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MODELLING AND ANALYSIS OF
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MODELLING AND ANALYSIS OF TOKEN-PASSING COMPUTER NETWORKS

By

Vernon J. Rego

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ABSTRACT

MODELLING AND ANALYSIS OF TOKEN-PASSING COMPUTER NETWORKS

By

Vernon J. Rego

This thesis is concerned with the performance modelling of token-passing protocols on local area computer networks. A token-passing system operating on a baseband cable may be viewed in a queueing-theoretic framework. The computer stations on the system represent queueing stations at which randomly arriving packets queue for service. A free token behaves as a single server who travels around the network, allowing each station that is visited a chance to make a transmission on the channel. The random time taken by a free token to cycle around an operational network, called the token's cycle-time, is an analytically useful measure. The derivation of cycle-time distributions for general asymmetric N-station systems is a primary contribution of this research. Both exact and approximate forms of this distribution are derived. The exact approach is based on general distributions for packet arrival, service, and token-passing times, while the approximate approach uses a Poisson arrival assumption.

In modelling performance, the exact and approximate cycle-time distributions lead to exact and approximate queueing measures, respectively. Exact measures are obtained by using Poisson packet

arrivals and semi-Markovian transmission times. Some results on the invariance and insensitivity of cycle-times are obtained, along with distributions for busy and vacation periods of the channel, with respect to each station. Approximate measures are obtained via an independence assumption that leads to single server queues with dependent service. This is due to serial cycle-time dependencies which arise at all but extreme loads. Methods of analyzing such queues, including schemes for estimating service-time variance, covariance matrices, marginal and joint cycle-time distributions, and correlation effects are discussed.

Under certain conditions, it is shown that limiting and asymptotically stationary cycle-time distributions exist. Conditions for stationarity and system stability are derived, and simple stability measures are introduced. Methods for obtaining the distribution of system throughput and a new fairness measure are described. The link between multiqueueing systems with different service and queue emptying disciplines is demonstrated with the aid of a complex service discipline for which approximate cycle-time distributions are derived.

Dedicated to

Stanislaus & Felicita

& the rest of the clan

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LIST OF SYMBOLS

S	set of N stations
W	set of N walks
i,j	station indices
$A_j(.)$	customer interarrival distribution, mean $1/\lambda$
$B_j(.)$	customer service distribution, means μ, β
$U_j(.)$	server's walk distribution, mean α
$S_j(.)$	server's switching distribution, mean γ
Q	Markov kernel of transition functions
P	Markov probability transition matrix
$\mathbf{P}^*$	reordered form of **P**
$\mathbf{P}_j$(s/d)	station j's scan/dep cycle view of **P**
$\mathbf{P}_i$	sub-matrix of **P**, $i = 0,1$
π_j	limiting Markov state probability, element of Π
ϕ_j	limiting semi-Markov state probability, element of Φ
Θ	set of N-bit binary vectors
p, q	probability constants
Z_n	random variable, representing state of Markov process at n^{th} transition
T_n	time instant corresponding to n^{th} transition
z,x,y	fixed states in Θ
$p(.,.)$	probability of corresponding transition
$p_{i,k}$	probability of corresponding transition
M, **R**	shift operators, for vector rotation

$\delta_j(k)$	k^{th} entry of vector j
$\upsilon_k(z)$	k^{th} entry of vector z
$\gamma_j(m,j)$	$(j+m-1)$ mod N
C	cycle-time random variable
$C(z)$	random cycle-time generated by service vector z
$V(z)$	service time contributions to cycle-time during cycle generated by vector z
$T_i(.,.)$	cycle-time w.r.t i for corresponding transition
J_i	time spent by server at station i
$\xi_i(.,.)$	probability that station i makes corresponding transition
$p(.,.,.)$	probability of specified number of arrivals
$F_X(.)$	distribution function of X
$d(z)$	decimal representation of binary number z
$\mathbf{N}$, $\mathbf{E}$	transition matrices, from Θ_0, Θ_1
$\mathbf{Q}'$	infinitesimal generator of Markov process
$\mathbf{T}$	nonsingular square matrix
$\mathbf{e}$	m-vector with only unit entries
$\mathbf{a}$	$2N$-bit initial vector
ν	vector of initial probabilities, m elements
S(t)	state of chain $\mathbf{N}$ at time t
π(t)	phase of chain $T_{S(t)}$ at time t
$\mathbf{T}'$	infinitesimal generator of process $\{S(t),\pi(t)\}$
$\mathbf{A}^*$	$2mN$ square block-partioned matrix
$\mathbf{T}^*$	$2mN$ square block-diagonal matrix
$\mathbf{D}$	matrix of service-times
$\mathbf{E}^*$	matrix describing reference station customer transitions

ζ	limiting vector of $\mathbf{E}^*$
η	vector of mean service-times
$\eta^{(2)}$	vector of second moments of service-times
ρ	traffic intensity
A(t)	matrix of transition functions for reference station customers
<u>A(s)</u>	Laplace-Stieltjes transform of A(t)
$A_k(x)$	probability of k arrivals in time x
$B_k(x)$	probability of k arrivals in vacation period plus exponential time
$\mathbf{x}$, $\mathbf{y}$	queue length distributions at scan instants and arbitrary time, respectively
$\{G_n\}$	sequence of increasing matrices, converging to G
G	stochastic matrix, solution to nonlinear matrix equation
$\mathbf{g}$	invariant vector of G
$\mathbf{X}(t)$	probability generating function for $\mathbf{x}$
$\mathbf{Y}(t)$	probability generating function for $\mathbf{y}$
$\mathbf{W}(s)$	Laplace Stieltjes transform for delay distribution
$\mathbf{W}(t)$	distribution of delay time
L	mean queue length
Π	matrix having all rows as ζ
U^*	throughput random variable
$H_{ij}(.)$	conditional transition time distribution function
$h_i(.)$	sojourn time in state i
$f_C(.)$	cycle-time density
$F_C(.)$	cycle-time distribution
$G(z)$	geometric transform of queue length density

$L[f]$	Laplace-Stieltjes transform of f
r_{0j}	probability that reference station is empty
X'	token's (mixture) holding time at station i
X^*	sum of all token (mixture) holding times at stations
Y^*	sum of all token passing (walk) times
a_{j0}	$p_{j0}\mu_0$
a_{j1}	$q_j\mu_1$
a_0	symmetric form of a_{j0}
a_1	symmetric form of a_{j1}
d_N	a_0^N
e_N	a_1^N
ξ, ζ	inversion coefficients, chapter V
$\{C_n\}$	sequence of random cycle-times
C_i	time for the i^{th} cycle
$p_j^{(i)}$	probability queue j nonempty during cycle i
$F_j^{(i)}$	holding time distribution of token in state j, during cycle i
$f_{i/j}$	conditional density, cycles i and j
$f_{i,j}$	joint density, cycles i and j
$\mathbf{p}^{(m)}$	N-vector, with entries $p_j^{(m)}$
p_1, q_1	probability constants
$\mathbf{p}$	$(p_1,\ldots\ldots,p_N)$
$\mathbf{q}$	$(q_1,\ldots\ldots,q_N)$
S_q	quasi-service time random variable
$\{X_n^{(i)}\}$	moving average of level i iterations
$\rho_k^{(i)}$	lag k correlations for level i iterations

c_{ij}	correlation between cycles i and j
$L_j(z)$	geometric transform for queue length distribution at station j
r_{nj}	queue length distribution at station j
L_j	mean queueing time
M_j	token-intervisit time
$E(C)$	mean cycle-time
$E_j(C)$	mean cycle-time conditioned on station j's service
λ_j^*	critical arrival rate for station j customers
Λ	variable vector of arrival rates
Λ^*	critical vector of arrival rates
$I(\Lambda^*)$	general index of stability
$I^S(\Lambda^*)$	index for stable system
D_j	cycle-time conditioned on station j's service
R_1	maximum of conditional cycle-times
R_2	minimum of conditional cycle-times
ω_i	ratio of i^{th} moments of conditional cycle-times
Ω_i	$a_i\omega_i$, for weights a_i
J, K	random times between transmit events, switch events, respectively
F_J, F_K	distribution functions for J, K, respectively
$\{S_n^*\}$	rarefied point process
$F^{(*i)}$	i-fold convolution of F
C_1, C_2	cycle-times for first and second cycles of ATP process
A	set of recently determined active stations
Y_j'	scheduled walk-time in ATP
a_i'	expected value of Y_i'

