SOME COMPUTATIONS AND APPLICATIONS OF HEEGAARD FLOER CORRECTION TERMS

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ABSTRACT

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In this dissertation we study some computations and applications of Heegaard Floer correction terms. In particular we explore the correction terms for the double covers of the three-sphere branched along the Whitehead doubles of knots. As a consequence we show that Whitehead double and iterated double of some classes of knots are independent in the smooth knot concordance group. We also compute the correction terms of non-trivial circle bundles over oriented surfaces and discuss how they can be applied to four-dimensional topology.

For my family.

ACKNOWLEDGMENTS

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Chapter 1

Introduction

Heegaard Floer theory, introduced by Ozsváth and Szabó early 2000, has played an important role in studying low-dimensional topology. It has provided a collection of invariants for many of the objects in low-dimensional topology. It originally defined homological invariants for closed oriented smooth 3-manifolds, and then was extended to knots and links, 3-manifolds with boundary and oriented smooth 4-manifolds. In particular, correction terms (or d-invariants), defined as a rational-valued grading of a certain element in the Floer homology group, carry essential information to connect three and four dimensional topology. Ozsváth and Szabó used the correction terms to give new proofs of Donaldson's diagonalization theorem and the Thom Conjecture, notable results originally shown using gauge theory. Moreover the correction terms can be used to explain the surprising distinction between the topological and smooth categories in 4-dimensional topology.

One important example appears in the study of knot concordance, which deals with how an oriented surface, bounded by a knot, is topologically (locally flat) or smoothly embedded in the 4-ball. In the theory of knot concordance, Whitehead doubles play an important role since they always bound a topologically embedded disk but many do not a smoothly embedded disk, i.e. these are topologically slice but not smoothly slice. It is thus important to understand their concordance properties as portrayed in the knot concordance group. For example, it is generally difficult to understand whether a collection of Whitehaed doubles forms a set of independent elements in this group.

In Chapter 3 we study this question for iterated Whitehead doubles and show that for each $m > 1$ the Whitehead double and once iterated Whitehead double of $(2,\, 2m+1)$ torus knots are not smoothly concordant and that indeed they generate a $\mathbb{Z} \oplus \mathbb{Z}$ summand in the subgroup of the smooth knot concordance group generated by topologically slice knots. Our main tool is the correction term for the double cover of S^3 branched along a knot. We also present some sufficient conditions for general knots to have this independence property. Additionally, for some classes of knots including $(p,\, q)$ torus knots, we give an algorithmic formula for testing it in terms of its Alexander polynomial, along with an implementation of a computer program.

In Chapter 4 we compute the correction terms of circle bundles over oriented surfaces. Our main tool for this computation is the integer surgery formula which relates the knot Floer homology of a knot to Heegaard Floer homology of the three-manifold obtained by a surgery of the knot. One of the key properties of the correction terms is that they give restrictions on the intersection form of 4-manifolds which are bounded by a 3-manifold. Hence, our computation would be applied to this direction, since the boundary of the tubular neighborhood of an embedded surface in a 4-manifold is diffeomorphic to a circle bundle.

Chapter 2

Heegaard Floer Theory

The purpose of this chapter is to overview some of the features and properties of Heegaard Floer theory and set up notations, used in the following chapters. Hence we skip the details of the constructions of Heegaard Floer homology, knot Floer homology and more.

2.1 Heegaard Floer homology

For a closed oriented 3-manifold Y equipped with a spin^c structure $\mathfrak{t}$, one can associate to it a relatively $\mathbb{Z}$-graded and filtered chain complex $CF^\infty(Y,\mathfrak{t})$, a finitely and freely generated $\mathbb{Z}[U,U^{-1}]$-module. In particular the filtration is given by the negative power of U, and U-multiplication lowers the homological grading by 2. The filtered chain homotopy type of $CF^\infty(Y,\mathfrak{t})$ is an invariant of $(Y,\mathfrak{t})$, called the Heegaard Floer chain complex. For more detailed and general exposition of the definition we refer to [18, 19].

We set $CF^-(Y,\mathfrak{t}) := CF^\infty(Y,t)\{i<0\}$, the subcomplex consisting of the elements in $CF^\infty(Y,t)$ whose filtration level i is less than 0, and also define the quotient complexes $CF^+(Y,\mathfrak{t}){:=}CF^\infty(Y,\mathfrak{t})\{i\geq 0\}$ and $\widehat{CF}(Y,\mathfrak{t}){:=}CF^\infty(Y,t)\{i=0\}$. The homology group of $CF^\infty(Y,\mathfrak{t})$ is denoted by $HF^\infty(Y,\mathfrak{t})$, and HF^-, HF^+ and $\widehat{HF}$ denote the homology of the other chain complexes, respectively. We refer to them as Heegaard Floer homologies of $(Y,\mathfrak{t})$, and sometimes denote all versions of homology groups together HF°. The various versions of Heegaard Floer homologies naturally fit into a long exact sequence:

$$\cdots \longrightarrow HF^-(Y,\,\mathfrak{t}) \xrightarrow{\iota} HF^\infty(Y,\,\mathfrak{t}) \xrightarrow{\pi} HF^+(Y,\,\mathfrak{t}) \xrightarrow{\delta} \cdots. \tag{2.1}$$

Additionally Heegaard Floer homologies of Y are acted upon by the exterior algebra $\Lambda^* H_1(Y;\,\mathbb{Z})/\mathrm{Tors}$, and the maps in the above long exact sequence are $\Lambda^* H_1(Y;\,\mathbb{Z})/\mathrm{Tors} \otimes \mathbb{Z}[U]$-equivariant.

We also define *reduced* Floer homology groups as

$$HF^-_{red} := \ker(\iota) \cong \mathrm{coker}(\pi) =: HF^+_{red}.$$

Note that the isomorphism is induced by the coboundary map of long exact sequence. Moreover, the image of the multiplication of U^d in HF^+ stabilizes for sufficiently large d, and we have

$$HF^+_{red} \cong HF^+/U^d \cdot HF^+(Y).$$

For example, it is known from [19] that $HF^\infty(S^3) \cong \mathbb{Z}[U,\,U^{-1}]$ (we usually drop the spin^c structure in the notation if there is a unique one), and for $Y = \#^n S^2 \times S^1$

$$HF^\infty(\#^n S^2 \times S^1,\,\mathfrak{t}_0) \cong \mathbb{Z}[U,\,U^{-1}] \otimes \Lambda^* H^1(Y;\,\mathbb{Z}),$$

where $\mathfrak{t}_0$ is the unique torsion spin^c structure over Y, and the $\Lambda^* H_1(Y;\,\mathbb{Z})$-action is induced by the contraction map.

2.2 Maps induced by cobordisms and the absolute grading

An oriented $n+1$ dimensional manifold W is called a *cobordism* from Y_1 to Y_2 if the boundary of W is the disjoint union of $-Y_1$ and Y_2. One of the key features of Heegaard Floer theory is that a 4-dimensional cobordism induces maps between Floer homology groups of the boundary 3-manifolds: they form a $(3+1)$ Topological Quantum Field Theory (TQFT). More precisely, if W is a cobordism from Y_1 to Y_2 and $\mathfrak{s}$ is a spin^c structure over W whose restriction to Y_i is $\mathfrak{t}_i$ for $i = 1, 2$, then W induces a commutative diagram between long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & HF^-(Y_1,\, \mathfrak{t}_1) \arrow[r, "\iota"] \arrow[d, "F^-_{W,\mathfrak{s}}"'] & HF^\infty(Y_1,\, \mathfrak{t}_1) \arrow[r, "\pi"] \arrow[d, "F^\infty_{W,\mathfrak{s}}"'] & HF^+(Y_1,\, \mathfrak{t}_1) \arrow[r, "\delta"] \arrow[d, "F^+_{W,\mathfrak{s}}"'] & \cdots \\
\arrow[r] & HF^-(Y_2,\, \mathfrak{t}_2) \arrow[r, "\iota"] & HF^\infty(Y_2,\, \mathfrak{t}_2) \arrow[r, "\pi"] & HF^+(Y_2,\, \mathfrak{t}_2) \arrow[r, "\delta"] & {}
\end{tikzcd} \tag{2.2}$$

In fact, Ozsváth and Szabó showed that the vertical maps are invariants of $(W, \mathfrak{s})$ up to an overall sign [22]. Moreover the vertical maps commute with the $\Lambda^* H_1/\mathrm{Tors}$-action in the sense that if $\gamma_1 \in H_1(Y_1\,;\mathbb{Z})/\mathrm{Tors}$ and $\gamma_1 \in H_1(Y_2\,;\mathbb{Z})/\mathrm{Tors}$ are homologous in W, then

$$F^\circ_{W,\mathfrak{s}}(\gamma_1 \cdot \xi) = \gamma_2 \cdot F^\circ_{W,\mathfrak{s}}(\xi).$$

The maps induced by cobordims allow us to assign a $\mathbb{Q}$-valued grading, called the *absolute grading*, on Heegaard Floer homology groups of Y equipped with a torsion spin^c structure $\mathfrak{t}$, which has the following properties [22]:

- The absolute grading respects the homological grading of $CF^\infty(Y)$.

- The absolute grading of a generator in $\widehat{HF}(S^3) \cong \mathbb{Z}$ is 0.

- If $(W, \mathfrak{s})$ is a spin^c cobordism from $(Y_1, \mathfrak{t}_1)$ to $(Y_2, \mathfrak{t}_2)$, then

$$gr(F^+_{W,\mathfrak{s}}(\xi)) - gr(\xi) = \frac{c_1^2(\mathfrak{s}) - 2\chi(W) - 3\sigma(W)}{4} \tag{2.3}$$

  for $\xi \in HF^+(Y_1, \mathfrak{t}_1)$. Here, c_1 is the first Chern class, χ is the Euler characteristic, σ is the signature of the intersection form of W.

We usually write down the absolute grading using a subscript with parenthesis, for example, $\widehat{HF}(S^3) \cong \mathbb{Z}_{(0)}$.

2.3 Correction terms

Let Y be a rational homology three-sphere with a spin^c structure, $\mathfrak{t}$. Then, the *correction term* of $(Y, \mathfrak{t})$ is defined to be the minimal absolute-grading of any non-torsion element in the image of $\pi\colon HF^\infty(Y, \mathfrak{t}) \to HF^+(Y, \mathfrak{t})$ in (2.1).

The correction terms can be generalized to three-manifolds whose HF^∞ has a standard form. More precisely, we call a manifold Y has *standard* HF^∞ if there is $\Lambda^* H_1(Y;\mathbb{Z})/\mathrm{Tors} \otimes \mathbb{Z}[U]$-module isomorphism

$$HF^\infty(Y, \mathfrak{t}) \cong \Lambda^* H^1(Y;\mathbb{Z})/\mathrm{Tors} \otimes \mathbb{Z}[U, U^{-1}]$$

for each torsion spin^c structure $\mathfrak{t}$ over Y, where the action of $\Lambda^* H_1(Y;\mathbb{Z})$ is given by the contraction. For example, $\#^n S^2 \times S^1$ has standard HF^∞, and any 3-manifolds with $b_1(Y) \leq 2$ are also known to have standard HF^∞ [19].

Let H be finitely generated, free abelian group and $\Lambda = \Lambda^*(H)$ denote the exterior algebra of H. For any Λ-module M, we denote the kernel of the action of Λ on M as

$$\mathcal{K}M := \{x \in M | v \cdot x = 0 \ \forall v \in H\},$$

and the quotient by the image of Λ as

$$\mathcal{Q}M := M/(\Lambda \cdot M).$$

Then, for a 3-manifold Y with standard HF^∞, we have the following maps induced from (2.1):

$$\mathcal{K}HF^\infty(Y, \mathfrak{t}) \xrightarrow{\mathcal{K}(\pi)} \mathcal{K}HF^+(Y, \mathfrak{t})$$

and

$$\mathcal{Q}HF^\infty(Y, \mathfrak{t}) \xrightarrow{\mathcal{Q}(\pi)} \mathcal{Q}HF^+(Y, \mathfrak{t}),$$

for each torsion spin^c structure $\mathfrak{t}$. We define the *bottom* and *top correction terms* of $(Y, \mathfrak{t})$ to be the minimal grading of any non-torsion element in the image of $\mathcal{K}(\pi)$ and $\mathcal{Q}(\pi)$ respectively. We denote them d_{bot} and d_{top}. In particular, they satisfy the following properties about conjugation of spin^c structure and orientation reversal [15, Proposition 4.2.][12]:

$$d(Y, \mathfrak{t}) = d(Y, \bar{\mathfrak{t}})$$

and

$$d_{bot}(Y, \mathfrak{t}) = -d_{top}(-Y, \mathfrak{t}).$$

The correction terms are invariants of the rational homology cobordism class of 3-manifolds:

Theorem 2.3.1. [15, Theorem 1.2.][12, Theorem 4.4.] *Let Y_1 and Y_2 be closed, oriented three-manifolds with standard HF^∞, and let W be a rational homology cobordism from Y_1 to Y_2 (meaning that the inclusions $i\colon Y_1 \to W$ and $i\colon Y_2 \to W$ induce isomorphisms on rational homology). Let $\mathfrak{s}$ be any spin^c structure over W whose restrictions $\mathfrak{t}_1 = \mathfrak{s}|_{Y_1}$ and $\mathfrak{t}_2 = \mathfrak{s}|_{Y_2}$ are both torsion. Then we have*

$$d_{bot}(Y_1,\, \mathfrak{t}_1) = d_{bot}(Y_2,\, \mathfrak{t}_2) \text{ and } d_{top}(Y_1,\, \mathfrak{t}_1) = d_{top}(Y_2,\, \mathfrak{t}_2).$$

Proof. By theorem [15, Theorem 9.1.], the induced maps $F^\infty_{W,\mathfrak{s}}\colon HF^\infty(Y_1, \mathfrak{t}_1) \to HF^\infty(Y_2,\, \mathfrak{t}_2)$ and $F^\infty_{-W,\mathfrak{s}}\colon HF^\infty(Y_2, \mathfrak{t}_2) \to HF^\infty(Y_1,\, \mathfrak{t}_1)$ are isomorphisms. Then the statement easily follows from the commutative diagram in (2.2) and the degree formula in (2.3). □

More generally, the correction terms of Y give restrictions on intersection forms of 4-manifolds which bounded by Y.

