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The Discretized Korteweg-de Vries Equation

By

Michael J. Nixon

A Dissertation

Submitted to
Michigan State University
in partial fulfillment of the requirements
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ABSTRACT

The Discretized Korteweg-de Vries Equation

By

Michael J. Nixon

In this dissertation, we study a discretized version of the (generalized) Korteweg-de Vries equation, $\partial_t u + \partial_x^3 u + u^4 \partial_x u = 0$. After a number of estimates, we utilize the Contraction Mapping Principle to prove the global well-posedness of this equation in a certain discrete Banach space. Our results are analogous to those of Kenig, Ponce, and Vega in the continuous setting. However, due to the nature of the Fourier multipliers, the proofs of several of these estimates in the discrete setting require new techniques. Our results yield a numerical procedure for computing the solution. We present a numerical algorithm which is based on successive iterations to obtain a fixed point guaranteed by the Contraction Mapping Principle. This fixed point is the desired solution which we demonstrate with several numerical experiments.

To my family

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Introduction

Let $\mathcal{S}'(\mathbf{R})$ be the set of tempered distributions on the real line. Let $\mathcal{B}$ denote a Banach space such that $\mathcal{B} \subset \mathcal{S}'(\mathbf{R})$. The following initial value problems which arise in the study of water waves are called the (generalized) *Korteweg-de Vries* equations,

$$(KdV)_k \begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0 & x \in \mathbf{R}, t \in \mathbf{R}, k \in \mathbf{N} \\ u|_{t=0} = u_0 \in \mathcal{B}. \end{cases}$$

We say that $(KdV)_k$ is *globally well-posed* in $\mathcal{B}$ if there exists a unique $u = u(x,t)$ such that the following holds:

- $u \in C((-\infty, +\infty); \mathcal{B})$, i.e., $u(\cdot, t) \in \mathcal{B}$ for all $t \in \mathbf{R}$ and $t \to u(\cdot, t)$ is continuous from $\mathbf{R}$ to $\mathcal{B}$.
- $u(\cdot, 0) = u_0$.
- For $t \in \mathbf{R}$, $\lim_{h \to 0} \frac{u(\cdot, t+h) - u(\cdot, t)}{h}$ exists as an element of $\mathcal{S}'(\mathbf{R})$. Define $\partial_t u(\cdot, t) = \lim_{h \to 0} \frac{u(\cdot, t+h) - u(\cdot, t)}{h}$.
- The nonlinear term $u^k(\cdot, t)\partial_x u(\cdot, t) \in \mathcal{S}'(\mathbf{R})$ for all $t \in \mathbf{R}$.
- u is a solution to $(KdV)_k$ in the distributional sense.
- The mapping $u_0 \to u$ is continuous from $\mathcal{B}$ to $C((-\infty, +\infty); \mathcal{B})$.

We say that $(KdV)_k$ is *locally well-posed* if the above holds with $C((-\infty, +\infty); \mathcal{B})$ replaced by $C((A_0, A_1); \mathcal{B})$ for some $A_0 < 0$ and $A_1 > 0$.

For $F \in \mathcal{S}'(\mathbf{R})$, let $\hat{F}$ be the *Fourier Transform* of F, defined by

$$\hat{F}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} F(x) e^{-ix\xi} dx.$$

For $s \in \mathbf{R}$, the *Sobolev Space* of order s, denoted $H^s(\mathbf{R})$, is the set of tempered distributions F for which $\hat{F}$ is a function and

$$||F||_{H^s}^2 = \int_{\mathbf{R}} |\hat{F}(\xi)|^2 (1+\xi^2)^s d\xi < \infty.$$

The *homogeneous Sobolev Space* of order s, denoted $\dot{H}^s(\mathbf{R})$, is the set of tempered distributions F for which $\hat{F}$ is a function and

$$||F||_{\dot{H}^s}^2 = \int_{\mathbf{R}} |\hat{F}(\xi)|^2 \xi^{2s} d\xi < \infty.$$

Obviously, $H^s(\mathbf{R}) \subset \dot{H}^s(\mathbf{R})$ for $s \geq 0$. It is well known that $H^s(\mathbf{R})$ and $\dot{H}^s(\mathbf{R})$ are Hilbert spaces with inner product in $H^s(\mathbf{R})$ defined by

$$< F, G >= \int_{\mathbf{R}} \hat{F}(\xi) \overline{\hat{G}}(\xi) (1+\xi^2)^s d\xi$$

and in $\dot{H}^s(\mathbf{R})$ by

$$< F, G >= \int_{\mathbf{R}} \hat{F}(\xi) \overline{\hat{G}}(\xi) \xi^{2s} d\xi.$$

In their 1976 paper [1], using energy methods, J. Bona and R. Scott proved that $(KdV)_k$ is locally well-posed in $H^s(\mathbf{R})$ for $s > 3/2$, for all $k \in \mathbf{N}$. By the Sobolev Imbedding Theorem for $s > 3/2$, $H^s(\mathbf{R}) \subset C^1(\mathbf{R})$, the set of all continuously differentiable functions on $\mathbf{R}$. Hence, this result is restrictive in the sense that it can not guarantee a solution to $(KdV)_k$ if the initial data u_0 has, say, a jump discontinuity. However, the homogeneity of the nonlinear term suggests that well-posedness in $H^s(\mathbf{R})$ may hold for smaller values of s. That is, if u solves $(KdV)_k$, then for $\lambda > 0$, so does

$$u_\lambda(x,t) = \lambda^{2/k} u(\lambda x, \lambda^3 t)$$

with initial data

$$u_\lambda(x,0) = \lambda^{2/k} u(\lambda x, 0).$$

Since we are using a scaling argument, let us assume that these solutions are in the homogeneous Sobolev space, $\dot{H}^s(\mathbf{R})$. Note that

$$||u_\lambda(x,0)||_{\dot{H}^s} = \lambda^{2/k}\lambda^s\lambda^{-1/2}||u(x,0)||_{\dot{H}^s}.$$

This suggests that the optimal s for $(KdV)_k$ is

$$s_k = \frac{1}{2} - \frac{2}{k}.$$

Note that $s_k \geq 0$ if and only if $k \geq 4$.

In their 1993 paper [5], Kenig, Ponce, and Vega proved that $(KdV)_k$ is globally well-posed in $\dot{H}^{s_k}(\mathbf{R})$ for $k \geq 4$ provided that the initial data is small, i.e., $||u_0||_{\dot{H}^{s_k}} \leq \delta$ for some $\delta > 0$. If the initial data is not small, they proved local well-posedness in $\dot{H}^{s_k}(\mathbf{R})$. Note that by Plancherel's Theorem, $\dot{H}^0(\mathbf{R}) = L^2(\mathbf{R})$, the set of square integrable functions on $\mathbf{R}$. Thus, for $k = 4$, the smallest value of k for which Kenig, Ponce, and Vega obtained well-posedness in $\dot{H}^{s_k}(\mathbf{R})$, we have global well-posedness in the familiar function space $L^2(\mathbf{R})$. Because of this, the remainder of this thesis will focus on the Korteweg-de Vries equation with $k = 4$ and with $\mathcal{B} = L^2(\mathbf{R})$, i.e.,

$$(KdV)_4 \begin{cases} \partial_t u + \partial_x^3 u + u^4 \partial_x u = 0 & x \in \mathbf{R}, t \in \mathbf{R} \\ u|_{t=0} = u_0 \in L^2(\mathbf{R}). \end{cases}$$

We now give a brief outline of the proof in [5] for $k = 4$. Consider the associated homogeneous linear equation

$$(H)\begin{cases} \partial_t w + \partial_x^3 w = 0 \\ w|_{t=0} = w_0 \in L^2(\mathbf{R}) \end{cases}$$

and the corresponding inhomogeneous equation

$$(IH)\begin{cases} \partial_t v + \partial_x^3 v = g \\ v|_{t=0} = 0. \end{cases}$$

One can easily verify that (H) is globally well-posed in $L^2(\mathbf{R})$ and the solution is given by

$$w(x,t) = W(t)w_0(x) = \int_{\mathbf{R}} e^{i(x\xi + t\xi^3)}\hat{w}_0(\xi)d\xi. \tag{1}$$

Note that $W(t)$ is a Fourier multiplier operator with multiplier $e^{it\xi^3}$, i.e.,

$$(W(t)\omega_0)\hat{}(\xi) = e^{it\xi^3}\hat{\omega}_0(\xi).$$

Using Duhamel's Principle, one can obtain that (IH) is globally well-posed in $L^2(\mathbf{R})$ for g such that $g(\cdot, t) \in L^2(\mathbf{R})$ for all $t \in \mathbf{R}$ with the solution given by

$$v(x,t) = \int_0^t W(t-t')g(\cdot, t')(x)dt'. \tag{2}$$

To see the connection with $(KdV)_4$, if

$$Su(x,t) = W(t)u_0(x) - \int_0^t W(t-t')(u^4\partial_x u)(\cdot, t')(x)dt', \tag{3}$$

then (1) and (2) imply that u solves $(KdV)_4$ if and only if $u = Su$, provided Su makes sense as a distribution. In other words, u is a solution of $(KdV)_4$ if and only if u is a fixed point of the nonlinear operator S.

More precisely, for $T \in (0, +\infty]$, define

$$\begin{aligned} Z_T &= \Big\{ v \in C((-T,T); L^2(\mathbf{R})) \cap L^\infty((-T,T); L^2(\mathbf{R})) : \\ &\qquad ||v||_{L^5_x L^{10}_T} < \infty \text{ and } ||\partial_x v||_{L^\infty_x L^2_T} < \infty \Big\}, \end{aligned}$$

where

$$||v||_{L^p_x L^q_T} = \left(\int_{\mathbf{R}} \left(\int_{-T}^{T} |v(x,t)|^q dt \right)^{p/q} dx \right)^{1/p}$$

for $1 \le p, q \le \infty$. If $q = \infty$, then

$$||v||_{L^p_x L^\infty_T} = \left(\int_{\mathbf{R}} \sup_{t \in [-T,T]} |v(x,t)|^p dx \right)^{1/p}$$

with a similar definition for $p = \infty$. The norm for these Banach spaces is

$$||v||_{Z_T} = \max\{ \sup_{t\in[-T,T]} ||v(\cdot,t)||_{L^2}; ||v||_{L^5_x L^{10}_T}; ||\partial_x v||_{L^\infty_x L^2_T} \}.$$

By proving various estimates with these mixed norms involving the terms

$$W(t)u_0(x) \tag{4}$$

and

$$\int_0^t W(t-t')(u^4 \partial_x u)(\cdot, t')(x) dt', \tag{5}$$

Kenig, Ponce, and Vega were able to show that given small initial data, S is a contraction mapping on $B_r = \{ v \in Z_\infty : ||v||_{Z_\infty} \le r \}$ for some $r > 0$. For arbitrary initial data, they were able to find a $T > 0$ for which S is a contraction on $B_r \subset Z_T$. Hence, the Contraction Mapping Principle guarantees a solution to $u = Su$ in this ball in Z_T.

Contraction mappings and fixed point iterations are useful tools in the numerical study of differential equations. Can the method developed in [5] be adapted to show that a discrete version of $(KdV)_4$ is well-posed? Is the solution a fixed point of some operator in some space? If so, does this lead to a method that can be implemented numerically? In this dissertation, we answer these questions in the affirmative.

The general outline of the proof for the discrete setting is analogous to that of the continuous setting. However, there are major differences that make the discrete proofs different and in some cases quite a bit more involved than their continuous counterparts.

The first step in this process is to discretize $(KdV)_4$ in a natural way. To his end, fix $h > 0$ and let us first consider the associated linear equation (H). Let $\omega(n,m)$ be a discrete function defined on $\mathbf{Z} \times \mathbf{Z}$. The obvious replacement for $\partial_x^3 u(x,t)$ is $\partial_{n,h}^3 \omega(n,m)$ (see Definition 1.6). However, the replacement of $\partial_t u$ is not so clear. Assuming $m > 0$, on the one hand, we could replace $\partial_t u(x,t)$ with

$$\frac{\omega(n,m+1) - \omega(n,m)}{h^3}$$

and (H) becomes

$$\frac{\omega(n,m+1) - \omega(n,m)}{h^3} + \partial_{n,h}^3 \omega(n,m) = 0. \tag{6}$$

Note, for homogeneity purposes we are letting h be the step size in the x-direction, while h^3 is the step size in the t-direction. On the other hand, we could replace $\partial_t u(x,t)$ with

$$\frac{\omega(n,m) - \omega(n,m-1)}{h^3}$$

and (H) becomes

$$\frac{\omega(n,m)-\omega(n,m-1)}{h^3}+\partial^3_{n,h}\omega(n,m)=0. \tag{7}$$

The scheme in (6) is referred to as an explicit scheme, because, given the values $\omega(n,m)$ at height m, one can solve explicitly for the values of $\omega(n,m+1)$ at height $m+1$. The other is called an implicit scheme since the values $\omega(n,m)$ at height m are determined implicitly by the values $\omega(n,m-1)$ at height $m-1$. It may seem that the explicit scheme is the more convenient of the two. However, considering the Fourier multiplier which corresponds to $e^{it\xi^3}$, we will see that the implicit scheme is the better choice. Here, the Fourier Transform takes functions defined on $\mathbf{Z}$ to functions defined on $[-\pi/h,\pi/h]$ (see Definition 1.2).

Let ω_0 be the initial data. If we discretize explicitly, taking the Fourier Transform of both sides in the first variable, (6) becomes

$$\frac{\hat{\omega}_h^{(1)}(\theta,m+1)-\hat{\omega}_h^{(1)}(\theta,m)}{h^3}+\frac{i\sin^3(h\theta)}{h^3}\hat{\omega}_h^{(1)}(\theta,m)=0,$$

which implies

$$\hat{\omega}_h^{(1)}(\theta,m+1)=(1-i\sin^3(h\theta))\hat{\omega}_h^{(1)}(\theta,m)$$

and hence,

$$\hat{\omega}_h^{(1)}(\theta,m)=(1-i\sin^3(h\theta))^m\hat{\omega}_0^{(1)}(\theta).$$

Notice the multiplier $(1-i\sin^3(h\theta))^m$ has magnitude $(1+\sin^6(h\theta))^{m/2}$, which blows up for $0<|\theta|<\pi/h$ as $m\to\infty$. Hence, by Plancherel's Theorem (Proposition 1.3), the norms of $\omega(\cdot,m)$ in the discrete space $l_h^2(\mathbf{Z})$ (see Definition 1.1) go to infinity with m. This is undesirable both mathematically and physically.

Applying the same argument to (7), it follows that

$$\hat{\omega}_h^{(1)}(\theta, m) = \frac{\hat{\omega}_0^{(1)}(\theta)}{(1 + i\sin^3(h\theta))^m}$$

(for the proof of this see §1.2). Note that the multiplier $(1 + i\sin^3(h\theta))^{-m}$ has magnitude which is bounded by one for all $\theta \in [-\pi/h, \pi/h]$ and hence, the norm of $\omega(\cdot, m)$ in $l_h^2(\mathbf{Z})$ will be bounded by $||\omega_0||_{l_h^2}$ for every m. Therefore, we choose the implicit scheme in (7) and the resulting multiplier

$$\frac{1}{1 + i\sin^3(h\theta)}. \tag{8}$$

The fact that the multiplier is not a pure exponential, like $e^{it\xi^3}$, causes difficulties in the proofs of the discrete case that do not arise in the continuous case. For example, consider the following estimate proved in [5], which is used in establishing that S is a contraction mapping.

Lemma 0.1 *Let* $w_0 \in L^2(\mathbf{R})$. *Then*

$$||\partial_x W(t) w_0||_{L_x^\infty L_t^2} \leq C||w_0||_{L^2},$$

where

$$||v||_{L_x^\infty L_t^2} = \sup_{x \in \mathbf{R}} \left(\int_{\mathbf{R}} |v(x,t)|^2 dt \right)^{1/2}.$$

Proof: Recall, $W(t)w_0(x) = \int_{\mathbf{R}} e^{i(x\xi + t\xi^3)} \hat{w}_0(\xi) d\xi$. It follows, after the change of variables $\xi^3 = \tau$, that

$$\begin{aligned}
\partial_x W(t) w_0(x) &= i \int_{\mathbf{R}} \xi e^{i(x\xi + t\xi^3)} \hat{w}_0(\xi) d\xi \\
&= \frac{i}{3} \int_{\mathbf{R}} e^{it\tau} e^{ix\tau^{1/3}} \tau^{-2/3+1/3} \hat{w}_0(\tau^{1/3}) d\tau.
\end{aligned}$$

Thus, using Plancherel's Theorem in the t-variable,

$$\begin{aligned} ||\partial_x W(t) w_0||^2_{L^2_t} &= \frac{1}{9}\int_{\mathbf{R}} \left| e^{ix\tau^{1/3}} \tau^{-1/3} \hat{w}_0(\tau^{1/3})\right|^2 d\tau \\ &= c\int_{\mathbf{R}} |\hat{w}_0(\xi)|^2 d\xi, \end{aligned}$$

again using $\xi = \tau^{1/3}$, which finishes the proof.

$\square$

This proof is simplified by the fact that the multiplier $e^{it\xi^3}$ has magnitude one for all $\xi \in \mathbf{R}$, and the fact that after the substitution $\tau = \xi^3$, the term $e^{it\tau}$ is just what is needed to regard the integral as a Fourier transform in the t-variable. The multiplier $(1 + i\sin^3(h\theta))^{-m}$ is not exactly of this form, so when we proceed as in the proof of Lemma 0.1 we encounter an "error" term after the change of variables. The proof will be reduced to the following inequality (see Proposition 2.9):

$$\sum_{m\in\mathbf{Z}} \left| \int_{-\pi}^{\pi} e^{-im\tau} \cos^{|m|}\tau \, g(\tau) d\tau \right|^2 \le c \int_{-\pi}^{\pi} |g(\tau)|^2 d\tau, \tag{9}$$

where the $\cos^{|m|}\tau$ is the anticipated extra term. If the proof was completely analogous to the proof of Lemma 0.1, we would not have this term and (9) would become a statement of Plancherel's Theorem involving Fourier series. Instead, the proof requires some additional techniques including a combinatorial identity (see the proof of Claim 2.11 following Proposition 2.9) pointed out to us by Bruce Sagan.

Next, consider $m < 0$. The implicit scheme in this case is

$$\frac{\omega(n, m+1) - \omega(n, m)}{h^3} + \partial^3_{n,h}\omega(n, m) = 0. \tag{10}$$

After applying the argument in §1.2, (10) becomes

$$\hat{\omega}_h^{(1)}(\theta, m) = \frac{\hat{\omega}_0^{(1)}(\theta)}{(1 - i\sin^3(h\theta))^{|m|}}$$

and the resulting multiplier is

$$\frac{1}{1 - i\sin^3(h\theta)}. \tag{11}$$

Notice that because of our need to control the size of the multiplier, we now have two multipliers, (8) for $m > 0$ and (11) for $m < 0$.

Recall that the Fourier multiplier in the continuous case associated with $W(t)$ is $e^{it\xi^3}$. Since

$$e^{i(t+s)\xi^3} = e^{it\xi^3}e^{is\xi^3},$$

the family of operators $\{W(t)\}_{t\in\mathbf{R}}$ forms a group under composition, i.e.,

$$W(t+s)w_0 = W(t)W(s)w_0 \quad \text{for all } s, t \in \mathbf{R}.$$

This plays an important role in the proof of the key estimate

$$\left\|\partial_x^2 \int_0^t W(t-t')g(\cdot,t')(x)dt'\right\|_{L_x^\infty L_t^2} \le c\|g\|_{L_x^1 L_t^2} \tag{12}$$

needed in [5]. If

$$(H_h(m)\eta_0)\hat{}(\theta) = \frac{(\hat{\eta}_0)_h(\theta)}{(1 + \operatorname{sgn}(m) i \sin^3(h\theta))^{|m|}},$$

then it follows for $m_1, m_2 \in \mathbf{Z}$ that

$$H_h(m_1 + m_2)\eta_0 \ne H_h(m_1)H_h(m_2)\eta_0$$

if $m_1 \cdot m_2 < 0$. This forces the corresponding proof in the discrete setting (see Lemma 1.17(ii)) to proceed along different lines than the proof of (12) in [5]. This necessitates the technically difficult proof of Proposition 2.19 in addition to the relatively simple proof of Proposition 2.17 (see Remark 2.20).

Despite the additional difficulties resulting from the multiplier in the discrete formulation, we are still able to obtain the full analog of the results in [5]. We now turn to the formal statement of the problem and our results. Since the nonlinear term of $(KdV)_4$ can be written as $\frac{1}{5}\partial_x(u^5)$, we discretize $(KdV)_4$ as follows:

$$(KdV)_4^d \begin{cases} \dfrac{\eta(n,m)-\eta(n,m-1)}{h^3} + \partial_{n,h}^3\eta(n,m) + \dfrac{1}{5}\partial_{n,h}(\eta^5)(n,m) = 0 & m > 0 \\ \eta(n,0) = \eta_0(n) & m = 0 \\ \dfrac{\eta(n,m+1)-\eta(n,m)}{h^3} + \partial_{n,h}^3\eta(n,m) + \dfrac{1}{5}\partial_{n,h}(\eta^5)(n,m) = 0 & m < 0, \end{cases}$$

where η_0 is the discrete initial data. If

$$H_h(m)\eta_0(n)$$

is the analog of (4) and solves the discrete version of (H) and

$$\frac{1}{5}\Lambda_h\partial_{n,h}(\eta^5)(n,m)$$

is the analog of (5) (see Definitions 1.12 and 1.14), then the analog of the operator S in (3) is

$$\Phi_{\eta_0}\eta(n,m) = H_h(m)\eta_0(n) - \frac{1}{5}\Lambda_h\partial_{n,h}(\eta^5)(n,m).$$

It will follow (see §1.4) that η solves $(KdV)_4^d$ if and only if

$$\eta = \Phi_{\eta_0}\eta.$$

Analogous to the space Z_∞ from [5], we will define a discrete space X_h (see Definition 1.16). The primary conclusion of this thesis is the following:

Main Result: *Suppose the initial data η_0 is small, i.e.,*

$$||\eta_0||_{l_h^2} = \left(h \sum_{n \in \mathbf{Z}} |\eta_0(n)|^2 \right)^{1/2} \leq \delta_0$$

for some $\delta_0 > 0$. Then there exists $r > 0$ such that the operator Φ_{η_0} is a contraction mapping in

$$B_r = \{\eta \in X_h : ||\eta||_{X_h} \leq r\},$$

with δ_0 and r independent of $h > 0$. Consequently, by the Contraction Mapping Principle, there exists a unique solution $\eta \in B_r$ of $(KdV)_4^d$. Moreover, $\eta \in l^\infty(l_h^2(\mathbf{Z}); \mathbf{Z})$ and the map

$$\eta_0 \to \eta$$

is continuous from $l_h^2(\mathbf{Z})$ to *X_h. Hence, $(KdV)_4^d$ is well-posed in X_h.*

This result lends itself to a numerical procedure for computing the solution to $(KdV)_4^d$. To do this, we first numerically implement the operator Φ_{η_0}. This will be done by using the coefficients $Q[n, m]$ which have been precomputed and stored (see §3.2). Then, by picking the initial guess to be the zero function on $\mathbf{Z} \times \mathbf{Z}$, we run the fixed point iteration. If the initial data is small enough, then the main result guarantees that the iterates $\Phi_{\eta_0}^{(n)}(0)$ converge to the solution of $(KdV)_4^d$ as $n \to \infty$. Also, because the contraction mapping constant is less than one, the convergence is exponential. The fact that the operator Φ_{η_0} and the iteration scheme are easy to implement is the motivation for this thesis.

This approach differs from the standard approach in several ways. The standard approach is to solve the difference equations iteratively going up one level at a time, while we work on the entire space $\mathbf{Z} \times \mathbf{Z}$ for each iteration. A disadvantage of our approach is the amount of computer storage required. Each global iteration has to be stored in order to define the next iteration, whereas when solving the difference equations level by level, one can discard the data at each level. However, the advantages of our method include the rapid computation of the entire solution on the entire grid, and the fact that we avoid problems with ill-conditioning and accumulation of round-off error. In addition, our approach guarantees certain size estimates on the iterations, namely the three norms of the space X_h (see Definition 1.16) including a bound on the l^2 norms of each level. Finally, note that our result holds for all $h > 0$ not just, for example, small h. This distinguishes our approach from another possible approach in which one tries to compare Φ_{η_0}, for small h, with S, which is known to be contraction map in the continuous setting. Thus what we have obtained is a true difference equation result.

In Chapter 1, we state several definitions which are used throughout this thesis. After this, we show that our choice of discretization yields a nonlinear discrete operator whose fixed point solves $(KdV)_4^d$. Then, we state the crucial estimates and define the discrete Banach space X_h in which we find our solution. Finally, with these estimates, we establish that we have a contraction mapping, which quickly leads to a solution of $(KdV)_4^d$. This part of the proof follows the same general outline as in [5]. Chapter 2 contains the proofs of the main estimates. It is here that techniques different from those in [5] are required to overcome the difficulties noted above related to the the multipliers in the discrete setting. Finally, results of various numerical experiments are presented in Chapter 3. An explanation of the numerical algorithm is included there as well.

CHAPTER 1

The Main Result

1.1 Preliminaries

Let $h > 0$. In the following definitions, let σ and ϱ denote functions defined on $\mathbf{Z}$ and ω denote a function defined on $\mathbf{Z} \times \mathbf{Z}$.

Definition 1.1 *$\sigma \in l_h^2(\mathbf{Z})$ if*

$$||\sigma||_{l_h^2} = (h \sum_{n \in \mathbf{Z}} |\sigma(n)|^2)^{1/2} < \infty.$$

If h represents the step size and $\sigma(n) = f(nh)$ for, say, $f \in C(\mathbf{R}) \cap L^2(\mathbf{R})$, then $||\sigma||_{l_h^2}^2$ represents a Riemann sum of $|f|^2$. Hence, $||\sigma||_{l_h^2} \sim ||f||_{L^2}$ for small h.