S' set of scheduled walk-times

S^* union of sets S and W

S^* union of sets S, W, and S', chapter VIII

$B_j'(.)$ distribution of $(Y_i' + X_i)$

T_j^* trip-time random variable

F_{Tj^*} distribution of trip-time

p_{0j} probability queue j is empty during first cycle

p_{1j} probability queue j has exactly one customer queued during first cycle

q_j $(1 - p_{0j} - p_{1j})$, chapter VIII

$\delta_0(.)$ dirac delta function

CHAPTER I

INTRODUCTION

In a local area network, channel usage is regulated by a protocol operating at the data-link layer, one of the seven layers of the ISO reference model [Zimm80]. A channel access protocol is an algorithm specifying how a distributed set of stations must share a common communication medium. Broadly speaking, these protocols are of two types: contention-based and conflict-free. Examples of the first type are the ALOHA [Abra70], CSMA [Toba74, KlTo75], and CSMA/CD [ToHu80]. Among these, the CSMA/CD is the most commonly used medium access control method for a local area network using a contention bus topology. The well known Ethernet local network [MeBo76, MeBTC77] is a CSMA/CD protocol that was developed and patented by Xerox. Performance models for the CSMA/CD protocols can be found in [ToHu80, Lams80]. The contention-free protocols based on token-passing methods or various reservation methods try to avoid the contention drawback. An excellent survey of these, including descriptions of several access, initialization, and synchronization methods, can be found in Penney and Baghdadi [PeBa79].

1.1 Simple Problem Statement

The operation of a token-passing protocol can be described independently of the local network topology in an abstract fashion. Consider a system of N independent queueing stations that are separated from one another by unequal distances. Customers arrive in a random fashion and queue at any one of the N stations for service. Arrivals between queues, as well as within queues are independent. In general, any two customers queued at a given station will possess the same arrival and service characteristics. This is not necessarily true of two customers queued at two different stations. We will always assume that there is no restriction on the length of a queue. A single server is required to visit the N queueing stations in a certain sequence, serving customers at each station. Within each station, customers are always served in a first-in first-out mode. Upon completing a station's service, the server takes a finite time to walk from one station to the next station in the sequence. We assume that the system is perfectly reliable at all times and that the server is in perpetual motion, either serving a customer at a queue, bypassing an empty queue, or walking between queues. Sometimes it is necessary to account for the time it takes the server to detect an empty queue and consequently bypass that queue. This time, called the station's *switching time*, does not include the time it takes the server to walk to the next queue.

The system is said to operate under a *simple service scheme* if the sequence of queueing stations visited by the server is strictly cyclic, i.e. $1, 2, \ldots, N, 1$. Simple service schemes may be either nonexhaustive or exhaustive. The simple nonexhaustive scheme called a *fair service discipline* is one in which service can be provided to at most one customer per station queue on each server visit to that

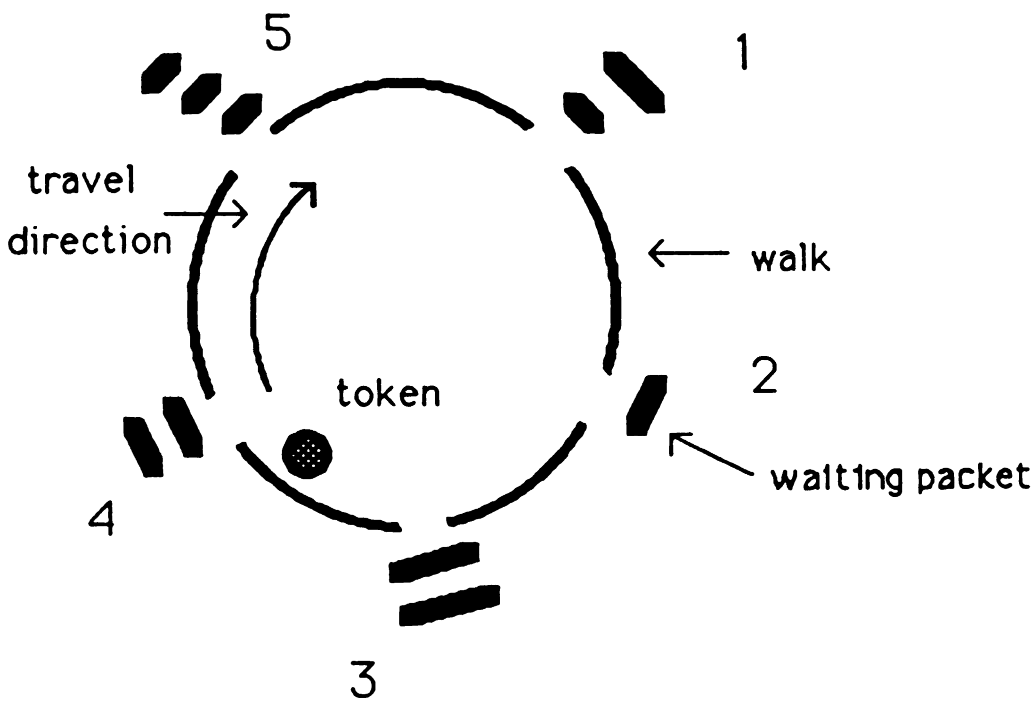


Fig.1 Conceptual view of Simple service schemes

station. Thus, a system may be completely described by specifying N, the service scheme, the arrival rates and service rates of customers at each queue, and the mean time it takes the server to walk from one queue to the next in the cyclic sequence. Given such information, it is instructive to obtain answers to both qualitative and quantitative questions regarding the behaviour of such a system under general conditions. The conceptual queueing model can be seen in Fig. 1. Given a system of N independent queues with a single server obeying the fair service discipline, and also given probability distributions describing customer interarrival and service times at the various queues and the random walk times between queues, the problem is to find methods of analysis that will enable us to obtain measures of queue lengths, response times, server utilization, effectiveness of the service discipline, improvements in service patterns, conditions under which queues remain stable etc. A formal definition of the problem is deferred to the material in chapter II.

1.2 Token-Passing Protocols

The token-passing (contention-free) protocols on ring and bus networks are two of the three access mechanisms presently being standardized by the IEEE Standards Committee [IEEE84a, IEEE84b]. In principle, the token-passing protocols, like the Newhall networks [FaNe69], use a token to regulate channel access. The station that has received the token is allowed to access the channel. If the station is ready to transmit and has a packet stored in its buffer, it immediately puts the packet onto the channel. Upon completing its transmission, a station passes the token in an orderly fashion to the next station on the channel. If a station has no packet to send when it acquires the token, it simply passes the token along to the next station. In either event, a cyclic token-passing sequence of stations is defined.

