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ELLIPTIC FUNCTIONS, THETA FUNCTION, AND SUBMANIFOLDS IN SPACE FORMS

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ABSTRACT

ELLIPTIC FUNCTIONS, THETA FUNCTION, AND SUBMANIFOLDS IN SPACE FORMS

By

Jie Yang

In the first part of the thesis(Chapter 1), we study slant surfaces in $\mathbb{C}^2$. The complete classification for all proper slant surfaces with constant Gaussian curvature and nonzero constant mean curvature in $\mathbb{C}^2$ is obtained in this part.

In 1993, B. Y. Chen introduced an important Riemannian invariant δ_M for a Riemannian n-manifold M^n, namely take the scalar curvature and subtract at each point the smallest sectional curvature. He proved that every submanifold M^n in a Riemannian space form $R^m(\epsilon)$ satisfies a sharp inequality:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}H^2 + (n+1)(n-2)\epsilon.$$

In the second part of the thesis (Chapter 2 and 3), first we classify hypersurfaces with constant mean curvature in a Riemannian space form which satisfy the equality case of the inequality. Next, by utilizing Jacobi's elliptic functions and Theta function we obtain the complete classification of conformally flat hypersurfaces in Riemannian space forms which satisfy the equality.

To my parents, my wife Ying and daughter Jessica

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TABLE OF CONTENTS

CHAPTER 0

INTRODUCTION

Let (N, g, J) denote an almost Hermitian manifold equipped with an almost complex structure J and almost Hermitian metric g. A submanifold M of N is called *slant* if its Wirtinger angle is constant. Complex submanifolds and totally real submanifolds are two special classes of slant submanifolds which have Wirtinger angle 0 and $\frac{\pi}{2}$, respectively. It is known that there exist ample examples of slant submanifolds other than complex and totally real submanifolds. The first part of the thesis studies slant surfaces in the complex 2-plane $\mathbb{C}^2$.

It is well-known that helical cylinders in $\mathbb{C}^2$ are flat proper slant surfaces with nonzero constant mean curvature. Conversely, we prove that every flat proper slant surface with nonzero constant mean curvature in $\mathbb{C}^2$ is an open portion of a helical cylinder.

Although it is known that there exist abundant examples of slant surfaces with constant mean curvature and constant Gaussian curvature in non-flat complex-space-forms. However, we prove that there do not exist proper slant surfaces in $\mathbb{C}^2$ with constant mean curvature and nonzero constant Gaussian curvature.

According to the well-known Nash imbedding theorem, every Riemannian n-manifold can be realized as a submanifold in a Riemannian space form, in particular, in a Euclidean space. For a submanifold in a Riemannian space form, Chen proved in 1993 a general inequality involving sectional curvature, scalar curvature and the squared mean curvature of the submanifold. Chen's inequality has some important applications. For example, it gives rise to the second Riemannian obstruction for a Riemannian manifold to admit a minimal isometric immersion into a Euclidean space. It also gives rise to an obstruction to Lagrangian isometric immersions of compact Riemannian manifolds with finite fundamental group π_1 into complex space forms.

Since Chen's inequality is very general and sharp, it is natural and interesting to understand submanifolds which satisfy the equality case of this inequality. Recently, there are several interesting papers which investigate submanifolds satisfying Chen's equality. In this thesis, we investigate the most fundamental case; namely, hypersurfaces satisfying Chen's equality. First, we give a complete classification of hypersurfaces with constant mean curvatures satisfying the equality. Next, by utilizing the Jacobi elliptic functions and the Theta function, we completely classify conformally flat hypersurfaces satisfying the equality.

Let M be an n-dimensional $(n > 2)$ hypersurface in a Riemannian space form $R^{n+1}(\epsilon), (\epsilon = 1, -1 \text{ or } 0)$ which satisfies the equality. We show that, if $\epsilon = 0$, M is either minimal or an open portion of a spherical hypercylinder; if $\epsilon = 1$, M is either totally geodesic or a tubular hypersurface with radius $\frac{\pi}{2}$ about a 2-dimensional minimal surface; and if $\epsilon = -1$, M is either totally geodesic, or an open portion of a tubular hypersurface with radius $\cosh^{-1}(\sqrt{2})$ about a 2-dimensional totally geodesic surface of $R^{n+1}(-1)$, or a "suitable tubular hypersurface" about a minimal surface in the de Sitter space-time $S_1^{n+1}(1)$.

In order to classify conformally flat hypersurfaces satisfying the equality, we need to define some special families of Riemannian manifolds: $P_a^n\,(a > 0)$, $C_a^n\,(a > 1)$, $D_a^n\,(0 < a < 1)$, F^n, L^n, $A_a^n\,(a > 0)$, $B_a^n\,(0 < a < 1)$, G^n, $H_a^n\,(a > 0)$, W_a^n, $(a > 0)$ and $Y_a^n(0 < a < 1)$ via warped products of $\mathbb{R}$ (or an open interval) and some Riemannian space form by some warp functions that may involve the Jacobi elliptic functions. For example, P_a^n, C_a^n, are the warped products:$I \times_{\mu_a} S^{n-1}(\frac{a^4-1}{4})$, $R \times_{\eta_a} H^{n-1}(\frac{a^4-1}{4})$, where $\mu_a = ak\,\mathrm{cn}(ax, \frac{\sqrt{a^2-1}}{\sqrt{2}a})$ and $\eta_a = \frac{a}{k}\mathrm{dn}(\frac{a}{k}x, \frac{\sqrt{2}a}{\sqrt{a^2+1}})$, and $\mathrm{cn}(ax, k)$, $\mathrm{dn}(\frac{a}{k}, k)$ are the Jacobi elliptic functions with modulus k. Topologically, S^n is the two point compactification of P_a^n, C_a^n as well as of B_a^n and the Riemannian metrics defined on P_a^n, C_a^n or B_a^n can be extended smoothly to S^n. Let $\hat{P}_a^n$, $\hat{C}_a^n$ and $\hat{B}_a^n$ denote the n-sphere together with the Riemannian metrics given by the smooth extensions of the metrics on P_a^n, C_a^n and B_a^n to S^n, respectively.

We prove that if M is a conformally flat hypersurface of a Riemannian space form which satisfies the basic equality, then either M is totally geodesic or M is an open portion of one these ten special families of Riemannian manifolds. Furthermore, we are able to determine these immersions explicitly. If the the ambient space is spherical, there exist three families of such hypersurfaces. One of the families is the immersion of $\hat{P}_a^3$ into $S^4(1)$ and its local expression involves the Jacobi elliptic functions and the Theta function. In order to get the expression, we have to solve a family of second order ODEs of Picard type whose coeffients involve the Jacobi elliptic functions, namely. $u''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)u'(x) - u(x) = 0$. We call such an equation a differential equation of Picard type since a similar equation was studied by E. Picard in 1879. However the method of Picard does not work for our equations, so we need to develop a new approach to obtain the general solutions for this type of ODEs. Our results seem to have independent interest by themselves. If the ambient spaces are hyperbolic, we are able to obtain the complete classification via nine families of immersions from the following Riemannian manifolds A_a^n, G^n, H_a^n, Y_a^n, L^3, $\hat{C}_a^3$, D_a^3, etc, to the ambient space. In order to establish the local expressions of the immersions of $\hat{C}_a^3$ and D_a^3, we need to solve two families of ODEs similar to the one mentioned above.

CHAPTER 1

SLANT SURFACES WITH CONSTANT MEAN CUEVATURE IN $\mathbf{C}^2$

In this chapter, we completely classify proper slant surfaces with constant Gaussian curvature and nonzero constant mean curvature in $\mathbb{C}^2$.

1.1 Introduction

Let M be a Riemannian n-manifold and $(\tilde{M}, g, J)$ an almost Hermitian manifold with almost complex structure J and almost Hermitian metric g. Let T_pM be the tangent space to M at p. An isometric immersion $f : M \to \tilde{M}$ is called holomorphic if at each point $p \in M$ we have $J(T_pM) = T_pM$. The immersion is called totally real if $J(T_pM) \subset T^{\perp}M$ for each $p \in M$, where $T^{\perp}M$ is the normal space of M in $\tilde{M}$ at p. For each nonzero vector X tangent to M at p, the angle $\gamma(X)$ between JX and T_pM is called the Wirtinger angle of X. The immersion $f : M \to \tilde{M}$ is said to be *slant* if $\gamma(X)$ is a constant (which is independent of the choice of $p \in M$ and

$X \in T_pM$. see [1] for details). The Wirtinger angle γ of a slant immersion is called the *slant angle.* Homolorphic and totally real immersions are slant immersions with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant immersion is said to be *proper slant* if it is neither holomorphic nor totally real.

The simpliest and most important examples of slant submanifolds are slant surfaces in $\mathbb{C}^2$, where $\mathbb{C}^2$ is the Euclidean 4-space $\mathbb{R}^4$ equipped with its canonical complex structure. In [2], B. Y. Chen constructed ample examples of such surfaces. He also proved that there is no proper slant surface in $\mathbb{C}^2$ with parallel mean curvature vector (cf. also [3]). Thus the following open problem proposed in [2] by B. Y. Chen is very interesting:

Problem: *Classify slant surfaces in $\mathbb{C}^2$ with nonzero constant mean curvature.*

It is known that helical cylinders in $\mathbb{C}^2$ are flat proper slant surfaces with nonzero constant mean curvature [2]. The first result of this chapter is to prove that the converse of this fact is also true. Namely we prove the following:

Theorem 1.1. *A flat proper slant surface with nonzero constant mean curvature in $\mathbb{C}^2$ is an open portion of a helical cylinder.*

B. Y. Chen and L. Vrancken show in [4] that there exist many proper slant surfaces with constant mean curvature or with constant Gaussian curvature in complex-space-forms. However, in this paper we prove the following nonexistence theorem:

Theorem 1.2. *There do not exist proper slant suraces with nonzero constant mean curvature and nonzero constant Gaussian curvature in $\mathbb{C}^2$.*

1.2 Preliminaries

Let $\mathbb{C}^2$ be $\mathbb{R}^4$ equipped with its complex structure J and M be a proper slant surface isometrically immersed in $\mathbb{C}^2$.

For any vector X tangent to M, set

$$JX = PX + FX$$

where PX and FX are respectively the tangential and normal components of JX. It is clear that P is an endomorphism of the tangent bundle TM and that F is a normal-bundle-valued 1-form on TM_p.

Let e_1 be an unit local vector field in TM. We choose a canonical orthonormal local frame e_1, e_2, e_3, e_4 such that

$$e_2 = (\sec\gamma)Pe_1, \qquad e_3 = (\csc\gamma)Fe_1, \qquad e_4 = (\csc\gamma)Fe_2.$$

Such an orthonomal frame is called an *adapted slant frame.*

Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the dual frame of e_1, e_2, e_3, e_4. Then the structure equations are given by

$$d\omega_A = -\omega_{AB} \wedge \omega_B, \quad d\omega_{AB} = -\omega_{AC} \wedge \omega_{CB}, \quad \omega_{AB} + \omega_{BA} = 0.$$

where A, B=1, 2, 3, 4. r, s=3, 4. i, j=1, 2. Restricting to M, $\omega_r = 0$. Then we have

$$\omega_{ri} = h^r_{ij}\omega_j, \qquad h^r_{ij} = h^r_{ji}.$$

A helical cylinder in $\mathbb{R}^4$ is defined by

$$x(u,v) = (u, k\cos v, mv, k\sin v) \tag{1.2.1}$$

where m and k are nonzero constants. With complex structure J_0 in $\mathbb{R}^4$, (1.2.1) defines a proper slant surface in $\mathbb{C}^2$ with slant angle $\cos^{-1}(\frac{m}{\sqrt{m^2+k^2}})$, where

$$J_0 : (x_1, x_2, x_3, x_4) \longmapsto (-x_3, -x_4, x_1, x_2)$$

If we choose

$$e_1 = \frac{1}{\sqrt{m^2+k^2}}(0, -k\sin v, m, k\cos v), \qquad e_2 = (1,0,0,0),$$
$$e_3 = (0, -\cos v, 0, -\sin v), \qquad e_4 = \frac{1}{\sqrt{m^2+k^2}}(0, -m\sin v, -k, m\cos v),$$

then e_1, e_2, e_3, e_4 form an adapted slant frame with respect to J_0, where e_3 is in the direction of mean curvature vector. The connection form of (1.2.1) is given by

$$(\omega_{AB}) = \begin{pmatrix} 0 & 0 & \frac{k}{m^2+k^2}\omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{k}{m^2+k^2}\omega_1 & 0 & 0 & -\frac{m}{m^2+k^2}\omega_1 \\ 0 & 0 & \frac{m}{m^2+k^2}\omega_1 & 0 \end{pmatrix}. \tag{1.2.2}$$

1.3 Basic Equations

Let M be a proper slant surface with constant mean curvature $\frac{c}{2} \neq 0$. We recall from [2] that

$$h^3_{12} = h^4_{11}, \qquad h^3_{22} = h^4_{12}, \tag{1.3.1}$$

$$\omega_{34} - \omega_{12} = \cot\gamma\{(\mathrm{tr}h^3)\omega_1 + (\mathrm{tr}h^4)\omega_2\}. \tag{1.3.2}$$

where γ is the slant angle. We can choose an adapted slant frame such that e_3 is parallel to the direction of mean curvature vector. Thus the shape operator A_3 and A_4 take the following forms:

$$A_3 = \begin{pmatrix} c-\lambda & \alpha \\ \alpha & \lambda \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \alpha & \lambda \\ \lambda & -\alpha \end{pmatrix}.$$

Let $a = c\cot\gamma > 0$. By (1.3.2), we have

$$\omega_{34} - \omega_{12} = a\omega_1.$$

Using the structure equations, we get $d\omega_1 = 0$. So, locally, there exists a variable x such that $dx = \omega_1$. Since

$$d\omega_1 = -\omega_{12} \wedge \omega_2,$$

we obtain that

$$\omega_{12} = \beta\omega_2, \qquad \text{and} \qquad \omega_{34} = \beta\omega_2 + \alpha\omega_1.$$

Assume $\omega_2 = f(x,y)dy$, where $f(x,y)$ is a C^2 function and nowhere zero on a neighborhood U of (0,0). So, on U, the metric tensor is

$$g = dx^2 + f^2(x,y)dy^2.$$

Since $\omega_{31} = (c-\lambda)\omega_1 + \alpha\omega_2$, we have

$$d\omega_{31} = (e_2\lambda + e_1\alpha - \alpha\beta)\omega_1 \wedge \omega_2.$$

On the other hand,

$$\begin{aligned} d\omega_{31} &= -\omega_{32} \wedge \omega_{21} - \omega_{34} \wedge \omega_{41} \\ &= (2\alpha\beta - a\lambda)\omega_1 \wedge \omega_2, \end{aligned}$$

So, by comparing these two expressions, we deduce that $e_2\lambda + e_1\alpha = 3\alpha\beta - \alpha\lambda$.

Similarly, we have $e_1\lambda - e_2\alpha = -c\beta + 3\lambda\beta + a\alpha$.

Let K denote Gaussian curvature of M, then

$$K\omega_1 \wedge \omega_2 = d\omega_{12} = (e_1\beta - \beta^2)\omega_1 \wedge \omega_2.$$

But

$$\begin{aligned} d\omega_{12} &= -\omega_{13} \wedge \omega_{32} - \omega_{14} \wedge \omega_{42} \\ &= (c\lambda - 2\lambda^2 - 2\alpha^2)\omega_1 \wedge \omega_2, \end{aligned}$$

therefore

$$K = e_1\beta - \beta^2 = c\lambda - 2\lambda^2 - 2\alpha^2.$$

Consequentely, we have

$$(1.3.3) \qquad \begin{cases} e_1\lambda - e_2\alpha = -c\beta + 3\lambda\beta + a\alpha, \\ e_2\lambda + e_1\alpha = 3\alpha\beta - a\lambda, \\ K = e_1\beta - \beta^2 = c\lambda - 2\lambda^2 - 2\alpha^2. \end{cases}$$

In particular, (1.3.3) implies $K \leq \frac{c^2}{8}$. Since $\omega_2 = f(x, y)dy$,

$$\begin{aligned}\frac{\partial f}{\partial x}dxdy &= d\omega_2 = -\omega_{21}\wedge\omega_1\\ &= -\beta f dxdy.\end{aligned}$$

Therfore $\beta = -\frac{\frac{\partial f}{\partial x}}{f}$, and

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= -\beta\frac{\partial f}{\partial x} - f\frac{\partial \beta}{\partial x}\\ &= \beta^2 f - f(\beta^2 + K)\\ &= -Kf.\end{aligned} \tag{1.3.4}$$

1.4 Proof of Theorem 1.1

Now, we consider the flat case, i.e. $K = 0$.

By (1.3.4), we have

$$f(x, y) = p(y)x + q(y)$$

on some open neighborhood U of (0, 0). Let α, β, γ as in the previous section.

First, if $\alpha \equiv 0$ and then $\lambda \equiv 0$ on U by (1.3.3) since $c \neq 0$. In this case, we can compute its connection form as follows:

$$\begin{aligned}&\omega_{12} = \beta\omega_2 = 0, \qquad \omega_{31} = h^3_{11}\omega_1 + h^3_{12}\omega_2 = (c-\lambda)\omega_1 + \alpha\omega_2 = c\omega_1,\\ &\omega_{34} = \beta\omega_2 + a\omega_1 = a\omega_1, \qquad \omega_{32} = h^3_{12}\omega_1 + h^3_{22}\omega_2 = \alpha\omega_1 + \lambda\omega_2 = 0,\\ &\omega_{41} = h^4_{11} + h^4_{12}\omega_2 = \alpha\omega_1 + \lambda\omega_2,\\ &\omega_{42} = h^4_{11}\omega_1 + h^4_{22}\omega_2 = \lambda\omega_1 - \alpha\omega_2 = 0.\end{aligned}$$

Thus we have

$$(\omega_{AB}) = \begin{pmatrix} 0 & 0 & -c\omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ c\omega_1 & 0 & 0 & a\omega_1 \\ 0 & 0 & -a\omega_1 & 0 \end{pmatrix}.$$

Therefore, we can chose m and k such that $c = \frac{k}{m^2+k^2}$. Let

$$\gamma = \cos^{-1} \frac{m}{\sqrt{m^2 + k^2}}.$$

By assumption,

$$a = c \cot \gamma = -\frac{m}{m^2 + k^2}.$$

Thus we get (1.2.1). i.e., M is helical cylinder on U. So we have proven the theorem in this case.

If there exists a point $z \in U$ such that $\alpha(z) \neq 0$, we can find an open subset V of U such that α is nowhere zero on V.

Lemma *Let V as be chosen above, then $p(y) \equiv 0$ on V.*

Proof. Let us assume that $p(y)$ does not vanish identically. Let $z' \in V$ be a point such that $p(z') \neq 0$, we can choose an open subset of V, say W, such that $p(y)$ is nowhere zero on W. Thus on W,

$$f(x, y) = p(y)(x + \frac{q(y)}{p(y)}).$$

So, without loss of generality (by replacing $p(y)dy$ by dY if necessary), we can assme

$$f(x, y) = x + g(y) \qquad \text{on} \quad W.$$

In this case, $\beta = -\frac{\frac{\partial f}{\partial x}}{f} = -\frac{1}{f}$. From (1.3.3), we have

$$
(1.4.1) \qquad \begin{cases} \dfrac{\partial\lambda}{\partial x} - \dfrac{1}{f}\dfrac{\partial\alpha}{\partial y} = \dfrac{c-3\lambda}{f} + a\alpha, \\ \dfrac{1}{f}\dfrac{\partial\lambda}{\partial y} + \dfrac{\partial\alpha}{\partial x} = -\dfrac{3\alpha}{f} - a\lambda, \\ (c-4\lambda)\dfrac{\partial\lambda}{\partial x} - 4\alpha\dfrac{\partial\alpha}{\partial x} = 0, \\ (c-4\lambda)\dfrac{\partial\lambda}{\partial y} - 4\alpha\dfrac{\partial\alpha}{\partial y} = 0. \end{cases}
$$

Solving (1.4.1) and noticing that $c\lambda - 2\lambda^2 - 2\alpha^2 = 0$, we have

$$
(1.4.2) \qquad \begin{aligned} \frac{\partial\lambda}{\partial x} &= \frac{4\alpha}{c}(\frac{\alpha}{f} + \alpha f), & \frac{\partial\lambda}{\partial y} &= -\frac{4\alpha}{c}(c - \lambda + a\alpha f), \\ \frac{\partial\alpha}{\partial x} &= \frac{c-4\lambda}{c}(\frac{\alpha}{f} + \alpha f), & \frac{\partial\alpha}{\partial y} &= -\frac{c-4\lambda}{c}(c - \lambda + a\alpha f). \end{aligned}
$$

Let $A = \frac{\alpha}{f} + a\alpha$, and $B = c - \alpha + a\alpha f$. Since $\frac{\partial^2 <}{\partial x \partial y} = \frac{\partial^2\lambda}{\partial y \partial x}$, we have

$$
\frac{4\alpha}{c}(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}) = 0
$$

Since $\alpha \neq 0$ on W,

$$
\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0.
$$

Using (1.4.1), this implies

$$
(1.4.3) \qquad P\alpha = Q\lambda - D,
$$

where

$$
(*) \qquad P = \frac{g'}{f^2} + 3a, \quad Q = \frac{3}{f} - a^2 f, \quad D = \frac{c}{f}.
$$

Therefore $P^2\alpha^2 = Q^2\lambda^2 - 2QD\lambda + D^2$. By (1.3.3), we have

$$2(P^2+Q^2)\lambda^2 - (cP^2+4QD)\lambda + 2D^2 = 0. \tag{1.4.4}$$

From (1.4.3),

$$\frac{\partial P}{\partial x}\alpha + P\frac{\partial \alpha}{\partial x} = \frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x} - \frac{\partial D}{\partial x}$$

By (1.4.4) and (1.4.1), this implies

$$(a^3f^4 - 2a^2g'f^2 - 2g' - ag'^2)\lambda - 2acf^2 = 0. \tag{1.4.5}$$

Thus (1.4.4) and (1.4.5) give

$$((ag')^2 + 3ag' + 2)(9a^2f^4 + 6ag'f^2 + g'^2) = 0. \tag{1.4.6}$$

So, if there exists a point $z_0 \in W$ such that $((ag')^2 + 3ag' + 2)(z_0) \neq 0$, we can choose an open neighborhood W' of z_0 in W such that $(ag')^2 + 3ag' + 2$ is nowhere zero on W'. Thus, by (1.4.6), we have

$$9a^2f^4 + 6ag'f^2 + g'^2 = 0 \qquad \text{on} \quad W',$$

i.e., $(x+g(y))^2 = f^2 = \frac{-g'(y)}{3a^2}$ on W'. This is a contradiction.

In the following, we assume

$$(ag')^2 + 3ag' + 2 = 0 \qquad \text{on} \quad W.$$

Thus, $g' = -\frac{1}{a}$ or $-\frac{2}{a}$ on W, then $f(x,y) = x + by + C$ on W, where $b = -\frac{1}{a}$ or $-\frac{2}{a}$.

Now, from (1.4.3), we have

$$\frac{\partial P}{\partial y}\alpha + P\frac{\partial \alpha}{\partial y} - \lambda\frac{\partial Q}{\partial y} - Q\frac{\partial \lambda}{\partial y} - \frac{\partial D}{\partial y} = 0.$$

By doing similarly computations as before, we have

$$(1.4.7)\quad \begin{aligned}&(-4afDP + 3cP^2 - 4DQ + acfPQ + cP_yQ + 4cQ^2 - cPQ_y)\lambda \\ &+2D^2 + acfDP + cD_yP - c^2P^2 - cDP_y - 4cDQ = 0\end{aligned}$$

Notice that, in this case

$$P = \frac{b}{f^2} + 3a, \qquad Q = \frac{3}{f} - a^2 f, \qquad D = \frac{c}{f}.$$

Substituting the above and (1.4.7) back in (1.4.4) and simplifying, we have

$$(1.4.8)\quad \begin{aligned} 3a^6c^4f^8 &+ (-8a^4c^4 - 38a^5c^4b - 18a^6c^4b^2)f^6 \\ &+ (-276a^2c^4 - 706a^3c^4b - 532a^4c^4b^2 - 120a^5c^4b^3)f^4 \\ &+ (36c^4 - 144ac^4b - 432a^2c^4b^2 - 338a^2c^4b^3 - 74a^4c^4b^4)f^2 \\ &- 20c^4b^2 - 66ac^4b^3 - 55a^2c^4b^4 - 12a^3c^4b^5 = 0.\end{aligned}$$

The leading term of (1.4.8) is $3a^6c^4 \neq 0$, and the other coefficients are constants. Thus we get $f \equiv$ constant on W. This contradicts $f = x + by + C$. So, we have completed the proof of the lemma.

Returning to the proof of Theorem 1, From the above lemma, we know $f(x, y) = q(y)$ on W. By $(*)$, we can see that P, Q and D are functions of y. Also $\beta = -\frac{\frac{\partial f}{\partial x}}{f} = 0$. Moreover, from (*), we see that P , Q and D can not be simultaneously zero at any point on W, otherwise we have $c = 0$ at this point. Thus λ is a fuction of y, and so is α. By using (1.3.3), we have

$$\frac{1}{f}\frac{d\alpha}{dy} = -a\alpha, \qquad \text{and} \qquad \frac{1}{f}\frac{d\lambda}{dy} = -a\lambda. \tag{1.4.9}$$

So, $\alpha = C_1\lambda$, where C_1 is some constant. By (1.4.1), we get

$$2(1 + C_1^2)\lambda^2 - c\lambda = 0$$

then

$$\lambda = 0, \qquad \text{or} \qquad \lambda = \frac{c}{2(C_1^2 + 1)}.$$

If $\lambda = 0$, we have $\alpha = 0$. This contradicts our assumption that $\alpha \neq 0$ on W.

If $\lambda = \frac{c}{2(C_1^2+1)} \neq 0$, by (1.4.9) and $g(y) \neq 0$, we get $a = 0$. This contradicts our assumption. Thus we have completed the proof of Theorem 1.1. □

1.5 Proof of Theorem 1.2 for K a Positive Constant

From (1.3.4), we have $\frac{\partial^2 f}{\partial x^2} = -l^2 f$. For simplicity, we assume $l = 1$, i.e. $K = 1$. Thus, in a neighborhood U of (0, 0),

$$\begin{aligned} f(x, y) &= g_1(y)\sin x + g_2(y)\cos x \\ &= \frac{1}{g_1^2 + g_2^2}\sin(x + g(y)). \end{aligned}$$

where $g(y) = \cos^{-1} g_1(y)$. Since $g_2(0) \neq 0$ (otherwise $f(0, 0) = 0$), therefore, without loss of generality, we may assume $f(x, y) = \sin(x + g(y))$.