Definition 1.2 *For $\theta \in [-\pi/h, \pi/h]$, let*

$$\hat{\sigma}_h(\theta) = h \sum_{k \in \mathbf{Z}} \sigma(k) e^{ikh\theta}.$$

Note that by Fourier inversion,

$$\sigma(n) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \hat{\sigma}_h(\theta)e^{-inh\theta}d\theta.$$

Proposition 1.3 *If $\sigma \in l_h^2(\mathbf{Z})$, then*

$$||\sigma||_{l_h^2} = \left(\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} |\hat{\sigma}_h(\theta)|^2 d\theta\right)^{1/2}.$$

Proof:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} |\hat{\sigma}_h(\theta)|^2 d\theta &= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \left(h\sum_{k\in\mathbf{Z}} \sigma(k)e^{ikh\theta}\right)\left(\overline{h\sum_{j\in\mathbf{Z}} \sigma(j)e^{ijh\theta}}\right) \\
&= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} h^2 \sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \sigma(k)\overline{\sigma(j)}e^{i(k-j)h\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} h^2 \sum_{k\in\mathbf{Z}} |\sigma(k)|^2 d\theta \\
&= h\sum_{k\in\mathbf{Z}} |\sigma(k)|^2 \\
&= ||\sigma||_{l_h^2}^2.
\end{aligned}$$

□

Definition 1.4 *For σ and $\varrho \in l_h^2(\mathbf{Z})$, let*

$$(\sigma * \varrho)(n) = h\sum_{k\in\mathbf{Z}} \sigma(k)\varrho(n-k)$$

be the convolution of σ and ϱ.

Remark 1.5 One can see as in the standard case with $h = 1$ that

$$(\sigma * \varrho)_h\hat{}(\theta) = h\sum_{n\in\mathbf{Z}}\left(h\sum_{k\in\mathbf{Z}} \sigma(k)\varrho(n-k)\right)e^{inh\theta}$$

$$\begin{aligned} &= h\sum_{k\in\mathbf{Z}}\sigma(k)e^{ikh\theta}h\sum_{n\in\mathbf{Z}}\varrho(n-k)e^{i(n-k)h\theta} \\ &= \hat{\sigma}_h(\theta)\hat{\varrho}_h(\theta) \end{aligned}$$

for $\theta \in [-\pi/h, \pi/h]$.

Definition 1.6 *Let*

$$\partial_h\sigma(n) = \frac{\sigma(n+1)-\sigma(n-1)}{2h}.$$

For $k \in \mathbf{N}$, define $\partial_h^k\sigma(n)$ inductively, i.e.,

$$\partial_h^k\sigma(n) = \partial_h\partial_h^{k-1}\sigma(n).$$

For example, $\partial_h^3\sigma(n) = \dfrac{\sigma(n+3)-3\sigma(n+1)+3\sigma(n-1)-\sigma(n-3)}{8h^3}$.

Proposition 1.7 *Let $\sigma \in l_h^2(\mathbf{Z})$. Then*

$$(\partial_h\sigma)\hat{}_h(\theta) = \frac{-i\sin(h\theta)}{h}\hat{\sigma}_h(\theta).$$

Proof: By definition,

$$\begin{aligned} (\partial_h\sigma)\hat{}_h(\theta) &= h\sum_{k\in\mathbf{Z}}\partial_h\sigma(k)e^{ikh\theta} \\ &= h\sum_{k\in\mathbf{Z}}\frac{\sigma(k+1)-\sigma(k-1)}{2h}e^{ikh\theta} \\ &= \frac{1}{2}\sum_{k\in\mathbf{Z}}\sigma(k+1)e^{ikh\theta} - \frac{1}{2}\sum_{k\in\mathbf{Z}}\sigma(k-1)e^{ikh\theta}. \end{aligned}$$

By setting $j = k+1$ in the first sum and setting $j = k-1$ in the second sum, we have

$$\begin{aligned}(\partial_h \sigma)\hat{}_h(\theta) &= \frac{1}{2}\sum_{j\in\mathbf{Z}} \sigma(j) e^{ijh\theta}(e^{-ih\theta} - e^{ih\theta}) \\ &= -i\frac{\sin(h\theta)}{h}\hat{\sigma}_h(\theta).\end{aligned}$$

□

By induction on k, it follows that

$$(\partial_h^k \sigma)\hat{}_h(\theta) = \left(-i\frac{\sin(h\theta)}{h}\right)^k \hat{\sigma}_h(\theta).$$

Definition 1.8 *For $\sigma \in l_h^2(\mathbf{Z})$, $\beta \in \mathbf{C}$, let*

$$D_h^\beta \sigma(n) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \left|\frac{\sin(h\theta)}{h}\right|^\beta \hat{\sigma}_h(\theta) e^{-inh\theta} d\theta.$$

Remark 1.9 Due to the homogeneity of the associated linear equation, if h is the step size in the x-direction, then h^3 will be the step size in the t-direction. This motivates the following definition.

Definition 1.10 *For $1 \le p, q \le \infty$, $\omega \in l_n^p l_{m,h}^q(\mathbf{Z}\times\mathbf{Z})$ if*

$$||\omega||_{l_n^p l_{m,h}^q} = \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}} |\omega(n,m)|^q\right)^{p/q}\right)^{1/p} < \infty.$$

If $p = \infty$, then

$$||\omega||_{l_n^\infty l_{m,h}^q} = \sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}} |\omega(n,m)|^q\right)^{1/q}$$

and if $q = \infty$, then

$$||\omega||_{l^p_n l^\infty_{m,h}} = \left(h \sum_{n \in \mathbf{Z}} \sup_{m \in \mathbf{Z}} |\omega(n,m)|^p \right)^{1/p}.$$

Definition 1.11 *Let*

$$\partial_{n,h}\omega(n,m) = \frac{\omega(n+1,m) - \omega(n-1,m)}{2h}$$

and define $\partial^k_{n,h}\omega(n,m)$ inductively as above.

1.2 Discretization of $(KdV)_4$

Recall our discretization of $(KdV)_4$,

$$(KdV)_4^d \begin{cases} \dfrac{\eta(n,m) - \eta(n,m-1)}{h^3} + \partial^3_{n,h}\eta(n,m) + \dfrac{1}{5}\partial_{n,h}(\eta^5)(n,m) = 0 & m > 0 \\ \eta(n,0) = \eta_0(n) & m = 0 \\ \dfrac{\eta(n,m+1) - \eta(n,m)}{h^3} + \partial^3_{n,h}\eta(n,m) + \dfrac{1}{5}\partial_{n,h}(\eta^5)(n,m) = 0 & m < 0. \end{cases}$$

We discretize the linear equation implicitly as well,

$$\begin{cases} \dfrac{\omega(n,m) - \omega(n,m-1)}{h^3} + \partial^3_{n,h}\omega(n,m) = 0 & m > 0 \\ \omega(n,0) = \omega_0(n) & m = 0 \\ \dfrac{\omega(n,m+1) - \omega(n,m)}{h^3} + \partial^3_{n,h}\omega(n,m) = 0 & m < 0. \end{cases} \tag{1.1}$$

Assume $m > 0$ and let $\omega(\cdot, m) \in l^2_h(\mathbf{Z})$ for all $m \in \mathbf{Z}$ be a solution to (1.1). By taking $\hat{}$ of both sides of (1.1) in the first variable, we have

$$\frac{\hat{\omega}^{(1)}_h(\theta,m) - \hat{\omega}^{(1)}_h(\theta,m-1)}{h^3} + \frac{i\sin^3(h\theta)}{h^3}\hat{\omega}^{(1)}_h(\theta,m) = 0,$$

with

$$\hat{\omega}_h^{(1)}(\theta, m) = h \sum_{n \in \mathbf{Z}} \omega(n, m) e^{inh\theta}.$$

This implies

$$\hat{\omega}_h^{(1)}(\theta, m) = \frac{\hat{\omega}_h^{(1)}(\theta, m-1)}{1 + i\sin^3(h\theta)},$$

and hence,

$$\hat{\omega}_h^{(1)}(\theta, m) = \frac{(\hat{\omega}_0)_h(\theta)}{(1 + i\sin^3(h\theta))^m}.$$

If $m < 0$, reasoning as above, it follows that

$$\hat{\omega}_h^{(1)}(\theta, m) = \frac{(\hat{\omega}_0)_h(\theta)}{(1 - i\sin^3(h\theta))^{|m|}}.$$

This leads to the following definition.

Definition 1.12 *For $m, n \in \mathbf{Z}$ and $\omega_0 \in l_h^2(\mathbf{Z})$, let*

$$H_h(m)\omega_0(n) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{e^{-inh\theta}(\hat{\omega}_0)_h(\theta)}{(1 + \operatorname{sgn}(m) i \sin^3(h\theta))^{|m|}} d\theta.$$

Remark 1.13 Note,

$$(H_h(m)\omega_0)\hat{}(\theta) = \frac{(\hat{\omega}_0)_h(\theta)}{(1 + \operatorname{sgn}(m) i \sin^3(h\theta))^{|m|}}.$$

Hence, by Plancherel's Theorem, $H_h(m) : l_h^2(\mathbf{Z}) \to l_h^2(\mathbf{Z})$ is bounded with operator norm less than or equal to 1. Also, $H_h(0) = I$ where I is the identity operator on $l_h^2(\mathbf{Z})$. Moreover, by the argument above and Fourier inversion, if

$$\omega(n, m) = H_h(m)\omega_0(n),$$

then ω is the unique solution to (1.1) for $\omega(\cdot, m) \in l_h^2(\mathbf{Z})$ for all $m \in \mathbf{Z}$.

1.3 A Discretized Version of Duhamel's Principle

Proceeding along the same path as [5], we now consider a discretized version of the inhomogeneous linear equation.

Definition 1.14 *For ω such that $\omega(\cdot, m) \in l_h^2(\mathbf{Z})$ for all $m \in \mathbf{Z}$, let*

$$\Lambda_h\omega(n,m) = \begin{cases} h^3 \sum_{j=1}^{m} H_h(m+1-j)\omega(\cdot,j)(n) & m > 0 \\ 0 & m = 0 \\ -h^3 \sum_{j=m}^{-1} H_h(m-1-j)\omega(\cdot,j)(n) & m < 0. \end{cases}$$

Proposition 1.15 *Let η be such that $\eta(\cdot, m) \in l_h^2(\mathbf{Z})$ for all $m \in \mathbf{Z}$. Let $\omega(n,m) = \Lambda_h\eta(n,m)$. Then ω solves*

$$\begin{cases} \dfrac{\omega(n,m) - \omega(n,m-1)}{h^3} + \partial_{n,h}^3\omega(n,m) = \eta(n,m) & m > 0 \\ \omega(n,0) = 0 & m = 0 \\ \dfrac{\omega(n,m+1) - \omega(n,m)}{h^3} + \partial_{n,h}^3\omega(n,m) = \eta(n,m) & m < 0. \end{cases}$$

Moreover, ω is unique for $\omega(\cdot, m) \in l_h^2(\mathbf{Z})$ for all $m \in \mathbf{Z}$.

Proof: The case $m = 0$ is trivial by definition of $\Lambda_h\eta$. Assume $m > 0$. Then

$$\begin{aligned} &\left(\frac{\omega(\cdot,m) - \omega(\cdot,m-1)}{h^3} + \partial_{n,h}^3\omega(\cdot,m)\right)^{\wedge}(\theta) \\ &= \Bigg(\sum_{j=1}^{m} H_h(m+1-j)\eta(\cdot,j) - \sum_{j=1}^{m-1} H_h(m-j)\eta(\cdot,j) \\ &\qquad + h^3 \sum_{j=1}^{m} \partial_{n,h}^3 H_h(m+1-j)\eta(\cdot,j)\Bigg)^{\wedge}(\theta) \\ &= \sum_{j=1}^{m} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}}\hat{\eta}_h^{(1)}(\theta,j) - \sum_{j=1}^{m-1} \frac{1}{(1+i\sin^3(h\theta))^{m-j}}\hat{\eta}_h^{(1)}(\theta,j) \\ &\qquad + i\sin^3(h\theta) \sum_{j=1}^{m} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}}\hat{\eta}_h^{(1)}(\theta,j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{m} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}} \hat{\eta}_h^{(1)}(\theta,j) \\
&\qquad -(1+i\sin^3(h\theta)) \sum_{j=1}^{m-1} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}} \hat{\eta}_h^{(1)}(\theta,j) \\
&\qquad +i\sin^3(h\theta) \sum_{j=1}^{m} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}} \hat{\eta}_h^{(1)}(\theta,j) \\
&= \frac{\hat{\eta}_h^{(1)}(\theta,m)}{1+i\sin^3(h\theta)} - i\sin^3(h\theta) \sum_{j=1}^{m-1} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}} \hat{\eta}_h^{(1)}(\theta,j) \\
&\qquad +i\sin^3(h\theta) \sum_{j=1}^{m} \frac{1}{(1+i\sin^3(h\theta))^{m+1-j}} \hat{\eta}_h^{(1)}(\theta,j) \\
&= \frac{1+i\sin^3(h\theta)}{1+i\sin^3(h\theta)} \hat{\eta}_h^{(1)}(\theta,m) \\
&= \hat{\eta}^{(1)}(\theta,m).
\end{aligned}$$

Thus,

$$\left(\frac{\omega(\cdot,m)-\omega(\cdot,m-1)}{h^3} + \partial_{n,h}^3 \omega(\cdot,m)\right)^{\hat{}}(\theta) = \hat{\eta}_h^{(1)}(\theta,m),$$

with $\hat{\eta}_h^{(1)}(\theta,m) = h \sum_{k\in\mathbf{Z}} \eta(k,m)e^{ikh\theta}$. The proposition follows after integrating both sides against $e^{in\theta}$. The proof for $m<0$ is very similar.

□

1.4 The Operator Φ_{η_0}

Our goal is the following: given $\eta_0 \in l_h^2(\mathbf{Z})$, find η such that $\eta(\cdot,m) \in l_h^2(\mathbf{Z})$ for all $m \in \mathbf{Z}$ and

$$\begin{cases} \eta(n,m) = H_h(m)\eta_0(n) - \frac{1}{5}\Lambda_h \partial_{n,h}(\eta^5)(n,m) \\ \eta(n,0) = \eta_0(n) \end{cases} \tag{1.2}$$

for $m,n \in \mathbf{Z}$. Combining Remark 1.13 and Proposition 1.15, such an η solves $(KdV)_4^d$. To find an η solving (1.2), we will prove that the operator

$$\Phi_{\eta_0}\eta(n,m) = H_h(m)\eta_0(n) - \frac{1}{5}\Lambda_h \partial_{n,h}(\eta^5)(n,m)$$

is a contraction mapping on a ball centered at the origin in a particular Banach space.

1.5 The Space X_h

We now define the discrete Banach space upon which we will show Φ_{η_0} is a contraction.

Definition 1.16 *Let*

$$X_h = \{\omega(n,m) \in l^\infty(l_h^2(\mathbf{Z}); \mathbf{Z}) : ||\omega||_{l_n^5 l_{m,h}^{10}} < \infty, ||\partial_{n,h}\omega||_{l_n^\infty l_{m,h}^2} < \infty\},$$

with

$$||\omega||_{X_h} = \max\{\sup_{m\in\mathbf{Z}} ||\omega(\cdot,m)||_{l_h^2}; ||\omega||_{l_n^5 l_{m,h}^{10}}; ||\partial_{n,h}\omega||_{l_n^\infty l_{m,h}^2}\}.$$

Obviously, X_h is a Banach space with the above norm. Using the estimates below, we can find the solution to (1.2) in this Banach space X_h. This will be done by proving that Φ_{η_0} is a contraction map from a small ball around the origin in the space X_h to itself. The estimates needed are the following:

Lemma 1.17 *For $h > 0$,*

$$i)\ ||\partial_{n,h} H_h(m)\eta_0||_{l_n^\infty l_{m,h}^2} \leq c||\eta_0||_{l_h^2}$$

and

$$ii)\ ||\partial_{n,h}^2 \Lambda_h \omega||_{l_n^\infty l_{m,h}^2} \leq c||\omega||_{l_n^1 l_{m,h}^2},$$

with c independent of $h > 0$.

Lemma 1.18 *For $h > 0$,*

$$i)\ ||H_h(m)\eta_0||_{l^5_n l^{10}_{m,h}} \leq c||\eta_0||_{l^2_h}$$

and

$$ii)\ ||\Lambda_h \omega||_{l^5_n l^{10}_{m,h}} \leq c||\omega||_{l^{5/4}_n l^{10/9}_{m,h}},$$

with c independent of $h > 0$.

Before proceeding, we need the following proposition.

Proposition 1.19 *If $\omega \in X_h$, then $\partial_{n,h}(\omega^5) \in l^{5/4}_n l^{10/9}_{m,h}$, with*

$$||\partial_{n,h}(\omega^5)||_{l^{5/4}_n l^{10/9}_{m,h}} \leq 5||\omega||^5_{X_h}.$$

Proof: By definition,

$$\begin{aligned}
\partial_{n,h}(\omega^5) &= \frac{\omega^5(n+1,m) - \omega^5(n-1,m)}{2h} \\
&= \frac{\omega(n+1,m) - \omega(n-1,m)}{2h} \Big(\omega^4(n+1,m) + \omega^3(n+1,m)\omega(n-1,m) \\
&\qquad + \omega^2(n+1,m)\omega^2(n-1,m) + \omega(n+1,m)\omega^3(n-1,m) + \omega^4(n-1,m)\Big) \\
&= \partial_{n,h}\omega(n,m) \Big(\omega^4(n+1,m) + \omega^3(n+1,m)\omega(n-1,m) \\
&\qquad + \omega^2(n+1,m)\omega^2(n-1,m) + \omega(n+1,m)\omega^3(n-1,m) + \omega^4(n-1,m)\Big).
\end{aligned}$$

Set $\nu(n,m) = \omega(n+1,m)$ and $\mu(n,m) = \omega(n-1,m)$. It follows that

$$\begin{aligned}
||\partial_{n,h}(\omega^5)||_{l^{5/4}_n l^{10/9}_{m,h}} &\leq \sum_{j=0}^{4} ||(\partial_{n,h}\omega)\nu^{4-j}\mu^j||_{l^{5/4}_n l^{10/9}_{m,h}} \\
&= \sum_{j=0}^{4} \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\partial_{n,h}\omega|^{10/9} |\nu^{4-j}\mu^j|^{10/9} \right)^{9/8} \right)^{4/5}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{4}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\omega|^2\right)^{5/8}\left(h^3\sum_{m\in\mathbf{Z}}|\nu^{4-j}\mu^j|^{5/2}\right)^{1/2}\right)^{4/5}\\
&\leq \sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\omega|^2\right)^{1/2}\sum_{j=0}^{4}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu^{4-j}\mu^j|^{5/2}\right)^{1/2}\right)^{4/5}\\
&\leq ||\omega||_{X_h}\sum_{j=0}^{4}A_j,
\end{aligned}
$$

with

$$
A_j = \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu^{4-j}\mu^j|^{5/2}\right)^{1/2}\right)^{4/5}.
$$

Note that the second inequality above follows from an application of Hölder's Inequality on the sum in m with $p = \frac{9}{5}$ and $q = \frac{9}{4}$.

Claim 1.20 *For $\nu, \mu \in X_h$ and $0 \leq j \leq 4$,*

$$
A_j \leq ||\nu||_{X_h}^{4-j}||\mu||_{X_h}^{j}.
$$

Proof:(of Claim 1.20) Fix j. Set $p = \dfrac{4}{4-j}$ and $q = \dfrac{4}{j}$ (p or $q = \infty$ when dividing by zero). Then using Hölder's Inequality on each sum, it follows that

$$
\begin{aligned}
A_j &\leq \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu|^{10}\right)^{\frac{4-j}{8}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu|^{10}\right)^{\frac{j}{8}}\right)^{4/5}\\
&\leq \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu|^{10}\right)^{1/2}\right)^{\frac{4-j}{5}}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu|^{10}\right)^{1/2}\right)^{\frac{j}{5}}\\
&\leq ||\nu||_{X_h}^{4-j}||\mu||_{X_h}^{j}.
\end{aligned}
$$

///

Since $||\nu||_{X_h} = ||\mu||_{X_h} = ||\omega||_{X_h}$, it follows that $A_j \leq ||\omega||^4_{X_h}$. Therefore,

$$||\partial_{n,h}(\omega^5)||_{l_n^{5/4} l_{m,h}^{10/9}} \leq 5||\omega||^5_{X_h},$$

which concludes the proof of Proposition 1.19.

□

1.6 Contraction Mapping Theorem

In this section, assuming Lemmas 1.17 and 1.18, we prove Φ_{η_0} is a contraction mapping on a ball centered about the origin in the Banach Space X_h.

Theorem 1.21 *There exists $\delta_0 > 0$ and $r > 0$ such that, if $B_r = \{\omega \in X_h : ||\omega||_{X_h} \leq r\}$ and $||\eta_0||_{l_h^2} \leq \delta_0$, then*

i) $\Phi_{\eta_0} : B_r \to B_r$ continuously and

ii) $||\Phi_{\eta_0}(\nu) - \Phi_{\eta_0}(\mu)||_{X_h} \leq \lambda||\nu - \mu||_{X_h}$, for some $\lambda < 1$ and $\nu, \mu \in B_r$,

with λ, δ_0, and r independent of $h > 0$.

Proof(i): Recall, the operator Φ_{η_0} has two terms, $H_h(m)\eta_0(n)$ and $\Lambda_h\partial_{n,h}(\eta^5)(n,m)$. In order to establish the existence of an $r > 0$ such that $\Phi_{\eta_0} : B_r \to B_r$, we need six estimates, three involving each term.

First, we focus on $H_h(m)\eta_0(n)$. Since the multiplier associated with the operator $H_h(m)$ is bounded by one, we have

$$\sup_{m\in\mathbf{Z}} ||H_h(m)\eta_0||_{l_h^2} \leq ||\eta_0||_{l_h^2} \leq \delta_0. \tag{1.3}$$

By Lemma 1.17(i),

$$||\partial_{n,h}H_h(m)\eta_0||_{l_n^\infty l_{m,h}^2} \leq c||\eta_0||_{l_h^2} \leq c\delta_0 \tag{1.4}$$

and by Lemma 1.18(i),

$$||H_h(m)\eta_0||_{l_n^5 l_{m,h}^{10}} \leq c||\eta_0||_{l_h^2} \leq c\delta_0, \tag{1.5}$$

with c independent of $h > 0$. Combining (1.3), (1.4), and (1.5), we can conclude $H_h(m)\eta_0 \in X_h$ and

$$||H_h(m)\eta_0||_{X_h} \leq c\delta_0, \tag{1.6}$$

with c independent of $h > 0$.