A token ring [IEEE84a, ECMA83a] is typically configured as a series of point-to-point cables between consecutive stations, with stations tapping onto the ring using active interfaces. The token is a unique signalling sequence of bits that circulates on the communication medium in one of two states. A station that wants to transmit listens to the network, and attempts to identify from each sequence of bits it receives the special bit pattern corresponding to the token. The last bit of the token's bit pattern may be either a *0* or a *1*, corresponding to *busy* and *free* token states, respectively. Any station that detects the token in the *free* state may capture the token, change it to a *busy* token (by inverting the last bit), and append a number of informative bits that go to make up a variable length *packet*. These bits include appropriate control and address fields, the data field set up by the logical link control sublayer, the frame check sequence, and the frame-ending delimiter. The busy-token and packet combination is read and forwarded one bit at a time (since the topology is point-to-point) by consecutive stations on the ring, and only the destination station copies each bit of the packet as it passes. In an attempt to detect a free token, each station must pass a sequence of bits, including its most recently acquired bit, through a pattern matching circuit. Only in the instance of a free token will the last bit be inverted. In any event, this token-detection mechanism causes a *1*-bit delay within each station. When a station's transmission is complete, the sending station performs certain tasks to ensure proper operation and then creates a new free token, which it passes on to the next station in the cyclic token-passing sequence. The sending station removes the packet from the ring when the cycle is complete.

The token-passing bus is conceptually very similar to the token-passing ring [Buxw84]. A token bus [IEEE84b, ECMA83b] is configured as a passive medium, generally a long unbranched trunk, with stations tapping onto it via stubs in a multidrop fashion. The bus topology does not impose a sequential ordering of stations, as in the case of the ring. Thus, the token is made to circulate on a logical ring instead of a physical one, with a sequence of station addresses defining the token's path. A bus operates in broadcast mode, in which a station's transmission can be heard by all stations on the bus. Token bus protocols take advantage of broadcast mechanisms in executing the difficult tasks of establishing and maintaining the logical ring. Each station on the ring is required to know its predecessor and successor. In steady-state, the protocol is seen to alternate between packet broadcasts to destination stations, and token-passing broadcasts.

Within the framework of token-passing just described, there is room for flexibility in protocol design. For example, in the token ring protocol, two modes of token operation are possible. The description above specifies that a new token is generated by the transmitting station *only after* the busy-token header of the packet is removed from the ring. This is called the *single token* rule [Buxw81]. In a *multiple* *token* strategy, the transmitting station may issue a token *before* it receives the previous busy-token, thus allowing for more than one token on the ring at the same time [BCJK82]. This reduces the idle time on the ring that occurs in the single token method. For rings with small delay (i.e., the time taken for a single bit to make a complete cycle on the ring), the performance of both methods is approximately the same [AnSc82]. However, since efficiency of the single token mode is generally close to that of the multiple token mode and its complexity is far less, especially when considering fault tolerance, the single token mode is more popular. In token-passing protocols, a more important design issue is the maximum length of time that a station is allowed to retain control of the transmission medium, i.e., a station's *channel-retention time*. In the following section, the channel-retention time of a token-owning station is identified with a finite number of consecutive packet transmissions that this station is allowed to make before relinquishing the token. Hence, this time will depend on both the time to transmit a packet, and the number of consecutive packet transmissions permitted for that station. In general, the channel-retention times of the different stations may be allowed to vary.

1.3 Classification Of Multiqueueing Systems

In queueing terminology, token-passing systems behave like multiqueues with a single shared server. The token represents the server and the station buffers with packets represent queues of customers. The set of stations forms a logical ring of independent queues. The terms *service scheme* or *service discipline* are used to describe the sequence in which the server attempts to serve stations. Recall that this idea was already introduced in section 1.1, where stations were serviced in the order *1, 2, ... N, 1, ...* etc.. Models of token-passing systems generally assume service schemes that belong to one of three categories. The first category consists of the early models that place no limit on any station's channel-retention time (*simple exhaustive* service). These models were given considerable attention in the literature. The second class consists of models dealing with systems exhibiting *simple nonexhaustive* service. One such scheme is the *fair service* discipline described in section 1.1. Another example is the *gated service* discipline, where the server only serves those customers that are found present in the queue when the server arrives at the queue. That is, new arrivals are required to wait for the next round of service. This service discipline has also attracted special attention in the literature, and probably so due to analytic difficulties that crop up with other nonexhaustive service patterns [Buxw84]. Ferguson and Aminetzah [FeAm85] suggest that the reason for the relatively large amount of research effort directed at performance modelling of exhaustive service systems is due to the inferiority of gated system waiting times in contrast to waiting times in exhaustive service systems. Yet another scheme is one in which the server is allowed to serve a different number of customers at each queue. Though this may lead to unfair service, distributing the shared server among queues with different demands in a manner that is sensitive to changes in station loads often improves server utilization.

The third class of models are those involving *complex service disciplines*. With such disciplines, it is no longer necessary for service to be strictly cyclic, nor is it necessary for the sequence of stations visited by the server to be static. A *complex adaptive nonexhaustive* service discipline is introduced in chapter VIII. The server attempts to adjust the path of the service cycle in order to account for imbalances in station loads. Complex disciplines generally involve service schemes that are not fixed, but change according to a criterion, such as the minimization of station response times. These disciplines are usually hard to analyze and have not received much attention in the literature. Additionally, conditions under which one kind of server behaviour can be shown to perform better (i.e., attain nearer optimality of the criterion) than another kind of server behaviour are difficult to obtain.

In order to further classify different service schemes, we define a station's channel-retention time in terms of an upper bound on the number of consecutive packet transmissions that the station is allowed at any one instance of token acquisition. This instant is defined to be the instant at which a station obtains and recognizes a *free* token. The bound is determined by the station's *queue emptying discipline*, or QED. The QED determines the maximum number of packets that can be transmitted by a station whenever the station acquires the token. The time taken by the token to make successive reappearances at a station is defined to be the *cycle-time* of the token with respect to the station. The token's arrival instant at a station during a cycle is called the *scan instant* of the station for that cycle. The instant that the token leaves a station during a cycle is called the *departure instant* of the station for that cycle. Observe that the token's cycle-time can be defined either with respect to scan instants or departure instants, and both definitions are consistent.

A QED associated with a scan instant is called an *s*-QED. With the discipline *s*-QED = *n*, the number of transmissions made by a station is *min(n,x)*, where *x* is the number of waiting packets recorded by the token at the station's scan instant. A QED not associated with a scan instant is called an *r*-QED, where *r* is used to denote *relaxation* of the scan condition. With the discipline *r*-QED = *n*, at most *n* transmissions are allowed. In this discipline, packets arriving after the scan instant but while previous arrivals are being served may still be transmitted within the batch of *n*. When *n* is finite, both QED's require that transmissions from the stations be *nonexhaustive*. The scheme *s*-QED = ∞ is really a nonexhaustive transmission mode since *x* is always finite and packets arriving after this scan instant have to wait for the next cycle. With *r*-QED = ∞, *exhaustive* transmissions are allowed. The above classification holds for the cases of both finite and infinite queues. If the queue capacity (buffer size) is finite, say some positive integer *b*, then the disciplines are specialized to *s*-QED(*b*) and *r*-QED(*b*). With *s*-QED(*b*) = *n*, the number of allowable transmissions is *min(b,n,x)*. Note that since $x \le b$, this number will always be the minimum of *x* and *n*. If the flow control process of a network node is clever enough to make use of finite (data-link level) buffer space even while packets are being transmitted from this node, then the scheme *r*-QED(*b*) is equivalent to *r*-QED = ∞. This is to say that a finite queue capacity does not preclude the possibility of new packet arrivals at a station while old packets are being transmitted, with arrivals and transmissions overlapping in a continuous stream.