Theorem 2.3.2. [15, Theorem 9.15.][12, Theorem 4.7.] *Let X be an oriented negative semi-definite 4-manifold bounded by a closed oriented manifold Y with standard HF^∞. Then, for any Spin^c structure $\mathfrak{s}$ over W whose restriction to Y is a torsion spin^c structure $\mathfrak{t}$, then we have the following inequality:*

$$c_1(\mathfrak{s})^2 + b_2^-(X) \leq 4d_{bot}(Y,\, \mathfrak{t}) + 2b_1(Y).$$

In addition, if the map $H_1(Y)/Tors \to H_1(X)/Tors$ induced by inclusion is injective, then *we have*

$$c_1(\mathfrak{s})^2 + b_2^-(X) \leq 4d_{top}(Y, \mathfrak{t}) - 2b_1(Y).$$

2.4 Knot Floer homology

A null-homologous knot K in an integral homology 3-sphere Y has an associated $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex $CFK^\infty(Y, K)$ which reduces to $CF^\infty(Y)$ after forgetting the second $\mathbb{Z}$ filtration. The U-multiplication decreases both of the filtration levels by 1. The filtered chain homotopy type of $CFK^\infty(Y, K)$ is an invariant of the knot and we refer it as the knot Floer invariant of (Y, K) [20, 25]. We denote by $CFK^\infty(Y, K)\{(i, j)\}$ the subgroup at (i, j)-filtration level in $CFK^\infty(Y, K)$ and define $\widehat{CFK}(Y, K){:=}CFK^\infty(Y, K)\{i = 0\}$. In particular, for a knot K in S^3 we abbreviate the notations by $CFK^\infty(K) := CFK^\infty(S^3, K)$ and $\widehat{CFK}(K) := \widehat{CFK}(S^3, K)$. It is an easy fact that $H_*(CFK^\infty(Y, K)) \cong HF^\infty(Y)$ and $H_*(\widehat{CFK}(Y, K)) \cong \widehat{HF}(Y)$. As a consequence, for a knot K in S^3, we obtain an induced sequence of maps:

$$\iota_K^m : H_*(\widehat{CFK}(S^3, K)_{\{j \leq m\}}) \to \widehat{HF}(S^3) \cong \mathbb{Z}.$$

An invariant τ for a knot K in S^3 is defined by

$$\tau(K){:=}\min\{m \in \mathbb{Z} |\ \iota_K^m \text{ is non-trivial}\}.$$

When working with coefficients in $\mathbb{F} \cong \mathbb{Z}/2\mathbb{Z}$, it is useful to visualize a knot Floer complex as a collection of dots and arrows lying in a grid in the plane. In a diagram, a dot in (i, j)-coordinate box represents an $\mathbb{F}$-generator in $CFK^\infty(K)\{(i, j)\}$, and an arrow represents the non-trivial map, $\mathbb{F} \to \mathbb{F}$. The differential is then the sum of the arrows, as a map of vector spaces. See Figure 2.1 for examples.

2.5 Staircase complexes

For a given $(n-1)$-tuple of positive integers $\mathbf{v}$, a *staircase complex of length* n, $\mathrm{St}(\mathbf{v})$, is defined as a finitely generated $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex over $\mathbb{F}$ with n generators, where the numbers in $\mathbf{v}$ are the length of arrows, which alternate horizontal and vertical starting at the top left generator and moving to the bottom right generator in alternating right and downward steps in a grid diagram. We also locate the top left dot on the vertical axis ($i = 0$) and the bottom right on the horizontal axis ($j = 0$) on the diagram. See [1] for more detail. For instance, complexes generated by St(1, 1, 1, 1) and St(1, 2, 2, 1) are shown in Figure 2.1.

The knot Floer invariant is a categorification of Alexander-Conway polynomial $\Delta_K(t)$, in the following sense:

$$\Delta_K(t) = \sum_{k \in \mathbb{Z}} \chi(\widehat{CFK}(K)_{\{j=k\}})t^k,$$

where $\Delta_K(t)$ is the symmetrized Alexander-Conway polynomial of K and χ is the Euler characteristic.

Conversely, for some classes of knots such as alternating knots and L-space knots (including torus knots), the knot Floer complex of a knot is determined by its Alexander-Conway polynomial [17, 21]. For an L-space knot, K, the Alexander-Conway polynomial has the form $\Delta_K(t) = \sum_{k=0}^{2m}(-1)^k t^{(n_k)}$, and $CFK^\infty(K)$ (with coefficients in $\mathbb{F}$) is generated by a stair complex,

$$CFK^\infty(K) \cong \mathrm{St}(n_{i+1} - n_i) \otimes \mathbb{F}[U, U^{-1}],$$

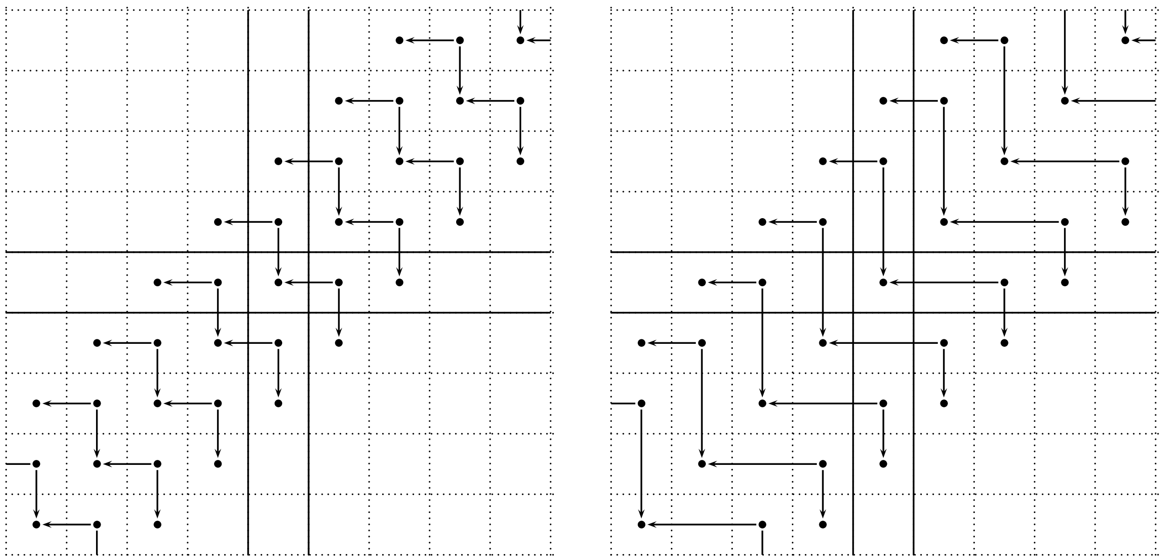


Figure 2.1: Diagrams of $CFK^{\infty}(T_{2,5})$ and $CFK^{\infty}(T_{3,4})$ generated by staircase complexes St(1, 1, 1, 1) and St(1, 2, 2, 1) respectively.

where i runs from 0 to $2m-1$ and U-multiplication is naturally extended: i.e. if x is a generator in (i, j)-filtration level, then $U^n x$ sits in $(i-n, j-n)$-filtration level, and $\partial(U^n x) = U^n \partial(x)$. Denote $\mathrm{St}(K) := \mathrm{St}(n_{i+1} - n_i)$.

For example, since the Alexander-Conway polynomial of $T_{2,2m+1}$, the $(2, 2m+1)$ torus knot, is $\sum_{i=-m}^{m} (-1)^i t^i$, $CFK^{\infty}(T_{2,2m+1})$ is generated by $\mathrm{St}(1, \cdots, 1)$ of length $2m+1$. The knot Floer complex of $T_{3,4}$ can be given by $\mathrm{St}(1, 2, 2, 1) \otimes \mathbb{F}[U, U^{-1}]$ from the fact that $\Delta_{T_{3,4}}(t) = t^{-3} - t^{-2} + 1 - t^2 + t^3$. See Figure 2.1. Accordingly, it is easily obtained that $\tau(T_{p,q}) = (p-1)(q-1)/2$ from its staircase complex, see also [16, Corollary 1.7.]. In particular, $\tau(T_{2,2m+1})$ is equal to m.

2.6 Knot Surgery and Heegaard Floer homology

Let K be a null-homologous knot in an integer homology 3-sphere Y. Then, there is a canonical Seifert framing on K, giving rise to a simple closed curve λ in the boundary of the tubular neighborhood of the knot K, $\mathrm{nb}(K)$, which meets the meridian μ in a single point.

Given an integer n, the three-manifold $Y_n(K)$ denotes the new manifold obtained by filling $Y - \mathrm{nd}(K)$ with a solid torus so that its meridian maps to $n\mu + \lambda$. We call this procedure an *integral surgery* of Y along K. There is a natural two-handle cobordism W from from Y to $Y_n(K)$, attained by attaching a two-handle to $Y \times [0, 1]$ along K with framing n. The cobordism W allows us to enumerate the torsion Spin^c structures over $Y_n(K)$. Note that the set of Spin^c structures over $Y_n(K)$ has one to one correspondence with $H^2(Y_n(K); \mathbb{Z})$ i.e. $\mathbb{Z}/n\mathbb{Z}$. If $\mathfrak{t}$ is a Spin^c structure of $Y_n(K)$, we assign $\mathfrak{t}$ to $[i] \in \mathbb{Z}/n\mathbb{Z}$ so that it satisfies

$$\langle c_1(\mathfrak{s}), [F]\rangle + n = 2i,$$

where $\mathfrak{s}$ is an extension of $\mathfrak{t}$ over W, and $F \subset W$ is a capped-off Seifert surface, obtained by the union of a fixed Seifert surface of K and the core of the two-handle attached.

One of the key tools to compute Heegaard Floer homology is the integer surgery formula that allows us calculate $HF^+(Y_n(K), [i])$ using the knot Floer homology of (Y, K). Fix a knot Floer complex $C := CFK^\infty(Y, K)$ and a surgery coefficient $n \neq 0$. We write the two factors of the $\mathbb{Z} \oplus \mathbb{Z}$ filtration as (i, j). For each $s \in \mathbb{Z}$, we define A_s^+ to be the quotient complex of C corresponding to the filtration $\max(i, j - s) \geq 0$, and B_s^+ corresponding to $i \geq 0$.

There are two natural chain maps $v_s : A_s^+ \to B_s^+$ and $h_s : A_s^+ \to B_{s+n}^+$. The vertical chain map v_s is induced by the natural projection. We define the horizontal chain map h_s to be projection onto $C\{j \geq s\}$, followed by the multiplication of U^s, followed by the chain homotopy equivalence J from $C\{j \geq 0\}$ to $C\{i \geq 0\}$ (induced from the construction of C).

Denote

$$\mathbb{A}^+ := \bigoplus_{s \in \mathbb{Z}} A_s^+ \quad \text{and} \quad \mathbb{B}^+ := \bigoplus_{s \in \mathbb{Z}} B_s^+,$$

$$
\begin{tikzcd}
\cdots \arrow[drr] & A^+_{-3} \arrow[d, dashed] \arrow[drr, dashed] & A^+_{-2} \arrow[d] \arrow[drr] & A^+_{-1} \arrow[d, dashed] \arrow[drr, dashed] & A^+_{0} \arrow[d] \arrow[drr] & A^+_{1} \arrow[d, dashed] \arrow[drr, dashed] & A^+_{2} \arrow[d] \arrow[drr] & A^+_{3} \arrow[d, dashed] & \cdots \\
\cdots & B^+_{-3} & B^+_{-2} & B^+_{-1} & B^+_{0} & B^+_{1} & B^+_{2} & B^+_{3} & \cdots
\end{tikzcd}
$$

Figure 2.2: A mapping cone complex, $\mathbb{X}^+(2)$.[1]

and let $D^+ : \mathbb{A}^+ \to \mathbb{B}^+$ be the map defined by

$$D^+(\{a_s\}_{s\in\mathbb{Z}}) = \{b_s\}_{s\in\mathbb{Z}}$$

where

$$b_s = h_{s-n}(a_{s-n}) + v_s(a_s).$$

Let $\mathbb{X}^+$ denote the mapping cone complex of D^+, i.e. $\mathbb{X}^+ := \mathbb{A}^+ \oplus \mathbb{B}^+$ and the boundary map is given by

$$\begin{pmatrix} \partial_{\mathbb{A}^+} & 0 \\ D^+ & \partial_{\mathbb{B}^+} \end{pmatrix}.$$

There is well-defined relative $\mathbb{Z}$-grading given to $\mathbb{X}^+(n)$. Suppose $n > 0$ and observe $B^+_s \cong CF^+(Y)$. Write $s = \sigma + l \cdot n$, where $0 \leq \sigma < n$. Then the grading of B^+_s is given by shifting the absolute grading of $CF^+(Y)$ by $2l\sigma + nl(l-1) - 1$. We assign the grading of A_s so that both v_s and h_s are homogeneous map of degree -1. For the grading of $\mathbb{X}^+(-n)$ $(n > 0)$, write $s = -(\sigma + l\cdot n)$ for $0 \leq \sigma < n$. Then the grading of B^+_s is given by shifting the absolute grading of $CF^+(Y)$ by $-2l\sigma - nl(l-1)$, and we extend it to A^+_s similarly.

Note that $\mathbb{X}^+(n)$ splits as a direct sum of complexes $\mathbb{X}^+_i(n)$, $i \in \mathbb{Z}/n\mathbb{Z}$, where $\mathbb{X}^+_i(n)$ consists of A^+_s and B^+_s with $s \equiv i \bmod n$. See Figure 2.2.

