \lambda\} P_n(\lambda, P_A, P_B, m_1, m_2) \\ &= \sum_{m_1=0}^n \Pr \{m_1 | n, P_A\} P_n(\lambda, P_A, P_B, m_1), \end{aligned}$$

where $P_n(\lambda, P_A, P_B, m_1, m_2)$ is the exact conditional power function



for fixed m_1 and m_2 , we obtained a partial check on the exact computations for $P_n(\lambda, P_A, P_B)$ by checking the values of the two conditional power functions, at certain selected points. In particular, we have that each of the power functions is equal to .05 for $\lambda = 1$. Finally for $P_A = .5$, $P_n(\lambda, P_A, P_B)$ should be symmetric about $\lambda = 1$, which provided one further check on the accuracy of the computations. All of the computed values were given to nine decimal places, and rounded off to five places in the table.

For $n = 10$, $P_{10}(\lambda, P_A, P_B)$ evaluated at $\lambda = 1$ was exact to nine decimal places for all 15 combinations of P_A , and P_B , for $n = 20$ it was accurate to 7 places, and for $n = 30$, there was 6-place accuracy. The partial check on the exact values obtained by computing the conditional power functions for certain selected values indicated that the exact computations were probably accurate to at least five decimal places for all λ , with 4-place accuracy virtually certain.

The table numbers indicate the value of n and P_A , with the first number giving the value of $\frac{n}{10}$, and the second number P_A . For example, A.2.4 gives values of $P_{20}(\lambda, .4, P_B)$.



Table A.1.1

$$P_{10}(\lambda, .1, P_B)$$

λ	$P_B = .1$
.4	.05043
.8	.05005
1.2	.05005
1.6	.05046
2.0	.05129
2.4	.05258
2.8	.05434
3.2	.05661
3.6	.05940
4.0	.06273
4.4	.06665
4.8	.07116
5.2	.07630
5.6	.08208
6.0	.08854
6.4	.09569
6.8	.10356
7.2	.11216
7.6	.12151
8.0	.13164
8.4	.14254
8.8	.15424
9.2	.16674
9.6	.18005

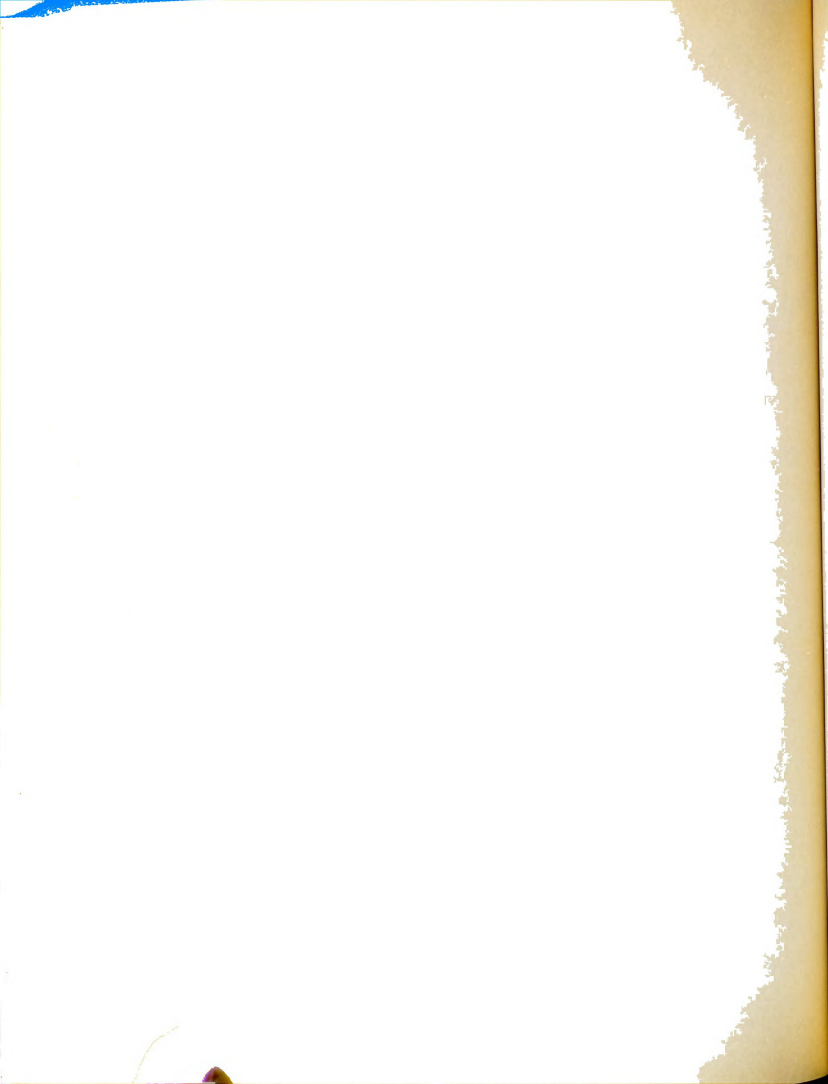


Table A.1.2

 $P_{10}(\lambda, .2, P_B)$

λ	$P_B = .2$	$P_B = .1$
.2	.06030	.05269
.4	.05588	.05154
.6	.05265	.05069
.8	.05068	.05018
1.0	.05000	.05000
1.2	.05070	.05018
1.4	.05288	.05073
1.6	.05665	.05167
1.8	.06212	.05302
2.0	.06943	.05480
2.2	.07874	.05702
2.4	.09018	.05970
2.6	.10390	.06287
2.8	.12006	.06655
3.0	.13878	.07075
3.2	.16020	.07551
3.4	.18442	.08084
3.6	.21150	.08677
3.8	.24149	.09331
4.0	.27437	.10048
4.2	.31006	.10832
4.4	.34843	.11683
4.6	.38926	.12603
4.8	.43221	.13595



Table A.1.3

 $F_{10}(\lambda, .3, P_B)$

λ	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.12249	.07968	.05715
.2	.10650	.07343	.05568
.3	.09277	.06794	.05437
.4	.08116	.06320	.05323
.5	.07151	.05920	.05226
.6	.06372	.05591	.05145
.7	.05771	.05335	.05082
.8	.05344	.05150	.05037
.9	.05086	.05038	.05009
1.0	.05000	.05000	.05000
1.1	.05088	.05039	.05009
1.2	.05356	.05156	.05038
1.3	.05811	.05355	.05086
1.4	.06472	.05640	.05154
1.5	.07344	.06011	.05243
1.6	.08444	.06483	.05353
1.7	.09790	.07051	.05485
1.8	.11398	.07723	.05639
1.9	.13287	.08505	.05817
2.0	.15474	.09402	.06018
2.1	.17974	.10421	.06244
2.2	.20802	.11567	.06495
2.3	.23968	.12845	.06773
2.4	.27478	.14263	.07077
2.5	.31334	.15824	.07409
2.6	.35527	.17533	.07769
2.7	.40040	.19396	.08159
2.8	.44847	.21414	.08579
2.9	.49905	.23592	.09030
3.0	.55155	.25930	.09513
3.1	.60521	.28428	.10029
3.2	.65903	.31085	.10578
3.3	.71175	.33898	.11162

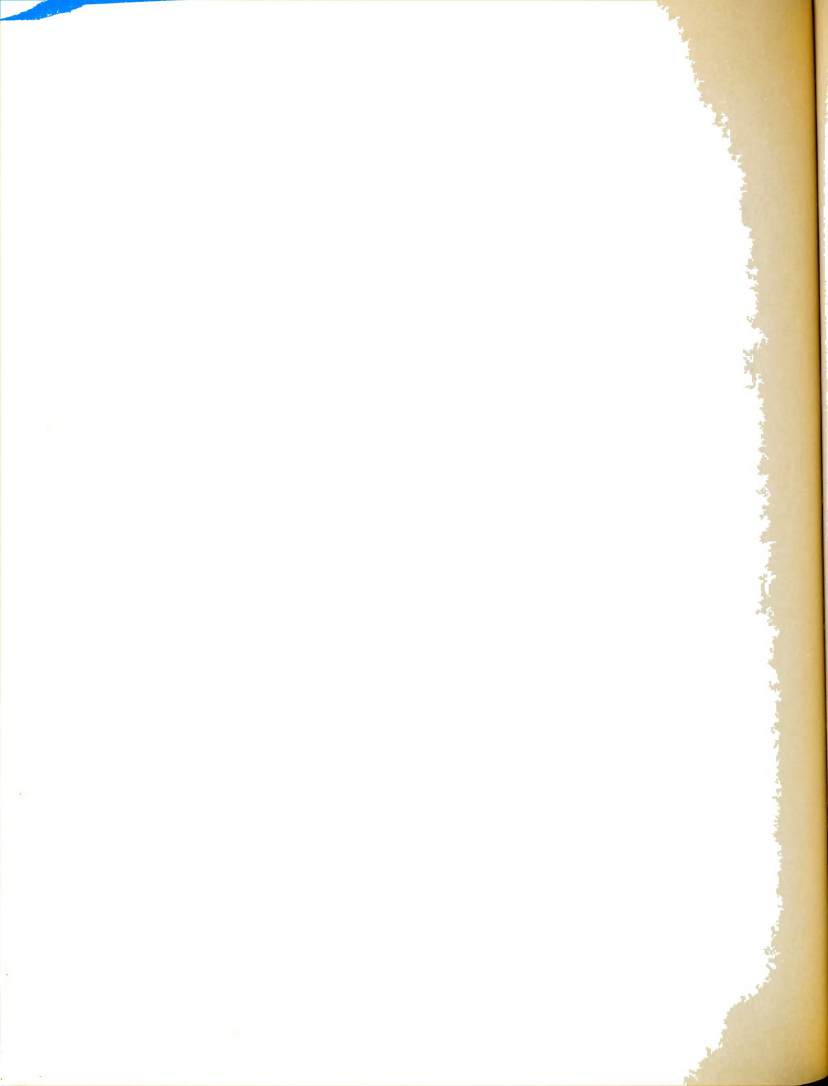


Table A.1.4

 $P_{10}(\lambda, .4, P_B)$

λ	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.33312	.19277	.10626	.06267
.2	.26580	.15980	.09390	.06000
.3	.20936	.13202	.08327	.05766
.4	.16309	.10896	.07424	.05563
.5	.12608	.09020	.06673	.05392
.6	.09738	.07535	.06067	.05251
.7	.07608	.06412	.05599	.05142
.8	.06141	.05624	.05267	.05063
.9	.05283	.05156	.05067	.05016
1.0	.05000	.05000	.05000	.05000
1.1	.05285	.05158	.05068	.05016
1.2	.06157	.05638	.05274	.05065
1.3	.07659	.06459	.05623	.05146
1.4	.09855	.07646	.06122	.05262
1.5	.12826	.09230	.06781	.05412
1.6	.16659	.11249	.07608	.05599
1.7	.21440	.13741	.08615	.05822
1.8	.27240	.16746	.09814	.06083
1.9	.34095	.20304	.11219	.06385
2.0	.41985	.24446	.12843	.06727
2.1	.50806	.29194	.14699	.07112
2.2	.60334	.34554	.16801	.07541
2.3	.70187	.40513	.19162	.08016
2.4	.79766	.47024	.21793	.08538

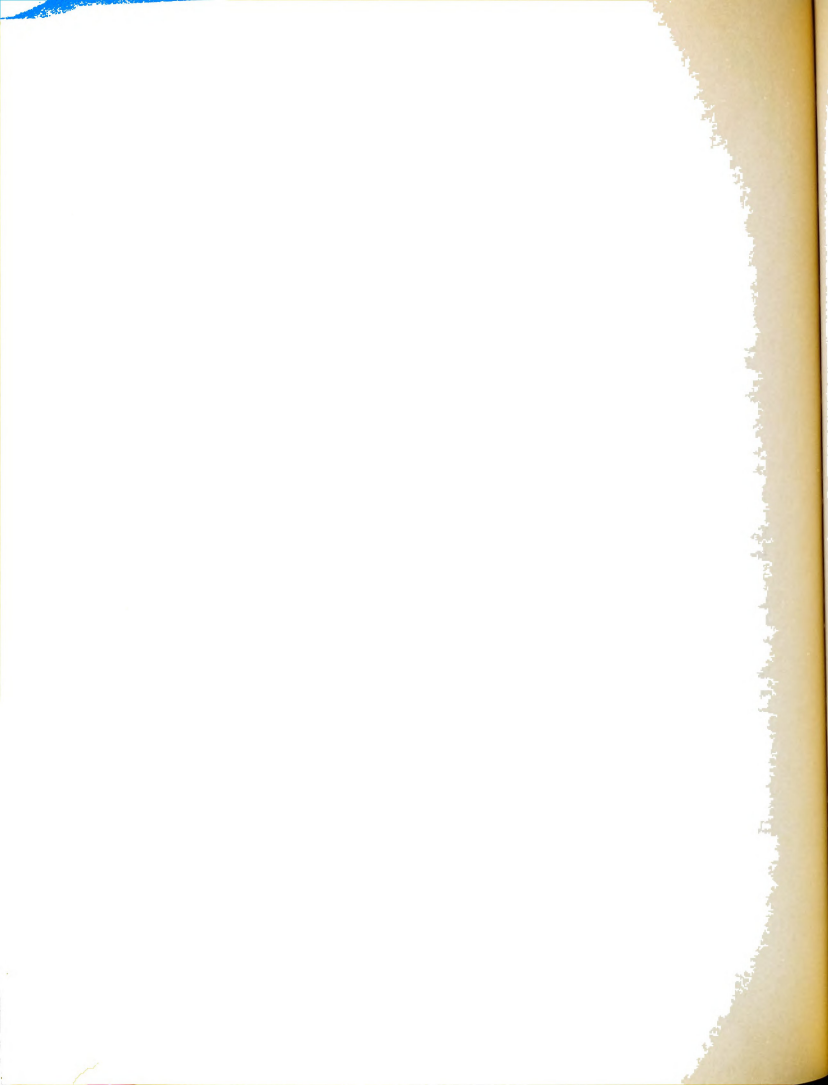


Table A.1.5

$$P_{10}(\lambda, .5, P_B)$$

λ	$P_B = .5$	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.80107	.52724	.29699	.14644	.07077
.2	.64688	.41557	.23832	.12437	.06627
.3	.49672	.31927	.18914	.10567	.06237
.4	.36521	.23957	.14878	.09008	.05903
.5	.25826	.17609	.11646	.07734	.05623
.6	.17650	.12749	.09137	.06724	.05397
.7	.11779	.09209	.07275	.05958	.05223
.8	.07897	.06821	.05994	.05422	.05099
.9	.05706	.05447	.05246	.05105	.05025
1.0	.05000	.05000	.05000	.05000	.05000
1.1	.05706	.05447	.05246	.05105	.05025
1.2	.07897	.06821	.05994	.05422	.05099
1.3	.11779	.09209	.07275	.05958	.05223
1.4	.17650	.12749	.09137	.06724	.05397
1.5	.25826	.17609	.11646	.07734	.05623
1.6	.36521	.23957	.14878	.09008	.05903
1.7	.49672	.31927	.18914	.10567	.06237
1.8	.64688	.41557	.23832	.12437	.06627
1.9	.80107	.52724	.29699	.14644	.07077

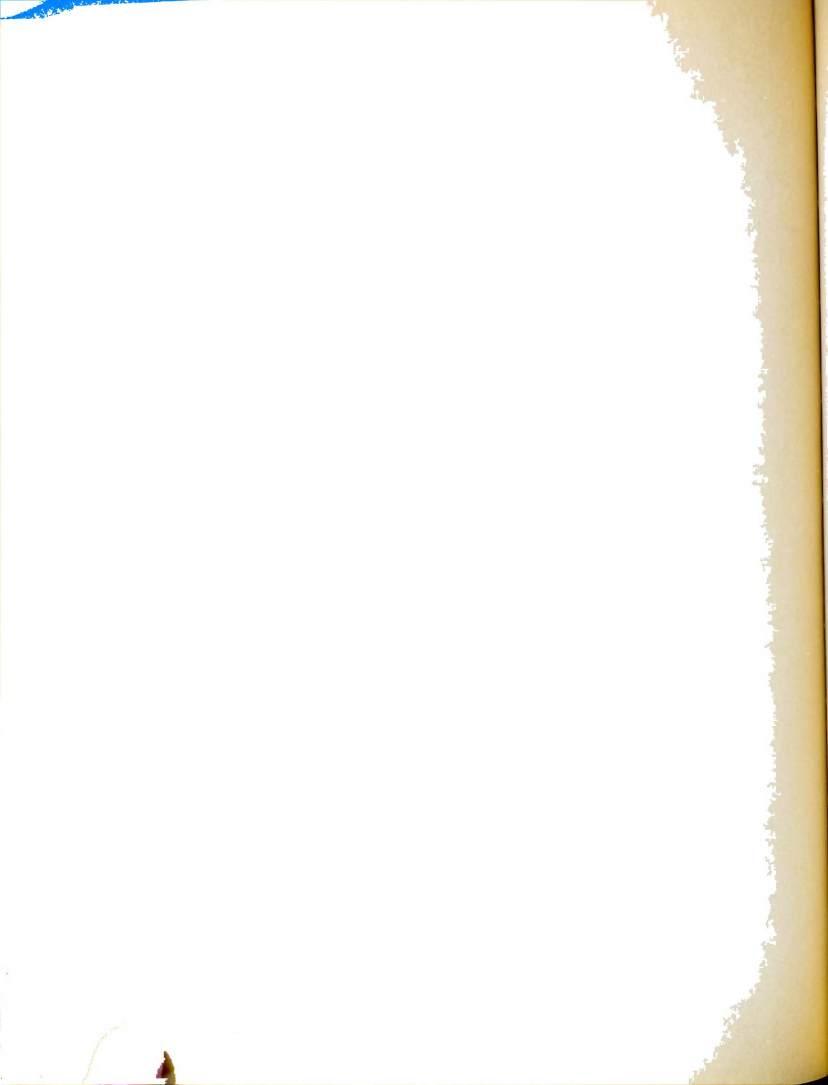


Table A.2.1

$$P_{20}(\lambda, .1, P_B)$$

λ	$P_B = .1$
.3	.05266
.6	.05089
.9	.05006
1.2	.05024
1.5	.05150
1.8	.05394
2.1	.05761
2.4	.06258
2.7	.06891
3.0	.07664
3.3	.08581
3.6	.09646
3.9	.10860
4.2	.12223
4.5	.13735
4.8	.15394
5.1	.17197
5.4	.19140
5.7	.21218
6.0	.23424
6.3	.25752
6.6	.28193
6.9	.30740
7.2	.33383
7.5	.36114
7.8	.38925
8.1	.41807
8.4	.44752
8.7	.47756
9.0	.50812
9.3	.53918
9.6	.57072
9.9	.60278

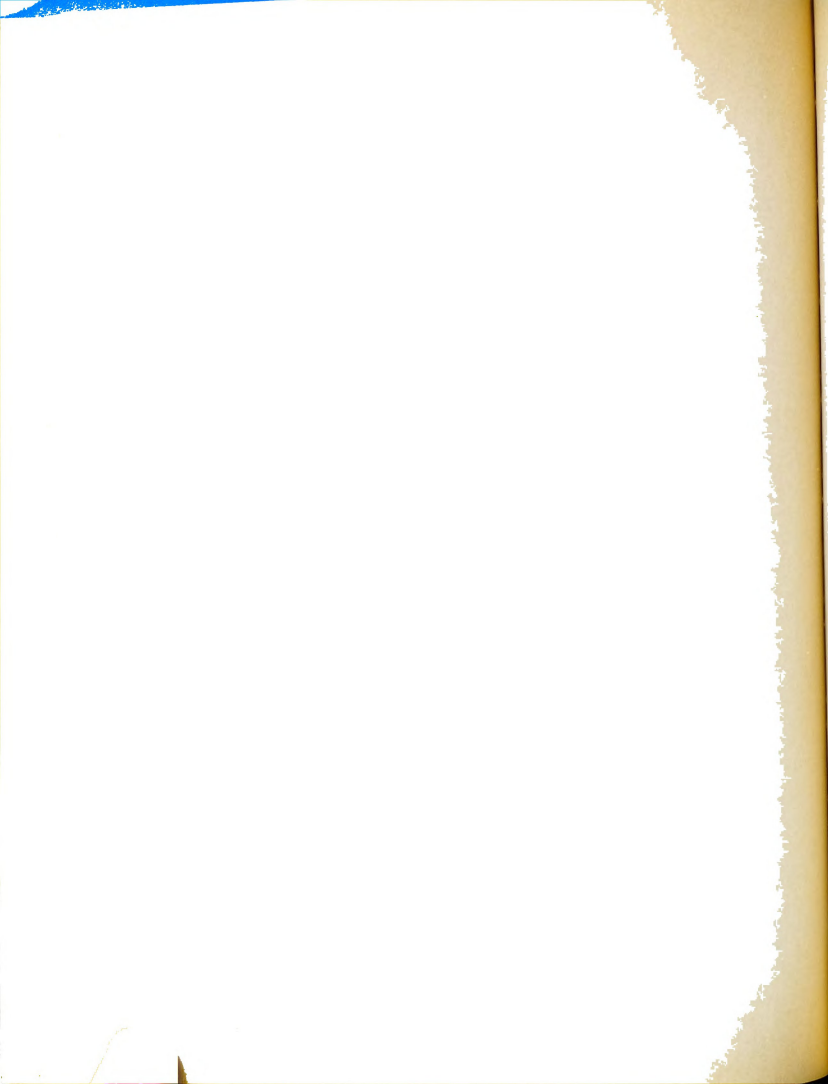


Table A.2.2

 $P_{20}(\lambda, .2, P_B)$

λ	$P_B = .2$	$P_B = .1$
.2	.08984	.06084
.4	.07248	.05624
.6	.06010	.05284
.8	.05257	.05073
1.0	.05000	.05000
1.2	.05271	.05077
1.4	.06115	.05315
1.6	.07585	.05726
1.8	.09731	.06321
2.0	.12593	.07113
2.2	.16190	.08111
2.4	.20519	.09324
2.6	.25539	.10759
2.8	.31177	.12421
3.0	.37323	.14312
3.2	.43831	.16431
3.4	.50531	.18776
3.6	.57237	.21339
3.8	.63762	.24111
4.0	.69934	.27078
4.2	.75613	.30223
4.4	.80712	.33527
4.6	.85212	.36966
4.8	.89173	.40517