Given that the service discipline and QED of a multiqueueing system remain fixed while the system is in operation for a period of interest, and also assuming that each station utilizes the same QED, these systems can be further classified in terms of station parameters. In describing multiqueues we use random times to model arrival times, service times, and walk times. In general, station i can be described in terms of n_i parameters, with n_i not necessarily equal to n_j, for $i \neq j$. In the present situation, $n_i = n_j = 3$.

If the customer arrival processes at all queues are identical, the system arrival process is called *symmetric*; otherwise, the arrival process is called *asymmetric*. This notion of symmetry applies also to the service distributions and walk distributions. The concept associates more readily to a *model* of a queueing system than the real process itself. Nevertheless, if a token-passing configuration possesses uniform traffic arrival characteristics we apply a *symmetric arrival* model, and otherwise, an *asymmetric arrival model*. Similarly, if all service requirements are identical, a *symmetric service* model is applied, and otherwise, an *asymmetric service* model. This holds also for the walk processes. A *symmetric* system is one in which each of the arrival, service, and walk distributions is the same for all stations on the network, thus requiring a model that is characterized by only three distributions. On the other hand, an *asymmetric* system is characterized by *3N* distributions. *Partially symmetric* systems are ones which lie somewhere in between the two extremes, with some stations characterized by common distributions while others differ.

1.4 Review Of Previous Work

A considerable amount of work on performance comparisons via modelling of the various channel access schemes including the token-passing schemes has been done in the past [ArSt82, Buxw81, LiHG82, Stuc83]. Most of the literature on protocol performance modelling assumes some combination of the characteristics of symmetry, s-QED = n, or r-QED = n, with either $n = \infty$, or $1 < n < \infty$. The more realistic asymmetric configurations with QED = 1 appears to have been largely neglected. One possible reason for this is the inherent difficulty associated with processes not exhibiting independence, and the computational intractability of asymmetric multiqueue systems possessing queues with customer interference.

One of the earliest formal versions of the multiqueueing problem was presented by Liebowitz [Lieb61]. The arrival distributions are assumed to be Poisson, walk times are finite, and service distributions are arbitrary. The system developed was symmetric in all distributions, and switching times were not considered. Liebowitz was interested in the stationary probability distribution of the the random number of customers found waiting (by the server) at any station at its scan instants. Utilizing the s-QED = ∞ discipline (gated service), an approximate solution was obtained. The result is an approximation because of an assumption of independence, namely that "stochastic processes within a particular queue are considered independent of the processes within the other queues". Liebowitz [Lieb68] cited his problem as an important queueing problem for which no exact solution had been obtained. Eisenberg [Eise72] discovered that the different queue states at scan times could be modelled as a Markov process and consequently proposed the first general solution to the problem. Transform solutions were obtained for customer waiting times and server intervisit-times at each queue. Unfortunately, Eisenberg's transform solutions turned out to be difficult to work with. More recently, Ferguson and Aminetzah [FeAm85] developed an exact solution to this problem, obtaining the mean customer waiting times (for each station) on a general asymmetric system. A closely related model in which the server uses the visit sequence $1,2,\ldots.,N,N-1,\ldots.2,1$ has been developed by Swartz [Swar81] with an application to disk service policy. Swartz obtains average queue contents and average station intervisit time. Another general model involving the scheme S-QED = 1, for both symmetric and asymmetric arrivals has been examined by Kuehn [Kueh79] in an approximate fashion, improving on a result of Hashida and Ohara [HaOh72]. Kuehn uses an independence assumption that leads to an approximate solution for mean queue lengths and waiting times. The solution does well in the low and high traffic regions but weakens in between, especially with an increasing number of queues or increasing variance in service times. However, Kuehn's presentation was a first view of the problem as a multiqueue problem subject to different queue emptying disciplines.

Many models have been proposed in the literature for variations on the multiqueue problem. The chief difference between the Hashida-Ohara/Kuehn model and these is that the variations often involve a simplifying assumption of dependence between the customer arrival and service processes. The most common assumption is one which restricts the station buffers to hold at most one packet at a time ($b = 1$), additionally requiring that a new packet may be generated only an independently random time after the present packet is transmitted. This reduces to a state dependent arrival process, or s-QED(1) = 1 (the machine interference problem). While this approach is a workable one, it follows that the generation of a new packet at a station depends on the availability of buffer space. In practice, a packet sent to a filled buffer signals the flow control process of the network layer to stop sending packets to the data-link layer. This packet is then lost to the data-link layer and must be regenerated. The time until a regenerated packet finds itself in the buffer is thus a random time that *depends* on the time some packet left the buffer via the transmission medium. If viewed in this fashion, the customer arrival times do not correspond to the points of a renewal process [Coxd62], since the interarrival times are not identically distributed and certainly not independent. The point to be noted is that the assumption described above will work under conditions of light load, but gradually tend to fail as the load is increased to the point where packet arrivals at the buffers are not independent.

One of the earliest applicable models of the s-QED(1) = 1 discipline in the machine interference context is given by Mack, Murphy, and Webb [MaMW57]. The model assumes a symmetric system, with constant walk and service times but zero switching times. The customer interarrival times (or machine running times) are exponentially distributed random variables. Expressions for the distribution of the number of failed machines per server cycle and server utilization were obtained. Mack also extends the results to the case of variable walk times [Mack57]. Using the same queueing configuration, Kaye [Kaye72] uses the Mack, Murphy, and Webb results to obtain an algorithm for the waiting time distribution of an arbitrary packet. An important point to be noted here is that Kaye was trying to model loop systems which typically allowed more than one packet to queue in a station's buffer. Thus, Kaye's approach was an approximation. However, Kaye conducted simulation experiments to argue that the number of packets lost in assuming that buffers could hold only one packet at a time was generally small. Bux [Buxw81] takes the same approximate approach in modelling the *S-QED = 1* token-passing system and justifies using the approximation with Kaye's simulation result on symmetric systems.

Another nonexhaustive service model, but in the setting of an M/M/1 system with two queues was supplied by Eisenberg [Eise79]. The solutions given require detailed calculations for the restricted two station model. Other variations on models for nonexhaustive transmissions have been proposed by Konheim [Konh76], Hamacher and Shedler [HaSh81], Wu and Chen [WuCh75], and Heyman [Heym83]. Heyman develops an approximation (independently of Kuehn's model) for mean packet delay in the multiqueue problem using simple nonexhaustive service, constant walk and service times, and zero switching times, with the intention of analyzing the performance of Fasnet [LiFl82].