[1]Observe that it splits as two summand $\mathbb{X}^+_0$ (containing solid arrows) and $\mathbb{X}^+_1$ (containing dotted arrows).

Finally the following theorem relates homology of the mapping cone complex with Heegaard Floer homology of $Y_n(K)$.

Theorem 2.6.1. [23](Integer surgery formula) *Let K be a null-homologous knot in an integer homology 3-sphere Y. Then, for any non-zero integer n and $[i] \in \mathbb{Z}/n\mathbb{Z}$, $HF^+(Y_n(K), [i])$ is isomorphic to the homology of the mapping cone $\mathbb{X}_i^+(n)$ of*

$$D_i^+ : \mathbb{A}_i^+ \to \mathbb{B}_i^+,$$

and the isomorphism from $\mathbb{X}_i^+(n)$ to $HF^+(Y_n(K), [i])$ is a homogeneous map of degree $d(n, i)$, where

$$d(n, i) = - \max_{\{s \in \mathbb{Z} | s \equi i \, (mod \, n)\}} \frac{1}{4} \left(1 - \frac{(n+2s)^2}{n} \right). \tag{2.4}$$

Moreover, for $k \in \mathbb{Z}$, the natural map induced by inclusion, $H_*(B_k^+) \to H_*(\mathbb{X}_{[k]}^+(n))$ is identified with the map

$$F^+_{W, \mathfrak{s}_k} : HF^+(Y) \to HF(Y_n(K), [k]),$$

where W is the natural 2-handle cobordism from Y to $Y_n(K)$ and $\mathfrak{s}_k$ is the spin^c structure of W satisfying the following equation:

$$\langle c_1(\mathfrak{s}_k), [F] \rangle + n = 2k,$$

where F is the capped-off Seifert surface of K, which is used for $[i]$.

Chapter 3

Iterated Whitehead doubles in the knot concordance group

The main goal of this chapter is to discuss how iterated Whitehead double knots are independent in the smooth knot concordance group. In Section 3.1, we give background and motivation of the problem and state our main results. We devote the next two sections to prove main theorems. In Section 3.4, we discuss how to generalize the theorems to broader classes of knots.

3.1 The knot concordance group and Whitehead doubles

A knotted circle K in S^3 is called *smoothly* (resp. *topologically*) *slice* if it bounds a smoothly (locally flat) embedded disk in B^4. Two knots K_1 and K_2 are called *smoothly* (*topologically*) *concordant* if $K_1 \# -K_2$ is smoothly (topologically) slice, where $-K$ is the mirror of K with reversed orientation. Modulo smooth concordance, the set of knots forms an abelian group, *the* (*smooth*) *knot concordance group*, $\mathcal{C}$. Note that every smoothly slice knot is topologically slice, but the converse is not true. We let $\mathcal{C}_{TS}$ be the subgroup of $\mathcal{C}$ generated by topologically slice knots.

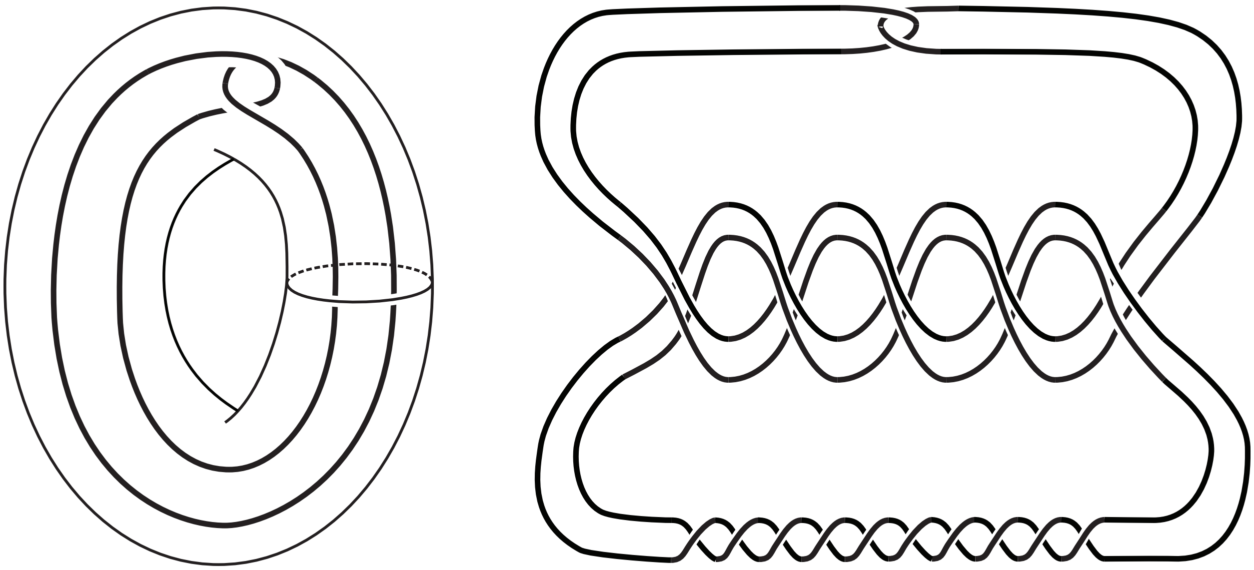


Figure 3.1: The pattern of the positively-clasped untwisted Whitehead double and the Whitehead double of the (2, 5) torus knot.[1]

The Whitehead double (positively-clasped untwisted) of a knot K, $D(K)$, is a satellite knot defined by the pattern in Figure 3.1. Whitehead doubles are interesting classes of knots in the study of the knot concordance. Any class of the Whitehead double of a knot is contained in $\mathcal{C}_{TS}$ since it has the same Alexander-Conway polynomial as the unknot and hence is topologically slice by a result of Freedman [4]. However, many of them are not smoothly slice, which show remarkable distinction between the smooth and topological categories in dimension four. It is thus important to understand their concordance properties as portrayed in the knot concordance group. It is also interesting to ask about the effect of D on $\mathcal{C}$, and there is a long-standing conjecture.

Conjecture 3.1.1. [11, Problem 1.38] *$D(K)$ is smoothly slice if and only if K is smoothly slice.*

Note that it is still unknown, as far as the author knows, if the conjecture is true even for some simple knots such as left-hand trefoil or figure-eight knot. One could study Whitehead

[1]The -5 extra full twists arise from untwisting the writhe of the projection of the (2, 5) torus knot.

doubles in $\mathcal{C}$ using homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$. The knot signature σ gives one such homomorphism. Unfortunately the signature, indeed any invariant of topological concordance group, is not effective homomorphism for Whitehead double knots, since it vanishes for these knots [4]. Heegaard Floer theory provides manifestly smooth concordance invariants, and some of which give homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$. One is the τ-invariant, defined using the knot Floer homology of Ozsváth-Szabó and Rasmussen [16, 20, 25]. Manolescu-Owens discovered another concordance invariant δ, twice the Heegaard Floer correction term (d-invariant) of the double cover of S^3 branched over a knot [14]. More recently, Peters studied another concordance invariant $d(S^3_1(K))$ given by the correction term of 1-surgery on $K \subset S^3$ [24]. In contrast to the other two invariants, dS^3_1 does not induce a homomorphism to $\mathbb{Z}$. See the survey paper [10] of Jabuka for some applications of Heegaard Floer theory to the concordance group. Rasmussen's s-invariant coming from Khovanov homology is also a powerful concordance homomorphism [26]. It is known that $-\sigma/2 = \tau = \delta = -s/2$ for alternating knots, but they differ in general: see [8], [14] and [13].

Even though there are many concordance invariants developed, most of them are inefficient for distinguishing Whitehead doubles in $\mathcal{C}$. The invariants $|\tau|$, $-dS^3_1/2$ and $|s/2|$ are known to be bounded above by the slice genus (four-ball genus) of the knot: the minimal genus of smoothly embedded surface in the 4-ball bounded by $K \subset \partial(B^4)$. Since the slice genus of $D(K)$ is at most one for any knot K, so are $|\tau|$, $-dS^3_1/2$ and $|s/2|$ of $D(K)$. Moreover, $\tau(D(K))$ is determined by $\tau(K)$ followed by the Theorem of Hedden below.

Theorem 3.1.2. [5, Theorem 1.5]

$$\tau(D(K)) = \begin{cases} 0, & \textit{for} \;\; \tau(K) \leq 0 \\ 1, & \textit{for} \;\; \tau(K) > 0 \end{cases}$$

In particular, $\tau(D^n(K))$ is identically either 0 or 1 for any $n \geq 1$ and is determined by $\tau(K)$, where $D^n(K)$ denote the nth iterated positively clapsed untwisted Whitehead double of K. Therefore, it is interesting to ask if it is possible to distinguish the $D^n(K)$'s in $\mathcal{C}$. Using δ-invariants, which are not constrained by the slice genus, we show:

Theorem 3.1.3. *For each $m \geq 2$, $D(T_{2,\,2m+1})$ and $D^2(T_{2,2m+1})$ are not smoothly concordant. In fact, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.*

In [14] it was computed that $\delta(D(T_{2,\,2m+1})) = -4m$. See also Section 3.4.2. Here, we show that $\delta(D^2(T_{2,\,2m+1})) = -4$ for any $m \geq 1$. A tool for our results is a computation of the infinity version of the knot Floer chain complex of $D(T_{2,\,2m+1})$:

Theorem 3.1.4. *For any $m \geq 1$, the chain complex $CFK^\infty(D(T_{2,\,2m+1}))$ is $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain homotopy equivalent to the chain complex $CFK^\infty(T_{2,\,3}) \oplus A$, where A is an acyclic complex.*

More generally, we have the following:

Theorem 3.1.5. *Suppose K is a knot in S^3. If $|\delta(D(K))| > 8$, then $D(K)$ and $D^n(K)$ are not smoothly concordant for each $n \geq 2$. If, in addition, $\tau(K) > 0$, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.*

Corollary 3.1.6. *Suppose K is an alternating knot in S^3. If $\tau(K) > 2$, then $D(K)$ and $D^n(K)$ are not smoothly concordant for each $n \geq 2$. In fact, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.*

Additionally, for some classes of knots including (p, q) torus knots, we give an algorithmic formula for testing it in terms of its Alexander-Conway polynomial along with an implementation of a computer program.

Recently, Cochran-Harvey-Horn suggested a bipolar filtration of $\mathcal{C}$ and the induced filtration of $\mathcal{C}_{TS}$ [3],

$$\{0\} \subset \cdots \subset \mathcal{T}_{n+1} \subset \mathcal{T}_n \subset \cdots \subset \mathcal{T}_0 \subset \mathcal{T} = \mathcal{C}_{TS}.$$

Since τ of $D(K)$ and $D^2(K)$ are nonzero for knots K in Theorem 3.1.3 and 3.1.5, both of them are contained in $\mathcal{T}/\mathcal{T}_0$ by [3, Corollary 4.9]. Therefore, their filtration cannot see the difference between $D(K)$ and $D^2(K)$. On the other hand, using those knots, we get the following corollary relating to the filtration. Let $\mathcal{C}_\Delta$ be the subgroup of $\mathcal{C}_{TS}$ generated by knots with trivial Alexander-Conway polynomial.

Corollary 3.1.7. *There is a $\mathbb{Z}^2$ summand of $\mathcal{C}_\Delta/\mathcal{C}_\Delta \cap \mathcal{T}_0$.*

Remark 3.1.8. Recently, in [9] Hom showed there is $\mathbb{Z}^\infty$ summand of $\mathcal{C}_{TS}$, but her technique cannot see the difference between the iterated Whitehead doubles in $\mathcal{C}$ either.

3.2 Infinity version of knot Floer complex of $D(T_{2,\,2m+1})$

Recently, Hedden-Kim-Livingston showed that $CFK^\infty(D(T_{2,\,3}))$ is chain homotopy equivalent to $CFK^\infty(T_{2,\,3}) \oplus A$ for some acyclic complex A, [6, Proposition 6.1.]. Also see [3, section 9.1]. Here, we will prove that the result can be generalized to the torus knots $T_{2,\,2m+1}$ for $m \geq 1$, and furthermore $CFK^\infty(D(T_{2,\,2m+1}))$ will be completely determined.

Before proving the theorem, recall the following useful lemma regarding how a basis change in a filtered chain complex over $\mathbb{F}$ affects the diagram of a knot Floer chain complex.

Lemma 3.2.1. [6, Lemma A.1.] *Let C_* be a knot Floer complex with a 2-dimensional arrow diagram D given by an $\mathbb{F}$-basis. Suppose that x and y are two basis elements of the same grading such that each of the i and j filtrations of x is not greater than that of y. Then the $\mathbb{Z} \oplus \mathbb{Z}$ filtered basis change given by $y' = y + x$ gives rise to a diagram D' of C_* which differs from D only at y and x as follows:*

- *Every arrow from some z to y in D adds an arrow from z to x in D'*
- *Every arrow from x to some w in D adds an arrow from y' to w in D'*

We use the above lemma for the purpose of removing certain boundary arrows in chain complexes over $\mathbb{F}$. For example, the proposition below will be useful for proving Theorem 3.1.4.

Proposition 3.2.2. *Suppose C is one of the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complexes over $\mathbb{F}$ given by the diagrams in Figure 3.2 with any possible combination of dotted arrows. Then all dotted arrows can be removed by a basis change.*

Proof. First, consider the complex (I). Suppose that

$$\partial a = b + c + Ax + By,$$

$$\partial b = d + Cz,$$

$$\text{and } \partial c = d + Dz$$

for some A, B, C and D in $\mathbb{F}$. Since $\partial^2 = 0$,

$$\begin{aligned} 0 = \partial^2 a &= \partial(b + c + Ax + By) \\ &= (A + B + C + D)z. \end{aligned} \tag{3.1}$$

Therefore, the coefficients have to satisfy the equation that $A + B + C + D$=0. Now, we consider every possible coefficient of A, B, C and D in $\mathbb{F}$ satisfying the equation and show that each case can be transformed to have A=B=C=D=0, as desired, after proper change of basis.