From (1.3.3) we have $c^2 \geq 8$. if $c^2 = 8$, we have $\alpha = 0$ and $\lambda = \frac{1}{4}$, and then $a = 0$.

This contradicts the assumption of $a > 0$. Thus we assume $c^2 - 8 > 0$.

Let $\theta = x + g(y)$, since $f = \sin\theta$, so $\beta = -\frac{\frac{\partial f}{\partial x}}{f} = -\cot\theta$. Thus, from (1.3.3), we have

$$(1.5.1)\qquad \begin{cases} \dfrac{\partial\lambda}{\partial x} - \dfrac{1}{\sin\theta}\dfrac{\partial\alpha}{\partial y} = (c-3\lambda)\cot\theta + a\alpha, \\ \dfrac{1}{\sin\theta}\dfrac{\partial\lambda}{\partial y} + \dfrac{\partial\alpha}{\partial x} = -3\alpha\cot\theta - a\lambda, \\ (c-4\lambda)\dfrac{\partial\lambda}{\partial x} - 4\alpha\dfrac{\partial\alpha}{\partial x} = 0, \\ (c-4\lambda)\dfrac{\partial\lambda}{\partial y} - 4\alpha\dfrac{\partial\alpha}{\partial y} = 0. \end{cases}$$

Solving (1.5.1) and taking (1.3.3) into account, we obtain

$$(1.5.2)\qquad \begin{cases} \dfrac{\partial\lambda}{\partial x} = -4A\alpha(\alpha\cot\theta + a\lambda - Ba), \\ \dfrac{\partial\lambda}{\partial y} = -4A\alpha((c-\lambda-3B)\cos\theta + a\alpha\sin\theta), \\ \dfrac{\partial\alpha}{\partial x} = (c-4\lambda)A(\alpha\cot\theta + a\lambda - Ba), \\ \dfrac{\partial\alpha}{\partial y} = -(c-4\lambda)A((c-\lambda-3B)\cos\theta + a\alpha\sin\theta), \end{cases}$$

where $A = \frac{c}{c^2-8}$ and $B = \frac{2}{c}$.

If $\alpha \equiv 0$ on U, we have $\lambda \equiv \frac{c}{4}$ on U, then $a = 0$ by (1.3.3). This contradicts our assumption.

If there exists a point $z \in U$ such that $\alpha(z) \neq 0$, we can find an open subset $V \subset U$ such that α nowhere zero on V. Thus, on V, by using $\frac{\partial^2\lambda}{\partial x\partial y} = \frac{\partial^2\lambda}{\partial y\partial x}$, (1.4.2) and (1.3 .3), we have

$$(1.5.3)\qquad P\alpha = Q\lambda - D$$

where

$$P = 3\alpha\cos\theta + g'\csc^2\theta = \frac{3a\sin^2\theta\cos\theta + g'}{\sin^2\theta}, \qquad D = \frac{-3B\sin^2\theta + c}{\sin\theta},$$

$$Q = \frac{(1-a^2)\sin^2\theta + 3\cos^2\theta}{\sin^2\theta} = \frac{3-(2+a^2)\sin^2\theta}{\sin\theta}.$$

Also, by (1.3.3), we have

$$(1.5.4) \qquad 2(P^2+Q^2)\lambda^2 + (P^2c + 4DQ)\lambda - 2D^2 - P^2.$$

From (1.5.3),

$$\frac{\partial P}{\partial x} + P\frac{\partial \alpha}{\partial x} = \frac{\partial Q}{\partial x}\lambda + Q\frac{\partial \lambda}{\partial x} - \frac{\partial D}{\partial x}.$$

Consider (1.3.3), (1.5.3) and (1.5.4). This implies

$$(1.5.5) \qquad K\lambda + F = 0,$$

where

$$\begin{aligned} K &= QP_x - AP^2ca + 4ABaQ^2 + 4AP^2Ba - PQ' + 4APD\cot\theta - 4AQDa \\ &\quad -AQPc\cot\theta, \\ F &= -P_xD - APDc\cot\theta - 4ABDQa + 2AQP\cot\theta + D_xP + 4AaD^2. \end{aligned}$$

Putting (1.5.5) back into (1.5.4) gives

$$(1.5.6) \qquad 2(P^2+Q^2)F^2 + (P^2c + 4DQ)FK + (2D^2 + P^2)K^2 = 0.$$

Let

$$u = \sin^2\frac{\theta}{2}, \qquad \text{and} \qquad t = g'(y).$$

Then (1.5.6) becomes the following polynomial equation in u,

$$b_{18}u^{18} + b_{17}u^{17} + \cdots + b_1 u + b_0 = 0, \tag{1.5.7}$$

where b_i are functions of a, c and t. That is to say that they are functions of y.

The followings are some coefficients we will use later:

$$\begin{aligned}
b_{18} &= 18874368(a-1)^2a^2(1+a)^2(a^2+c^2-9)\cdot \\
&\quad \cdot(576 - 64c^2 - 88a^2c^2 + 8a^4c^2 + 9a^2c^4), \\
b_{17} &= 169869312(a-1)^2a^2(1+a)^2(a^2+c^2-9)\cdot \\
&\quad \cdot(576 - 64c^2 - 88a^2c^2 + 8a^4c^2 + 9a^2c^4), \\
b_{13} &= 49152a(192471552a - 492549120a^3 + 331043328a^5 - 30965760a^7 \\
&\quad -14684544ac^2 + 38083584a^3c^2 + 23807616a^5c^2 - 56238336a^7c^2 \\
&\quad +9461760a^9c^2 - 430080a^{11}c^2 - 3647616ac^4 + 5267976a63c^4 \\
&\quad -10946376a^5c^4 + 10212216a^7c^4 - 886200a^9c^4 + 298368ac^6 \\
&\quad -17913a^3c^6 + 138138a^5c^6 - 418593a^7c^6 + 2688a^3c^8 + 37527a^5c^8 \\
&\quad -3939840t + 8607744a^2t - 4953600a^4t451584a^6t + 834048c^2t \\
&\quad -755392a^2c^2t - 4953600a^4t + 451584a^6c^2t - 195008a^8c^2t \\
&\quad +6272a^{10}c^2t - 39424c^4t - 133576a^2c^4 + 469344a^4c^4t \\
&\quad -7293432a^6c^4t + 20416a^8c^4t - 512c^6t + 10902a^2c^6t - 26721a^4c^6t \\
&\quad +14063a^6c^6t + 162a^2c^8 - 81a^4c^8t, \\
b_{16} &= 1179648(a-1)a^2(a+1)(a^2+c^2-9)(-32716 + 334080a^2 + 34048c^2
\end{aligned}$$

$13536a^2c^2 - 55680a^4c^2 + 4640a^6c^2 + 256c^4 - 4916a^2c^4 + 5200a^4c^4$

$-27a^2c^6.$

We will show that, for any a and c, the coefficients of (1.5.7) cannot be identically zero simultaneously on W. If this is true, let b_k be the biggest i, $(0 \leq i \leq 18)$, such that b_k is not identically zero on W . Thus there exists an open subset W' of W such that b_k is nowhere zero on W', So, on W' we have

$$u^k + \frac{b_{k-1}}{b_k}u^{k-1} + \cdots + \frac{b_1}{b_k}u + \frac{b_0}{b_k} = 0,$$

with all coefficients as function of y and leading coefficient 1. Thus, we can write $u = F(y)$ on W', i.e., $\sin^2 \frac{x+g(y)}{2} = F(y)$ on W', where $F(y)$ is some function of y. This is a contradiction.

Thus, to prove the theorem, it is sufficient to prove that, for any b and c, the coeficients b_i, $i = 0, 1, 2, \cdots, 18$, can not be identically zero simultaneously on W.

Case 1. $a \neq 1$ and $a^2+c^2 \neq 9$. In this case, b_{18} and b_{17} can not be zero simultaneously.

Otherwise, we have

(1.5.8) $$576 - 64c^2 - 88a^2c^2 + 8a^4c^2 + 9a^2c^4 = 0,$$

(1.5.9) $$-327168 + 334080a^2 + 34048c^2 + 13536a^2c^2 - 55680a^4c^2$$
$$+4640a^6c^2 + 256c^4 - 4916a^2c^4 + 5200a^4c^4 - 27a^2c^6 = 0.$$

Solve (1.5.8) for c^2 gives

(1.5.10) $$c^2 = \frac{64 + 88a^2 - 8a^4 \pm \sqrt{-20736a^2 + (-64 - 88a^2 + 8a^4)^2}}{18a^2}.$$

Putting the above back into (1.5.8), we get

$$143327232(a^2-4)(a^2-1)^4a^6 = 0.$$

Thus $a^2 = 4$ since $a^2 \neq 1$ and $a \neq 0$. Now put $a^2 = 4$ back into (1.5.10), we get $c^2 = 4$. This contradicts the assumption $c^2 > 8$.

Case 2. $a = 1$. In this case, we have $b_{18} = b_{17} = b_{16} = b_{15} = b_{14} = 0$. But

$$\begin{aligned} b_{13} &= 49152(165888t - 103680c^2t + 23328c^4t - 2268c^2t + 81c^8t \\ &= 3981312(c-2)(c+2)(c^2-8)^3t. \end{aligned}$$

So if $b_{13} \equiv 0$ on W, we have $t \equiv 0$ on W since $c^2 > 8$. Thus (5.7) becomes

$$\begin{aligned} &(-6115295232 + 4586471424c^2 - 1242169344c^4 + 143327232c^6 - 59719682c^8)u^{12} \\ &+ ... + (47775744c^2 - 25837056c^4 + 4064256c^6 - 198144c^8)u^4 \\ &+ (373248c^4 - 82944c^6 + 4608c^8)u^3 = 0. \end{aligned}$$

where

$$b_{12} = -5971968(c^2-4)^2(c^2-8)^2,$$

which is not zero since $c^2 > 8$. Therefore b_{12} and b_{13} can not be identically zero simultaneously on W.

Case 3. $a^2 + c^2 = 9$. In this case, $c^2 = 9 - a^2$. Substituting into (1.5.7), it becomes

$$\begin{aligned} &(-1179648a^6 + 2359296a^8 - 1179648a^{10})u^{12} + ... \\ &+ (-3276288a^6 + 1317888a^8 - 105984a^{10})u^4 \\ &+ (373248a^6 - 829944a^8 + 4608a^{10})u^3 = 0. \end{aligned}$$

But $b_{12} = -1179648(a^2-1)^2a^6$. Thus if $b_{12} = 0$, $a = 1$ since $a > 0$. So $c^2 = 9 - a^2 = 8$. This contradicts the assumption $c^2 > 8$. Thus $b_{12} \neq 0$. □

1.6 Proof of Theorem 1.2 for K a Negative Constant

Now we consider $K = -l^2 \neq 0$. For simplicity, we assume $l = 1$, i.e. $K = -1$.

As we did in the last section, we have $\frac{\partial f}{\partial x} = -\beta f$ and $\frac{\partial^2 f}{\partial x^2} = f$, and thus

$$f(x, y) = g_1(y)e^x + g_2(y)e^{-x}$$

in a neighborhood U of (0,0) and $f \neq 0$ on U.

Since g_1 and g_2 are not simutaneously vanished at 0, otherwise $f(0,0) = 0$. Thus without loss of generality, we assume $g(0) \neq 0$. Therefore there exists an open subset V of U such that $g_1(y)$ is nowhere zero on V, so on V

$$f = g_1(y)(e^x + \frac{g_2(y)}{g_1(y)}e^{-x}).$$

So, without loss of generality, by replacing $g_1(y)dy$ by dY if necessary, we can assume$f(x, y) = e^x + g(y)e^{-x}$. Therefore

$$\beta = -\frac{\frac{\partial f}{\partial x}}{f} = \frac{-e^x + ge^{-x}}{e^x + ge^{-x}}.$$

From (1.3.3) and

$$c\lambda - 2\lambda^2 - 2\alpha^2 = -1, \tag{1.6.1}$$

we have

$$(1.6.2)\qquad \begin{cases} \dfrac{\partial\lambda}{\partial x} = 4A\alpha(2a + ac\lambda^2 - \beta a\alpha), \\ \dfrac{\partial\lambda}{\partial y} = -4A\alpha(c\beta\lambda + ac\alpha - 6\beta - \beta c^2)f, \\ \dfrac{\partial\alpha}{\partial x} = A(c-4\lambda)(2a + ac\lambda - c\beta\alpha), \\ \dfrac{\partial\alpha}{\partial y} = -A(c-4\lambda)(c\beta\lambda + ac\alpha - 6\beta - \beta c^2)f. \end{cases}$$

where $A = \frac{1}{c^2+8}$. If $\alpha \equiv 0$ on V, by (1.6.1) we have

$$\lambda = \frac{c \pm \sqrt{c^2+4}}{4} \quad \text{on} \quad V,$$

which is a nozero constant. So by (1.3.3), we have $a \equiv 0$. This contraducts with our assumption.

If there exists a point $z \in V$ such that $\alpha(z) \neq 0$, we can find an open subset W of V such that α is never zero on W. Thus, on W, $\frac{\partial^2\lambda}{\partial x\partial y} = \frac{\partial^2\lambda}{\partial y\partial x}$ and (1.3.3) gives

$$(1.6.3)\qquad P\alpha = Q\lambda - D,$$

where

$$\begin{aligned} P &= -(acf\beta + 3acf\beta - c\frac{\partial\beta}{\partial y} + ac\frac{\partial f}{\partial x}), \\ Q &= (3cf\beta^2 - a^2cf + \frac{\partial f}{\partial x}\beta c + fc\frac{\partial\beta}{\partial x}, \\ D &= c^2f\beta^2 + (6+c^2)\frac{\partial}{\partial x}(f\beta). \end{aligned}$$

From (1.3.3) and (1.6.3), we have

$$(1.6.4)\qquad 2(P^2+Q^2)\lambda^2 - (cP^2 - 4QD)\lambda + 2D^2 - P^2 = 0.$$

On the other hand, from (1.6.3),

$$\frac{\partial P}{\partial x}\alpha + P\frac{\partial \alpha}{\partial x} = \frac{\partial Q}{\partial x}\lambda + Q\frac{\partial \lambda}{\partial x} - \frac{\partial D}{\partial x}.$$

By (1.6.1), (1.6.4), this implies

$$G\lambda + F = 0, \tag{1.6.5}$$

where

$$\begin{aligned} G &= P_xQ - PQA\beta c^2 - 8aQ^2A - 4DPA\beta c + 4DQAac + P^2Aac^2 \\ &\quad -8P^2Aa - PQ_x + 2QA\beta c^2 \\ F &= -P_xD + PDA\beta c^2 + 8aQDA + 2QPA\beta c + 2P^2Aac + PQ_x. \end{aligned}$$

Combining (1.6.4) and (1.6.6), we have

$$2(P^2+Q^2)F^2 + (P^2c + 4DQ)FG + (2D^2 - P^2)G^2 = 0. \tag{1.6.6}$$

Let

$$u = e^x \qquad \text{and} \qquad t = g'(y).$$

Then (1.6.6) becomes

$$b_{36}u^{36} + b_{35}u^{35} + \ldots + b_1u + b_0 = 0, \tag{1.6.7}$$

where

$$b_0 = 18a^2A^2c^4(5184 + 15552a^2 + 8640a^4 + 576a^6 + 64a^2c^2$$

$$-1872a^4c^2 - 2016a^6c^2 - 240a^8c^2 - 8a^{10}c^2 - 81a^2c^4 - 567a^4c^4$$
$$-459a^6c^4 - 29a^8c^4 - 27a^4c^6 - 27a^6c^6)g^{18}$$
$$b_1 = 9a^3A^2c^5(576 + 576a^2 - 72a^2c^2 - 128a^4c^2 - 8a^6c^2 - 9a^2c^4 - 15a^4c^4)g^{17},$$
$$b_{36} = 9a^2A^2c^4(10368 + 21888a^2 + 12672a^4 + 1152a^6 - 1152c^2 + 1424a^2c^2$$
$$800a^4c^2 - 2112a^6c^2 - 352a^8c^2 - 16a^{10}c^2 + 32c^4 - 34a^2c^4 - 6a^4c^4$$
$$-646a^6c^4 - 58a^8c^4 + 27a^4c^6 - 54a^6c^6),$$
$$b_{35} = 9a(a^2-2)A^2c^5(-576 - 576a^2 + 32c^2 - 24a^2c^2 + 96a^4c^2$$
$$+8a^6c^2 - 3a^2c^4 + 15a^4c^4).$$

By the same argument as in the last section, it is sufficient to prove that, for any a and c, the coefficients of (1.6.7) can not simultaneously be identically zero on V.

Case 1. $g \equiv 0$ on V. In this case, $t = g' \equiv 0$ on V . and (1.6.7) becomes

$$\bar{b}_0 + \bar{b}_1 u + \bar{b}_2 u^2 = 0. \tag{1.6.8}$$

where

$$\bar{b}_0 = (a^2-2)^2c^2(8 - a^2c^2), \qquad \bar{b}_1 = \frac{a}{9A^2c^4}b_{35}, \qquad \bar{b}_2 = \frac{1}{9A^2c^4}b_{36}.$$

Also it is easy to compute

$$P = 3acu \qquad Q = (2-a^2)cu, \qquad \text{and} \qquad D = -6u.$$

We will prove $\bar{b}_i$, $i = 1, 2, 3$, can not be zero simultaneously.

Subcase 1.1. $a^2 = 2$. By $P\alpha = Q\lambda - D$, we have $3acu\alpha = 6u$, i.e.,

$$(1.6.8) \qquad \alpha = \frac{2}{ac} = \frac{\sqrt{2}}{c} \quad \text{on} \quad V.$$

by (1.6.1), we have $\lambda \equiv$ constant on V. But $\beta = -\frac{e^x}{e^x} = -1$. Fro m(1.3.3), we get $3\alpha + a\lambda = 0$. By using (1.6.8), this implies

$$(1.6.9) \qquad \lambda = -\frac{3}{c}.$$

Putting (1.6.8) into (1.6.1) gives $c^2 + 11 = 0$. This is a contradiction.

Subcase 1.2. $a^2c^2 = 8$ but $a^2 \neq 2$. In this case, put $a^2 = \frac{8}{c^2}$ in $\bar{b}_1$ We have

$$\bar{b}_1 = \frac{8}{c^4}(512 + 192c^2 + 24c^4 + c^6)$$

which is never zero.

Case 2. There exists $z \in V$ such that $g(z) \neq 0$. Thus we can find an open subset V' of V such that $g(y)$ is nowhere zero on V'. We will prove that b_0, b_1, b_{35} and b_{36} can not be zero simultaneously on V'.

To this end, we assme $b_0 = b_1 = b_{35} = b_{36} = 0$. Then by $b_{35} = 0$, we have

$$(1.6.10) \qquad a^2 = 2,$$

or

$$(1.6.11) \qquad -576 - 576a^2 + 32c^2 - 24a^2c^2 + 96a^4c^2 + 8a^6c^2 - 3a^2c^4 + 15a^4c^4 = 0.$$

Subcase 2.1. If (1.6.10) is true, we put $a^2 = 2$ back to $b_{36} = 0$, it gives

$$324(11 + c^4)(32 - 8c^2 - c^4) = 0.$$

Thus $c^2 = \frac{-8+8\sqrt{3}}{2} \approx 2.9282$. On the other hand, put $a^2 = 2$ in $b_1 = 0$ gives

$$6(-288 + 120c^2 + 13c^4) = 0,$$

so $c^2 = \frac{-120+24\sqrt{51}}{26} \approx 1.9767$. This is a contradiction.

Subcase 2.2. if (1.6.11) is true. From $b_1 = 0$, we have

$$576 + 576a^2 - 72a^2c^2 - 128a^4c^2 - 8a^6c^2 - 9a^2c^4 - 15a^4c^4 = 0. \tag{1.6.12}$$

Then (1.6.11)+(1.6.12) gives $4c^2(-8 + 24a^2 + 8a^4 + 3a^2c^2) = 0$, then

$$c^2 = \frac{8(1 - 3a^2 - a^4)}{3a^2}.$$

Put the above into (1.6.12) and $b_{36} = 0$, we have

$$3 - 34a^2 - 13a^4 + 10a^6 + 4a^8 = 0, \tag{1.6.13}$$

and

$$16 - 309a^2 + 1768a^4 + 520a^6 - 992a^8 - 293a^{10} + 144a^{12} + 46a^{14} = 0. \tag{1.6.14}$$

Solving (1.6.13), we have

$$a^2 \approx 0.08523 \qquad \text{or} \qquad a^2 \approx 1.8009$$

Putting the above back into the left hand side of (1.6.14) to

$$\text{LHS} \quad \text{of} \quad (1.6.14) \approx 2.7704 \quad \text{or} \quad -15.3325$$

respectively. Both give contradictions. □

CHAPTER 2

HYPERSURFACES WITH CONSTANT MEAN CURVATURE SATISFYING CHEN'S EQUALITY

In this chapter, we will completely classify hypersurfaces in real space forms with constant mean curvature satisfying Chen's equality.

2.1 A Remannian Invariant and Chen's Equality

According to the well-known Nash imbedding theorem, every Riemannian n-manifold admits an isometric immersion into the Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$. In general, there exist enormously many isometric immersions from a Riemannian manifold into Euclidean spaces if no restriction on the codimension is made. Associated to submanifold of a Riemannian manifold there are several extrinsic invariants beside its intrinsic invariants. Among intrinsic invariants, sectional curvature and scalar curvature are the most fundamental ones. On the other hand, among extrinsic invariants, the mean curvature function and shape operator are most fundamental.

One of the most fundamental problems in submanifold theory is to obtain simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold and to find applications. Many famous results in differential geometry such as the isoperimetric inequality and Gauss-Bonnet's theorem, among others, are results in this direction.

Let M^n be an n-dimensional Riemannian manifold. In 1993, Chen introduced an important Riemannian invariant δ_M of M^n by $\delta_M(p) = \tau(p) - \inf K(p)$, where $\inf K$ is the function which assigs to each $p \in M^n$ the infimum of $K(\pi)$, where π runs over all 2-planes in T_pM and τ is defined by $\tau = \sum_{i<j} K(e_i \wedge e_j)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM^n. For $n = 2$, this invariant vanishes trivially. If M^n is any submanifold immsersed in an m-dimensional Riemannian space form $R^m(\epsilon)$ of constant sectional curvature ϵ, Chen proved in [7] a sharp inequality involving Chen invariant δ_M and the squared mean curvature H^2, namely

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} H^2 + (n+1)(n-2)\epsilon. \tag{2.1.1}$$

Inequality (2.1.1) is known as Chen's inequality and has some important applications, for example, it gives rise to the second Riemannian obstruction for a Riemannian manifold to admit a minimal isometric immersion into a Euclidean space. It also gives rise to an obstruction to Lagrangian isometric immersions from compact Riemannian manifolds with finite fundamental group π_1 into complex space forms (see [9] for details).

Since (2.1.1) is a very general and sharp inequality, it is natural and important to investigate and to understand submanifolds in a Riemannian space form which satisfy the equality case of Chen's inequality, which is known as Chen's equality:

$$\delta_M = \frac{n^2(n-2)}{2(n-1)}H^2 + (n+1)(n-2)\epsilon. \tag{2.1.2}$$

Submanifolds satisfying this basic equality were studied recently in many papers (cf. for instant, [7],[8],[9],[11],[12],[14],[19],[22],[23]). In this respect, we would like to point out in particular that 3-dimensional totally real submanifolds satisfying Chen's equality in the nearly Kähler 6-sphere $S^6(1)$ have been completely classified in [23] by F. Dillen and L. Vrancken; and minimal hypersurfaces in non-flat Riemannian space forms satisfying Chen's equality were classified completely in [14] by B. Y. Chen and L. Vrancken; roughly speaking they proved that a non-totally geodesic minimal hypersurface of $S^{n+1}(1)$ satisfies equality (2.1.2) if and only if it is a tubular hypersurface with radius $\frac{\pi}{2}$ about a 2-dimensional minimal surface in $S^{n+1}(1)$ and a non-totally geodesic minimal hypersurface of the hyperbolic $(n+1)$-space $H^{n+1}(-1)$ satisfies equality (2.1.2) if and only if it is a "suitable tubular hypersurface" about a minimal surface in the de Sitter space-time $S_1^{n+1}(1)$ (cf. [14] for details).

We will investigate the most fundamental case; namely hypersurfaces satisfying Chen's equality. We will deal with hypersurfaces with constant mean curvature in this chapter and conformally flat hypersurfaces in the next chapter. Since Chen's equality is trivial when $n = 2$, we will consider n-dimensional submanifolds for $n > 2$.

2.2 Main Results

Theorem 2.1 *A hypersurface M^n $(n > 2)$ of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ with constant mean curvature satisfies equality* (2.1.2) *if and only if either M^n is minimal or M^n is an open portion of a spherical hypercylinder $\mathbb{R} \times S^{n-1}(r)$.*

Theorem 2.2 *Let M^n $(n > 2)$ be a hypersurface with constant mean curvature in the* *sphere* $S^{n+1}(1)$. *Then* M^n *satisfies equality* (2.1.2) *if and only if one of the following two cases occurs.*

1. M^n *is a totally geodesic hypersurface.*

2. *There is an open dense subset* U *of* M^n *and a non-totally geodesic, isometric, minimal immersion* $\phi : B^2 \to S^{n+1}(1)$ *from a surface* B^2 *into* $S^{n+1}(1)$ *such that* U *is an open subset of the unit normal bundle* NB^2 *defined by*

$$N_pB^2 = \{\xi \in T_{\phi(p)}S^{n+1}(1) | < \xi, \xi >= 1 \text{ and } < \xi, \phi_*(T_pB^2) \ = 0\}.$$

Let $\mathbb{E}_1^{n+2}$ denote the $(n+2)$-dimensional Minkowski space-time with the Lorenzian metric $g = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$. Recall that the unit hyperbolic space $H^{n+1}(-1)$ and the unit de Sitter space-time $S_1^{n+1}(1)$ are isometrically imbedded in $\mathbb{E}_1^{n+2}$ respectively in the following standard ways:

$$H^{n+1}(-1) = \{\mathbf{x} = (x_1, \ldots, x_{n+2}) \in \mathbb{E}_1^{n+2} \mid < \mathbf{x}, \mathbf{x} >= -1\},$$

$$S_1^{n+1}(1) = \{\mathbf{x} \in \mathbb{E}_1^{n+2} | < \mathbf{x}, \mathbf{x} >= 1\}.$$

Theorem 2.3 *Let* M^n $(n > 2)$ *be a hypersurface with constant mean curvature in the hyperbolic space* $H^{n+1}(-1)$. *Then* M^n *satisfies equality* (2.1.2) *if and only if one of the following three cases occurs.*

1. M^n *is a totally geodesic hypersurface.*

2. M^n *is a tubular hypersurface with radius* $r = \coth^{-1}(\sqrt{2})$ *about a 2-dimensional totally geodesic surface of* $H^{n+1}(-1)$.