In the case of exhaustive transmissions and symmetric arrivals, nearly all of the literature on protocol comparisons relies somewhat heavily on the results of Konheim and Meister [KoMe74]. These results are based on a discrete time approach using scan times and yield expressions for mean queue length and virtual waiting time for a stationary system. A review of the literature indicates that *performance models* for *s*-QED = *1* based token-passing protocols and ordered access protocols sometimes make questionable use [Buxw81, ChLL82, LiHG82] of the Konheim and Meister result. This result is meant for *roll-call polling* type systems or *exhaustive service* systems, where stations transmit until their buffers are emptied. Bux [Buxw81] argues that the exhaustive model and Kaye's model [Kaye72] (where arrivals and service are dependent) are equivalent since stations on large systems (more that *50* nodes) rarely see more than a single packet queued at any given time. The exhaustive model is then used to analyze the performance of token-passing ring and bus protocols. It must be noted that these two protocols are generally implemented with an *s*-QED = *1*, which differs considerably from the service schemes in [KoMe74] (*r*-QED = ∞) and [Kaye72] (*s*-QED(*1*) = *1*, or state dependent arrivals). The above reasoning thus equates one model to a second model with the intention of analyzing a third. Though the three models resemble one another, they also have differences that can lead to strong arguments as to the justifiability of equating them. The load that a station places on the system, i.e., the ratio of the station's mean arrival rate to the mean rate of service it receives from the system, plays a part in making one model appear to behave as well as another. The system load is defined to be the sum of the individual station loads. The key to the dilemma lies in the viability of the statement "exhaustive service is equivalent to *s*-QED(*1*) = *1*". The models are equivalent if and only if the statement (call it **X**) is true. Define a *heavily loaded* station to be one whose average buffer content exceeds one (e.g., 1.01). Let the number of such heavily loaded stations (on the average) be H, and let γ and κ be the mean queue lengths for the heavily loaded stations, and the whole system, respectively. Define L to be the load ratio L = $(\gamma * H)/(\kappa * N)$. Viewing **X**(L) as a Bernoulli random variable whose parameter L is a function of the parameters of the network, it would be appropriate to say that **X** has a higher probability of being true for smaller values of L. As L approaches 1, it is more and more likely that **X**(L) will be false due to packet loss. It is possible to envisage situations where N is large but only a small fraction of these stations is responsible for most of the network traffic. Thus a low average system load on a large network, and a high average system load on a small network are both situations that must be suspect. **X** is certainly not true (with high probability) for heavily biased and highly loaded asymmetric systems. An analysis of such asymmetric polling systems is provided by Swartz [Swar80] as a generalization of the Konheim and Meister result. In fact, Swartz's approach uses the same embedded Markov chain as Eisenberg [Eise72] and yields an exact expression for mean waiting time at each station.

At this stage, it is worthwhile to point out that except for [Lieb61], [Kaye72], and [MaMW57], the solutions reviewed above are restricted to either mean waiting times, mean queue lengths, or both. It is uncommon to find results in the literature that present explicit forms for queueing and waiting time distributions, whether exact or approximate.

1.5 Summary Of Contributions

The research effort involved in writing this dissertation was motivated by a need for performance models of token-passing systems. In particular, we are interested in asymmetric token-passing systems where a station is allowed to transmit at most one packet at each time that it acquires a free token and is ready to transmit (i.e., *fair* service systems). In section 1.1 we described how such N station token-passing systems can be viewed as multiqueues sharing a single server, where the service discipline used is s-QED = 1. We attempt to analyze a more general model, where the time taken by a free token to bypass each station that is *not ready* to transmit (e.g., with an empty queue) is not negligible. This time, called a station's *switching time*, is typically a small fraction of a station's service time. Thus, each station is characterized by *four* probability distributions describing its arrival, service and switching times, as well as the random time its takes the server to walk to the following station. For the most part, we focus our attention on purely asymmetric systems, thus requiring an analysis with $4N$ distributions.

This work was initially undertaken as an exploratory project, with the objective of understanding token-passing system behaviour subject to simple service schemes and various QEDs. When it was discovered that symmetric and asymmetric s-QED = 1 systems (a proposed standard token-passing QED, important for its fairness) presented an enigma to performance modellers, the focus of the problem shifted to an analysis of precisely this service scheme and QED. Consider a strictly asymmetric N-station token-passing system operating under this QED. Assuming that steady state operation of such a system will exist under certain conditions, at any random instant in time the system will be in one of 2^N states. Each state can be thought of as a binary vector whose entries differentiate between empty and nonempty stations. Additionally, the server's position in the logical ring at this instant is important. Exponential complexity is one problem that shows itself immediately. This means that as N increases, simulations of such systems can be expected to be very expensive and time consuming. Additionally, the large number of states makes analysis more difficult. Another problem is one involving dependence. Since a queue can hold a number of customers at the same time, it becomes necessary to account for the effect that a certain queue length observed at a certain time may have on the succeeding server cycles. Still another problem is that of dependence between queues. A customer requiring a large service time at one queue will effect the following queues visited by the server, including the customer's own queue during the next cycle.

The literature reviewed in the last section does provide a few applicable models, but these are either approximate in themselves (e.g., due to assumptions of independence), or models of other queueing systems (borrowed-models) applied to token-passing queues. As pointed out in section 1.4, the latter approach leads to an entirely different kind of approximation which we label as borrowed-model approximations. Frequently, assumptions of independence are known to fail. But the conditions under which models assuming independence fail to perform well can be established far more readily than conditions under which borrowed-model approximations fail. For example, a model assuming independence is almost sure to fail in situations where strong dependence is an intrinsic part of the system being modelled. To detect conditions under which independence fails will usually require an identification of parameters causing dependence. On the other hand, to detect conditions under which a borrowed-model approximation fails will require an examination of the effects of the various hypotheses under which the borrowed model was analyzed. Since an approximation is good only as far as the conditions of its reliability are known, approximate models assuming independence are generally safer to use than borrowed-models.

The approach taken in modelling the S-QED = 1 token-passing protocol is to both make assumptions of independence, as well as to account for dependence by resorting to Markov formalism. In approximating, an independence assumption was chosen over the alternate approach of applying borrowed-models such as the machine interference model, or the exhaustive service model. The approximation arises due to an assumption that at steady state, the status (i.e., nonempty or empty) of every pair of queues in the system is independent. The entire research contribution can be divided into two parts. One part is the approximate method based on M/G/1 queues, introduced for multiqueues by

Paul Kuehn [Kueh79]. This is called the *service probability*, or SP approach. The other part is an exact analysis, called the *service vector*, or SV approach. The latter view is of token-passing multiqueues in the framework of Markov queues with semi-Markovian service times. The latter problem was originally studied by Neuts [Neut66] and Cinlar [Cinl67] in the context of single server queues.

The service probability methods that we introduce chronologically precede the exact methods. A first step in solving the problem was obtaining the distribution of server cycle-time for symmetric and asymmetric systems. This was motivated by Kuehn's [Kueh79] reference to this as an open problem. Due to the independence assumption, the cycle-time distribution by itself is of limited use in solving for queueing distributions. The problem is not directly amenable to an M/G/1 type analysis since the cycle-time random variable does not really correspond to the general service time random variable of an M/G/1 queueing system. However, given the form of the cycle time distribution, it is possible to approximate the effects of correlation between consecutive cycles. Kuehn recognized that an i.i.d (independent and identically distributed) service time random variable model would underestimate measures of customer waiting-times. Consequently, Kuehn introduced two kinds of cycle-times with the intention of increasing cycle-time variance and thereby obtained better waiting time measures. For the most part, the approximate methods that we suggest are variations on Kuehn's method. An attempt is made to account for correlation effects between neighbouring cycles. We include criteria under which station queues are stable, indices of stability, the existence of stationarity, applications of rarefactions, and a new definition of fairness. The same ideas are applied in determining the cycle-time distributions of the server in a complex nonexhaustive service discipline. In the latter analysis the idea of maximization of the entropy functional is used as a method of obtaining steady-state distributions.