- If A=B=1 and C=D=0, change the basis by b'=$b + x$, c'=$c + y$ and d'=$d + z$.
- If A=C=1 and B=D=0, change the basis by b'=$b + x$.
- If A=D=1 and B=C=0, change the basis by b'=$b + x$ and d'=$d + z$.
- If B=C=1 and A=D=0, change the basis by c'=$c + y$ and d'=$d + z$.
- If B=D=1 and A=C=0, change the basis by c'=$c + y$.
- If C=D=1 and A=B=0, change the basis by d'=$d + z$.
- If A=B=C=D=1, change the basis by b'=$b + x$ and c'=$c + y$.

Similar argument is applied to remove any combination of possible dotted arrows in the complexes (II) and (III). □

Proof of Theorem 4. Let D be $D(T_{2,\,2m+1})$ for $m \geq 1$. Theorem 1.2 of [5] together with the

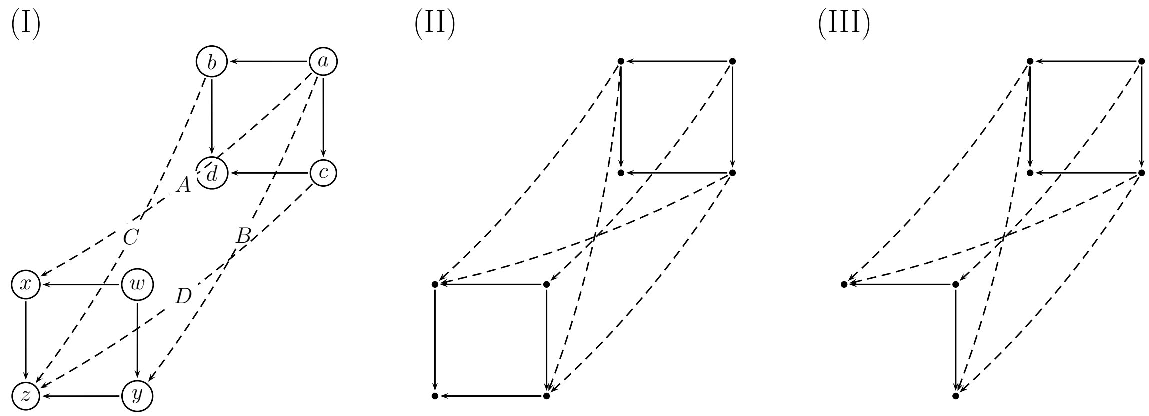


Figure 3.2: Any possible combination of dotted arrows can be removed by a basis change.

computation of $\widehat{CFK}(T_{2,\,2m+1})$ shows that

$$\widehat{HFK}_*(D,\,j) = \begin{cases} \mathbb{F}^{2m}_{(0)} \oplus \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-3)} \oplus \cdots \oplus \mathbb{F}^2_{(-2m+1)}, & j=1 \\ \mathbb{F}^{4m-1}_{(-1)} \oplus \mathbb{F}^4_{(-2)} \oplus \mathbb{F}^4_{(-4)} \oplus \cdots \oplus \mathbb{F}^4_{(-2m)}, & j=0 \\ \mathbb{F}^{2m}_{(-2)} \oplus \mathbb{F}^2_{(-3)} \oplus \mathbb{F}^2_{(-5)} \oplus \cdots \oplus \mathbb{F}^2_{(-2m-1)}, & j=-1 \\ 0 & otherwise. \end{cases}$$

We assign an $\mathbb{F}$-basis to each summand in the direct decomposition as below:

$$\widehat{HFK}_*(D,\,j) = \begin{cases} \langle x^0_1,\,\cdots,\,x^0_{2m}\rangle \oplus \langle u^{-1}_{1,1},\,u^{-1}_{1,2}\rangle \oplus \cdots \oplus \langle u^{-2m+1}_{m,1},\,u^{-2m+1}_{m,2}\rangle & j=1 \\ \langle y^{-1}_1,\,\cdots,\,y^{-1}_{4m-1}\rangle \oplus \langle v^{-2}_{1,1},\cdots,\,v^{-2}_{1,4}\rangle \oplus \cdots \oplus \langle v^{-2m}_{m,1},\cdots,\,v^{-2m}_{m,4}\rangle & j=0 \\ \langle z^{-2}_1,\,\cdots,\,z^{-2}_{2m}\rangle \oplus \langle w^{-3}_{1,1},\,w^{-3}_{1,2}\rangle \oplus \cdots \oplus \langle w^{-2m-1}_{m,1},\,w^{-2m-1}_{m,2}\rangle & j=-1 \\ 0 & otherwise, \end{cases}$$

where the superscript of a generator represents its absolute grading.

Since $\widehat{HFK}_*(D)$ is homotopic equivalent to the $\widehat{CFK}(D)$ [25, Lemma 4.5], we assume that $CFK^\infty(D)_{\{(0,j)\}} = \widehat{HFK}_*(D, j)$ and $CFK^\infty(D)_{\{(i,j)\}} \cong U^{-i}CFK^\infty(D)_{\{(0,j)\}} = \widehat{HFK}_*(D, j-i)$. Now, we investigate all differentials in $CFK^\infty(D)$ by using the facts, $\partial^2 = 0$, $H_*(\widehat{CFK}(D)) \cong \widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$ and $H_*(CFK^\infty(D)) \cong HF^\infty(S^3) \cong \mathbb{F}[U, U^{-1}]$.

First, note that there are no components of the boundary maps between generators of the same (i, j)-filtration since they would reduce $\widehat{HFK}_*(D)$. Thus, we can decompose the boundary maps ∂ to the vertical, horizontal and diagonal components, $\partial = \partial_V + \partial_H + \partial_D$. Also, we remark that it is enough to determine boundary maps of $\mathbb{F}[U, U^{-1}]$-generators in CFK^∞ because the boundary map is U-equivariant.

Exactly the same argument as [6, Proposition 6.1] can be used to determine ∂_V and ∂_H i.e. by the fact that $H_*(\widehat{CFK}(D)) = \mathbb{F}_{(0)}$ and using grading consideration, after changing basis, we can assume that

$$\begin{aligned}
&\partial_V(x_k^d) = y_{k-1}^{d-1} && \text{for } k = 2, \cdots, 2m\\
&\partial_V(y_{2m+l-1}^{d-1}) = z_l^{d-2} && \text{for } l = 1, \cdots, 2m.\\
&\partial_V(u_{p,i}^{d-1}) = v_{p,i}^{d-2} \text{ and } \partial_V(v_{p,i+2}^{d-2}) = w_{p,i}^{d-3} && \text{for } p = 1, \cdots, m \text{ and } i = 1, 2,
\end{aligned}$$

for $d \in 2\mathbb{Z}$ and ∂_V's of other elements are trivial. Analogously, since $H_*(CFK^\infty(D)_{\{j=0\}})$ is isomorphic to $\widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$ and $\partial^2 = 0$, the horizontal boundary components can be assumed as following:

$$\begin{aligned}
&\partial_H(z_k^d) = y_{k-1}^{d-1} && \text{for } k = 2, \cdots, 2m\\
&\partial_H(y_{2m+l-1}^{d-1}) = x_l^{d-2} && \text{for } l = 1, \cdots, 2m.\\
&\partial_H(w_{p,i}^{d-1}) = v_{p,i}^{d-2} \text{ and } \partial_H(v_{p,i+2}^{d-2}) = u_{p,i}^{d-3} && \text{for } p = 1, \cdots, m \text{ and } i = 1, 2,
\end{aligned}$$

and ∂_H's of other elements are trivial. We drop $\mathbb{F}[U, U^{-1}]$ coefficients of generators since they are canonically determined by the grading consideration. See Figure 3.3 for a diagram. In fact we will show that we can assume there are no ∂_D components for any elements as shown in Figure 3.3.

We can split $CFK^\infty(D)$ as following disjoint subsets:

$$
\begin{aligned}
&A^d_{p,\,i} := \{v^d_{p,\,i+2},\, u^{d-1}_{p,\,i},\, w^{d-1}_{p,\,i},\, v^{d-2}_{p,\,i}\}\\
&B^d_q := \{y^{d-1}_{2m+q},\, x^{d-2}_{q+1},\, z^{d-2}_{q+1},\, y^{d-3}_q\}\\
&\text{and } C^d := \{y^{d-1}_{2m},\, x^{d-2}_1,\, z^{d-2}_1\},
\end{aligned}
$$

for $1 \leq p \leq m$, $1 \leq q \leq 2m-1$, $i = 1,\, 2$ and $d \in 2\mathbb{Z}$. Note that any arrows between subsets must be diagonal. Disregarding the diagonal arrows between subsets, each complex of A's and B's has four generators with square-shaped grid diagram, and each complex of C's has three generators which looks like $St(1,1)$. Therefore, if we remove all arrows between subsets i.e. $CFK^\infty(D)$ is a direct sum of A's, B's and C's, the theorem follows.

Define a subset of A as $A'^d_{p,\,i} := \{v^d_{p,\,i+m},\, u^{d-1}_{p,\,i},\, w^{d-1}_{p,\,i}\}$. Due to grading constraints on the filtered complex, we observe the following:

- ∂_D of any generator in $A'^d_{p,\,i}$ has components only in $A^d_{k,\,j}$, B^d_q and C^d for $k < p$, $j = 1,\, 2$, and $q = 1, \cdots, 2m-1$ (i.e. diagonal arrows between A's going from higher to smaller first index.)

- ∂_D of the generators in B's and C's are zero.

These observations allow us to apply Corollary 3.2.2 inductively to remove all diagonal arrows in the complex.

We start to remove any diagonal arrows from $A'^d_{m,\,1}$. First, we remove all diagonal arrows

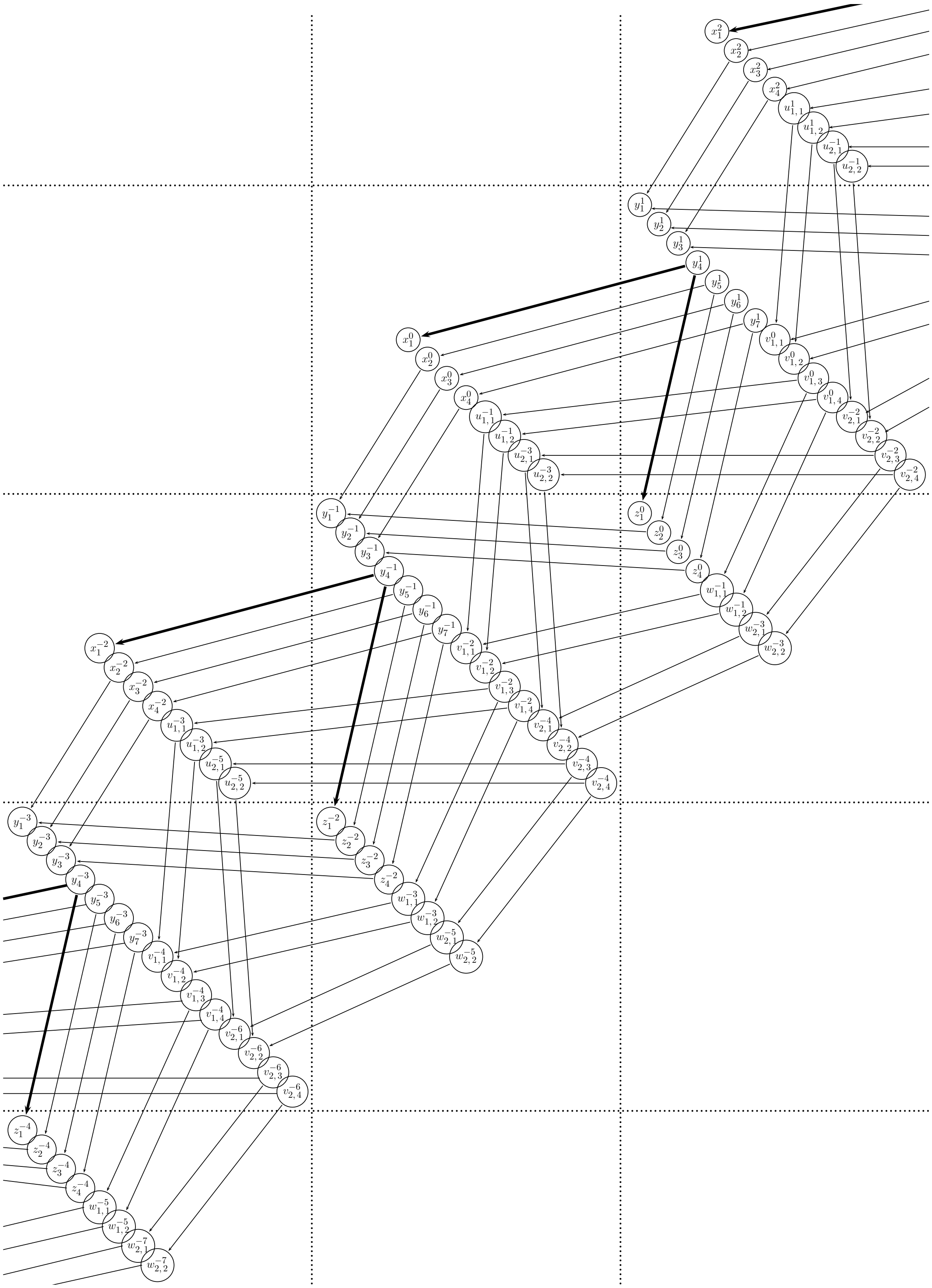


Figure 3.3: A diagram of $CFK^{\infty}(D(T_{2,5}))$.

going from $A'^d_{m,1}$ to $A^d_{m-1,1}$, using basis-change of the case (I) of Corollary 3.2.2. Differently from the corollary, there can be other components in ∂_D of elements in $A'^d_{m,1}$, not in $A^d_{m-1,1}$. However, considering the grading constraints again, one can easily check that the other components cannot induce z component of $\partial^2 a$ in the Equation (3.1), hence the equation that $A + B + C + D = 0$ in the proof of the corollary still holds and we remove diagonal arrows using a basis-change in the corollary.