Next, consider the nonlinear term $\Lambda_h\partial_{n,h}(\eta^5)(n,m)$. Fix $m > 0$. For a sequence η_0, let

$$S_m^*\eta_0(n,j) = H_h(j-m-1)\partial_{n,h}\eta_0(n)\chi_{1\leq j\leq m}.$$

Again, by Lemma 1.17(i),

$$||S_m^*\eta_0||_{l_n^\infty l_{j,h}^2} \leq c||\eta_0||_{l_h^2},$$

with c independent of $m > 0$. By duality, $S_m : l_n^1 l_{j,h}^2(\mathbf{Z}\times\mathbf{Z}) \to l_h^2(\mathbf{Z})$ is bounded with bounds independent of $m > 0$ where

$$\begin{aligned} S_m\omega(n) &= h^3\sum_{j=1}^{m} H_h(m+1-j)\partial_{n,h}\omega(\cdot,j)(n) \\ &= \Lambda_h\partial_{n,h}\omega(n,m). \end{aligned}$$

Therefore,

$$||\Lambda_h\partial_{n,h}\omega(\cdot,m)||_{l_h^2} \leq c||\omega||_{l_n^1 l_{j,h}^2}, \tag{1.7}$$

with c independent of $m > 0$ and $h > 0$. Letting $\omega = \eta^5$ in (1.7) and taking the supremum over $m > 0$, it follows that

$$\begin{aligned}
\sup_{m>0} ||\Lambda_h \partial_{n,h}(\eta^5)(\cdot, m)||_{l_h^2} &\leq c||\eta^5||_{l_n^1 l_{j,h}^2} \\
&= c||\eta||^5_{l_n^5 l_{j,h}^{10}} \\
&\leq c||\eta||^5_{X_h}.
\end{aligned}$$

The same proof can be used for $m < 0$ (by definition, $\Lambda_h \partial_{n,h}(\eta^5)(n, 0) = 0$). Hence,

$$\sup_{m\in\mathbf{Z}} ||\Lambda_h \partial_{n,h}(\eta^5)(\cdot, m)||_{l_h^2} \leq c||\eta||^5_{X_h}. \tag{1.8}$$

Next, by Lemma 1.17(ii),

$$\begin{aligned}
||\partial_{n,h}\Lambda_h \partial_{n,h}(\eta^5)||_{l_n^\infty l_{m,h}^2} &= ||\partial^2_{n,h}\Lambda_h(\eta^5)||_{l_n^\infty l_{m,h}^2} \\
&\leq c||\eta^5||_{l_n^1 l_{m,h}^2} \\
&= c||\eta||^5_{l_n^5 l_{m,h}^{10}} \\
&\leq c||\eta||^5_{X_h}.
\end{aligned} \tag{1.9}$$

Finally, by Lemma 1.18(ii) and Proposition 1.19,

$$\begin{aligned}
||\Lambda_h \partial_{n,h}(\eta^5)||_{l_n^5 l_{m,h}^{10}} &\leq c||\partial_{n,h}(\eta^5)||_{l_n^{5/4} l_{m,h}^{10/9}} \\
&\leq 5c||\eta||^5_{X_h}.
\end{aligned} \tag{1.10}$$

Thus, if $\eta \in X_h$, combining (1.8), (1.9), and (1.10), it follows $\Lambda_h \partial_{n,h}(\eta^5) \in X_h$ as well. Furthermore,

$$||\Lambda_h \partial_{n,h}(\eta^5)||_{X_h} \leq c||\eta||^5_{X_h}, \tag{1.11}$$

with c independent of $h > 0$. To conclude the proof of (i), if $\eta \in X_h$ with $||\eta||_{X_h} \leq r$, then by (1.6) and (1.11) there exists c_1 and c_2 such that

$$||\Phi_{\eta_0}\eta||_{X_h} \leq c_1\delta_0 + c_2 r^5.$$

If δ_0 and r are chosen small enough, then

$$||\Phi_{\eta_0}\eta||_{X_h} \leq c_1\delta_0 + c_2 r^5 \leq r$$

which implies

$$\Phi_{\eta_0} : B_r \to B_r.$$

Proof(ii): Let ν and μ be in $B_r \subset X_h$. To prove Φ_{η_0} is a contraction mapping on B_r for some $r > 0$, we need three estimates, one for each norm in the definition of $||\cdot||_{X_h}$. First,

$$\begin{aligned}\sup_{m>0} ||\Phi_{\eta_0}\nu(\cdot, m) - \Phi_{\eta_0}\mu(\cdot, m)||_{l_h^2} &= \frac{1}{5}\sup_{m>0} ||\Lambda_h \partial_{n,h}(\nu^5 - \mu^5)(\cdot, m)||_{l_h^2} \\ &= \frac{1}{5}\sup_{m>0} ||h^3 \sum_{j=1}^{m} H_h(m+1-j)\partial_{n,h}(\nu^5 - \mu^5)(\cdot, j)||_{l_h^2} \\ &\leq c||\nu^5 - \mu^5||_{l_n^1 l_{m,h}^2},\end{aligned}$$

where the last inequality follows from (1.7) with ω replaced with $\nu^5 - \mu^5$. As before, this can be proven for $m \leq 0$ as well. Thus,

$$\begin{aligned}&\sup_{m\in\mathbf{Z}} ||\Phi_{\eta_0}\nu(\cdot, m) - \Phi_{\eta_0}\mu(\cdot, m)||_{l_h^2} \leq c||(\nu-\mu)(\nu^4 + \nu^3\mu + \nu^2\mu^2 + \nu\mu^3 + \mu^4)||_{l_n^1 l_{m,h}^2} \\ &= ch\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|(\nu-\mu)(\nu^4 + \nu^3\mu + \nu^2\mu^2 + \nu\mu^3 + \mu^4)\right|^2\right)^{1/2}\end{aligned}$$

$$\begin{aligned}
&\le ch \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\nu-\mu|^{10} \right)^{1/10} \cdot \left(h^3 \sum_{m\in\mathbf{Z}} \left|\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4\right|^{5/2} \right)^{2/5} \\
&\le c \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\nu-\mu|^{10} \right)^{1/2} \right)^{1/5} \\
&\qquad \cdot \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4\right|^{5/2} \right)^{1/2} \right)^{4/5} \\
&\le c||\nu-\mu||_{X_h}(A_0+A_1+A_2+A_3+A_4),
\end{aligned}$$

with

$$A_j = \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\nu^{4-j}\mu^j|^{5/2} \right)^{1/2} \right)^{4/5}$$

for $0 \le j \le 4$. By Claim 1.20, $A_j \le r^4$. Therefore,

$$\sup_{m\in\mathbf{Z}} ||\Phi_{\eta_0}\nu(\cdot,m) - \Phi_{\eta_0}\mu(\cdot,m)||_{l_h^2} \le cr^4||\nu-\mu||_{X_h}. \tag{1.12}$$

Note,

$$\partial_{n,h}(\alpha\beta)(n,m) = \partial_{n,h}\alpha(n,m)\beta(n+1,m) + \alpha(n-1,m)\partial_{n,h}\beta(n,m).$$

Thus,

$$\begin{aligned}
||\Phi_{\eta_0}\nu - \Phi_{\eta_0}\mu||_{l_n^5 l_{m,h}^{10}} &= \frac{1}{5}||\Lambda_h\partial_{n,h}(\nu^5-\mu^5)||_{l_n^5 l_{m,h}^{10}} \\
&\le c||\partial_{n,h}(\nu^5-\mu^5)||_{l_n^{5/4} l_{m,h}^{10/9}} \\
&= c \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\partial_{n,h}\left[(\nu-\mu)(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)\right]\right|^{10/9} \right)^{9/8} \right)^{4/5} \\
&\le c \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\partial_{n,h}(\nu-\mu)(n,m) \right.\right. \\
&\qquad \left.\left. \cdot(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)(n+1,m)\right|^{10/9} \right)^{9/8} \right)^{4/5}
\end{aligned}$$

$$
\begin{aligned}
&+c\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|(\nu-\mu)(n-1,m)\right.\right.\\
&\qquad\qquad\left.\left.\cdot\partial_{n,h}(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)(n,m)\right|^{10/9}\right)^{9/8}\Bigg)^{4/5}\\
&= \quad I+II,
\end{aligned}
$$

where the first inequality follows from Lemma 1.18(ii). By applying Hölder's inequality to the sum on m with $p=\frac{9}{5}$ and $q=\frac{9}{4}$, it follows that

$$
\begin{aligned}
I &\leq c\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}(\nu-\mu)(n,m)|^2\right)^{5/8}\right.\\
&\qquad\left.\cdot\left(h^3\sum_{m\in\mathbf{Z}}\left|(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)(n+1,m)\right|^{5/2}\right)^{1/2}\right)^{4/5}\\
&\leq c\sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}(\nu-\mu)(n,m)|^2\right)^{1/2}\\
&\qquad\cdot\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)(n,m)\right|^{5/2}\right)^{1/2}\right)^{4/5}\\
&\leq c||\nu-\mu||_{X_h}(A_0+A_1+A_2+A_3+A_4)\\
&\leq cr^4||\nu-\mu||_{X_h}, \qquad\qquad (1.13)
\end{aligned}
$$

with the last inequality following from Claim 1.20. Next, we consider II. By applying Hölder's Inequality to both the sum in m with $p=9$ and $q=\frac{9}{8}$ and to the the sum in n with $p=4$ and $q=\frac{4}{3}$, we have

$$
\begin{aligned}
II &\leq \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu-\mu|^{10}\right)^{1/2}\right)^{1/5}\\
&\qquad\cdot\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|\partial_{n,h}(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)\right|^{5/4}\right)^{4/3}\right)^{3/5}
\end{aligned}
$$

$$\leq \; ||\nu-\mu||_{X_h} \sum_{j=0}^{4} \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\partial_{n,h}(\nu^{4-j}\mu^j)|^{5/4} \right)^{4/3} \right)^{3/5}$$

$$\leq \; ||\nu-\mu||_{X_h} \sum_{j=0}^{4} \left[\left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\partial_{n,h}(\nu^{4-j})(n,m)\mu^j(n+1,m)\right|^{5/4} \right)^{4/3} \right)^{3/5} \right.$$

$$\left. + \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\nu^{4-j}(n-1,m)\partial_{n,h}(\mu^j)(n,m)\right|^{5/4} \right)^{4/3} \right)^{3/5} \right]. \qquad (1.14)$$

Claim 1.22 *Let ν and μ be in X_h. Then for an integer j such that $0 \leq j \leq 4$,*

$$\left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\nu^{4-j}\partial_{n,h}(\mu^j)\right|^{5/4} \right)^{4/3} \right)^{3/5} \leq j||\nu||_{X_h}^{4-j}||\mu||_{X_h}^{j}.$$

Proof: Fix j such that $1 \leq j \leq 4$ (the case $j = 0$ is trivial). Using Hölder's Inequality on the sum in m with $p_1 = \dfrac{8}{4-j}$ and $q_1 = \dfrac{8}{j+4}$, then using Hölder's Inequality again on the sum in n with $p_2 = \dfrac{3}{4-j}$ and $q_2 = \dfrac{3}{j-1}$ ($p_i, q_i = \infty$ for $i = 1, 2$ when dividing by zero), it follows that

$$\left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\nu^{4-j}\partial_{n,h}(\mu^j)\right|^{5/4} \right)^{4/3} \right)^{3/5}$$

$$\leq \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\nu|^{10} \right)^{\frac{4-j}{6}} \cdot \left(h^3 \sum_{m\in\mathbf{Z}} |\partial_{n,h}(\mu^j)|^{\frac{10}{j+4}} \right)^{\frac{j+4}{6}} \right)^{3/5}$$

$$\leq \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} |\nu|^{10} \right)^{1/2} \right)^{\frac{4-j}{5}} \cdot \left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left|\partial_{n,h}(\mu^j)\right|^{\frac{10}{j+4}} \right)^{\frac{j+4}{2j-2}} \right)^{\frac{j-1}{5}}$$

$$\leq ||\nu||_{X_h}^{4-j} \cdot$$

$$\left(h \sum_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left(\sum_{k=0}^{j-1} |\partial_{n,h}\mu(n,m)||\mu^{j-1-k}(n+1,m)||\mu^k(n-1,m)| \right)^{\frac{10}{j+4}} \right)^{\frac{j+4}{2j-2}} \right)^{\frac{j-1}{5}}$$

$$\leq \quad ||\nu||_{X_h}^{4-j} \cdot$$

$$\sum_{k=0}^{j-1}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\mu(n,m)|^{\frac{10}{j+4}}|\mu(n+1,m)|^{\frac{10(j-1-k)}{j+4}}|\mu(n-1,m)|^{\frac{10k}{j+4}}\right)^{\frac{j+4}{2j-2}}\right)^{\frac{j-1}{5}}$$

$$= \quad ||\nu||_{X_h}^{4-j}\sum_{k=0}^{j-1}B_{j,k},$$

with

$$B_{j,k} = \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\mu(n,m)|^{\frac{10}{j+4}}|\mu(n+1,m)|^{\frac{10(j-1-k)}{j+4}}|\mu(n-1,m)|^{\frac{10k}{j+4}}\right)^{\frac{j+4}{2j-2}}\right)^{\frac{j-1}{5}}.$$

Fix k. Again, apply Hölder's Inequality to the sum on m with $p_1 = \dfrac{j+4}{5}$ and $q_1 = \dfrac{j+4}{j-1}$ and to the sum on n with $p_2 = \dfrac{j-1}{j-1-k}$ and $q_2 = \dfrac{j-1}{k}$. This gives

$$\begin{aligned}
B_{j,k} &\leq \left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\mu|^2\right)^{\frac{5}{2j-2}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu(n+1,m)|^{\frac{10(j-1-k)}{j-1}}|\mu(n-1,m)|^{\frac{10k}{j-1}}\right)^{1/2}\right)^{\frac{j-1}{5}} \\
&\leq \sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\mu|^2\right)^{1/2}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu(n+1,m)|^{\frac{10(j-1-k)}{j-1}}|\mu(n-1,m)|^{\frac{10k}{j-1}}\right)^{1/2}\right)^{\frac{j-1}{5}} \\
&\leq ||\mu||_{X_h}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu(n+1,m)|^{10}\right)^{\frac{j-1-k}{2(j-1)}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu(n-1,m)|^{10}\right)^{\frac{k}{2(j-1)}}\right)^{\frac{j-1}{5}} \\
&\leq ||\mu||_{X_h}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu|^{10}\right)^{1/2}\right)^{\frac{j-1-k}{5}}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\mu|^{10}\right)^{1/2}\right)^{\frac{k}{5}} \\
&\leq ||\mu||_{X_h}^{j}.
\end{aligned}$$

Therefore,

$$\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|\nu^{4-j}\partial_{n,h}(\mu^j)\right|^{5/4}\right)^{4/3}\right)^{3/5} \leq j||\nu||_{X_h}^{4-j}||\mu||_{X_h}^{j},$$

which concludes Claim 1.22.

///

Notice the second term of (1.14) can be bounded as above by switching the roles of ν and μ. Hence,

$$II \leq cr^4 ||\nu - \mu||_{X_h}.$$

Combining this with (1.13), it follows that

$$||\Phi_{\eta_0}\nu - \Phi_{\eta_0}\mu||_{l^5_n l^{10}_{m,h}} \leq cr^4 ||\nu - \mu||_{X_h}. \tag{1.15}$$

Finally,

$$\begin{aligned}
||\partial_{n,h}(\Phi_{\eta_0}\nu - \Phi_{\eta_0}\mu)||_{l^\infty_n l^2_{m,h}} &= \frac{1}{5}||\partial^2_{n,h}\Lambda_h(\nu^5 - \mu^5)||_{l^\infty_n l^2_{m,h}} \\
&\leq c||\nu^5 - \mu^5||_{l^1_n l^2_{m,h}} \\
&= ch\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|(\nu-\mu)(\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4)\right|^2\right)^{1/2} \\
&\leq ch\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu-\mu|^{10}\right)^{1/10}\cdot\left(h^3\sum_{m\in\mathbf{Z}}\left|\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4\right|^{5/2}\right)^{2/5} \\
&\leq c\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\nu-\mu|^{10}\right)^{1/2}\right)^{1/5} \\
&\qquad\cdot\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}\left|\nu^4+\nu^3\mu+\nu^2\mu^2+\nu\mu^3+\mu^4\right|^{5/2}\right)^{1/2}\right)^{4/5} \\
&\leq c||\nu-\mu||_{X_h}(A_0+A_1+A_2+A_3+A_4) \\
&\leq cr^4||\nu-\mu||_{X_h},
\end{aligned} \tag{1.16}$$

where the first inequality follows from Lemma 1.17(ii) and the last follows from Claim 1.20. Finally, (1.12), (1.15), and (1.16) imply that for ν and $\mu \in B_r$,

$$||\Phi_{\eta_0}\nu - \Phi_{\eta_0}\mu||_{X_h} \leq cr^4||\nu - \mu||_{X_h}.$$

By picking r small enough, i.e., $cr^4 = \lambda < 1$, it follows that Φ_{η_0} is a contraction map on B_r, which concludes the proof of Theorem 1.21(ii).

$\square$

1.7 Well-Posedness of $(KdV)_4^d$ in X_h

If the initial data is small, the Contraction Mapping Principle guarantees the existence of a unique $\eta \in B_r \subset X_h$ such that $\eta = \Phi_{\eta_0}\eta$. By definition of Φ_{η_0}, η solves (1.2). Also, by definition of the space X_h, $\eta \in l^\infty(l^2(\mathbf{Z})\,;\mathbf{Z})$. Moreover, we have the following theorem.

Theorem 1.23 *Let δ_0 and r be as in Theorem 1.21. Let $\eta_0 \in l_h^2(\mathbf{Z})$ such that $||\eta_0||_{l_h^2} \leq \delta_0$ and let $\eta \in B_r$ be the unique solution of $(KdV)_4^d$ guaranteed by Theorem 1.21. Then the map*

$$\eta_0 \to \eta$$

is continuous from $B(0;\delta_0)$ to B_r with $B(0;\delta_0) = \{\sigma : ||\sigma||_{l_h^2} \leq \delta_0\}$.

Proof: Fix $h > 0$. Let $\omega_0, \eta_0 \in B(0;\delta_0)$. Then the unique solutions to $(KdV)_4^d$ in B_r with initial data ω_0 and η_0 are ω and η, respectively, which satisfy

$$\omega(n,m) = H_h\omega_0(n) - \frac{1}{5}\Lambda_h\partial_h(\omega^5)(n,m)$$

and

$$\eta(n,m) = H_h\eta_0(n) - \frac{1}{5}\Lambda_h\partial_h(\eta^5)(n,m).$$

Thus,

$$\omega(n,m) - \eta(n,m) = H_h(m)(\omega_0 - \eta_0)(n) - \frac{1}{5}\Lambda_h\partial_h(\omega^5 - \eta^5)(n,m),$$

which implies

$$\begin{aligned} ||\omega - \eta||_{X_h} &\le ||H_h(m)(\omega_0 - \eta_0)||_{X_h} + c||\Lambda_h \partial_h(\omega^5 - \eta^5)||_{X_h}. \\ &= I + II. \end{aligned}$$

It follows from the proof of Theorem 1.21 that

$$I \le c||\omega_0 - \eta_0||_{l_h^2},$$

and

$$II \le cr^4||\omega - \eta||_{X_h} \le \lambda||\omega - \eta||_{X_h},$$

with $\lambda < 1$ (by the choice of r). Hence,

$$||\omega - \eta||_{X_h} \le c||\omega_0 - \eta_0||_{l_h^2} + \lambda||\omega - \eta||_{X_h}$$

which implies

$$||\omega - \eta||_{X_h} \le \frac{c}{1-\lambda}||\omega_0 - \eta_0||_{l_h^2}.$$

This concludes the proof of Theorem 1.23.

$\square$

Remark 1.24 By definition of X_h, the map $\eta_0 \to \eta$ is continuous from $B(0;\delta_0)$ to $l^\infty(l^2(\mathbf{Z})\,;\mathbf{Z})$.

CHAPTER 2

The Estimates

In this chapter, we will prove the estimates stated in the previous chapter, namely Lemmas 1.17 and 1.18.

2.1 A Discrete Version of Fractional Integration

Before proceeding, we need the following lemma.

Lemma 2.1 *Let k be a sequence such that*

$$|k(n)| \leq \frac{c}{|n|^{1/2}+1}.$$

Let

$$T(b) = b * k$$

for a sequence b. Then

$$T : l^p(\mathbf{Z}) \to l^q(\mathbf{Z})$$

is bounded if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

Proof: Fix $\lambda > 0$. Let $\mu > 0$ which is yet to be determined. Set $k_1 = k \cdot \chi_{\{0 \leq |n| \leq \mu\}}$, $k_2 = k \cdot \chi_{\{|n| > \mu\}}$, and let $b \in l^p(\mathbf{Z})$ with $||b||_{l^p} = 1$. Then $b * k = b * k_1 + b * k_2$, both of which are well defined. This leads to

$$\begin{aligned} \#\{n : |(b * k)(n)| > 2\lambda\} &\leq \#\{n : |(b * k_1)(n)| > \lambda\} + \#\{n : |(b * k_2)(n)| > \lambda\} \\ &= I_\lambda + II_\lambda. \end{aligned} \tag{2.1}$$

Now,

$$I_\lambda \leq \frac{||b * k_1||_{l^p}^p}{\lambda^p} \leq \frac{||k_1||_{l^1}^p}{\lambda^p} \leq \frac{c\mu^{p/2}}{\lambda^p}. \tag{2.2}$$

Let $p' = \dfrac{p}{p-1}$ be the conjugate exponent of p. Then

$$||b * k_2||_{l^\infty} \leq ||k_2||_{l^{p'}} \leq c_1 \mu^{-1/q},$$

since $p < 2$ hence, $p' > 2$. Pick μ such that $c_1\mu^{-1/q} = \lambda$, i.e., $\mu = c_2\lambda^{-q}$. Thus,

$$\#\{n : |(b * k_2)(n) > \lambda\} = 0, \tag{2.3}$$

which implies $II_\lambda = 0$. Therefore, combining (2.1), (2.2), and (2.3), it follows that

$$\#\{n : |(b * k)(n)| > 2\lambda\} \leq \frac{c\mu^{p/2}}{\lambda^p} + \frac{c}{\lambda^q} = \frac{c}{\lambda^q}.$$

Hence, T is weak-type (p, q) for $1 \leq p < q < \infty$ with $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{2}$. By the Marcinkiewicz Interpolation Theorem, T is strong-type (p, q) for $1 < p < 2$ and q such that $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{2}$.

$\square$

2.2 A Lemma Involving Fractional Differentiation

We now proceed with proving the main estimates. To this end, we make use of the operators D^β and complex interpolation. The following lemma provides half of the necessary results.

Lemma 2.2 *For $\gamma \in \mathbf{R}$,*

$$i)\ ||D_h^{-1/4+i\gamma} H_h(m)\eta_0||_{l_n^4 l_{m,h}^\infty} \le c||\eta_0||_{l_h^2}$$

and

$$ii)\ ||D_h^{-1/2+i\gamma}\Lambda_h\omega||_{l_n^4 l_{m,h}^\infty} \le C_\gamma||\omega||_{l_n^{4/3} l_{m,h}^1},$$

with $C_\gamma = c(|\gamma|+1)$ where c is independent of γ and $h > 0$.

Remark 2.3 Before proving (i), we check to see if the homogeneity is correct. Assume (i) is true for $h = 1$. Then

$$\begin{aligned}
||D_h^{-1/4+i\gamma} H_h(m)\eta_0||_{l_n^4 l_{m,h}^\infty} &= \left(h\sum_{n\in\mathbf{Z}}(\sup_{m\in\mathbf{Z}}|D_h^{-1/4+i\gamma}H_h(m)\eta_0(n)|)^4\right)^{1/4} \\
&= \left(h\sum_{n\in\mathbf{Z}}\left(\sup_{m\in\mathbf{Z}}\frac{1}{2\pi}\left|\int_{-\pi/h}^{\pi/h}\left|\frac{\sin(h\theta)}{h}\right|^{-1/4+i\gamma}\frac{(\hat{\eta}_0)_h(\theta)e^{-inh\theta}}{(1+\operatorname{sgn}(m)i\sin^3(h\theta))^{|m|}}d\theta\right|\right)^4\right)^{1/4} \\
&= h^{1/2}\left(\sum_{n\in\mathbf{Z}}\left(\sup_{m\in\mathbf{Z}}\frac{1}{2\pi}\left|\int_{-\pi/h}^{\pi/h}|\sin(h\theta)|^{-1/4+i\gamma}\frac{(h\sum_{k\in\mathbf{Z}}\eta_0(k)e^{ikh\theta})e^{-inh\theta}}{(1+\operatorname{sgn}(m)i\sin^3(h\theta))^{|m|}}d\theta\right|\right)^4\right)^{1/4} \\
&= h^{1/2}\left(\sum_{n\in\mathbf{Z}}\left(\sup_{m\in\mathbf{Z}}\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}|\sin\theta|^{-1/4+i\gamma}\frac{(\hat{\eta}_0)_1(\theta)e^{-in\theta}}{(1+\operatorname{sgn}(m)i\sin^3\theta)^{|m|}}d\theta\right|\right)^4\right)^{1/4} \\
&\le ch^{1/2}(\sum_{n\in\mathbf{Z}}|\eta_0(n)|^2)^{1/2} \\
&= c(h\sum_{n\in\mathbf{Z}}|\eta_0(n)|^2)^{1/2}.
\end{aligned}$$

Proof(i): By Remark 2.3, the case $h = 1$ implies all other cases with $h > 0$. Hence, we can assume $h = 1$ and drop the subscript notation. Note that

$$
\begin{aligned}
&\sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} D^{-1/4+i\gamma}H(m)\eta_0(n)\overline{\omega(n,m)} \\
&\quad = \sum_{m\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin\theta|^{-1/4+i\gamma}\frac{\hat{\eta}_0(\theta)}{(1+\mathrm{sgn}(m)i\sin^3\theta)^{|m|}}\overline{\hat{\omega}^{(1)}(\theta,m)}d\theta \\
&\quad = \sum_{m\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\eta}_0(\theta)\left(\overline{|\sin\theta|^{-1/4-i\gamma}\frac{\hat{\omega}^{(1)}(\theta,m)}{(1-\mathrm{sgn}(m)i\sin^3\theta)^{|m|}}}\right)d\theta \\
&\quad = \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \eta_0(n)\overline{D^{-1/4-i\gamma}H(-m)\omega(\cdot,m)(n)}.
\end{aligned}
$$

To prove (i), by duality, we need to show that

$$
\|\sum_{m\in\mathbf{Z}} D^{-1/4+i\gamma}H(m)\omega(\cdot,m)\|_{l^2_n} \le c\|\omega\|_{l^{4/3}_n l^1_m}, \tag{2.4}
$$

with c independent of γ. The square of the left hand side of (2.4) equals

$$
\begin{aligned}
&\sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}} D^{-1/4+i\gamma}H(m)\omega(n,m)\overline{D^{-1/4+i\gamma}H(\nu)\omega(n,\nu)} \\
&\quad = \sum_{m\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin\theta|^{-1/2}\frac{\hat{\omega}^{(1)}(\theta,m)\overline{\hat{\omega}^{(1)}(\theta,\nu)}}{(1+\mathrm{sgn}(m)i\sin^3\theta)^{|m|}(1-\mathrm{sgn}(\nu)i\sin^3\theta)^{|\nu|}}d\theta \\
&\quad = \sum_{m\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\omega}^{(1)}(\theta,m)|\sin\theta|^{-1/2}T(m,\nu)(\theta)\overline{\hat{\omega}^{(1)}(\theta,\nu)}d\theta \\
&\quad = \sum_{m\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \omega(n,m)D^{-1/2}T_{m,\nu}\overline{\omega(n,\nu)} \\
&\quad = \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \omega(n,m)\sum_{\nu\in\mathbf{Z}} D^{-1/2}T_{m,\nu}\overline{\omega(n,\nu)},
\end{aligned} \tag{2.5}
$$

where the operator $T_{m,\nu}$ is the operator with Fourier multiplier $T(m,\nu)(\theta)$ on the first variable with

$$
\begin{aligned}
T(m,\nu)(\theta) &= (1+\mathrm{sgn}(m)i\sin^3\theta)^{-|m|}(1-\mathrm{sgn}(\nu)i\sin^3\theta)^{-|\nu|} \\
&= \frac{1}{(1\pm i\sin^3\theta)^{\rho}}\frac{1}{(1+\sin^6\theta)^{r}},
\end{aligned}
$$

where $\rho \geq 0$ and $r \geq 0$ depend on m and ν. By (2.5) and duality, we need to show that

$$||\sum_{\nu\in\mathbf{Z}} D^{-1/2}T_{m,\nu}\omega(n,\nu)||_{l^4_n l^\infty_m} \leq c||\omega||_{l^{4/3}_n l^1_m}. \tag{2.6}$$

Now,

$$\begin{aligned} \left|\sum_{\nu\in\mathbf{Z}} D^{-1/2}T_{m,\nu}\omega(n,\nu)\right| &= \left|\sum_{\nu\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}|\sin\theta|^{-1/2}T(m,\nu)(\theta)\hat{\omega}^{(1)}(\theta,\nu)d\theta\right| \\ &= \left|\sum_{\nu\in\mathbf{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}|\sin\theta|^{-1/2}T(m,\nu)(\theta)\sum_{k\in\mathbf{Z}}\omega(k,\nu)e^{ik\theta}d\theta\right| \\ &\leq \sum_{k\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)\theta}|\sin\theta|^{-1/2}T(m,\nu)(\theta)d\theta\right| |\omega(k,\nu)|. \end{aligned}$$

Claim 2.4

$$\left|\int_{-\pi}^{\pi} e^{in\theta}|\sin\theta|^{-1/2+i\gamma}T(m,\nu)(\theta)d\theta\right| \leq \frac{C_\gamma}{|n|^{1/2}+1},$$

with $C_\gamma = c(|\gamma|+1)$ and c independent of m, ν, and $n \in \mathbf{Z}$.