The service vector methods are a recent development and consequently, there is scope for improvement. For the first time, this problem has been placed in a queueing framework that is both appropriate (unlike M/G/1 queues) as well as familiar to readers of the more advanced queueing literature. The key to this solution lies in a Markov renewal matrix and the corresponding embedded Markov matrix. Unfortunately, for an N station system, the required square matrix is of size 2^N. This matrix enables us to study transitions between cycles of various types and the effects of varying station parameters on the system. Again, the cycle-time distribution is obtained as a first step. In this case the cycle time random variable is station dependent, and the distribution obtained is functionally exact. Additionally, it is shown that the cycle-time distribution is unique. Otherwise stated, this is really an invariance result. In the interests of brevity we restrict our discussion only to an analysis, for the most part. We reserve much of the application for later work. Given the transition matrix, it becomes possible to embed our problem into single server queueing systems with semi-Markovian service and thereby solve for restricted mean waiting time and mean queue length for a given station.

The chief assumptions made in the various sections of the analyses may be summarized as follows. In the approximate methods the assumption is one of independence between different queues, as explained above. In obtaining cycle-time distributions under independence, exponential random variables are used. Actually, the exponential restriction is required only for customer interarrival times. Similarly, in the service vector methods, only the customer arrivals are required to be Poisson. All other distributions may be arbitrary. In all cases we conveniently assume existence of the first two moments of every distribution involved. Besides these, the only other assumptions are that the system is perfectly reliable, transmission or server behaviour is flawless, and queues whose steady-state distributions we seek satisfy a given stability criterion. The stability criterion as stated holds for GI/G/1 queues.

1.6 Organization Of The Thesis

The use of queueing theory in the study of communication models or related applications is so widespread that any survey of the literature for applicable models must almost surely be incomplete. The review presented in this chapter demonstrates a collection of useful problems in the realm of multiqueues that has not been given due attention. We focus our interest on a special member of this class and in the course of the dissertation, indicate the many difficulties that arise in the analysis. The token-passing protocol is formally described as a multiqueuing problem in a probabilistic context in chapter II. The parameters of the queueing system are introduced and interpreted in terms of token-passing systems. We take the opportunity to briefly review the definition of a Markov renewal process in terms of required notation.

The two approaches used in studying the multiqueueing problem form the two major portions of the dissertation. In chapter III, we introduce the exact SV methods for cycle-time distributions and obtain invariance and insensitivity results. In chapter IV, these results are put to use in determining useful queuing measures. Additionally, we define a random variable to represent the system's channel utilization. The results in chapter III are basically applications of Markov processes and numerical analysis, and the results of Chapter IV rely on PH-distributions [Neut81]. The results are merely stated in the text, and all proofs are given in the appendix.

In chapter V we introduce the approximate SP approach, or the model based on an independence assumption. The cycle-time distribution is derived for asymmetric and symmetric systems. Interestingly enough, both the SV as well as the SP approach yield cycle-times whose distributions are (finite) mixtures. In chapter VI the approximate cycle-time time random variable is further analyzed for an application in the study of the queueing process at a fixed station. For the most part, this analysis focuses the effects of cycle-time dependency on the queueing process a station.

In chapter VII some conditions (necessary and sufficient) are presented under which the token (on systems whose packet arrival rates are constant for the different stations) sees stable queues of packets. A new definition of fairness is given. This definition relies on information that is in distributional form. With such information, we will be able to vary parameters to obtain fair protocols, or even compare two (or more) systems, to decide which is more fair. Additionally, we present an application of rarefactions to a point process obtained via the token's departure instants at a station. Under heavy or light traffic, the point process is approximately alternating renewal and yields the (approximate) distributions of token busy and idle periods with respect to a fixed station.

Chapter VIII deals with a (non-existent) token-passing scheme based on a *complex* service discipline. The motivating idea is that under certain conditions, an adaptive token (made to adapt in a simple fashion to states of the network) will perform better than a non-adaptive one. Using an approximate approach, we obtain distributions for cycle-times associated with this system. In chapter IX we summarize and conclude our study and also outline some interesting problems in the shape of future work.

CHAPTER II

A QUEUEING MODEL FOR TOKEN-PASSING SCHEMES

In developing a stochastic model for token-passing on buses and rings we resort to the theory of Markov processes. In particular, we use a class of random processes called Markov renewal processes. This class is useful for modelling complex systems and has the advantage of including many of the standard processes that are popular for modelling. The theory of Markov renewal processes has been known since the pioneering work by Levy [Levy54] and Smith [Smit55]. Its popularity is chiefly attributed to the two papers by Pyke [Pyke61a, Pyke61b] though the theory was already being applied in inventory models and queueing models even before this time [Fabe61, Finc59]. In the following two sections, we present a formal definition of the token-passing model and a brief review of Markov renewal processes. In addition, we relate the parameters of the model to the parameters of token-passing systems.

2.1 The Multiqueue And Cyclic Server Model

The token-passing protocol can be viewed in terms of the *multiqueue and cyclic server* model [ReNi84]. The MQCS model is a system of N independent buffers, chained together to form a ring by sections of varying cable lengths, as depicted in Fig. 2. Packet arrivals at station j are generated by some process with interarrival distribution