After applying the basis-change, two types of newer diagonal arrows will be added due to arrows coming to $A'^d_{m,1}$ and arrows going from $A^d_{m-1,1}$. First note that there are no diagonal arrows coming to $A^d_{m,1}$ by the observations above (m is the greatest index for A.) Secondly, a diagonal arrow from $A^d_{m-1,1}$ to some generator adds an arrow going from $A^d_{m,1}$ to the generator after basis-change, but note that these arrows are going to the subsets $A^d_{p,i}$ with $p < m-1$ and $i = 1, 2$, which we will remove later.

Now, we similarly change the basis for removing diagonal arrows from $A'^d_{m,1}$ to $A^d_{m-1,2}$, $A^d_{m-2,1}$, $A^d_{m-2,2} \cdots$, $A^d_{1,1}$, and $A^d_{1,2}$ in sequence. Then, case (II) and (III) of the corollary will be applied to remove arrows from $A'^d_{m,1}$ to B^d_q's and C^d. The induction ends with removing any ∂_D from $A'^d_{m,1}$, since there are no diagonal arrows from $A^d_{1,i}$'s, B^d_q's and C^d.

Then, we remove ∂_D of $A'^d_{m,2}$, $A'^d_{m-1d,1}$, $A'^d_{m-1,2}, \cdots$, $A'^d_{1,1}$, and $A'^d_{1,2}$ likewise. After removing the diagonal arrows from $A'_{p,i}$ for all $p = 1, \cdots, m$ and $i = 1, 2$, the only remaining non-trivial ∂_D are ones of $v_{p,1}$ and $v_{p,2}$. It is easy to see that ∂_D's of $v_{p,1}$ and $v_{p,2}$ also vanish: $0 = \partial^2(u_{p,i}) = \partial(v_{p,i})$. Thus, we may assume that ∂_D's of CFK^∞ are all zero. $\square$

3.3 δ-invariant of $D^2(T_{2,2m+1})$ and proof of Theorem 3.1.3

First, we present a lemma that relates the δ-invariant of a Whitehead double to dS^3_1.

Lemma 3.3.1. *For any knot K, $\delta(D(K)) = 2dS^3_1(K\#K^r)$.*

Proof. Let K_p denote the 3-manifold obtained by p-surgery of S^3 along a knot K. Manolescu-Owens showed that $d(K_{-1/2}) = d(K_{-1})$ for any knot in the proof of [14, Proposition 6.2]; in fact both of them equal $2h_0(K)$, where $h_0(K)$ is an invariant defined by Rasmussen in [25]. Thus, using the behaviour of d-invariants under orientation reversal [15, Propostion 4.2], we have

$$\begin{aligned} d(K_{1/2}) &= -d(\overline{K}_{-1/2}) \\ &= -d(\overline{K}_{-1}) \\ &= d(K_1), \end{aligned}$$

where $\overline{K}$ is the mirror image of K.

Recall that the double cover of S^3 branched over $D(K)$, $\Sigma(D(K))$, can be obtained by $1/2$-surgery along $K\#K^r$ in S^3, where K^r is the knot K with its orientation reversed, see [14, Proposition 6.1]. From the definition of δ-invariant,

$$\begin{aligned} \delta(K) &:= 2d(\Sigma(K)) \\ &= 2d((K\#K^r)_{1/2}) \\ &= 2d((K\#K^r)_1). \end{aligned}$$

$\square$

Proof of Theorem 3.1.3. By applying the lemma above to $D^2(T)$, we have

$$\delta(D^2(T)) = 2d((D(T)\#D(T)^r)_1),$$

where $T = T_{2,\,2m+1}$.

To compute $d(K_1)$, it suffices to understand $CFK^\infty(K)$ [24]. Let us recall the algorithm.

Pick ξ, a generator of $H_*(CFK^\infty(K)_{\{i=0\}}) \cong \mathbb{F}$. Note that any generator become zero in $CFK^\infty_{\{i\geq 0 \text{ or } j\geq 0\}}$ by the multiplication by high enough power of U. Then

$$d(K_1) = -2\min\{n \geq 0 : [U^{n+1}\xi] = 0 \in H_*(CFK^\infty(K)_{\{i\geq 0 \text{ or } j\geq 0\}})\}. \tag{3.2}$$

Observe that $d(K_1)$ is derived from a direct summand of $CKF^\infty(K)$ containing a generator of $H_*(CFK^\infty(K)_{\{i=0\}}) \cong \mathbb{F}$. In particular, if $CFK^\infty(K)$ and $CFK^\infty(K')$ are differ only by an acyclic complex, then $d(K_1) = d(K_1')$.

Now, let us understand $CFK^\infty(D(T)\#D(T)^r)$. Since the knot Floer complex is unchanged under the orientation reversal [20, Proposition 3.9.] and by the connected sum formula for knot Floer complexes in [20, Theorem 7.1.],

$$CFK^\infty(D(T)\#D(T)^r) \cong CFK^\infty(D(T)) \otimes CFK^\infty(D(T)).$$

Thus, $d((D(T)\#D(T)^r)_1)$ equals to $d((T_{2,3}\#T_{2,3})_1)$ by Theorem 3.1.4. It is computed that $d((T_{2,3}\#T_{2,3})_1) = -2$ as an example of the computer program, dCalc in [24], (it can also be computed by Proposition 3.4.2) so that $\delta(D^2(T)) = -4$. On the other hand, $\delta(D(T)) = -4m$ from the computation of $\tau(T)$ stated in Section 2.5 and the fact that $\delta(D(K)) = -4\max\{\tau(K), 0\}$ for alternating knot K [14, Theorem 1.5.]. The first part of Theorem 3.1.3 follows.

Recall that $\delta \equiv \sigma/2 \mod 4$ [14, (2.1)] and $\sigma = 0$ for any knot in $\mathcal{C}_{TS}$. Consider the homomorphism $\psi = (\tau, \delta/4) : \mathcal{C}_{TS} \to \mathbb{Z} \oplus \mathbb{Z}$. Since $\psi(D(T)) = (1, -m)$ and $\psi(D^2(T)) = (1, -1)$, ψ is surjective if $m \geq 2$. Therefore, $\mathcal{C}_{TS}$ has a $\mathbb{Z}^2$ summand generated by $D(T)$ and $D^2(T)$.

$\square$

Proof of Corollary 3.1.7. By [3, Corollary 4.9, Corollary 6.11] both τ and δ vanish for the knots in $\mathcal{C}_\Delta \cap \mathcal{T}_0$. Now, consider the induced homomorphism $(\tau, \delta/4) : \mathcal{C}_\Delta/\mathcal{C}_\Delta \cap \mathcal{T}_0 \to \mathbb{Z} \oplus \mathbb{Z}$, and the subjectivity of it can be shown by the knots, $D(T)$ and $D^2(T)$. $\square$

3.4 Generalization of the result

In this section we discuss how to generalize the result for $T_{2,2m+1}$ to other knots. First, we use a genus-bound property of the concordance invariant dS^3_1 to show that, provided that $|\delta(D(K))| > 8$, $D(K)$ and $D^n(K)$ are not smoothly concordant for each $n \geq 2$. Secondly, we present formulas to compute $dS^3_1(K)$ and $\delta(D(K)) = 2dS^3_1(K\#K)$ for a given staircase complex of K introduced in Section 2.5.

3.4.1 Genus bound and proof of Theorem 3.1.5

Proof of Theorem 3.1.5. It is shown in [24, Theorem 1.5.] that $-dS^3_1(K)/2$ is a lower bound for the slice genus, $g^*(K)$, of K, and note that the slice genus of $D(K)$ is at most 1 for any knot K. Hence, for $n \geq 2$,

$$-\delta(D^n(K)) = -2dS^3_1(D^{n-1}(K)\#D^{n-1}(K)^r) \leq 4g^*(D^{n-1}(K)\#D^{n-1}(K)^r) \leq 8.$$

Therefore, if $\delta(D(K)) < -8$, equivalently $dS^3_1(K\#K) < -4$, $D(K)$ and $D^n(K)$ are not smoothly concordant. (According to [14, Theorem 1.5], $\delta(D(K))$ is nonpositive for any knot K.)

If $\tau(K) > 0$, both $\tau(D(K))$ and $\tau(D^n(K))$ are 1 by Theorem 3.1.2. Now one can prove the second part of the theorem by considering the surjective homomorphism $(\tau, \delta/4) : \mathcal{C}_{TS} \to$

$\mathbb{Z} \oplus \mathbb{Z}$. □

Proof of Corollary 3.1.6. This is obtained by Theorem 3.1.5 together with Theorem [14, Theorem 1.5.]. □

Remark 3.4.1. Since $\delta(D(T_{2,\,2m+1})) = -4m$, Theorem 3.1.3 is a special case of Theorem 3.1.5 for $m \geq 3$, but we have shown it for the case $m = 2$ as well by computing $\delta(D^2(T_{2,\,2m+1})) = -4$.

3.4.2 $dS^3_1(K)$ and $\delta(D(K))$ of a staircase complex

If a knot admits a knot Floer complex generated by a staircase complex (equivalently, L-space knots), then its d-invariant can be easily obtained. For the d-invariants of higher surgery coefficients, we refer to [1, Section 4.2.].

Proposition 3.4.2. *Suppose the knot Floer complex of K can be given by a staircase complex $St(K)$, then*

$$d(S^3_1(K)) = -2 \min_{(i,\,j)\in \mathrm{Vert}(\mathrm{St}(K))} \max\{i,\, j\}$$

$$\delta(D(K))/2 = d(S^3_1(K\#K)) = -2 \min_{(i,\,j),\,(k,\,l)\in \mathrm{Vert}(\mathrm{St}(K))} \max\{i+k,\, j+l\},$$

where $\mathrm{Vert}(\mathrm{St}(K))$ *is the set of the coordinates of the generators of* $\mathrm{St}(K)$.

Proof. Suppose that $CFK^\infty(K)$ is generated by $\mathrm{St}(K)$, then the top left element in $\mathrm{St}(K)$ represents the generator of $H_*(CFK^\infty(K)_{\{i=0\}}) \cong \mathbb{F}$: say ξ. The chain complex $\mathrm{St}(K)$ has the form $0 \to \mathbb{F}^k_{(+1)} \to \mathbb{F}^{k+1}_{(0)} \to 0$. Observe that any non-trivial generator η of $\mathbb{F}^{k+1}_{(0)}$ is homologous to ξ. For η with $(i,\, j)$-coordinates, $U^{\max\{i,\,j\}+1}\eta$ lies in the subcomplex $CFK^\infty(K)_{\{i<0 \text{ and } j<0\}}$, whereas $U^{\max\{i,\,j\}}\eta$ does not. Note also that since $\mathrm{St}(K)$ is a

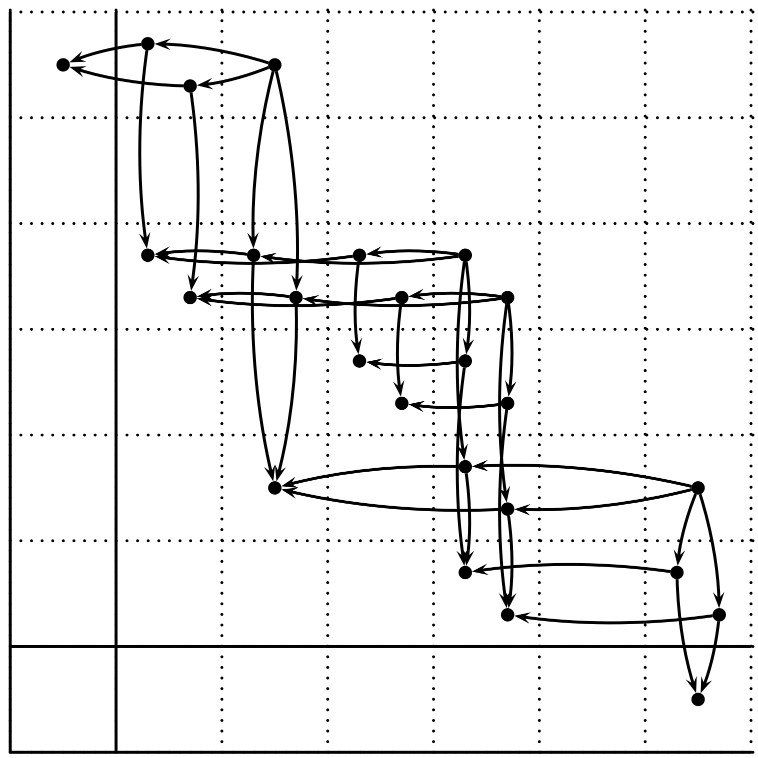


Figure 3.4: A grid diagram of the complex $\mathrm{St}(T_{3,\,4}) \otimes \mathrm{St}(T_{3,\,4})$.

$\mathbb{Z} \oplus \mathbb{Z}$-filtered complex, $\min_{(i,\,j) \in \mathrm{Vert}(\mathrm{St}(K))} \max\{i,\, j\}$ is realized by the elements in $\mathbb{F}^{k+1}_{(0)}$, not ones in $\mathbb{F}^{k}_{(+1)}$. Hence, the first formula follows from the Equation (3.2).