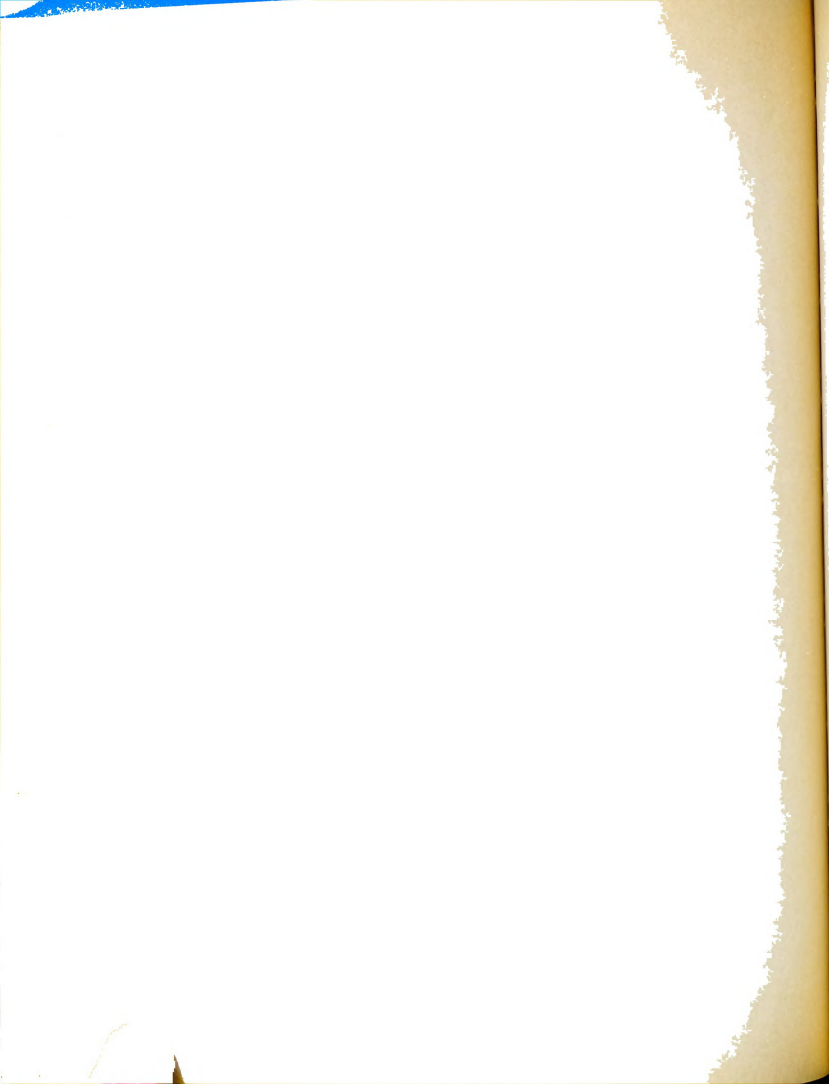


Table A.2.3

$$P_{20}(\lambda, .3, P_B)$$

λ	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.34262	.17163	.07892
.2	.27224	.14383	.07291
.3	.21375	.12040	.06761
.4	.16609	.10080	.06301
.5	.12811	.08491	.05910
.6	.09868	.07216	.05587
.7	.07683	.06242	.05333
.8	.06176	.05552	.05150
.9	.05292	.05138	.05038
1.0	.05000	.05000	.05000
1.1	.05293	.05141	.05039
1.2	.06186	.05571	.05157
1.3	.07708	.06303	.05359
1.4	.09897	.07353	.05647
1.5	.12791	.08739	.06027
1.6	.16420	.10478	.06502
1.7	.20792	.12584	.07076
1.8	.25892	.15069	.07754
1.9	.31666	.17938	.08539
2.0	.38024	.21191	.09438
2.1	.44830	.24817	.10452
2.2	.51913	.28799	.11586
2.3	.59067	.33106	.12844
2.4	.66072	.37698	.14227
2.5	.72700	.42527	.15739
2.6	.78745	.47530	.17380
2.7	.84038	.52639	.19153
2.8	.88469	.57778	.21056
2.9	.91998	.62866	.23089
3.0	.94663	.67820	.25251
3.1	.96577	.72560	.27538
3.2	.97910	.77009	.29948
3.3	.98864	.81100	.32475

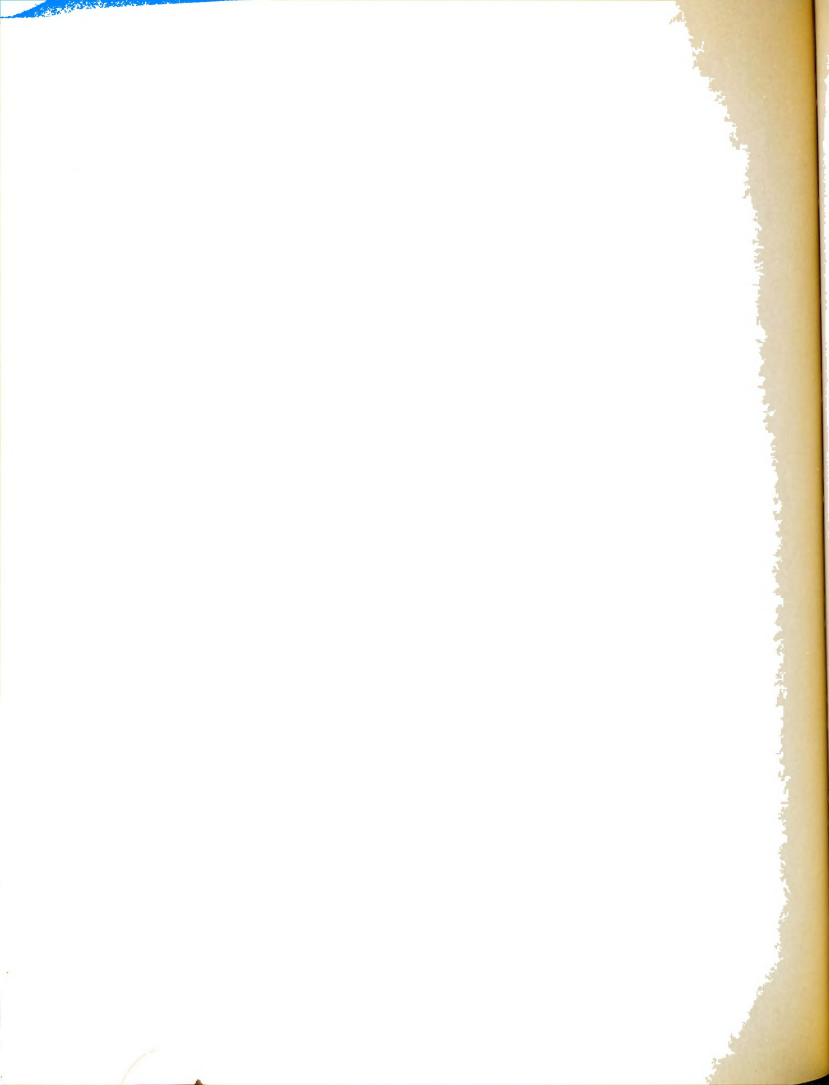


Table A.2.4

$$P_{20}(\lambda, .4, P_B)$$

λ	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.80606	.56085	.28613	.10424
.2	.66881	.44407	.22988	.09234
.3	.52824	.34230	.18303	.08211
.4	.39746	.25710	.14468	.07342
.5	.28514	.18851	.11395	.06618
.6	.19544	.13552	.09000	.06033
.7	.12893	.09663	.07211	.05581
.8	.08400	.07023	.05972	.05259
.9	.05832	.05498	.05242	.05065
1.0	.05000	.05000	.05000	.05000
1.1	.05829	.05498	.05244	.05066
1.2	.08374	.07017	.05986	.05268
1.3	.12781	.09625	.07256	.05610
1.4	.19201	.13408	.09090	.06101
1.5	.27674	.18440	.11531	.06748
1.6	.38016	.24748	.14622	.07561
1.7	.49724	.32280	.18400	.08551
1.8	.61969	.40870	.22885	.09729
1.9	.73686	.50226	.28078	.11107
2.0	.83781	.59933	.33955	.12695
2.1	.91418	.69479	.40456	.14505
2.2	.96293	.78313	.47487	.16549
2.3	.98758	.85918	.54914	.18836
2.4	.99666	.91905	.62566	.21375

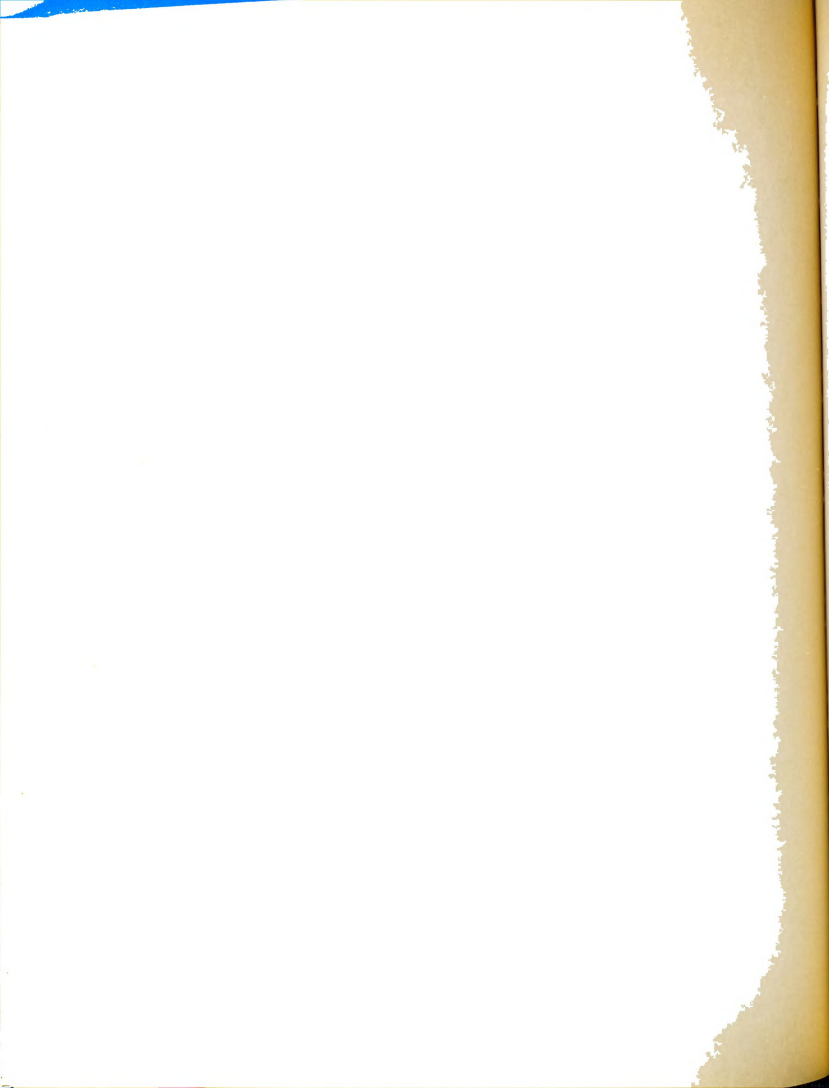


Table A.2.5

$$P_{20}(\lambda, .5, P_B)$$

λ	$P_B = .5$	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.99776	.94459	.75763	.43360	.14390
.2	.97491	.84922	.62362	.34434	.12233
.3	.89875	.71369	.49071	.26800	.10409
.4	.75844	.55819	.36927	.20465	.08890
.5	.57672	.40553	.26593	.15373	.07652
.6	.39250	.27353	.18367	.11428	.06671
.7	.23853	.17173	.12265	.08517	.05928
.8	.13056	.10224	.08134	.06530	.05409
.9	.06943	.06271	.05768	.05377	.05102
1.0	.05000	.05000	.05000	.05000	.05000
1.1	.06943	.06271	.05768	.05377	.05102
1.2	.13056	.10223	.08134	.06530	.05401
1.3	.23853	.17173	.12265	.08517	.05928
1.4	.39250	.27352	.18366	.11428	.06671
1.5	.57672	.40552	.26592	.15373	.07652
1.6	.75844	.55818	.36927	.20465	.08890
1.7	.89875	.71368	.49070	.26800	.10409
1.8	.97491	.84922	.62362	.34434	.12233
1.9	.99776	.94459	.75762	.43360	.14390

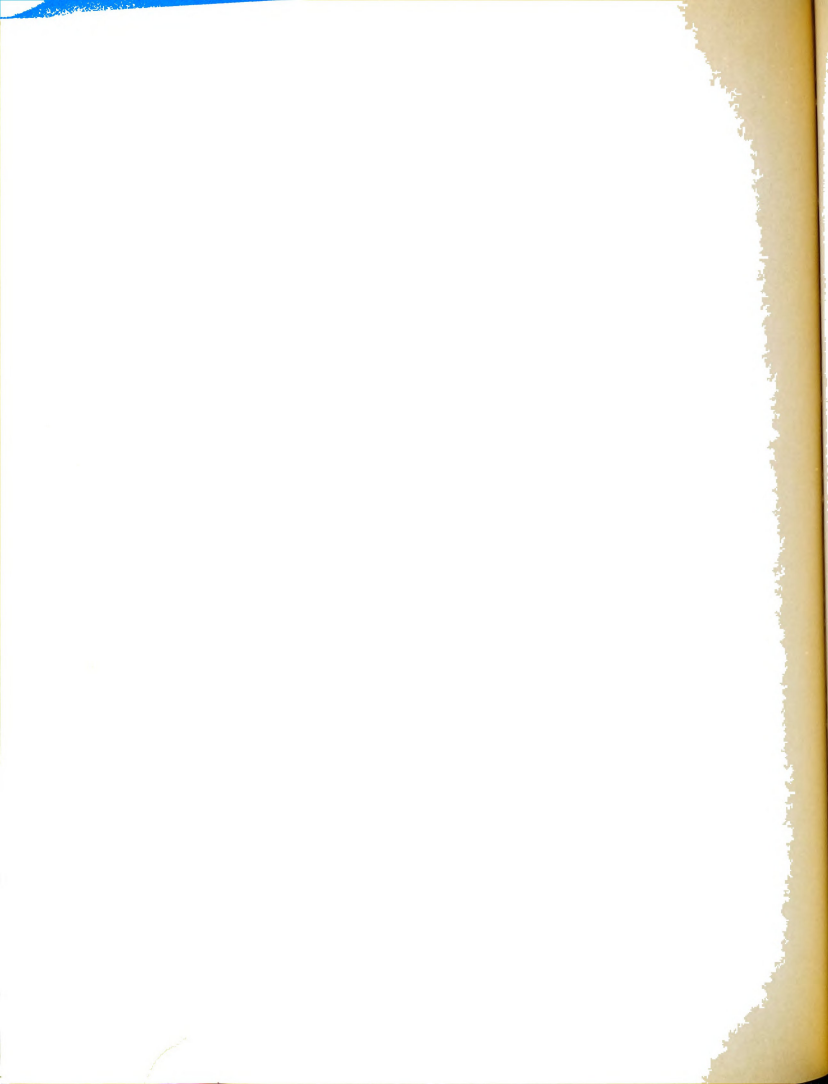


Table A.3.3

$$P_{30}(\lambda, .3, P_B)$$

λ	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.60156	.31493	.11263
.2	.47297	.25013	.09906
.3	.36179	.19696	.08735
.4	.26957	.15400	.07736
.5	.19602	.11993	.06900
.6	.13972	.09360	.06219
.7	.09872	.07403	.05689
.8	.07107	.06055	.05308
.9	.05517	.05261	.05077
1.0	.05000	.05000	.05000
1.1	.05514	.05263	.05079
1.2	.07078	.06063	.05323
1.3	.09738	.07421	.05740
1.4	.13558	.09365	.06337
1.5	.18561	.11918	.07124
1.6	.24721	.15097	.08110
1.7	.31926	.18903	.09304
1.8	.39967	.23318	.10713
1.9	.48544	.28298	.12345
2.0	.57286	.33775	.14203
2.1	.65791	.39652	.16293
2.2	.73672	.45813	.18613
2.3	.80609	.52118	.21162
2.4	.86392	.58422	.23935
2.5	.90936	.64574	.26925
2.6	.94289	.70429	.30119
2.7	.96602	.75861	.33504
2.8	.98086	.80765	.37061
2.9	.98973	.85070	.40770
3.0	.99467	.88738	.44607
3.1	.99727	.91764	.48545
3.2	.99862	.94178	.52556
3.3	.99936	.96034	.56609