Remark 2.5 Assume Claim 2.4 for the moment. It follows that

$$\sup_{m\in\mathbf{Z}}|\sum_{\nu\in\mathbf{Z}} D^{-1/2}T_{m,\nu}\omega(n,\nu)| \leq \sum_{k\in\mathbf{Z}}\frac{c}{|k-n|^{1/2}+1}\sum_{\nu\in\mathbf{Z}}|\omega(k,\nu)|.$$

By Lemma 2.1,

$$||\sup_{m\in\mathbf{Z}}\sum_{\nu\in\mathbf{Z}} D^{-1/2}T_{m,\nu}\omega(n,\nu)||_{l^4_n} \leq c||\sum_{\nu\in\mathbf{Z}}|\omega(n,\nu)|\,||_{l^{4/3}_n} = c||\omega||_{l^{4/3}_n l^1_\nu},$$

which is (2.6). Hence, we can conclude the proof of Lemma 2.2(i) once we have proven Claim 2.4.

Proof:(of Claim 2.4) Obviously, the left hand side of the claim is bounded independent of γ, m, ν, and n. Hence, we can assume $n \neq 0$. Let $\lambda_1(\theta) \in C_0^\infty(\mathbf{R})$ supported in $[-\pi/4, \pi/4]$ and $\lambda_1 \equiv 1$ on $[-\pi/8, \pi/8]$. Set $\lambda_3(\theta) = \lambda_1(\theta - \pi)$ and $\lambda_2 = 1 - \lambda_1 - \lambda_3$. Then

$$
\begin{aligned}
&\left| \int_{-\pi}^{\pi} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} T(m,\nu)(\theta) d\theta \right| \\
&\leq \left| \int_{-\pi/4}^{\pi/4} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} \lambda_1(\theta) T(m,\nu)(\theta) d\theta \right| \\
&\qquad + \left| \int_{\pi/8<|\theta|<7\pi/8} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} \lambda_2(\theta) T(m,\nu)(\theta) d\theta \right| \\
&\qquad + \left| \int_{3\pi/4}^{5\pi/4} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} \lambda_3(\theta) T(m,\nu)(\theta) d\theta \right| \\
&= I_{n,\gamma} + II_{n,\gamma} + III_{n,\gamma}.
\end{aligned}
$$

First, consider $II_{n,\gamma}$. Since $\left| \frac{d}{d\theta} |\sin\theta|^{-1/2+i\gamma} \right| \leq c(1+|\gamma|)|\sin\theta|^{-3/2}$, after integrating by parts, we obtain

$$
\begin{aligned}
II_{n,\gamma} &= \frac{1}{|n|} \left| \int_{\pi/8<|\theta|<7\pi/8} \frac{d}{d\theta}(e^{in\theta}) |\sin\theta|^{-1/2+i\gamma} \lambda_2(\theta) T(m,\nu)(\theta) d\theta \right| \\
&= \frac{1}{|n|} \left| \int_{\pi/8<|\theta|<7\pi/8} e^{in\theta} \frac{d}{d\theta} \Big(|\sin\theta|^{-1/2+i\gamma} \lambda_2(\theta) T(m,\nu)(\theta) \Big) d\theta \right| \\
&\leq \frac{1}{|n|} \left| \int_{\pi/8<|\theta|<7\pi/8} e^{in\theta} \frac{d}{d\theta} \Big(|\sin\theta|^{-1/2+i\gamma} \Big) \lambda_2(\theta) T(m,\nu)(\theta) d\theta \right| \\
&\qquad + \frac{1}{|n|} \left| \int_{\pi/8<|\theta|<7\pi/8} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} \lambda_2'(\theta) T(m,\nu)(\theta) d\theta \right| \\
&\qquad + \frac{1}{|n|} \left| \int_{\pi/4<|\theta|<3\pi/4} e^{in\theta} |\sin\theta|^{-1/2+i\gamma} \frac{d}{d\theta} \lambda_2(\theta) T(m,\nu)(\theta) d\theta \right| \\
&\leq \frac{C_\gamma}{|n|} \int_{\pi/8<|\theta|<7\pi/8} |\sin\theta|^{-3/2} d\theta + \frac{c}{|n|} + \frac{c}{|n|} \int_{\pi/8<|\theta|<7\pi/8} \left| \frac{d}{d\theta}(T(m,\nu)(\theta) \right| d\theta,
\end{aligned}
$$

where $C_\gamma = c(|\gamma|+1)$. Thus,

$$
\begin{aligned}
II_{n,\gamma} \leq \frac{C_\gamma}{|n|} &+ \frac{c}{|n|} \int_{\pi/8<|\theta|<7\pi/8} \left| \frac{d}{d\theta} \left(\frac{1}{(1 \pm i \sin^3\theta)^\rho} \right) \right| d\theta \\
&+ \frac{c}{|n|} \int_{\pi/8<|\theta|<7\pi/8} \left| \frac{d}{d\theta} \left(\frac{1}{(1+\sin^6\theta)^r} \right) \right| d\theta.
\end{aligned} \tag{2.7}
$$

The second integral in (2.7) is easily bounded independent of r by a bounded variation argument. For the first integral in (2.7),

$$\begin{aligned}
\frac{c}{|n|}\int_{\pi/8<|\theta|<7\pi/8}\left|\frac{d}{d\theta}\left(\frac{1}{(1\pm i\sin^3\theta)^\rho}\right)\right|d\theta &= \frac{c}{|n|}\int_{\pi/8<|\theta|<7\pi/8}\frac{3\rho\sin^2\theta|\cos\theta|}{(1+\sin^6\theta)^{(\rho+1)/2}}d\theta \\
&\leq \frac{c}{|n|}\int_{\pi/8<|\theta|<7\pi/8}\frac{6(\frac{\rho-1}{2})|\sin^5\theta\cos\theta|}{(1+\sin^6\theta)^{(\rho+1)/2}}d\theta \\
&= \frac{c}{|n|}\int_{\pi/8<|\theta|<7\pi/8}\left|\frac{d}{d\theta}\left(\frac{1}{(1+\sin^6\theta)^{(\rho-1)/2}}\right)\right|d\theta \\
&\leq \frac{c}{|n|}
\end{aligned}$$

for $\rho\neq 1$ (with a similar proof for $\rho=1$) which is again bounded independent of ρ using a bounded variation argument. Thus,

$$II_{n,\gamma}\leq \max\left(\frac{C_\gamma}{|n|},c\right)\leq \frac{C_\gamma}{|n|^{1/2}+1}. \tag{2.8}$$

Next,

$$\begin{aligned}
I_{n,\gamma} &= \left|\int_{-\pi/4}^{\pi/4}e^{in\theta}|\sin\theta|^{-1/2+i\gamma}\lambda_1(\theta)\frac{e^{\pm i\rho\arctan(\sin^3\theta)}}{(1+\sin^6\theta)^{r+\rho/2}}d\theta\right| \\
&= \left|\int_{-\pi/4}^{\pi/4}e^{i(n\theta\pm\rho\arctan(\sin^3\theta))}\lambda_1(\theta)\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}d\theta\right|,
\end{aligned}$$

with $q=r+\rho/2$. Let φ_0 be a C^∞ function such that $\varphi_0\equiv 1$ on $|\theta|\geq 2$ and $\mathrm{supp}\varphi_0\subset\{|\theta|\geq 1\}$. Then

$$\begin{aligned}
I_{n,\gamma} \leq\ & \left|\int_{-\pi/4}^{\pi/4}e^{i\psi_{\rho,n}(\theta)}\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)\varphi_0(|n|\theta)d\theta\right| \\
&+\left|\int_{-\pi/4}^{\pi/4}e^{i\psi_{\rho,n}(\theta)}\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)(1-\varphi_0(|n|\theta))d\theta\right|,
\end{aligned} \tag{2.9}$$

with

$$\psi_{\rho,n}(\theta)=n\theta\pm\rho\arctan(\sin^3\theta).$$

Since $\sin\theta \sim \theta$ for $\theta \in [-\pi/4, \pi/4]$, the second integral in (2.9) is easily seen to be less than $\frac{c}{|n|^{1/2}+1}$. We now focus on the first integral in (2.9). Set

$$A_{\rho,n} = \{\theta : |\psi'_{\rho,n}(\theta)| \leq |n|/2\}$$

and

$$B_{\rho,n} = \{\theta : |\psi'_{\rho,n}(\theta)| \leq |n|/3\}.$$

Let $\zeta_1, \zeta_2 \in C^\infty$ with $\zeta_1 + \zeta_2 = 1$, $\zeta_1 \equiv 1$ on $B_{\rho,n}$ and $\text{supp}\zeta_1 \subset A_{\rho,n}$. Note, ζ_1 and ζ_2 can be chosen such that ζ_1' and ζ_2' change sign only a finite number of times. If $\rho = 0$, then $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 1$. The first integral in (2.9) equals

$$\begin{aligned}
&\left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)} \frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q} \lambda_1(\theta)\varphi_0(|n|\theta)d\theta\right| \\
&\leq \left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)} \frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q} \lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)d\theta\right| \\
&\qquad + \left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)} \frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q} \lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta)d\theta\right| \\
&= I_{n,\gamma}^{(1)} + I_{n,\gamma}^{(2)}.
\end{aligned}$$

Note, $\psi'_{\rho,n}(\theta) = n \pm (\rho\arctan(\sin^3\theta))'$ with $(\rho\arctan(\sin^3\theta))' \sim \rho\theta^2$ and $\psi''_{\rho,n}(\theta) = (\rho\arctan(\sin^3\theta))'' \sim \rho\theta$ for $\theta \in [-\pi/4, \pi/4]$, both of which can be easily verified. Since $|\psi'| \leq \frac{|n|}{3}$ on the set $A_{\rho,n}$, it follows $\rho\theta^2 \sim |n|$ on $\text{supp}\zeta_1$. Hence,

$$|\theta| \sim \frac{\sqrt{|n|}}{\sqrt{\rho}}$$

for $\theta \in \text{supp}\zeta_1$. Since $\psi''_{\rho,n}(\theta) \sim \rho\theta$, there exists $c > 0$ such that

$$|\psi''_{\rho,n}(\theta)| \geq c\sqrt{|n|\rho} \tag{2.10}$$

on

$$\text{supp}\zeta_1 \subset \left\{\theta : c_1\sqrt{\frac{|n|}{\rho}} \le |\theta| \le c_2\sqrt{\frac{|n|}{\rho}}\right\}. \tag{2.11}$$

Before proceeding, we need the following lemma [10, pg. 342].

Lemma(van der Corput) *Suppose ϕ is C^∞ and compactly supported. Let ψ be a real-valued function so that, for some $k \in \mathbf{N}$, $k \ge 2$,*

$$|D^k\psi| \ge C_k > 0$$

throughout the support of ϕ. Then

$$\left|\int_{-\infty}^{\infty} e^{i\psi(x)}\phi(x)dx\right| \le \frac{c}{C_k^{1/k}} \cdot (||\phi||_{L^\infty} + ||D\phi||_{L^1}).$$

Applying van der Corput's Lemma to $I_{n,\gamma}^{(1)}$ with $k = 2$ and appealing to (2.10) and (2.11), it follows that

$$I_{n,\gamma}^{(1)} \le \frac{c}{|n|^{1/4}\rho^{1/4}}\left(||F||_{L^\infty(\text{supp}\zeta_1)} + \int_{\text{supp}\zeta_1} |F'(\theta)|d\theta\right), \tag{2.12}$$

with

$$F(\theta) = \frac{|\sin\theta|^{-1/2+i\gamma}\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)}{(1+\sin^6\theta)^q}.$$

Note that

$$|F(\theta)| \le c\frac{\rho^{1/4}}{|n|^{1/4}} \tag{2.13}$$

on suppζ_1. Furthermore,

$$
\begin{aligned}
|F'(\theta)| &\leq C_\gamma |\sin\theta|^{-3/2}\left|\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)}{(1+\sin^6\theta)^q}\right| + |\sin\theta|^{-1/2}\left|\left(\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)}{(1+\sin^6\theta)^q}\right)'\right| \\
&\leq C_\gamma|\theta|^{-3/2} + c\frac{\rho^{1/4}}{|n|^{1/4}}\left|\left(\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)}{(1+\sin^6\theta)^q}\right)'\right|
\end{aligned}
$$

for $\theta \in$ suppζ_1. Therefore,

$$
\begin{aligned}
\int_{\mathrm{supp}\zeta_1} |F'(\theta)|d\theta &\leq C_\gamma \int_{c_1\sqrt{\frac{|n|}{\rho}}\leq|\theta|} |\theta|^{-3/2}d\theta + c\frac{\rho^{1/4}}{|n|^{1/4}}\int_{|\theta|\leq\pi/4}\left|\left(\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_1(\theta)}{(1+\sin^6\theta)^q}\right)'\right|d\theta \\
&\leq C_\gamma\frac{\rho^{1/4}}{|n|^{1/4}} \qquad (2.14)
\end{aligned}
$$

again using a bounded variation argument. Finally, combining (2.12), (2.13), and (2.14), we have

$$
I^{(1)}_{n,\gamma} \leq \frac{C_\gamma}{|n|^{1/2}+1}.
$$

Next, we need to show

$$
I^{(2)}_{n,\gamma} \leq \frac{C_\gamma}{|n|^{1/2}+1}, \qquad (2.15)
$$

with $C_\gamma = c(|\gamma|+1)$ and c independent of ρ, q, and n. It follows that

$$
\begin{aligned}
I^{(2)}_{n,\gamma} &= \left|\int_{-\pi/4}^{\pi/4}\frac{1}{\psi'_{\rho,n}(\theta)}\frac{d}{d\theta}\left(e^{i\psi_{\rho,n}(\theta)}\right)\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta)d\theta\right| \\
&= \left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)}\frac{d}{d\theta}\left(\frac{1}{\psi'_{\rho,n}(\theta)}\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta)\right)d\theta\right| \\
&\leq \left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)}\frac{\psi''_{\rho,n}(\theta)}{(\psi'_{\rho,n}(\theta))^2}\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta))d\theta\right| \\
&\quad + C_\gamma\left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)}\frac{\cos\theta}{\psi'_{\rho,n}(\theta)}\frac{|\sin\theta|^{-3/2+i\gamma}}{(1+\sin^6\theta)^q}\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta))d\theta\right| \\
&\quad + \left|\int_{-\pi/4}^{\pi/4} e^{i\psi_{\rho,n}(\theta)}\frac{|\sin\theta|^{-1/2+i\gamma}}{\psi'_{\rho,n}(\theta)}\left(\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta)}{(1+\sin^6\theta)^q}\right)'d\theta\right|.
\end{aligned}
$$

Recall, $\psi'_{\rho,n}(\theta) = n \pm \rho(\arctan(\sin^3\theta))'$ with $\rho(\arctan(\sin^3\theta))' \sim \rho\theta^2$ on $[-\pi/4, \pi/4]$. Since $|\psi'_{\rho,n}(\theta)| \geq \frac{|n|}{3}$ on suppζ_2, it follows that $|\psi'_{\rho,n}(\theta)| \geq c\rho\theta^2$ on suppζ_2. This and the fact that $|\psi''_{\rho,n}(\theta)| \leq c\rho|\theta|$ implies

$$\begin{aligned} I^{(2)}_{n,\gamma} &\leq \frac{C}{|n|}\int_{1/|n|\leq|\theta|} \frac{\rho|\theta|}{\rho\theta^2}|\theta|^{-1/2}d\theta + \frac{C_\gamma}{|n|}\int_{1/|n|\leq|\theta|}|\theta|^{-3/2}d\theta \\ &\quad + \frac{c}{|n|^{1/2}}\int_{-\pi/4}^{\pi/4}\left|\left(\frac{\lambda_1(\theta)\varphi_0(|n|\theta)\zeta_2(\theta)}{(1+\sin^6\theta)^q}\right)'\right| d\theta \\ &\leq \frac{C_\gamma}{|n|^{1/2}}. \end{aligned}$$

Since $|I^{(2)}_{n,\gamma}| \leq c$, this finishes the proof of (2.15) which implies

$$I_{n,\gamma} \leq \frac{C_\gamma}{|n|^{1/2}+1}.$$

The estimate for $III_{n,\gamma}$ can be obtained from the estimate for $I_{n,\gamma}$ using a change of variable, namely $\tau = \theta - \pi$. These facts and (2.8) conclude the proof of Claim 2.4 and hence concludes the proof of Lemma 2.2(i).

Proof:(ii) We need to show that

$$||D_h^{-1/2+i\gamma}\Lambda_h\omega||_{l^4_n l^\infty_{m,h}} \leq C_\gamma||\omega||_{l^{4/3}_n l^1_{m,h}},$$

with $C_\gamma = c(|\gamma|+1)$. Suppose $m > 0$. Then, by Claim 2.4,

$$\begin{aligned} &|D_h^{-1/2+i\gamma}\Lambda_h\omega(n,m)| \\ &= \left| h^3\sum_{j=1}^{m}\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-inh\theta}\left|\frac{\sin(h\theta)}{h}\right|^{-1/2+i\gamma}\frac{\hat{\omega}_h^{(1)}(\theta,j)}{(1+\mathrm{sgn}(m+1-j)i\sin^3(h\theta))^{|m+1-j|}}d\theta\right| \\ &= h^{1/2}\left|\sum_{k\in\mathbf{Z}} h^3\sum_{j=1}^{m}\omega(k,j)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)\theta}\frac{|\sin\theta|^{-1/2+i\gamma}}{(1+i\sin^3(\theta))^{m+1-j}}d\theta\right| \end{aligned}$$

$$\begin{aligned}
&\leq \; h^{1/2}h^3 \sum_{k\in\mathbf{Z}}\sum_{j=1}^{m} |\omega(k,j)| \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)\theta} \frac{|\sin\theta|^{-1/2+i\gamma}}{(1+i\sin^3(\theta))^{m+1-j}} d\theta \right| \\
&\leq \; C_\gamma h^{1/2}h^3 \sum_{k\in\mathbf{Z}}\sum_{j=1}^{m} |\omega(k,j)| \frac{1}{|k-n|^{1/2}+1},
\end{aligned}$$

with $C_\gamma = c(|\gamma|+1)$ and independent of m. The same proof works for $m \leq 0$. Therefore,

$$\sup_{m\in\mathbf{Z}} |D_h^{-1/2+i\gamma}\Lambda_h\omega(n,m)| \leq C_\gamma h^{7/2} \sum_{k\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} |\omega(k,j)| \frac{1}{|k-n|^{1/2}+1} .$$

Hence, by Lemma 2.1,

$$\begin{aligned}
||D_h^{-1/2+i\gamma}\Lambda_h\omega||_{l^4_n l^\infty_{m,h}} &= (h\sum_{n\in\mathbf{Z}}(\sup_{m\in\mathbf{Z}} |D_h^{-1/2+i\gamma}\Lambda_h\omega|)^4)^{1/4} \\
&\leq C_\gamma h^{1/4}h^{7/2}(\sum_{n\in\mathbf{Z}}(\sum_{m\in\mathbf{Z}} |\omega(n,m)|)^{4/3})^{3/4} \\
&= C_\gamma ||\omega||_{l^{4/3}_n l^1_{m,h}}
\end{aligned}$$

and this concludes the proof of Lemma 2.2(ii).

□

2.3 Proof of Lemma 1.17

The following lemma implies Lemma 1.17.

Lemma 2.6 *For $\gamma \in \mathbf{R}$,*

$$i) \; ||D_h^{1+i\gamma}H_h(m)\eta_0||_{l^\infty_n l^2_{m,h}} \leq c||\eta_0||_{l^2_h}$$

and

$$ii) \; ||D_h^{2+i\gamma}\Lambda_h\omega||_{l^\infty_n l^2_{m,h}} \leq C_\gamma ||\omega||_{l^1_n l^2_{m,h}},$$

with $C_\gamma = c(|\gamma|+1)$ and c independent of γ and $h > 0$.

Remark 2.7 To see how this implies Lemma 1.17, first consider Lemma 1.17(ii) which says

$$||\partial^2_{n,h}\Lambda_h\omega||_{l^\infty_n l^2_{m,h}} \leq C_\gamma ||\omega||_{l^1_n l^2_{m,h}}.$$

The multiplier associated with $\partial^2_{n,h}$ is

$$\frac{-\sin^2(h\theta)}{h^2},$$

which is a constant multiple of the multiplier

$$\frac{|\sin(h\theta)|^2}{h^2}$$

associated with D_h^2. Hence, Lemma 1.17(ii) reduces to Lemma 2.6(ii) with $\gamma = 0$.

As for Lemma 1.17(i) which says

$$||\partial_{n,h}H_h(m)\eta_0||_{l^\infty_n l^2_{m,h}} \leq c||\eta_0||_{l^2_h},$$

the multiplier associated with $\partial_{n,h}$ is

$$\frac{-i\sin(h\theta)}{h}$$

while the multiplier associated with D_h^1 is

$$\frac{|\sin(h\theta)|}{h}.$$

Let $M(h\theta) = -i\mathrm{sgn}(h\theta)$ for $\theta \in [-\pi/h, \pi/h]$. Hence,

$$\frac{-i\sin(h\theta)}{h} = M(h\theta)\frac{|\sin(h\theta)|}{h}$$

which implies for $m \in \mathbf{Z}$

$$\partial_{n,h} H_h(m)\eta_0 = D_h^1 H_h \tilde{T}_h \eta_0,$$

where $\tilde{T}_h$ is the Fourier multiplier on $[-\pi/h, \pi/h]$ associated with $M(h\theta)$. Since $\tilde{T}_h : l_h^2(\mathbf{Z}) \to l_h^2(\mathbf{Z})$ is bounded (with norm less than one for all $h > 0$), Lemma 2.6(i) implies Lemma 1.17(i).