Although $CFK^{\infty}(K\#K)$ is not generated by a staircase complex, it can be constructed from the tensor complex, $\mathrm{St}(K) \otimes \mathrm{St}(K)$ by the connected-sum formula [20, Theorem 7.1]. The coordinates of the generators in $\mathrm{St}(K) \otimes \mathrm{St}(K)$ are given by the sums of a pair of coordinates of the generators of $\mathrm{St}(K)$. The complex $\mathrm{St}(K) \otimes \mathrm{St}(K)$ has the form $0 \to \mathbb{F}^{k^2}_{(+2)} \to \mathbb{F}^{2k(k+1)}_{(+1)} \to \mathbb{F}^{(k+1)^2}_{(0)} \to 0$, and the generators with (0)-grading are homologous to the generator of $H_*(CFK^{\infty}(K\#K)_{\{i=0\}}) \cong \mathbb{F}$. See Figure 3.4 for the tensor complex of two copies of St(1, 2, 2, 1). Therefore, we get the second formula similarly. $\square$

For example, since $\mathrm{St}(T_{2,\,2m+1}) = (1, \cdots, 1)$ of length $2m+1$, one can compute that $\delta(D(T_{2,\,2m+1})) = -4m$ again. In the case of (3, 4) torus knot, $\mathrm{St}(T_{3,\,4}) = (1,\, 2,\, 2,\, 1)$, and so

$$\mathrm{Vert}(T_{3,4}) = \{(0, 3), (1, 3), (1, 1), (3, 1), (3, 0)\}.$$

Thus $\delta(D(T_{3,4})) = -8$, and hence we cannot figure out if $D(T_{3,4})$ and $D^2(T_{3,4})$ are concordant, using Theorem 3.1.5. Note that there are many knots such that $\delta(D(K)) = -8$: for example, any knot K whose CFK^∞ is generated by $\mathrm{St}(1, n, n, 1)$. However, one can similarly apply the arguments in Section 3.2 to show $CFK^\infty(D(T_{3,4})) \cong CFK^\infty(T_{2,3}) \oplus A$ for some acyclic complex A. Therefore, $\delta(D^2(T_{2,3})) = -4$, and we conclude that $D(T_{3,4})$ and $D^2(T_{3,4})$ are not concordant.

For the right-handed trefoil knot, as far as the author knows, all concordance invariants of $D(T_{2,3})$ and $D^2(T_{2,3})$ are same, so it is still mysterious if $D(T_{2,3})$ and $D^2(T_{2,3})$ are smoothly concordant.

Conjecture 3.4.3. *$D(T_{2,3})$ and $D^2(T_{2,3})$ are not smoothly concordant.*

Remark 3.4.4. This conjecture is possibly approached using gauge-theoretic invariants. See [7].

Implementation

We wrote a C++ program computing $dS^3_1(K)$ and $\delta(D(K))$ for a (p, q) torus knot or a staircase complex of K. You may download the source file in the author's webpage. In order to algorithmically obtain the staircase complex (equivalently Alexander-Conway polynomial) of a (p, q) torus knot in the program, we used the subsemigroup of $\mathbb{N}$ generated by p and q, see [2].

Chapter 4

Correction terms of circle bundles over oriented surfaces

In this chapter we discuss the correction terms (or d-invariants) of non-trivial circle bundles over oriented surfaces with coefficients in $\mathbb{F}\cong\mathbb{Z}/2\mathbb{Z}$. Let $Y(g, n)$ denote the circle bundle over genus g oriented surface with Euler number n. Earlier, in [15], Ozsváth and Szabó computed the bottom d-invariant of $Y(g, n)$ for higher Euler number n with respect to g, and reproved the Thom conjecture as a consequence. In [24] Peters studied the top and bottom d-invariants of $Y(g, n)$ when $n = \pm 1$. Our goal here is to completely calculate top and bottom d-invariants of $Y(g, n)$ for any $g > 0$ and $n \neq 0$ and discuss some applications of them.

Observe that the manifold $Y(g, n)$ can be obtained from n-framed surgery on $\#^{2g}S^2\times S^1$ along the Borromean knot K (see Figure 4.1). Hence there is a natural two-handle cobordism W from $\#^{2g}S^2\times S^1$ to $Y(g, n)$, obtained by attaching a two-handle to $(\#^{2g}S^2\times S^1)\times I$ along K with framing n. After fixing a Seifert surface of K, we enumerate a torsion spin^c-structure $\mathfrak{t}$ over $Y(g, n)$ to $[i] \in \mathbb{Z}/n\mathbb{Z}$ by the way in Section 2.6.

Proposition 4.0.5. *The manifold $Y(g, n)$ has standard HF^∞ if $n \neq 0$.*

Proof. Suppose $n < 0$. Let W be the cobordism above and $\mathfrak{s}$ be a spin^c structure of W whose restriction to $\#^{2g}S^2 \times S^1$ and $Y(g, n)$ are $\mathfrak{t}_0$ (the unique torsion spin^c-structure over $\#^{2g} S^2 \times S^1$) and $[i]$ respectively. Then since W has negative definite intersection form, it follows from Ozsváth-Szabó [15, Proposition 9.4.] that the induced map $F^{\infty}_{W,\mathfrak{s}}$ is an isomorphism:

$$F^{\infty}_{W,\mathfrak{s}}\colon HF^{\infty}(\#^{2g} S^2 \times S^1,\, \mathfrak{t}_0) \xrightarrow{\cong} HF^{\infty}(Y(g,\, n),\, [i]),$$

where $\mathfrak{t}_0$ is the unique torsion Spin^c structure over $\#^{2g} S^2 \times S^1$. Indeed, after identifying $H_1(Y(g,\, n);\, \mathbb{Z})/\mathrm{Tors}$ with $H_1(\#^{2g} S^2 \times S^1;\, \mathbb{Z})$ so that the corresponding elements are homologous in W, $F^{\infty}_{W,\mathfrak{s}}$ is a $\Lambda^*(H_1/\mathrm{Tors}) \otimes \mathbb{Z}[U]$-module isomorphism.

If n is positive, by the duality property of Heegaard Floer homology under orientation reversal (note that $-Y(g,\, n) \cong Y(g,\, -n)$), we can show that $Y(g,\, n)$ has also standard HF^{∞} for each torsion Spin^c structure. Alternatively, we may consider the cobordism $-W$ from $Y(g,\, n)$ to $\#^{2g} S^2 \times S^1$. $\square$

For $n < 0$, we have the following commutative diagram:

$$\begin{array}{ccccc}
HF^{\infty}(\#^{2g} S^2 \times S^1,\, \mathfrak{t}_0) & \xrightarrow{\pi_*} & HF^{+}(\#^{2g} S^2 \times S^1,\, \mathfrak{t}_0) & \xleftarrow{=} & H_*(B^+) \\
\Big\downarrow F^{\infty}_{W,\mathfrak{s}} = \cong & & \Big\downarrow F^{+}_{W,\mathfrak{s}} & & \Big\downarrow \iota \\
HF^{\infty}(Y(g,\, n),\, [i]) & \xrightarrow{\pi_*} & HF^{+}(Y(g,\, n),\, [i]) & \xleftarrow{\cong} & H_*(\mathbb{X}^+_i(n))
\end{array} \tag{4.1}$$

Note that the left square of the diagram is induced from the cobordism W from $\#^{2g} S^2 \times S^1$ to $Y(g,\, n)$, and the right square come from the integral surgery formula for the Borromean knot in $\#^{2g} S^2 \times S^1$. In particular we are interested in the bottom-left horizontal map π_* for the d-invariants of $Y(g,\, n)$.

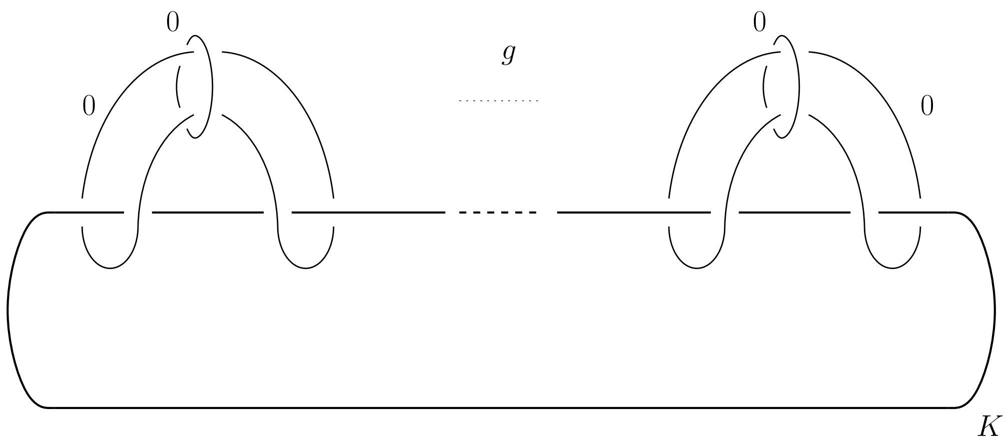


Figure 4.1: The Borromean knot in $\#^{2g}S^2 \times S^1$.

4.1 Heegaard Floer homology of the circle bundles over surfaces

In this section we study HF^+ of $Y(g, n)$ with coefficients in $\mathbb{F} \cong \mathbb{Z}/2\mathbb{Z}$. In [23, Section 5.2], Ozsváth and Szabó calculated HF^+_{red} of $Y(g, n)$ (with coefficients in $\mathbb{F}$) applying the integer surgery formula to the knot K in $\#^{2g}S^2 \times S^1$. We adapt their arguments but focus more on HF^+/HF^+_{red} (the image of HF^∞) to pursue an aim of the next section, computing d-invariants.

The knot Floer complex of $(\#^{2g}S^2 \times S^1, K)$ was calculated in Ozsváth-Szabó [20] and as given by the following:

$$C := CFK^\infty(\#^{2g}S^2 \times S^1, K) \cong \Lambda^* H^1(\Sigma_g; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}], \tag{4.2}$$

the $\mathbb{Z} \oplus \mathbb{Z}$-filtration is given by

$$C\{i, j\} = \Lambda^{g-i+j} H^1(\Sigma_g; \mathbb{Z}) \otimes U^{-i},$$

				$U^{-1}\Lambda^4_{(4)}$	$U^{-2}\Lambda^3_{(5)}$	$U^{-3}\Lambda^2_{(6)}$
			$\Lambda^4_{(2)}$	$U^{-1}\Lambda^3_{(3)}$	$U^{-2}\Lambda^2_{(4)}$	$U^{-3}\Lambda^1_{(5)}$
		$U\Lambda^4_{(0)}$	$\Lambda^3_{(1)}$	$U^{-1}\Lambda^2_{(2)}$	$U^{-2}\Lambda^1_{(3)}$	$U^{-3}\Lambda^0_{(4)}$
	$U^2\Lambda^4_{(-2)}$	$U\Lambda^3_{(-1)}$	$\Lambda^2_{(0)}$	$U^{-1}\Lambda^2_{(1)}$	$U^{-2}\Lambda^0_{(2)}$	
$U^3\Lambda^4_{(-4)}$	$U^2\Lambda^3_{(-3)}$	$U\Lambda^2_{(-2)}$	$\Lambda^1_{(-1)}$	$U^{-1}\Lambda^0_{(0)}$		
$U^3\Lambda^3_{(-5)}$	$U^2\Lambda^2_{(-4)}$	$U\Lambda^1_{(-3)}$	$\Lambda^0_{(-2)}$			
$U^3\Lambda^2_{(-6)}$	$U^2\Lambda^1_{(-5)}$	$U\Lambda^0_{(-4)}$				

Figure 4.2: A knot Floer complex of the Borromean knot in $\#^4 S^2 \times S^1$ and the quotient complex A_1^+ shaded.

the group $C\{i, j\}$ is supported in the grading $(i+j)$, and all the differentials are trivial. See Figure 4.2 for a diagram for $CFK^\infty(\#^4 S^2 \times S^1, K)$.

Under the identification $H_1(\#^{2g} S^2 \times S^1; \mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z})$ and $HF^\infty(\#^{2g} S^2 \times S^1, \mathfrak{t}_0) \cong C$, the action of $\Lambda^* H_1(\Sigma; \mathbb{Z})$ on C is given by the formula:

$$\gamma.(\xi \otimes U^l) = (\iota_\gamma \xi) \otimes U^l + PD(\gamma) \wedge \xi \otimes U^{l+1}, \tag{4.3}$$

for $\gamma \in H_1(\Sigma_g\, ; \mathbb{Z})$, where ι_γ denotes contraction.

After applying the integer surgery formula to CFK^∞, we claim the following.

Theorem 4.1.1. *Let $Y(g, n)$ denote the circle bundle over a genus g oriented surface with*

Euler number $n \neq 0$. Then, for any choice of $[i] \in \mathbb{Z}/n\mathbb{Z}$, let k be an integer with minimal absolute value among all integers congruent to i modulo n. Under the identification $H_1(Y(g, n); \mathbb{Z})/Tors \cong H_1(\Sigma_g; \mathbb{Z})$, there are $\Lambda^* H_1(\Sigma_g; \mathbb{Z}) \otimes \mathbb{F}[U]$-module isomorphisms,

$$\begin{aligned} & HF^+(Y(g, n), [i]; \mathbb{F}) \cong A_k^+ \oplus HF_{red}^+ && \text{if } n > 0 \\ \text{and} \quad & HF^+(Y(g, n), [i]; \mathbb{F}) \cong B^+\{j \geq k\} \oplus HF_{red}^+ && \text{if } n < 0, \end{aligned}$$

where A_k^+ and $B^+\{j \geq k\}$ are the quotient complexes of $CFK^\infty(\#^{2g} S^2 \times S^1, K; \mathbb{F})$ corresponding to $\max(i, j-k) \geq 0$ and $\min(i, j-k) \geq 0$ respectively.

Proof. Although $\mathrm{Spin}^c(\#^{2g} S^2 \times S^1) \cong \mathbb{Z}^{2g}$, $HF^+(\#^{2g} S^2 \times S^1)$ is nontrivial only for the unique torsion spin^c structure $\mathfrak{t}_0$ by the adjunction inequality [19]. Hence, even though the integer surgery formula in Section 2.6 is stated for integral homology 3-sphere, we can apply it to our situation (c.f. [23, Theorem 4.10.]).