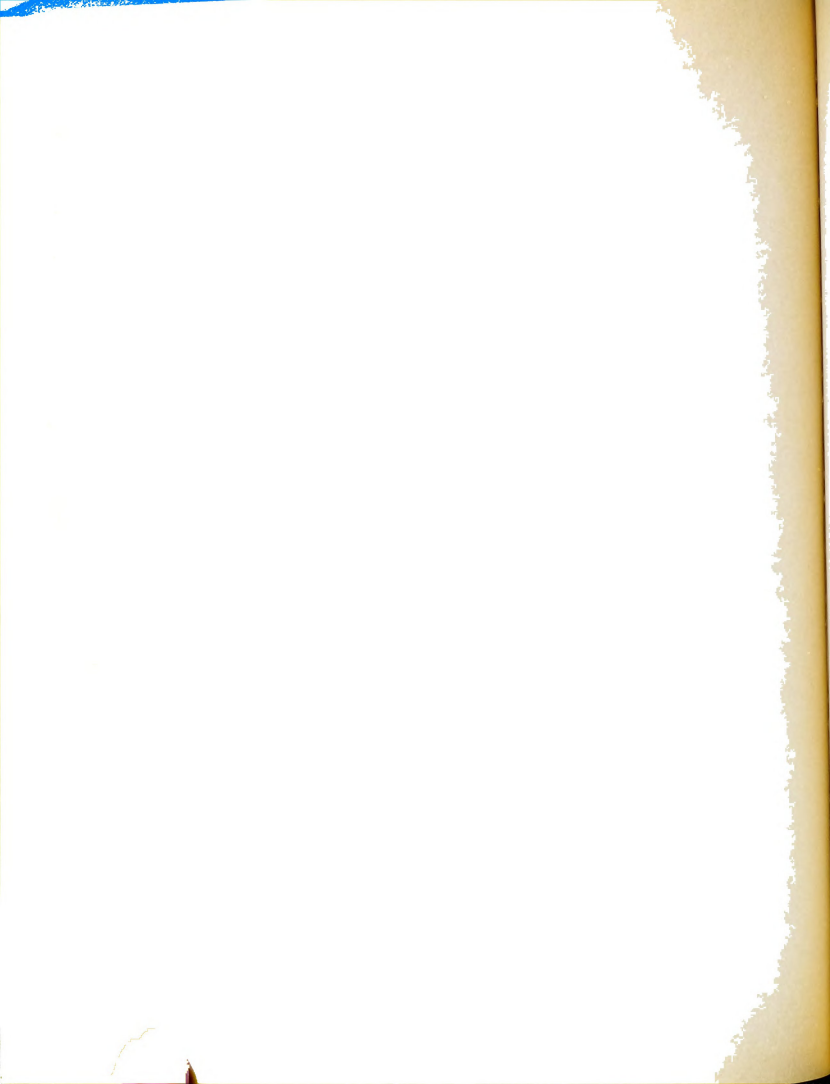


Table A.3.4

$$P_{30}(\lambda, .4, P_B)$$

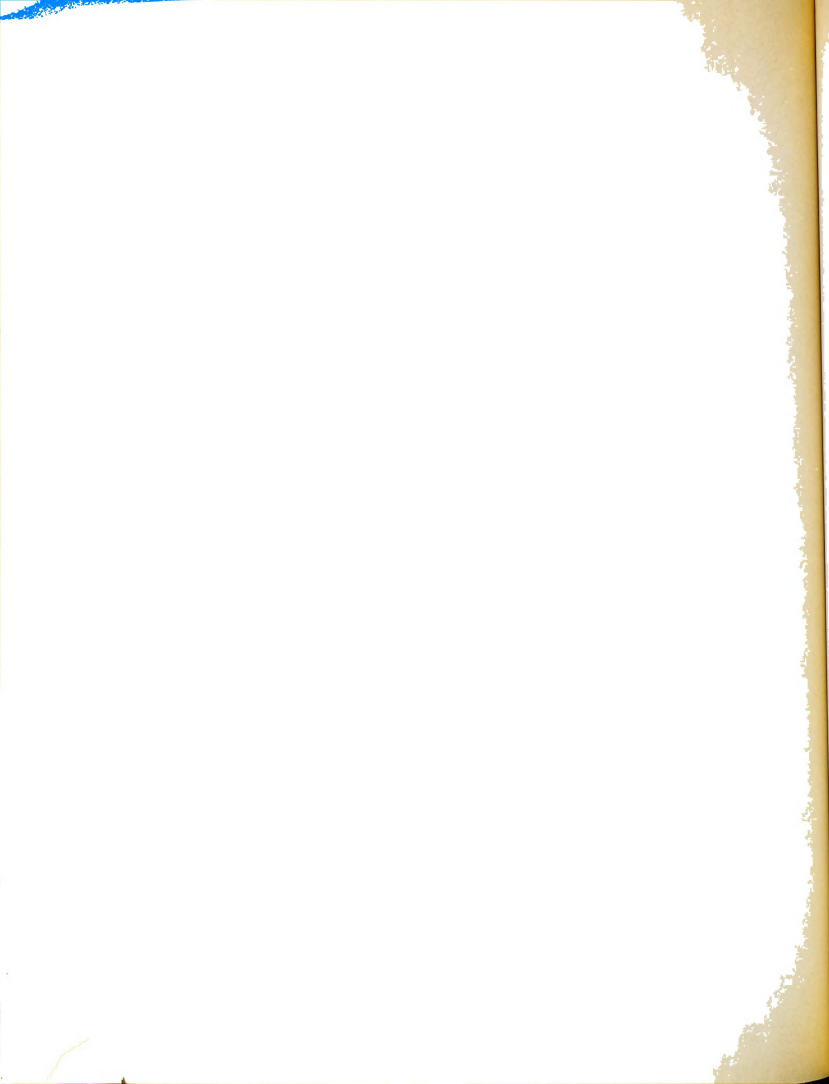
λ	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.95914	.82329	.51971	.17364
.2	.87368	.68411	.40857	.14499
.3	.74398	.54045	.31307	.12096
.4	.58794	.40623	.23610	.10106
.5	.42955	.29079	.17411	.08487
.6	.28966	.19868	.12654	.07204
.7	.18063	.13054	.09173	.06230
.8	.10598	.08463	.06811	.05544
.9	.06359	.05846	.05446	.05136
1.0	.05000	.05000	.05000	.05000
1.1	.06353	.05841	.05445	.05138
1.2	.10542	.08417	.06803	.05560
1.3	.17818	.12854	.09126	.06280
1.4	.28245	.19262	.12483	.07318
1.5	.41343	.27624	.16934	.08695
1.6	.55911	.37693	.22505	.10436
1.7	.70185	.48938	.29165	.12567
1.8	.82343	.60570	.36804	.15110
1.9	.91155	.71664	.45220	.18087
2.0	.96420	.81339	.54125	.21512
2.1	.98895	.88962	.63156	.25395
2.2	.99751	.94293	.71905	.29737
2.3	.99952	.97519	.79958	.34528
2.4	.99983	.99144	.86950	.39747



Table A.3.5

$$P_{30}(\lambda, .5, P_B)$$

λ	$P_B = .5$	$P_B = .4$	$P_B = .3$	$P_B = .2$	$P_B = .1$
.1	.99999	.99565	.93964	.70478	.26299
.2	.99889	.96969	.84401	.57284	.21245
.3	.98485	.89698	.71078	.44751	.17016
.4	.92299	.76575	.55793	.33608	.13545
.5	.78054	.59101	.40683	.24282	.10759
.6	.57085	.40716	.27511	.16923	.08591
.7	.35181	.24866	.17287	.11484	.06978
.8	.18073	.13532	.10278	.07801	.05865
.9	.08131	.07061	.06285	.05687	.05214
1.0	.05000	.05000	.05000	.05000	.05000
1.1	.08131	.07061	.06285	.05687	.05214
1.2	.18073	.13532	.10278	.07801	.05865
1.3	.35181	.24866	.17287	.11484	.06978
1.4	.57085	.40716	.27511	.16923	.08591
1.5	.78054	.59101	.40683	.24282	.10759
1.6	.92299	.76575	.55793	.33608	.13545
1.7	.98485	.89698	.71078	.44751	.17016
1.8	.99889	.96969	.84401	.57284	.21245
1.9	.99999	.99565	.93964	.70478	.26299



Appendix B. Exact Power for the 2 x 2 Comparative Trial.

In the 2 x 2 comparative trial, one set of marginal totals is fixed, say m_1 and $n - m_1$, which we may regard as the sizes of two independent random samples from two different populations A_1 and A_2 . The probability that an observation from A_1 falls in B_1 is p_1 , $i = 1, 2$. The null hypothesis states that $p_1 = p_2$, and the alternative hypothesis is that

$p_1 \neq p_2$. If we put $p_1 = \lambda P_B$ and $p_2 = \frac{P_B(1 - \lambda P_A)}{1 - P_A}$, then

the probability that n_1 of the observations from population A_1 fall in B_1 and $m_2 - n_1$ of the observations from A_2 fall in B_1 is the conditional probability, given m_1 , of the probability distribution in the double dichotomy. We thus use two notations for the power function for the test that $p_1 = p_2$, against alternatives $p_1 \neq p_2$, namely, $P_n(\lambda, P_A, P_B, m_1)$ and $P_n(p_1, p_2, m_1)$.

It is convenient for tabulating purposes to give tables for $P_n(p_1, p_2, m_1)$ where $p_1, p_2 = .1, .2, .3, \dots, .9$, $p_2 \leq p_1$. In Appendix D, when we compare power among the three cases corresponding to the number of restrictions on the marginal totals m_1 and m_2 , it will be convenient to tabulate $P_n(\lambda, P_A, P_B, m_1)$.

As in appendix A, we have divided this appendix into three sections, B.1, B.2, and B.3, corresponding to $n = 10, 20$, and 30 respectively. The tables in these sections contain exact values of $P_n(p_1, p_2, m_1)$ for $m_1 = 1, 2, \dots, \frac{n}{2}$, and all combinations of p_1, p_2 with $p_2 \leq p_1$. These values are rounded to



five decimal places, and are virtually certain to be accurate to this number of figures.

The table numbers indicate the value of m_1 and n , with the first number being $\frac{n}{10}$ and the second the value of m_1 .

For example, table B.3.11 contains computed values of $n = 30$, $m_1 = 11$.

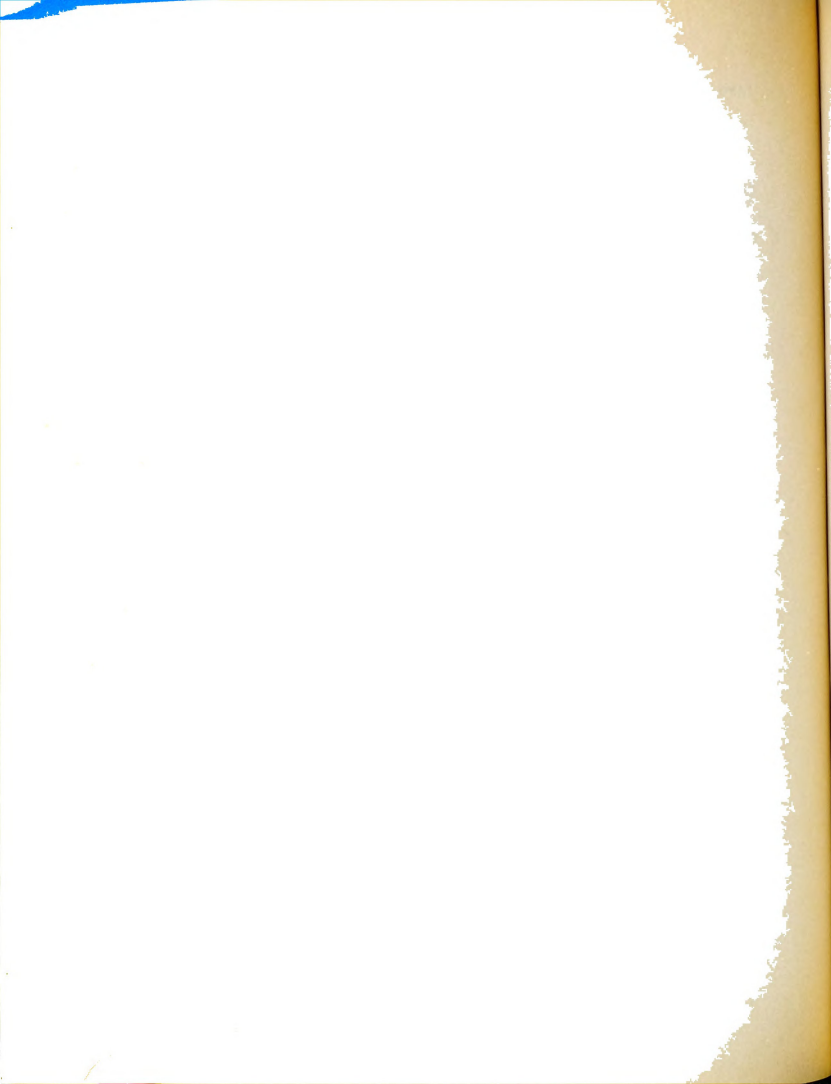




Table B.1.3

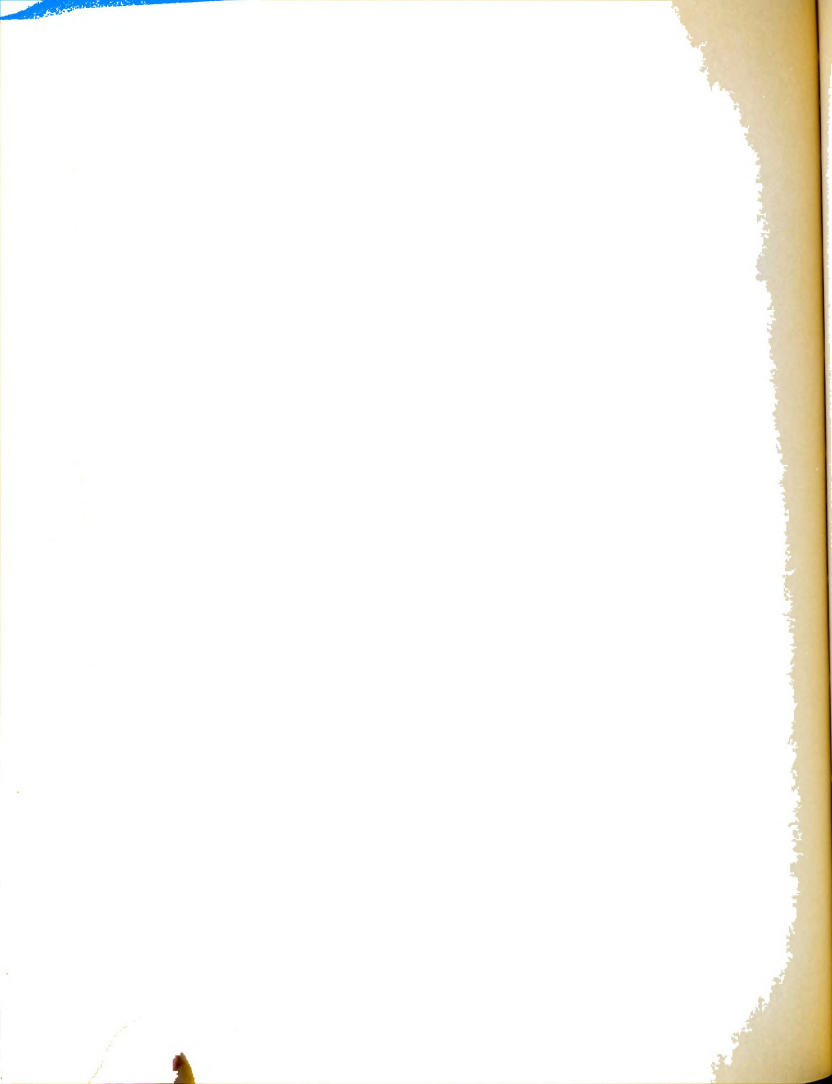
$$P_{10}(p_1, p_2, 3)$$

[illegible]



Table B.1.4

$$P_{10}(p_1, p_2, l)$$
[illegible]





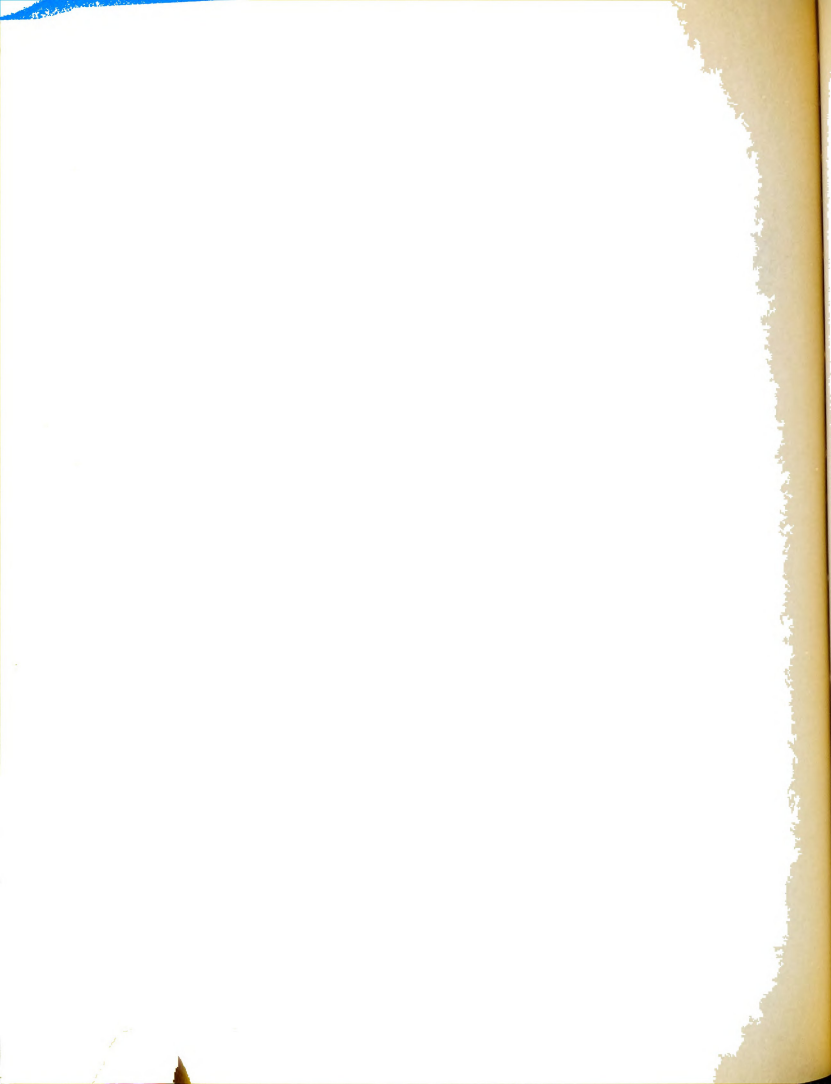






Table B.2.5

$$P_{20}(p_1, p_2, 5)$$
[illegible]

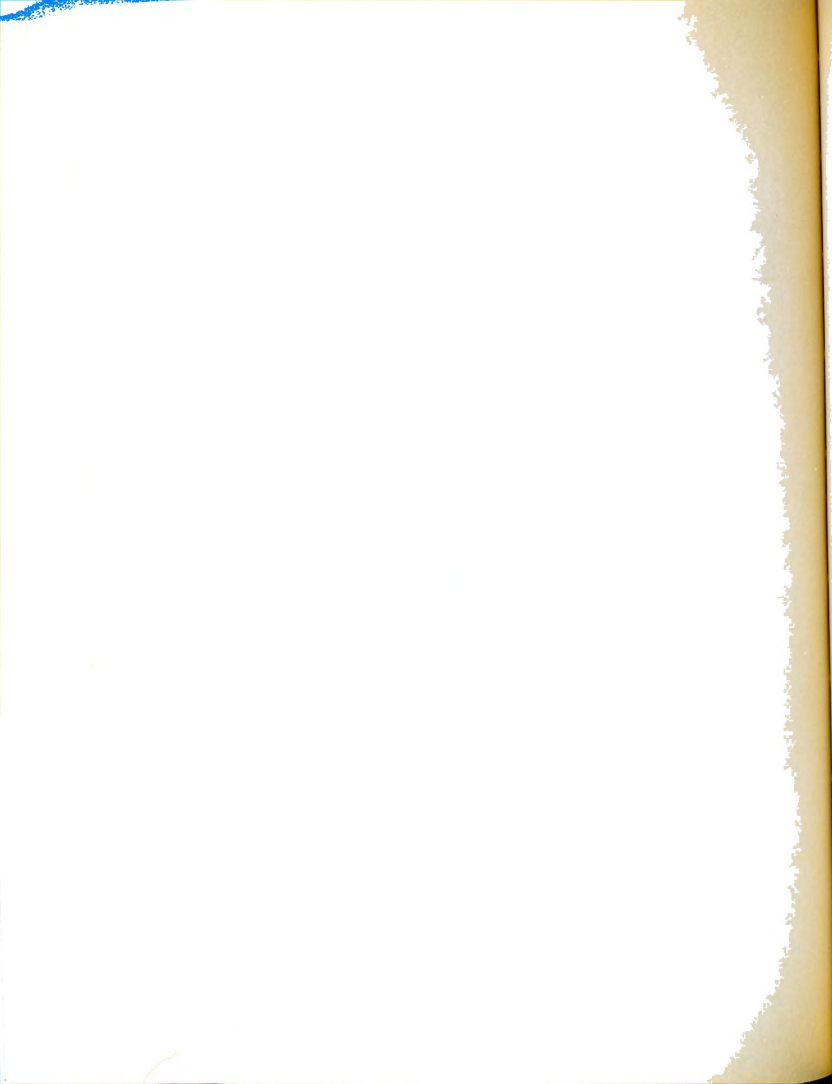
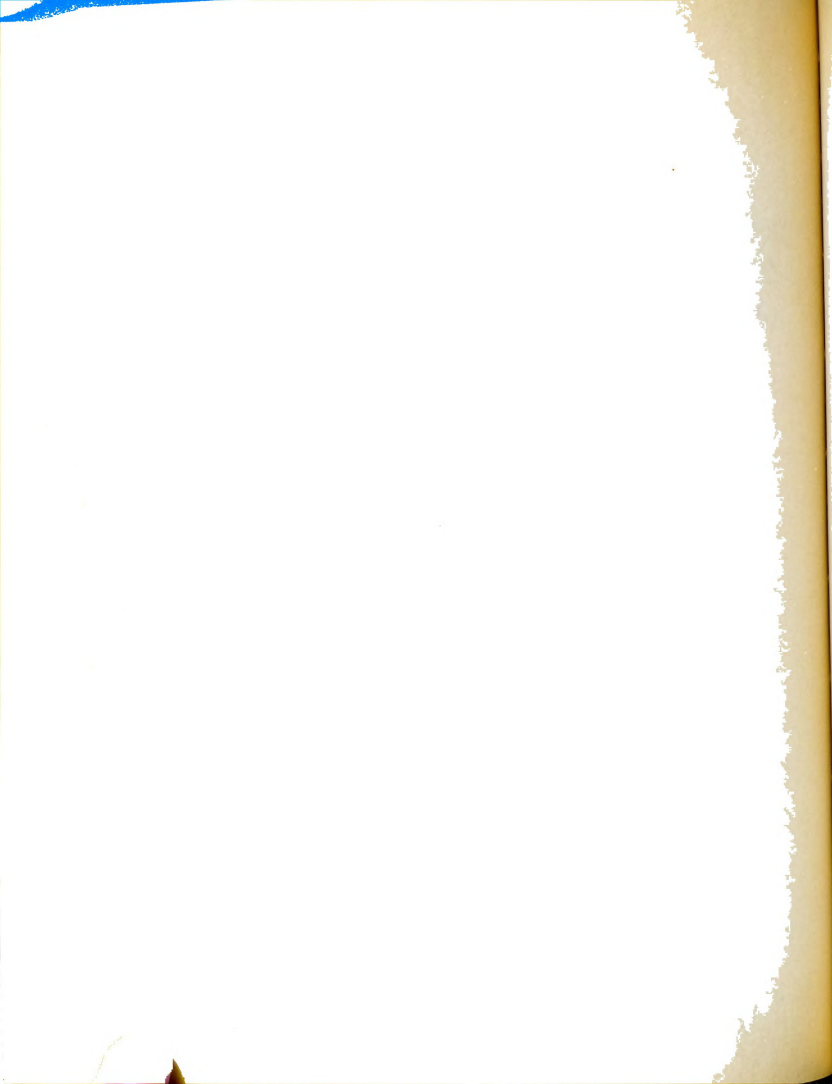