3. *There is an open dense subset* U *of* M^n *and a non-totally geodesic, isometric, minimal immersion* $\phi : B^2 \to S_1^{n+1}$ *from a surface* B^2 *into the de Sitter space-*

time $S_1^{n+1}(1)$ *such that* U *is an open subset of the unit normal bundle* NB^2 *of* B^2 *defined by*

$$N_pB^2 = \{\xi \in T_{\phi(p)}S_1^{n+1}(1) | < \xi, \xi >= -1 \text{ and } < \xi, \phi_*(T_pB^2) >= 0\}.$$

2.3 Prelimilaries

Assume that M^n is a hypersurface in real-space-form $R^{n+1}(\epsilon)$, ϵ = 1, 0, or -1. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C \leq n+1; \qquad 1 \leq i, j, k \leq n; \qquad 3 \leq \alpha, \beta, \gamma \leq n.$$

Denote by $A = A_\xi$ the shape operator of M^n in $R^{n+1}(\epsilon)$ with respect to a unit normal vector ξ and by h the second fundamental form M^n in $R^{n+1}(\epsilon)$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A with respect to orthonormal eigenvector field $e_1, e_2, ..., e_n$, i.e., we have

(2.3.1) $$Ae_i = \lambda_i e_i.$$

If M^n satisfies the Chen's equality (2.1.2), then, by rearranging $e_1, e_2, ..., e_n$ if necessary, we have ([7])

(2.3.2) $$\lambda_1 = a, \qquad \lambda_2 = \mu - a, \qquad \lambda_\alpha = \mu,$$

Let $\omega^1, \omega^2, ..., \omega^n$ denote the dual frame of $e_1, e_2, ..., e_n$. Then Cartan's structure equations give

$$(2.3.3) \qquad d\omega^i = -\omega^i_j \wedge \omega^j, \qquad d\omega^i_j = \omega^i_k \wedge \omega^k_j + (\epsilon + \lambda_i\lambda_j)\omega^i \wedge \omega^j,$$

where (ω^A_B) are the connection forms. From (2.3.3) and Codazzi equation, we have

$$(2.3.4) \qquad \begin{cases} e_i\lambda_j = (\lambda_i - \lambda_j)\omega^j_i(e_j) \\ (\lambda_j - \lambda_k)\omega^k_j(e_i) = (\lambda_i - \lambda_k)\omega^k_i(e_j) \end{cases}$$

for distinct i, j, k.

Let Γ^k_{ij} be defined by $\nabla_{e_i}e_j = \sum_{k=1}^n \Gamma^k_{ij}e_k$, where ∇ is the Levi-Civita connection on M^n. Then we have

$$\omega^k_i = \sum_k \Gamma^k_{ij}\omega^j, \qquad \Gamma^k_{ij} = \omega^k_i(e_j), \qquad \Gamma^k_{ij} = -\Gamma^i_{kj}.$$

In this way, (2.3.4) becomes

$$(2.3.5) \qquad \begin{cases} e_i\lambda_j = (\lambda_i - \lambda_j)\Gamma^i_{ij} \\ (\lambda_j - \lambda_k)\Gamma^k_{ji} = (\lambda_i - \lambda_k)\Gamma^k_{ij} \end{cases}$$

for distinct i, j, k. Let H denote the mean curvature of M^n in $R^{n+1}(\epsilon)$. Then $H = \dfrac{n-1}{n}\mu$.

In the following, Let V denote the open subset of M^n on which M has exactly three distinct principle curvatures, i.e.

$$(2.3.6) \qquad V = \{x \in M | a(x) \neq 0, a(x) \neq \mu, a(x) \neq \frac{\mu}{2}\},$$

where a and μ are given by (2.3.2). From (2.3.2) and (2.3.5) we obtain the following lemma.

Lemma 2.1 *Let M^n be a hypersurface in a Riemannian space form $R^{n+1}(\epsilon)$. If M^n satisfies (2.1.2), then on V we have*

$$(2.3.7)\quad \begin{cases} \Gamma^2_{1\alpha} = \frac{a}{2a-\mu}\Lambda_\alpha, & \Gamma^1_{\alpha 2}\frac{a}{\mu-a}\Lambda_\alpha, \quad \Gamma^\beta_{1\alpha} = \Gamma^\beta_{2\alpha} = 0, \quad \alpha = \beta \\ \Gamma^\alpha_{\alpha\alpha} = \frac{e_2\mu}{a-\mu}, & \Gamma^\alpha_{2\alpha} = -\frac{e_2\mu}{a}, \quad \Gamma^2_{12} = -\frac{e_1(\mu-a)}{\mu-2a}, \\ \Gamma^1_{21} = \frac{e_2 a}{\mu-2a}, & \Gamma^2_{\alpha 2} = \frac{e_\alpha(\mu-a)}{a}, \quad \Gamma^1_{\alpha 1} = \frac{e_\alpha a}{\mu-a}. \end{cases}$$

$$(2.3.8)\quad \begin{cases} \omega^1_\alpha = \frac{e_\alpha}{\mu-a}\omega^1 + \frac{a\Lambda_\alpha}{\mu-a}\omega^2 + \frac{e_1\mu}{\mu-a}\omega^a l. \\ \omega^1_2 = \frac{e_2 a}{\mu-2a}\omega^1 + \frac{e_1(\mu-a)}{\mu-2a}\omega^2 + \frac{a}{\mu-2a}\sum_{\alpha=3}^{n}\Lambda_\alpha\omega^\alpha, \\ \omega^2_\alpha = \Lambda_\alpha\omega^1 + \frac{e_\alpha(\mu-a)}{a}\omega^2 + \frac{e_2\mu}{a}\omega^\alpha, \end{cases}$$

where $\Lambda_\alpha = \Gamma^2_{\alpha 1}$.

Proof. It is clear by (2.3.5). □

We also need the following lemma.

Lemma 2.2 *Let M^n be a hypersurface in $R^{n+1}(\epsilon)$. If M^n satisfies equation (2.1.2), then on V we have*

$$(2.3.9)\quad \begin{aligned} & e_1\left(\frac{e_1\mu}{\mu-a}\right) - e_\alpha\left(\frac{e_\alpha a}{\mu-a}\right) - \frac{(e_\alpha a)^2}{(\mu-a)^2} - \frac{2a\Lambda_\alpha^2}{\mu-2a} - \frac{(e_1\mu)^2}{(\mu-a)^2} \\ &= -\frac{(e_2 a)(e_2\mu)}{(\mu-2a)a} - \frac{1}{\mu-a}\sum_{\beta\neq\alpha}^{n}(e_\beta a)\Gamma^\beta_{\alpha\alpha} + a\mu + \epsilon, \end{aligned}$$

$$(2.3.10)\quad \begin{aligned} & e_2\left(\frac{e_1\mu}{\mu-a}\right) - e_\alpha\left(\frac{a\Lambda_\alpha}{\mu-a}\right) + \frac{a^2\Lambda_\alpha(\epsilon_\alpha a)}{(\mu-a)^2(\mu-2a)} + \frac{e_\alpha(a-\mu)\Lambda_\alpha}{\mu-a} \\ & -\frac{e_\alpha(\mu-a)}{\mu-2a}\Lambda_\alpha - \frac{(e_1\mu)(\epsilon_2\mu)}{(\mu-a)a} = \frac{e_1(a-\mu)(e_2\mu)}{(\mu-2a)a} - \frac{1}{\mu-a}\sum_{\alpha\neq\beta}^{n} a\Lambda_\beta\Gamma^\beta_{\alpha\alpha}, \end{aligned}$$

$$
\begin{aligned}
(2.3.11)\qquad & e_1(\frac{a\Lambda_\alpha}{\mu-a}) - e_2(\frac{e_\alpha a}{\mu-a}) - \frac{(e_\alpha a)(e_2 a)}{(\mu-a)(\mu-2a)} - \frac{\mu\Lambda_\alpha e_1(\mu-a)}{(\mu-a)(\mu-2a)}, \\
& + \frac{(\mu-2a)\Lambda_\alpha e_1\mu}{(\mu-a)^2} = \frac{(e_2 a)e_\alpha(\mu-a)}{a(\mu-2a)} - \sum_{\alpha\neq\beta}^{n}(\frac{\epsilon_\beta a}{\mu-a}\Gamma^\beta_{\alpha 2} - \frac{a\Lambda_\beta}{\mu-a}\Gamma^\beta_{\alpha 1}).
\end{aligned}
$$

Proof. By applying the second Cartan's structure equation and by using Lemma 2.1, we may compute $d\omega^1_\alpha$ in two different ways. Formulas (2.3.9), (2.3.10) and (2.3.11) are obtained by comparing the coefficients of $\omega^1\wedge\omega^\alpha, \omega^2\wedge\omega^\alpha$, and $\omega^1\wedge\omega^2$ of $d\omega^1_\alpha$, respectively. □

Similarly, by computing $d\omega^\alpha_2$ in two different ways and by comparing the coefficients of $\omega^1\wedge\omega^\alpha, \omega^2\wedge\omega^\alpha, \omega^1\wedge\omega^2$ of $d\omega^2_\alpha$, respectively, we obtain

Lemma 2.3 *Let M^n be a hypersurface in a Riemainnian space form $R^{n+1}(\epsilon)$. If M^n satisfies equality (2.1.2), then on V we have*

$$
\begin{aligned}
(2.3.12)\qquad & e_2(\frac{e_2\mu}{a}) + e_\alpha(\frac{e_\alpha(a-\mu)}{a}) + \frac{2a^2\Lambda^2_\alpha}{(\mu-a)(\mu-2a)} - \frac{(e_\alpha(\mu-a))^2}{a^2} - \frac{(e_2\mu)^2}{a^2}, \\
& = \frac{e_1(\mu-a)(e_1\mu)}{(\mu-a)(\mu-2a)} - \sum_{\alpha\neq\beta}^{n}\frac{e_\beta(\mu-a)}{a}\Gamma^\beta_{\alpha\alpha} + (\mu-a)\mu + \epsilon,
\end{aligned}
$$

$$
\begin{aligned}
(2.3.13)\qquad & e_1(\frac{e_2\mu}{a}) - e_\alpha\Lambda_\alpha - \frac{\Lambda_\alpha e_\alpha a}{\mu-a} - \frac{\mu-a}{a(\mu-2a)}\Lambda_\alpha e_\alpha(\mu-a) - \frac{(e_2\mu)(e_1\mu)}{a(\mu-a)} \\
& = \frac{(e_2 a)(e_1\mu)}{(\mu-a)(\mu-2a)} - \frac{a\Lambda_\alpha e_2 a}{(\mu-a)(\mu-2a)} - \sum_{\alpha\neq\beta}^{n}\Lambda_\beta\Gamma^\beta_{\alpha\alpha},
\end{aligned}
$$

$$
(2.3.14)\quad \begin{aligned} & e_1(\frac{e_\alpha(\mu-a)}{a}) - e_2\Lambda_\alpha - \frac{\mu\Lambda_\alpha e_2 a}{(\mu-a)(\mu-2a)} - \frac{(e_1(\mu-a))(e_\alpha(\mu-a))}{a(\mu-2a)} \\ & + \frac{(\mu-2a)\Lambda_\alpha(e_2\mu)}{a(\mu-a)} = -\frac{(e_1(\mu-a))(e_\alpha a)}{(\mu-a)(\mu-2a)} - \sum_{\alpha\neq\beta}^{n}(\Lambda_\beta\Gamma_{\alpha 2}^{\beta} - \frac{e_\beta(\mu-a)}{a}\Gamma_{\alpha 1}^{\beta}). \end{aligned}
$$

2.4 Proof of Theorems 2.1, 2.2 and 2.3

Let M^n be a hypersurface with constant mean curvature H in a Riemannian space form $R^{n+1}(\epsilon)$ ($n > 2, \epsilon = 1, -1$, or 0) which satisfies Chen's equality (2.1.2). If M^n is not minimal, then $\mu = \frac{n}{n-1}H$ is a nonzero constant. In this case, (2.3.9) and (2.3.12) reduce respectively to

$$
(2.4.1)\quad \begin{aligned} & (a-\mu)(e_\alpha e_\alpha a) - 2(e_\alpha a)^2 = \frac{2a\Lambda_\alpha^2(\mu-a)^2}{\mu-2a} + (\epsilon+\mu a)(\mu-a)^2 \\ & - \sum_{\beta\neq\alpha}^{n}(e_\beta a)\Gamma_{\alpha\alpha}^{\beta}(\mu-a), \end{aligned}
$$

$$
(2.4.2)\quad \begin{aligned} & a(e_\alpha e_\alpha a) - 2(e_\alpha a)^2 = -\frac{2a^4\Lambda_\alpha^2}{(\mu-a)(\mu-2a)} + (\epsilon+(\mu-a))a^2 \\ & + a\sum_{\beta\neq\alpha}^{n}(e_\beta a)\Gamma_{\alpha\alpha}^{\beta}. \end{aligned}
$$

From (2.4.1) and (2.4.2) we obtain on V that

$$
(2.4.3)\quad \begin{aligned} & e_\alpha e_\alpha a = -\frac{2a\Lambda_\alpha^2(\mu^2-3\mu a+3a^2)}{(\mu-a)(\mu-2a)} + (\mu-2a)(a^2-a\mu-\epsilon) \\ & + \sum_{\alpha\neq\beta}^{n}(e_\beta a)\Gamma_{\alpha\alpha}^{\beta}, \\ & (e_\alpha a)^2 = -a^2(\Lambda_\alpha^2 + (\mu-a)^2) - \frac{\epsilon}{2}a(\mu-a). \end{aligned}
$$

Since μ is constant, (2.3.13) reduces to

$$e_\alpha \Lambda_\alpha = \frac{\Lambda_\alpha (e_\alpha a)(\mu^2 - 3a\mu + 2a^2)}{(\mu - a)(\mu - 2a)} + \sum_{\alpha \neq \beta}^{n} \Lambda_\beta \Gamma^\beta_{\alpha\alpha}. \tag{2.4.4}$$

Now, by applying Lemma 2.1 and Cartan's structure equations, we can compute $d\omega^1_\alpha$ and $d\omega^2_\alpha$ in two dofferent ways. Afetr that, by comparing the coefficients of $\omega^1 \wedge \omega^\beta$ and $\omega^2 \wedge \omega^\beta$ with $\beta \neq \alpha$ in the formulas of $d\omega^1_\alpha$ and of ω^2_α so obtained, we find respectively the following two formulas:

$$e_\beta e_\alpha a + \frac{2(e_\alpha a)(e_\beta a)}{\mu - a} + \frac{2a(\mu - a)\Lambda_\alpha \Lambda_\beta}{\mu - 2a} = \sum_{\gamma=3}^{n} (e_\gamma a) \Gamma^\gamma_{\alpha\beta},$$

$$e_\beta e_\alpha a - \frac{2(e_\alpha a)(e_\beta a)}{a} + \frac{2a^3 \Lambda_\alpha \Lambda_\beta}{(\mu - a)(\mu - 2a)} = \sum_{\gamma=3}^{n} (e_\gamma a) \Gamma^\gamma_{\alpha\beta}.$$

By taking the difference of these two equations, we obtain on V that

$$(e_\alpha a)(e_\beta a) = -a^2 \Lambda_\alpha \Lambda_\beta, \qquad \alpha \neq \beta. \tag{2.4.5}$$

Now, by applying (2.4.3), we may also obtain

$$-(e_\alpha a)^2 = a^2(\Lambda^2_\alpha + (\mu - a)^2) + \frac{\epsilon}{2} a(\mu - a),$$

$$-(e_\beta a)^2 = a^2(\Lambda^2_\beta + (\mu - a)^2) + \frac{\epsilon}{2} a(\mu - a).$$

By taking the difference of these two equations, we get

$$(e_\alpha a)^2 - (e_\beta a)^2 = a^2(\Lambda^2_\beta - \Lambda^2_\alpha). \tag{2.4.6}$$

Combining (2.4.5) and (2.4.6), we obtain, on V,

$$(e_\alpha a)^2 = a^2 \Lambda_\beta^2, \qquad \alpha \neq \beta. \tag{2.4.7}$$

Case (i): $\epsilon = 1$, $R^{n+1}(1) = S^{n+1}(1)$. If M^n is non-minimal, follow the exact arguments as in Case (iii) by replacing $\epsilon = -1$ by $\epsilon = 1$, we know that a is constant on each component of V. Therefore (2.4.7) implies $\Lambda_\alpha = 0$. Then the second identity of (2.4.3) implies that V is an empty set. Thus, M^n is an isoparametric hypersurface of $R^{n+1}(1)$ with at most two distinct principle curvatures given either by $\lambda_1 = 0, \lambda_2 = \dots = \lambda_n = \mu$ or by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu, \lambda_3 = \dots = \lambda_n = \mu$. Both cases are impossible according to a well-known result of E. Cartan [5]. Consequently, M^n is minimal in $S^{n+1}(1)$. Let U denote the open subset of M^n consisting of non-totally geodesic points. Then U is an open dense subset of M^n. Now, by a result of [7], U has relative nullity $n-2$. Thus, by applying a result of Dajczer and Gromoll (Lemma 2.2 of [18]), the Gauss image B^2 of U is a minimal surface in the unit shpere. Consequently, U is an open subset of unit bundle NB^2 defined in Theorem 2.2.

Case (ii): $\epsilon = 0, R^{n+1}(0) = \mathbb{E}^{n+1}$. If M^n is non-minimal, then the second equation in (2.4.3) implies $a = 0$ or $a = \mu$ on V which contradicts the definition of V. Thus V is an empty set. Hence, M is a non-minimal isoparametric hypersurface with at most two distinct principle curvatures given either by $\lambda_1 = 0, \lambda_2 = \dots = \lambda_n = \mu$ or by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu, \lambda_3 = \dots = \lambda_n = \mu$. It is well-known that the first case occurs if and only if M^n is an open portion of a spherical hypercylinder $\mathbb{R} \times S^{n-1}(r)$ for some $r > 0$. It is also known that the latter case cannot occur.

Case (iii): $\epsilon = -1, R^{n+1}(-1) = H^{n+1}(-1)$. First we assume that M^n is non-minimal. We claim that the function a is constant on each component of V. We divide the proof of this claim into three cases.

Case (iii-1): $n \geq 5$. From (2.4.7) we have

$$(2.4.8) \qquad (e_3a)^2 = (e_4a)^2 = ... = (e_na)^2 = a^2\Lambda_\alpha^2, \qquad \alpha = 3, ..., n.$$

Without loss of generality, we may assume $e_3a = a\lambda_3$. By using (2.4.5), we have $e_4a = -a\Lambda_4$ and $e_5a = -a\Lambda_5$ which imply

$$(e_4a)(e_5a) = a^2\Lambda_4\Lambda_5.$$

Thus $e_\alpha a = \Lambda_\alpha = 0$ on V by (2.4.5) and (2.4.8). Hence, by applying the first equation in (2.4.3), we obtain $a^2 - a\mu - \epsilon = 0$ on V. Since μ is constant, this implies that a is constant on each component of V.

Case (iii-2): $n = 4$. from (2.4.5) and (2.4.7) we may assume, without loss of generality, that $e_3a = a\Lambda_4$ and $e_4a = -a\Lambda_3$. By differentiating the second equation of (2.4.3) with respect to e_3, we get

$$(2.4.9) \qquad \begin{aligned} &-2(e_3a)(e_3e_3a) = 2(e_3a)a(\Lambda_3^2 + (\mu - a)^2) + a^2(2\Lambda_3(e_3\Lambda_3) \\ &-2(\mu - a)(e_3a) + \tfrac{\epsilon}{2}(\mu - 2a)e_3a. \end{aligned}$$

On the other hand, since $\Lambda_4 = e_3(\ln a)$, (2.4.4) yields

$$e_3\Lambda_3 = \frac{(e_3a)\Lambda_3(\mu^2 - 3a\mu + 4a^2)}{(\mu - 2a)(\mu - a)a} + \frac{e_3a}{a}\Gamma_{33}^4.$$

Substituting this into (2.4.9), we find

$$(2.4.10) \qquad \begin{aligned} &-(e_3a)(e_3e_3a) = (e_3a)(\frac{2a\Lambda_3^2(\mu^2 - 3a\mu + 3a^2)}{(\mu - a)(\mu - 2a)} + a(\mu - 2a)(\mu - a) \\ &+\frac{\epsilon}{4}(\mu - 2a) - (e_4a)\Gamma_{33}^4). \end{aligned}$$

If e_3a does not vanish identically on V, there exists an open set $W \subset V$ such that

e_3a is nowhere zero on W. Thus, on W, (2.4.8) becomes

$$\begin{aligned} -e_3e_3a = {} & \frac{2a\Lambda_3^2(\mu^2 - 3a\mu + 3a^2)}{(\mu - a)(\mu - 2a)} + a(\mu - a)(\mu - 2a) \\ & + \frac{\epsilon}{4}(\mu - 2a) - (e_4a)\Gamma_{33}^4. \end{aligned} \tag{2.4.11}$$

Combining (2.4.3) and (2.4.11), we obtain $\mu = 2a$ which implies that a is ocnstant on W. This contradicts the assumption that $e_3a \neq 0$ on W. So $e_3a \equiv 0$ on V. A similar argument yields $e_4a \equiv 0$ on V. Thus by applying (2.4.5), we know that Λ_3 vanishes identically on V. Hence, by applying (2.4.3) again, we have $a^2 - a\mu + \frac{\epsilon}{2} = 0$ on V. Consequently, a is constant on each component of V.

Case (iii-3): $n = 3$. From (2.4.3), we have

$$\begin{aligned} -2(e_3a)(e_3e_3a) = {} & a^2(2\Lambda_3(e_3\Lambda_3) - 2(e_3a)(\mu - a)) \\ & + 2a(e_3a)(\Lambda_3^2 + (\mu - a)^2) + \tfrac{\epsilon}{2}(e_3a)(\mu - a). \end{aligned}$$

By using (2.4.4), this equation becomes

$$\begin{aligned} -2(e_3a)(e_3e_3a) = {} & (e_3a)\{\frac{4a\Lambda_3^2(\mu^2 - 3a\mu + 3a^2)}{(\mu - a)(\mu - 2a)} \\ & + 2a(\mu - a)(\mu - 2a) + \frac{\epsilon}{2}(\mu - 2a)\}. \end{aligned} \tag{2.4.12}$$

We claim that $e_3a \equiv 0$ on V. Since, otherwise there exists an open subset W of V such that e_3a is nowhere vanished on W. Then, on W, (2.4.12) becomes

$$-2(e_3e_3a) = \frac{4a\Lambda_3^2(\mu^2 - 3a\mu + 3a^2)}{(\mu - a)(\mu - 2a)} + 2a(\mu - a)(\mu - 2a) + \frac{\epsilon}{2}(\mu - 2a). \tag{2.4.13}$$

Combining this with the first equation in (2.4.3), we conclude that a is constant on$W$ which is a contradiction. Therefore, e_3a vanishes identically on W. Consequently, from the second equation of (2.4.3), we obtain

$$a^2\Lambda_3^2 = -a^2(\mu - a)^2 - \frac{\epsilon}{2}a(\mu - a). \tag{2.4.14}$$

On the other hand, from the first equation in (2.4.3), we have

$$-\frac{2a\Lambda_3^2(\mu^2 - 3a\mu + 3a^2)}{(\mu - a)(\mu - 2a)} + (\mu - 2a)(a^2 - a\mu - \epsilon) = 0. \tag{2.4.15}$$

(2.4.14) and (2.4.15) imply that a satisfies a polynomial equation of degree 4 with constant coefficients on V. Hence, a is constant on each component of V.

Therefore, we know that, for any dimesion $n > 2$, every component of V is an isoparametric hypersurface of the hyperpolic space. On the other hand, according to a well-known result of Cartan, every isoparametric hypersurface of $H^{n+1}(-1)$ has at most two distinct principle curvatures. Therfore we conclude that each component of V has at most two distinct principle curvatures. This contradicts the definition of V. Thus, V must be the empty set. Consequently, M^n is an isoparametric hypersurface of $H^{n+1}(-1)$ with exactly two distinct principle curvatures. Therefore, by applying Cartan's classfication theorem of isoparametric hypersurfaces in hyperbolic spaces, M^n is an open set of the Riemannian product of $H^2(-\frac{1}{2})$ and $S^{n-2}(1)$ isometrically imbeded in the hyperbolic $(n+1)$-space in the standard way. Such a hypersurface is a tubular hypersurface with radius $r = \coth^{-1}(\sqrt{2})$ about a 2-dimensional totally geodesic surface (cf. [8]).

Now, assume M^n is a non-totally geodesic minimal hypersurface. Let U denote the open subset of M^n consisting of non-totally geodesic points. Then U is an open dense subset of M^n. Now, by a result of [7], U has relative nullity $n - 2$. In this case, by applying an argument similar to spherical case, we may conclude that the Gauss image B^2 of U is a minimal surface in the unit de Sitter space-time $S_1^{n+1}(1)$.

Consequently, U is an open subset of the unit bundle NB^2 defined in Theorem 2.3.

The converses are easy to verify. □

CHAPTER 3

COMFORMALLY FLAT HYPERSURFACES SATISFYING CHEN'S EQUALITY

In this chapter, we will study conformally flat hypersurfaces satisfying Chen's equality (2.1.2) in Riemannian space forms. By utilizing the Jacobi elliptic functions and the Theta function we obtain the complete classification of such hypersurfaces.

3.1 Main Results

In order to state our results, we recall three families of Riemannian manifolds, $P_a^n(a > 1), C_a^n(a > 1), D_a^n(0 < a < 1)$ and the two exceptional spaces F^n, L^n, first introduced by Chen in [10].

Let $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$ and $\mathrm{sn}(u, k)$ denote the three main Jacobi's elliptic functions with modulus k. The nine other elliptic functions $\mathrm{nd}(u, k)$, $\mathrm{nc}(u, k)$, $\mathrm{ns}(u, k)$, $\mathrm{sc}(u, k)$, $\mathrm{cd}(u, k)$, $\mathrm{ds}(u, k)$, $\mathrm{cs}(u, k)$, $\mathrm{dc}(u, k)$, $\mathrm{sd}(u, k)$ are defined by taking reciprocals and quotients. For example, $\mathrm{sd}(u, k)= \mathrm{sn}(u, k)/\mathrm{dn}(u, k)$, $\mathrm{nd}(u, k)=1/\mathrm{dn}(u, k)$ (cf. [24] and the next section for details).