We recall the notation for the integer surgery formula in Section 2.6. Note that, once we have CFK^∞ in hand, the only non-combinatorial part in applying the furmula is finding the chain homotopy equivalence from $C\{j \geq 0\}$ to $C\{i \geq 0\}$. However, in the case of $CFK^\infty(\#^{2g} S^2 \times S^1, K)$ with coefficients in $\mathbb{F}$, Ozsváth-Szabó showed that the map is necessarily described as follows.

Let $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$ be a symplectic basis of $H^1(\Sigma_g; \mathbb{Z})$ i.e. so that the intersection matrix is given as the direction sum of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We define a $\Lambda^* H_1(\Sigma_g; \mathbb{Z}) \otimes \mathbb{Z}[U]$-module automorphism of C by extending the formula

$$J(\omega \otimes U^{-i}) := (*I\omega) \otimes U^{-j}, \tag{4.4}$$

for $\omega \otimes U^{-i} \in C\{i, j\}$, where $*$ is the Hodge star operator, and I is an induced map by $I(\alpha_i)=-\beta_i$ and $I(\beta_i)=\alpha_i$. Observe that J sends a homogeneous element in $\{i, j\}$ to a homogeneous element in $\{j, i\}$. Hence J induces a $\Lambda^* H_1(\Sigma_g;\mathbb{Z})\otimes\mathbb{F}[U]$ module isomorphism $\overline{J}$ from $\overline{C}\{j \geq 0\}$ to $\overline{C}\{i \geq 0\}$, where $\overline{C}$ denotes $\mathbb{F}$-reduction of C, i.e. $\overline{C} \cong C \otimes \mathbb{F}$. Moreover $\overline{J}$ is the only such isomorphism [23, Proposition 5.2]. Now, we work with $\mathbb{F}$-coefficients.

It follows from the integer surgery formula that $HF^+(Y(g, n), [i]; \mathbb{F})$ is isomorphic to the homology of the mapping cone complex $\mathbb{X}_i^+(n)$ of $\overline{C}$. Since all differentials of $\overline{C}$ are trivial, $H_*(\mathbb{X}_i^+(n))$ determined by the kernel and cokernel of $D_i^+ \colon \mathbb{A}_i^+ \to \mathbb{B}_i^+$.

Suppose $n > 0$, and let k be an integer with minimal absolute value among all integers congruent to i modulo n. There is a natural projection chain map, $\Pi : \mathbb{X}_i^+(n) \to A_k^+$. We can easily check that Π is surjective on homology from a diagram chasing. Specifically, an element $a_k \in A_k$ can be extended to a sequence $\mathbf{a} = \{a_{k+sn}\}_{s\in\mathbb{Z}}$ in the homology of $\mathbb{X}_i^+(n)$ (in fact, $\ker(D_i^+)$) as follows: we choose $a_{k+(s+1)n} \in A_{k+(s+1)n}^+$ and $a_{k-(s+1)n} \in A_{k-(s+1)n}^+$ inductively on $s \geq 0$ so that

$$h_{k+sn}^+(a_{k+sn}) + v_{k+(s+1)n}^+(a_{k+(s+1)n}) = 0 \in B_{k+(s+1)n}^+$$

and

$$v_{k-sn}^+(a_{k-sn}) + h_{k-(s+1)n}^+(a_{k-(s+1)n}) = 0 \in B_{k-sn}^+.$$

This can be done since v_{k+sn}^+ and h_{k-sn}^+ are surjective for all positive integer s. Moreover $\mathbf{a}$ is finitely supported from the fact that the maps v_{k+sn}^+ and h_{k-sn}^+ are injective for all sufficiently large s.

We claim that Π induces an isomorphism between the image of U^d in $H_*(\mathbb{X}_i^+(n))$ and

A_k^+ for all sufficiently large d :

$$\Pi_*\colon U^d H_*(\mathbb{X}_i^+(n)) \subset H_*(\mathbb{X}_i^+(n)) \xrightarrow{\cong} A_k^+ \tag{4.5}$$

The subjectivity of Π_* above directly follows from the subjectivity of both Π on homology and the U-multiplication map in A_k^+. Before we show the injectivity of the map, we observe the followings:

- All homogeneous elements of $H_*(\mathbb{X}_i^+(n))$ in sufficiently large degrees have non-trivial component in A_k^+. This can be seen since in all sufficiently large degree h_{k-sn}^+ and v_{k+sn}^+ are injective for all positive s.

- For an element m of a $\mathbb{F}[U]$-module M, we define a *length* of m, $L_M(m)$, as the largest integer l with the property that $U^l \cdot m \neq 0$. If $\mathbf{c}$ is an element in $H_*(\mathbb{X}_i^+(n))$ which has a homogeneous component c_k in A_k, then

  $$L_{A_k^+}(c_k) \geq L_{A_{k+sn}^+}(c_{k+sn}), \text{ for any } s \in \mathbb{Z},$$

  where c_{k+sn} is the component of in A_{k+sn}^+. This can be shown by chasing the arrows in $\mathbb{X}_i^+(n)$ and the fact that k is chosen as an integer with the minimal absolute value in $i + n\mathbb{Z}$. See Figure 4.3 for instance.

Suppose $\mathbf{a}$ is an element in $U^d H_*(\mathbb{X})$, then there is an element $\mathbf{b}$ such that $U^d\mathbf{b} = \mathbf{a}$ for some large d. By the first observation, $\mathbf{b}$ has non-trivial component in A_k^+. Thus if $\Pi_*(\mathbf{a}) = 0 \in A_k$, then $\mathbf{a}$ must be trivial by the second observation. This verifies the isomorphism of Π_* in (4.5).

If n is negative, we can similarly prove that the natural projection map Π : $\mathbb{X}_i^+ \rightarrow$

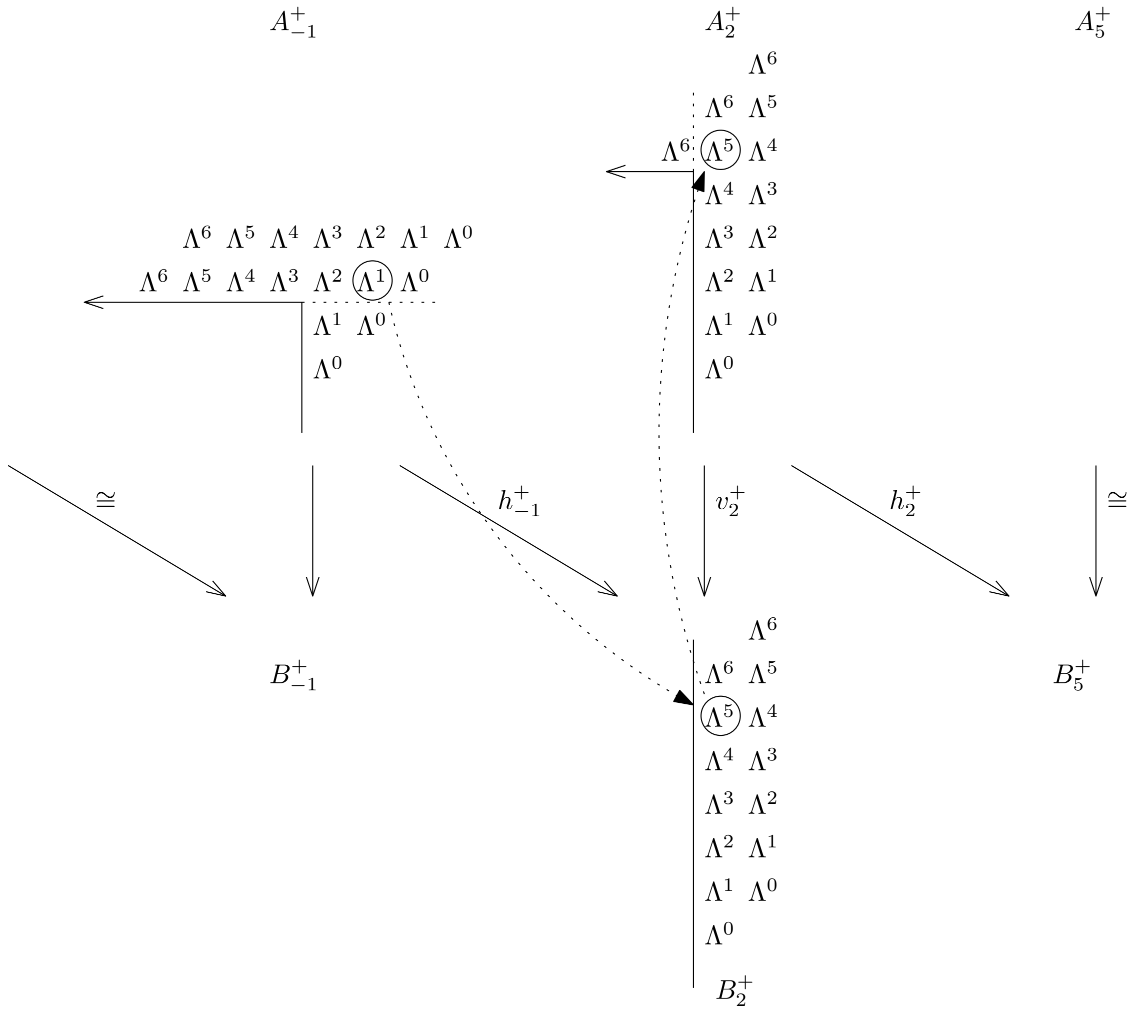


Figure 4.3: A mapping cone complex $\mathbb{X}_2^+(3)$.[1]

$B_k^+\{j \geq k\}$ induces an isomorphism

$$\Pi_* : U^d H_*(\mathbb{X}_i^+(n)) \subset H_*(\mathbb{X}_i^+(n)) \to B_k^+\{j \geq k\},$$

for all sufficiently large d. A key observation is that if an element in $\mathbb{B}_i^+$ has a non-trivial component in $B_k^+\{j \geq k\}$, then it cannot be an image of D^+ from a finitely supported element in $\mathbb{A}_i^+$.

[1]The dotted arrows show a diagram chasing for a homogeneous element in A^+_{-1}. Also we can see that the length of elements in the circled groups in A^+_{-1} is greater than the length of them in A^+_2

$\square$

If n is even and i is an integer congruent to $\frac{n}{2}$ modulo n, then there could be two different choice of k ($\pm\frac{n}{2}$) in Theorem 4.1.1. However we can easily check that A^+_k and A^+_{-k} (or $B^+\{j \geq k\}$ and $B^+\{j \geq -k\}$) are isomorphic as $\Lambda^* H_1(\Sigma_g)/\mathrm{Tor} \otimes \mathbb{F}[U]$-module via the map J defined in (4.4).

4.2 Correction terms

Fix $g > 0$. Let C^∞ be the $\mathbb{Z}$-graded $\Lambda^* H_1(\Sigma_g;\mathbb{Z}) \otimes \mathbb{F}[U]$-module obtained by reducing CFK^∞ in (4.2) modulo 2,

$$C^\infty \cong \Lambda^* H^1(\Sigma_g;\mathbb{Z}) \otimes \mathbb{F}[U, U^{-1}].$$

We first examine $\mathcal{K}C^\infty := \{x \in C^\infty | v \cdot x = 0\ \forall v \in H_1\}$ and $\mathcal{Q}C^\infty := C^\infty/(H_1.C^\infty)$. Let $\{a_i, b_i\}_{i=1}^g$ be a symplectic basis of $H_1(\Sigma_g;\mathbb{F})$, and $\{\alpha_i, \beta_i\}_{i=1}^g$ be the canonical dual basis for $\{a_i, b_i\}_{i=1}^g$, which forms a basis of $H^1(\Sigma_g;\mathbb{F})$. We denote $v_i = \alpha_i \wedge \beta_i \in \Lambda^2 H^1(\Sigma_g;\mathbb{F})$, and let I_s be the subset of the canonical generators of $\Lambda^{2s} H^1(\Sigma_g;\mathbb{F})$, which contains all generators having only $v_i = \alpha_i \wedge \beta_i$ factors, i.e.

$$I_s := \{v_{i_1} \wedge \cdots \wedge v_{i_s} | 0 < i_1 < i_2 < \cdots < i_s \leq g\} \text{ and } I_0 := \{1\}.$$

Lemma 4.2.1.

$$\mathcal{K}C^\infty = \mathbb{F}[U, U^{-1}]\langle \sum_{s=0}^{g} \sum_{v \in I_s} v \otimes U^s \rangle.$$

Proof. Let K denote the generator of the right hand side. It is computed directly applying the formula of H_1-action in (4.3) that

$$a_1.(v_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_s} \otimes U + v_{i_2} \wedge \cdots \wedge v_{i_s}) = 0$$

for any $1 < i_2 < \cdots < i_s \leq g$. From this it is easily verified that $a_1.K = 0$.

Indeed, by the symmetry of K (we can interchange the role of α_i and β_i, or v_i and v_j), K is annihilated by any H_1-action. The statement then follows from the fact that $\mathcal{K}C^\infty$ has rank 1 (c.f. [12, Lemma 2.4.]). $\square$

Lemma 4.2.2.

$$\mathcal{Q}C^\infty \cong \mathbb{F}[U, U^{-1}]\langle v \otimes U^s \rangle_{v \in I_s,\, 0 \leq s \leq g}/\mathcal{R},$$

where the relation $\mathcal{R}$ is given as that all generators are equal to each other.

Proof. Let $L := \{v \otimes U^{s-g}\}_{v \in I_s,\, 0 \leq s \leq g}$, and observe that it is a subset of the canonical $\mathbb{F}$-generators of $C^\infty_{(-g)} \cong \Lambda^0 \oplus U^1\Lambda^2 \oplus \cdots \oplus U^g\Lambda^{2g}$. Suppose ξ is a generator in the complement of L on $C^\infty_{(-g)}$. Then ξ necessarily contains either α_k or β_k, but not both of them for some k. We claim that ξ is a image of H_1-action. Without loss of generality, say $\xi = \alpha_1 \wedge \gamma$ for some γ which does not contain β_1. We can easily check that $\xi = b_1.(v_1 \wedge \gamma)$ from (4.3). (Note that if γ contains β_1, then $b_1.(v_1 \wedge \gamma)$ become trivial.) Therefore, L can be a generating set of $\mathcal{Q}C^\infty$.