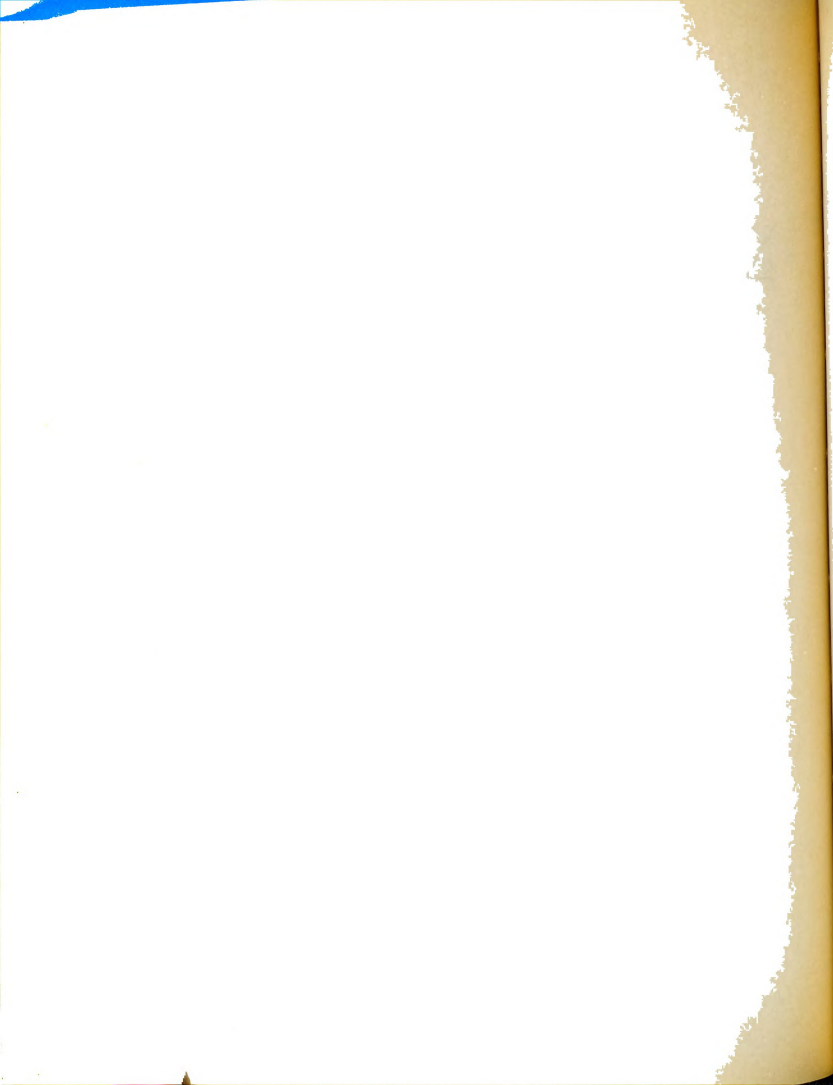
Table B.2.6

$$P_{20}(p_1, p_2, 6)$$
[illegible]



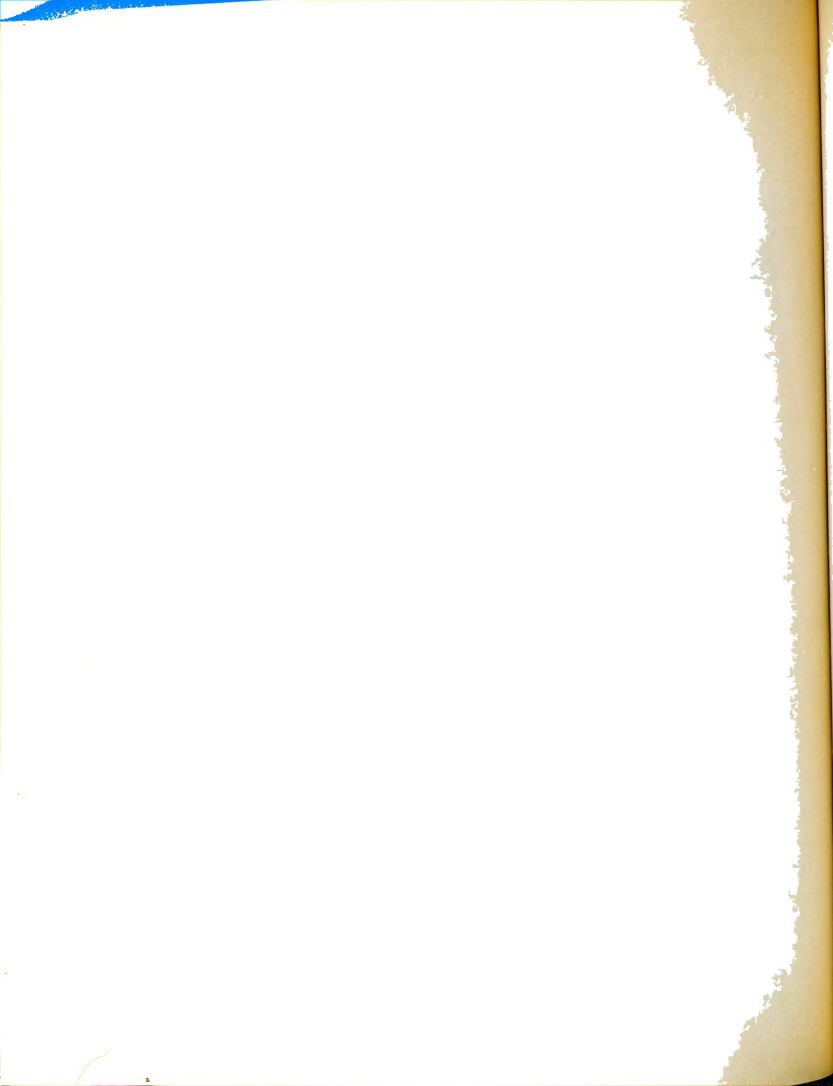




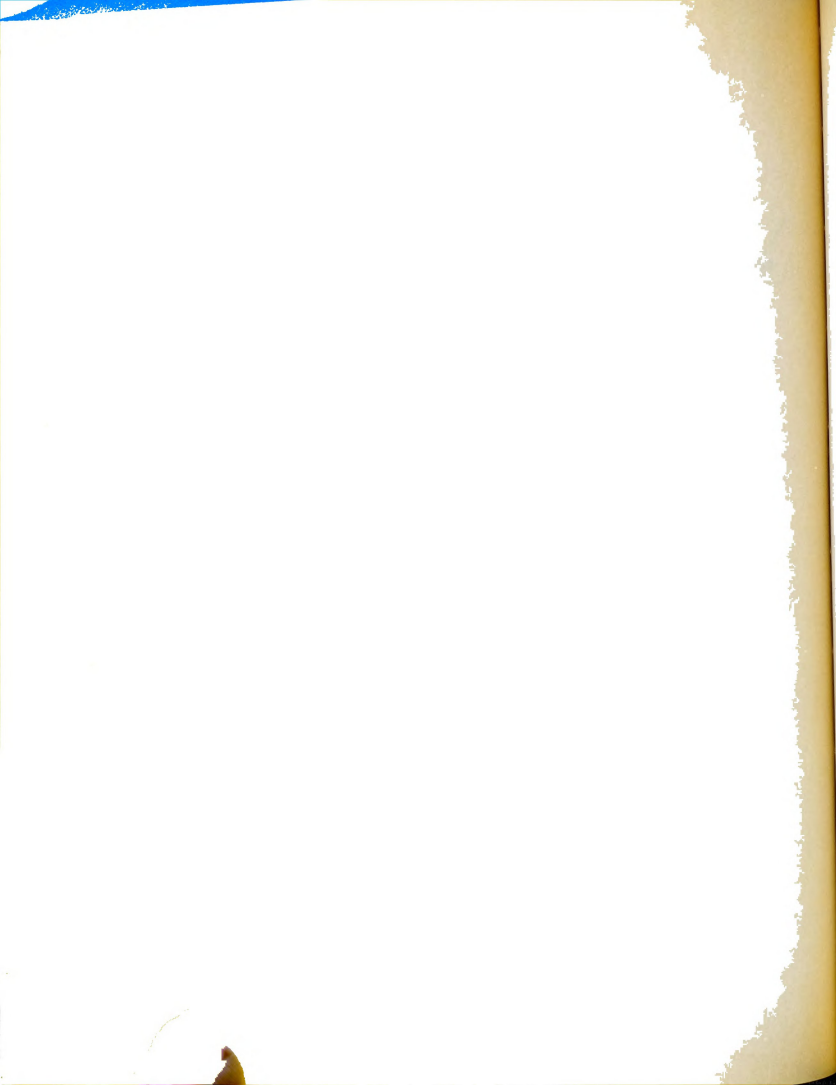


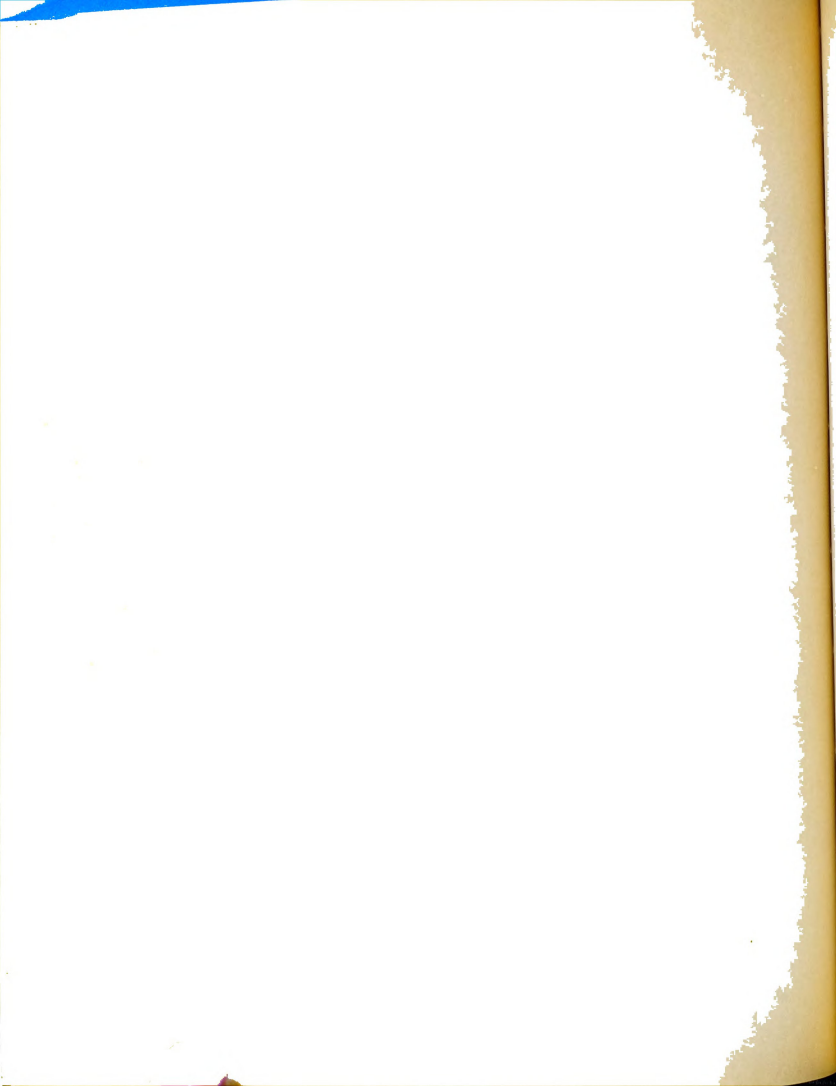


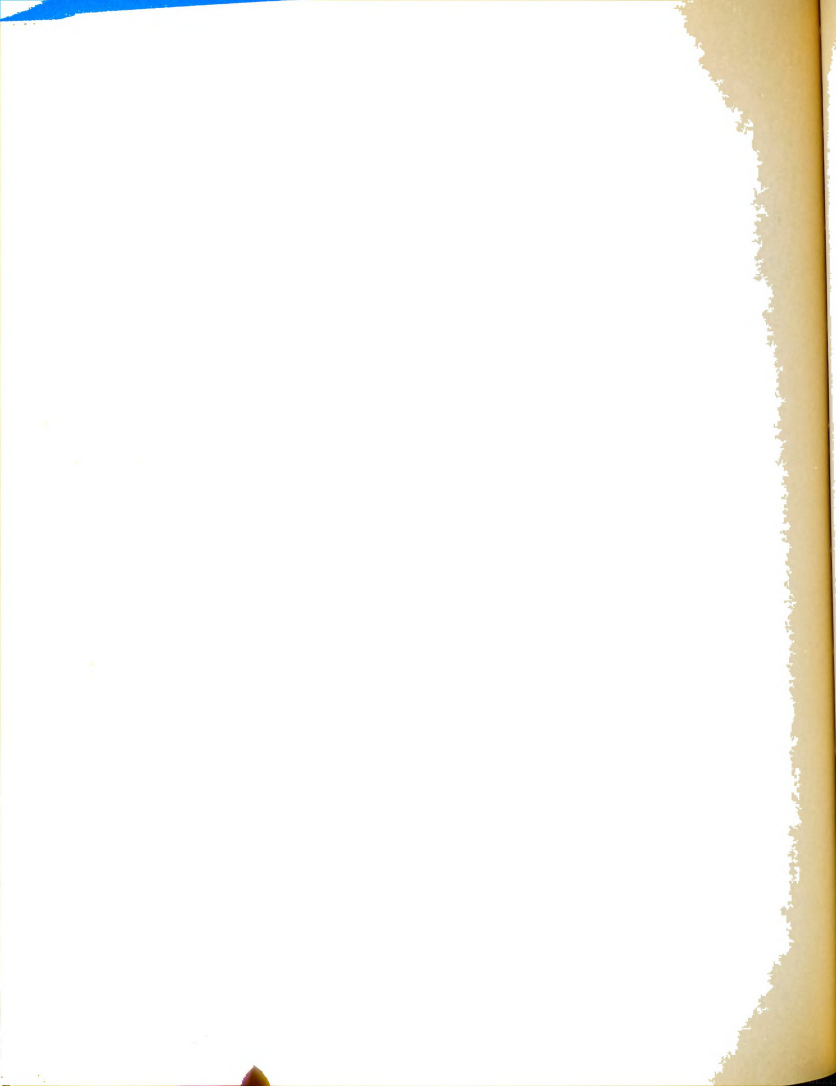














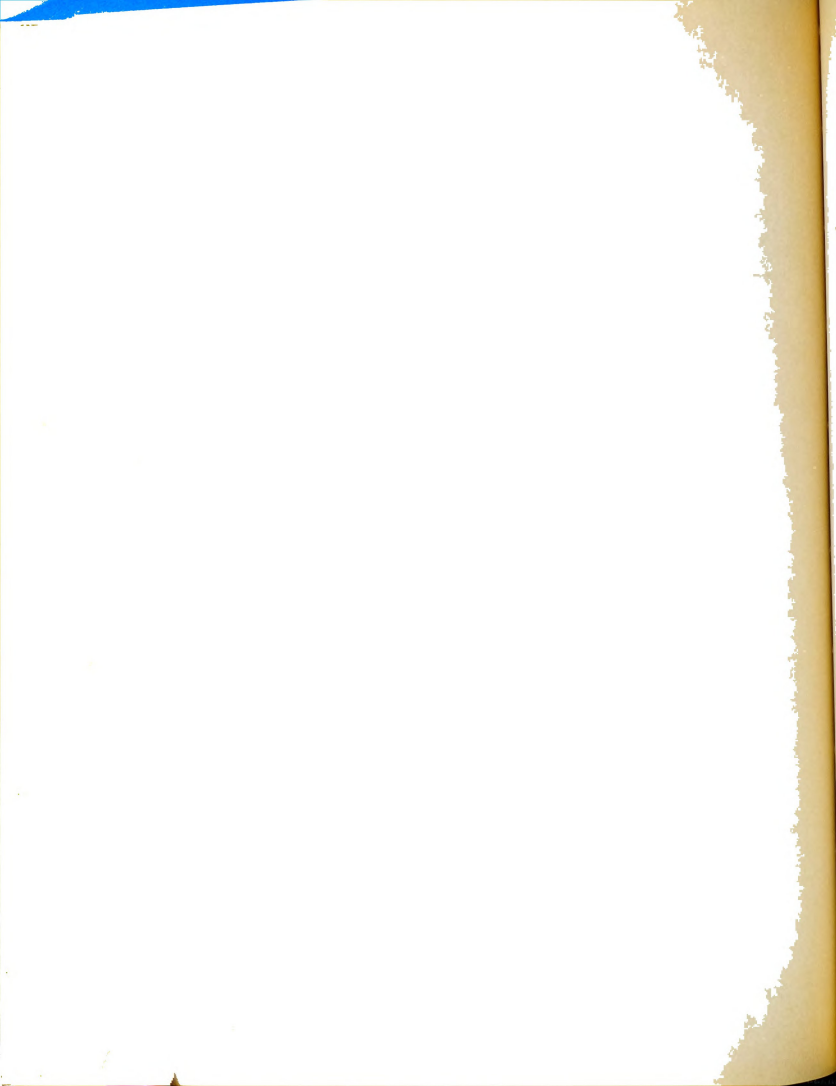


Table B.3.11

$$P_{30}(p_1, p_2, ll)$$

[illegible]

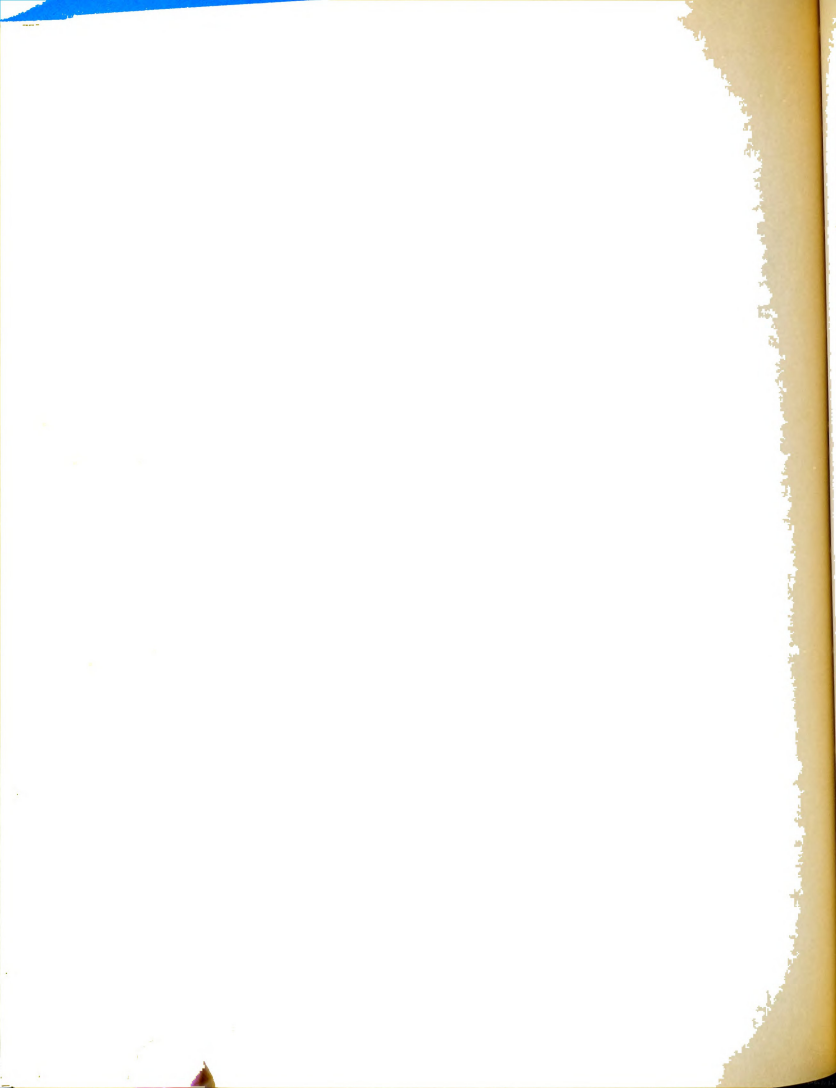


Table B.3.12

$$P_{30}(p_1, p_2, 12)$$
[illegible]



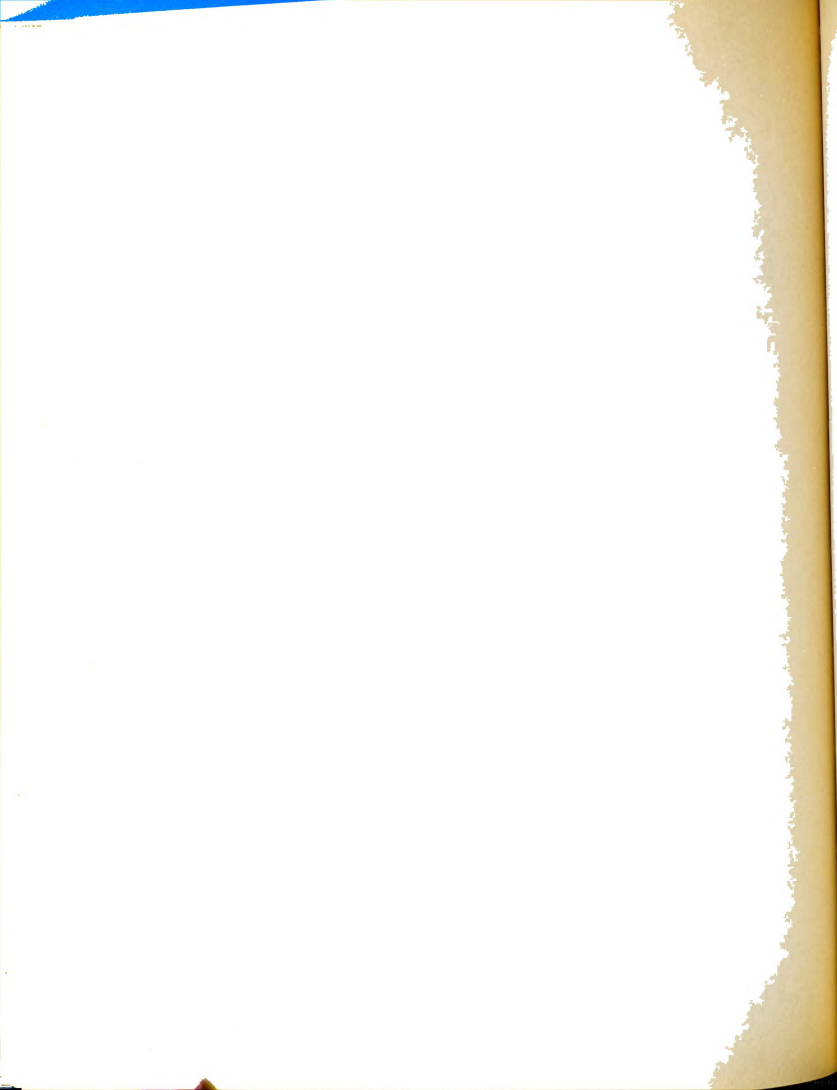


Table B.3.14

$$P_{30}(p_1, p_2, 14)$$
[illegible]

Table B.3.15

$$P_{30}(p_1, p_2, 15)$$
[illegible]



Appendix C. Exact Power in the 2 x 2 Independence Trial.