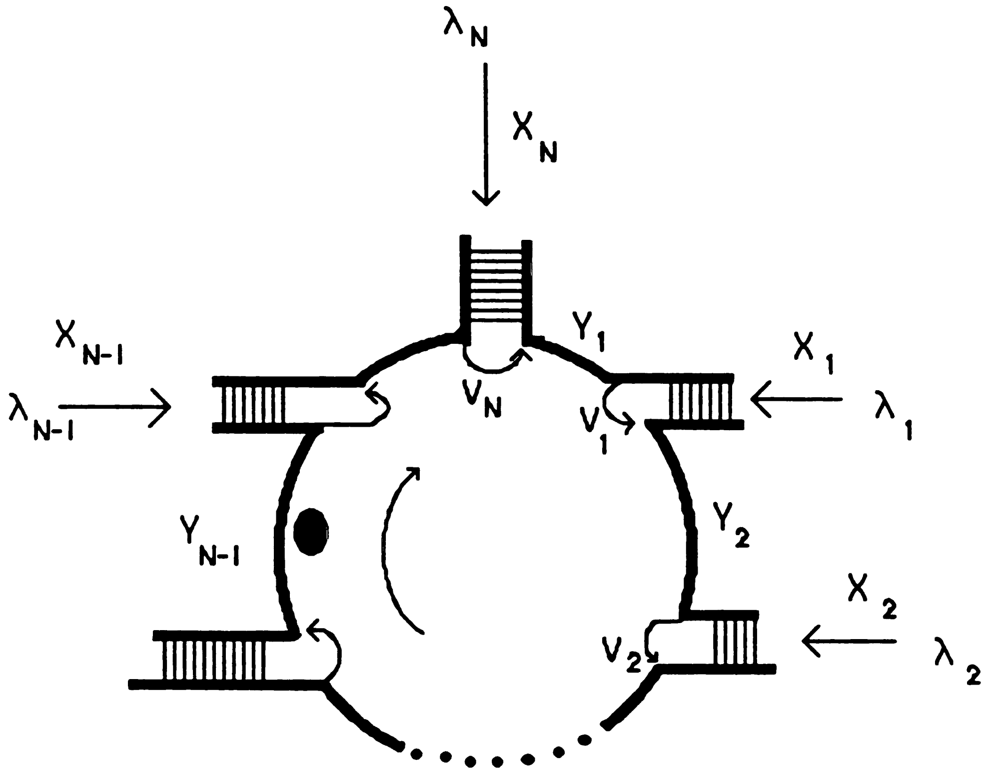


Fig.2 Formal view of token-passing queues

given by $A_j(t) = Pr(I_j \leq t)$, where I_j is the interarrival time random variable at station j, $j \in S = \{1,2,\ldots,N\}$. Let us label the *walk* between station $(j-1)$ and station j as W_j, $j \in S$. In our notation $(j-1)$ indicates the station just prior to station j, and $(j+1)$ indicates the station immediately after j on the path of the token. The station index immediately after j is obtained by computing $j \bmod N + 1$. If the circulating token finds a waiting packet at the buffer of station j, a transmission of random length X_j ensues, with probability distribution function $B_j(t) = Pr(X_j \leq t)$. If not, it *switches* from walk W_j to walk W_{j+1}, taking a random time V_j, with distribution function $S_j(t) = Pr(V_j \leq t)$. In any event, after leaving station j, the token spends a random time Y_{j+1} in walk $W_{j+1} \in W = \{W_1, W_2, \ldots, W_N\}$. Y_j has distribution function given by $U_j(t) = Pr(Y_j \leq t)$, $j \in S$.

We use the term distribution to denote the cumulative probability distribution, while the term density is reserved for the probability density function. For analytic convenience, we assume that all distributions have finite first and second moments. Observe the following points regarding standard queueing notation. For a fixed index j, the queue at station j is a single server queue. The fact that the server is unable to give uninterrupted service to queue j does not change a *j-customer's* view of this queue as a single server queue. Thus, queue j is really a GI/G/1 queue, where the customer interarrival distribution is given by $A_j(t)$. Though customers at station j spend only a random time X_j in service, the GI/G/1 approach requires the service time random variable to be the length of time that the server spends away from station j from the time the last customer's service began.

The time that the server spends away from queue j may be treated as a server vacation. Several traditional models usually take this vacation time to be a random variable which is independent of the general service time distribution. Clearly, the presence of dependency between queues makes this approach an approximation in our case. If we combine the vacation time with the random time that the server spends at station j, we obtain a single random variable that may be viewed as station j's service-time random variable. This random time is the cycle-time of the server as seen from station j.

It is convenient to introduce a random variable that describes the amount of time that elapses starting from the instant that the server finds station j empty to the instant that the server finds a customer queued at this station. This time is called a *vacation period* of the server, since a station j customer arriving in this period will find the server unavailable. Though queue j remains a single server queue in the cycle-time context, it can no longer be treated as a GI/G/1 queue. This is because the consecutive customer service-times (i.e., cycle-times) will no longer be i.i.d. To be precise, the queue must be labelled as a GI/GD/1 queue, where the "GD" denotes a *dependent* service-time random variable. If this dependence can be placed in a semi-Markovian setting, we will have a GI/SM/1 queue with vacationing server at station j. If we visualize the entire system of N queues as a single queue, the notation $GI^{[MQ]}/GD/1$ or $GI^{[MQ]}/SM/1$ can be used for the appropriate situations. In order to avoid any confusion when referring to any particular view, we will always describe the queueing configuration.

The notation MQCS/*S*-QED = *1* is used to described the entire *N*-station system with respect to the queue emptying discipline that is of major interest to us.

Due to conceptual similarities, a token ring and a token bus can both be analyzed with the same performance model. However, due to their structural differences, the choice of certain system parameters will differ. On a ring, the walk time between two consecutive stations is comprised of the signal propagation delay between the two stations and the token-delay inherent to the two stations in creating and receiving the token [Buxw84]. The first kind of delay is in the order of *5* μ-sec per *Km* of cable, and the second kind of delay (station latency) is in the order of *1* bit time. In comparison, the walk time on a token bus is made up of three kinds of delays. The first kind is caused by the transmission of the token, the second due to the signal propagation delay between the stations, and the third due to the delay caused by the token-receiving station before it transmits either a token or a packet. Here, the first delay requires a time equivalent to the transmission time for *152* bits [Buxw84], the second is precisely the maximum end-to-end propagation delay of the cable, and the third is typically in the order of *1* bit. Thus, in terms of throughput and delay, a ring scheme performs better than a corresponding token-passing bus [Stuc83].

As token-passing service disciplines become more *complex*, it will be more convenient (or perhaps necessary) to view token related delays as being of two kinds. One kind will involve station delays, i.e., pattern matching circuit delays, delays between token receiving and reaction times, delays in scheduling near optimal paths etc. The other class of delays will be due to signal propogation, token-transmission time, or even delays caused by an unreliable server (such as a lost token). With a view towards this type of generality, it was decided to incorporate the notion of *switching times* in the MQCS model. These random times are intended to serve as station related delays. The second class of delays are represented by the random walk times between stations.

Performing the analysis is a first step toward acquiring tools for understanding system behaviour as a function of system parameters. In an average sense, only two basic characteristics of a local network, i.e., propagation delay and data rate, set an upper bound on performance, independent of the channel access protocol [Stal84]. This is especially true if performance is defined in terms of system averages. In our model, since we are interested in a more detailed view of individual stations' characteristics, we choose to model systems with variable propagation delay between pairs of stations, as well as variable switching times. Introducing such randomness is one way to obtain somewhat more realistic and detailed measurements about queueing processes, such as distributions of packet delay, minimum and maximum packet queue-sizes, etc. Such information is useful for large systems with asymmetric traffic characteristics and constraints on packet-queueing time or buffer sizes. Once again, we will only be interested in the steady-state behaviour of token-passing systems with a simple service scheme, *S*-QED = *1* and FIFO within queues, flawless message transfer, and a perfectly reliable physical configuration.

2.2 Markov Renewal Processes

In this section we present a brief review of Markov renewal processes and then interpret our problem in terms of notation that we develop as we proceed. In order to properly define a Markov renewal process we need a sequence of pairs of random variables from our system. Let Z_n be a random variable taking values in a finite set S^* for n in the set of non-negative integers I^+. Let T_n be another random variable such that for each $Z_n \in S^*$, T_n takes values in the non-negative real numbers R^+. The complete probability space (Ω, F, Pr) is defined with

$$Z_n : \Omega \;\text{----->}\; S^*,$$
$$T_n : \Omega \;\text{----->}\; R^+$$

such that

$$0 = T_0(\omega) \le T_1(\omega) \le \ldots. \le T_n(\omega) \le \ldots., \quad for\ \omega \in \Omega.$$

The sequence of pairs $\{(Z_n, T_n)\}$ is called a Markov renewal process if

$$Pr(\, Z_{n+1} = j \,,\, T_{n+1} - T_n \le \; t \,|\; Z_0, \ldots\ldots, Z_n \,,\, T_0, \ldots\ldots, T_n \,)$$
$$= Pr(\, Z_{n+1} = j \,,\, T_{n+1} - T_n \le \; t \;|\; Z_n \,). \tag{1}$$