Remark 2.8 Before proving Lemma 2.6(i), we again check to see if the homogeneity is correct. Assume (i) is true for $h = 1$. Then

$$\begin{aligned}
&||D_h^{1+i\gamma} H_h(m)\eta_0||_{l_n^\infty l_{m,h}^2} \\
&= \sup_{n\in\mathbf{Z}} \left(h^3 \sum_{m\in\mathbf{Z}} \left| \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-inh\theta} \left| \frac{\sin(h\theta)}{h} \right|^{1+i\gamma} \frac{(\hat{\eta}_0)_h(\theta)}{(1+\operatorname{sgn}(m) i \sin^3(h\theta))^{|m|}} d\theta \right|^2 \right)^{1/2} \\
&= h^{1/2} \sup_{n\in\mathbf{Z}} \left(\sum_{m\in\mathbf{Z}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \Big(\sum_{k\in\mathbf{Z}} \eta_0(k) e^{ik\theta}\Big) \frac{|\sin\theta|^{1+i\gamma}}{(1+\operatorname{sgn}(m) i \sin^3(\theta))^{|m|}} d\theta \right|^2 \right)^{1/2} \\
&\le c h^{1/2} \Big(\sum_{n\in\mathbf{Z}} |\eta_0(n)|^2\Big)^{1/2} = c||\eta_0||_{l_h^2}.
\end{aligned}$$

Proof(i): As before, assume $h = 1$ and drop the notation. For $m, n \in \mathbf{Z}$,

$$\begin{aligned}
D^{1+i\gamma} H(m)\eta_0(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \frac{|\sin\theta|^{1+i\gamma} \hat{\eta}_0(\theta)}{(1+\operatorname{sgn}(m) i \sin^3\theta)^{|m|}} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{in\theta} \frac{|\sin\theta|^{1+i\gamma} \hat{\eta}_0(\theta)}{(1+\operatorname{sgn}(m) i \sin^3\theta)^{|m|}} d\theta \\
&\quad + \frac{1}{2\pi} \int_{\pi/4 \le |\theta| \le 3\pi/4} e^{in\theta} \frac{|\sin\theta|^{1+i\gamma} \hat{\eta}_0(\theta)}{(1+\operatorname{sgn}(m) i \sin^3\theta)^{|m|}} d\theta \\
&\quad + \frac{1}{2\pi} \int_{3\pi/4}^{5\pi/4} e^{in\theta} \frac{|\sin\theta|^{1+i\gamma} \hat{\eta}_0(\theta)}{(1+\operatorname{sgn}(m) i \sin^3\theta)^{|m|}} d\theta \\
&= I(n,m) + II(n,m) + III(n,m).
\end{aligned}$$

First,

$$
\begin{aligned}
\sup_{n\in\mathbf{Z}}\left(\sum_{m\in\mathbf{Z}}|II(n,m)|^2\right)^{1/2} &\le c\left(\sum_{m\in\mathbf{Z}}\int_{\pi/4\le|\theta|\le 3\pi/4}\frac{|\hat{\eta}_0(\theta)|^2}{(1+\sin^6\theta)^{|m|}}d\theta\right)^{1/2} \\
&= c\left(\int_{\pi/4\le|\theta|\le 3\pi/4}|\hat{\eta}_0(\theta)|^2\left(\sum_{m\in\mathbf{Z}}\frac{1}{(1+\sin^6\theta)^{|m|}}\right)d\theta\right)^{1/2} \\
&\le c\left(\int_{-\pi}^{\pi}|\hat{\eta}_0(\theta)|^2 d\theta\right)^{1/2} \\
&= c(\sum_{n\in\mathbf{Z}}|\eta_0(n)|^2)^{1/2}. \qquad (2.16)
\end{aligned}
$$

Next, we need to show

$$
\sup_{n\in\mathbf{Z}}\left(\sum_{m\in\mathbf{Z}}|I(n,m)|^2\right)^{1/2} \le c\left(\sum_{n\in\mathbf{Z}}|\eta_0(n)|^2\right)^{1/2}. \qquad (2.17)
$$

Note that

$$
\begin{aligned}
\sup_{n\in\mathbf{Z}}\left(\sum_{m\in\mathbf{Z}}|I(n,m)|^2\right)^{1/2} &= c\sup_{n\in\mathbf{Z}}\left(\sum_{m\in\mathbf{Z}}\left|\int_{-\pi/4}^{\pi/4}\frac{f_{n,\gamma}(\theta)}{(1+\operatorname{sgn}(m)i\sin^3\theta)^{|m|}}d\theta\right|^2\right)^{1/2} \\
&= c\sup_{n\in\mathbf{Z}}\left(\sum_{m\in\mathbf{Z}}\left|\int_{-\pi/4}^{\pi/4}e^{-im\arctan(\sin^3\theta)}\frac{f_{n,\gamma}(\theta)}{(1+\sin^6\theta)^{|m|/2}}d\theta\right|^2\right)^{1/2}, \qquad (2.18)
\end{aligned}
$$

with

$$
f_{n,\gamma}(\theta) = e^{in\theta}|\sin\theta|^{1+i\gamma}\hat{\eta}_0(\theta).
$$

To motivate the proof of (2.17), consider Lemma 0.1 of [5] which is the continuous analog of Lemma 2.6(i) (with ∂_x replaced with $D^{1+i\gamma}$). The next logical step would be to follow the proof of Lemma 0.1 by transforming the Fourier series variable from n to m via a change in variables. Hence, by setting $\tau = \arctan(\sin^3\theta)$, i.e.,

$$
\theta = \arcsin(\tan^{1/3}\tau) = h(\tau)
$$

and

$$d\theta = \frac{1}{3}\frac{\sec^2\tau}{\tan^{2/3}\tau\sqrt{1-\tan^{2/3}\tau}}d\tau = h'(\tau)d\tau,$$

(2.18) becomes

$$c\left(\sum_{m\in\mathbf{Z}}\left|\int_{-b}^{b} e^{-im\tau}\cos^{|m|}\tau\tilde{f}_{n,\gamma}(\tau)h'(\tau)d\tau\right|^2\right)^{1/2}, \tag{2.19}$$

with

$$\tilde{f}_{n,\gamma}(\tau) = f_{n,\gamma}(h(\tau))$$

and $b = \arctan(\sin^3(\pi/4))$. Since $h'(\tau) \sim \tau^{-2/3}$, it follows that

$$\begin{aligned}
\left(\int_{-b}^{b}|\tilde{f}_{n,\gamma}(\tau)h'(\tau)|^2 d\tau\right)^{1/2} &\leq c\left(\int_{-b}^{b}|\tilde{f}_{n,\gamma}(\tau)|^2\tau^{-2/3}h'(\tau)d\tau\right)^{1/2} \\
&= c\left(\int_{-\pi/4}^{\pi/4}|f_{n,\gamma}(\theta)|^2(\arctan(\sin^3\theta))^{-2/3}d\theta\right)^{1/2} \\
&= c\left(\int_{-\pi/4}^{\pi/4}|\hat{\eta}_0(\theta)|^2\frac{\sin^2\theta}{(\arctan(\sin^3\theta))^{2/3}}d\theta\right)^{1/2} \\
&\leq c\left(\int_{-\pi/4}^{\pi/4}|\hat{\eta}_0(\theta)|^2 d\theta\right)^{1/2}. \qquad (2.20)
\end{aligned}$$

Thus, (2.17) has been reduced to showing the following proposition.

Proposition 2.9

$$\sum_{m\in\mathbf{Z}}\left|\int_{-\pi}^{\pi} e^{-im\tau}\cos^{|m|}\tau\, g(\tau)d\tau\right|^2 \leq c\int_{-\pi}^{\pi}|g(\tau)|^2 d\tau.$$

Remark 2.10 By setting $g(\tau) = \tilde{f}_{n,\gamma}(\tau)h'(\tau)\chi_{\{-b\leq\tau\leq b\}}$ in (2.19), with (2.20), Proposition 2.9 implies (2.17).

Proof: Suppose $g(\tau) \in L^2[-\pi,\pi]$. Let a_k be the k^{th} Fourier coefficient of g, i.e., $g(\tau) = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}} a_k e^{ik\tau}$. Assume $m \geq 0$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{-im\tau}\cos^m\tau\, g(\theta)d\theta &= \int_{-\pi}^{\pi} e^{-im\tau}\left(\frac{e^{i\tau}+e^{-i\tau}}{2}\right)^m g(\tau)d\tau \\
&= 2^{-m}\int_{-\pi}^{\pi}(1-e^{-i2\tau})^m g(\tau)d\tau \\
&= 2^{-m}\int_{-\pi}^{\pi}\sum_{j=0}^{m}\binom{m}{j}e^{-i2j\tau}g(\tau)d\tau \\
&= 2^{-m}\sum_{j=0}^{m}\binom{m}{j}a_{2j} \\
&= \sum_{j=0}^{m} B_{m,j}a_{2j},
\end{aligned}$$

with $B_{m,j} = \frac{1}{2^m}\binom{m}{j}$ for $m \geq 0$ and $0 \leq j \leq m$.

Claim 2.11 *For all* $m \geq 1$,

$$i)\ \sum_{k=0}^{m} B_{m,k} = 1$$

and for all $k \geq 0$,

$$ii)\ \sum_{m=k}^{\infty} B_{m,k} = 2.$$

Proof:(of Claim 2.11) To prove (i), observe

$$\sum_{k=0}^{\infty} B_{m,k} = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}1^k 1^{m-k} = \frac{1}{2^m}(1+1)^m = 1.$$

To prove (ii), consider $g_k(x) = \left(\frac{1}{1-x}\right)^k$. It follows that $\frac{g_k^{(m)}(0)}{m!} = \binom{m+k-1}{k-1}$. This implies that for $|x| < 1$,

$$\left(\frac{1}{1-x}\right)^{k+1} = \sum_{m=0}^{\infty}\binom{m+k}{k}x^m.$$

Therefore,

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{m=k}^{\infty} \binom{m}{k} x^m.$$

Setting $x = 1/2$ completes the proof of (ii).

///

By Schur's Lemma (see e.g., [2, pg. 394]),

$$\begin{aligned}
\left(\sum_{m\geq 0}\left|\int_{-\pi}^{\pi} e^{-im\tau}\cos^{|m|}\tau\, g(\tau)d\tau\right|^2\right)^{1/2} &= \left(\sum_{m\geq 0}|\sum_{j=0}^{m} B_{m,j}a_{2j}|^2\right)^{1/2}\\
&\leq \sqrt{2}\left(\sum_{j\in\mathbf{Z}}|a_j|^2\right)^{1/2}\\
&= \sqrt{2}\left(\int_{-\pi}^{\pi}|g(\tau)|^2 d\tau\right)^{1/2}.
\end{aligned}$$

After making the change of variables $\varphi = -\theta$ we obtain a similar estimate with $\sum\limits_{m\geq 0}$ replaced by $\sum\limits_{m\leq 0}$ and this concludes the proof of Proposition 2.9 and hence, concludes the proof of (2.17). Notice that the same estimate works with $(1+\mathrm{sgn}(m)i\sin^3\theta)^{-|m|}$ replaced by $(1-\mathrm{sgn}(m)i\sin^3\theta)^{-|m|}$, setting $\phi = \theta - \pi$ yields

$$\sup_{n\in\mathbf{Z}}(\sum_{m\in\mathbf{Z}}|III(n,m)|^2)^{1/2} \leq c(\sum_{n\in\mathbf{Z}}|\eta_0(n)|^2)^{1/2}.$$

This fact along with (2.16) and (2.17) conclude the proof of Lemma 2.6(i).

□

The following is a useful corollary of the proof of (2.17).

Corollary 2.12

$$\left(\sum_{m\in\mathbf{Z}}\left|\int_{-\pi/4}^{\pi/4}\frac{f(\theta)}{(1+\mathrm{sgn}(m)i\sin^3\theta)^{|m|}}d\theta\right|^2\right)^{1/2} \leq \left(\int_{-\pi/4}^{\pi/4}\frac{|f(\theta)|^2}{\sin^2\theta}d\theta\right)^{1/2}.$$

The following proposition is a consequence of Lemma 2.6(i) via duality.

Proposition 2.13

$$||h^3 \sum_{m\in\mathbf{Z}} D_h^{1+i\gamma} H_h(-m)\omega(\cdot,m)||_{l_h^2} \leq c||\omega||_{l_n^1 l_{m,h}^2},$$

with c independent of $\gamma \in \mathbf{R}$ and $h > 0$.

Before we prove Lemma 2.6(ii), we define three functions on $\mathbf{Z} \times \mathbf{Z}$.

Definition 2.14

$$A_h\omega(n,m) = \begin{cases} h^3 \sum_{j\neq m+1} H_h(m+1-j)\omega(\cdot,j)(n) & \textit{if } m > 0 \\ 0 & \textit{if } m = 0 \\ -h^3 \sum_{j\neq m-1} H_h(m-1-j)\omega(\cdot,j)(n) & \textit{if } m < 0. \end{cases}$$

Definition 2.15

$$B_h\omega(n,m) = \begin{cases} h^3 \sum_{j\neq m+1} \operatorname{sgn}(m+1-j)H_h(m+1-j)\omega(\cdot,j)(n) & \textit{if } m > 0 \\ 0 & \textit{if } m = 0 \\ h^3 \sum_{j\neq m-1} \operatorname{sgn}(m-1-j)H_h(m-1-j)\omega(\cdot,j)(n) & \textit{if } m < 0. \end{cases}$$

Definition 2.16

$$E_h\omega(n,m) = \begin{cases} h^3 \sum_{j=-\infty}^{0} H_h(m+1-j)\omega(\cdot,j)(n) & \textit{if } m > 0 \\ 0 & \textit{if } m = 0 \\ -h^3 \sum_{j=0}^{\infty} H_h(m-1-j)\omega(\cdot,j)(n) & \textit{if } m < 0. \end{cases}$$

It follows that

$$B_h\omega = 2\Lambda_h\omega - A_h\omega + 2E_h\omega,$$

which implies

$$D_h^{2+i\gamma} B_h \omega = 2D_h^{2+i\gamma} \Lambda_h \omega - D_h^{2+i\gamma} A_h \omega + 2D_h^{2+i\gamma} E_h \omega. \tag{2.21}$$

Proposition 2.17

$$||D_h^{2+i\gamma} E_h \omega||_{l_n^\infty l_{m,h}^2} \leq c||\omega||_{l_n^1 l_{m,h}^2},$$

with c independent of γ and $h > 0$.

Remark 2.18 For the last time, we check the homogeneity. Assume Proposition 2.17 is true for $h = 1$ and let $m > 0$. Then

$$\begin{aligned}
&\sup_{n\in\mathbf{Z}} \left(h^3 \sum_{m>0} \left| h^3 \sum_{j=-\infty}^{0} \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-inh\theta} \left| \frac{\sin(h\theta)}{h} \right|^{2+i\gamma} \frac{\hat{\omega}_h^{(1)}(\theta, j)}{(1 + i\sin^3(h\theta))^{m+1-j}} d\theta \right|^2 \right)^{1/2} \\
&= h^{3/2} h^3 h^{-2} \left(\sum_{m>0} \left| \sum_{j=-\infty}^{0} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} |\sin\theta|^{2+i\gamma} \frac{\hat{\omega}^{(1)}(\theta, j)}{(1 + i\sin^3\theta)^{m+1-j}} d\theta \right|^2 \right)^{1/2} \\
&\leq chh^{3/2} \sum_{n\in\mathbf{Z}} \left(\sum_{m\in\mathbf{Z}} |\omega(n,m)|^2 \right)^{1/2} \\
&= c||\omega||_{l_n^1 l_{m,h}^2}.
\end{aligned}$$

The same proof works for $m < 0$. Combining these and the case $h = 1$ implies all other cases for $h > 0$.

Proof: Assume $h = 1$ and assume $m > 0$. Then

$$D^{2+i\gamma} E\omega(n, m) = D^{1+i\gamma} H(m) \left[H(1) \left(\sum_{j\in\mathbf{Z}} D^1 H(-j)[\chi_{\{j\leq 0\}} \omega(\cdot, j)] \right) \right].$$

This implies

$$
\begin{aligned}
\|\chi_{\{m\geq 0\}}D^{2+i\gamma}E\omega\|_{l^\infty_n l^2_m} &\leq c\left\|H(1)\left(\sum_{j\in\mathbf{Z}} D^1 H(-j)[\chi_{\{j\leq 0\}}\omega(\cdot,j)]\right)\right\|_{l^2_n} && (2.22)\\
&\leq c\|\sum_{j\in\mathbf{Z}} D^1 H(-j)[\chi_{\{j\leq 0\}}\omega(\cdot,j)]\|_{l^2_n} && (2.23)\\
&\leq c\|\chi_{\{m\leq 0\}}\omega(n,m)\|_{l^1_n l^2_m} && (2.24)\\
&\leq c\|\omega\|_{l^1_n l^2_m},
\end{aligned}
$$

where (2.22) follows from Lemma 2.6(i), (2.23) follows from fact that $H(1)$ is bounded on $l^2(\mathbf{Z})$, and finally (2.24) follows from Proposition 2.13. The case $m < 0$ is similar. Combining these two cases and applying the triangle inequality finishes the proof of Proposition 2.17.

□

Proposition 2.19

$$\|D_h^{2+i\gamma}A_h\omega\|_{l^\infty_n l^2_{m,h}} \leq c\|\omega\|_{l^1_n l^2_{m,h}},$$

with c independent of $\gamma \in \mathbf{R}$ and $h > 0$.

Remark 2.20 If the set of operators $\{H_h(m)\}_{m\in\mathbf{Z}}$ did form a group under composition, i.e.,

$$H_h(m_1 + m_2) = H_h(m_1)H_h(m_2) \tag{2.25}$$

for all $m_1, m_2 \in \mathbf{Z}$, then the proof of Proposition 2.19 would proceed exactly as the proof of Proposition 2.17, which used the fact that (2.25) is true if m_1 and m_2 are the same sign. However, because of our choice of discretization, we do not have the group structure in general. Hence, our proof will require different methods.

Proof: By a similar homogeneity argument as in Remark 2.18, we can assume $h = 1$. Suppose $m > 0$. Then

$$
\begin{aligned}
D^{2+i\gamma}A\omega(n,m) &= \sum_{j\neq m+1} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}|\sin\theta|^{2+i\gamma}\frac{\hat{\omega}^{(1)}(\theta,j)}{(1+\operatorname{sgn}(m+1-j)i\sin^3\theta)^{|m+1-j|}}d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}|\sin\theta|^{2+i\gamma}(1-i\sin^3\theta)\left(\sum_{j=m+2}^{\infty}(1-i\sin^3\theta)^{m-j}\hat{\omega}^{(1)}(\theta,j)\right)d\theta \\
&\quad +\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta}\left(\sum_{j=-\infty}^{m}\frac{\hat{\omega}^{(1)}(\theta,j)}{(1+i\sin^3\theta)^{m-j}}\right)d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\int_{-\pi}^{\pi} e^{-im\varphi}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}+\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)\hat{\omega}(\theta,\varphi)d\varphi d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{-\pi}^{\pi} e^{-im\varphi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}+\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)\hat{\omega}(\theta,\varphi)d\theta d\varphi \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\varphi}\sum_{j\in\mathbf{Z}} K_{\epsilon,\gamma}(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi)d\varphi,
\end{aligned}
$$

with

$$K_{\epsilon,\gamma}(m,\varphi) = \frac{1}{2\pi}\int_{\epsilon\leq|\theta|\leq\pi-\epsilon} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}+\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta,$$

$$\hat{\omega}^{(2)}(j,\varphi) = \sum_{k\in\mathbf{Z}}\omega(j,k)e^{ik\varphi},$$

and

$$\hat{\omega}(\theta,\varphi) = \sum_{j\in\mathbf{Z}}\hat{\omega}^{(2)}(j,\varphi)e^{ij\varphi}.$$

Before proceeding with the proof of Proposition 2.19, we need the following definition and a few claims concerning it.

Definition 2.21 *For $\varphi \neq 0$, let*

$$K_{\gamma}(m,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}+\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta.$$

Note, $\lim_{\epsilon\to 0} K_{\epsilon,\gamma}(m,\varphi) = K_\gamma(m,\varphi)$ for all $m \in \mathbf{Z}$ and for all $\varphi \in [-\pi,\pi]$ with $\varphi \neq 0$. Since $K_{\epsilon,\gamma}(m,0) = 0$ for all $\epsilon > 0, m \in \mathbf{Z}$, and $\gamma \in \mathbf{R}$, set $K_\gamma(m,0) = 0$. Hence,

$$\lim_{\epsilon\to 0} K_{\epsilon,\gamma}(m,\varphi) = K_\gamma(m,\varphi)$$

for all $m \in \mathbf{Z}$ and $\varphi \in [-\pi,\pi]$.

Claim 2.22 *There exists C such that*

$$||K_{\epsilon,\gamma}(m,\cdot)||_{L^2[-\pi,\pi]} \leq C,$$

with C independent of $\epsilon, m,$ and γ.

Proof:(of Claim 2.22) By definition,

$$\begin{aligned}
K_{\epsilon,\gamma}(m,\cdot)\hat{}(l) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_{\epsilon,\gamma}(m,\varphi)e^{-il\varphi}d\varphi \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta - e^{i\varphi}} + \frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta - e^{-i\varphi}}\right) d\theta\, e^{-il\varphi}d\varphi \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}|\sin\theta|^{2+i\gamma}\left[\frac{1}{1+i\sin^3\theta}\left(\frac{1}{1-\frac{e^{i\varphi}}{1+i\sin^3\theta}}\right)\right. \\
&\qquad \left. + \frac{e^{-i2\varphi}}{1-i\sin^3\theta}\left(\frac{1}{1-\frac{e^{-i\varphi}}{1-i\sin^3\theta}}\right)\right] d\theta\, e^{-il\varphi}d\varphi \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}|\sin\theta|^{2+i\gamma}\left[\frac{1}{1+i\sin^3\theta}\sum_{k=0}^{\infty}\frac{e^{ik\varphi}}{(1+i\sin^3\theta)^k}\right. \\
&\qquad \left. + \frac{e^{-i2\varphi}}{1-i\sin^3\theta}\sum_{k=0}^{\infty}\frac{e^{-ik\varphi}}{(1-i\sin^3\theta)^k}\right] e^{-il\varphi}d\theta d\varphi \\
&= \frac{1}{4\pi^2}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}|\sin\theta|^{2+i\gamma}\int_{-\pi}^{\pi}\sum_{k=0}^{\infty}\left(\frac{e^{i(k-l)\varphi}}{(1+i\sin^3\theta)^{k+1}} + \frac{e^{-i(k+2+l)\varphi}}{(1-i\sin^3\theta)^{k+1}}\right) d\varphi d\theta \\
&= \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta & \text{if } l \geq 0 \\ 0 & \text{if } l = -1 \\ \dfrac{1}{2\pi}\displaystyle\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1-i\sin^3\theta)^{|l|-1}}d\theta & \text{if } l \leq -2. \end{cases}
\end{aligned}$$

By Plancherel's Theorem,

$$
\begin{aligned}
||K_{\epsilon,\gamma}(m,\cdot)||_{L^2[-\pi,\pi]} &= \left(\sum_{l\in\mathbf{Z}} |K_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l)|^2\right)^{1/2} \\
&\leq \left(\sum_{l\leq -2} |K_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l)|^2\right)^{1/2} + \left(\sum_{l\geq 0} |K_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l)|^2\right)^{1/2} \qquad (2.26)
\end{aligned}
$$

First, consider the second sum in (2.26). Assuming ϵ is small,

$$
\begin{aligned}
\left(\sum_{l\geq 0} |K_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l)|^2\right)^{1/2} &= \left(\sum_{l\geq 0}\left|\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta\right|^2\right)^{1/2} \\
&\leq \left(\sum_{l\geq 0}\left|\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi/4} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta\right|^2\right)^{1/2} \\
&\quad + \left(\sum_{l\geq 0}\left|\frac{1}{2\pi}\int_{\pi/4<|\theta|<3\pi/4} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta\right|^2\right)^{1/2} \\
&\quad + \left(\sum_{l\geq 0}\left|\frac{1}{2\pi}\int_{\epsilon<|\theta-\pi|<\pi/4} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta\right|^2\right)^{1/2} \\
&= I + II + III.
\end{aligned}
$$

By Jensen's Inequality,

$$
\begin{aligned}
II \leq c\left(\sum_{l\geq 0}\int_{\pi/4}^{3\pi/4}\frac{|\sin\theta|^4}{(1+\sin^6\theta)^{l+1}}d\theta\right)^{1/2} &= c\left(\int_{\pi/4}^{3\pi/4}\sum_{l\geq 0}\frac{|\sin\theta|^4}{(1+\sin^6\theta)^{l+1}}d\theta\right)^{1/2} \\
&= c\left(\int_{\pi/4}^{3\pi/4}\frac{1}{\sin^2\theta}d\theta\right)^{1/2} \\
&\leq C,
\end{aligned}
$$

with C independent of m, ϵ,and γ. Next, turning to I, it follows that

$$
I = c\left(\sum_{l\geq 0}\left|\int_{-\pi/4}^{\pi/4}\frac{1}{(1+i\sin^3\theta)^l}f_{\epsilon,m,\gamma}(\theta)d\theta\right|^2\right)^{1/2},
$$

with $f_{\epsilon.m,\gamma}(\theta) = \dfrac{e^{-im\theta}|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta}\chi_{\{\epsilon<|\theta|<\pi/4\}}$. By Corollary 2.12,

$$\begin{aligned} I \leq c\left(\int_{-\pi/4}^{\pi/4}\frac{|f_{\epsilon,m,\gamma}(\theta)|^2}{\sin^2\theta}d\theta\right)^{1/2} &\leq c\left(\int_{-\pi/4}^{\pi/4}\frac{\sin^4\theta}{1+\sin^6\theta}\frac{1}{\sin^2\theta}d\theta\right)^{1/2} \\ &= c\left(\int_{-\pi/4}^{\pi/4}\frac{\sin^2\theta}{1+\sin^6\theta}d\theta\right)^{1/2} \\ &\leq C, \end{aligned}$$

with C independent of m, ϵ, and γ. Furthermore, III can be reduced to I (but with $1 - i\sin^3\theta$ in place of $1 + i\sin^3\theta$ which can be handled similarly to I). The first sum in (2.26) can be bounded by C independently of ϵ, m, and γ using similar arguments as above.

///

Claim 2.23

$$\lim_{\epsilon\to 0} K_{\epsilon,\gamma}(m,\cdot) = K_\gamma(m,\cdot)$$

in $L^2[-\pi,\pi]$, *uniformly in* m.