In the 2 x 2 independence trial, both sets of marginal totals are fixed. The classic example of an experiment in which the marginal totals should be so regarded is Fisher's tea-tasting experiment described earlier in section 1.6. The appropriate probability distribution is obtained conditionally from the multinomial distribution, and is given by (1.2.5) in Chapter 1. The exact power function for the test of independence is a conditional power function, for fixed m_1 and m_2 , and is given by

$$P_n(t | m_1, m_2) = \frac{\sum_{n_1 \in W(m_1, m_2)} \Pr \{n_1 | m_1, m_2, n\} t^{n_1}}{\sum_j \Pr \{j | m_1, m_2, n\} t^j}, \quad 0 < t < \infty.$$

In order to relate $P_n(t | m_1, m_2)$ to the exact power function in the double dichotomy, we put

$$t = \frac{\lambda(1 - P_A - P_B + \lambda P_A P_B)}{(1 - \lambda P_A)(1 - \lambda P_B)},$$

and to relate it to the power function in the 2 x 2 comparative trial, we set

$$t = \frac{p_1(1 - p_2)}{p_2(1 - p_1)}$$

Exact values of $P_n(t | m_1, m_2)$ were computed using a distinctly different program than that used for the previous two cases. The



accuracy of these values was checked by direct hand computation for several cases, which indicated that the values computed were accurate to a minimum of five decimal places in every case given in the tables. The machine computed results were given to 11 decimal places, but were rounded off to five places. When $t = 1$, $P_n(t | m_1, m_2)$ is equal to .05. This served as a check also on the values computed on the digital computer. For all combinations of m_1, m_2 , and n for which computations were made, the computed value of $P_n(t | m_1, m_2)$ for $t = 1$ was accurate to 10 decimal places, which again indicated that the values of power given in this appendix are certainly correct to five decimal places.

The tables in this appendix are divided into three sections, C.1, C.2, and C.3, corresponding to $n = 10, 20$, and 30 . Since there are many combinations of m_1 and m_2 for each value of n , exact values for $P_n(t | m_1, m_2)$ are given for selected combinations of m_1 and m_2 .

It was convenient to tabulate the power function as a function of t rather than as a function of λ . For this reason it is necessary first to determine t from the relation

$$t = \frac{\lambda(1 - P_A - P_B) + \lambda P_A P_B}{\lambda(1 - \lambda P_A)(1 - \lambda P_B)}$$

where $P_A = \frac{m_1}{n}$, $P_B = \frac{m_2}{n}$. Then exact power may be found by looking in the table for which m_1, m_2 , and n are the marginal totals and sample size. More values of power in the 2×2 independence trial may be found in Appendix D.



Table C.1.1

 $P_{10}(t | 5, m_2)$

t	$m_2 = 5$	$m_2 = 4$	$m_2 = 3$	$m_2 = 2$
00.02	.71781	.82859	.27223	.10714
00.03	.62896	.75971	.25985	.10466
00.04	.55944	.69941	.24835	.10229
00.05	.50346	.64626	.23763	.10003
00.075	.40157	.53773	.21383	.09482
00.10	.33254	.45480	.19362	.09018
00.125	.28253	.38981	.17629	.08603
00.15	.24459	.33783	.16133	.08231
00.175	.21481	.29557	.14832	.07898
00.20	.19084	.26072	.13696	.07597
00.25	.15478	.20716	.11818	.07083
00.30	.12915	.16855	.10349	.06664
00.40	.09591	.11830	.08261	.06042
00.50	.07629	.08883	.06923	.05625
00.60	.06428	.07098	.06064	.05350
00.70	.05697	.06019	.05524	.05174
00.80	.05273	.05398	.05207	.05069
00.90	.05061	.05089	.05046	.05015
01.00	.05000	.05000	.05000	.05000
01.10	.05050	.05072	.05038	.05013
01.20	.05182	.05265	.05138	.05046
01.30	.05377	.05550	.05285	.05095
01.40	.05620	.05906	.05467	.05155
01.50	.05900	.06318	.05676	.05223
02.00	.07629	.08883	.06923	.05625
03.00	.11595	.14858	.09545	.06429
04.00	.15478	.20716	.11818	.07083
05.00	.19084	.26071	.13696	.07597
06.00	.22399	.30873	.15246	.08005
07.00	.25446	.35157	.16538	.08333
08.00	.28253	.38981	.17629	.08603
09.00	.30848	.42404	.18559	.08828
10.00	.33254	.45480	.19362	.09018
15.00	.43086	.57057	.22133	.09649
20.00	.50346	.64626	.23763	.10003
25.00	.55944	.69941	.24835	.10229
30.00	.60399	.73873	.25593	.10386
40.00	.67049	.79297	.26593	.10589
50.00	.71781	.82859	.27223	.10714
60.00	.75322	.85376	.27657	.10800
70.00	.78072	.87249	.27973	.10862
80.00	.80270	.88696	.28215	.10909



Table C.1.2

 $P_{10}(t | l, m_2)$

t	$m_2 = 4$	$m_2 = 3$	$m_2 = 2$
00.02	.31458	.16970	.08722
00.03	.29937	.16489	.08590
00.04	.28528	.16031	.08463
00.05	.27220	.15592	.08340
00.075	.24331	.14577	.08050
00.10	.21893	.13664	.07784
00.125	.19816	.12842	.07539
00.15	.18031	.12098	.07314
00.175	.16487	.11424	.07107
00.20	.15144	.10813	.06916
00.25	.12935	.09749	.06579
00.30	.11217	.08863	.06293
00.40	.08786	.07505	.05845
00.50	.07236	.06555	.05526
00.60	.06240	.05898	.05304
00.70	.05614	.05460	.05155
00.80	.05243	.05187	.05063
00.90	.05055	.05043	.05014
01.00	.05000	.05000	.05000
01.10	.05045	.05037	.05012
01.20	.05166	.05139	.05046
01.30	.05346	.05293	.05096
01.40	.05570	.05490	.05159
01.50	.05831	.05721	.05233
02.00	.07152	.07215	.05690
03.00	.11228	.10823	.06702
04.00	.14979	.14396	.07609
05.00	.18499	.17651	.08368
06.00	.21760	.20551	.09000
07.00	.24777	.23121	.09528
08.00	.27570	.25400	.09975
09.00	.30162	.27428	.10356
10.00	.32573	.29241	.10684
15.00	.42478	.35984	.11817
20.00	.49828	.40327	.12482
25.00	.55505	.43347	.12918
30.00	.60023	.45566	.13225
40.00	.66767	.48606	.13630
50.00	.71562	.50589	.13884
60.00	.75146	.51985	.14059
70.00	.77928	.53020	.14187
80.00	.80149	.53819	.14284



Table C.1.3

t	$P_{10}(t 3, 3)$	$P_{10}(t 3, 2)$	$P_{10}(t 2, 2)$
00.02	.10479	.07354	.06357
00.03	.10299	.07283	.06321
00.04	.10124	.07215	.06287
00.05	.09955	.07148	.06253
00.075	.09555	.06988	.06172
00.10	.09186	.06839	.06094
00.125	.08843	.06698	.06020
00.15	.08526	.06566	.05950
00.175	.08232	.06443	.05883
00.20	.07959	.06327	.05819
00.25	.07471	.06117	.05702
00.30	.07050	.05933	.05596
00.40	.06375	.05633	.05417
00.50	.05878	.05407	.05276
00.60	.05520	.05242	.05169
00.70	.05272	.05127	.05091
00.80	.05113	.05053	.05039
00.90	.05027	.05012	.05009
01.00	.05000	.05000	.05000
01.10	.05024	.05011	.05009
01.20	.05090	.05042	.05033
01.30	.05191	.05089	.05071
01.40	.05324	.05149	.05122
01.50	.05482	.05222	.05184
02.00	.06561	.05700	.05625
03.00	.09455	.06892	.06882
04.00	.12664	.08088	.08333
05.00	.15880	.09179	.09812
06.00	.18989	.10147	.11250
07.00	.21948	.11000	.12619
08.00	.24743	.11752	.13910
09.00	.27375	.12417	.15119
10.00	.29849	.13008	.16250
15.00	.40185	.15178	.20903
20.00	.47945	.16549	.24305
25.00	.53949	.17491	.26880
30.00	.58722	.18176	.28892
40.00	.65822	.19106	.31825
50.00	.70844	.19707	.33858
60.00	.74581	.20127	.35349
70.00	.77470	.20438	.36488
80.00	.79770	.20676	.37387



Table C.2.1

 $P_{20}(t | 10, m_2)$

t	$m_2 = 10$	$m_2 = 8$	$m_2 = 6$	$m_2 = 4$
00.02	.97393	.95526	.85200	.51582
00.03	.94258	.91540	.79039	.48869
00.04	.90501	.87179	.73535	.46352
00.05	.86438	.82719	.68592	.44012
00.075	.76099	.71998	.58213	.38835
00.10	.66501	.62502	.49995	.34466
00.125	.58054	.54363	.43361	.30747
00.15	.50770	.47460	.37920	.27560
00.175	.44537	.41613	.33398	.24811
00.20	.39210	.36652	.29601	.22428
00.25	.30744	.28813	.23630	.18533
00.30	.24477	.23031	.19217	.15528
00.40	.16200	.15398	.13322	.11333
00.50	.11336	.10900	.09778	.08699
00.60	.08407	.08179	.07597	.07033
00.70	.06648	.06541	.06266	.05998
00.80	.05642	.05601	.05495	.05392
00.90	.05143	.05134	.05110	.05088
01.00	.05000	.05000	.05000	.05000
01.10	.05117	.05109	.05090	.05072
01.20	.05428	.05401	.05331	.05262
01.30	.05889	.05831	.05685	.05542
01.40	.06466	.06370	.06127	.05889
01.50	.07135	.06995	.06636	.06287
02.00	.11336	.10900	.09778	.08699
03.00	.21198	.20008	.16895	.13902
04.00	.30744	.28813	.23630	.18533
05.00	.39210	.36652	.29601	.22428
06.00	.46507	.43455	.34817	.25684
07.00	.52738	.49316	.39870	.28422
08.00	.58054	.54363	.43361	.30747
09.00	.62599	.58722	.46878	.32740
10.00	.66501	.62502	.49995	.34466
15.00	.79505	.75458	.61396	.40462
20.00	.86438	.82719	.68592	.44012
25.00	.90501	.87179	.73535	.46352
30.00	.93056	.90111	.77136	.48009
40.00	.95929	.93604	.82030	.50200
50.00	.97393	.95526	.85200	.51582
60.00	.98224	.96695	.87419	.52533
70.00	.98732	.97460	.89060	.53228
80.00	.99062	.97986	.90322	.53757



Table C.2.2

 $P_{20}(t | 8, m_2)$

t	$m_2 = 8$	$m_2 = 6$	$m_2 = 4$
00.02	.86891	.87554	.25687
00.03	.81285	.82113	.24821
00.04	.76196	.77121	.23995
00.05	.71558	.72529	.23207
00.075	.61597	.62551	.21390
00.10	.53485	.54328	.19767
00.125	.46780	.47484	.18313
00.15	.41174	.41737	.17006
00.175	.36439	.36875	.15828
00.20	.32408	.32734	.14765
00.25	.25968	.26127	.12929
00.30	.21127	.21178	.11417
00.40	.14552	.14501	.09125
00.50	.10535	.10465	.07544
00.60	.08033	.07975	.06463
00.70	.06490	.06454	.05747
00.80	.05588	.05571	.05303
00.90	.05132	.05128	.05070
01.00	.05000	.05000	.05000
01.10	.05110	.05105	.05060
01.20	.05405	.05387	.05225
01.30	.05844	.05806	.05475
01.40	.06398	.06332	.05794
01.50	.07044	.06944	.06170
02.00	.11154	.10809	.08620
03.00	.20961	.19958	.14657
04.00	.30550	.28887	.20761
05.00	.39090	.36863	.26396
06.00	.46459	.43787	.31457
07.00	.52755	.49744	.35963
08.00	.58123	.54864	.39970
09.00	.62709	.59275	.43540
10.00	.66643	.63091	.46733
15.00	.79709	.76079	.58597
20.00	.86635	.83278	.66204
25.00	.90674	.87662	.71470
30.00	.93204	.90524	.75324
40.00	.96035	.93909	.80580
50.00	.97471	.95757	.83994
60.00	.98282	.96876	.86388
70.00	.98776	.97604	.88160
80.00	.99096	.98104	.89524

Table C.2.3

t	$P_{20}(t 6,6)$	$P_{20}(t 6,4)$	$P_{20}(t 4,4)$
00.02	.33466	.14384	.09543
00.03	.32200	.14081	.09428
00.04	.30998	.13787	.09315
00.05	.29857	.13502	.09204
00.075	.27244	.12825	.08936
00.10	.24932	.12197	.08680
00.125	.22879	.11614	.08436
00.15	.21049	.11073	.08203
00.175	.19413	.10570	.07981
00.20	.17945	.10102	.07770
00.25	.15438	.09263	.07378
00.30	.13395	.08538	.07024
00.40	.10342	.07372	.06421
00.50	.08269	.06514	.05945
00.60	.06869	.05895	.05580
00.70	.05949	.05467	.05313
00.80	.05384	.05194	.05134
00.90	.05088	.05045	.05032
01.00	.05000	.05000	.05000
01.10	.05076	.05040	.05030
01.20	.05282	.05151	.05115
01.30	.05593	.05322	.05250
01.40	.05989	.05542	.05429
01.50	.06456	.05805	.05648
02.00	.09469	.07555	.07212
03.00	.16767	.12028	.11689
04.00	.23997	.16709	.16821
05.00	.30557	.21165	.21973
06.00	.36357	.25286	.26891
07.00	.41453	.29058	.31477
08.00	.45931	.32502	.35708
09.00	.49881	.35648	.39591
10.00	.53379	.38526	.43147
15.00	.66084	.49826	.56962
20.00	.73949	.57647	.66193
25.00	.79224	.63367	.72657
30.00	.82969	.67729	.77369
40.00	.87862	.73938	.83667
50.00	.90859	.78144	.87594
60.00	.92844	.81181	.90219
70.00	.94234	.83477	.92068
80.00	.95248	.85274	.93423



Table C.3.1

 $P_{30}(t | 15, m_2)$

t	$m_2 = 15$	$m_2 = 14$	$m_2 = 13$	$m_2 = 12$	$m_2 = 11$
00.02	.99810	.99679	.99703	.99497	.99259
00.03	.99224	.98960	.98996	.98609	.98012
00.04	.98173	.97779	.97784	.97257	.96192
00.05	.96672	.96170	.96093	.95494	.93903
00.075	.91217	.90681	.90234	.89732	.86848
00.10	.84256	.83922	.83012	.82839	.78945
00.125	.76726	.76687	.75356	.75567	.71034
00.15	.69231	.69488	.67837	.68390	.63549
00.175	.62110	.62618	.60759	.61576	.56680
00.20	.55533	.56229	.54265	.55263	.50488
00.25	.44203	.45101	.43149	.44303	.40071
00.30	.35185	.36112	.34349	.35475	.31937
00.40	.22610	.23355	.22118	.22972	.20722
00.50	.14978	.15474	.14705	.15256	.13937
00.60	.10350	.10640	.10208	.10324	.09808
00.70	.07580	.07727	.07513	.07672	.07325
00.80	.06002	.06061	.05977	.06040	.05905
00.90	.05223	.05236	.05217	.05231	.05210
01.00	.05000	.05000	.05000	.05000	.05000
01.10	.05182	.05193	.05178	.05189	.05164
01.20	.05668	.05707	.05651	.05693	.05604
01.30	.06388	.06469	.06353	.06439	.06253
01.40	.07293	.07425	.07234	.07375	.07068
01.50	.08345	.08533	.08258	.08461	.08013
02.00	.14978	.15474	.14705	.15256	.13937
03.00	.30282	.31171	.29576	.30629	.27553
04.00	.44203	.45101	.43149	.44303	.40071
05.00	.55533	.56229	.54265	.55263	.50488
06.00	.64429	.64860	.63058	.63797	.58895
07.00	.71345	.71521	.69950	.70412	.65630
08.00	.76726	.76687	.75356	.75567	.71034
09.00	.80937	.80729	.79622	.79620	.75397
10.00	.84256	.83922	.83012	.82839	.78945
15.00	.93254	.92701	.92396	.91825	.89351
20.00	.96672	.96170	.96093	.95494	.93903
25.00	.98173	.97779	.97784	.97257	.96192
30.00	.98926	.98620	.98649	.98208	.97464
40.00	.99563	.99376	.99413	.99113	.98713
50.00	.99810	.99679	.99703	.99497	.99259
60.00	.99899	.99809	.99836	.99688	.99535
70.00	.99937	.99888	.99901	.99793	.99690
80.00	.99952	.99922	.99935	.99856	.99782



Table C.3.2

 $P_{30}(t | 15, m_2)$

t	$m_2 = 10$	$m_2 = 9$	$m_2 = 8$	$m_2 = 7$	$m_2 = 6$
00.02	.99467	.97416	.97623	.90784	.87206
00.03	.98534	.94839	.95200	.86432	.81654
00.04	.97120	.91806	.92307	.82267	.76578
00.05	.95279	.88502	.89118	.78292	.71928
00.075	.89296	.79818	.80596	.69181	.61875
00.10	.82195	.71316	.72115	.61191	.53643
00.125	.74766	.63453	.64186	.54215	.46826
00.15	.67488	.56374	.57001	.48132	.41124
00.175	.60625	.50089	.50597	.42829	.36315
00.20	.54303	.44547	.44940	.38200	.32227
00.25	.43404	.35404	.35605	.30612	.25722
00.30	.34693	.28364	.28431	.24772	.20857
00.40	.22445	.18705	.18648	.16708	.14303
00.50	.14935	.12837	.12761	.11748	.10343
00.60	.10346	.09239	.09180	.08671	.07904
00.70	.07584	.07058	.07023	.06790	.06415
00.80	.06005	.05803	.05788	.05700	.05554
00.90	.05223	.05179	.05175	.05156	.05123
01.00	.05000	.05000	.05000	.05000	.05000
01.10	.05183	.05146	.05143	.05128	.05101
01.20	.05670	.05536	.05526	.05467	.05369
01.30	.06392	.06111	.06091	.05968	.05765
01.40	.07297	.06830	.06799	.06593	.06259
01.50	.08348	.07663	.07620	.07313	.06829
02.00	.14935	.12837	.12761	.11748	.10343
03.00	.29934	.24586	.24595	.21628	.18283
04.00	.43404	.35404	.35605	.30612	.25722
05.00	.54303	.44547	.44940	.38200	.32227
06.00	.62857	.52098	.52646	.44517	.37830
07.00	.69533	.58314	.58974	.49786	.42654
08.00	.74766	.63453	.64186	.54215	.46826
09.00	.78899	.67729	.68506	.57972	.50459
10.00	.82195	.71316	.72115	.61191	.53643
15.00	.91463	.82729	.83472	.72089	.65000
20.00	.95279	.88502	.89118	.78292	.71928
25.00	.97120	.91806	.92307	.82267	.76578
30.00	.98114	.93868	.94279	.85023	.79912
40.00	.99063	.96199	.96484	.88585	.84367
50.00	.99467	.97416	.97623	.90784	.87206
60.00	.99667	.98130	.98287	.92275	.89174
70.00	.99778	.98584	.98707	.93351	.90617
80.00	.99845	.98891	.98990	.94165	.91721