We define

$$(3.1.1) \qquad \mu_a = ak\mathrm{cn}(ax, k), \qquad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a}, \qquad a > 0,$$

$$(3.1.2) \qquad \eta_a = \frac{a}{k}\mathrm{dn}(\frac{a}{k}x, k), \qquad k = \frac{\sqrt{2}a}{\sqrt{a^2+1}}, \qquad 0 < a < 1,$$

$$(2.1.3) \qquad \rho_a = ak\mathrm{cn}(ax, k), \qquad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a}, \qquad a > 1.$$

Let $S^{n+1}(c)$ and $H^{n+1}(-c)$ denote the n-sphere with constant sectional curvature c and the hyperbolic n-space with constant sectional curvature $-c$, respectively. For $n > 2, P_a^n, D_a^n$ and C_a^n are the Riemannian n-manifolds given by the warped product manifolds $I \times_{\mu_a} S^{n-1}(\frac{a^4-1}{4}), \mathbb{R} \times_{\eta_a} H^{n-1}(\frac{a^4-1}{4})$ and $I \times_{\rho_a} S^{n-1}(\frac{a^4-1}{4})$ with warp functions μ_a, η_a and ρ_a, respectively, where I denote the open interval on which the corresponding warp function is positive. The two exceptional spaces F^n and L^n are the warped product manifolds $\mathbb{R} \times_{1/\sqrt{2}} H^{n-1}(-\frac{1}{4})$ and $\mathbb{R} \times_{\mathrm{sech}(x)} \mathbb{R}^{n-1}$, respectively.

D_a^n, F^n and L^n are complete Riemannian n-manifolds, but P_a^n and C_a^n are not complete. Topologically, S^n is the two point compactfication of both P_a^n and C_a^n. From [13] we know that the Riemannian metrics defined on P_a^n and C_a^n can be extended smoothly to their two point compactifications S^n. We denote by $\hat{P}_a^n$ and $\hat{C}_a^n$ the sphere S^n together with the Riemannian metrics given by the smooth extensions of the metrics on P_a^n and C_a^n to S^n, respectively. We remark that P_a^n, D_a^n, C_a^n are indeed isometric to the wraped products n-manifolds $I \times_{f_1} S^{n-1}(1), I \times_{f_2} H^{n-1}(-1), I \times_{f_3} S^{n-1}(1)$ with warped functions $2\rho_a/(a^2+1), 2\mu_a/(a^2-1), 2\eta_a/(a^2+1)$, respectively.

Let $A_a^n(a > 1), B_a^n(0 < a < 1), G^n, H_a^n(a > 0)$ and $Y_a^n(0 < a < 1)$ denote respectively the following warped product manifolds:

$$\mathbb{R} \times_{\sqrt{a^2-1}\cosh x} S^{n-1}(1), \qquad (-\frac{1}{2},\frac{1}{2}) \times_{\sqrt{1-a^2}\cos x} S^{n-1}(1),$$

$$\mathbb{R} \times_{\cosh x} \mathbb{E}^{n-1}, \qquad \mathbb{R} \times_{\sqrt{a^2+1}\cosh x} H^{n-1}(-1), \qquad (0,\infty) \times_{\sqrt{1-a^2}\sinh x} S^{n-1}(1).$$

When $n=2$, the second factor S^{n-1} or H^{n-1} in each of the wraped product manifolds will be replaced either by $S^1(1)$ or by $\mathbb{R}$. The geometry of A_a^2, G^2 and H_a^2 is similar in the sense that one can be obtained from the others by applying some suitable scalings on the first factor $\mathbb{R}$.

Clearly, A_a^n, G^n, H_a^n are complete Riemannian manifolds. Topologically, S^n is the two point compactification of B_a^n. As for P_a^n and C_a^n, the warped metric on B_a^n can be extended smoothly to its two point compactification, a fact that follows from (3.1.6). We denote by $\hat{B}_a^n$ the shere S^n together with the Riemannian metrics on S^n extended from the metric on B_a^n.

For $n \geq 2$ and any real number $a > 0$, there is a well-known Lagrangian immersion from the unit n-sphere S^n into a complex Euclidean n-space $\mathbb{C}^n$ defined by

$$w_a(y_0, y_1, ..., y_n) = \frac{a}{1+y_0^2}(y_1, ..., y_n {\cdot} y_0y_1, ..., y_0y_n), \tag{3.1.4}$$

where $y_0^2 + y_1^2 + ... + y_n^2 = 1$. The immersiom w_a, due to Whitney, has a unique self-intersection point $w_a(-1,0,...0) = w_a(1,0....,0)$. The S^n together with the metric induced from Whitney's immersion w_a, denoted by W_a^n, is called a *Whitney n-sphere.*

In this chapter, we first sharpen a result of [22] (Propositon 2) to the following

Theorem 3.1. *Let* $\mathbf{x} : M^n \to \mathbb{E}^{n+1} (n > 2)$ *be an isometric immersion of a conformally flat n-manifold into a Euclidean (n+1)-space. Then it satisfies Chen's equality* (2.1.2) *if and only if one of the following four cases occurs:*

1. *M^n is totally geodesic.*

2. M^n is an open portion of a spherical hypercylinder $S^{n-1} \times \mathbb{R}$.

3. M^n is an open portion of a round hypercone.

4. $n = 3$ and M^3 is an open portion of a Whitney 3-sphere W_a^3 for some $a > 0$ and, up to rigid motions, the immersion $\mathbf{x} : W_a^3 \to \mathbb{E}^4$ is given by

(3.1.5)
$$\mathbf{x}(x, y_1, y_2, y_3) = \left(\frac{1}{2}\int_0^x \mathrm{sd}^2\left(\frac{\sqrt{2}}{a}x\right)dx, ay_1\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right), ay_2\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right), ay_3\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\right),$$

where $y_1^2 + y_2^2 + y_3^2 = \frac{1}{2}$ and $k = \frac{1}{\sqrt{2}}$ is the modulus of the Jacobi elliptic functions.

Our main results in this chapter are the following.

Theorem 3.2. *Let $\mathbf{x} : M^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2} (n > 2)$ be an isometric immersion of a conformally flat n-manifold. Then it satisfies Chen's equality* (2.1.2) *if and only if one of the following three cases occurs.*

1. M^n is an open portion of $S^n(1)$ and the immersion $\mathbf{x} : M^n \to S^{n+1}(1)$ is totally geodesic.

2. M^n is an open portion of $\hat{B}_a^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2}$ given by

$$\mathbf{x}(x, y_1, ..., y_n) = (\sin x, a\cos x, \sqrt{1-a^2}y_1 \cos x, ..., \sqrt{1-a^2}y_n \cos x) \tag{3.1.6}$$

with $y_1^2 + y_2^2 + ... + y_n^2 = 1$.

3. $n = 3, M^3$ is an open portion of $\hat{P}_a^3$ for some $a > 1$ and, up to rigid motions, the immersion $\mathbf{x} : \hat{P}_a^3 \to S^4(1) \subset \mathbb{E}^5$ is given by

$$\mathbf{x}(x, y_1, y_2, y_3) = \frac{1}{ak'}(y_1\mathrm{cn}(ax), y_2\mathrm{cn}(ax), y_3\mathrm{cn}(ax), \jmath\cos\mathcal{X}, \jmath\sin\mathcal{X}), \tag{3.1.7}$$

where $\jmath = \sqrt{a^2k'^2 - cn^2(ax)}, y_1^2 + y_2^2 + y_3^2 = 1, \gamma = \mathrm{sn}^{-1}(\frac{\sqrt{-1}}{ak^2})$, and $k = \sqrt{a^2-1}/(\sqrt{2}a), k' = \sqrt{a^2+1}/(\sqrt{2}a)$ are the modulus and the complementary *modulus of Jacobi's elliptic functions respectively, and*

$$\mathcal{X} = \frac{k'}{k}x + \frac{\sqrt{-1}}{2}(\ln \frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2aZ(\gamma)x)),$$

where $\Theta(u) = \Theta(u,k)$ *is the Theta function and* $Z(u) = Z(u,k)$ *is the Zeta function.*

Theorem 3.3. *Let* $\mathbf{x} : M^n \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2} (n > 2)$ *be an isometric immersion of a conformally flat n-manifold. Then it satisfies Chen's equality* (2.1.2) *if and only if one of the following nine cases occurs.*

1. M^n *is an open portion of* $H^n(-1)$ *and the immersion* $\mathbf{x} : M^n \to H^{n+1}(-1)$ *is totally geodesic.*

2. M^n *is an open portion of* A_a^n *for some* $a > 1$ *and, up to rigid motions, the immersion* $\mathbf{x} : A_a^n \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ *is given by*

   (3.1.8)
   $$\mathbf{x}(x, y_1, ..., y_n) = (a\cosh x, \sinh x, \sqrt{a^2-1}y_1 \cosh x, ..., \sqrt{a^2-1}y_n \cosh x)$$

   with $y_1^2 + y_2^2 + ... + y_n^2 = 1$.

3. M^n *is an open portion of* G^n *and, up to rigid motions, the immersion* $\mathbf{x} : G^n \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ *is given by*

   $$\begin{aligned} \mathbf{x}(x, u_2, ..., u_n) = ((1 + \frac{u_2^2 + ... + u_n^2}{2}) \cosh x, \\ \frac{u_2^2 + ... + u_n^2}{2} \cosh x, \sinh x, u_2 \cosh x, ..., u_n \cosh x). \end{aligned} \quad (3.1.9)$$

4. M^n *is an open portion of* H_a^n *for some* $a > 0$*, and, up to rigid motions, the*

immersion $\mathbf{x} : H^n_a \to H^{n+1}(-1) \subset \mathbb{E}^{n+2}_1$ *is given by*

(3.1.10)

$$\mathbf{x}(x, y_1, ..., y_n) = (\sqrt{a^2+1}y_1 \cosh x, ..., \sqrt{a^2+1}y_n \cosh x, a \cosh x, \sinh x)$$

with $y_1^2 - y_2^2 - ... - y_n^2 = 1$.

5. M^n *is an open portion of* Y^n_a *for some* $0 < a < 1$, *and, up to rigid motions, the immersion* $\mathbf{x} : Y^n_a \to H^{n+1}(-1) \subset \mathbb{E}^{n+2}_1$ *is given by*

(3.1.11)

$$\mathbf{x}(x, y_1, ..., y_n) = (\cosh x, a \sinh x, \sqrt{1-a^2}y_1 \sinh x, ..., \sqrt{1-a^2}y_n \sinh x)$$

with $y_1^2 + y_2^2 + ... + y_n^2 = 1$.

6. $n = 3, M^n$ *is an open portion of* F^3 *and, up to rigid motions, the immersion* $\mathbf{x} : F^3 \to H^4(-1) \subset \mathbb{E}^5_1$ *is given by*

(3.1.12) $$\mathbf{x}(x, u, v) = (\sqrt{2} \cosh u \cosh v, \sqrt{2} \cosh u \sinh v, \sqrt{2} \sinh u, \cos x, \sin x).$$

7. $n = 3, M^n$ *is an open portion of* L^3 *and, up to rigid motions, the immersion* $\mathbf{x} : L^3 \to H^4(-1) \subset \mathbb{E}^5_1$ *is given by*

(3.1.13)

$$\mathbf{x}(x, u, v) = \mathrm{sech} x(x^2 + u^2 + v^2 + \cosh^2 x + \frac{1}{4}, x^2 + u^2 + v^2 + \cosh^2 x - \frac{1}{4}, x, u, v).$$

8. $n = 3, M^n$ *is an open portion of* $\hat{C}^3_a$ *for some* $a > 1$ *and, up to rigid motions, the immersion* $\mathbf{x} : \hat{C}^3_a \to H^4(-1) \subset \mathbb{E}^5_1$ *is given by*

(3.1.14) $$\mathbf{x}(x, y_1, y_2, y_3) \frac{1}{ak'}(\ell \cosh \mathcal{Y}, \ell \sinh \mathcal{Y}, y_1 \mathrm{cn}(ax), y_2 \mathrm{cn}(ax), y_3 \mathrm{cn}(ax))$$

with $y_1^2 + y_2^2 + y_3^2 = 1$, *where* $\ell = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}$,

$$\mathcal{Y} = \frac{k'}{k}x - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - aZ(\gamma)x,$$

where $\gamma = \mathrm{sn}^{-1}(1/(ak^2)), k = \sqrt{a^2+1}/(\sqrt{2}a)$ *and* $k' = \sqrt{a^2-1}/(\sqrt{2}a)$.

9. $n = 3, M^n$ *is an open portion of* D_a^3 *and, up to rigid motions, the immersion* $\mathbf{x} : D_a^3 \to H^4(-1) \subset \mathbb{E}_1^5$ *is given by* $\mathbf{x}(x,u,v) =$

(3.1.15)

$$\frac{1}{ak'}(k\mathrm{dn}(\frac{a}{k}x)\cosh u\cosh v, k\mathrm{dn}(\frac{a}{k}x)\cosh u\sinh v, k\mathrm{dn}(\frac{a}{k}x)\sinh u, \wp\cos\mathcal{Z}, \wp\sin\mathcal{Z}),$$

where $k = \sqrt{2}a/\sqrt{1+a^2}$ *and* $k' = \sqrt{1-a^2}/\sqrt{1+a^2}$, $\gamma = \mathrm{sn}^{-1}(k/a)$, $\wp = \sqrt{k^2\mathrm{dn}^2(\frac{a}{k}x) - a^2k'^2}$, *and*

$$\mathcal{Z} = k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x-\gamma)}{\Theta(\frac{a}{k}x+\gamma)} - \sqrt{-1}\frac{a}{k}Z(\gamma)x.$$

3.2 The Jacobi Elliptic Functions, Theta Function and Zeta Function

We review very briefly some known facts on Jacobi's elliptic functions, Theta function and Zeta function for later use (see [1], [24] or [26] for details).

Let θ be the temperature at time t at any point in a solid material whose conducting properties are uniform and isotropic. If ρ is the material's density, s its specific heat, and k its thermal conductivity, θ satisfies the heat conduction equation: $\kappa\nabla^2\theta = \partial\theta/\partial t$, where $\kappa = k/s\rho$ is the diffusivity. In the special case where there is no variation of temperation in the x- and y-directions, the heat flow is everywhere parallel to the z-axis and the heat equation reduced to the form:

$$\kappa \frac{\partial^2 \theta}{\partial z^2} = \frac{\partial \theta}{\partial t}, \qquad \theta = \theta(z,t). \tag{3.2.1}$$

Consider the boundary conditions: $\theta(0,t) = \theta(\pi,t) = 0$ and $\theta(z,0) = \pi\delta(z-\pi/2)$ for $0 < z < \pi$, where $\delta(z)$ is Dirac's unit impulse function. Then the solution of the boundary value problem is given by

$$\theta(z,t) = 2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)^2\kappa t}\sin(2n+1)z. \tag{3.2.2}$$

By writing $e^{-4\kappa t} = q$, the solution of (3.2.2) assume the form

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z, \tag{3.2.3}$$

which is the first of the four theta functions. For simplicity, we shall often suppress the dependance on q.

If one changes the boundary conditions to $\partial\theta/\partial z = 0$ on $z = 0$ and $z = \pi$ with $\theta(z,0) = \pi\delta(z-\pi/2)$ for $0 < z < \pi$, then the corresponding solution of the boundary value problem of the heat equation (3.2.1) is given by

$$\theta_4(z) = \theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{3.2.4}$$

The theta function $\theta_1(z)$ of (3.2.3) is periodic with period 2π. Incrementing z by $\frac{1}{2}\pi$ yields the second theta function:

$$\theta_2(z) = \theta_2(z,q) = \theta_1(z+\frac{\pi}{2},q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z. \tag{3.2.5}$$

Similarly, incrementing z by $\frac{\pi}{2}$ for θ_4 yields the third theta function:

$$\theta_3(z) = \theta_3(z,q) = \theta_4(z+\frac{\pi}{2},q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz. \tag{3.2.6}$$

The four theta functions $\theta_1, \theta_2, \theta_3, \theta_4$ can be extended to complex values for z and q such that $|q| < 1$.

The elliptic functions $\mathrm{sn}u, \mathrm{cn}u$ and $\mathrm{dn}u$ are defined as ratios of theta functions:

$$\mathrm{sn}u = \frac{\theta_3(0)\theta_1(z)}{\theta_2(0)\theta_3(z)}, \qquad \mathrm{cn}u = \frac{\theta_4(0)\theta_2(z)}{\theta_2(0)\theta_4(z)}, \qquad \mathrm{dn}u = \frac{\theta_4(0)\theta_3(z)}{\theta_3(0)\theta_4(z)}, \tag{3.2.7}$$

where $z = u/\theta_3^2(0)$. Define parameters k and k' by

$$k = \theta_2^2(0)/\theta_3^2(0), \qquad k' = \theta_4^2(0)/\theta_3^2(0)$$

which are called the modulus and the complementary modulus of the elliptic function. k and k' satisfy $k^2 + k'^2 = 1$. When it is required to state the modulus explicitly, the elliptic functions of Jacobi will be written $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$.

The elliptic functions $\mathrm{sn}u, \mathrm{cn}u$ and $\mathrm{dn}u$ satisfy the following relations:

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1, \qquad \mathrm{dn}^2u + k^2\mathrm{sn}^2u = 1, \qquad k^2\mathrm{cn}^2u + k'^2 = \mathrm{dn}^2u, \tag{3.2.8}$$

$$\mathrm{sn}(u+v) + \mathrm{sn}(u-v) = \frac{2\mathrm{sn}u\mathrm{cn}v\mathrm{dn}v}{1 - k^2\mathrm{sn}^2u\mathrm{sn}^2v}, \tag{3.2.9}$$

$$\mathrm{sn}'(u) = \mathrm{cn}(u)\mathrm{dn}(u), \quad \mathrm{cn}'(u) = -\mathrm{sn}(u)\mathrm{dn}(u), \quad \mathrm{dn}'(u) = -k^2\mathrm{sn}(u)\mathrm{cn}(u). \tag{3.2.10}$$

The Theta function, $\Theta(u)$, and the Zeta function, $\mathrm{Z}(u)$, are defined by

$$\Theta(u) = \theta_4(\frac{\pi u}{2K}), \qquad \mathrm{Z}(u) = \frac{d}{du}(\ln \theta_4), \qquad K = \frac{\pi}{2}\theta_3^2(0), \tag{3.2.11}$$

and satisfy the following identities:

$$\mathrm{Z}(u+v) = \mathrm{Z}(u) + \mathrm{Z}(v) - k^2 \mathrm{sn}u\mathrm{sn}v\mathrm{sn}(u+v), \tag{3.2.12}$$

$$\mathrm{Z}(u) = \frac{\Theta'(u)}{\Theta(u)}, \tag{3.2.13}$$

From (3.2.12) we have

$$k^2 \mathrm{sn}u\mathrm{sn}v[\mathrm{sn}(u+v) + \mathrm{sn}(u-v)] = \mathrm{Z}(u-v) - \mathrm{Z}(u+v) + 2\mathrm{Z}(v). \tag{3.2.14}$$

3.3 Two Lemmas

Lemma 3.1. *Let $\tilde{W}_a^n$ denote the warped product manifold $I \times_{\frac{a}{\sqrt{2}}\mathrm{sd}(\frac{\sqrt{2}}{a}x, \frac{1}{\sqrt{2}})} S^{n-1}(1)$, with the warped product metric given by*

$$g = dx^2 + \frac{a^2}{2}\mathrm{sd}^2(\frac{\sqrt{2}}{a}x, \frac{1}{\sqrt{2}})g_0, \tag{3.3.1}$$

where I denotes the largest open interval containing 0 such that $\mathrm{sd}(\frac{\sqrt{2}}{a}x)$ is nowhere zero on I and g_0 is the standard metric on the unit $(n-1)$-sphere. Then the Whitney n-sphere W_a^n is topologically the two point compactification of $\tilde{W}_a^n$. Moreover, the metric on W_a^n is the smooth extension of the warped product metric on $\tilde{W}_a^n$ to its two point compactification.

Proof. Let $\tilde{S}^n$ denote the unit n-sphere with the north and south poles, $\{N, S\}$, being removed and let $\{u_1, u_2, ..., u_n\}$ denote the spherical coordinate system on $\tilde{S}^n$ given by

$$(3.3.2) \qquad \begin{aligned} &y_0 = \cos u_1, \qquad y_1 = \sin u_1 \cos u_2, ... \\ &y_{n-1} = \sin u_1 ... \sin u_{n-1} \cos u_n, \qquad y_n = \sin u_1 ... \sin u_{n-1} \sin u_n. \end{aligned}$$

From (3.1.4) and (3.3.2) we know that the metric induced from Whitney's immersion w_a on $\tilde{S}^n$ is given by

$$(3.3.3) \qquad g = (\frac{a^2}{1+\cos^2 u_1})du_1^2 + (\frac{a^2 \sin^2 u_1}{1+\cos^2 u_1})g_0.$$

Put

$$(3.3.4) \qquad x(u_1) = \int_0^{u_1} \frac{a}{\sqrt{1+\cos^2 t}} dt.$$

Then

$$(3.3.5) \qquad x(u_1) = \frac{a}{\sqrt{2}} \int_0^{u_1} \frac{a}{\sqrt{1-\frac{1}{2}\sin^2 t}} dt = \frac{a}{\sqrt{2}} \mathrm{sn}^{-1}(\sin u_1, \frac{1}{\sqrt{2}}).$$

Thus

$$(3.3.6) \qquad \sin u_1 = \mathrm{sn}(\frac{\sqrt{2}}{a}x, \frac{1}{\sqrt{2}}).$$

From (3.2.8), (3.3.3), (3.3.4) and (3.3.6), we obtain (3.3.1). This shows that $\tilde{W}_a^n$ is the $\tilde{S}^n$ endowed with the metric induced from Whitney's immersion w_a. Hence, topologically, the Whitney n-sphere W_a^n is two point compactification of $\tilde{W}_a^n$ and, moreover, the metric on W_a^n is the smooth extension of the warped product metric on $\tilde{W}_a^n$ to its two point compactification. □

The following Lemma is important in this chapter.

Lemma 3.2. *Let $M^3 \subset R^4(\epsilon)$ be a conformally flat hypersurface of a Riemannian space form $R^4(\epsilon)$ satisfying Chen's equality* (2.1.2), *then M^3 has at most two distinct principle curvatures.*

Proof. Assume M^3 is a conformally flat hypersurface in a 4-dimensional Riemannian space form $R^4(\epsilon)$ satisfying equality (2.1.2).

Let L be the symmetric 2-tensor defined by

$$L = -\mathrm{Ric} + \frac{\tau}{2} g, \tag{3.3.7}$$

where Ric, τ and g denote respectively the Ricc tensor, the scalar curvature and the metric tensor of M^3. Then by a result of H. Weyl we have

$$(\nabla_Y L)(Z, W) = (\nabla_Z L)(Y, W), \tag{3.3.8}$$

for vectors Y, Z, W tangent to M^3. Let $\lambda_1 = a, \lambda_2 = \mu - a, \lambda_3 = \mu$ be the principle curvatures of M^3 with their corresponding principle directions e_1, e_2, e_3 given as in Section 2.3. Thus, from the equation of Gauss and (3.3.8), we have

$$\begin{cases} (\lambda_j^2 - \lambda_i^2)\Gamma_{ij}^j = 3(e_i H)\lambda_j - \frac{1}{2} e_i \tau - e_i \lambda_j^2, \\ (\lambda_j^2 - \lambda_k^2)\Gamma_{ji}^k = (\lambda_i^2 - \lambda_k^2)\Gamma_{ij}^k, \end{cases} \tag{3.3.9}$$

$$\tau = 3\epsilon + \mu^2 + a\mu - a^2, \tag{3.3.10}$$

for distinct $i, j, k (i, j, k = 1, 2, 3)$, where $H = \frac{2}{3}\mu$ is the mean curvature function.

Equation (3.3.9) and (3.3.10) imply

$$2(e_i\mu)\lambda_j - \frac{1}{2}e_i(\mu^2 + a\mu - a^2) - e_i\lambda_j^2 = (\lambda_j^2 - \lambda_i^2)\Gamma_{ij}^j = -(\lambda_i + \lambda_j)e_i\lambda_j.$$

Thus

$$(\lambda_i - \lambda_j)e_i\lambda_j = (-2\lambda_j + \mu + \frac{1}{2}a)e_i\mu + (\frac{1}{2}\mu - a)e_i a.$$

By taking (i, j) equal to (1, 2), (2, 3) and (3, 1) respectively, we obtain

(3.3.11) $$(\mu - 2a)e_1 a = a e_1 \mu,$$

(3.3.12) $$(\mu - 2a)e_2 a = (2\mu - 3a)e_2\mu,$$

(3.3.13) $$\mu e_3 a = (2\mu - 3a)e_3\mu.$$

Let V denote the open subset of M^3 on which M^3 has exactly three distinct principle curvatures, i.e., V is given by (2.3.6). Suppose that V is not empty. In this remaining part of the proof, we shall work on this non-empty subset to obtain a contradiction.

From (2.3.5) and (3.3.9) we obtain

(3.3.14) $$\Gamma_{jk}^i = 0,$$

for distinct i, j, k. Equation (3.3.14) implies

(3.3.15) $$\omega_2^1 = \frac{e_2 a}{\mu - 2a}\omega_1 + \frac{e_1(\mu - a)}{\mu - 2a}\omega_2.$$

By taking the exterior derivative of (3.3.15) and by applying Cartan's structure equations, we obtain the following formulas on V by comparing the corresponding coefficients in the resulting formula of $d\omega_2^1$.

(3.3.16)
$$\begin{aligned} &e_1(\frac{e_1(\mu-a)}{\mu-2a}) - e_2(\frac{e_2 a}{\mu-2a}) - \frac{(e_2 a)^2}{(\mu-2a)^2} - \frac{(e_1(\mu-a))^2}{(\mu-2a)^2} \\ &= \frac{(e_3 a) e_3(\mu-a)}{a(\mu-a)} + a(\mu-a) + \epsilon, \end{aligned}$$

(3.3.17)
$$e_3(\frac{e_2 a}{\mu-2a}) + \frac{(e_2 a)(e_3 a)}{(\mu-a)(\mu-2a)} = -\frac{(e_3 a)(e_2 \mu)}{a(\mu-a)},$$

(3.3.18)
$$e_3(\frac{e_1(\mu-a)}{\mu-2a}) + \frac{e_1(\mu-a)(e_3(\mu-a)}{a(\mu-2a)} = \frac{(e_1\mu) e_3(\mu-a)}{a(\mu-a)}.$$

By (3.3.13) we have $\Lambda_3 = \Gamma_{31}^2 = 0$. Thus, by applying (2.3.10), (2.3.13), (3.3.11), (3.3.12) and (3.3.13), we find

(3.3.19)
$$e_1 e_2 \mu = \frac{2\mu - 3a}{(\mu-2a)^2}(e_1\mu)(e_2\mu),$$

(3.3.20)
$$e_2 e_2 \mu = \frac{3a - \mu}{(\mu-2a)^2}(e_1\mu)(e_2\mu).$$

On the other hand, by Lemma 2.1, (3.3.11) and (3.3.12), we have

$$[e_1, e_2]\mu = -\Gamma_{21}^1 e_1\mu + \Gamma_{12}^2 e_2\mu = -\frac{e_2 a}{\mu-2a} e_1\mu - \frac{e_1(\mu-a)}{\mu-2a} e_2\mu = -\frac{3}{\mu-2a}(e_1\mu)(e_2\mu).$$

Combining this with (3.3.19) and (3.3.20) we find

(3.3.21)
$$(e_1\mu)(e_2\mu) = 0.$$

Therefore, $(e_1\mu)(e_2\mu) = 0$ on V.