Remark 2.24 Assume Claim 2.23 for the moment. If $\omega \in l^1_n l^2_m$, then

$$\begin{aligned} &\lim_{\epsilon\to 0}\left|\int_{-\pi}^{\pi} e^{-im\varphi}\left(\sum_{j\in\mathbf{Z}}[K_{\epsilon,\gamma}(n-j,\varphi) - K_\gamma(n-j,\varphi)]\hat{\omega}^{(2)}(j,\varphi)\right)d\varphi\right| \\ &\leq \lim_{\epsilon\to 0}\int_{-\pi}^{\pi}\sum_{j\in\mathbf{Z}}|K_{\epsilon,\gamma}(n-j,\varphi) - K_\gamma(n-j,\varphi)||\hat{\omega}^{(2)}(j,\varphi)|d\varphi \\ &= \lim_{\epsilon\to 0}\sum_{j\in\mathbf{Z}}\int_{-\pi}^{\pi}|K_{\epsilon,\gamma}(n-j,\varphi) - K_\gamma(n-j,\varphi)||\hat{\omega}^{(2)}(j,\varphi)|d\varphi \\ &\leq \lim_{\epsilon\to 0}\sum_{j\in\mathbf{Z}}\| K_{\epsilon,\gamma}(n-j,\cdot) - K_\gamma(n-j,\cdot)\|_{L^2[-\pi,\pi]}\cdot\|\hat{\omega}^{(2)}(j,\cdot)\|_{L^2[-\pi,\pi]} \\ &= 0. \end{aligned}$$

Therefore,

$$D^{2+i\gamma}A\omega(n,m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\varphi}\sum_{j\in\mathbf{Z}} K_\gamma(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi)d\varphi. \tag{2.27}$$

Proof:(of Claim 2.23) Let $\epsilon_2 > \epsilon_1 > 0$ with $\epsilon_2 < \pi/4$ and assume $l \geq 0$. Then

$$\begin{aligned}&(K_{\epsilon_2,\gamma}(m,\cdot) - K_{\epsilon_1,\gamma}(m,\cdot))\hat{}(l)\\ &= \frac{1}{2\pi}\left(\int_{\epsilon_1<|\theta|<\epsilon_2} + \int_{\pi-\epsilon_2<|\theta|<\pi-\epsilon_1}\right) e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta.\end{aligned} \tag{2.28}$$

By Corollary 2.12,

$$\left(\sum_{l\geq 0}\left|\int_{\epsilon_1<|\theta|<\epsilon_2}\frac{e^{-im\theta}|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta}\frac{1}{(1+i\sin^3\theta)^l}d\theta\right|^2\right)^{1/2} \leq c\left(\int_{\epsilon_1<|\theta|<\epsilon_2}\frac{\sin^2\theta}{1+\sin^6\theta}d\theta\right)^{1/2}$$

which goes to 0 as $\epsilon_1, \epsilon_2 \to 0$. Again, a change of variables gives us the same result for the second integral of (2.28). It follows that $\{K_{\epsilon,\gamma}(m,\cdot)\hat{}(l)\}_{l\geq 0}$ is uniformly Cauchy in $l^2(\mathbf{Z}^+)$ as $\epsilon \to 0$ independent of $m \in \mathbf{Z}$. A similar argument would give us the same result for $l < 0$. Therefore, $\{K_{\epsilon,\gamma}(m,\cdot)\hat{}(j)\}_{j\in\mathbf{Z}}$ is uniformly Cauchy in $l^2(\mathbf{Z})$ independent of $m \in \mathbf{Z}$. Hence, by Plancherel's Theorem, $K_{\epsilon,\gamma}(m,\cdot)$ converges in $L^2[-\pi,\pi]$ as $\epsilon \to 0$, uniformly in m. Since $K_{\epsilon,\gamma}(m,\cdot) \to K_\gamma(m,\cdot)$ pointwise, $K_{\epsilon,\gamma}(m,\cdot) \to K_\gamma(m,\cdot)$ in $L^2[-\pi,\pi]$ uniformly in m. Furthermore, it follows that

$$K_\gamma(m,\cdot)\hat{}(l) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1+i\sin^3\theta)^{l+1}}d\theta & \text{if } l \geq 0\\ 0 & \text{if } l = -1\\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}}{(1-i\sin^3\theta)^{|l|-1}}d\theta & \text{if } l \leq -2.\end{cases} \tag{2.29}$$

///

Claim 2.25 *The Fourier coefficients of $K_0(0,\cdot)$ are absolutely summable, i.e.,*

$$\sum_{l\in\mathbf{Z}} |K_0(0,\cdot)\hat{}(l)| \leq C.$$

Proof: Suppose $\gamma = 0$ and $l \geq 0$. Let $\psi_1 \in C_0^\infty$ with $\psi_1 \equiv 1$ on $(-\pi/8, \pi/8)$ and $\mathrm{supp}\psi_1 \subset (-\pi/4, \pi/4)$. Let $\psi_2(\theta) = \psi_1(\theta - \pi)$ and $\psi_3 = 1 - \psi_1 - \psi_2$. From (2.29) it follows that

$$\begin{aligned}
K_0(0,\cdot)\hat{}(l) &= c\int_{-\pi/4}^{\pi/4} \psi_1(\theta)\frac{\sin^2\theta}{(1+i\sin^3\theta)^{l+1}}d\theta \\
&\quad + c\int_{\pi/8<|\theta|<7\pi/8} \psi_3(\theta)\frac{\sin^2\theta}{(1+i\sin^3\theta)^{l+1}}d\theta \\
&\quad + c\int_{3\pi/4}^{5\pi/4} \psi_2(\theta)\frac{\sin^2\theta}{(1+i\sin^3\theta)^{l+1}}d\theta \\
&= I(l) + II(l) + III(l).
\end{aligned}$$

First,

$$\begin{aligned}
\sum_{l=0}^{\infty}|II(l)| &= \sum_{l=0}^{\infty}\left|\int_{\pi/8<|\theta|<7\pi/8}\psi_3(\theta)\frac{\sin^2\theta}{(1+i\sin^3\theta)^{l+1}}d\theta\right| \\
&\leq c\int_{\pi/8<|\theta|<7\pi/8}\sum_{l=0}^{\infty}\frac{1}{(1+\sin^6\theta)^{l/2}}d\theta \\
&\leq C.
\end{aligned} \tag{2.30}$$

By the same change of variables as before, namely

$$\theta = \arcsin(\tan^{1/3}\tau) = h(\tau),$$

it follows that

$$\sum_{l=0}^{\infty}|I(l)| = \sum_{l=0}^{\infty}\left|\int_{-\pi/4}^{\pi/4}\frac{\psi_1(\theta)\sin^2\theta}{1+i\sin^3\theta}\frac{1}{(1+i\sin^3\theta)^l}d\theta\right|$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\left|\int_{-\pi/4}^{\pi/4} \frac{e^{-il\arctan(\sin^3\theta)}}{(1+\sin^6\theta)^{l/2}} \frac{\psi_1(\theta)\sin^2\theta}{1+i\sin^3\theta} d\theta\right| \\
&= \sum_{l=0}^{\infty}\left|\frac{1}{3}\int_{-b}^{b} e^{-il\tau}\cos^l\tau \frac{\psi_1(h(\tau))}{1+i\tan\tau}\frac{\sec^2\tau}{\sqrt{1-\tan^{2/3}\tau}} d\tau\right| \\
&= \sum_{l=0}^{\infty}\left|\frac{1}{3}\int_{-b}^{b} e^{-il\tau}\cos^l\tau\, g(\tau) d\tau\right| \\
&= \sum_{l=0}^{\infty}\left|\frac{1}{3}\int_{-\pi}^{\pi} e^{-il\tau}\cos^l\tau \sum_{k\in\mathbf{Z}} a_k e^{ik\tau} d\tau\right|,
\end{aligned}
\tag{2.31}
$$

with

$$b = \arctan(\sin^3(\pi/4)),$$

$$g(\tau) = \frac{\psi_1(h(\tau))}{1+i\tan\tau}\frac{\sec^2\tau}{\sqrt{1-\tan^{2/3}\tau}},$$

and

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\tau)e^{-ik\tau}d\tau.$$

Before proceeding with the proof of Claim 2.25, we need the following definition, proposition, corollary, and a theorem of Bernstein.

Definition 2.26 *A function f defined on Π is in $Lip^{\alpha}(\Pi)$ for $0 < \alpha \le 1$ if there exists $C > 0$ such that for $x, y \in \Pi$,*

$$|f(x) - f(y)| \le C|x-y|^{\alpha},$$

with distance measured on the unit circle.

Proposition 2.27 *Let $0 < \alpha \le 1$ and suppose f is a continuous function defined on Π which is differentiable for $x \in \Pi\backslash\{0\}$. If there exists C such that*

$$|f'(\theta)| \le C|\theta|^{\alpha-1},$$

then $f \in Lip^{\alpha}(\Pi)$.

Proof: We need only concern ourselves with x and y near the origin, say, $x, y \in [-\pi/4, \pi/4]$. Suppose $|x| < |y|$. If neither x nor y equal 0 and they are on the same side of the origin, then $f(y) - f(x) = \pm \int_x^y f'(t)dt$, which implies

$$\begin{aligned}
|f(y) - f(x)| &\leq \pm \int_x^y |f'(t)|dt \\
&\leq \pm C \int_x^y |t|^{\alpha-1} dt \\
&= C_\alpha(|y|^\alpha - |x|^\alpha) \\
&\leq C_\alpha |y - x|^\alpha.
\end{aligned}$$

The last inequality follows from the fact that the function $\kappa_\alpha(x) = |x|^\alpha \in Lip^\alpha(\Pi)$. Now, if $x = 0$, then

$$f(y) - f(0) = \pm \int_0^y f'(t)dt$$

since f' has an integrable singularity at the origin and f is continuous there as well. Hence, the argument above works when either x or y are zero. Finally, suppose x and y are on opposite sides of the origin with x negative. Then

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f(0)| + |f(0) - f(x)| \\
&\leq C \int_0^y |t|^{\alpha-1} dt + C \int_x^0 |t|^{\alpha-1} dt \\
&\leq C_\alpha(|y|^\alpha + |x|^\alpha) \\
&\leq \tilde{C}_\alpha |y + |x||^\alpha \\
&= \tilde{C}_\alpha |y - x|^\alpha,
\end{aligned}$$

where the last inequality comes from the equivalence of the l^1 and $l^{1/\alpha}$ norms on $\mathbf{R}^2$. This concludes the proof of Proposition 2.27.

$\square$

Recall,

$$g(\tau) = \frac{\psi_1(h(\tau))\sec^2\tau}{1 + i\tan\tau}\frac{1}{\sqrt{1 - \tan^{2/3}\tau}} = g_1(\tau)\cdot g_2(\tau),$$

with

$$h(\tau) = \arcsin(\tan^{1/3}\tau).$$

Since ψ_1 is smooth, compactly supported, and $\psi_1 \equiv 1$ in neighborhood of the origin, it follows that $g_1 \in C_0^\infty(\Pi)$. Next,

$$g_2'(\tau) = -\frac{1}{3}(1 - \tan^{2/3}\tau)^{-3/2}\tan^{-1/3}\tau\sec^2\tau,$$

which implies

$$|g_2'(\tau)| \leq C|\tau|^{-1/3}$$

on $\mathrm{supp}\psi_1(h(\tau))$.

Corollary 2.28

$$g \in Lip^{2/3}(\Pi).$$

Proof: Since g_1 is smooth and $|g_2'(\tau)| \leq C|\tau|^{-1/3}$, we have

$$|g'(\tau)| \leq C|\tau|^{-1/3}.$$

Since g is continuous, Proposition 2.27 implies that $g \in Lip^{2/3}(\Pi)$.

///

The following is a theorem of Bernstein [4, pg. 32].

Theorem 2.29 *If*

$$f \in Lip^{\alpha}(\Pi)$$

for some $\alpha \in (1/2, 1]$, *then the Fourier coefficients of* f *are absolutely summable, i.e.,*

$$\sum_{n\in\mathbf{Z}} |\hat{f}(n)| \leq C.$$

Corollary 2.28 implies that

$$\sum_{k\in\mathbf{Z}} |a_k| < \infty$$

where a_k is the k^{th} Fourier coefficient of

$$g(\tau) = \frac{\psi_1(h(\tau))\sec^2\tau}{1 + i\tan\tau}\frac{1}{\sqrt{1-\tan^{2/3}\tau}} = g_1(\tau)\cdot g_2(\tau).$$

By the proof of Proposition 2.9, (2.31) becomes

$$\sum_{l=0}^{\infty}\left|\sum_{j=0}^{l} B_{l,j}a_{2j}\right|, \tag{2.32}$$

with $B_{l,j} = \dfrac{1}{2^l}\dbinom{l}{j}$. After interchanging the order of summation and applying Claim 2.11(ii), (2.32) becomes

$$\begin{aligned}
\sum_{l=0}^{\infty}|I(l)| &\leq \sum_{j=0}^{\infty}|a_{2j}|\sum_{l=j}^{\infty}B_{l,j} \\
&= 2\sum_{j=0}^{\infty}|a_{2j}| \\
&\leq c\sum_{j\in\mathbf{Z}}|a_j| \\
&\leq C. \qquad (2.33)
\end{aligned}$$

Again, by a change of variables, $\sum_{l=0}^{\infty} |III(l)| \leq C$. With (2.30) and (2.33), this implies

$$\sum_{l \geq 0} |K_0(0,\cdot)\hat{}(l)| < \infty.$$

The proof showing $\sum_{l<0} |(K_0(0,\cdot))\hat{}(l)| < \infty$ is nearly identical. Therefore,

$$\sum_{l \in \mathbf{Z}} |K_0(0,\cdot)\hat{}(l)| < \infty, \tag{2.34}$$

which concludes the proof of Claim 2.25.

///

Finally, to finish the proof of Proposition 2.19, observe that (2.34) implies $K_0(0,\varphi)$ is a continuous function of φ, which implies that there exists C such that

$$|K_0(0,\varphi)| \leq C. \tag{2.35}$$

Claim 2.30 *There exists C such that*

$$|K_\gamma(m,\varphi)| \leq C,$$

with C independent of m, φ, and γ.

Remark 2.31 Claim 2.30 will finish the proof of Proposition 2.19. To see why this is the case, assume there exists a C such that $|K_\gamma(m,\varphi)| \leq C$ independently of m, φ, and γ. Hence, by (2.27),

$$||D^{2+i\gamma} A\omega||_{l_n^\infty l_m^2} = c \sup_{n \in \mathbf{Z}} \left(\sum_{m \in \mathbf{Z}} \left| \int_{-\pi}^{\pi} e^{-im\varphi} \sum_{j \in \mathbf{Z}} K_\gamma(n-j,\varphi) \hat{\omega}^{(2)}(j,\varphi) d\varphi \right|^2 \right)^{1/2}$$

$$
\begin{aligned}
&= c \sup_{n\in\mathbf{Z}} \left(\int_{-\pi}^{\pi} \left| \sum_{j\in\mathbf{Z}} K_\gamma(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi) \right|^2 d\varphi \right)^{1/2} \\
&\le c \sup_{n\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} \left(\int_{-\pi}^{\pi} \left| K_\gamma(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi) \right|^2 d\varphi \right)^{1/2} \\
&\le C \sum_{j\in\mathbf{Z}} \left(\int_{-\pi}^{\pi} |\hat{\omega}^{(2)}(j,\varphi)|^2 d\varphi \right)^{1/2} \\
&= C \sum_{j\in\mathbf{Z}} \left(\sum_{k\in\mathbf{Z}} |\omega(j,k)|^2 \right)^{1/2} \\
&= C \|\omega\|_{l^1_n l^2_m}.
\end{aligned}
$$

Proof:(of Claim 2.30) Recall,

$$
\begin{aligned}
K_0(0,\varphi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\sin^2\theta}{1+i\sin^3\theta - e^{i\varphi}} + \frac{\sin^2\theta e^{-i2\varphi}}{1-i\sin^3\theta - e^{-i\varphi}} \right) d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\sin^2\theta}{1+i\sin^3\theta - e^{i\varphi}} + \frac{\sin^2\theta}{1-i\sin^3\theta - e^{-i\varphi}} \right) d\theta \\
&\qquad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2\theta(e^{-i2\varphi}-1)}{1-i\sin^3\theta - e^{-i\varphi}} d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(1-\cos\varphi)\sin^2\theta}{(1-\cos\varphi)^2 + (\sin^3\theta - \sin\varphi)^2} d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2\theta(e^{-i2\varphi}-1)}{1-i\sin^3\theta - e^{-i\varphi}} d\theta \\
&= I + II.
\end{aligned}
$$

First, for $\pi/4 \le |\varphi| \le \pi$, II is trivially bounded. If $|\varphi| < \pi/4$,

$$
\begin{aligned}
|II| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|e^{-i2\varphi}-1|\sin^2\theta}{|1-\cos\varphi + i(\sin^3\theta - \sin\varphi)|} d\theta \\
&\le C\int_{\{\theta:|\sin^3\theta|\le 2|\sin\varphi|\}} \frac{|\varphi|\sin^2\theta}{|1-\cos\varphi|} d\theta + C\int_{\{\theta:|\sin^3\theta|>2|\sin\varphi|\}} \frac{|\varphi|\sin^2\theta}{|\sin^3\theta|} d\theta \\
&\le C\int_{\{\theta:|\theta|\le c|\varphi|^{1/3}\}} \frac{\theta^2}{|\varphi|} d\theta + C\int_{\{\theta:|\theta-\pi|\le c|\varphi|^{1/3}\}} \frac{(\theta-\pi)^2}{|\varphi|} d\theta \\
&\qquad + C\int_{\{\theta:|\sin^3\theta|>2|\sin\varphi|\}} \frac{|\varphi|\sin^2\theta}{|\sin\varphi|} d\theta \\
&\le C\int_{\{\theta:|\theta|\le c|\varphi|^{1/3}\}} \frac{\theta^2}{|\varphi|} d\theta + C \\
&\le C. \qquad (2.36)
\end{aligned}
$$

Notice

$$\frac{(1-\cos\varphi)\sin^2\theta}{(1-\cos\varphi)^2+(\sin^3\theta-\sin\varphi)^2}\geq 0 \tag{2.37}$$

for all $\theta, \varphi \in [-\pi, \pi]$. Combining (2.35) and (2.36), it follows $I = |I| \leq C$. Hence, (2.37) implies

$$\int_{-\pi}^{\pi}\frac{(1-\cos\varphi)\sin^2\theta}{(1-\cos\varphi)^2+(\sin^3\theta-\sin\varphi)^2}d\theta \leq C \tag{2.38}$$

Finally, (2.36) and (2.38) imply

$$\begin{aligned}
|K_\gamma(m,\varphi)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}+\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta\right| \\
&\leq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}|\sin\theta|^{i\gamma}\left(\frac{\sin^2\theta}{1+i\sin^3\theta-e^{i\varphi}}+\frac{\sin^2\theta}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta\right| \\
&\quad + \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(e^{-i2\varphi}-1)}{1-i\sin^3\theta-e^{-i\varphi}}d\theta\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-\cos\varphi)\sin^2\theta}{(1-\cos\varphi)^2+(\sin^3\theta-\sin\varphi)^2}d\theta + C \\
&\leq C,
\end{aligned}$$

with C independent of m, φ, and γ. This concludes the proof of Claim 2.30 which concludes the proof of Proposition 2.19.

□

Proposition 2.32

$$||D_h^{2+i\gamma}B_h\omega||_{l_n^\infty l_{m,h}^2} \leq C_\gamma ||\omega||_{l_n^1 l_{m,h}^2},$$

with $C_\gamma = c(1+|\gamma|)$ and c independent of $h > 0$.

Proof: By a similar homogeneity argument as in Remark 2.18, we can assume $h = 1$. Suppose $m > 0$. Then

$$D^{2+i\gamma}B\omega(n,m) = \sum_{j\neq m+1}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}|\sin\theta|^{2+i\gamma}\frac{\operatorname{sgn}(m+1-j)\hat{\omega}^{(1)}(\theta,j)}{(1+\operatorname{sgn}(m+1-j)i\sin^3\theta)^{|m+1-j|}}d\theta$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta}\left(\sum_{j=-\infty}^{m}\frac{\hat{\omega}^{(1)}(\theta,j)}{(1+i\sin^3\theta)^{m-j}}\right)d\theta \\
&\quad -\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}|\sin\theta|^{2+i\gamma}(1-i\sin^3\theta)\left(\sum_{j=m+2}^{\infty}(1-i\sin^3\theta)^{m-j}\hat{\omega}^{(1)}(\theta,j)\right)d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\int_{-\pi}^{\pi} e^{-im\varphi}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}-\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)\hat{\omega}(\theta,\varphi)d\varphi d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{-\pi}^{\pi} e^{-im\varphi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-in\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}-\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)\hat{\omega}(\theta,\varphi)d\theta d\varphi \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\varphi}\left(\sum_{j\in\mathbf{Z}}\tilde{K}_{\epsilon,\gamma}(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi)\right)d\varphi,
\end{aligned}
$$

with

$$
\tilde{K}_{\epsilon,\gamma}(m,\varphi)=\frac{1}{2\pi}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}}-\frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta.
$$

Now, as with $K_{\epsilon,\gamma}(m,\varphi)$ in Claim 2.22, it follows that

$$
\begin{aligned}
&\tilde{K}_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l) \\
&= \frac{1}{4\pi^2}\int_{\epsilon<|\theta|<\pi-\epsilon} e^{-im\theta}|\sin\theta|^{2+i\gamma}\int_{-\pi}^{\pi}\sum_{k=0}^{\infty}\left(\frac{e^{i(k-l)\varphi}}{(1+i\sin^3\theta)^{k+1}}-\frac{e^{-i(k+2+l)\varphi}}{(1-i\sin^3\theta)^{k+1}}\right)d\varphi d\theta \\
&= \operatorname{sgn}(l+1)K_{\epsilon,\gamma}(m,\cdot)\hat{\ }(l).
\end{aligned}
$$

It follows, as in Claim 2.23, that $\{\tilde{K}_{\epsilon,\gamma}(m,\cdot)\hat{\ }(j)\}_{j\in\mathbf{Z}}$ is uniformly bounded and uniformly Cauchy in $l^2(\mathbf{Z})$ as $\epsilon\to 0$ independent of $m\in\mathbf{Z}$. Again, by Plancherel's Theorem, $\tilde{K}_{\epsilon,\gamma}(m,\cdot)$ converges in $L^2[-\pi,\pi]$ uniformly in m.

Definition 2.33 *Let*

$$
\tilde{K}_{\gamma}(m,\cdot)=\lim_{\epsilon\to 0}\tilde{K}_{\epsilon,\gamma}(m,\cdot)
$$

in $L^2[-\pi,\pi]$.

It follows, as in Remark 2.24, that

$$D^{2+i\gamma}B\omega(n,m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\varphi}\sum_{j\in\mathbf{Z}}\tilde{K}_\gamma(n-j,\varphi)\hat{\omega}^{(2)}(j,\varphi)d\varphi$$

where, for $\varphi \neq 0$,

$$\begin{aligned}
\tilde{K}_\gamma(m,\varphi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}} - \frac{|\sin\theta|^{2+i\gamma}e^{-i2\varphi}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\left(\frac{|\sin\theta|^{2+i\gamma}}{1+i\sin^3\theta-e^{i\varphi}} - \frac{|\sin\theta|^{2+i\gamma}}{1-i\sin^3\theta-e^{-i\varphi}}\right)d\theta \\
&\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(1-e^{-i2\varphi})}{1-i\sin^3\theta-e^{-i\varphi}}d\theta \\
&= \frac{-i}{\pi}\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(\sin^3\theta-\sin\varphi)}{2-2\cos\varphi-\sin^2\varphi+(\sin^3\theta-\sin\varphi)^2}d\theta \\
&\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(1-e^{-i2\varphi})}{1-i\sin^3\theta-e^{-i\varphi}}d\theta \\
&= \tilde{W}_\gamma(m,\varphi)+\tilde{V}_\gamma(m,\varphi).
\end{aligned}$$

To finish the proof of the proposition, we need to show that there exists C such that

$$|\tilde{K}_\gamma(m,\varphi)| \leq C(|\gamma|+1)$$

independently of $m \in \mathbf{Z}$, for almost every $\varphi \in [-\pi,\pi]$ and γ (see Remark 2.31). Note that

$$|\tilde{V}_\gamma(m,\varphi)| \leq C,$$

by absolute value estimates and (2.36). Thus, we need to show

$$|\tilde{W}_\gamma(m,\varphi)| \leq C(|\gamma|+1), \tag{2.39}$$

with C independent of γ, m,and for almost every φ. Note, the integrand of $\tilde{W}_0(0,\varphi)$ is not of constant sign. Hence, we can not use the same methods used in the proof of

Proposition 2.19.

Definition 2.34 *Let*

$$W_{\epsilon,\gamma}(m,\varphi) = \frac{-i}{\pi}\int_{\{\theta:\epsilon<|\sin\theta-\sin^{1/3}\varphi|\}} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(\sin^3\theta-\sin\varphi)}{2-2\cos\varphi-\sin^2\varphi+(\sin^3\theta-\sin\varphi)^2}d\theta.$$

Obviously, $\lim_{\epsilon\to 0} W_{\epsilon,\gamma}(m,\varphi) = \tilde{W}_\gamma(m,\varphi)$ for $\varphi \neq 0$.

Claim 2.35 *There exists C_γ such that*

$$|W_{\epsilon,\gamma}(m,\varphi)| \leq C_\gamma$$

for all $m \in \mathbf{Z}$ and $\varphi \neq 0$, where $C_\gamma = C(|\gamma|+1)$ with C independent of ϵ, m, γ, and $\varphi \neq 0$.

Claim 2.35 implies (2.39).