Table C.3.3

t	$P_{30}(t 11, m_2)$			$P_{30}(t 13, m_2)$	
	$m_2 = 11$	$m_2 = 12$	$m_2 = 10$	$m_2 = 13$	$m_2 = 12$
00.02	.99704	.99278	.97696	.99288	.99457
00.03	.98999	.98062	.95364	.98087	.98494
00.04	.97789	.96284	.92587	.96332	.97031
00.05	.96103	.94048	.89530	.94120	.95130
00.075	.90262	.87136	.81355	.87278	.88976
00.10	.83063	.79367	.73185	.79570	.81720
00.125	.75430	.71561	.65495	.71810	.74174
00.15	.67929	.64146	.58470	.64425	.66822
00.175	.60867	.57317	.52154	.57612	.59921
00.20	.54382	.51141	.46528	.51441	.53591
00.25	.43275	.40703	.37132	.40989	.42733
00.30	.34471	.32508	.29805	.32764	.34104
00.40	.22218	.21137	.19634	.21321	.22041
00.50	.14776	.14210	.13397	.14330	.14685
00.60	.10253	.09973	.09552	.10045	.10203
00.70	.07539	.07413	.07214	.07451	.07512
00.80	.05988	.05942	.05866	.05958	.05977
00.90	.05220	.05210	.05193	.05214	.05217
01.00	.05000	.05000	.05000	.05000	.05000
01.10	.05180	.05173	.05159	.05176	.05177
01.20	.05662	.05635	.05584	.05649	.05649
01.30	.06376	.06323	.06214	.06352	.06347
01.40	.07273	.07188	.07006	.07239	.07223
01.50	.08320	.08195	.07926	.08273	.08238
02.00	.14923	.14570	.13726	.14832	.14592
03.00	.30201	.29373	.27170	.30094	.29081
04.00	.44123	.42948	.39632	.44046	.42202
05.00	.55465	.54095	.50059	.55425	.52944
06.00	.64375	.62923	.58503	.64366	.61483
07.00	.71303	.69847	.65284	.71317	.68228
08.00	.76695	.75280	.70735	.76734	.73573
09.00	.80913	.79566	.75139	.80951	.77837
10.00	.84239	.82972	.78724	.84282	.81266
15.00	.93254	.92393	.89246	.93292	.91063
20.00	.96673	.96098	.93849	.96704	.95157
25.00	.98175	.97791	.96161	.98200	.97127
30.00	.98928	.98655	.97446	.98948	.98182
40.00	.99587	.99416	.98705	.99588	.99163
50.00	.99785	.99706	.99255	.99818	.99556
60.00	.99857	.99835	.99533	.99879	.99745
70.00	.99940	.99902	.99688	.99950	.99836
80.00	.99952	.99937	.99782	.99957	.99897



Table C.3.4

$$P_{30}(t | 12, m_2)$$

t	$m_2 = 12$	$m_2 = 9$	$m_2 = 6$
00.02	.98089	.91472	.75464
00.03	.96093	.87393	.71550
00.04	.93663	.83457	.67892
00.05	.90937	.79674	.64471
00.075	.83433	.70901	.56833
00.10	.75683	.63092	.50312
00.125	.68200	.56185	.44713
00.15	.61226	.50096	.39882
00.175	.54855	.44736	.35692
00.20	.49106	.40019	.32044
00.25	.39366	.32205	.26058
00.30	.31661	.26122	.21422
00.40	.20831	.17619	.14928
00.50	.14121	.12325	.10836
00.60	.09961	.09010	.08233
00.70	.07421	.06966	.06602
00.80	.05950	.05773	.05634
00.90	.05213	.05173	.05144
01.00	.05000	.05000	.05000
01.10	.05176	.05144	.05120
01.20	.05649	.05528	.05444
01.30	.06354	.06101	.05929
01.40	.07244	.06821	.06542
01.50	.08283	.07660	.07258
02.00	.14893	.12958	.11806
03.00	.30299	.25336	.22490
04.00	.44364	.36990	.32610
05.00	.55803	.46930	.41328
06.00	.64762	.55144	.48633
07.00	.71706	.61870	.54718
08.00	.77092	.67377	.59797
09.00	.81292	.71905	.64061
10.00	.84593	.75650	.67666
15.00	.93477	.87053	.79359
20.00	.96811	.92323	.85470
25.00	.98265	.95079	.89077
30.00	.98993	.96657	.91398
40.00	.99595	.98254	.94132
50.00	.99818	.98975	.95644
60.00	.99910	.99347	.96582
70.00	.99925	.99559	.97210
80.00	.99956	.99688	.97656



Table C.3.5

t	$P_{30}(t 9,9)$	$P_{30}(t 9,6)$	$P_{30}(t 6,6)$
00.02	.88917	.29359	.13802
00.03	.83966	.28408	.13547
00.04	.79363	.27497	.13299
00.05	.75077	.26624	.13056
00.075	.65584	.24592	.12476
00.10	.57567	.22758	.11931
00.125	.50753	.21096	.11419
00.15	.44928	.19589	.10938
00.175	.39921	.18220	.10485
00.20	.35599	.16973	.10060
00.25	.28585	.14799	.09285
00.30	.23230	.12985	.08601
00.40	.15861	.10194	.07473
00.50	.11313	.08234	.06612
00.60	.08466	.06876	.05972
00.70	.06706	.05964	.05517
00.80	.05674	.05394	.05218
00.90	.05152	.05091	.05052
01.00	.05000	.05000	.05000
01.10	.05126	.05079	.05047
01.20	.05466	.05298	.05180
01.30	.05974	.05630	.05388
01.40	.06615	.06058	.05661
01.50	.07365	.06564	.05992
02.00	.12147	.09887	.08295
03.00	.23560	.18144	.14623
04.00	.34589	.26456	.21606
05.00	.44217	.34027	.28403
06.00	.52340	.40702	.34710
07.00	.59112	.46523	.40436
08.00	.64745	.51586	.45581
09.00	.69440	.55996	.50184
10.00	.73369	.59849	.54297
15.00	.85626	.73271	.69258
20.00	.91462	.80961	.78234
25.00	.94558	.85755	.83948
30.00	.96345	.88941	.87773
40.00	.98144	.92781	.92368
50.00	.98943	.94919	.94882
60.00	.99354	.96230	.96384
70.00	.99584	.97092	.97343
80.00	.99713	.97689	.97986



Appendix D. Comparison of Power Functions.

The three exact power functions $P_n(\lambda, P_A, P_B)$, $P_n(\lambda, P_A, P_B, m_1)$, and $P_n(\lambda, P_A, P_B, m_1, m_2)$ are compared in this appendix. More precisely, tables are given comparing the three power functions simultaneously, with $P_n(\lambda, P_A, P_B, m_1)$ evaluated at $m_1 = nP_A$, and $P_n(\lambda, P_A, P_B, m_1, m_2)$ evaluated at $m_1 = nP_A$ and $m_2 = nP_B$. Comparisons are made for $n = 10, 20$, and 30 , with various combinations of P_A and P_B . The first column gives the dependence parameter λ , and then values of power for the double dichotomy, 2×2 comparative trial, and the 2×2 independence trial are given in the next three columns.

In the last column values of the normal approximations (3.1.4) and (3.3.5), which are identical for the particular choices of P_A, P_B, m_1 , and m_2 in these tables, are given. This last set of figures may be used to evaluate the adequacy of either (3.1.4) as an approximation to $P_n(\lambda, P_A, P_B, m_1, m_2)$, or (3.3.5) as an approximation to $P_n(\lambda, P_A, P_B)$.

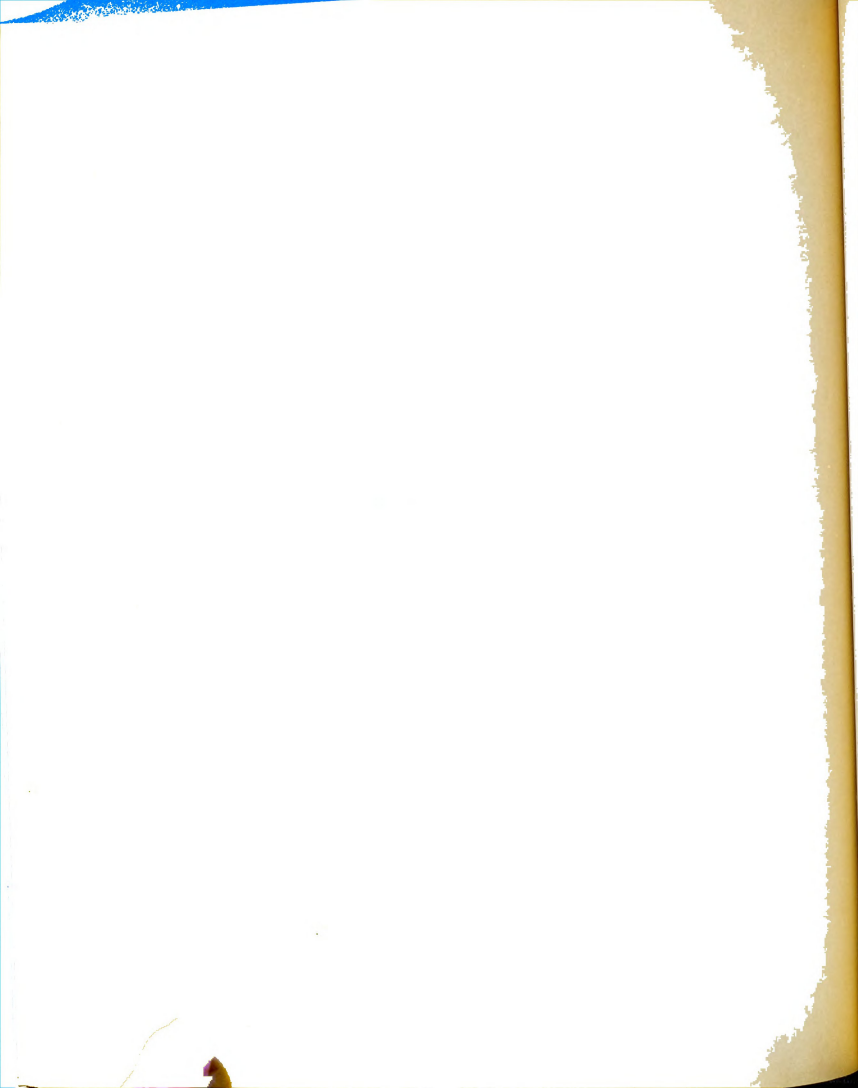


Table D.1.1

λ	I	II	III
.1		.93284	.80107
.2	.80466	.78609	.64688
.3	.62018	.61666	.49672
.4	.44707	.45653	.36521
.5	.30848	.32107	.25826
.6	.20593	.21537	.17650
.7	.13368	.13874	.11779
.8	.08598	.08791	.07897
.9	.06888	.05923	.05706
1.0	.05000	.05000	.05000
1.1	.06888	.05923	.05706
1.2	.08598	.08791	.07897
1.3	.13368	.13874	.11779
1.4	.20593	.21537	.17650
1.5	.30848	.32107	.25826
1.6	.44707	.45653	.36521
1.7	.62018	.61666	.49672
1.8	.80466	.78609	.64688
1.9		.93284	.80107

I. $P_{10}(\lambda, .5, .5, .5, .5)$ III. $P_{10}(\lambda, .5, .5)$ II. $P_{10}(\lambda, .5, .5, .5, .5)$

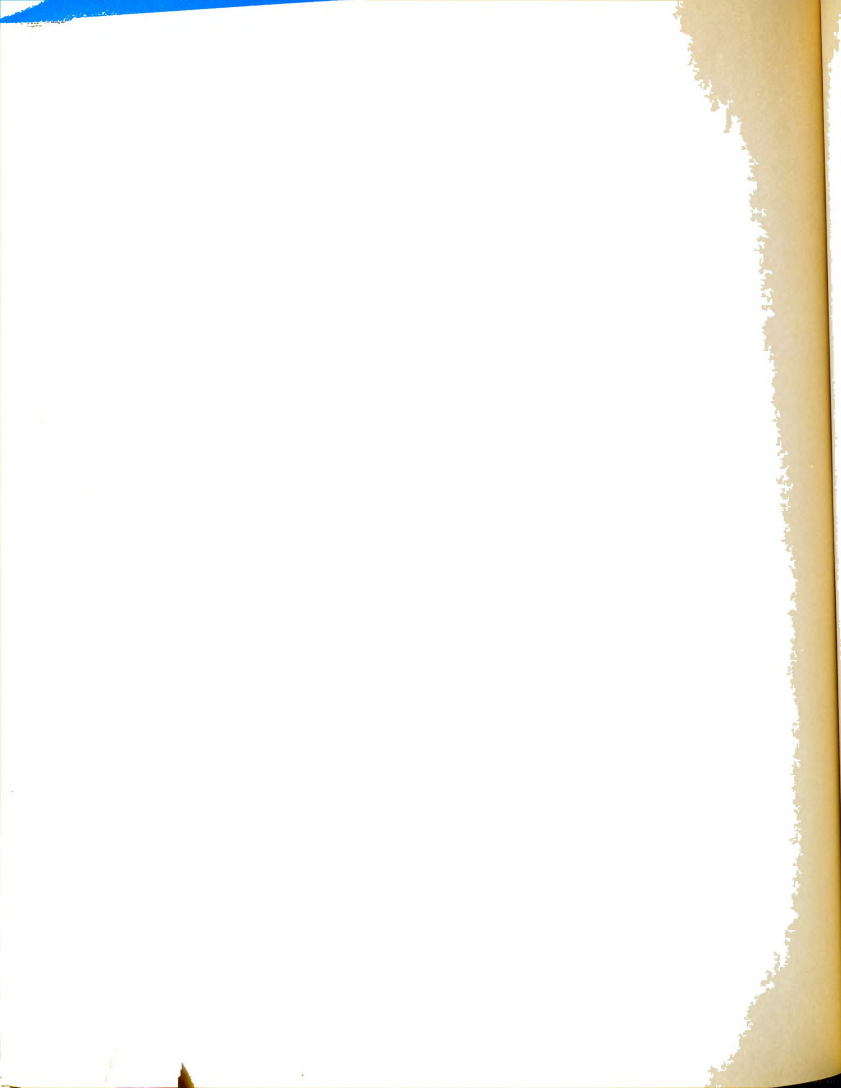


Table D.1.2

λ	I	II	III
.1	.30922	.41038	.33312
.2	.26623	.32312	.26580
.3	.22250	.25084	.20936
.4	.18054	.19211	.16309
.5	.14248	.14543	.12608
.6	.10993	.10937	.09738
.7	.08396	.08267	.07608
.8	.06516	.06430	.06141
.9	.05380	.05355	.05283
1.0	.05000	.05000	.05000
1.1	.05385	.05358	.05285
1.2	.06557	.06451	.06157
1.3	.08563	.08334	.07659
1.4	.11481	.11079	.09855
1.5	.15433	.14779	.12826
1.6	.20582	.19526	.16659
1.7	.27126	.25406	.21440
1.8	.35262	.32474	.27240
1.9	.45094	.40732	.34095
2.0	.56489	.50104	.41985
2.1	.68852	.60392	.50806
2.2	.80978	.71242	.60334
2.3	.91173	.82083	.70187
2.4	.97796	.92069	.79766

I. $P_{10}(\lambda, .4, .4, .4, .4)$ III. $P_{10}(\lambda, .4, .4)$ II. $P_{10}(\lambda, .4, .4, .4)$

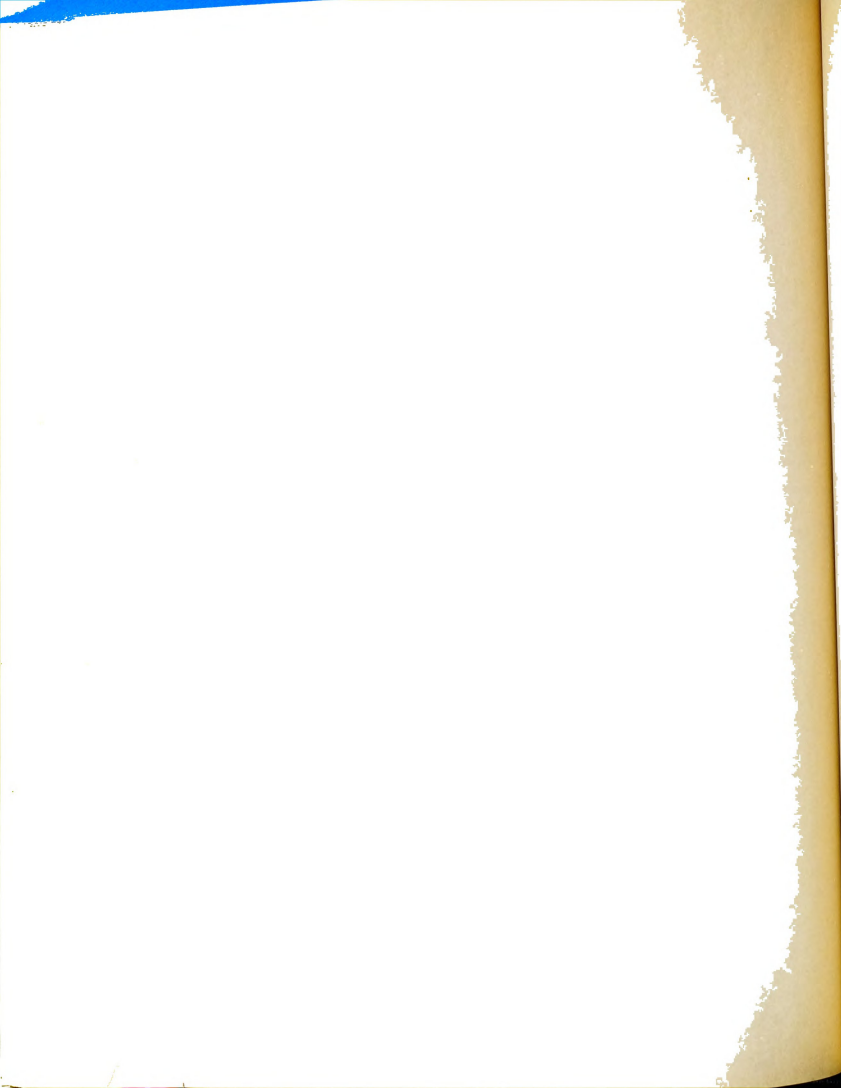


Table D.1.3

λ	I	II	III
.1	.10065	.12472	.12249
.2	.09263	.10891	.10650
.3	.08469	.09509	.09277
.4	.07704	.08320	.08116
.5	.06880	.07315	.07151
.6	.06346	.06491	.06372
.7	.05799	.05846	.05771
.8	.05375	.05380	.05344
.9	.05099	.05096	.05086
1.0	.05000	.05000	.05000
1.1	.05110	.05099	.05088
1.2	.05462	.05403	.05356
1.3	.05814	.05924	.06097
1.4	.07056	.06677	.06472
1.5	.08387	.07677	.07344
1.6	.10146	.08944	.08444
1.7	.12390	.10497	.09790
1.8	.15181	.12360	.11398
1.9	.18585	.14556	.13287
2.0	.22662	.17111	.15474
2.1	.27464	.20053	.17974
2.2	.33022	.23409	.20802
2.3	.39334	.27211	.23968
2.4	.46350	.31488	.27478
2.5	.53949	.36271	.31334
2.6	.61929	.41593	.35527
2.7	.70002	.47485	.40040
2.8	.77802	.53979	.44847
2.9	.84918	.61106	.49905
3.0	.90952	.68896	.55155
3.1	.95574	.77377	.60521
3.2	=====	.86576	.65903
3.3	=====	.96518	.71175