The relation $\mathcal{R}$ can be obtained by generalizing the following computation:

$$a_1.(\alpha_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_s} \otimes U) = v_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_s} \otimes U + v_{i_2} \wedge \cdots \wedge v_{i_s}.$$

$\square$

For example, $\mathcal{K}HF^{\infty}(Y(2,\,n),\,i)$ is generated by

$$1 + U\alpha_1\beta_1 + U\alpha_2\beta_2 + U^2\alpha_1\beta_1\alpha_2\alpha_2,$$

and $\mathcal{Q}HF^{\infty}(Y(2,\,n),\,i)$ is given by

$$\mathbb{F}[U,\,U^{-1}]\langle 1,\,U\alpha_1\beta_1,\,U\alpha_2\beta_2,\,U^2\alpha_1\beta_1\alpha_2\beta_2\rangle/\{1{=}U\alpha_1\beta_1{=}U\alpha_2\beta_2{=}U^2\alpha_1\beta_1\alpha_2\beta_2\},$$

where we abbreviate notations of the tensor and wedge.

Now we are ready to compute d-invariants of $Y(g,\,n)$. Let $d(Y,\mathfrak{t};\mathbb{F})$ denote d-invariants corresponding to Floer homology of $(Y,\mathfrak{t})$ with coefficients in $\mathbb{F}$.

Theorem 4.2.3. *Let $Y(g,\,n)$ denote the circle bundle over genus g oriented surface with Euler number n. Then, for any choice of $[i] \in \mathbb{Z}/n\mathbb{Z}$, let k be an integer with minimal absolute value among all integers congruent to i modulo n. If $n > 0$*

$$d_{bot}(Y(g,\,n),\,[i];\,\mathbb{F}) = -d_{top}(Y(g,\,-n),\,[i];\,\mathbb{F}) = d(n,\,i) - g,$$

and

$$\begin{aligned} d_{top}(Y(g,\,n),\,[i];\,\mathbb{F}) &= -d_{bot}(Y(g,\,-n),\,[i];\,\mathbb{F}) \\ &= \begin{cases} d(n,\,i) + g & \text{if } |k| \geq g \\ d(n,\,i) + |k| & \text{if } |k| < g \text{ and } g+k \text{ is even} \\ d(n,\,i) + |k| - 1 & \text{if } |k| < g \text{ and } g+k \text{ is odd,} \end{cases} \end{aligned}$$

where

$$d(n,\,i) = -\max_{\{s\in\mathbb{Z}|s\equiv i\,(mod\,n)\}} \frac{1}{4}\left(1 - \frac{(n+2s)^2}{n}\right).$$

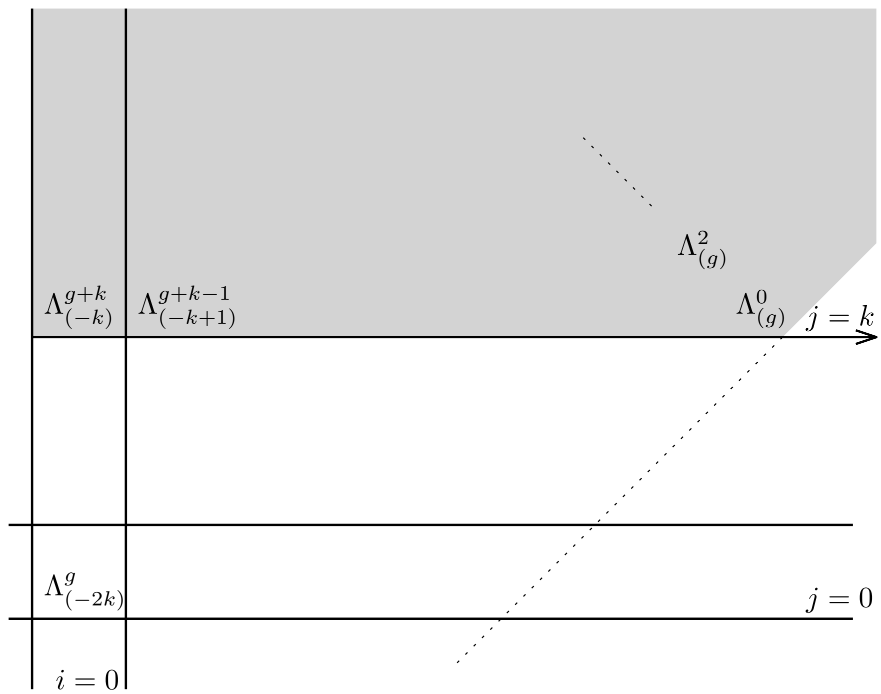
Proof. Fix g, $n < 0$ and $[i] \in \mathbb{Z}/n\mathbb{Z}$, and let k be an integer with minimal absolute value among all integers congruent to i modulo n. Let W be the natural two-handle cobordism from $\#^{2g}S^2 \times S^1$ to $Y(g,\,n)$, and $\mathfrak{s}$ be the Spin^c structure of the cobordism W corresponding to k. Then we have the following diagram restricting the diagram (4.1) to the image of HF^∞ and including the statement of Theorem 4.1.1.

$$\begin{array}{ccccc}
HF^\infty(\#S^2\times S^1,\,\mathfrak{t}_0) & \xrightarrow{\pi_*} & HF^+(\#S^2\times S^1,\,\mathfrak{t}_0) & \xleftarrow{\cong} & B_k^+ \\
{\scriptstyle F^\infty_{W,\mathfrak{s}}=\cong}\Big\downarrow & & {\scriptstyle F^+_{W,\mathfrak{s}}}\Big\downarrow & & {\scriptstyle \Pi}\Big\downarrow \\
HF^\infty(Y(g,\,n),\,i) & \xrightarrow{\pi_*} & \mathrm{im}(\pi_*) \subset HF^+(Y(g,\,n),\,i) & \xleftarrow{\cong} & B_k^+\{j \geq k\}
\end{array}$$

Let us identify HF^∞'s of the left column with C^∞. As we showed in Proposition 4.0.5, they are $\Lambda^* H_1(\Sigma_g) \otimes \mathbb{F}[U]$-module isomorphic to the "standard" HF^∞ (in which H_1/Tors-action is simply given by the contraction map). However, the identification of $HF^\infty(Y(g,\,n),\,[i])$ with C^∞ is more appropriate in our purpose since $HF^+(Y(g,\,n),\,[i])$ is described as a quotient complex of CFK^∞ in the previous section.

Then it follows from chasing the diagram that the bottom-left horizontal map π_* can be identified with the natural projection map π from C^∞ to $B_k^+\{j \geq k\}$. Observe that the generator K of $\mathcal{K}C^\infty$ of Lemma 4.2.1 is a sum whose summands are contained in each wedge product of even degree. Hence d_{bot} will be the minimal grading among all wedge product of even degree in $B_k^+\{j \geq k\}$. It can be easily verified from the following commuting diagram:

$$\begin{array}{ccc}
\mathcal{K}C^\infty & \xrightarrow{\mathcal{K}(\pi)} & \mathcal{K}B_k^+\{j \geq k\} \\
{\scriptstyle \iota}\Big\downarrow & & {\scriptstyle \iota}\Big\downarrow \\
C^\infty & \xrightarrow{\pi} & B_k^+\{j \geq k\}
\end{array}$$



B_k^+, $k \geq 0$

Figure 4.4: A digram of B_k^+ and $B_k^+\{j \geq k\}$ shaded.

In particular if $|k| < g$, the minimal grading of even wedge product of $B_k^+\{j \geq k\}$ is realized in $B_k^+\{0, k\}$ if $g+k$ is even, and $B_k^+\{1, k\}$ if $g+k$ is odd (see Figure 4.4.) Recall that a $\mathbb{Z}$-grading was given to the mapping cone complex $\mathbb{X}^+(n)$ in Section 2.6. Hence it is easy to check from it that $B_k^+\{0, k\}$ and $B_k^+\{1, k\}$ are supported in the grading $-|k|$ and $-|k|+1$ respectively. If $|k| \geq g$, then the minimal even wedge is in $B_k^+\{g+k, k\}$ if $k > 0$ and $B_k^+\{0, -g\}$ if $k < 0$, which are supported in the grading $-g$.

On the other hand, $\mathcal{Q}C^\infty$ is generated by elements in each wedge product of even degree, and they are related identically. And hence, if $\mathcal{Q}B_k^+\{j \geq k\}_{(p)}$ has non-trival image of $\mathcal{Q}C^\infty_{(p)}$ then $B_k^+\{j \geq k\}_{(p)}$ must contain all wedge product of even degree. Otherwise the image of $\mathcal{Q}C^\infty_{(p)}$ become trivial factored by $\mathcal{R}$. We observe that $B_k^+\{j \geq k\}_{(g)}$ contains every even wedge product, but $B_k^+\{j \geq k\}_{(g-2)}$ does not. Hence, since the quotient maps are $\mathbb{F}[U]$-equivariant, d_{top} is realized at grading g.

Finally, the statement follows from the fact that the isomorphism from the homology of mapping cone complex to HF^+ is of degree $d(n, i)$, and the property of d-invariants under orientation reversal. Remark that, instead of using the property of orientation reversal, we achieve the same results for the d-invariants of $Y(g, n)$ $(n > 0)$ by identifying the map from HF^∞ to HF^+ with the projection map from C^∞ to A_k^+, similarly to the negative n case.

□

4.3 Applications

Recall that the d-invariants of Y can provide restrictions to the intersection forms of negative semi-definite 4-manifolds bounded by Y. Hence there is an immediate corollary of Theorem 2.3.2.

Corollary 4.3.1. *Let W be an oriented negative semi-definite 4-manifold bounded by $Y(g, n)$. Then, for any Spin^c structure $\mathfrak{s}$ over W whose restriction to $Y(g, n)$ is $[i]$, we have the following inequality:*

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4d_{bot}(Y(g, n), [i]) + 4g.$$

In addition, if the map $H_1(Y)/Tors \to H_1(W)/Tors$ induced by inclusion is injective, then we have

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4d_{top}(Y(g, n), [i]) - 4g.$$

Moreover the corollary can be applied to closed 4-manifolds. Let X be an oriented closed 4-manifold and Σ be an embedded genus g surface in X with $[\Sigma] \cdot [\Sigma] = n$. Then X can be

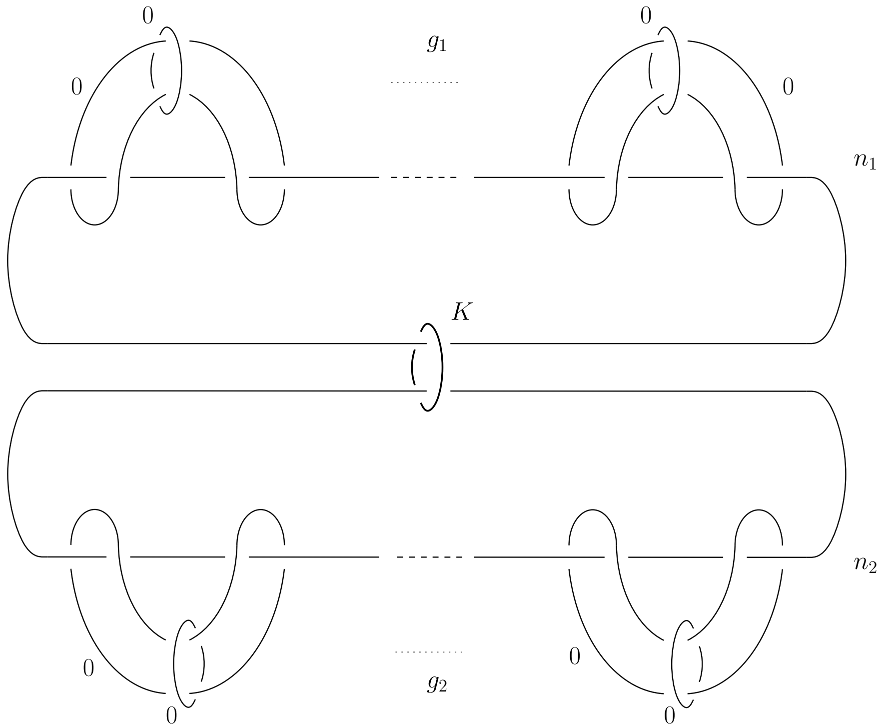


Figure 4.5: A surgery digram of $Y(g_1, n_1)\#Y(g_2, n_2)$

separated as two parts, the tubular neighborhood of Σ and its complement W,

$$X = \mathrm{nb}(\Sigma) \cup W.$$

Observe that the boundary of W is diffeomorphic to $-Y(g, n)$, so we may apply Corollary 4.3.1 if W is negative semi-definite.

In a different direction, our computation together with the surgery exact sequence can be applied to compute Heegaard Floer homology of interesting 3-manifolds. For instance, let Y be the connected sum of two circle bundles over surfaces, $Y = Y(g_1, n_1)\#Y(g_2, n_2)$ and K be the knot described in Figure 4.5. We can easily verify that $Y_0(K)$ is indeed diffeomorphic to $Y(g_1+g_2, n_1+n_2)$ by a handle-slide and handle-cancellation. By blowing-down K, $Y_1(K)$ is diffeomorphic to the manifold obtained by the plumbing of two disk-bundles over surfaces. Then they fit into the long exact sequence [19, Section 9],

$$\cdots \longrightarrow HF^+(Y) \longrightarrow HF^+(Y_0) \longrightarrow HF^+(Y_1) \longrightarrow \cdots.$$

Observe that the first two groups in the sequence can be described from the computation in this chapter and the connected sum formula of HF° in [19, Theorem 6.2.]. Therefore we may expect the other one, Heegaard Floer homology of the manifolds obtained from plumbing, can be computed by analyzing the long exact sequence. Moreover, the correction terms of the manifolds are expected to give another application in 4-dimensional topology.

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