Proof:(of Claim 2.35) We can assume both ϵ and $|\varphi|$ are small. Fix $\varphi \neq 0$. Set $\tau = \sin\varphi$ and $r = 2-2\cos\varphi-\sin^2\varphi$. It follows that $|\tau| \sim |\varphi|$ and $r = (1-\cos\varphi)^2 \sim c\varphi^4$. Thus,

$$\begin{aligned}
\pi|W_{\epsilon,\gamma}(m,\varphi)| &\leq \left|\int_{|\theta|<\pi/4,\epsilon<|\sin\theta-\tau^{1/3}|} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(\sin^3\theta-\tau)}{r+(\sin^3\theta-\tau)^2}d\theta\right| \\
&\quad + \left|\int_{\pi/4<|\theta|<3\pi/4,\epsilon<|\sin\theta-\tau^{1/3}|} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(\sin^3\theta-\tau)}{r+(\sin^3\theta-\tau)^2}d\theta\right| \\
&\quad + \left|\int_{3\pi/4<\theta<5\pi/4,\epsilon<|\sin\theta-\tau^{1/3}|} e^{-im\theta}\frac{|\sin\theta|^{2+i\gamma}(\sin^3\theta-\tau)}{r+(\sin^3\theta-\tau)^2}d\theta\right| \\
&= I_{\epsilon,\gamma}(m,\varphi) + II_{\epsilon,\gamma}(m,\varphi) + III_{\epsilon,\gamma}(m,\varphi).
\end{aligned}$$

For small φ (recall τ is small), clearly

$$II_{\epsilon,\gamma}(m,\varphi) \le C, \tag{2.40}$$

with C independent of ϵ, φ, m, and γ. Next, consider $I_{\epsilon,\gamma}(m,\varphi)$. By setting $\xi = \sin\theta$, we have

$$I_{\epsilon,\gamma}(m,\varphi) = \left| \int_{|\arcsin\xi|<\pi/4, \epsilon<|\xi-\tau^{1/3}|} e^{-im\arcsin\xi} \frac{\xi^2|\xi|^{i\gamma}(\xi^3-\tau)}{r+(\xi^3-\tau)^2} \frac{d\xi}{\sqrt{1-\xi^2}} \right|.$$

By a second change of variable, namely $\xi = \tau^{1/3}x$,

$$I_{\epsilon,\gamma}(m,\varphi) = \left| \int_{|x|<\frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{x^2|x|^{i\gamma}(x^3-1)}{r'+(x^3-1)^2} \frac{dx}{\sqrt{1-\tau^{2/3}x^2}} \right|,$$

with $\epsilon' = \tau^{-1/3}\epsilon$ and $r' = \tau^{-2}r \sim c\varphi^2$. Note that $r' > 0$. Let $\zeta_0 \in C_0^\infty(\mathbf{R})$ be an even function so that $\zeta_0 \equiv 1$ for $|x| \le 2$ and $\mathrm{supp}\zeta_0 \subset \{|x| \le 3\}$. Also, let $\zeta_1 = 1 - \zeta_0$. Then

$$\begin{aligned} I_{\epsilon,\gamma}(m,\varphi) &\le \left| \int_{|x|<\frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{x^2|x|^{i\gamma}(x^3-1)}{r'+(x^3-1)^2} \frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ &\quad + \left| \int_{|x|<\frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{x^2|x|^{i\gamma}(x^3-1)}{r'+(x^3-1)^2} \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ &= A_{\epsilon',\gamma}(m,\tau) + B_{\epsilon',\gamma}(m,\tau). \end{aligned}$$

First, consider $A_{\epsilon',\gamma}(m,\tau)$. It follows that

$$\begin{aligned} A_{\epsilon',\gamma}(m,\tau) &\le \left| \int_{|x|\le 4, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \left(\frac{x^2|x|^{i\gamma}(x^3-1)}{r'+(x^3-1)^2} \right.\right. \\ &\qquad \left.\left. - \frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2} \right) \frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ &\quad + \left| \int_{|x|\le 4, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2} \frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ &= A^{(1)}_{\epsilon',\gamma}(m,\tau) + A^{(2)}_{\epsilon',\gamma}(m,\tau), \end{aligned}$$

with

$$
\begin{aligned}
&A^{(1)}_{\epsilon',\gamma}(m,\tau) \\
&\leq C\int_{|x|\leq 4}\left|\frac{r'(x-1)^2(x^3+2x^2+3x+3)+3(x-1)^4(2x^3+3x^2+3x+1)}{(r')^2+r'(x-1)^2(x^2+x+1)^2+9r'(x-1)^2+9(x-1)^4(x^2+x+1)^2}\right|dx \\
&\leq C\int_{|x|\leq 4}\left(\frac{|x^3+2x^2+3x+3|}{(x^2+x+1)^2}+\frac{|2x^3+3x^2+3x+1|}{3(x^2+x+1)^2}\right)dx \\
&\leq C. \qquad (2.41)
\end{aligned}
$$

Next, let $a \in \mathbf{R}$ such that $a = -\arcsin(\tau^{1/3})$, which implies $|a| \sim |\tau|^{1/3}$. Then

$$
\begin{aligned}
A^{(2)}_{\epsilon',\gamma}(m,\tau) &\leq \left|\int_{|x|\leq 4,\epsilon'<|x-1|<\frac{1}{|ma|}}(e^{-im\arcsin(\tau^{1/3}x)}-e^{ixma})\frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2}\frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}}dx\right| \\
&\quad+\left|\int_{|x|\leq 4,\epsilon'<|x-1|<\frac{1}{|ma|}}e^{ixma}\frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2}\frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}}dx\right| \\
&\quad+\left|\int_{|x|\leq 4,\max(\frac{1}{|ma|},\epsilon')<|x-1|}e^{-im\arcsin(\tau^{1/3}x)}\frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2}\frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}}dx\right| \\
&= A^{(3)}_{\epsilon',\gamma}(m,\tau)+A^{(4)}_{\epsilon',\gamma}(m,\tau)+A^{(5)}_{\epsilon',\gamma}(m,\tau).
\end{aligned}
$$

After applying the Mean Value Theorem to the real and imaginary parts of $s(x) = e^{-im\arcsin(\tau^{1/3}x)} - e^{ixma}$, it follows that

$$
\begin{aligned}
A^{(3)}_{\epsilon',\gamma}(m,\tau) &= \left|\int_{|x|\leq 4,\epsilon'<|x-1|<\frac{1}{|ma|}}\left(\frac{m\tau^{1/3}}{\sqrt{1-\tau^{2/3}z_x^2}}\sin(im\arcsin(\tau^{1/3}z_x))-ma\sin(z_x ma)\right)\right. \\
&\qquad\left.\cdot(x-1)\frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2}\frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}}dx\right| \\
&\quad+\left|\int_{|x|\leq 4,\epsilon'<|x-1|<\frac{1}{|ma|}}\left(\frac{m\tau^{1/3}}{\sqrt{1-\tau^{2/3}y_x^2}}\cos(im\arcsin(\tau^{1/3}y_x))+ma\cos(y_x ma)\right)\right. \\
&\qquad\left.\cdot(x-1)\frac{3|x|^{i\gamma}(x-1)}{r'+9(x-1)^2}\frac{\zeta_0(x-1)}{\sqrt{1-\tau^{2/3}x^2}}dx\right| \\
&\leq c|ma|\int_{|x-1|<\frac{1}{|ma|}}dx \\
&\leq C, \qquad (2.42)
\end{aligned}
$$

where z_x and y_x are between 1 and x. Let $\psi \in C_0^\infty$ with $\psi \equiv 1$ for $|x| \leq 1/4$ and $\mathrm{supp}\psi \subset \{|x| \leq 1/2\}$. By changing variables,

$$
\begin{aligned}
A^{(4)}_{\epsilon',\gamma}(m,\tau) &= \left| \int_{|x|\leq 3, \epsilon'<|x|<\frac{1}{|ma|}} e^{ixma} \frac{3|x+1|^{i\gamma}x}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
&\leq \left| \int_{|x|\leq 3, \epsilon'<|x|<\frac{1}{|ma|}} e^{ixma} \frac{3x|x+1|^{i\gamma}\psi(x+1)}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
&\quad + \left| \int_{|x|\leq 3, \epsilon'<|x|<\frac{1}{|ma|}} e^{ixma} \frac{3x|x+1|^{i\gamma}[1-\psi(x+1)]}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
&\leq C + C \left| \int_{\mathbf{R}} e^{ixma} \left(\frac{x}{r'+9x^2} \chi_{\{\epsilon'<|x|<\frac{1}{|ma|}\}}(x) \right) \right. \\
&\qquad \left. \cdot \left(\frac{|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} \right) dx \right| \\
&= C + C \left| \int_{\mathbf{R}} e^{ixma} \hat{f}_{\epsilon',m,a,r'}(x) \hat{g}_{\gamma,\tau}(x) dx \right| \\
&= C + C |(f_{\epsilon',m,a,r'} * g_{\gamma,\tau})(ma)|, \qquad (2.43)
\end{aligned}
$$

with

$$
\hat{f}_{\epsilon',m,a,r'}(x) = \frac{x}{r'+9x^2} \chi_{\{\epsilon'<|x|<\frac{1}{|ma|}\}}(x)
$$

and

$$
\hat{g}_{\gamma,\tau}(x) = \frac{|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}}.
$$

Claim 2.36 *There exists C such that*

$$
||f_{\epsilon',m,a,r'}||_{L^\infty} \leq C,
$$

with C independent of $\epsilon', m, a,$ and r'.

Proof: By Fourier inversion,

$$
|f_{\epsilon',m,a,r'}(y)| = c \left| \int_{\mathbf{R}} e^{ixy} \hat{f}_{\epsilon',m,a,r'}(x) dx \right|
$$

$$
\begin{aligned}
&= c\left|\int_{\epsilon'<|x|<\frac{1}{|ma|}} e^{ixy}\frac{x}{r'+9x^2}dx\right| \\
&= c\left|\int_{\epsilon'<|x|<\frac{1}{|ma|}} \sin(xy)\frac{x}{r'+9x^2}dx\right| \\
&\leq c\int_{|x|<\frac{1}{|y|}} |y|\frac{x^2}{r'+9x^2}dx \\
&\qquad +\frac{c}{|y|}\left|\int_{|x|>\frac{1}{|y|},\epsilon'<|x|<\frac{1}{|ma|}} \frac{d}{dx}(\cos(xy))\frac{x}{r'+9x^2}dx\right| \\
&\leq C+\frac{1}{|y|}\left|\int_{|x|>\frac{1}{|y|},\epsilon'<|x|<\frac{1}{|ma|}} \cos(xy)\frac{r'-9x^2}{(r'+9x^2)^2}dx\right|+c \\
&\leq C+\frac{c}{|y|}\int_{|x|>\frac{1}{|y|}} \frac{1}{x^2}dx \\
&\leq C,
\end{aligned}
$$

where C and c are independent of ϵ', m, a, and r' (c is the boundary term).

///

Claim 2.37 *There exists C such that*

$$||g_{\gamma,\tau}||_{L^1} \leq C(|\gamma|+1),$$

with C independent of γ and τ (small).

Remark 2.38 Since

$$||f_{\epsilon' m,a,r'} * g_{\gamma,\tau}||_{L^\infty} \leq ||f_{\epsilon' m,a,r'}||_{L^\infty} \cdot ||g_{\gamma,\tau}||_{L^1},$$

Claims 2.36 and 2.37 imply that

$$|f_{\epsilon' m,a,r'} * g_{\gamma,\tau}(ma)| \leq C(|\gamma|+1).$$

With (2.43) this implies

$$A^{(4)}_{\epsilon',\gamma}(m,\tau) \leq C(|\gamma|+1). \tag{2.44}$$

Proof:(of Claim 2.37) Assume τ is small. Recall,

$$\hat{g}_{\gamma,\tau}(x) = \frac{|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}},$$

which is smooth and compactly supported for all γ. By Fourier inversion,

$$g_{\gamma,\tau}(y) = \int_{\mathbf{R}} e^{ixy}\hat{g}_{\gamma,\tau}(x)dx,$$

which implies there exists C which is independent of γ and small τ such that $|g_{\gamma,\tau}(y)| \le C$ for all $y \in \mathbf{R}$. Also, one can check that there exists C such that

$$\left|\frac{d^2}{dx^2}\hat{g}_{\gamma,\tau}(x)\right| \le C$$

for small $|\gamma|$ ($|\gamma| \le 1$) and that

$$\left|\frac{d^2}{dx^2}\hat{g}_{\gamma,\tau}(x)\right| \le C|\gamma|^2$$

for large $|\gamma|$ ($|\gamma| > 1$). Thus,

$$\begin{aligned}
\int_{\mathbf{R}} |g_{\gamma,\tau}(y)|dy &= \int_{|y|\le\max(|\gamma|,1)} |g_{\gamma,\tau}(y)|dy \\
&\quad + \int_{|y|\ge\max(|\gamma|,1)} \left|\frac{1}{y^2}\int_{\mathbf{R}} e^{ixy}\frac{d^2}{dx^2}(\hat{g}_{\gamma,\tau})(x)dx\right| dy \\
&\le C\max(|\gamma|,1) + C\max(|\gamma|^2,1)\int_{|y|\ge\max(|\gamma|,1)} \frac{1}{y^2}dy \\
&\le C\max(|\gamma|,1) \\
&\le C(|\gamma|+1),
\end{aligned}$$

and this concludes the claim.

///

Finally, assuming $m \neq 0$ and by change of variables, $A^{(5)}_{\epsilon',\gamma}(m,\tau)$ becomes

$$
\begin{aligned}
&\left| \int_{\max(\epsilon', \frac{1}{|ma|})<|x|<3} e^{-im\arcsin(\tau^{1/3}[x+1])} \frac{3|x+1|^{i\gamma}x}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
= \; & \left| \int_{\max(\epsilon', \frac{1}{|ma|})<|x|<3} e^{-im\arcsin(\tau^{1/3}[x+1])} \frac{3x|x+1|^{i\gamma}\psi(x+1)}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
& + \left| \int_{\max(\epsilon', \frac{1}{|ma|})<|x|<3} e^{-im\arcsin(\tau^{1/3}[x+1])} \frac{3x|x+1|^{i\gamma}[1-\psi(x+1)]}{r'+9x^2} \frac{\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
\leq \; & C + c \left| \int_{\max(\epsilon', \frac{1}{|ma|})<|x|<3} \frac{\sqrt{1-\tau^{2/3}(x+1)^2}}{m\tau^{1/3}} \right. \\
& \qquad \left. \cdot \frac{d}{dx}\left(e^{-im\arcsin(\tau^{1/3}[x+1])}\right) \frac{x}{r'+9x^2} \frac{|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0(x)}{\sqrt{1-\tau^{2/3}(x+1)^2}} dx \right| \\
\leq \; & C + \frac{c}{|m\tau^{1/3}|} \left| \int_{\max(\epsilon', \frac{1}{|ma|})<|x|<3} e^{-im\arcsin(\tau^{1/3}[x+1])} \right. \\
& \qquad \left. \cdot \frac{d}{dx}\left(\frac{x}{r'+9x^2}|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0(x)\right) dx \right| + \frac{c|ma|}{|m\tau^{1/3}|} \\
\leq \; & C + \frac{c}{|m\tau^{1/3}|} \int_{\frac{1}{|ma|}<|x|} \left| \frac{d}{dx}\left(\frac{x}{r'+9x^2}\right) \right| dx \\
& + \frac{c}{|m\tau^{1/3}|} \int_{\frac{1}{|ma|}<|x|<3} \left| \frac{x}{r'+9x^2} \right| \left| \frac{d}{dx}(|x+1|^{i\gamma}[1-\psi(x+1)]\zeta_0)(x) \right| dx + c \\
\leq \; & C + \frac{c}{|m\tau^{1/3}|} \int_{\frac{1}{|ma|}<|x|} \frac{1}{x^2} dx + \frac{c(|\gamma|+1)|ma|}{|m\tau^{1/3}|} \\
\leq \; & C(|\gamma|+1).
\end{aligned} \tag{2.45}
$$

Therefore, combining (2.42), (2.44), and (2.45), it follows that

$$A^{(2)}_{\epsilon',\gamma}(m,\tau) \leq C(|\gamma|+1).$$

With (2.41), this implies

$$A_{\epsilon',\gamma}(m,\tau) \leq C(|\gamma|+1). \tag{2.46}$$

Next, we need to show

$$B_{\epsilon',\gamma}(m,\tau) \le C(|\gamma|+1), \tag{2.47}$$

with C independent of ϵ', γ, m, and τ. Using similar methods as above,

$$\begin{aligned} B_{\epsilon',\gamma}(m,\tau) \le\ & \left| \int_{|x|\le \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \left(\frac{x^2|x|^{i\gamma}(x^3-1)}{r'+(x^3-1)^2} \right.\right. \\ & \left.\left. - \frac{|x|^{i\gamma}(x-1)}{r'+(x-1)^2} \right) \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ & + \left| \int_{|x|\le \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{|x|^{i\gamma}(x-1)}{r'+(x-1)^2} \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ =\ & B^{(1)}_{\epsilon',\gamma}(m,\tau) + B^{(2)}_{\epsilon',\gamma}(m,\tau). \end{aligned}$$

Since $r' \sim c\varphi^2$ is small, $\sqrt{1-\tau^{2/3}x^2} \ge c$ for $|x| \le \frac{\sqrt{2}}{2}\tau^{-1/3}$, and $\text{supp}\zeta_1(x-1) \subset \{|x|>2\}$, it follows that

$$\begin{aligned} B^{(1)}_{\epsilon',\gamma}(m,\tau) \le\ & c\int_{|x|\ge 2} \frac{|r'x^2(x^3-1) - r'(x-1)|}{(x-1)^2(x^3-1)^2} dx \\ & + c\int_{|x|\ge 2} \frac{|x^2(x-1)^2(x^3-1) - (x-1)(x^3-1)^2|}{(x-1)^2(x^3-1)^2} dx \\ \le\ & c\int_{|x|\ge 2} \frac{|x|^5}{x^8} dx + c\int_{|x|\ge 2} \frac{x^6}{x^8} dx \\ \le\ & C. \end{aligned} \tag{2.48}$$

Next,

$$\begin{aligned} B^{(2)}_{\epsilon',\gamma}(m,\tau) \le\ & \left| \int_{|x|\le \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|<\frac{1}{|ma|}} (e^{-im\arcsin(\tau^{1/3}x)} - e^{ixma}) \frac{|x|^{i\gamma}(x-1)}{r'+(x-1)^2} \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ & + \left| \int_{|x|\le \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x-1|<\frac{1}{|ma|}} e^{ixma} \frac{|x|^{i\gamma}(x-1)}{r'+(x-1)^2} \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ & + \left| \int_{|x|\le \frac{\sqrt{2}}{2}\tau^{-1/3}, \max(\epsilon',\frac{1}{|ma|})<|x-1|} e^{-im\arcsin(\tau^{1/3}x)} \frac{|x|^{i\gamma}(x-1)}{r'+(x-1)^2} \frac{\zeta_1(x-1)}{\sqrt{1-\tau^{2/3}x^2}} dx \right| \\ =\ & B^{(3)}_{\epsilon',\gamma}(m,\tau) + B^{(4)}_{\epsilon',\gamma}(m,\tau) + B^{(5)}_{\epsilon',\gamma}(m,\tau). \end{aligned}$$

It can be shown

$$B^{(3)}_{\epsilon',\gamma}(m,\tau) \leq C \tag{2.49}$$

using similar techniques as those used in (2.42). By changing variables,

$$\begin{aligned} B^{(4)}_{\epsilon',\gamma}(m,\tau) &= \left| \int_{|x+1|\leq \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x|<\frac{1}{|ma|}} e^{ixma} \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right| \\ &\leq \left| \int_{|x|\leq \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x|<\frac{1}{|ma|}} e^{ixma} \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right| + C, \end{aligned}$$

with C arising as a result of the change in region of integration (recall τ is small). Hence,

$$\begin{aligned} B^{(4)}_{\epsilon',\gamma}(m,\tau) &\leq \left| \int_{|x|\leq \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x|<\frac{1}{|ma|}} \cos(xma) \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right| \\ &\quad + \left| \int_{|x|\leq \frac{\sqrt{2}}{2}\tau^{-1/3}, \epsilon'<|x|<\frac{1}{|ma|}} \sin(xma) \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right| + C \\ &= I + II + C. \end{aligned}$$

First, assuming τ is small,

$$II \leq c|ma| \int_{|x|<\frac{1}{|ma|}} \frac{x^2}{r'+x^2} dx \leq C. \tag{2.50}$$

Next, using the fact that ζ_1 is even,

$$\begin{aligned} I &= \left| \int_{\epsilon'<x\leq \min(\frac{\sqrt{2}}{2}\tau^{-1/3}, \frac{1}{|ma|})} \cos(xma) \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right. \\ &\quad \left. + \int_{\max(\frac{-\sqrt{2}}{2}|\tau^{-1/3}|, -\frac{1}{|ma|})\leq x<-\epsilon'} \cos(xma) \frac{|x-1|^{i\gamma} x}{r'+x^2} \frac{\zeta_1(x)}{\sqrt{1-\tau^{2/3}(x-1)^2}} dx \right| \end{aligned}$$

$$= \left| \int_{\epsilon' < x \le \min(\frac{\sqrt{2}}{2}\tau^{-1/3}, \frac{1}{|ma|})} \cos(xma) \frac{x}{r' + x^2} \right.$$
$$\left. \cdot \left(\frac{|x-1|^{i\gamma}}{\sqrt{1 - \tau^{2/3}(x-1)^2}} - \frac{|x+1|^{i\gamma}}{\sqrt{1 - \tau^{2/3}(x+1)^2}} \right) \zeta_1(x) dx \right|$$
$$= \left| \int_{\epsilon' < x \le \min(\frac{\sqrt{2}}{2}\tau^{-1/3}, \frac{1}{|ma|})} \cos(xma) \frac{x}{r' + x^2} (f_\gamma(x+1) - f_\gamma(x-1)) \zeta_1(x) dx \right|,$$

with

$$f_\gamma(x) = \frac{|x|^{i\gamma}}{\sqrt{1 - \tau^{2/3} x^2}}$$

and

$$f_\gamma'(x) = \frac{i\gamma |x|^{i\gamma}}{x\sqrt{1 - \tau^{2/3}x^2}} + \frac{-2|x|^{i\gamma}\tau^{2/3}x}{(1 - \tau^{2/3}x^2)^{3/2}}.$$

Using the fact that for a complex-valued C^1 function f and $x_1, x_2 \in \mathbf{R}$,

$$|f(x_1) - f(x_2)| \le 2 \sup_{y \in [x_1, x_2]} |f'(y)||x_1 - x_2|,$$

I is bounded by

$$2 \int_{2 \le |x| \le \frac{\sqrt{2}}{2}\tau^{-1/3}} \frac{1}{|x|} \sup_{|x-y| \le 1} |f_\gamma'(y)| dx$$

since $\mathrm{supp}\zeta_1 \subset \{|x| \ge 2\}$. For $2 \le |x| \le \frac{\sqrt{2}}{2}\tau^{-1/3}$ and y such that $|x - y| \le 1$, we have $|y| \sim |x|$. Hence

$$\sup_{|y-x| \le 1} |f_\gamma'(y)| \le \frac{c|\gamma|}{|x|} + c\tau^{2/3}|x|.$$

Therefore

$$I \le c|\gamma| \int_{2 \le |x|} \frac{1}{x^2} dx + c\tau^{2/3} \int_{0 \le x \le \frac{\sqrt{2}}{2}\tau^{-1/3}} dx \le C(|\gamma| + 1), \tag{2.51}$$

since τ is small. It follows by (2.50) and (2.51) that

$$B^{(4)}_{\epsilon', \gamma}(m, \tau) \le C(|\gamma| + 1). \tag{2.52}$$

Finally, by changing variables once more, assuming $m \neq 0$, and integrating by parts (as in (2.45)), $B^{(5)}_{\epsilon',\gamma}(m,\tau)$ becomes

$$\begin{aligned}
& c\left|\int_{|x+1|\leq\frac{\sqrt{2}}{2}\tau^{-1/3},\max(\epsilon',\frac{1}{|ma|})<|x|} \frac{1}{m\tau^{1/3}}\frac{d}{dx}\left(e^{-im\arcsin(\tau^{1/3}[x+1])}\right)\frac{x}{r'+x^2}|x+1|^{i\gamma}\zeta_1(x)dx\right| \\
\leq\; & \frac{c}{|m\tau^{1/3}|}\int_{\frac{1}{|ma|}<|x|}\left|\frac{d}{dx}\left(\frac{x}{r'+x^2}\right)\right|dx \\
& \qquad +\frac{c}{|m\tau^{1/3}|}\int_{\frac{1}{|ma|}<|x|}\left|\frac{x}{r'+x^2}\right|\left|\frac{d}{dx}\left(|x+1|^{i\gamma}\zeta_1(x)\right)\right|dx+\frac{c\tau^{1/3}}{|ma|} \\
\leq\; & C+\frac{c|\gamma|}{|m\tau^{1/3}|}\int_{\max(2,\frac{1}{|ma|})\leq|x|}\left|\frac{x}{r'+x^2}\right|\frac{1}{|x+1|}|\zeta_1(x)|dx \\
& \qquad +\frac{c}{|m\tau^{1/3}|}\int_{\frac{1}{|ma|}<|x|}\left|\frac{x}{r'+x^2}\right||\zeta_1'(x)|dx \\
\leq\; & c\frac{|\gamma|\min(1/2,|ma|)}{|m\tau^{1/3}|}+C \\
\leq\; & C(|\gamma|+1). \qquad (2.53)
\end{aligned}$$

Together, (2.48), (2.49), (2.52), and (2.53) imply (2.47). Combining this with (2.46), we conclude that

$$I_{\epsilon,\gamma}(m,\varphi)\leq C(|\gamma|+1), \tag{2.54}$$

with C independent of $\epsilon,\gamma,m,$ and φ. Setting $\theta' = \theta+\pi$, it follows $III_{\epsilon,\gamma}(m,-\varphi) = I_{\epsilon,\gamma}(m,\varphi)$, which implies $III_{\epsilon,\gamma}(m,\varphi) \leq C(|\gamma|+1)$ with C independent of $\epsilon,\gamma,m,$ and φ. This, (2.40), and (2.54) conclude the proof of Claim 2.35 which concludes the proof of Proposition 2.32.

□

Recall,

$$D_h^{2+i\gamma}B_h\omega = 2D_h^{2+i\gamma}\Lambda_h\omega - D_h^{2+i\gamma}A_h\omega + 2D_h^{2+i\gamma}E_h\omega.$$

Combining Propositions 2.17, 2.19, and 2.32 finishes the proof of Lemma 2.6(ii), and hence completes the proof of Lemma 1.17(ii).

□

2.4 Discrete Complex Interpolation

We are now in a position to prove the final crucial estimate, namely Lemma 1.18. Recall the statement of Lemma 1.18(i),

$$||H_h(m)\eta_0||_{l^5_n l^{10}_{m,h}} \leq c||\eta_0||_{l^2_h}, \tag{2.55}$$

with c independent of $h > 0$.

To prove (2.55), consider the analytic family of operators

$$T_z\eta_0(n) = D_h^{-z/4} D_h^{(1-z)} H_h(m)\eta_0(n), \qquad 0 \leq \mathrm{Re} z \leq 1,$$

with $\eta_0 \in l^2_h(\mathbf{Z})$. If $z = i\gamma$, then

$$||T_z\eta_0||_{l^\infty_n l^2_{m,h}} = ||D^{1-i5/4\gamma}\eta_0||_{l^\infty_n l^2_{m,h}} \leq C||\eta_0||_{l^2_h}, \tag{2.56}$$

by Lemma 2.6(i). If $z = 1 + i\gamma$, then

$$||T_z\eta_0||_{l^4_n l^\infty_{m,h}} = ||D^{-1/4-i5/4\gamma} H_h(m)\eta_0||_{l^4_n l^\infty_{m,h}} \leq C||\eta_0||_{l^2_h}, \tag{2.57}$$

by Lemma 2.2(i). Combining (2.56) and (2.57) with Stein's analytic interpolation theorem [7] and letting $x = 4/5$ implies (2.55) which finishes the proof of Lemma 1.18(i).