I. $P_{10}(\lambda, .3, .3, 3, 3, 3)$ III. $P_{10}(\lambda, .3, .3, 3)$ II. $P_{10}(\lambda, .3, .3, 3, 3)$



Table D.2.1

λ	I	II	III
.1	1.00000	.99917	.99776
.2	.99088	.98260	.97491
.3	.93853	.91642	.89875
.4	.81232	.78338	.75844
.5	.62599	.60202	.57672
.6	.42593	.41219	.39250
.7	.25595	.25044	.23853
.8	.13729	.13584	.13056
.9	.07101	.07072	.06943
1.0	.05000	.05000	.05000
1.1	.07101	.07072	.06943
1.2	.13729	.13584	.13056
1.3	.25595	.25044	.23853
1.4	.42593	.41219	.39250
1.5	.62599	.60202	.57672
1.6	.81232	.78338	.75844
1.7	.93853	.91642	.89875
1.8	.99088	.98260	.97491
1.9	1.00000	.99917	.99776

I. $P_{20}(\lambda, .5, .5, 10, 10)$ III. $P_{20}(\lambda, .5, .5)$ II. $P_{20}(\lambda, .5, .5, 10)$



Table D.2.2

λ	I	II	III
.1	.83067	.79297	.75763
.2	.66586	.65873	.62362
.3	.51490	.52151	.49071
.4	.38373	.39360	.36927
.5	.27520	.28330	.26593
.6	.18980	.19476	.18367
.7	.12640	.12877	.12265
.8	.08316	.08400	.08134
.9	.05816	.05833	.05768
1.0	.05000	.05000	.05000
1.1	.05816	.05833	.05768
1.2	.08316	.08400	.08134
1.3	.12640	.12877	.12265
1.4	.18980	.19476	.18367
1.5	.27520	.28330	.26593
1.6	.38373	.39360	.36927
1.7	.51490	.52151	.49071
1.8	.66586	.65873	.62362
1.9	.83067	.79297	.75763

I. $P_{20}(\lambda, .3, .5, 6, 10)$ III. $P_{20}(\lambda, .3, .5)$ II. $P_{20}(\lambda, .3, .5, 10)$

Table D.2.3

λ	I	II	III
.1	.84901	.85130	.80606
.2	.69467	.71252	.66881
.3	.54656	.56531	.52824
.4	.41244	.42576	.39746
.5	.29767	.30471	.28514
.6	.20510	.20759	.19544
.7	.13528	.13547	.12893
.8	.08721	.08678	.08400
.9	.05922	.05899	.05832
1.0	.05000	.05000	.05000
1.1	.05941	.05895	.05829
1.2	.08871	.08639	.08374
1.3	.14024	.13385	.12781
1.4	.21618	.20284	.19200
1.5	.31681	.29357	.27674
1.6	.43861	.40357	.38016
1.7	.57300	.52681	.49724
1.8	.70664	.65368	.61969
1.9	.82416	.77215	.73686
2.0	.91273	.87041	.83781
2.1	.96712	.94025	.91418
2.2	.99200	.98018	.96293
2.3	.99912	.99638	.98758
2.4		.99984	.99666

I. $P_{20}(\lambda, .4, .4, 8, 8)$ III. $P_{20}(\lambda, .4, .4)$ II. $P_{20}(\lambda, .4, .4, 8)$

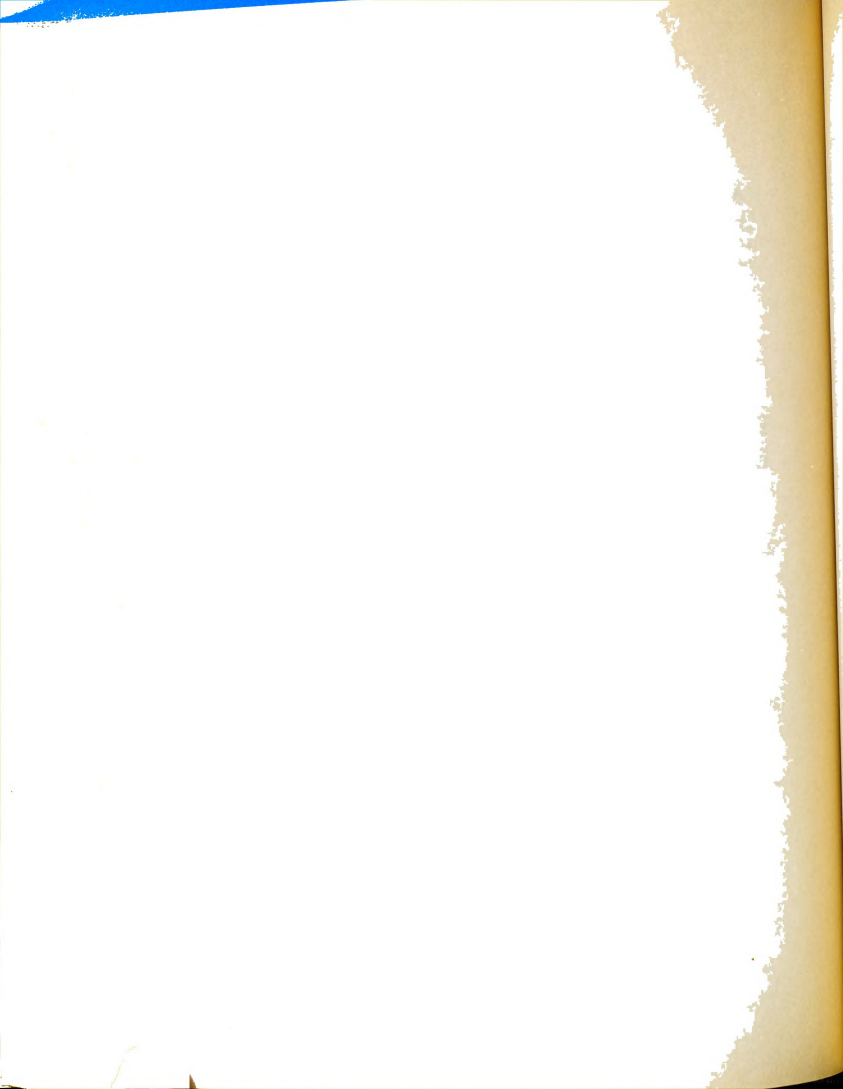


Table D.2.4

λ	I	II	III
.1	.80315	.61344	.56085
.2	.62577	.48458	.44407
.3	.47265	.37202	.32430
.4	.34589	.27780	.25710
.5	.24525	.20207	.18851
.6	.16882	.14371	.13552
.7	.11384	.10099	.09663
.8	.07732	.07208	.07023
.9	.05664	.05543	.05498
1.0	.05000	.05000	.05000
1.1	.05653	.05541	.05498
1.2	.07632	.07193	.07017
1.3	.11022	.10024	.09625
1.4	.15936	.14124	.13408
1.5	.22466	.19565	.18440
1.6	.30608	.26465	.24748
1.7	.40205	.34445	.32280
1.8	.50891	.43600	.40870
1.9	.62080	.53480	.50226
2.0	.73004	.63602	.59932
2.1	.82809	.73382	.69479
2.2	.90714	.82200	.78313
2.3	.96199	.89501	.85918
2.4	.99167	.94895	.91905

I. $P_{20}(\lambda, .3, .4, 6, 8)$ III. $P_{20}(\lambda, .3, .4)$ II. $P_{20}(\lambda, .3, .4, 6)$



Table D.2.5

λ	I	II	III
.1	.30596	.36726	.34262
.2	.25407	.28980	.27224
.3	.20728	.22617	.21375
.4	.16616	.17471	.16609
.5	.13106	.13380	.12811
.6	.10216	.10231	.09868
.7	.07953	.07886	.07683
.8	.06324	.06267	.06176
.9	.05344	.05315	.05292
1.0	.05000	.05000	.05000
1.1	.05344	.05317	.05293
1.2	.06398	.06284	.06186
1.3	.08199	.07933	.07708
1.4	.10783	.10305	.09897
1.5	.14177	.13442	.12791
1.6	.18391	.17374	.16420
1.7	.23409	.22113	.20792
1.8	.29184	.27642	.25892
1.9	.35631	.33903	.31666
2.0	.42627	.40796	.38024
2.1	.50011	.48174	.44830
2.2	.57589	.55841	.51967
2.3	.65142	.63559	.59067
2.4	.72434	.71063	.66072
2.5	.79224	.78076	.72700
2.6	.85276	.84336	.78745
2.7	.90383	.89618	.84038
2.8	.94385	.93774	.88469
2.9	.97210	.96754	.91998
3.0	.98916	.98630	.94663
3.1	.99722	.99595	.96577
3.2	—	.99943	.97910
3.3	—	1.00000	.98864

I. $P_{20}(\lambda, .3, .3, 6, 6)$ III. $P_{20}(\lambda, .3, .3)$ II. $P_{20}(\lambda, .3, .3, 6)$



Table D.3.1

λ	I	II	III	IV
.1	1.00000	1.00000	.99999	1.00000
.2	.99929	.99928	.99889	.99997
.3	.99130	.98788	.98485	.99501
.4	.94201	.93215	.92299	.94695
.5	.80937	.79571	.78054	.80522
.6	.59762	.58686	.57085	.58527
.7	.36830	.36323	.35181	.35677
.8	.20028	.18612	.18073	.18049
.9	.08983	.08263	.08131	.07832
1.0	.05000	.05000	.05000	.05000
1.1	.08983	.08263	.08131	.07832
1.2	.20028	.18612	.18073	.18049
1.3	.36830	.36323	.35181	.35677
1.4	.59762	.58686	.57085	.58527
1.5	.80937	.79571	.78054	.80522
1.6	.94201	.93215	.92299	.94695
1.7	.99130	.98788	.98485	.99501
1.8	.99929	.99928	.99889	.99997
1.9	1.00000	1.00000	.99999	1.00000

I. $P_{30}(\lambda, .5, .5, 15, 15)$ II. $P_{30}(\lambda, .5, .5, 15)$ III. $P_{30}(\lambda, .5, .5)$

IV. Normal Approximation



Table D.3.2

λ	I	II	III
.1	.99834	.99717	.99565
.2	.98146	.97587	.96969
.3	.92423	.90988	.89698
.4	.80775	.78372	.76575
.5	.63796	.60946	.59101
.6	.44675	.42173	.40716
.7	.27391	.25754	.24866
.8	.14682	.13925	.13532
.9	.07343	.07156	.07061
1.0	.05000	.05000	.05000
1.1	.07343	.07156	.07061
1.2	.14682	.13925	.13532
1.3	.27391	.25754	.24866
1.4	.44675	.42173	.40716
1.5	.63796	.60946	.59101
1.6	.80775	.78372	.76572
1.7	.92423	.90988	.89698
1.8	.98146	.97587	.96969
1.9	.99834	.99717	.99565

I. $P_{30}(\lambda, .4, .5, 12, 15)$ III. $P_{30}(\lambda, .4, .5)$ II. $P_{30}(\lambda, .4, .5, 15)$

Table D.3.3

λ	I	II	III	IV
.1	.97462	.96884	.95914	.99584
.2	.89547	.89000	.87368	.92299
.3	.76891	.76312	.74398	-----
.4	.61317	.60586	.58794	.59429
.5	.45190	.44364	.42955	-----
.6	.30659	.29912	.28966	.28187
.7	.19122	.18596	.18063	-----
.8	.11101	.10832	.10598	.10049
.9	.06493	.06419	.06359	-----
1.0	.05000	.05000	.05000	.05000
1.1	.06509	.06417	.06353	-----
1.2	.11224	.10799	.10542	.12797
1.3	.19578	.18415	.17818	-----
1.4	.31299	.29309	.28245	.29294
1.5	.45941	.42922	.41343	-----
1.6	.61705	.57914	.55911	.58387
1.7	.76298	.72366	.70185	-----
1.8	.87673	.84365	.82343	.85915
1.9	.94896	.92723	.91155	-----
2.0	.98453	.97407	.96420	.98542
2.1	.99706	.99382	.98895	-----
2.2	.99953	.99940	.99751	.99995
2.3	1.00000	1.00000	.99952	-----
2.4	1.00000	1.00000	.99983	1.00000

I. $P_{30}(\lambda, .4, .4, 12, 12)$ III. $P_{30}(\lambda, .4, .4)$ II. $P_{30}(\lambda, .4, .4, 12)$

IV. Normal Approximation



Table D.3.4

λ	I	II	III
.1	.77941	.64201	.60156
.2	.47297	.50126	.47297
.3	.43930	.38103	.36179
.4	.31821	.28227	.26957
.5	.22500	.20409	.19602
.6	.15572	.14454	.13972
.7	.10655	.10131	.09872
.8	.07412	.07220	.07107
.9	.05585	.05545	.05517
1.0	.05000	.05000	.05000
1.1	.05568	.05542	.05514
1.2	.07272	.07188	.07078
1.3	.10140	.09990	.09738
1.4	.14219	.14006	.13558
1.5	.19536	.19266	.18561
1.6	.26067	.25738	.24721
1.7	.33705	.33302	.31926
1.8	.42238	.41736	.39967
1.9	.51346	.50716	.48544
2.0	.60619	.59838	.57286
2.1	.69589	.68660	.65791
2.2	.77793	.76751	.73672
2.3	.84833	.83751	.80609
2.4	.90450	.89424	.86392
2.5	.94558	.93685	.90936
2.6	.97269	.96613	.94289
2.7	.98834	.98416	.96602
2.8	.99605	.99383	.98086
2.9	=====	.99814	.98973
3.0	=====	.99962	.99467
3.1	=====	.99997	.99727
3.2	=====	1.00000	.99862
3.3	=====	1.00000	.99936

I. $P_{30}(\lambda, .3, .3, 9, 9)$ III. $P_{30}(\lambda, .3, .3)$ II. $P_{30}(\lambda, .3, .3, 9)$



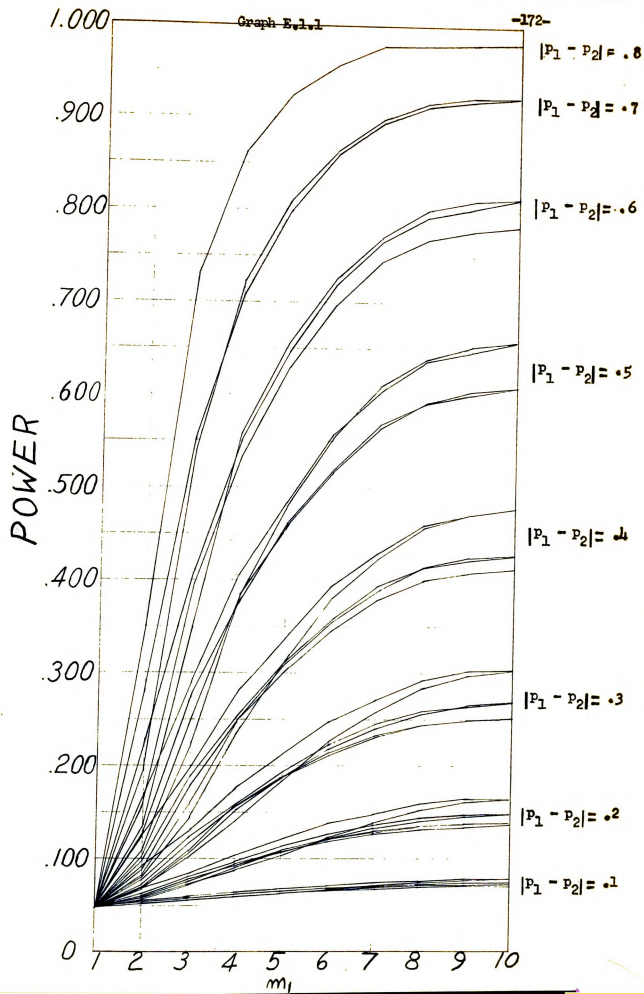
Appendix E. Approximations to Power in the 2 x 2 Comparative Trial.

We discussed the power for the test of independence in the 2 x 2 comparative Trial in section 3.2, and gave tables of exact power in Appendix B, and in Appendix D. In E.1 of this appendix, three graphs are given, two of which illustrate how the power changes as m_1 varies from 0 to $\frac{n}{2}$, and the third graph contains some power curves for $m_1 = 15$ and $n = 30$. We fix p_2 and then plot $P_{30}(p_1, p_2, 15)$ as a function of p_1 . This is done for $p_2 = .1, .2, .3, \dots, .9$.

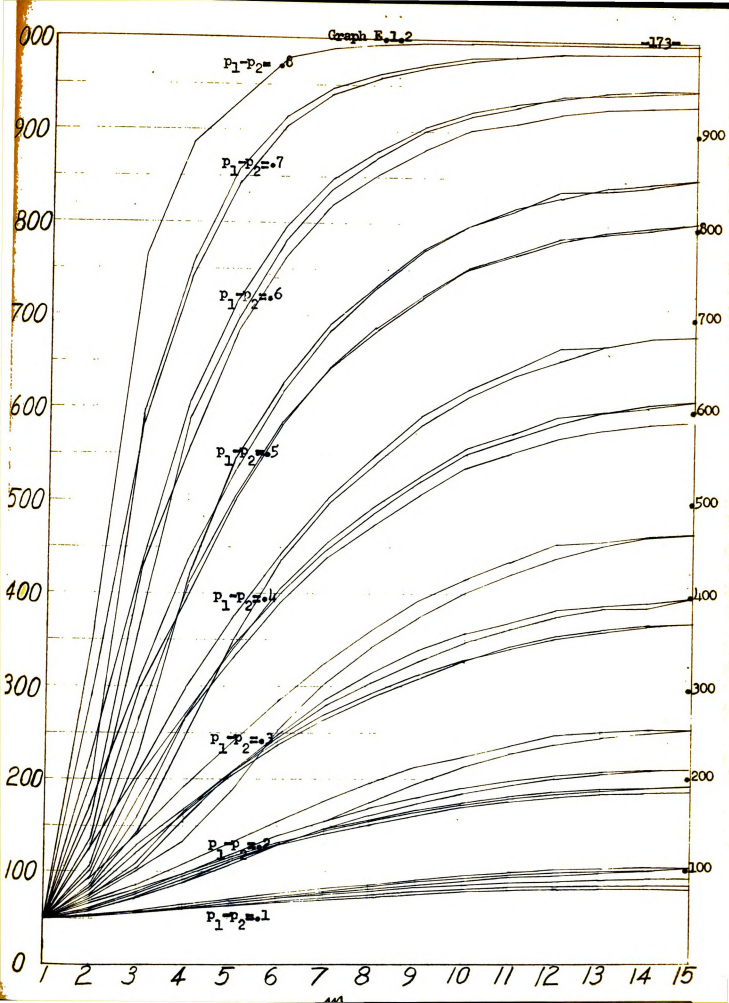
In E.2, Sillitto's approximation, given by (3.2.6) and Patnaik's 2nd approximation, given by (1.7.18), are compared with the exact values of power in the 2 x 2 comparative trial, for the two special cases $m_1 = 10, n = 20$, and $m_1 = 15, n = 30$. These values occur in groups of four for each p_1, p_2 combination, with the top number being the exact one, the second and third are obtained from Sillitto's and Patnaik's approximations respectively, and the fourth number is $P_n(t \mid m_1, \gamma)$, $\gamma = m_1 p_1 + (n - m_1) p_2$, and

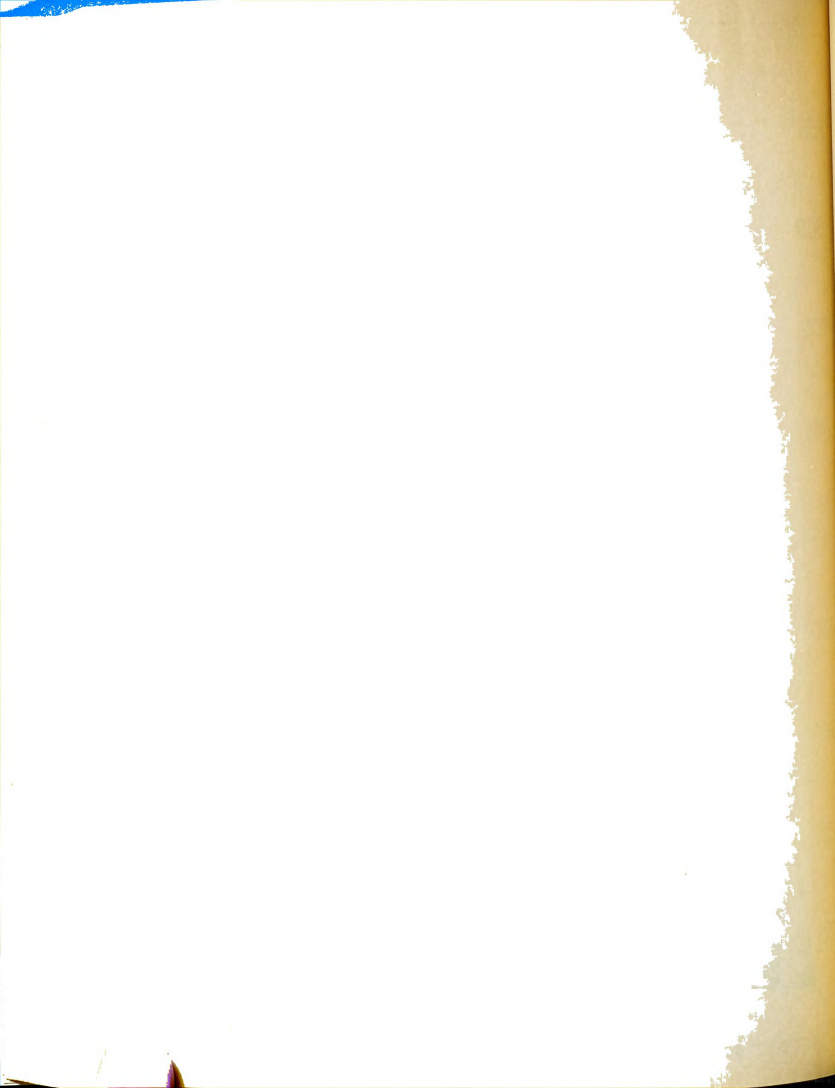
$$t = \frac{p_1 q_2}{p_2 q_1}. \text{ Not all } P_n(t \mid m_1, \gamma) \text{ values are given for}$$

each p_1, p_2 combination.