If $e_2\mu \not\equiv 0$ on V, there exists an open subset $W \subset V$ on which $e_2\mu \neq 0$. On W, we have $e_1\mu = 0$ identically. Moreover, from (2.3.11) and (3.3.17) we have

$$(2\mu - 3a)e_2e_3\mu = -\frac{3a^2 - 3a\mu + 2\mu^2}{a\mu}(e_2\mu)(\epsilon_3\mu), \tag{3.3.22}$$

$$(2\mu - 3a)e_3e_2\mu = \frac{6a^2 - 3a\mu - 2\mu^2}{a\mu}(e_2\mu)(e_3\mu). \tag{3.3.23}$$

Combining the above two equations we obtain

$$(2\mu - 3a)[e_2, e_3]\mu = \frac{3(2\mu - 3a)}{\mu}(e_2\mu)(e_3\mu). \tag{3.3.24}$$

On the other hand, we also have

$$[e_2, e_3]\mu = \Gamma_{23}^2 e_2\mu - \Gamma_{32}^3 e_3\mu = -\frac{e_3(\mu - a)}{a}e_3\mu - \frac{e_2\mu}{a}e_3\mu = -\frac{3}{\mu}(e_2\mu)(e_3\mu).$$

Therefore, we find

$$(2\mu - 3a)(e_2\mu)(e_3\mu) = 0. \tag{3.3.25}$$

If $(e_2\mu)(e_3\mu) \not\equiv 0$ on W, then on an open subset W' of W on which $(e_2\mu)(e_3\mu) \neq 0$, we have

$$2\mu - 3a = 0. \tag{3.3.26}$$

Since $2\mu - 3a = 0$ on W', (3.3.12) and (3.3.13) imply $e_2a = e_3a = 0$ on W'. On the other hand, since $e_1\mu = 0$ on W, (3.3.11) yields $e_1a = 0$, Therefore, we obtain $e_1\mu = e_2\mu = e_3\mu = 0$ which is a contradiction. Consequently, we must have $e_3\mu \equiv 0$

on W, from which we obtain $(e_1 a)(e_3 a) \equiv 0$ on W by virtue of (3.3.13). Thus, by (3.3.15) and (3.3.12), we obtain

$$(2\mu - 3a)(e_2\mu)^2 = a(\mu - 2a)^2(a\mu + \epsilon). \tag{3.3.27}$$

On the other hand, from (3.3.11), we know that $2\mu - 3a \neq 0$ on W. Thus, there is an open subset $W_1 \subset W$ on which $2\mu - 3a \neq 0$. On W_1, we have (3.3.27) yields

$$(e_2\mu)^2 = \frac{a(a\mu + \epsilon)(\mu - 2a)^2}{2\mu - 3a}. \tag{3.3.28}$$

By using (3.3.12) and (3.3.28) we obtain

$$\begin{aligned} e_2e_2\mu = &\frac{1}{(2\mu - 3a)^2}[-6a^5 - 15a^4\mu + 38a^3\mu^2 - 23a^2\mu^3 + 4a\mu^4 \\ &+\epsilon(-16a^3 + 27a^2\mu - 14a\mu^2 + 2\mu^3)]. \end{aligned} \tag{3.3.29}$$

On the other hand, (3.3.12) and (3.3.16) imply that on W_1 we have

$$\frac{2\mu - 3a}{(\mu - 2a)^2}e_2e_2\mu + \frac{12\mu^2 - 33a\mu + 23a^2}{(\mu - 2a)^2}(e_2\mu)^2 = -a(\mu - a) - \epsilon.$$

By using (3.3.28), the above equation becomes

$$\begin{aligned} -e_2e_2\mu = &\frac{1}{(2\mu - 3a)^2}[12a^5 - 9a^4\mu - 2a^3\mu^2 - a^2\mu^3 + 2a\mu^4 \\ &+\epsilon(11a^3 - 13a^2\mu + a\mu^2 + 2\mu^3)]. \end{aligned} \tag{3.3.30}$$

Combining (3.3.30) and (3.3.29) we get

$$6a^3 - 12a^2\mu + 6a\mu^2 + \epsilon(4\mu - 5a) = 0. \tag{3.3.31}$$

Differentiating (3.3.31) and applying (3.3.12) we find

$$-30a^3 + 72a^2\mu - 54a\mu^2 + 12\mu^3 + \epsilon(7a - 6\mu) = 0. \tag{3.3.32}$$

Also, by combining (3.3.31) and (3.3.32) we get

$$6a^2\mu + 6\mu^3 - 12a\mu^2 + \epsilon(7\mu - 9a) = 0. \tag{3.3.33}$$

The above implies

$$a = \frac{9\epsilon + 12\mu^2 \pm \sqrt{81 + 48\epsilon\mu^2}}{12\mu}. \tag{3.3.34}$$

By substituting (3.3.34) into (3.3.32) we conclude that μ must satisfy the following polynormial equation with constant coefficients:

$$162 + 120\mu^2 + 88\epsilon\mu^4 + 32\mu^6 = \pm\sqrt{3}(9 + 8\epsilon\mu + 8\mu^4)\sqrt{27 + 16\epsilon\mu^2} \tag{3.3.35}$$

which is impossible since otherwise μ is locally a constant function. Thus we must have $e_2\mu \equiv 0$.

Similarly, we may prove that $e_1\mu \equiv 0$, too. Hence, by applying (3.3.11) - (3.3.13), we obtain

$$e_1 a = e_2 a = e_1\mu = e_2\mu = 0. \tag{3.3.36}$$

Now we claim that $e_3\mu \equiv 0$ on V, also. In fact, otherwise there exists an open subset $O_1 \subset V$ on which $e_3\mu \neq 0$.

If $2\mu - 3a = 0$ on V, then (3.3.13) implies $e_3 a = 0$. Thus, by (3.3.16), we get $a(\mu - a) + \epsilon = 0$. Thus $2\mu^2 + 9\epsilon = 0$ which is impossible.

If $\mu = 3a$, then (3.3.13) yields $\mu e_3\mu = 0$, which is also a contradiction. Hence, we have $2\mu \neq 3a$ and $\mu \neq 3a$ on V.

Now, from (3.3.16) we get

$$\text{(3.3.37)} \qquad \frac{(2\mu - 3a)(\mu - 3a)}{\mu^2 a(\mu - a)}(e_3\mu)^2 = a(\mu - a) + \epsilon.$$

Thus there exists an open subset $O_2 \subset O_1$ on which

$$\text{(3.3.38)} \qquad (e_3\mu)^2 = \frac{(a(\mu - a) + \epsilon)\mu^2 a(\mu - a)}{(2\mu - 3a)(\mu - 3a)}.$$

On the other hand, (2.3.9) and (3.3.13) imply

$$\text{(3.3.39)} \qquad -e_3\left(\frac{(2\mu - 3a)e_3\mu}{\mu(\mu - a)}\right) - \frac{(e_3\mu)^2(2\mu - 3a)^2}{(\mu - a)^2\mu^2} = a\mu + \epsilon.$$

Equations (3.3.38) and (3.3.39) yield

$$\text{(3.3.40)} \qquad \begin{aligned} e_3e_3\mu = {} & \frac{\mu}{(\mu - 3a)(2\mu - 3a)^2}[6a^5 - 15a^3\mu^2 + 11a^2\mu^3 - 2a\mu^4 \\ & +\epsilon(3a^3 - 15a^2\mu + 11a\mu^2 - 2\mu^3)]. \end{aligned}$$

Similarly, by applying (2.3.12), (3.3.13) and (3.3.38), we have

$$\text{(3.3.41)} \qquad \begin{aligned} e_3e_3\mu = {} & \frac{\mu}{\mu}(2\mu - 3a)(\mu - 3a)^2[6a^5 - 30a^4\mu + 45a^3\mu^2 + 26a^2\mu^3 \\ & +5a\mu^4 + \epsilon(3a^3 + 6a^2\mu - 10a\mu^2 + 3\mu^3)]. \end{aligned}$$

Summing the above two equations yield

$$\text{(3.3.42)} \quad \mu^2(\mu - 2a)[-24a^4 + 48a^3\mu - 30a^2\mu^2 + 6a\mu^3 + \epsilon(15a^2 - 15a\mu + 4\mu^2)] = 0.$$

On the other hand, from (3.3.14), we know that the distribution $\mathcal{F}^\perp$ spanned by e_1, e_2 is integrable. Also, the distribution $\mathcal{F}$ spanned by e_3 is clearly integrable. Therefore, there exists a local coordinate system $\{x_1, x_2, x_3\}$ such that $e_3 = \partial/\partial t$, where $t = x_3$. Hence, by applying (3.3.36), we know that both a and μ depend only on t. Therefore, (3.3.13) yields

$$\mu \frac{da}{d\mu} = 2\mu - 3a. \tag{3.3.43}$$

By solving (3.3.43) we obtain

$$a = \frac{\mu}{2} + C\mu^{-3} \tag{3.3.44}$$

for some constant C. By substituting the above into (3.3.42), we know that μ must satisfies a polynomial equation with constant coefficients. Therefore, $e_3\mu = 0$ on O_2 which is a contradiction.

Consequently, both μ and a are constants on each component of V. Hence, by (2.3.9), (2.3.12) and (3.3.16), we get $a\mu = (\mu - a)\mu = -\epsilon$ which is clearly impossible. Therefore, we know that V is an empty set. Hence, M^3 has at most two distinct principle cuevatures. This complete the proof of the Lemma. □

3.4 Proof of Theorem 3.1

If $\mathbf{x} : M^n \to \mathbb{E}^{n+1}$ is an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies Chen's equality (2.1.2). Then, by Lemma 3.2 and a well-known result of E. Cartan and J. A. Schouten on confromally flat hypersurfaces (cf. p154 of [6]), we know that M^n has a principle curvature with multiplicity at least $n - 1$ for $n = 3$ as well as for $n \geq 4$. Thus, by applying (2.3.2), we know that either (i) the principle curvatures of M^n are given by $\lambda_1 = 0, \lambda_2 = \ldots = \lambda_n = \mu$, or (ii) $n = 3$ and then the principle curvatures of M^3 are given by $\lambda_1 = \lambda_2 = \mu/2, \lambda_3 = \mu \neq 0$.

If Case (i) occurs, M^n is either totally geodesic, or an open portion of spherical hypercylinder, or an open portion of a round hypercone (cf. [22]).

Now, we assume case (ii) occurs. Denote by U the open subset of M^3 on which the mean curvature function is nonzero. Then U is a non-empty open subset of M^3. We shall work on U unless mentioned otherwise.

From (2.3.5) we obtain

$$\Gamma_{31}^2 = \Lambda_3 = 0, \qquad e_1\mu = e_2\mu = 0, \qquad \Gamma_{33}^1 = \Gamma_{33}^2 = 0. \tag{3.4.1}$$

Denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. By (3.4.1) we know that the integral curves of $\mathcal{F}^\perp$ are geodesics and the distribution $\mathcal{F}$ is integrable. Consequently, there exist local coordinate systems $\{x, u, v\}$ such that $\mathcal{F}$ is spanned by $\{\partial/\partial u, \partial/\partial v\}$ and $e_3 = \partial/\partial x$.

From (3.4.1) we know that μ depends only on x, i.e., $\mu = \mu(x)$. Also, from (2.3.8) and (3.4.1), we have

$$\omega_3^1 = \frac{\mu'(x)}{\mu(x)}\omega^1, \qquad \omega_3^2 = \frac{\mu'(x)}{\mu(x)}\omega^2. \tag{3.4.2}$$

Using the above we obtain

$$\nabla_{e_j} e_3 = \frac{\mu'(x)}{\mu(x)} e_j, \qquad j = 1, 2. \tag{3.4.3}$$

Therefore, each integral submanifold of $\mathcal{F}$ is an extrinsic sphere of $\mathbb{E}^4$. Hence, by applying a result of Hiepko [21] (cf. also Remark 2.1 of [20]), we know that M^3 is locally the warped product $I \times_{f(x)} S^2(1)$, where $f(x)$ is a suitable warped function.

So, the metric of M^3 is given by

$$g = dx^2 + f^2(x)g_0, \tag{3.4.4}$$

where g_0 is the standard metric of $S^2(1)$. In particular, if we choose the spherical coordinate system $\{\theta, \phi\}$ for $S^2(1)$, we have

$$g = dx^2 + f^2(x)(d\phi^2 + \cos^2\phi d\theta^2). \tag{3.4.5}$$

Applying the above equation, we obtain

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \qquad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial \theta} = \frac{f'}{f}\frac{\partial}{\partial \theta}, \qquad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial \phi} = \frac{f'}{f}\frac{\partial}{\partial \phi}, \\ &\nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \phi} = -ff'\frac{\partial}{\partial x}, \qquad \nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \theta} = -\tan\theta\frac{\partial}{\partial \theta}, \\ &\nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} = -ff'\cos^2\phi\frac{\partial}{\partial x} + \sin\phi\cos\phi\frac{\partial}{\partial \phi}. \end{aligned} \tag{3.4.6}$$

By computing $d\omega_3^1$ and by using (3.4.2) and Cartan's structure equations, we find

$$\mu''(x) + \frac{\mu^3(x)}{2} = 0. \tag{3.4.7}$$

Integrating once (3.4.7) yields

$$4\mu'^2 + \mu^4 = \frac{4}{a^4}, \tag{3.4.8}$$

for some real number $a > 0$.

Now, we claim that U is dense in M^3. If it is not, then $M^3 - U$ has nonempty interior. From the definition of U, we know that the squared mean curvature function, μ and μ' vanish identically on the interior of $M^3 - U$. On the other hand, (3.4.8) says that this is impossible due to the continuity of the squared mean curvature function. Thus, U must be an open dense subset of M^3.

Solving (3.4.8) yields $\mu = \frac{\sqrt{2}}{a}\mathrm{sd}(\frac{\sqrt{2}}{a}x + b, \frac{1}{\sqrt{2}})$ for some constant b. By applying a translation in x if necessary, we obtain

$$\mu(x) = \frac{\sqrt{2}}{a}\mathrm{sd}(\frac{\sqrt{2}}{a}x, k), \qquad k = \frac{1}{2}. \tag{3.4.9}$$

From (3.4.5) and from our assumption on the the principle curvatures, we know that the second fundamental form h of M^3 in $\mathbb{E}^4$ satisfies

$$\begin{aligned} &h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \mu\xi, \quad h(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) = \frac{1}{2}\mu f^2 \xi, \quad h(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = \frac{1}{2}\mu f^2 \cos^2\phi \xi, \\ &h(\frac{\partial}{\partial x}, \frac{\partial}{\partial \phi}) = h(\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}) = h(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}) = 0, \end{aligned} \tag{3.4.10}$$

where ξ is a unit normal vector field of M^3 in $\mathbb{E}^4$ and μ is given in (3.4.9).

Applying (3.4.6), (3.4.10) and the equation of Codazzi, we may obtain $\mu' f = \mu f'$. Therefore,

$$f(x) = c\mu(x), \tag{3.4.11}$$

for some nonzero constant c.

On the other hand, using (3.4.6) we can compute the sectional curvature K_{23} of the plane section spanned by $\{\partial/\partial\phi, \partial/\partial\theta\}$. We also can compute K_{23} by using the equation of Gauss. By comparing these two different expressions of K_{23}, we find $\mu^4 f^2 = 4(1 - f'^2)$. Thus by using (3.4.9) and (3.4.11), we find

$$f(x) = \frac{\sqrt{2}}{a}\mathrm{sd}(\frac{\sqrt{2}}{a}x, k), \qquad k = \frac{1}{2}. \tag{3.4.12}$$

From (3.4.5) and (3.4.12) and Lemma 3.2 we know that M^3 is an open portion of the Whitney 3-sphere W_a^3 for some $a > 0$.

(3.4.5), (3.4.6), (3.4.9), (3.4.12) and the formula of Gauss imply that the immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2 \mathbf{x}}{\partial \theta^2} = \frac{\sqrt{2}}{a}\mathrm{sd}(\frac{\sqrt{2}}{a}x)\xi, \tag{3.4.13}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \theta^2} = -\frac{a}{\sqrt{2}}\mathrm{cd}(\frac{\sqrt{2}}{a}x)\mathrm{sd}(\frac{\sqrt{2}}{a}x)\mathrm{nd}(\frac{\sqrt{2}}{a}x)\frac{\partial \mathbf{x}}{\partial x} + \frac{a}{2\sqrt{2}}\mathrm{sd}^3(\frac{\sqrt{2}}{a}x)\xi, \tag{3.4.14}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \theta^2} = \cos^2\phi\frac{\partial^2 \mathbf{x}}{\partial \phi^2} + \sin\phi\cos\phi\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.4.15}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \phi} = \frac{\sqrt{2}}{a}\mathrm{cs}(\frac{\sqrt{2}}{a}x)\mathrm{nd}(\frac{\sqrt{2}}{a}x)\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.4.16}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \theta} = \frac{\sqrt{2}}{a}\mathrm{cs}(\frac{\sqrt{2}}{a}x)\mathrm{nd}(\frac{\sqrt{2}}{a}x)\frac{\partial \mathbf{x}}{\partial \theta}, \tag{3.4.17}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \theta \partial \phi} = -\tan\phi\frac{\partial \mathbf{x}}{\partial \theta}. \tag{3.4.18}$$

Solving (3.4.18) yields

$$\mathbf{x}(x,\theta,\phi) = B(x,\theta)\cos\phi + C(x,\phi), \tag{3.4.19}$$

for some function $B(x,\theta), C(x,\phi)$ of two variables. Substituting (3.4.19) into (3.4.16) yields

$$\frac{\partial B}{\partial x} = \frac{\sqrt{2}}{a}\mathrm{cs}(\frac{\sqrt{2}}{a}x)\mathrm{nd}(\frac{\sqrt{2}}{a}x)B,$$

(3.4.20)

$$\frac{\partial C}{\partial x \partial \phi} = \frac{\sqrt{2}}{a}\mathrm{cs}(\frac{\sqrt{2}}{a}x)\mathrm{nd}(\frac{\sqrt{2}}{a}x)\frac{\partial C}{\partial \phi}.$$

Solving the above system yields

(3.4.21) $$B(x,\theta) = G(\theta)\mathrm{sd}(\frac{\sqrt{2}}{a}x), \qquad C(x,\phi) = F(\phi)\mathrm{sd}(\frac{\sqrt{2}}{a}x) + H(x).$$

Combining (3.4.19) and (3.4.21) gives

(3.4.22) $$\mathbf{x}(x,\phi,\theta) = G(\theta)\mathrm{sd}(\frac{\sqrt{2}}{a}x)\cos\phi + F(\phi)\mathrm{sd}(\frac{\sqrt{2}}{a}x) + H(x).$$

By taking the partial derivatives of (3.4.14) and (3.4.15) with respect to ϕ and θ respectively, we find

(3.4.23) $$F'''(\phi) + F'(\phi) = 0, \qquad G'''(\theta) + G'(\theta) = 0.$$

From (3.4.22) and (3.4.23) we find

(3.4.24) $$\mathbf{x}(x,\phi,\theta) = \mathrm{sd}(\frac{\sqrt{2}}{a}x)(c_1\cos\phi\cos\theta + c_2\cos\phi\sin\theta + c_3\cos\phi + c_4\sin\phi) + A(x),$$

where $c_1, ..., c_4$ are constant vectores.

Substituting (3.4.24) into (3.4.14) and using (3.4.13) yields

(3.4.26)

$$\mathbf{x}(x,\phi,\theta) = \mathrm{sd}(\frac{\sqrt{2}}{a}x)(c_1\cos\phi\cos\theta + c_2\cos\phi\sin\theta + c_3\cos\theta + c_4\sin\phi) + c_5\int \mathrm{sd}^3(\frac{\sqrt{2}}{a}x)dx,$$

where $c_1, ..., c_5$ are constant vectors in $\mathbb{E}^4$.

By choosing suitable initial conditions, we obtain (3.1.5) from (3.4.26). Consequently, up to rigid motion of $\mathbb{E}^4$, the immersion $\mathbf{x}$ is given by (3.1.5).

The converse can be verified by straight-forward computation. □

3.5 Exact Solutions of Differential Equations of Picard Type

For the proof of Theorem 3.2 and Theorem 3.3, we need the exact solutions of some differential equations with Jacobi's elliptic functions in their coefficients. The results obtained in this section seem to be of independent interest in themselves.

Proposition 5.1. *For any real number $a > 0$, the general solution of the second order differential equation:*

$$y''(x) + 2a\mathrm{sc}(ax)\mathrm{nd}(ax)y'(x) - y(x) = 0 \tag{3.5.1}$$

is given by $y(x) = c_1y_1(x) + c_2y_2(x)$ with

$$y_1(x) = \sqrt{a^2k'^2 - \mathrm{cn}^2(ax)}\cos\Big(\frac{k'}{k}x + \frac{\sqrt{-1}ak'^2}{2\sqrt{a^2k^4+1}}(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2aZ(\gamma)x)\Big),$$

and

$$y_2(x) = \sqrt{a^2k'^2 - \mathrm{cn}^2(ax)}\sin\Big(\frac{k'}{k}x + \frac{\sqrt{-1}ak'^2}{2\sqrt{a^2k^4+1}}(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2aZ(\gamma)x)\Big),$$

where $k = \sqrt{a^2-1}/(\sqrt{2}a)$ and $k' = \sqrt{a^2+1}/(\sqrt{2}a)$ are the modulus and the complementary modulus of the Jocobi elliptic functions, Θ the Theta function, Z the Zeta function, and $\gamma = \mathrm{sn}^{-1}(\sqrt{-1}/(ak^2))$.

Proof. The trick to solve (3.5.1) is to make two key transformations. First we make the transformation:

$$y(x) = f(x)\exp(\sqrt{-1}g(x)), \tag{3.5.2}$$

where $f(x)$ and $g(x)$ are real-valued functions. Then

$$y' = (f' + \sqrt{-1}fg')\exp(\sqrt{-1}g), \tag{3.5.3}$$

$$y'' = (f'' - f(g')^2 + \sqrt{-1}(2f'g' + fg''))\exp(\sqrt{-1}g). \tag{3.5.4}$$

substituting (3.5.3) and (3.5.4) into (3.5.1) we get by taking the imaginary part

$$f(x)g''(x) + 2(f'(x) + a\mathrm{sc}(ax)\mathrm{dn}(ax)f(x))g'(x) = 0, \tag{3.5.5}$$

and

$$f''(x) - f(x)(g'(x))^2 + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)f'(x) - f(x) = 0. \tag{3.5.6}$$

Equation (3.5.5) can be written as

$$(\ln g'(x))' = -2(\ln f(x))' + 2(\ln \mathrm{cn}(ax))' \tag{3.5.7}$$

which yields

$$g'(x) = \frac{\alpha \mathrm{cn}^2(ax)}{f^2(x)}, \tag{3.5.8}$$

for some constant α.

Substituting (3.5.8) into (3.5.6) we obtain a second order nonlinear equation:

$$f''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)f'(x) - f(x) = \frac{\alpha^2\mathrm{cn}^4(ax)}{f^3(x)}. \tag{3.5.9}$$

We make the second transformation by putting

$$f(x) = \sqrt{h(x)}, \qquad u = \mathrm{cn}^2(ax). \tag{3.5.10}$$

From (3.5.9) and (3.5.10) we obtain another nonlinear equation:

$$\begin{aligned} &a^2\mathrm{cn}^2(ax)\mathrm{dn}^2(ax)\mathrm{sn}^2(ax)(2h(u)h''(u) - (h'(u))^2) \\ &+a^2(k^2\mathrm{cn}^2(ax)\mathrm{sn}^2(ax) - \mathrm{dn}^2(ax))h(u)h'(u) = h^2(u) + \alpha^2\mathrm{cn}^4(ax), \end{aligned} \tag{3.5.11}$$

which, by (3.2.8), is equivalent to the following nonlinear equation:

$$\begin{aligned} &2a^2u(1-u)(k^2u + k'^2)h(u)h''(u) - a^2(k^2u^2 + k'^2)h(u)h'(u) \\ &-a^2(k^2u + k'^2)(u - u^2)h'^2(u) = h^2(u) + \alpha^2u^2. \end{aligned} \tag{3.5.12}$$

If $h = b - cu$ is a linear function in u with constant coefficients, then (3.5.12) becomes

$$b(b - a^2ck'^2) + 2c(a^2ck'^2 - b)u + (\alpha^2 + c^2 - a^2bck^2 + a^2c^2(k^2 - k'^2))u^2 = 0. \tag{3.5.13}$$

It is straight-forward to verify that (3.5.13) holds if and only if $\alpha = ca^2kk'$ and $b = a^2ck'^2$. This shows taht $h(u) = c(a^2k'^2 - u)$ is a solution of (3.5.12).

If we choose $c = 1$, we obtain $\alpha = a^2kk'$ and

$$f(x) = \sqrt{a^2k'^2 - \mathrm{cn}^2(ax)}. \tag{3.5.14}$$

By computing (3.5.8) and (3.5.14) we obtain

$$g(x) = \int_0^x \frac{a^2kk'\text{cn}^2(ax)}{a^2k'^2 - \text{cn}^2(ax)}dx. \tag{3.5.15}$$

On the other hand, from (3.2.8) and (3.2.9) we have

$$\begin{aligned}
\frac{a^2kk'\text{cn}^2(ax)}{a^2k'^2 - \text{cn}^2(ax)} &= \frac{k'}{k} - \frac{a^2k'^3\text{sn}^2(ax)}{k(a^2k^2 + \text{sn}^2(ax))} \\
&= \frac{k'}{k} - \frac{ak'^2}{\sqrt{a^2k^4+1}}\left(\frac{\text{cn}(\gamma)\text{dn}(\gamma)\text{sn}^2(ax)}{1 - k^2\text{sn}^2(\gamma)\text{sn}^2(ax)}\right) \\
&= \frac{k'}{k} - \frac{ak'^2}{2\sqrt{a^2k^4+1}}(\text{sn}(ax+\gamma) + \text{sn}(ax-\gamma))\text{sn}(ax) \\
&= \frac{k'}{k} + \frac{\sqrt{-1}a^2k'^2}{2\sqrt{a^2k^4+1}}(k^2(\text{sn}(ax+\gamma) + \text{sn}(ax-\gamma))\text{sn}(\gamma)\text{sn}(ax)),
\end{aligned} \tag{3.5.16}$$

where $\text{sn}(\gamma) = \sqrt{-1}/(ak^2), \text{dn}(\gamma) = k'/k, \text{cn}(\gamma) = k'^2/k^2$.