Next, recall the statement of Lemma 1.18(ii),

$$||\Lambda_h\omega||_{l_n^5 l_{m,h}^{10}} \leq c||\omega||_{l_n^{5/4} l_{m,h}^{10/9}}, \tag{2.58}$$

with c independent of $h > 0$. Similar to the proof of (2.55), consider the analytic family of operators

$$T_z\omega = D_h^{-z/2} D_h^{2(1-z)} \Lambda_h\omega, \qquad 0 \leq \mathrm{Re} z \leq 1,$$

with ω defined on $\mathbf{Z} \times \mathbf{Z}$ and compactly supported. If $z = i\gamma$, then

$$||T_z\omega||_{l_n^\infty l_{m,h}^2} = ||D^{2-i5\gamma/2}\Lambda_h\omega||_{l_n^\infty l_{m,h}^2} \leq C_\gamma||\omega||_{l_n^1 l_{m,h}^2}, \tag{2.59}$$

by Lemma 2.6(ii). If $z = 1 + i\gamma$, then

$$||T_z\omega||_{l_n^4 l_{m,h}^\infty} = ||D^{-1/2-i5\gamma/2}\Lambda_h\omega||_{l_n^4 l_{m,h}^\infty} \leq C_\gamma||\omega||_{l_n^{4/3} l_{m,h}^1}, \tag{2.60}$$

by Lemma 2.2(ii). In both (2.59) and (2.60), $C_\gamma = c(1 + |\gamma|)$ with c independent of γ and $h > 0$. Again, applying Stein's analytic interpolation theorem finishes the proof of (2.58) which concludes the proof of Lemma 1.18(ii).

□

CHAPTER 3

Numerical Results

In this chapter, we discuss the numerical implementation of the fixed point iteration with the operator Φ_{η_0}.

3.1 The Cutoff Function $\gamma_h(n, m)$

The iterates of the contraction mapping Φ_{η_0} are defined on the entire grid, $\mathbf{Z} \times \mathbf{Z}$. To go from one iteration to the next, the entire iteration is needed. This is obviously not feasible numerically. Hence, we need to introduce a "smooth" cutoff function which is zero for n or m large. One may think we can simply set the iterates equal to zero for large n or m. However, this may adversely affect the norm estimates of the contraction mapping. This is due to the fact that if ω is defined on $\mathbf{Z} \times \mathbf{Z}$ and

$$\chi_{N,M}(n,m) = \begin{cases} 1 & \text{if } |n| \le N, |m| \le M \\ 0 & \text{else,} \end{cases}$$

then it is not necessarily the case that

$$||\partial_{n,h}(\chi_{N,M}\omega)||_{l_n^\infty l_{m,h}^2} \le c||\partial_{n,h}\omega||_{l_n^\infty l_{m,h}^2},$$

with c independent of $h > 0$. In fact, it is quite the contrary as $h \to 0$. Thus, we need to introduce a cutoff function which is one for $|n| \le N$ and $|m| \le M$, but decays slowly to zero for large n.

Let $h > 0$ be small. Suppose we would like a solution of $(KdV)_4^d$ for $(n, m) \in \mathbf{Z} \times \mathbf{Z}$ such that $|nh| \le N$ and $|mh^3| \le M$ where h is chosen so that N is an integer multiple of h and M is an integer multiple of h^3. Let $\gamma_h(n, m)$ be a piecewise linear function such that

$$\gamma_h \equiv 1 \quad \text{for } |nh| \le N \text{ and } |mh^3| \le M$$

and

$$\text{supp}\gamma_h = \{(n, m) \in \mathbf{Z} \times \mathbf{Z} : |nh| \le 2Nh^{-1/5}, |mh^3| \le M\}.$$

Obviously, γ_h can be chosen so that

$$\sup_{n,m \in \mathbf{Z}} |\partial_{n,h} \gamma_h| \le \frac{ch^{1/5}}{N} \tag{3.1}$$

for small h. Note that we do not need any decay in the m-direction since none of the three norms of X_h involves differences in the m-direction.

Definition 3.1 *For a discrete function η defined on $\mathbf{Z} \times \mathbf{Z}$, let*

$$\tilde{S}\eta(n, m) = \gamma_h(n, m)\eta(n, m).$$

Recall the definition of Φ_{η_0},

$$\Phi_{\eta_0}\eta(n, m) = H_h(m)\eta_0(n) - \frac{1}{5}\Lambda_h \partial_{n,h}(\eta^5)(n, m),$$

with $\eta_0 \in l_h^2(\mathbf{Z})$.

Theorem 3.2 *There exists $\delta_0 > 0$ and $r > 0$ such that, if $B_r = \{\omega \in X_h :$ $||\omega||_{X_h} \leq r\}$ and $||\eta_0||_{l_h^2} \leq \delta_0$, then*

i) $\tilde{S}\Phi_{\eta_0} : B_r \to B_r$ continuously and

ii) $||\tilde{S}\Phi_{\eta_0}(\nu) - \tilde{S}\Phi_{\eta_0}(\mu)||_{X_h} \leq \lambda||\nu - \mu||_{X_h}$, for some $\lambda < 1$ and $\nu, \mu \in B_r$,

with λ, δ_0, and r independent of $h > 0$.

Proof(i): First, we would like to determine a bound for the operator norm of $\tilde{S}$ on X_h. Obviously,

$$||\tilde{S}\eta||_{l_n^5 l_{m,h}^{10}} \leq ||\eta||_{l_n^5 l_{m,h}^{10}} \tag{3.2}$$

and

$$\sup_{m\in\mathbf{Z}} ||\tilde{S}\eta(\cdot, m)||_{l_h^2} \leq \sup_{m\in\mathbf{Z}} ||\eta(\cdot, m)||_{l_h^2}. \tag{3.3}$$

Note that

$$\partial_{n,h}(\gamma_h\eta)(n,m) = \gamma_h(n+1,m)\partial_{n,h}\eta(n,m) + \partial_{n,h}\gamma_h(n,m)\eta(n-1,m).$$

Then

$$\begin{aligned}
||\partial_{n,h}(\gamma_h\eta)||_{l_n^\infty l_{m,h}^2} &\leq ||\partial_{n,h}\eta||_{l_n^\infty l_{m,h}^2} + \sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\partial_{n,h}\gamma_h(n,m)|^2|\eta(n-1,m)|^2\right)^{1/2} \\
&\leq ||\partial_{n,h}\eta||_{l_n^\infty l_{m,h}^2} + \sup_{n\in\mathbf{Z}}\left(h^3\sum_{|m|\leq Mh^{-3}}|\partial_{n,h}\gamma_h(n,m)|^{5/2}\right)^{2/5}\left(h^3\sum_{m\in\mathbf{Z}}|\eta(n-1,m)|^{10}\right)^{1/10} && (3.4) \\
&\leq ||\partial_{n,h}\eta||_{l_n^\infty l_{m,h}^2} + \frac{ch^{1/5}M^{2/5}}{N}\sup_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\eta(n-1,m)|^{10}\right)^{1/10} && (3.5) \\
&\leq ||\partial_{n,h}\eta||_{l_n^\infty l_{m,h}^2} + \tilde{c}\left(h\sum_{n\in\mathbf{Z}}\left(h^3\sum_{m\in\mathbf{Z}}|\eta(n,m)|^{10}\right)^{1/2}\right)^{1/5} \\
&\leq (1+\tilde{c})||\eta||_{X_h},
\end{aligned}$$

with $\tilde{c}$ depending on N and M. Note that (3.4) follows from Hölder's inequality and (3.5) follows from (3.1). By this estimate, (3.2), and (3.3), we have

$$||\tilde{S}\eta||_{X_h} \leq (1+\tilde{c})||\eta||_{X_h}. \tag{3.6}$$

Recall that from the proof of Theorem 1.21, we were able to show that

$$||\Phi_{\eta_0}\eta||_{X_h} \leq c\delta_0 + Cr^5$$

for $\eta \in B_r$. By choosing r and δ_0 small enough, we have

$$c\delta_0 + Cr^5 \leq \frac{r}{1+\tilde{c}}.$$

Hence,

$$||\tilde{S}\Phi_{\eta_0}\eta||_{X_h} \leq (1+\tilde{c})\frac{r}{1+\tilde{c}} = r$$

for $\eta \in B_r$, which implies

$$\tilde{S}\Phi_{\eta_0} : B_r \to B_r.$$

Proof(ii): Since $\tilde{S}$ is linear, (3.6) implies that if $\mu, \nu \in X_h$, then

$$||\tilde{S}\Phi_{\eta_0}\mu - \tilde{S}\Phi_{\eta_0}\nu||_{X_h} \leq (1+\tilde{c})||\Phi_{\eta_0}\mu - \Phi_{\eta_0}\nu||_{X_h}. \tag{3.7}$$

From the proof of Theorem 1.21(ii), if $\mu, \nu \in B_r$, then

$$||\Phi_{\eta_0}\mu - \Phi_{\eta_0}\nu||_{X_h} \leq Cr^4||\mu - \nu||_{X_h}. \tag{3.8}$$

If we choose r small enough so that $Cr^4 < \dfrac{1}{1+\tilde{c}}$, then (3.7) and (3.8) imply

$$||\tilde{S}\Phi_{\eta_0}\mu - \tilde{S}\Phi_{\eta_0}\nu||_{X_h} \leq \lambda||\mu - \nu||_{X_h},$$

with $\lambda = Cr^4(1+\tilde{c}) < 1$.

$\square$

Remark 3.3 In the definition of γ_h, if we fix $m \in \mathbf{Z}$ such that $|mh^3| \leq M$, then $\gamma_h(n,m) = 0$ only if $|nh| > 2Nh^{-1/5}$. This was necessary to insure (3.1), which led to (3.6) independent of $h > 0$. If we assume h is bounded below, then the $h^{-1/5}$ can be discarded, i.e., we can select γ_h with $\gamma_h(n,m) = 0$ only if $|nh| > 2N$.

Hence, $\tilde{S}\Phi_{\eta_0}$ is a contraction mapping on $B_r \subset X_h$ for some $r > 0$. By the Contraction Mapping Principle, there exists a unique $\tilde{\eta} \in B_r$ such that

$$\tilde{S}\Phi_{\eta_0}\tilde{\eta} = \tilde{\eta}$$

which implies

$$\gamma_h\Phi_{\eta_0}\tilde{\eta} = \tilde{\eta}.$$

Let $A = N/h \in \mathbf{N}$. If $(n,m) \in \mathbf{Z} \times \mathbf{Z}$ such that $|n| \leq A$ and $|mh^3| \leq M$, then

$$\Phi_{\eta_0}\tilde{\eta}(n,m) = \tilde{\eta}(n,m). \tag{3.9}$$

By definition of the operator Φ_{η_0}, (3.9) implies that $\tilde{\eta}$ solves $(KdV)^d_4$ for $-A+3 \leq n \leq A-3$ and $|mh^3| \leq M$. Since the definition of $\partial^3_{n,h}\omega(n,m)$ includes $\omega(n+3,m)$ and $\omega(n-3,m)$, it is necessary to have n satisfy $-A+3 \leq n \leq A-3$ so that when the linear operator (as in (1.1)) is applied to both sides of (3.9), we maintain equality.

3.2 The Coefficients $Q[n, m]$

In this section, we now turn to the actual implementation of the fixed point iteration. Fix $h > 0$ and let $\eta_0 \in l_h^2(\mathbf{Z})$. Recall the definition of the operator $H_h(m)$,

$$\begin{aligned}
H_h(m)\eta_0(n) &= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{e^{-inh\theta}(\hat{\eta_0})_h(\theta)}{(1+\operatorname{sgn}(m)i\sin^3(h\theta))^{|m|}}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-inh\theta}e^{-im\arctan(\sin^3(h\theta))}\frac{h\sum_{k\in\mathbf{Z}}\eta_0(k)e^{ikh\theta}}{(1+\sin^6(h\theta))^{|m|/2}}d\theta \\
&= \sum_{k\in\mathbf{Z}}\eta_0(k)\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-i(n-k)h\theta}e^{-im\arctan(\sin^3(h\theta))}\frac{1}{(1+\sin^6(h\theta))^{|m|/2}}d\theta \\
&= \sum_{k\in\mathbf{Z}}\eta_0(k)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-k)\theta-im\arctan(\sin^3(\theta))}\frac{1}{(1+\sin^6\theta)^{|m|/2}}d\theta \\
&= \sum_{k\in\mathbf{Z}}\eta_0(k)\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos((n-k)\theta+m\arctan(\sin^3(\theta)))}{(1+\sin^6\theta)^{|m|/2}}d\theta.
\end{aligned}$$

Definition 3.4 *For $n, m \in \mathbf{Z}$, let*

$$Q[n,m] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos(n\theta+m\arctan(\sin^3(\theta)))}{(1+\sin^6\theta)^{|m|/2}}d\theta.$$

It follows that

$$H_h(m)\eta_0(n) = \sum_{k\in\mathbf{Z}}\eta_0(k)Q[n-k,m]. \tag{3.10}$$

We can represent $\Lambda_h\omega(n,m)$ using these coefficients as well. Note that $Q[n,m]$ does not depend on h. These coefficients can be computed and stored for future use.

To begin, we choose the zero function as our starting point. Independent of r, this guarantees that we are starting in B_r. With $Q[n,m]$ precomputed for n and m

such that $|nh| \leq 2Nh^{-1/5}$ and $|mh^3| \leq M$, we compute the first iteration, $\tilde{\omega}_1$. Let

$$\begin{aligned}
\omega_1(n,m) &= \Phi_{\eta_0}(0)(n,m) \\
&= H_h(m)\eta_0(n) - \frac{1}{5}\Lambda_h\partial_{n,h}(0)(n,m) \\
&= H_h(m)\eta_0(n) \\
&= \sum_{k\in\mathbf{Z}} \eta_0(k)Q[n-k,m].
\end{aligned}$$

It follows that ω_1 solves the associated discrete linear equation. To obtain the first iteration, we need to multiply by the cutoff, i.e.,

$$\tilde{\omega}_1(n,m) = \gamma_h(n,m)\cdot\omega_1(n,m) = \tilde{S}\Phi_{\eta_0}(0)(n,m),$$

which still solves the linear equation for n and m such that $-A+3 \leq n \leq A-3$ and $|mh^3| \leq M$. Since $\tilde{\omega}_1$ is the solution to the associated linear equation, we check the computation's accuracy by means of maximum relative error,

$$\max_{|n|\leq A-3}\ \max_{0<m\leq Mh^{-3}} \left| \frac{\frac{\tilde{\omega}_1(n,m)-\tilde{\omega}_1(n,m-1)}{h^3} + \partial^3_{n,h}\tilde{\omega}_1(n,m)}{\tilde{\omega}_1(n,m)} \right| \tag{3.11}$$

and similarly for $m < 0$. We can also check how well it solves $(KdV)^d_4$ by considering

$$\max_{|n|\leq A-3}\ \max_{0<m\leq Mh^{-3}} \left| \frac{\frac{\tilde{\omega}_1(n,m)-\tilde{\omega}_1(n,m-1)}{h^3} + \partial^3_{n,h}\tilde{\omega}_1(n,m) + \frac{1}{5}\partial_{n,h}(\tilde{\omega}_1^5)(n,m)}{\tilde{\omega}_1(n,m)} \right| \tag{3.12}$$

and similarly for $m < 0$.

The second iteration $\tilde{\omega}_2$ is computed as above, i.e.,

$$\tilde{\omega}_2(n,m) = \gamma_h(n,m)\cdot\Phi_{\eta_0}\tilde{\omega}_1(n,m) = \tilde{S}\Phi_{\eta_0}\tilde{\omega}_1(n,m).$$

By induction,

$$\tilde{\omega}_k(n,m) = \gamma_h(n,m)\cdot\Phi_{\eta_0}\tilde{\omega}_{k-1}(n,m) = \tilde{S}\Phi_{\eta_0}\tilde{\omega}_{k-1}(n,m)$$

for $k \in \mathbf{N}$. At each stage, we compute the maximum relative error by replacing $\tilde{\omega}_1$ with $\tilde{\omega}_k$ in (3.12) for $m > 0$ and $m < 0$. Finally, we compute the X_h-norm of each iteration. Recall, this involves three size estimates,

$$||\tilde{\omega}_k||_1 = ||\tilde{\omega}_k||_{l^5_n l^{10}_{m,h}} = \left(h \sum_{|n|\le 2Nh^{-6/5}} \left(h^3 \sum_{|m|\le Mh^3} |\tilde{\omega}_k(n,m)|^{10}\right)^{1/2}\right)^{1/5},$$

$$||\tilde{\omega}_k||_2 = ||\partial_{n,h}\tilde{\omega}_k||_{l^\infty_n l^2_{m,h}} = \max_{|n|\le 2Nh^{-6/5}} \left(h^3 \sum_{|m|\le Mh^3} |\partial_{n,h}\tilde{\omega}_k(n,m)|^2\right)^{1/2},$$

and

$$||\tilde{\omega}_k||_3 = \sup_{m\in\mathbf{Z}} ||\tilde{\omega}_k(\cdot,m)||_{l^2_h} = \max_{|m|\le Mh^3} \left(h \sum_{|n|\le 2Nh^{-6/5}} |\tilde{\omega}_k(n,m)|^2\right)^{1/2}.$$

3.3 A Few Examples

In this section, we give a few examples of the convergence and divergence of the operator Φ_{η_0}. Using Mathematica, we have computed $Q[n,m]$ for $|n| \le 32$ and $|m| \le 32$. Hence, our iterates will be defined on $|n| \le 16$ and $|m| \le 32$. In the following examples, the step size h will be no smaller than 0.1. By Remark 3.3, we can set

$$\gamma_h(n,m) = \begin{cases} 1 & |n|\le 8, |m|\le 32 \\ \frac{n}{8}+2 & -16\le n\le -8, |m|\le 32 \\ -\frac{n}{8}+2 & 8\le n\le 16, |m|\le 32 \\ 0 & \text{else.} \end{cases}$$

This implies that we only consider the points $\{(n,m) : |n| \leq 5 \text{ and } |m| \leq 32\}$ for the solution of $(KdV)_4^d$. Since we are limited in the number of coefficients available, we will choose our initial data in each example to be compactly supported near the origin.

Example 3.5 Suppose our initial data is given by the function

$$u_0(x) = \begin{cases} 0.5x + 0.5 & -0.5 \leq x \leq 0 \\ -0.5x + 0.5 & 0 \leq x \leq 0.5 \\ 0 & |x| \geq 0.5. \end{cases}$$

Note that u_0 is an even function. It is easy to see that if $w(x,t)$ solves the associated linear problem (H) (see pg. 4), then so does $w(-x,-t)$. Furthermore, if $u(x,t)$ solves $(KdV)_4$, then $u(-x,-t)$ does as well. By uniqueness, $u(-x,-t) = u(x,t)$. In fact, this symmetry property should be true for all the iterates of the operator S in (3), since the right-hand side of (IH) (see pg. 4), namely $g(x,t)$, will have the property that $-g(x,t) = g(-x,-t)$. Recall that we choose the zero function as our starting point. It follows that

$$S^{(n)}(0)(x,t) = S^{(n)}(0)(-x,-t) \tag{3.13}$$

for all $x, t \in \mathbf{R}$ and $n \in \mathbf{N}$.

Let $h = 1$. To begin, we need to discretize our initial data. We do so by choosing the values of u_0 at the integers. Thus, our discrete initial data is

$$\eta_0(n) = \begin{cases} 0.5 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

In this example, we run four successive iterations. Recall that the first iteration solves the associated linear equation. To measure the accuracy of the computation, we compute the the maximum relative error of the first iteration with respect to the linear equation using (3.11) with $A = 8$ and $M = 32$. The result is 8.62601×10^{-11}. All three norms of each iteration and the maximum relative error (MRE) of each iteration with respect to the $(KdV)_4^d$ equation are listed in Figure 3.6. For the last three iterations, we include the distance of the current iteration from the previous one measured in the X_h-norm under the column with heading "distance". As in the continuous setting, we have

$$(\tilde{S}\Phi_{\eta_0})^{(k)}(0)(n,m) = (\tilde{S}\Phi_{\eta_0})^{(k)}(0)(-n,-m) \tag{3.14}$$

for all n and m on our grid and $k \in \mathbf{N}$. One can see from the norms that these iterations are converging very rapidly. Since the relative error is becoming extremely small, we can conclude that the iterations are converging to a solution of $(KdV)_4^d$.

iteration	MRE	norm one	norm two	norm three	distance
1	0.00094209	0.516778	0.409208	0.5	
2	0.0000615788	0.516887	0.408461	0.5	0.03331
3	3.87118×10^{-7}	0.51688	0.408455	0.5	0.0000105811
4	2.43126×10^{-9}	0.51688	0.408455	0.5	5.08566×10^{-8}

Figure 3.6

Example 3.7 Let the initial data u_0 be as in Example 3.5. However, in this example, we set $h = 0.5$. If $\eta_0(n) = u_0(nh)$, then

$$\eta_0(n) = \begin{cases} 0.5 & n = 0 \\ 0 & n \neq 0, \end{cases}$$

which is the same discrete initial data as in Example 3.5. One might think that we should get the same iterations as well. However, recall that the operator Λ_h is dependent upon h. This influences each iteration.

As in the previous example, we compute the first four iterations (starting from the zero function) and their relevant data (see Figure 3.8). Once again, these iterations have the property mentioned in (3.14). By considering the distance estimates between successive iterations and the size of the maximum relative error, one can see that the iterations are converging exponentially to the solution of $(KdV)_4^d$ guaranteed by Theorem 3.2. See Figure 3.8.

iteration	MRE	norm one	norm two	norm three	distance
1	0.0188406	0.365417	0.289354	0.353553	
2	0.0000305815	0.365431	0.289222	0.353553	0.0231647
3	4.78587×10^{-8}	0.365431	0.289222	0.353553	4.6644×10^{-7}
4	6.90206×10^{-10}	0.365431	0.289222	0.353553	7.68505×10^{-10}

Figure 3.8

Example 3.9 Suppose our initial data is given by the function

$$u_0(x) = \begin{cases} 1 & -1 \le x \le 0 \\ -1 & 0 \le x \le 1 \\ 0 & |x| \ge 1. \end{cases}$$

Unlike the previous example, this function is discontinuous. However, it does have finite left and right-hand limits for all $x \in \mathbf{R}$. Assuming $h = 1$, let

$$\eta_0(n) = \frac{1}{2}\left(\lim_{\delta \to 0^+} u_0(n+\delta) + \lim_{\delta \to 0^+} u_0(n-\delta)\right),$$

i.e.,

$$\eta_0(n) = \begin{cases} 0.5 & n = -1 \\ 0 & n = 0 \\ -0.5 & n = 1 \\ 0 & \text{other} \end{cases}$$

for $n \in \mathbf{Z}$. Also, note that u_0 is an odd function. In this case, if $w(x,t)$ solves the associated linear equation (H), then so does $-w(-x,-t)$ and if $u(x,t)$ solves $(KdV)_4$, then so does $-u(-x,-t)$. Hence, $u(-x,-t) = u(x,t)$. By a similar argument as before, this is also true of all the iterates of the operator S.

As in the previous example, our initial guess is the zero function. However, in this example, we compute the first seven iterations. This data illustrates the odd reflexivity noted above. Notice that for iterations five through seven, the only column which changes is the "distance" column which becomes extremely small (see Figure 3.10). This implies that by the fourth iteration, we are very close to the fixed point of our operator. Apparently, the best we can hope for in this example in terms of the relative error is 1.04304×10^{-8}.

iteration	MRE	norm one	norm two	norm three	distance
1	0.0099275	0.587994	0.737931	0.707107	
2	0.000044212	0.588019	0.738561	0.707107	0.036962
3	1.35837×10^{-7}	0.588019	0.738563	0.707107	5.61944×10^{-6}
4	1.04304×10^{-8}	0.588019	0.738563	0.707107	2.61774×10^{-8}
5	1.04309×10^{-8}	0.588019	0.738563	0.707107	1.37671×10^{-10}
6	1.04309×10^{-8}	0.588019	0.738563	0.707107	6.09476×10^{-13}
7	1.04309×10^{-8}	0.588019	0.738563	0.707107	2.2745×10^{-15}

Figure 3.10

Example 3.11 Again consider the initial data from Example 3.9, but with $h = 0.1$. Hence,

$$\eta_0(n) = \begin{cases} 0.5 & n = -10 \\ 1 & -9 \leq n \leq -1 \\ 0 & n = 0 \\ -1 & 1 \leq n \leq 9 \\ -0.5 & n = 10 \\ 0 & \text{other.} \end{cases}$$

Again, we compute seven iterations. Unlike the previous examples, the maximum relative error of the first iteration is quite large, approximately 5.9. This is mainly due to the larger data which can be seen from the norms in Figure 3.12 and the smaller h. However, by the seventh iteration the maximum relative error is once again very small.

iteration	MRE	norm one	norm two	norm three	distance
1	5.90663	0.93733	0.834393	1.33463	
2	2.00782	0.932334	0.838427	1.31646	0.186061
3	0.0562563	0.932087	0.838494	1.31646	0.00255853
4	0.00457199	0.932068	0.838484	1.31646	0.0000613827
5	0.000312503	0.932067	0.838483	1.31646	4.0051×10^{-6}
6	9.41672×10^{-6}	0.932067	0.838483	1.31646	2.27996×10^{-7}
7	2.4938×10^{-6}	0.932067	0.838483	1.31646	1.31978×10^{-8}

Figure 3.12

Example 3.13 In the previous example, the initial data was larger than in previous examples. This seemed to lead to slower convergence of the iterations. Can the initial data be too large to yield convergence? With $h = 1$, we consider the following initial data,

$$\eta_0(n) = \begin{cases} 0 & |n| > 5 \\ n & -5 \le n \le 5. \end{cases}$$

After two iterations, it is evident that the iterations in this example are diverging in the space X_h (see Figure 3.14). This suggests that the condition in Theorem 1.21 that the initial data be small is necessary.

iteration	MRE	norm one	norm two	norm three	distance
1	128.373	8.07398	6.46922	10.4881	
2	1.44893×10^{12}	850.296	1000.54	1071.73	1070.25

Figure 3.14

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