1.000

Graph E.1.3

-174-

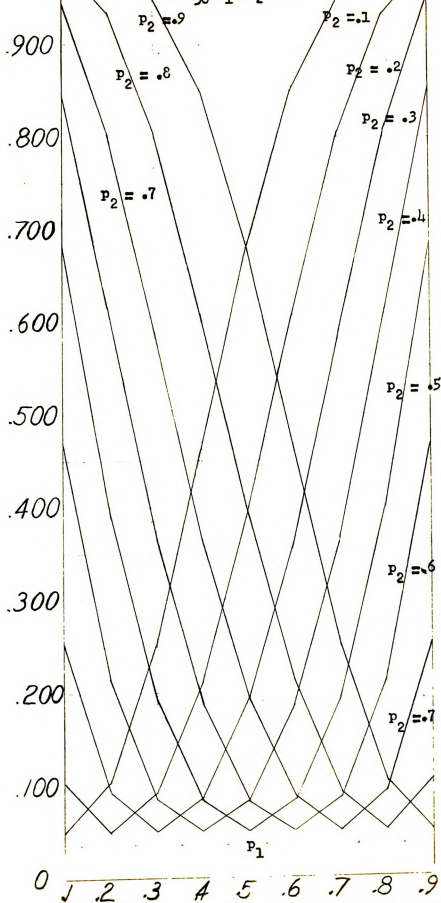
 $P_{30}(p_1, p_2, 15)$ 

Table E.2.1

Comparisons of Power in the 2 x 2 Comparative Trial

$P_1 \backslash P_2$	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.0500 .0500 .0500 .0500	.0783 .0973 .0977	.1644 .2109 .2273	.3036 .3683 .4114	.4773 .5452 .6143 .4688	.6569 .7137 .7930	.8124 .8493 .9148	.9231 .9397 .9785	.9826 .9856 .9983 .9909
.2		.0500 .0500 .0500 .0500	.0749 .0813 .0820	.1483 .1671 .1717	.2684 .3015 .3162	.4275 .4722 .5018 .4346	.6078 .6550 .6999	.7834 .8206 .8703 .8123	.9231 .9397 .9785
.3			.0500 .0500 .0500 .0500	.0722 .0756 .0761	.1389 .1512 .1535	.2524 .2780 .2860	.4122 .4525 .4736 .4259	.6078 .6550 .6999	.8124 .8493 .9148
.4				.0500 .0500 .0500 .0500	.0709 .0736 .0741	.1358 .1468 .1482 .1373	.2524 .2780 .2860	.4275 .4722 .5018 .4346	.6569 .7137 .7930
.5					.0500 .0500 .0500 .0500	.0709 .0736 .0741	.1389 .1512 .1535	.2684 .3015 .3162	.4773 .5452 .6143 .4688
.6						.0500 .0500 .0500 .0500	.0722 .0756 .0761	.1483 .1671 .1717	.3036 .3683 .4114
.7	First No.: $P_{20}(p_1, p_2, 10)$						.0500 .0500 .0500	.0749 .0813 .0820	.1644 .2109 .2273
.8	Second No.: Sillitto's Approx.								
	Third No.: Patnaik's Approx.								
	Fourth No.: $P_{20}(t 10, Y)$							.0500 .0500 .0500 .0500	.0783 .0973 .0997
.9									.0500 .0500 .0500 .0500



Table E.2.2

Comparisons of Power in the 2 x 2 Comparative Trial

$P_1 \backslash P_2$	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	.0500 .0500 .0500 .0500	.1040 .1216 .1245	.2535 .2927 .3163	.4646 .5117 .5663 .6773	.6820 .7193 .7872	.8516 .8714 .9251	.9500 .9560 .9828 .9595	.9848 .9905 .9982	.9993 .9991 1.0000 .9993
.2		.0500 .0500 .0500 .0500	.0916 .0974 .0982	.2110 .2279 .2339	.3954 .4220 .4411 .6380	.6095 .6387 .6715 .6380	.8020 .8240 .8598	.9322 .9411 .9668 .9420	.9898 .9905 .9982
.3			.0500 .0500 .0500 .0500	.0856 .0888 .0892	.1920 .2036 .2069 .2037	.3667 .3897 .4006	.5869 .6161 .6345 .5976	.8020 .8240 .8598	.9500 .9560 .9828 .9595
.4				.0500 .0500 .0500 .0500	.0830 .0855 .0862	.1861 .1968 .1993 .2003	.3667 .3896 .4006	.6095 .6387 .6715 .6380	.8516 .8714 .9251
.5					.0500 .0500 .0500 .0500	.0830 .0855 .0862	.1920 .2036 .2069 .2037	.3954 .4220 .4411	.6820 .7193 .7872 .6773
.6						.0500 .0500 .0500 .0500	.0856 .0888 .0892	.2110 .2279 .2339	.4646 .5117 .5663
.7							.0500 .0500 .0500 .0500	.0916 .0974 .0982	.2535 .2927 .3163
.8								.0500 .0500 .0500 .0500	.1040 .1216 .1245
.9									.0500 .0500 .0500 .0500

First No.: $P_{30}(p_1, p_2, 15)$

Second No.: Sillitto's Approx.

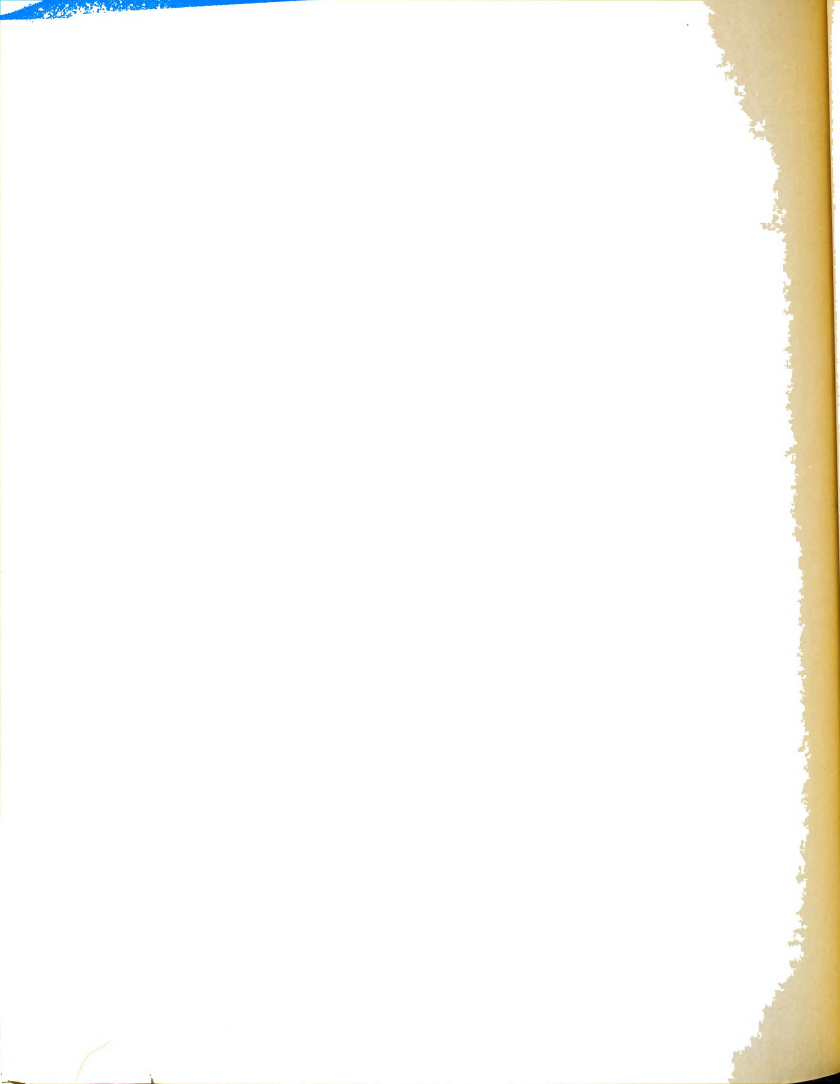
Third No.: Patnaik's Approx.

Fourth No.: $P_{30}(t | 15, \delta')$

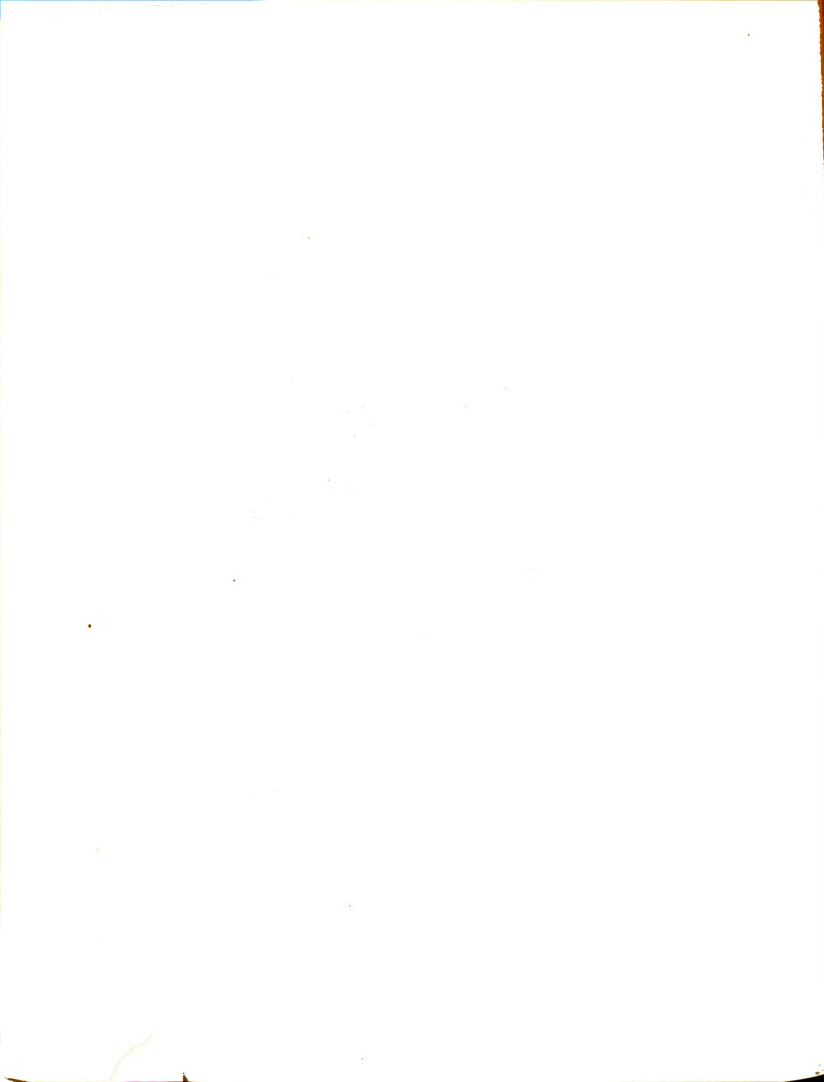


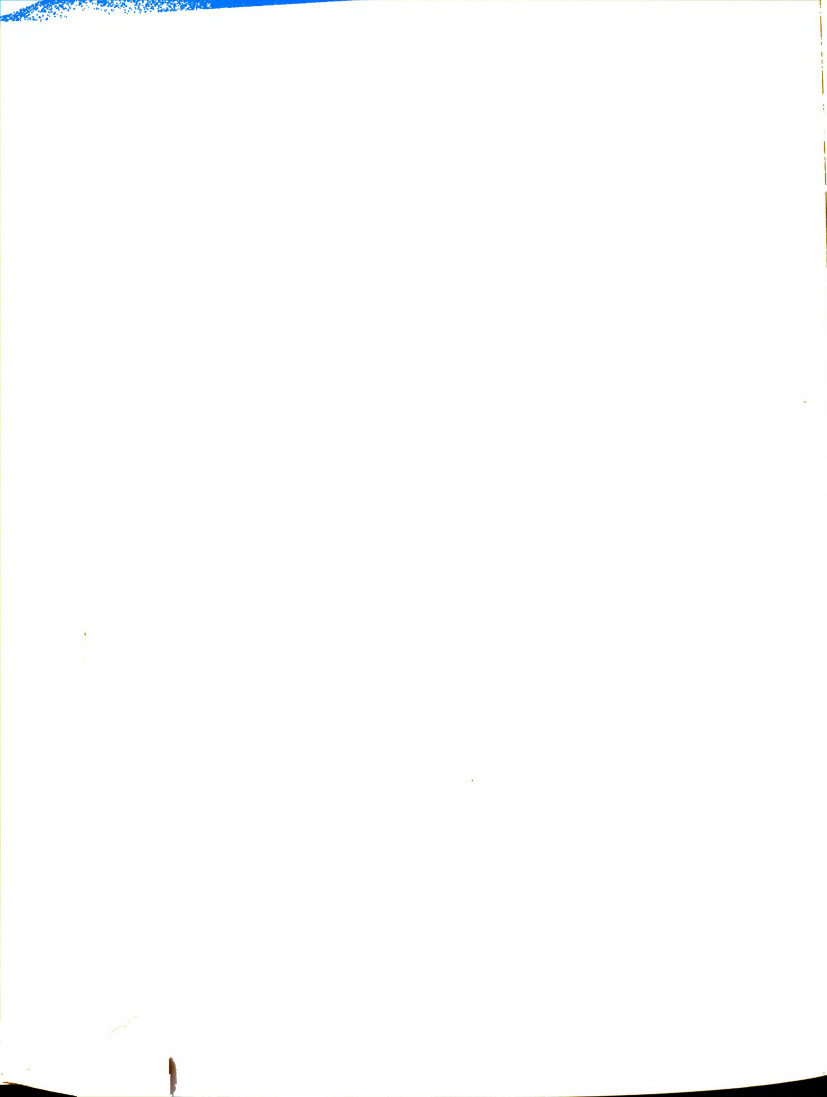
BIBLIOGRAPHY

1. Armsen, P., "A new form of table for significance tests in a 2 x 2 Contingency Table," Biometrika, Vol. 42 (1955), pp. 494-511.
2. Barnard, G. A., "Statistical inference," Journal Royal Statistical Society, Vol. 11 (1949), Series B, pp. 115-139.
3. Barnard, G. A., "Significance tests for 2 x 2 tables," Biometrika, Vol. 34 (1947), pp. 123-138.
4. Cochran, W. G., "The χ^2 -test of goodness of fit," Annals Mathematical Statistics, Vol. 23 (1952), pp. 315-345.
5. Feller, W., "An Introduction to Probability Theory, and Its Applications," 2nd edition, John Wiley and Sons, New York, 1957.
6. Finney, D. J., "The Fisher-Yates test of significance in 2 x 2 contingency tables," Biometrika, Vol. 35, (1948), pp. 145-156.
7. Fisher, R. A., "Statistical Methods for Research Workers," 2nd edition, Oliver and Boyd Ltd., Edingurgh, 1934.
8. Fisher, R. A., "The logic of inductive inference," Journal Royal Statistical Society, Vol. 98 (1935), pp. 39-54.
9. Katz, Leo, "The test of the hypothesis of no association in the four-fold table in light of the Neyman-Pearson Theory," Unpublished memorandum, Michigan State University, 1942.
10. Latscha, R., "Tests of Significance in 2 x 2 tables: extension of Finney's Table," Biometrika, Vol. 40 (1953), pp. 74-86.
11. Lehmann, E. L., "Significance level and power," Annals of Mathematical Statistics, Vol. 29 (1958), pp. 1167-1176.
12. Loeve, M., "Probability Theory," D. Van Nostrand Co., Inc., 1950.
13. Moore, P. G., "A test for randomness in a sequence of two alternatives involving a 2 x 2 table," Biometrika, Vol. 36 (1949), pp. 305-316.
14. Patnaik, P. B., "The power function of the test for the difference between two proportions in a 2 x 2 table," Biometrika, Vol. 35, (1949), pp. 157-175.



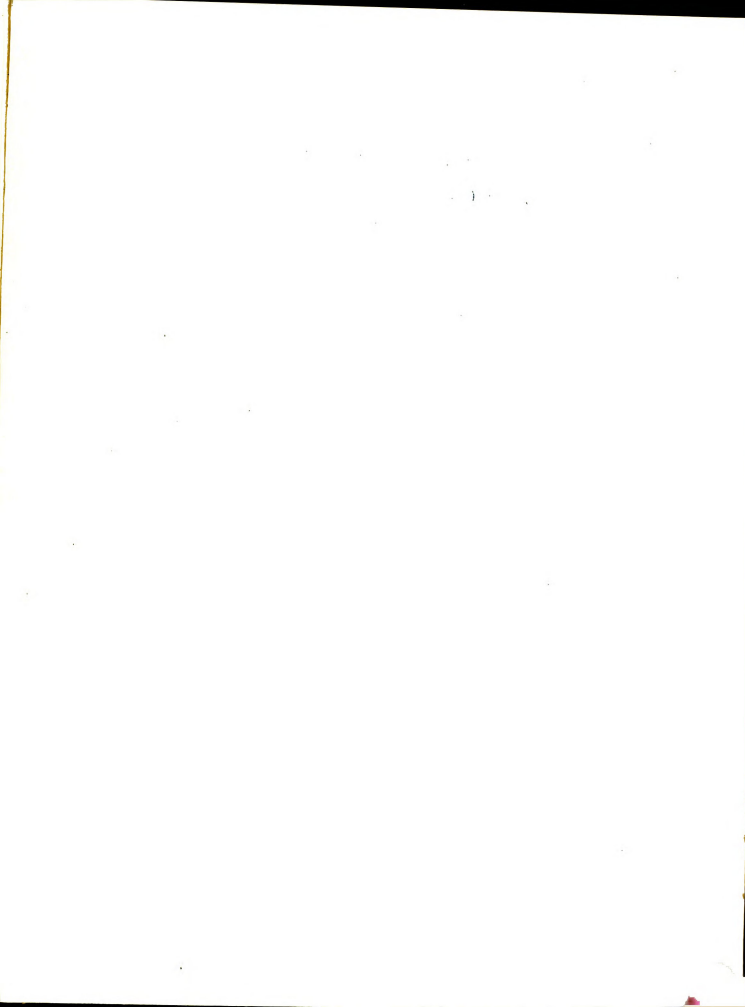
15. Patnaik, P. B., "The Non-Central χ^2 - and F-distributions and their applications," Biometrika, Vol. 36 (1949), pp. 202-232.
16. Pearson, E. S., "The choice of statistical tests illustrated on the interpretation of data classed in a 2 x 2 table," Biometrika, Vol. 34 (1947, pp. 139-167.
17. Pearson, K., "On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling," Philosophical Magazine Series 5, Vol. 50 (1900), pp. 157-172.
18. Riordan, J., "An Introduction to Combinatorial Analysis," John Wiley and Sons, Inc., New York, 1958.
19. Rokhar, A. E., "The effect of hypergeometric probability distribution on the design of sampling plans for small lot sizes," Master's Thesis Lehigh University, 1958.
20. Sekar, C. C., Agarwala, S. P., and Chakraborty, P. N., "On the power function of a test of significance for the difference between two proportions," Sankhya, Vol. 15, (1955), pp. 381-390.
21. Sillitto, G. P., "Note on approximation to the power function of the 2 x 2 comparative trial," Biometrika, Vol. 36, (1949), pp. 347-352.
22. Snow, C., "Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory," National Bureau of Standards Applied Mathematics Series 19, 1952.
23. Steck, G. P., "Limit Theorems for Conditional Distributions," (1957), University of California Publications in Statistics, Vol. 2, No. 12, pp. 237-284.
24. Sverdrup, E., "Similarity, unbiasedness, minimaxibility and admissibility of statistical test procedures," Skandinavisk Aktuarietidskrift, Vol. 36 (1953), pp. 64-86.
25. Tocher, K. D., "Extension of the Neyman - Pearson Theory of tests to discontinuous variates," Biometrika, Vol. 37 (1950), pp. 130-144.
26. Yates, F., "Contingency tables involving small numbers and the χ^2 -test," Supplement, Journal Royal Statistical Society, Vol. 1 (1934), pp. 217-235.





[REDACTED]

Page 1



MICHIGAN STATE UNIV. LIBRARIES



31293011037169