Combining, (3.2.14) and (3.5.16), we obtain

$$\frac{a^2kk'\text{cn}^2(ax)}{a^2k'^2 - \text{cn}^2(ax)} = \frac{k'}{k} + \frac{\sqrt{-1}a^2k'^2}{2\sqrt{a^2k^4+1}}\big(\text{Z}(ax-\gamma) - \text{Z}(ax+\gamma) + 2\text{Z}(\gamma)\big). \tag{3.5.17}$$

From (3.5.15) - (3.5.17), we find

$$\begin{aligned}
g(x) &= \frac{k'}{k}x + \frac{\sqrt{-1}a^2k'^2}{2\sqrt{a^2k^4+1}}\int_0^x (\text{Z}(ax-\gamma)\text{Z}(ax+\gamma) + 2\text{Z}(\gamma))dx \\
&= \frac{k'}{k}x + \frac{\sqrt{-1}ak'^2}{2\sqrt{a^2k^4+1}}\Big(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2a\text{Z}(\gamma)x\Big),
\end{aligned} \tag{3.5.18}$$

where we applied (3.2.13) and the fact that Θ is an even function. Therefore, by (3.5.2), (3.3.14) and (3.5.18), we conclude that the functions y_1, y_2 defined in Proposition 5.1 are independent solutions of (3.5.1). Consequently, the general solution of (3.5.1) is given by the linear combination of y_1, y_2. □

Proposition 5.2. *For any real number $a > 1$, the general solution of*

$$y''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)y'(x) + y(x) = 0 \tag{3.5.19}$$

is given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with

$$y_1(x) = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}\cosh\Big(\frac{k'}{k} - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - aZ(\gamma)x\Big)$$

and

$$y_2(x) = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}\sinh\Big(\frac{k'}{k} - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - aZ(\gamma)x\Big)$$

where $k = \sqrt{a^2+1}/(\sqrt{2}a)$ and $k' = \sqrt{a^2-1}/(\sqrt{2}a)$ are the modulus and the complementary modulus of the Jacobi elliptic functions and $\gamma = \mathrm{sn}^{-1}(1/ak^2))$.

Proof. This can be proved by using the same trick given in the proof of Proposition 5.1. After making the key transformation (3.5.2) for (3.5.19), we obtain (3.5.5) and

$$f''(x) - f(x)(g'(x))^2 + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)f'(x) + f(x) = 0. \tag{3.5.20}$$

Solving (3.5.5) yields

$$g'(x) = \frac{\alpha\mathrm{cn}^2(ax)}{f^2(x)}, \tag{3.5.21}$$

for some constant α. By substituting (3.5.21) into (3.5.20) we obtain

$$f''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)f'(x) + f(x) = \frac{\alpha^2\mathrm{cn}^4(ax)}{f^3(x)}. \tag{3.5.22}$$

After making the second key transformation $f(x) = \sqrt{h(u)}$ with $u = \mathrm{cn}^2(ax)$ for

(3.5.22), we find

$$2a^2u(1-u)(k^2u+k'^2)h(u)h''(u) - a^2(k^2u^2+k'^2)h(u)h'(u)$$
$$-a^2(k^2u+k'^2)(u-u^2)h'^2(u) = -h^2(u)+\alpha^2u^2. \tag{3.5.23}$$

It is then straight-forward to verify that a linear function $h = b - cu$ is a solution of (3.5.23) if and only if $b = -a^2ck'^2$ and $\alpha^2 = c^2(1-a^4)/4$. In particular, if we choose $c = -1$, we obtain

$$f(x) = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}, \qquad g'(x) = \frac{\sqrt{-1}a^2kk'\mathrm{cn}^2(ax)}{a^2k'^2+\mathrm{cn}^2(ax)}. \tag{3.5.24}$$

On the other hand, from (3.2.8), (3.2.9) we have

$$\frac{a^2kk'\mathrm{cn}^2(ax)}{a^2k'^2+\mathrm{cn}^2(ax} = \frac{k'}{k} - \frac{a^2k'^2\mathrm{sn}^(ax)}{k(a^2k^2-\mathrm{sn}^2(ax))}$$
$$= \frac{k'}{k} - \frac{\mathrm{cn}(\gamma)\mathrm{dn}(\gamma)\mathrm{sn}^2(ax)}{1-k^2\mathrm{sn}^2(ax)\mathrm{sn}^2(ax)}$$
$$= \frac{k'}{k} - \frac{1}{2}(\mathrm{sn}(ax+\gamma)+\mathrm{sn}(ax-\gamma))\mathrm{sn}(ax)$$
$$= \frac{k'}{k} - \frac{a}{2}(k^2(\mathrm{sn}(ax+\gamma)+\mathrm{sn}(ax-\gamma))\mathrm{sn}(\gamma)\mathrm{sn}(ax)). \tag{3.5.25}$$

where $\mathrm{sn}(\gamma) = 1/(ak^2), \mathrm{dn}(\gamma) = k'/k, \mathrm{cn}(\gamma) = (k'/k)^2$. Combining (3.2.14) and (3.4.25), we obtain

$$\frac{a^2kk'\mathrm{cn}^2(ax)}{a^2k'^2+\mathrm{cn}^2(ax)} = \frac{k'}{k} - \frac{a}{2}(\mathrm{Z}(ax-\gamma) - \mathrm{Z}(ax+\gamma) + 2\mathrm{Z}(\gamma)). \tag{3.5.26}$$

From (3.5.24) and (3.5.26), we have

$$
\begin{aligned}
g(x) &= \frac{k'}{k}x - \frac{a}{2}\int_0^x (\mathrm{Z}(ax-\gamma) - \mathrm{Z}(ax+\gamma) + 2\mathrm{Z}(\gamma))dx \\
&= \frac{k'}{k}x - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - a\mathrm{Z}(\gamma)x,
\end{aligned}
\tag{3.5.27}
$$

where we applied (3.2.13) and the fact that Θ is an even function. Therefore, by (3.5.2), (3.5.24) and (3.5.27) we conclude that

$$
z_1(x) = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}\exp\Big(-\frac{k'}{k}x + \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + a\mathrm{Z}(\gamma)x\Big)
$$

is a solution of (3.5.19). By applying the method of reduction of order, we know that

$$
z_2(x) = \sqrt{a^2k'^2 + \mathrm{cn}^2(ax)}\exp\Big(\frac{k'}{k}x - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - a\mathrm{Z}(\gamma)x\Big)
$$

is a second independent solution of (3.5.19). Consequently, the functions y_1, y_2 defined in Proposition 5.2 are two independent solutions of (3.5.19). Hence, the general solution of (3.5.19) is given by the linear combination of y_1, y_2. □

Proposition 5.3. *For any real number* $0 < a < 1$, *the general solution of*

$$
y''(x) + 2ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)y'(x) + y(x) = 0 \tag{3.5.28}
$$

is given by $y(x) = c_1y_1(x) + c_2y_2(x)$ *with*

$$
y_1(x) = \sqrt{k^2\mathrm{dn}^2(\frac{a}{k}x) - a^2k'^2}\cos\Big(k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x-\gamma)}{\Theta(\frac{a}{k}x+\gamma)} - \sqrt{-1}\frac{a}{k}Z(\gamma)x\Big)
$$

and

$$
y_2(x) = \sqrt{k^2\mathrm{dn}^2(\frac{a}{k}x) - a^2k'^2}\sin\Big(k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x-\gamma)}{\Theta(\frac{a}{k}x+\gamma)} - \sqrt{-1}\frac{a}{k}Z(\gamma)x\Big)
$$

where $k = \sqrt{2}a/sqrt1 + a^2$ *and* $k' = \sqrt{1-a^2}/\sqrt{1+a^2}$ *are the modulus and the complementary modulud of the Jocobi's elliptic functions and* $\gamma = \mathrm{sn}^{-1}(k/a)$.

Proof. This can also be proved by using the same trick. After making the key transformation (3.5.2) for (3.5.28), we obtain

$$f(x)g''(x) + 2(f'(x) + ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)f(x))g'(x) = 0, \tag{3.5.29}$$

and

$$f''(x) - f(x)(g'(x))^2 + 2ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)f'(x) + f(x) = 0. \tag{3.5.30}$$

Solving (3.5.29) yields

$$g'(x) = \frac{\alpha \mathrm{dn}^2(\frac{a}{k}x)}{f^2}, \tag{3.5.31}$$

for some constant α. By substituting (3.5.31) into (3.5.30) we obtain

$$f''(x) + 2ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)f'(x) + f(x) = \frac{\alpha^2 \mathrm{dn}^4(\frac{a}{k}x)}{f^3(x)}. \tag{3.5.32}$$

After making the second key transformation $f(x) = \sqrt{h(u)}$ with $u = \mathrm{dn}^2(ax/k)$ for (3.5.30), we find

$$\begin{aligned} &2a^2u(1-u)(u-k'^2)h(u)h''(u) + a^2(k'^2 - u^2)h(u)h'(u) \\ &-a^2(u-k'^2)(u-u^2)h'^2(u) = -k^2h^2(u) + \alpha^2k^2u^2. \end{aligned} \tag{3.5.33}$$

It is then straight-forward to verify that a linear function $h = b + cu$ is a solution of (3.5.33) if and only if $b = -a^2ak'^2/k^2$ and $\alpha^2 = a^2k'^2$. In particular, if we choose

$c = k^2, \alpha = a^2 k'^2$, we obtain

$$f(x) = \sqrt{k^2 \mathrm{dn}^2(\frac{a}{k}x) - a^2 k'^2}, \qquad g'(x) = \frac{a^2 k' \mathrm{dn}^2(\frac{a}{k}x)}{k^2 \mathrm{dn}^2(\frac{a}{k}x) - a^2 k'^2}. \tag{3.5.34}$$

On the other hand, from (3.2.8), (3.2.9) we have

$$\begin{aligned} \frac{a^2 k' \mathrm{dn}^2(\frac{a}{k}x)}{k^2 \mathrm{dn}^2(\frac{a}{k}x) - a^2 k'^2} &= k' - \sqrt{-1}k^2 \frac{\mathrm{cn}(\gamma)\mathrm{dn}(\gamma)\mathrm{sn}^2(\frac{a}{k}x)}{1 - k^2 \mathrm{sn}^2(\frac{a}{k}x)} \\ &= k' - \sqrt{-1}\frac{a}{2k}(k^2(\mathrm{sn}(\frac{a}{k}x + \gamma) + \mathrm{sn}(\frac{a}{k}x - \gamma)\mathrm{sn}(\gamma)\mathrm{sn}(\frac{a}{k}x)), \end{aligned} \tag{3.5.35}$$

where $\mathrm{sn}(\gamma) = k/a, \mathrm{dn}(\gamma) = k'^2, \mathrm{cn}(\gamma) = \sqrt{-1}k'$. Combining (3.2.14) and (3.5.33), we obtain

$$\frac{a^2 k' \mathrm{dn}^2(\frac{a}{k}x)}{k^2 \mathrm{dn}^2(\frac{a}{k}x) - a^2 k'^2} = k' - \sqrt{-1}\frac{a}{2k}(\mathrm{Z}(\frac{a}{k}x - \gamma) - \mathrm{Z}(\frac{a}{k}x + \gamma) + 2\mathrm{Z}(\gamma)). \tag{3.5.36}$$

From (3.5.21), (3.5.24) and (3.5.34), we have

$$\begin{aligned} g(x) &= k'x - \sqrt{-1}\frac{a}{2k}\int_0^x (\mathrm{Z}(\frac{a}{k}x - \gamma) - \mathrm{Z}(\frac{a}{k}x + \gamma) + 2\mathrm{Z}(\gamma)x)dx \\ &= k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x - \gamma)}{\Theta(\frac{a}{k}x + \gamma)} - \sqrt{-1}\frac{a}{k}\mathrm{Z}(\gamma)x. \end{aligned} \tag{3.5.37}$$

Therefore, by (3.5.2), (3.5.34) and (3.5.37), we conclude that the functions y_1, y_2 defined in Proposition 5.3 are independent solutions of (3.5.28). Hence, the general solution of (3.5.28) is given by the linear combinatin of y_1, y_2. □

Corollary 1. *For any real number $a > 0$, the general solution of*

$$z''(x) = (a^2 + 1)\mathrm{nc}^2(ax, k)z, \qquad k = \frac{\sqrt{a^2 - 1}}{\sqrt{2}a} \tag{3.5.38}$$

is given by $z(x) = c_1 \mathrm{cn}(ax, k) y_1(x) + c_2 \mathrm{cn}(ax, k) y_2(x)$ where $y_1(x), y_2(x)$ are defined *in Proposition 5.1.*

Proof. This follows from Proposition 5.1 and the fact that equation (3.5.38) can be obtained from (3.5.1) by making the transformation $y(x) = \mathrm{cn}(ax,k)z(x)$. □

Corollary 2. *For any real number $a > 1$, the general solution of*

$$z''(x) = (a^2-1)\mathrm{nc}^2(ax,k)z, \qquad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a} \tag{3.5.39}$$

is given by $z(x) = c_1\mathrm{cn}(ax,k)y_1(x) + c_2\mathrm{cn}(ax,k)y_2(x)$ where $y_1(x), y_2(x)$ are defined in Proposition 5.2.

Proof. This follows from Proposition 5.2 and the fact that equation (3.5.39) can be obtained from (3.5.19) by making the transformation $y(x) = \mathrm{cn}(ax,k)z(x)$. □

Corollary 3. *For any real number $0 < a < 1$, the general solution of*

$$z''(x) = (a^2-1)\mathrm{nd}^2(\frac{a}{k}x,k)z, \qquad k = \frac{\sqrt{2}a}{\sqrt{1+a^2}} \qquad k' = \frac{\sqrt{1-a^2}}{\sqrt{1+a^2}} \tag{3.5.40}$$

is given by $z(x) = c_1\mathrm{dn}(ax/k,k)y_1(x) + c_2\mathrm{dn}(ax/k,k)y_2(x)$ where y_1, y_2 are defined in Proposition 5.3.

Proof. This follows from Proposition 5.3 and the fact that equation (3.5.40) can be obtained from (3.5.28) by making the transformation $y(x) = \mathrm{dn}(ax/k,k)z(x)$. □

Remark 1. Conversely, since

$$\begin{aligned} &\frac{d}{du}(\ln\frac{\Theta(u-\gamma)}{\Theta(u+\gamma)} = \mathrm{Z}(u-\gamma) - \mathrm{Z}(au-\gamma) \\ &= k^2\mathrm{sn}(u)\mathrm{sn}(\gamma)(\mathrm{sn}(u-\gamma) + sn(u+\gamma)) - 2\mathrm{Z}(\gamma) \\ &= \frac{2k^2\mathrm{cn}(\gamma)\mathrm{dn}(\gamma)\mathrm{sn}(\gamma)\mathrm{sn}^2(u)}{1-k^2\mathrm{sn}^2(\gamma)\mathrm{sn}^2(u)} - 2\mathrm{Z}(\gamma), \end{aligned} \tag{3.5.41}$$

it is straight-forward to verify that two functions y_1, y_2 defined in Proposition 5.1 (respectively, in Proposition 5.2 and 5.3) are indeed independent solutions of (3.5.1) (respectively, of (3.5.19) and of (3.5.28)).

Remark 2. In 1879, E. Picard discovered in [25] a method for solving the differential equation:

$$y''(x) + nk^2\mathrm{cn}(x)\mathrm{sd}(x)y'(x) + \alpha y(x) = 0, \tag{3.5.42}$$

where n is a positive integer and α a constant.

Although equation (3.5.42) is quite similar to the equations (3.5.1), (3.5.19) and (3.5.28), unfortunely, Picard's method does not apply to these equations.

3.6 Proof of Theorem 3.2

Let $\mathbf{x} : M^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2}$ be an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies Chen's equality (2.1.2). Then, by Lemma 3.2 and a well-known result of E. Cartan and J. A. Schouton on conformally flat hypersurfaces(cf. p.154 of [6]), we know that M^n has a principle curvature with multiplicity at least $n-1$ for $n = 3$ as well as for $n \geq 4$. Thus, by applying (2.3.2), we know that either (i) the principle curvatures of M^n are given by $\lambda_1 = 0, \lambda_2 = ... = \lambda_n = \mu$, or (ii) $n = 3$ and the principle curvature of M^3 are given by $\lambda_1 = \lambda_2 = \mu/2, \lambda_3 = \mu \neq 0$. We treat these two cases separately.

Case (i): $\lambda_1 = 0$ and $\lambda_2 = ... = \lambda_n = \mu$. If $\mu = 0$ identically, then M^n is totally geodesic in $S^{n+1}(1)$.

Now, suppose that M^n is not totally in $S^{n+1}(1)$. Denote by U the open subset of M^n on which the mean curvature function is nonzero. Then U is non-empty. We will prove in the following that U is the whole manifold M^n in this case.

We denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the distributions on the open subset U spanned by $\{e_1\}$ and $\{e_2, ..., e_n\}$, respectively.

Now, by putting $j = 1, i = 2, 3, ..., n$ in (2.3.5), we get $\omega_1^i(e_1) = 0$ which implies that integral curves of $\mathcal{D}$ are geodesics.

Also by using the first equation of (2.3.5), we find

$$e_2\mu = ... = e_n\mu = 0. \tag{3.6.1}$$

It is easy to see from (2.3.4) that

$$\omega_1^i(e_j) = 0, \qquad 1 \leq i \neq j \leq n, \tag{3.6.2}$$

which implies that $\mathcal{D}^\perp$ is integrable. Consequently, there exist local coordinate systems $\{x_1, x_2, ..., x_n\}$ such that $\partial/\partial x_2, ..., \partial/\partial x_n$ span $\mathcal{D}^\perp$ and $e_1 = \partial/\partial x$ with $x = x_1$. From (3.6.1), we know that μ depends only on x, i.e., $\mu = \mu(x)$.

Choosing $i = 1$ for the first equation in (2.3.4), we get $\mu'(x) = -\mu(x)\omega_1^j(e_j)$ for any $j \geq 2$. Thus

$$\nabla_{e_j} e_1 = \sum_{k=2}^{n} \omega_1^k(e_j)e_k = \omega_1^j(e_j)e_j = -(\ln \mu)' e_j. \tag{3.6.3}$$

Using (3.6.3) we know that each integral submanifold of $\mathcal{D}^\perp$ is an extrinsic sphere of $S^{n+1}(1)$, i.e., it is a totally umbilical submanifold with nonzero parallel mean curvature vector in $S^{n+1}(1)$. Hence, the distribution $\mathcal{D}^\perp$ is a spherical distribution. Therefore, by applying a result of Hiepko [21] (cf. Remark 2.1 of [20]), we know that U is locally the warped product $I \times_{f(x)} S^{n-1}(1)$, where $f(x)$ is a suitable warped function. Therefore, the metric on U is given by

$$g = dx^2 + f^2(x)g_0, \tag{3.6.4}$$

where g_0 is the metric of $S^{n-1}(1)$. In particular, if we choose the spherical coordinate system $\{u_2, ..., u_n\}$ on $S^{n-1}(1)$, then we have

$$g = dx^2 + f^2(x)(du_2^2 + \cos^2 u_2 du_3^2 + ... + \cos^2 u_2 ... \cos^2 u_{n-1} du_n^2). \tag{3.6.5}$$

By (3.6.5) we obtain

(3.6.6)

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \qquad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial u_k} = \frac{f'}{f} \frac{\partial}{\partial u_k}, \qquad \nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = -ff' \frac{\partial}{\partial x},$$

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = -(\tan u_i) \frac{\partial}{\partial u_j}, \qquad 2 \le i < j,$$

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = -(ff' \prod_{\ell=2}^{j-1} \cos^2 u_\ell) \frac{\partial}{\partial x} + \sum_{k=2}^{j-1} (\frac{\sin 2u_k}{2} \prod_{\ell=k+1}^{j-1} \cos^2 u_\ell) \frac{\partial}{\partial u_k}, 2 \le i, j, k \le n,$$

From (3.6.5) and the assumption on the principle curvature, we know that the second fundamental form h of M^n in $S^{n+1}(1)$ satisfies

(3.6.7)

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0, \qquad h(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}) = \mu f^2 \xi,$$

$$h(\frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_3}) = \mu f^2 \cos^2 u_2 \xi, \quad ..., \quad h(\frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_n}) = \mu f^2 \prod_{k=2}^{n-1} \cos^2 u_k \xi,$$

$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial u_j}) = 0, \qquad h(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = 0, \qquad 2 \le i \ne j \le n,$$

where ξ is a unit normal vector field of M^n in $S^{n+1}(1)$.

Let $\tilde{\nabla} h$ denote the covariant derivative of the second fundamental form h. Then by the equation of Codazzi, we have

$$(\tilde{\nabla}_{\frac{\partial}{\partial x}} h)(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}) = (\tilde{\nabla}_{\frac{\partial}{\partial u_2}} h)(\frac{\partial}{\partial x}, \frac{\partial}{\partial u_2}). \tag{3.6.8}$$

From (3.6.6) - (3.6.8)we obtain $\mu' f = -\mu f'$. Therefore

$$\mu(x) = \frac{a}{f(x)} \tag{3.6.9}$$

for some nonzero constant a.

Now, by applying (3.6.6), we know that the sectional curvature K_{12} and K_{23} of the plane section spanned by $\partial/\partial x, \partial/\partial u_2$ and $\partial/\partial u_2, \partial/\partial u_3$ are equal to $-f''/f$ and $(1-f'^2)/f^2$, respectively. On the hand, from equation of Gauss and our assumption on principle curvatures, we have $K_{12} = 1$ and $K_{23} = 1 + \mu^2$. Therefore, by combining these facts with (3.6.9), we obtain

$$f'' + f = 0, \qquad f'^2 + f^2 = 1 - a^2, \tag{3.6.10}$$

which implies in particular that $0 < a < 1$.

Solving the first differential equation in (3.6.10) yields $f(x) = C\cos(x+b)$ for some constants b, and C. By applying a translation in x if necessary, we have

$$f(x) = C\cos x. \tag{3.6.11}$$

Substituting (3.6.11) into the second differential equation in (3.6.10), we obtain $C = \sqrt{1-a^2}$. Consequently, we have

$$f(x) = \sqrt{1-a^2}\cos x, \qquad \mu(x) = \frac{a}{\sqrt{1-a^2}}\sec x, \tag{3.6.12}$$

where a is a constant satisfying $0 < a < 1$.

(3.6.12) implies $\mu \geq a/\sqrt{1-a^2} > 0$ on U. Hence, by the continuity of the squared mean curvature function $H^2 = (n-1)^2\mu^2/n^2$, we know that the mean curvature function is nowhere zero and therefore U is the whole manifold M^n. Consequently, (3.6.4) and (3.6.12) imply that M^n is isometric to an open portion of the Riemannian manifold $\hat{B}^n_a$ which was defined in Section 3.1.

By applying (3.6.5), (3.6.6), (3.6.7), (3.6.12) and the formula of Gauss, we conclude that the isometric immersion $\mathbf{x} : M^n \to S^{n-1}(1) \subset \mathbb{E}^{n+2}$ satisfies the following system of partial differential equations:

(3.6.13) $$\frac{\partial^2 \mathbf{x}}{\partial x^2} = -\mathbf{x},$$

(3.6.14) $$\frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = -(\tan x)\frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 2, 3, ..., n,$$

(3.6.15) $$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = -(\tan u_i)\frac{\partial \mathbf{x}}{\partial u_j}, \qquad 2 \le i < j,$$

(3.6.16 $$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = \frac{1}{2}(1-a^2)\sin 2x\frac{\partial \mathbf{x}}{\partial x} + a\sqrt{1-a^2}\cos x\xi - ((1-a^2)\cos^2 x)\mathbf{x},$$

(3.6.17) $$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = (\cos^2 u_j)\frac{\partial^2 \mathbf{x}}{\partial u_j^2} + \frac{1}{2}(\sin 2u_j)\frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 2, ..., n-1.$$

Solving (3.6.13) yields

(3.6.18) $$\mathbf{x} = P(u_2, .., u_n)\sin x + Q(u_2, .., u_n)\cos x,$$

foe some $\mathbb{E}^{n+1}$-valued functions $P = P(u_2, ..., u_n)$ and $Q = Q(u_2, ..., u_n)$. Substituting (3.6.18) into (3.6.14), we know that P is a constant vector, we denote it by c_1. Thus

(3.6.19) $$\mathbf{x} = c_1 \sin x + Q(u_2, ..., u_n)\cos x,$$

Substituting (3.6.19) into (3.6.15) with $i = 2$, we obtain

$$\frac{\partial^2 Q}{\partial u_j \partial u_2} + (\tan u_2)\frac{\partial Q}{\partial u_j} = 0, \qquad j = 3, ..., n \tag{3.6.20}$$

which implies

$$\frac{\partial Q}{\partial u_2} + (\tan u_2) Q = \tilde{\phi}_2(u_2), \tag{3.6.21}$$

for some function $\tilde{\phi}_2 = \tilde{\phi}_2(u_2)$. Therefore, by solving (3.6.21), we have

$$Q = \phi_2(u_2) + Q_3(u_3, ..., u_n) \cos u_2 \tag{3.6.22}$$

for some function $\phi_2 = \phi_2(u_2)$ and $Q_3 = Q_3(u_3, ..., u_n)$.

Similarly, by substituting (3.6.19) and (3.6.22) into (3.6.15) with $i = 3$ and $j > 3$, we find

$$Q_3 = \phi_3(u_3) + Q_4(u_4, ..., u_n) \cos u_3, \tag{3.6.23}$$

for some function $\phi_3 = \phi_3(u_2)$ and $Q_4 = Q_4(u_4, ..., u_n)$. Repeating such procedure $n - 2$ times, we obtain

$$\begin{aligned} Q &= \phi_2(u_2) + Q_3(u_3, ..., u_n) \cos u_2, \\ Q_3 &= \phi_3(u_3) + Q_4(u_4, ..., u_n) \cos u_3, \\ Q_4 &= \phi_4(u_4) + Q_5(u_5, ..., u_n) \cos u_4, \\ &\vdots \\ Q_{n-1} &= \phi_{n-1}(u_{n-1}) + \phi_n(u_n) \cos u_{n-1}, \end{aligned} \tag{3.6.24}$$

with $\phi_n(u_n) = Q_n(u_n)$. Substituting (3.6.24) into (3.6.19) we get

$$\begin{aligned} \mathbf{x} &= c_1 \sin x + \phi_2(u_2) \cos x + \phi_3(u_3) \cos u_2 \cos x + ... \\ &+\phi_{n-1}(u_{n-1}) \cos u_2 ... \cos u_{n-2} \cos x + \phi_n(u_n) \cos u_2 ... \cos u_{n-1} \cos x. \end{aligned} \tag{3.6.25}$$

Substituting (3.5.25) into (3.5.17) with $j = n-1$, we obtain

$$\phi_n''(u_n) + \phi_n(u_n) = \cos u_{n-1}\phi_{n-1}'' + \sin u_{n-1}\phi_{n-1}'(u_{n-1}), \tag{5.6.26}$$

which implies that

$$\phi_n''(u_n) + \phi_n(u_n) = k_n, \tag{3.6.27}$$

$$\cos u_{n-1}\phi_{n-1}''(u_{n-1}) + \sin u_{n-1}\phi_{n-1}'(u_{n-1}) = k_n, \tag{3.6.28}$$

for some constant vector k_n. Solving (3.6.27) yields

$$\phi_n = c_{n+1} \sin u_n + c_{n+2} \cos u_n + k_n, \tag{3.5.29}$$

for some constant vectors c_{n+1}, c_{n+2}. Combining (3.6.25) and (3.6.29) we obtain

$$\begin{aligned} \mathbf{x} &= c_1 \sin x + \phi_2(u_2) \cos x + \phi_3(u_3) \cos u_2 \cos x \\ &+\phi_{n-1}(u_{n-1}) \cos u_2 ... \cos u_{n-1} \cos x + \\ &+c_{n+1} \cos u_2 ... \cos u_{n-1} \sin u_n \cos x + c_{n+2} \cos u_2 ... \cos u_n \cos x, \end{aligned} \tag{3.6.30}$$

for some constant vectors $c_1, ..., c_{n+2}$.

Now, we choose the initial condition at 0=(0, ..., 0) as follows:

$$\begin{aligned} &\mathbf{x}(0) = a\epsilon_2 + \sqrt{1-a^2}\epsilon_{n+2}, && \frac{\partial \mathbf{x}}{\partial x}(0) = \epsilon_1, \\ &\frac{\partial \mathbf{x}}{\partial u_2}(0) = \sqrt{1-a^2}\epsilon_2, \quad \ldots && \frac{\partial \mathbf{x}}{\partial u_n} = \sqrt{1-a^2}\epsilon_{n+1}, \end{aligned} \tag{3.6.32}$$

where $\{\epsilon_1, ..., e_{n+2}\}$ is the natural coordinate basis of $\mathbb{E}^{n+2}$. Then, by applying (3.6.31) and (3.6.32), we obtain

$$c_1 = \epsilon_1, \quad e_2 = a\epsilon_2, \quad c_3 = \sqrt{1-a^2}\epsilon_3, \quad ..., \quad c_{n+2} = \sqrt{1-a^2}\epsilon_{n+2}. \tag{3.6.33}$$

Consequently, by (3.6.31) and (3.6.33) we conclude that, up to rigid motions of $\mathbb{E}^{n+2}$, the immersion $\mathbf{x}$ is given by (3.1.5) in Theorem 3.1. It is clear from (3.1.5) that the immersion $\mathbf{x}$ can be extended to the two point compactification $\hat{B}^n_a$ of B^n_a.

Case (ii): $\lambda_1 = \lambda_2 = \mu/2$ and $\lambda_3 = \mu \neq 0$. In this case (2.3.5) yields

$$\Gamma^2_{31} = \Lambda_3 = 0, \quad \epsilon_1\mu = e_2\mu = 0, \quad \Gamma^1_{33} = \Gamma^2_{33} = 0. \tag{3.6.34}$$

Denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. By (3.6.34) we know that the integral curves of $\mathcal{F}^\perp$ are geodesics and the distribution $\mathcal{F}$ is integrable. Consequently, there exist a local coordinate system$\{x, u, v\}$ such that $\mathcal{F}$ is spanned by $\{\partial/\partial u, \partial/\partial v\}$ and $e_3 = \partial/\partial x$.

From (3.6.34) we know that μ depends only on x, i.e., $\mu = \mu(x)$. Also, from (2.3.8) and (3.6.34), we have

$$\omega^1_3 = \frac{\mu'(x)}{\mu(x)}\omega^1, \qquad \omega^2_3 = \frac{\mu'(x)}{\mu(x)}\omega^2. \tag{3.6.35}$$

Using (3.6.35) we obtain

$$\nabla_{e_i} e_3 = \frac{\mu'(x)}{\mu(x)} e_j, \quad j = 1, 2. \tag{3.6.36}$$

Therefore, each integral submanifold of $\mathcal{F}$ is an extrisic sphere of $S^4(1)$. Hence, by applying a result of Heipko, we know that M^3 is locally the warped product $I \times_{f(x)} S^2(1)$, where $f(x)$ is a suitable warp function. In particular, if we choose the spherical coordinate system $\{\theta, \phi\}$ for $S^2(1)$, we have

$$g = dx^2 + f^2(x)(d\phi^2 + \cos^2 \phi d\theta^2). \tag{3.6.37}$$

By computing $d\omega_3^1$ and by using (3.6.35) and Cartan's structure equations, we find

$$\mu''(x) + \frac{\mu^3(x)}{2} + \mu(x) = 0. \tag{3.6.38}$$

Let $\psi = 2\mu(x)$. Then (3.6.38) becomes $\psi''(x) + 2\psi^3(x) + \psi(x) = 0$. Hence, by applying Lemma 5.3 of [10], we obtain

$$\mu(x) = \sqrt{2(a^2 - 1)}\mathrm{cn}(ax, k), \qquad k = \frac{\sqrt{a^2 - 1}}{\sqrt{2}a}, \tag{3.6.39}$$

where $a > 1$ is real number.

Now, by applying (3.6.37), (3.6.39) and the equation of Codazzi, we obtain $\mu' f = \mu f'$. Therefore

$$f(x) = c\mu(x), \tag{3.6.40}$$

for some non-zero constant c.

On the other hand, using (3.6.37) we can compute the sectional curvature K_{23} of the plane section spanned by $\{\partial/\partial\phi, \partial/\partial\theta\}$. On the other hand, we may also compute K_{23} by using the equation of Gauss. Comparing the two different expressions of K_{23} so obtained, yields

$$c^2\mu'^2 = 1 - c^2\mu^2 - \frac{1}{4}c^2\mu^4. \tag{3.6.41}$$

Substituting (3.6.39) into (3.6.41) we have $c^2 = 1/(a^4 - 1)$. Therefore

$$f = \frac{\sqrt{2}}{\sqrt{a^2+1}}\mathrm{cn}(ax,k), \qquad \mu = \sqrt{2(a^2-1)}\mathrm{cn}(ax,k), \qquad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a}. \tag{3.6.42}$$

(5.6.37) and (5.6.42) imply that M^3 is an open portion of the warped product manifold $I \times_{\sqrt{2}/\sqrt{a^2+1}\mathrm{cn}(ax)} S^2(1)$ which is isometric to P_a^3, first introduced by B. Y. Chen in [10] (also see [13]).

(3.6.37), (3.6.39), (3.6.42) and the formula of Gauss imply that the immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2\mathbf{x}}{\partial x^2} = \sqrt{2(a^2-1)}\mathrm{cn}(ax,k)\xi - \mathbf{x}, \tag{3.6.43}$$

$$\begin{aligned}\frac{\partial^2\mathbf{x}}{\partial\phi^2} &= \frac{2a}{a^2+1}\mathrm{cn}(ax)\mathrm{dn}(ax)\mathrm{sn}(ax)\frac{\partial\mathbf{x}}{\partial x}\\ &+\frac{\sqrt{2(a^2-1)}}{a^2+1}\mathrm{cn}^3(ax)\xi - \frac{2}{a^2+1}\mathrm{cn}^2(ax)\mathbf{x},\end{aligned} \tag{3.6.44}$$

$$\begin{aligned}\frac{\partial^2\mathbf{x}}{\partial\theta^2} &= \frac{2a}{a^2+1}\cos^2\phi\,\mathrm{cn}(ax)\mathrm{dn}(ax)\mathrm{sn}(ax)\frac{\partial\mathbf{x}}{\partial x} + \sin\phi\cos\phi\frac{\partial\mathbf{x}}{\partial\phi}\\ &+\frac{\sqrt{2(a^2-1)}}{a^2+1}\cos^2\phi\,\mathrm{cn}^3(ax)\xi - \frac{2}{a^2+1}\cos^2\phi\,\mathrm{cn}^2(ax)\mathbf{x},\end{aligned} \tag{3.6.45}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \phi} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.6.46}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \theta} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)\frac{\partial \mathbf{x}}{\partial \theta}, \tag{3.6.47}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \theta \partial \phi} = -\tan\phi\frac{\partial \mathbf{x}}{\partial \theta}. \tag{3.6.48}$$

By taking the partial derivative of (3.6.44) with respect to ϕ and by applying the equation of Weingarten, we obtain

$$\frac{\partial^3 \mathbf{x}}{\partial \phi^3} = -\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.6.49}$$

by virtue of (3.2.8), (3.2.10) and

$$k^2 = \frac{a^2-1}{2a^2}, \qquad k'^2 = \frac{a^2+1}{2a^2}. \tag{3.6.50}$$

Solving (3.6.49) yields

$$\mathbf{x}(x,\phi,\theta) = A(x,\theta)\sin\phi + B(x,\theta)\cos\phi + C(x,\theta), \tag{3.6.51}$$

for some $\mathbb{E}^5_1$-valued functions A, B, C of two variables. Substituting (3.6.51) into (3.6.46) yields

$$\frac{\partial A}{\partial x} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)A, \qquad \frac{\partial B}{\partial x} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)B. \tag{3.6.52}$$

Solve the above, we have

$$A(x,\theta) = E(\theta)\mathrm{cn}(ax), \qquad B(x,\theta) = D(\theta)\mathrm{cn}(ax) \tag{3.6.53}$$

for some $\mathbb{E}^5_1$-valued functions E, D. Thus,

$$\mathbf{x}(x,\phi,\theta) = E(\theta)\mathrm{cn}(ax)\sin\phi - D(\theta)\mathrm{cn}(ax)\cos\phi + C(x,\theta). \tag{3.6.54}$$

Substituting (3.6.54) into (3.6.47) yields

$$\frac{\partial^2 C}{\partial x \partial \theta} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)\frac{\partial C}{\partial \theta}. \tag{3.6.55}$$

By solving (3.6.55) we find $C(x,\theta) = G(\theta)\mathrm{cn}(ax) + K(x)$ for some $\mathbb{E}^5_1$-valued functions $G(\theta)$ and $K(x)$. Thus, (3.5.54) gives

$$\mathbf{x}(x,\phi,\theta) = E(\theta)\mathrm{cn}(ax)\sin\phi + D(\theta)\mathrm{cn}(ax)\cos\phi + G(\theta)\mathrm{cn}(ax) + K(x). \tag{3.6.56}$$

Substituting (3.6.56) into (3.6.48) yields $E'(\theta) = G'(\theta) = 0$. Thus, E and G are constant vectors in $\mathbb{E}^5$. Consequently, from (3.6.56) we know that $\mathbf{x}$ takes the form:

$$\mathbf{x}(x,\phi,\theta) = c_1\mathrm{cn}(ax)\sin\phi + D(\theta)\mathrm{cn}(ax)\cos\phi + F(x). \tag{3.6.57}$$

where c_1 ia a constant vector. From (3.6.43) we have

$$\xi = \frac{1}{\sqrt{2(a^2-1)}\mathrm{cn}(ax)}\left(\frac{\partial^2 \mathbf{x}}{\partial x^2} + \mathbf{x}\right). \tag{3.6.58}$$

By (3.6.45), (3.6.57), (3.6.58) and a long computation, we obtain

$$F''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)F'(x) - F(x) = 0, \tag{5.6.59}$$

$$D''(\theta) + D(\theta) = 0, \tag{3.6.60}$$

by virtue of (3.2.8), (3.2.10) and (3.6.50). Solving (3.6.60) yields

$$D(\theta) = c_2 \cos\theta + c_3 \sin\theta, \tag{3.6.61}$$

for some constant vectors c_2, c_3 in $\mathbb{E}^5$. Therefore, by applying Proposition 5.1, (3.6.57) and (3.6.61), we have

$$\begin{aligned} \mathbf{x}(x,\phi,\theta) &= c_1 \sin\phi \operatorname{cn}(ax) + c_2 \cos\phi \cos\theta \operatorname{cn}(ax) + c_3 \cos\phi \sin\theta \operatorname{cn}(ax) \\ &+ c_4\sqrt{a^2k'^2 - \operatorname{cn}^2(ax)} \cos\Big(\frac{k'}{k}x + \frac{\sqrt{-1}ak'^2}{2\sqrt{a^2k^4+1}}\Big(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2a\mathrm{Z}(\gamma)x\Big)\Big) \\ &+ c_5\sqrt{a^2k'^2 - \operatorname{cn}^2(ax)} \sin\Big(\frac{k'}{k}x + \frac{\sqrt{-1}ak'^2}{2\sqrt{a^2k^4+1}}\Big(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2a\mathrm{Z}(\gamma)x\Big)\Big), \end{aligned} \tag{3.6.62}$$

where Θ and Z are the Theta function and Zeta functions.

Now, if we choose the initial conditions at 0=(0, 0, 0) as follows:

$$\mathbf{x}(0) = \frac{1}{ak'}(\epsilon_2 + \epsilon_4), \quad \frac{\partial \mathbf{x}}{\partial x}(0) = \epsilon_5, \quad \frac{\partial \mathbf{x}}{\partial \phi}(0) = \frac{1}{ak'}\epsilon_1, \quad \frac{\partial \mathbf{x}}{\partial \theta}(0) = \frac{1}{ak'}\epsilon_3, \tag{3.6.63}$$

where $\{\epsilon_1, ..., \epsilon_5\}$ is the standard basis of $\mathbb{E}^5$, then we obtain

$$c_\alpha = \frac{1}{ak'}\epsilon_\alpha, \quad \alpha = 1, ..., 5. \tag{3.6.64}$$

Therefore, up to rigid motions, the immersion $\mathbf{x}$ is given by (2.1.7) in Theorem 3.2.

The converse can be verified by straigh-forward but long computations. □

3.7 Proof of Theorem 3.3

Let $\mathbf{x} : M^n \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ be an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies Chen's equality (2.1.2). By Lemma 3.2 and a result of Cartan and Shouten, either (a) the principle curvatures of M^n are given by $\lambda_1 = 0, \lambda_2 = ... = \lambda_n = \mu$, or (b) $n = 3$ and the principle cuevatures of M^3 are given by $\lambda_1 = \lambda_2 = \mu/2$, $\lambda_3 = \mu \neq 0$. We treat these two cases separately.

Case (a): $\lambda_1 = 0$ and $\lambda_2 = ... = \lambda_n = \mu$.

If $\mu = 0$ identically, then M^n is totally geodesic. This yields Case 1 of Theorem 3.3.

Now, suppose that M^n is not totally geodesic in $H^{n+1}(-1)$. Denote by U the open subset of M^n on which the mean curvature function is nonzero. We shall work on this open subset of M^n unless mentioned otherwise. Clearly, $\mu \neq 0$ on U.

We denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the distributions on the open subset U apanned by $\{e_1\}$ and $\{e_2, ..., e_n\}$, respectively. Then, as in the proof of Theorem 3.2, we can prove that integral curves of $\mathcal{D}$ are geodesics and $\mathcal{D}^\perp$ is integrable. Thus, there exist a local coordinate system $\{x_1, x_2, ..., x_n\}$ such that $\partial/\partial x_2, ..., \partial/\partial x_n$ span $\mathcal{D}^\perp$ and $e_1 = \partial/\partial x$ with $x = x_1$. Also, using (2.3.5) we can prove that μ depends only on x, i.e., $\mu = \mu(x)$.

If we choose $i = 1$ for the first equation of (2.3.4), we get $\mu'(x) = -\mu(x)\omega_1^j(e_j)$ for any $j \geq 2$. Thus

$$\nabla_{e_j} e_1 = \sum_{k=2}^{n} \omega_1^k(e_j)e_k = \omega_1^j(e_j)e_j = (\ln \mu)' e_j. \tag{3.7.1}$$

Hence, each integral submanifold of $\mathcal{D}^\perp$ is an extrinsic sphere of $H^{n+1}(-1)$, i.e., the distribution $\mathcal{D}^\perp$ is spherical. Therefore, U is locally a warped product manifold $I \times_{f(x)} N^{n-1}(\bar{c})$, where $f(x)(> 0)$ is the warp function and $N^{n-1}(\bar{c})$ ia a Riemannian space form of constant sectional curvature $\bar{c}$.

Since $N^{n-1}(\bar{c})$ is of constant curvature, it is conformally flat. Thus, there exists a local coordinate system $\{u_2, ..., u_n\}$ such that metric tensor of $N^{n-1}(\bar{c})$ is given by

$$g_0 = E^2(du_2^2 + du_3^2 + ... + du_n^2). \tag{3.7.2}$$

With respect to the coordinate system $\{x, u_2, ..., u_n\}$ on $I \times_{f(x)} N^{n-1}(\bar{c})$, we have

$$g = dx^2 + f^2 E^2 (du_2^2 + ... + du_n^2). \tag{3.7.3}$$

From the above, we have

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial u_i} = \frac{f'}{f} \frac{\partial}{\partial u_i}, \quad \nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \frac{E_i}{E} \frac{\partial}{\partial u_j} + \frac{E_j}{E} \frac{\partial}{\partial u_i}, \\ &\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_i} = -ff'E^2 \frac{\partial}{\partial x} + \frac{E_i}{E} \frac{\partial}{\partial u_i} - \sum_{k \neq i} \frac{E_k}{E} \frac{\partial}{\partial u_k}, \end{aligned} \tag{3.7.4}$$

for distinct $i, j, k (2 \leq i, j, k \leq n)$, where $E_i = \partial E / \partial u_i$.

Codazzi's equation, (3.7.4) and our assumption on principle curvatures imply

$$\mu = \frac{a}{f} \tag{3.7.5}$$

for some constant $a > 0$. Also, from (3.7.4), it follows that the sectional curvatures K_{12} and K_{23} of M^n associated with the plane section spanned by $\partial/\partial x, \partial/\partial u_2$ and $\partial/\partial u_2, \partial/\partial u_3$ are given by $-f''/f$ and $(\bar{c} - f'^2)/f^2$, respectively. On the other hand, Gauss equation yields $K_{12} = -1$ and $K_{23} = -1 + \mu^2$. Comparing these facts with (3.7.5), we get

$$f'' - f = 0, \qquad f^2 - f'^2 = a^2 - \bar{c}. \tag{3.7.6}$$

Solving the first equation of (3.7.6) yields $f(x) = c_1 \cosh x + c_2 \sinh x$, where c_1, c_2 are constants, which we can write as either $f(x) = \alpha \sinh(x+b)$ or $f(x) = \alpha \cosh(x+b)$, for some constant α, b. Thus , by applying a translation in x if necessary, we have $f(x) = \alpha \sinh x$ or $f(x) = \alpha \cosh x$. We consider these two cases separately.

Case (1): $f(x) = \alpha \cosh x$. In this case, the second equation of (3.7.6) yields

$\alpha^2 = a^2 - \bar{c}$. Thus, after applying a scaling on E if necessary, we have

$$(3.7.7) \qquad \bar{c} = 1, \quad f(x) = \sqrt{a^2 - 1}\cosh x, \quad \mu = \frac{a}{\sqrt{a^2 - 1}}\text{sech}x, \quad a > 1,$$

or

$$(3.7.8) \qquad \bar{c} = 0, \quad f(x) = \cosh x, \quad \mu = \text{sech}x,$$

or

$$(3.7.9) \qquad \bar{c} = -1, \quad f(x) = \sqrt{a^2 + 1}\cosh x, \quad \mu = \frac{a}{\sqrt{a^2 + 1}}\text{sech}x, \quad a > 0.$$

Case (1-i): $\bar{c} = 1, f(x) = \sqrt{a^2 - 1}\cosh x, \mu = \frac{a}{\sqrt{a^2-1}}\text{sech}x, a > 1$. In this case, the open subset U is the whole manifold M^n. Therefore, M^n is an open portion of the warped product manifold $A^n_a = \mathbb{R} \times_{\sqrt{a^2-1}\cosh x} S^{n-1}(1)$. By choosing spherical coordinates $\{u_2, ..., u_n\}$ on $S^{n-1}(1)$, we obtain (3.7.3) and (3.7.4) with

$$(3.7.10) \qquad f = \sqrt{a^2 - 1}\cosh x, \qquad \mu = \frac{a}{\sqrt{a^2 - 1}}\text{sech}x.$$

Therefore, the equation of Gauss implies that the isometric immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$(3.7.11) \qquad \frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x},$$

$$(3.7.12) \qquad \frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = \tanh x \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2....,n,$$

(3.7.13) $$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = -\tan u_i \frac{\partial \mathbf{x}}{\partial u_j}, \quad 2 \leq i < j,$$

(3.7.14) $$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = \frac{1-a^2}{2} \sinh 2x \frac{\partial \mathbf{x}}{\partial x} + a\sqrt{a^2-1} \cosh x \xi + ((a^2-1)\cosh^2 x)\mathbf{x},$$

(3.7.15) $$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = \cos^2 u_j \frac{\partial^2 \mathbf{x}}{\partial u_j^2} + \frac{1}{2} \sin 2u_j \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, ..., n-1.$$

where ξ is a unit normal vector field of M^n in $H^{n+1}(-1)$.

Solving equation (3.7.11) yields

(3.7.16) $$\mathbf{x} = P(u_2, ..., u_n) \sinh x + Q(u_2, ..., u_n) \cosh x,$$

for some $\mathbb{E}_1^{n+1}$-valued functions $P = P(u_2, ..., u_n)$ and $Q = Q(u_2, ..., u_n)$. By substituting (3.7.16) into (3.7.12), we know that P is a constant vector, denoted by c_1. Thus

(3.7.17) $$\mathbf{x} = c_1 \sinh x + Q(u_2, ..., u_n) \cosh x.$$

Substituting (3.7.17) into (3.7.13) with $i = 2$ yields

(3.7.18) $$\frac{\partial Q}{\partial u_2} + (\tan u_2) Q = \tilde{\phi}_2(u_2),$$

for some function $\tilde{\phi}_2 = \tilde{\phi}_2(u_2)$. Therefore, after solving (3.7.18), we obtain

(3.7.19) $$Q = \phi_2(u_2) + Q_3(u_3, ..., u_n) \cos u_2$$

for some functions $\phi_2 = \phi_2(u_2)$ and $Q_3 = Q_3(u_3, ..., u_n)$. Repeating this procedure

$n-2$ times, we obtain

$$Q = \phi_2(u_2) + Q_3(u_3, ..., u_n)\cos u_2,$$
$$(3.7.20) \qquad Q_3 = \phi_3(u_3) + Q_4(u_4, ..., u_n)\cos u_3,$$
$$\vdots$$
$$Q_{n-1} = \phi_{n-1}(u_{n-1}) + \phi_n(u_n)\cos u_{n-1},$$

where $\phi_n(u_n) = Q_n(u_n)$.

Substituting (3.7.20) into (3.7.17), we find

$$(3.7.21) \qquad \begin{aligned} &\mathbf{x} = c_1 \sinh x + \phi_2(u_2)\cosh x + \phi_3(u_3)\cos u_2 \cosh x + ... \\ &+\phi_{n-1}(u_{n-1})\cos u_2 ... \cos u_{n-2}\cosh x + \phi_n(u_n)\cos u_2 ... \cos u_{n-1}\cosh x. \end{aligned}$$

Now, by applying (3.7.21) and (3.7.15), we may obtain in the same way as given in the proof of Theorem 3.1 that

$$(3.7.22) \qquad \begin{aligned} &\mathbf{x} = c_1 \sinh x + c_2 \cosh x + c_3 \sin u_2 \cosh x + ... \\ &+c_{n+1}\cos u_2 ... \cos u_{n-1}\sin u_n \cosh x + c_{n+2}\cos u_2 ... \cos u_n \cosh x, \end{aligned}$$

for some constant vectors $c_1, ..., c_{n+2}$.

If we choose suitable initial conditions for $\mathbf{x}, \partial\mathbf{x}/\partial x, \partial\mathbf{x}/\partial u_2, ..., \partial\mathbf{x}/\partial u_n$ at 0=(0, ..., 0), we will obtain (3.1.8) from (3.7.22). Consequently, up to rigid motions, the immersion $\mathbf{x}$ is given by (3.1.8).

Case (1-ii): $\bar{c} = 0, f(x) = \cosh x, \mu = \operatorname{sech} x$. Again, the open subset U is the whole manifold M^n. Thus, M^n ia an open portion of the warped product manifold $G^n = \mathbb{R} \times_{\cosh x} \mathbb{E}^{n-1}$. Hence, the metric tensor of M^n is given by

$$(3.7.23) \qquad g = dx^2 + \cosh^2 x(du_2^2 + ... + du_n^2).$$

In this case, the equation of Gauss implies that the isometric immersion $\mathbf{x}$ satisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x}, \tag{3.7.24}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = \tanh x \frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 2, ..., n, \tag{3.7.25}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = 0, \qquad 2 \leq i < j, \tag{3.7.26}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_j^2} = \sinh x \cosh x \frac{\partial \mathbf{x}}{\partial x} + \cosh x \xi + \cosh^2 x \mathbf{x}, \quad j = 2, ..., n. \tag{3.7.27}$$

After solving this system, we obtain

$$\mathbf{x}(x, u_2, ..., u_n) = c_1 \sinh x + (\alpha_2 u_2^2 + ... + \alpha_n^2 u_n^2 + \beta_2 u_2 + ... + \beta_n u_n + \gamma) \cosh x, \tag{3.7.28}$$

for some constant vectors $c_1, \alpha_2, ..., \alpha_n, \beta_2, ..., \beta_n, \gamma$.

If we choose suitable initial conditions for $\mathbf{x}, \partial\mathbf{x}/\partial x, \partial\mathbf{x}/\partial u_2, ..., \partial\mathbf{x}/\partial u_n$, we may obtain (3.1.9) from (3.7.28). Thus, up to rigid motions, the immersion is given by (3.1.9).

Case (1-iii): $\bar{c} = -1, f(x) = \sqrt{a^2+1}\cosh x, \mu = a\text{sech}x/\sqrt{a^2+1}, a > 0$. Again, the open subset U is the whole manofold M^n and M^n is an open portion of the warped product manifold $H_a^n = \mathbb{R} \times_{\sqrt{a^2+1}\cosh x} H^{n-1}(-1)$. Thus, the metric tensor of M^n is given by

$$g = dx^2 + (a^2+1)(\cosh^2 x) g_0, \tag{3.7.29}$$

where

$$g_0 = du_2^2 + \sinh^2 u_2(du_3^2 + \cos^2 u_3 du_4^2 + ... + \cos^2 u_3 ... \cos^2 u_{n-1} du_n^2). \tag{3.7.30}$$

If we apply (3.7.29), (3.7.30), and our assumption on the second fundamental form and the equation of Gauss, we know that the isometric immersion **x** satisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x}, \tag{3.7.31}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = \tanh x \frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 2, ..., n, \tag{3.7.32}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2 \partial u_j} = \coth u_2 \frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 3, ..., n, \tag{3.7.33}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = -\tan u_j \frac{\partial \mathbf{x}}{\partial u_i}, \qquad 3 \leq i < j \leq n, \tag{3.7.34}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = -\frac{a^2+1}{2} \sinh 2x \frac{\partial \mathbf{x}}{\partial x} + a\sqrt{a^2+1} \cosh x \xi + (a^2+1) \cosh^2 x \mathbf{x}, \tag{3.7.35}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_3^2} = \sinh^2 u_2 \frac{\partial^2 \mathbf{x}}{\partial u_2^2} - \sinh x \cosh x \frac{\partial \mathbf{x}}{\partial u_2}, \tag{3.7.36}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = \cos^2 u_j \frac{\partial^2 \mathbf{x}}{\partial u_j^2} - \sin u_j \cos u_j \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, ..., n-1. \tag{3.7.37}$$

After solving this system in the same way as in Case (1-i) we obtain

$$\begin{aligned}\mathbf{x} = {} & c_1 \sinh x + c_2 \cosh x + c_3 \cosh u_2 \cosh x + c_4 \sinh u_2 \cos u_3 \sinh x \\ & +\ldots + c_{n+1} \sinh u_2 \cos u_3 \ldots \cos u_{n-1} \sin u_n \cosh x \\ & + c_{n+2} \sinh u_2 \cos u_3 \ldots \cos u_n \cosh x,\end{aligned} \tag{3.7.38}$$

for some constant vectors $c_1, \ldots, c_{n+2}$.

If we choose suitable initial condition for $\mathbf{x}, \partial\mathbf{x}/\partial x, \partial\mathbf{x}/\partial u_2, \ldots, \partial\mathbf{x}/\partial u_n$, we will obtain (3.1.10) from (3.7.38).

Case (2): $f(x) = \alpha \sinh x$. In this case, (3.7.6) yields $\bar{c} = \alpha^2 + a^2$. Thus, $\bar{c} > 0$. By applying a scaling on E if necessary, we have $\bar{c} = 1$ and $\alpha = \sqrt{1 - a^2}$. In summary, we have

$$\bar{c} = 1, \quad f(x) = \sqrt{1 - a^2} \sinh x, \quad \mu = \frac{a}{\sqrt{1 - a^2}} \operatorname{csch} x, \quad 0 < a < 1. \tag{3.7.39}$$

From (3.7.39) and continuity, we know that U is a dense open subset of M^n. Moreover, locally, U is an open subset of the warped product manifold Y_a^n. Thus the metric tensor of M^n is given by

$$g = dx^2 + (1 - a^2) \sinh^2 x (du_2^2 + \cos^2 u_2 du_3^2 + \ldots + (\prod_{j=2}^{n-1} \cos^2 u_j) du_n^2). \tag{3.7.40}$$

Therefore, by applying the equation of Gauss, we know that the isometric immersion $\mathbf{x}$ satisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x}, \tag{3.7.41}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = \coth x \frac{\partial \mathbf{x}}{\partial u_j}, \qquad j = 2, \ldots, n, \tag{3.7.42}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2 \partial u_j} = -\tanh u_2 \frac{\partial \mathbf{x}}{\partial u_j}, \qquad 2 \le i < j \tag{3.7.43}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = \frac{a^2-1}{2} \sinh 2x \frac{\partial \mathbf{x}}{\partial x} + a\sqrt{1-a^2} \sinh x \xi + (1-a^2)\sinh^2 x \mathbf{x}, \tag{3.7.44}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = \cos^2 u_j \frac{\partial^2 \mathbf{x}}{\partial u_j^2} + \sin u_j \cos u_j \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, ..., n-1. \tag{3.7.45}$$

After solving this system in the same way as in Case (1-i) we obtain

$$\begin{aligned} \mathbf{x} &= c_1 \cos x + c_2 \sinh x + c_3 \sin u_2 \sinh x + ... \\ &+ c_{n+1} \cos u_2 ... \cos u_{n-1} \sin u_n \sinh x + c_{n+2} \cos u_2 ... \cos u_n \sinh x, \end{aligned} \tag{3.7.46}$$

for some constant vectores $c_1, ..., c_{n+2}$.

If we choose suitable initial conditions for $\mathbf{x}, \partial\mathbf{x}/\partial x, \partial\mathbf{x}/\partial u_2, ..., \partial\mathbf{x}/\partial u_n$ at 0=(0, ..., 0), we will obtain (3.1.11) from (3.7.46). Therefore, up to rigid motions, the immersion is given by (3.1.11).

Case (b): $n = 3, \lambda_1 = \lambda_2 = \mu/2, \lambda_3 = \mu \neq 0$. Let $\mathcal{F}$ and $\mathcal{F}^\perp$ denote the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. Then as Case (ii) in the proof of Theorem 3.1, we can prove that the integral curves of $\mathcal{F}^\perp$ are geodesics and the distribution $\mathcal{F}$ is a spherical integrable distribution. Thus, there exist a local coordinate system $\{x, u, v\}$ such that $\mathcal{F}$ is spanned by $\{\partial/\partial u, \partial/\partial v\}$ and $e_3 = \partial/\partial x$. As in the proof of Theorem 3.1, we may also prove that $\mu = \mu(x)$ depends only on x.

Again, according to a result of Hiepko, M^3 is locally a warped product manifold $I \times_{f(x)} N^2(\bar{c})$, where $f(x) > 0$ is the warp function and $N^2(\bar{c})$ is a surface of constant curvature $\bar{c}$. So, the metric of M^3 can be written as

$$g = dx^2 + f^2(x) E^2 (du^2 + dv^2), \tag{3.7.47}$$

where $\{u, v\}$ is an isothermal coordinate system on $N^2(\bar{c})$. Applying (3.7.47) we obtain

(3.7.48)
$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial u} = \frac{f'}{f}\frac{\partial}{\partial u}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial v} = \frac{f'}{f}\frac{\partial}{\partial v},$$
$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v} = \frac{E_v}{E}\frac{\partial}{\partial u} + \frac{E_u}{E}\frac{\partial}{\partial v},$$
$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} = -ff'E^2\frac{\partial}{\partial x} + \frac{E_u}{E}\frac{\partial}{\partial u} - \frac{E_v}{E}\frac{\partial}{\partial v},$$
$$\nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = -ff'E^2\frac{\partial}{\partial x} - \frac{E_u}{E}\frac{\partial}{\partial u} + \frac{E_v}{E}\frac{\partial}{\partial v}.$$

From (3.7.47) and the hypothesis on the principle curvatures, we know that the second fundamental form h of M^3 in $H^4(-1)$ satisfies

(3.7.49)
$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \mu\xi, \quad h(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \frac{1}{2}\mu f^2E^2\xi, \quad h(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = \frac{1}{2}\mu f^2E^2\xi$$
$$h(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}) = h(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}) = h(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = 0,$$

where ξ is a unit normal vector field of M^3 in $H^4(-1)$.

From (3.7.48), (3.7.49) and equations of Gauss and Codazzi, we find

(3.7.50)
$$f(x) = \alpha\mu(x),$$

(3.7.51)
$$\mu''(x) + \frac{\mu^3(x)}{2} - \mu(x) = 0,$$

(3.7.52)
$$K_{13} = -1 + \frac{\mu^2}{2} = -\frac{f''}{f}, \qquad K_{12} = -1 + \frac{\mu^2}{4} = \frac{\bar{c} - f'^2}{f^2},$$

where α is a nonzero real number and K_{12}, K_{13} are sectional curvatures of M^3 of plane sections spanned by $\{e_1, e_2\}, \{e_1, e_3\}$, respectively.

Put $\psi(x) = 2\mu(x)$. Then (3.7.51) becomes $\psi''(x) + 2\psi^3(x) - \psi(x) = 0$. Hence, by applying Lemma 5.4 of [10], we know that $\mu = \mu(x)$ is one of the following functions:

(1) $\mu = \sqrt{2}$,

(2) $\mu = 2\text{sech}(x)$,

(3) $\mu(x) = \sqrt{2(a^2+1)}\text{cn}(ax, \dfrac{\sqrt{a^2+1}}{\sqrt{2}a}), \quad a > 1$,

(4) $\mu(x) = \sqrt{2(a^2+1)}\text{cn}(\dfrac{\sqrt{a^2+1}}{\sqrt{2}}x, \dfrac{\sqrt{2}a}{\sqrt{a^2+1}}), \quad 0 < a < 1$.

We consider these four cases separately.

Case (b-1): $\mu = \sqrt{2}, f = \sqrt{2}\alpha$. In this case, (3.7.52) yields $\bar{c} = -\alpha^2 < 0$. By choosing $\bar{c} = -1$, we obtain $\alpha = 1, f = \sqrt{2}$. Therefore, $U = M^3$ and M^3 is an open portion of the warped product manifold $\mathbb{R} \times_{\sqrt{2}} H^2(-1)$ which is isometric to F^3. Therefore, if we choose the hyperbolic coordinate system $\{\phi, \theta\}$ on $H^2(-1)$, we obtain

$$g = dx^2 + 2(d\phi^2 + \cosh^2\phi d\theta^2). \tag{3.7.53}$$

By (3.7.53), $\mu = \sqrt{2}$ and formula of Gauss, we know that the immersion satisfies the following system:

$$\begin{aligned}
&\frac{\partial^2 \mathbf{x}}{\partial x^2} = \sqrt{2}\xi + \mathbf{x}, \qquad \frac{\partial^2 \mathbf{x}}{\partial \phi^2} = \sqrt{2}\xi + 2\xi,\\
&\frac{\partial^2 \mathbf{x}}{\partial \theta^2} = -\sinh\phi\cosh\phi\frac{\partial \mathbf{x}}{\partial \phi} + \sqrt{2}\cosh^2\phi\xi + 2\cosh^2\phi\mathbf{x},\\
&\frac{\partial^2 \mathbf{x}}{\partial x\partial\theta} = \frac{\partial^2 \mathbf{x}}{\partial x\partial\phi} = 0, \qquad \frac{\partial^2 \mathbf{x}}{\partial\phi\partial\theta} = \tanh\phi\frac{\partial \mathbf{x}}{\partial\theta}.
\end{aligned} \tag{3.7.54}$$

After solving (3.7.54), we obtain

$$\mathbf{x}(u, \phi, \theta) = c_1\cos x + c_2\sin x + c_3\sinh\phi + c_4\cosh\theta + c_5\cosh\phi\sinh\theta.$$

for some constant vectors $c_1, c_2, ..., c_5$. Thus, by choosing suitable initial conditions, we obtain (3.1.12).

Case (b-2): $\mu = 2\text{sech}x, f = 2\alpha\text{sech}x$. In this case, (3.7.52) yields $\bar{c} = 0$. By choosing $\alpha = 1/2$, we obtain $f = \text{sech}x$. Therefore, M^3 is an open portion of the warped product manifold $\mathbb{R} \times_{\text{sech}x} \mathbb{E}^2$ which is isometric to F^3. Thus

$$g = dx^2 + \text{sech}^2 x(du^2 + dv^2). \tag{3.7.55}$$

By (3.7.55), $\mu = 2\text{sech}x$ and formula of Gauss, we know that the immersion satisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = 2\text{sech}x\xi + \mathbf{x}, \tag{3.7.56}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u^2} = \text{sech}^2 x \tanh x \frac{\partial \mathbf{x}}{\partial x} + \text{sech}^3 x\xi + \text{sech}^2 x\mathbf{x}, \tag{3.7.57}$$

$$\frac{\partial^2 \mathbf{x}}{\partial v^2} = \text{sech}^2 x \tanh x \frac{\partial \mathbf{x}}{\partial x} + \text{sech}^3 x\xi + \text{sech}^2 x\mathbf{x}, \tag{3.7.58}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial u} = -\tanh x \frac{\partial \mathbf{x}}{\partial u}, \qquad \frac{\partial^2 \mathbf{x}}{\partial x \partial v} = -\tanh x \frac{\partial \mathbf{x}}{\partial v}, \tag{3.7.59}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u \partial v} = 0. \tag{3.7.60}$$

Solving (3.7.59) yields

$$\mathbf{x}(x, u, v) = A(x) + B(u, v)\text{sech}x, \tag{3.7.61}$$

for some $\mathbb{E}_1^5$-valued functions $A(x), B(u,v)$. Substituting (3.7.61) into (3.7.60), we have $B(u,v) = C(u) + D(v)$ for some $\mathbb{E}_1^5$-valued functions $C(u), D(v)$. Thus

$$\mathbf{x}(x,u,v) = A(x) + (C(u) + D(v))\text{sech}x. \tag{3.7.62}$$

Substituting (3.7.62) into (3.7.57) and (3.7.58) yields

$$2C''(u) = \text{sech}x(A''(x) + 2\tanh x A'(x) + A(x)), \tag{3.7.63}$$

$$2D''(v) = \text{sech}x(A''(x) + 2\tanh x A'(x) + A(x)), \tag{3.7.64}$$

which imply that there exists a constant vector c_1 such that

$$\begin{aligned} &C''(u) = D''(v) = 2c_2, \\ &A''(x) + 2\tanh x A'(x) + A(x) = 4c_1 \cosh x. \end{aligned} \tag{3.7.65}$$

Solving (3.7.65) yields

$$\begin{aligned} &C(u) = c_1 u^2 + c_2 u + b_1, \qquad D(v) = c_1 v^2 + c_3 v + b_2, \\ &A(x) = \text{sech}x(c_1(x^2 - \frac{\cosh 2x}{2}) + c_4 x + b_3) \end{aligned} \tag{3.7.66}$$

for some constant vectors $c_2, c_3, c_4, b_1, b_2, b_3$. Combining (3.7.62) and (3.7.66) yields

$$\mathbf{x}(u,\phi,\theta) = \text{sech}x[c_1(x^2 + u^2 + v^2 - \cosh^2 x) + c_2 u + c_3 v + c_4 x + c_5] \tag{3.7.67}$$

for some constant vector c_5. Thus, by chosing suitable initial conditions, we will obtain (3.1.12) from (3.7.67). Consequently, up to rigid motions, the immersion $\mathbf{x}$ is given by (3.1.12).

Case (b-3): $\mu = \sqrt{2(a^2+1)}\mathrm{cn}(ax, \sqrt{a^2+1}/(\sqrt{2}a)), a > 1$. In this case, (3.7.52) yields $\bar{c} = \alpha^2(\alpha^4 - 1) > 0$. By choosing $\bar{c} = 1$, we have

$$(3.7.68)\quad \mu(x) = 2ak\mathrm{cn}(ax, k), \quad f = \frac{1}{ak'}\mathrm{cn}(ax, k), \quad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a}, \quad k' = \frac{\sqrt{a^2-1}}{\sqrt{2}a}.$$

Thus, locally, U is an open portion of the warped product manifold C_a^3. If we choose the spherical coordinate system $\{\phi, \theta\}$ on $S^2(1)$, we have

$$(3.7.69)\qquad g = dx^2 + \frac{\mathrm{cn}^2(ax, k)}{a^2k'^2}(d\phi^2 + \cos^2\phi d\theta^2).$$

By (3.7.68), (3.7.69) and formula of Gauss, we know that the immersion satisfies the following system:

$$(3.7.70)\qquad \frac{\partial^2 \mathbf{x}}{\partial x^2} = 2ak\mathrm{cn}(ax)\xi + \mathbf{x},$$

$$(3.7.71)\qquad \frac{\partial^2 \mathbf{x}}{\partial \phi^2} = \frac{1}{ak'^2}\mathrm{cn}(ax)\mathrm{dn}(ax)\mathrm{sn}(ax)\frac{\partial \mathbf{x}}{\partial x} + \frac{k\mathrm{cn}^3(ax)}{ak'^2}\xi + \frac{\mathrm{cn}^2(ax)}{a^2k'^2}\mathbf{x},$$

$$(3.7.72)\qquad \frac{\partial^2 \mathbf{x}}{\partial \theta^2} = \cos^2\phi\frac{\partial^2 \mathbf{x}}{\partial \phi^2} + \sin\phi\cos\phi\frac{\partial \mathbf{x}}{\partial \phi},$$

$$(3.7.73)\qquad \frac{\partial^2 \mathbf{x}}{\partial x \partial \phi} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)\frac{\partial \mathbf{x}}{\partial \phi},$$

$$(3.7.74)\qquad \frac{\partial^2 \mathbf{x}}{\partial x \partial \theta} = -a\mathrm{sc}(ax)\mathrm{dn}(ax)\frac{\partial \mathbf{x}}{\partial \theta},$$

$$\frac{\partial^2 \mathbf{x}}{\partial\phi\partial\theta} = -\tan\phi\frac{\partial \mathbf{x}}{\partial\theta}. \tag{3.7.75}$$

By taking the partial derivative of (3.7.71) with respect to ϕ, by applying (3.2.8), (3.7.73) and Weingarten equation, we obtain $\partial^3\mathbf{x}/\partial\phi^3 + \partial\mathbf{x}/\partial\phi = 0$, from which we get

$$\mathbf{x}(x,\phi,\theta) = A(x,\theta)\sin\phi + B(x,\theta)\cos\phi + C(x,\theta), \tag{3.7.76}$$

for some $\mathbb{E}^5_1$-valued functions A, B, C. Substituting (3.7.76) into (3.7.73) and (3.7.74) yields

$$A(x,\theta) = \mathrm{cn}(ax)D(\theta), \qquad B(x,\theta) = \mathrm{cn}(ax)E(\theta), \qquad C(x,\theta) = \mathrm{cn}(ax)F(\theta) + G(x)$$

for some $\mathbb{E}^5_1$-valued functions D, E, F and G. Thus, we obtain

$$\mathbf{x} = D(\theta)\mathrm{cn}(ax)\sin\phi + E(\theta)\mathrm{cn}(ax)\cos\phi + F(\theta)\mathrm{cn}(ax) + G(x). \tag{3.7.77}$$

Substituting (3.7.77) into (3.7.75) yields $D'(\theta) = F'(\theta) = 0$. Thus, (3.7.77) implies

$$\mathbf{x} = c_1\mathrm{cn}(ax)\sin\phi + E(\theta)\mathrm{cn}(ax)\cos\phi + K(x) \tag{3.7.78}$$

for some constant vector c_1 and $\mathbb{E}^5_1$-valued function $K(x)$.

By substituting (3.7.78) into (3.7.72) and by applying (3.2.8) and (3.7.70), we find $E''(\theta) + E(\theta) = 0$. Thus, $E(\theta) = c_2\cos\theta + c_3\sin\theta$, for some constant vectors c_2, c_3. Substituting this into (3.7.77) yields

$$\mathbf{x} = c_1\mathrm{cn}(ax)\sin\phi + c_2\mathrm{cn}(ax)\cos\phi\cos\theta + c_3\mathrm{cn}(ax)\sin\phi\cos\theta + K(x). \tag{3.7.79}$$

Substituting (3.7.79) into (3.7.70), we have

$$K''(x) + 2a\mathrm{sc}(ax)\mathrm{dn}(ax)K'(x) + K(x) = 0. \tag{3.7.80}$$

Therefore, by applying (3.7.79), (3.7.80) and Proposition 5.2, we know that $\mathbf{x}$ takes the form:

$$\begin{aligned} \mathbf{x}(x,\phi,\theta) &= c_1\mathrm{cn}(ax)\sin\phi + c_2\mathrm{cn}(ax)\cos\phi\cos\theta + c_3\mathrm{cn}(ax)\sin\phi\cos\theta \\ &+c_4\sqrt{a^2k'^2+\mathrm{cn}^2(ax)}\cosh\left(\frac{k'}{k}x - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - aZ(\gamma)x\right) \\ &+c_5\sqrt{a^2k'^2+\mathrm{cn}^2(ax)}\sinh\left(\frac{k'}{k}x - \frac{1}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} - aZ(\gamma)x\right), \end{aligned} \tag{3.7.81}$$

where $k = \sqrt{a^2+1}/(\sqrt{2}a)$ and $k' = \sqrt{a^2-1}/(\sqrt{2}a)$ are the modulus and the complementary modulus of the Jacobi elliptic functions and $\gamma = \mathrm{sn}^{-1}(1/(ak^2))$. Now, by choosing suitable initial conditions, we obtain (3.1.14) from (3.7.81). Therefore, up to rigid motions, the immersion is given by (3.1.14).

Case (b-4): $\mu(x) = \sqrt{2(1+a^2)}\mathrm{cn}(\sqrt{1+a^2}x/\sqrt{2}, \sqrt{2}a/\sqrt{1+a^2}), 0 < a < 1$. In this case, (3.7.52) yields $\bar{c} = -4a^2\alpha^2k'^2/k^2 < 0$. By choosing $\bar{c} = -1$, we obtain

$$\mu(x) = \frac{2a}{k}\mathrm{dn}(\frac{a}{k}x, k), \quad f = \frac{k}{ak'}\mathrm{dn}(\frac{a}{k}x, k), \quad k = \frac{\sqrt{2}a}{\sqrt{1+a^2}}, \quad k' = \frac{\sqrt{1-a^2}}{\sqrt{1+a^2}}. \tag{3.7.82}$$

Thus, $U = M^3$ which is an open portion of the warped product manifold $D^3_a = \mathbb{R} \times_{(k/(ak'))\mathrm{dn}(ax/k)} H^2(-1)$. If we choose the hyperbolic coordinate system $\{\phi, \theta\}$ on $H^2(-1)$, we get

$$g = dx^2 + \frac{k^2\mathrm{dn}^2(\frac{a}{k}x)}{a^2k'^2}(d\phi^2 + \cosh^2\phi d\theta^2). \tag{3.7.83}$$

By (3.7.88) and (3.7.89) and formula of Gauss, we know that the immersionsatisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \frac{2a}{k}\mathrm{dn}(\frac{a}{k}x)\xi + \mathbf{x}, \tag{3.7.84}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \phi^2} = \frac{k^3}{ak'^2}\mathrm{cn}(\frac{a}{k}x)\mathrm{dn}(\frac{a}{k}x)\mathrm{sn}(\frac{a}{k}x)\frac{\partial \mathbf{x}}{\partial x} + \frac{k\mathrm{dn}^3(\frac{a}{k}x)}{ak'^2}\xi + \frac{k^2\mathrm{dn}^2(\frac{a}{k}x)}{a^2k'^2}\mathbf{x}, \tag{3.7.85}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \theta^2} = \cosh^2\phi\frac{\partial^2 \mathbf{x}}{\partial \phi^2} - \sinh\phi\cosh\phi\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.7.86}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \phi} = -ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.7.87}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial \theta} = -ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)\frac{\partial \mathbf{x}}{\partial \theta}, \tag{3.7.88}$$

$$\frac{\partial^2 \mathbf{x}}{\partial \phi \partial \theta} = \tanh\phi\frac{\partial \mathbf{x}}{\partial \theta}. \tag{3.7.89}$$

Solving (3.7.89) yields

$$\mathbf{x}(x, \phi, \theta) = B(x, \theta)\cosh\phi + C(x, \phi). \tag{3.7.90}$$

for some $\mathbb{E}^5_1$-valued functions $B(x, \theta), C(x, \phi)$. Substituting (3.7.90) into (3.7.87) yields

$$\frac{\partial B}{\partial x} = -ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)B, \tag{3.7.91}$$

$$\frac{\partial^2 C}{\partial x \partial \phi} = -ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)\frac{\partial \mathbf{x}}{\partial \phi}, \tag{3.7.92}$$

Solving the above two equations, we get

$$B(x,\theta) = G(\theta)\mathrm{dn}(\frac{a}{k}x), \qquad C(x,\phi) = K(\phi)\mathrm{dn}(\frac{a}{k}x) + W(x). \tag{3.7.93}$$

for some $\mathbb{E}^5_1$-valued functions $G(\theta), K(\phi), W(x)$. Thus, we obtain

$$\mathbf{x}(x,\phi,\theta) = G(\theta)\mathrm{dn}(\frac{a}{k}x)\cosh\phi + K(\phi)\mathrm{dn}(\frac{a}{k}x) + W(x). \tag{3.7.94}$$

By subsituting (3.7.94) into (3.7.85) and (3.7.86), by applying (3.2.8) and (3.7.84), we have

$$G''(\theta) - G(\theta) = 0, \quad K''(\phi) - K(\phi) = 0, \tag{3.7.95}$$

$$W''(x) + 2ak\mathrm{cn}(\frac{a}{k}x)\mathrm{sd}(\frac{a}{k}x)W'(x) + W(x) = 0. \tag{3.7.96}$$

Therefore, by applying (3.7.94) - (3.7.96) and Proposition 5.2, we know that $\mathbf{x}$ takes the form:

$$\begin{aligned} \mathbf{x} = {} & c_1\mathrm{dn}(\frac{a}{k}x)\cosh\phi\cosh\theta + c_2\mathrm{dn}(\frac{a}{k}x)\cosh\phi\sinh\theta \\ & + c_3\mathrm{dn}(\frac{a}{k}x)\cosh\phi + c_4\mathrm{dn}(\frac{a}{k}x)\sinh\phi \\ & + c_5\sqrt{k^2\mathrm{dn}^2(\frac{a}{k}x) - a^2k'^2}\cos(k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x-\gamma)}{\Theta(\frac{a}{k}x+\gamma)} - \sqrt{-1}\frac{a}{k}\mathrm{Z}(\gamma)x) \\ & + c_6\sqrt{k^2\mathrm{dn}^2(\frac{a}{k}x) - a^2k'^2}\sin(k'x - \frac{\sqrt{-1}}{2}\ln\frac{\Theta(\frac{a}{k}x-\gamma)}{\Theta(\frac{a}{k}x+\gamma)} - \sqrt{-1}\frac{a}{k}\mathrm{Z}(\gamma)x), \end{aligned} \tag{3.7.97}$$

where $k = \sqrt{2}a/\sqrt{1+a^2}$ and $k' = \sqrt{1-a^2}/\sqrt{1+a^2}$ are the modulus and the complementary modulus of the Jacobi's elliptic functions and $\gamma = \mathrm{sn}^{-1}(k/a)$. Now, by choosing suitable initial conditions, we obtain (3.1.15) from (3.7.97). Therefore, up to rigid motions, the immersion is given by (3.1.15).

The converse can be verified by straight-forward long computation. □

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