We take $T_0 = 0$ for the rest of the discussion and suppose that $Pr(\, Z_0 = k \,)$ is given, for some $k \in S^*$. The sequence $\{T_{n+1} - T_n\}$ forms a sequence of *dependent* random varables, because $(T_{n+1} - T_n)$ depends on Z_{n+1} and Z_n for every $n \in I^+$. We will assume that $\{T_n,\, n \ge 0\}$ is *persistent*, i.e. $Pr(\, T_{n+1} - T_n < \infty \,) = 1$, for all $n \in I^+$. Additionally, to exclude the possibility of the process passing through infinitely many states in zero time, we assume that $Pr(\ T_{n+1} - T_n \neq 0\ ) = 1$ for all $n \in \mathbf{I}^+$. It can be shown [Pyke61a] that given $\{Z_n\}$, the $\{T_{n+1}-T_n\}$ are mutually conditionally independent and satisfy the properties,

(P1) the process $\{Z_n\ ,\ n \in \mathbf{I}^+\}$ forms a Markov chain satisfying

$$Pr(\ Z_n = j \mid Z_0, \ldots, Z_{n-1}\ ) = Pr(\ Z_n = j \mid Z_{n-1} = i\ )$$

(P2) $\{T_n \mid Z_n = z\}$ forms a renewal process for every fixed $z \in \mathbf{S}^*$.

Let **Q** be a square matrix with entry $Q_{ij}(t) \leq 1$ being a nondecreasing right continuous point function for $i,j \in \mathbf{S}^*$. **Q** is called a semi-Markov kernel over $\mathbf{S}^*$ if, for each $j \in \mathbf{S}^*$, the following properties are satisfied.

$$Q_{ij}(t) = 0 \qquad \text{for } t \leq 0, \text{ and}$$

$$\sum_{j \in \mathbf{S}^*} Q_{ij}(+\infty) = 1 \qquad \text{for } i \in \mathbf{S}^*. \tag{2}$$

Observe that the Q_{ij}'s define the joint conditional probability

$$Q_{ij}(t) = Pr(\ Z_{n+1} = j\ ,\ T_{n+1} - T_n \leq t \mid Z_n = i\ ) \tag{3}$$

Equation (3) says given that the present state is i, $Q_{ij}(t)$ describes the probability of making a transition to state j after spending at most time t in state i. The kernel **Q** serves (for a Markov renewal process) a purpose analagous to the function of the transition matrix **P** in a Markov process. Thus **Q** is really a matrix of transition functions for the process. In simulating the process, the states of the Markov chain must first be genetrated with the aid of **P**. Given that a transition has been made from state i to state j, $Q_{ij}(t)$ defines the distribution of the random time that elapses before the next jump is triggered. The probability transition kernel of the embedded Markov chain is defined as $\mathbf{P} = \mathbf{Q}(\infty)$.

In our application, we are mainly interested in the limiting probability distribution for the semi-Markov process $\{Z(t)\}$. This describes the limiting probability ϕ_j of finding the server in any particular state $j, j \in \mathbf{S}^*$, provided the process is stationary. The service probability approach (see chapters V and VIII) used in the derivation of cycle-time distributions requires an invariance result for semi-Markov processes in order for the distribution $\{\phi_j, j \in \mathbf{S}^*\}$ to exist. Such a result is already known, first provided by Arjas, Nummelin, and Tweedie [ArNT78], and later McDonald [McDo85], for different conditions. A similar result, much simpler and restricted to our special case, is obtained in Theorems 5.1 and 8.4. The essence of the result is an invariant probability measure for the sequence of probability transition kernels $\mathbf{P}^{(k)}$.

In our service probability and service vector applications, a particle makes a sequence of transitions in a finite space $\mathbf{S}^*$ according to a sequence of probability transition kernels $\{\mathbf{P}^{(n)}; n=1,..\infty\}$. If the particle is in state i after $(k-1)$ transitions, it goes to state j with a probability $p_{ij}^{(k)}$ on the k^{th} transition. If the distribution of the random time spent by the particle between transitions depends, in general, on the transition number, the current state, and the next state, then the process can be seen to behave as a semi-Markov process with a varying probability transition kernel. If the kernel **Q** of the renewal process is known, for each probability matrix **P**, and if an invariant probability measure Δ can be shown to exist for **P**, then the classical limit theorems for semi-Markov and semi-regenerative processes are shown to be robust enough to accomodate our case [McDo85]. Let μ_x be the mean time spent by the process in state x, $x \in \mathbf{S}^*$. McDonald showed that under certain mixing conditions,

$$\lim_{t \to \infty} P\{Z(t)=z\} = \frac{\Delta(z)\mu_z}{\sum_{x \in \mathbf{S}^*} \Delta(x)\mu_x} \qquad (4)$$

CHAPTER III

CYCLE-TIME DISTRIBUTIONS VIA SERVICE VECTORS

In the following analysis, a method for deriving the server's cycle-time distribution for a general, asymmetric MQCS/*s*-QED = *1* system is presented. The assumption that arrivals are Poisson is not crucial to the cycle-time analysis, and all input distributions may be general. We only require that a certain chain of events satisfy the Markov property, and this follows by construction. The cycle-time distribution is defined with respect to a given station which we call a reference station. Without loss of generality we take station *j* to be the reference station.

There are two ways in which we can measure the cycle-time of the server at the reference station. If an observer is positioned at the departure point of station *j* (i.e., the point at which the server exits from this station), and the observer is required to report to us the random times between server reappearances at this point, then we obtain cycle-times of departure type, or *dep-cycles*. If the observer is positioned at the reference station's scan point (i.e., the point at which the server enters this station, to check for possible customers), then we obtain cycle-times of scan type, or *scan-cycles*. The length of a dep-cycle is the random time between two successive appearances of the server at the departure point of the reference station. Correspondingly, a scan-cycle length is defined as the random time between two successive appearances of the server at the reference station's scan point.

In the analysis that follows, we derive the distributions of the cycle-times of *scan* type. This is based on simple probabilistic arguments and Markov renewal theory. In addition, we obtain a result of some interest. It is shown that the cycle-times of *dep* type measured at station j possess the same distribution as the cycle-times of *scan* type measured at station $(j) \bmod N + 1$. Also, it is shown that the cycle-times of scan type (or dep type) have a unique distributional form, independent of the reference station j. Putting these two results together, we obtain the interesting observation that all N dep-cycles and all N scan-cycles have the same distributional form, i.e., the system's cycle-time distribution is unique, and independent of the point at which the measurement is made.

Section 3.1 introduces an external view of the system, in which an observer positioned at the reference station's scan point behaves as a particle in a Markov renewal process. In section 3.2, the structure of the Markov renewal process is exploited in order obtain the Markov matrix describing the observer's steady-state transitions. Once this is known, solving for the invariant vector inevitably leads to a steady-state form of the cycle-time distribution. This distribution is obtained in the form of a finite mixture and is functionally exact. Section 3.3 contains some results on the invariance of cycle-times and the insensitivity of the cycle-time distribution. The last result is somewhat surprising in that the exact cycle-time distribution of the server can be reduced to a function of only the mean conditional and unconditional cycle-times.

3.1 The Markov Chain Of Vector Transfers

Assume that the queueing process at station j is stationary and that the system is operating at steady-state. Let C be the random time that elapses between two consecutive vector transfers from the server to the observer. The limiting distribution of this random time is our focus of interest. The mechanics of the approach that we use to arrive at this can briefly be explained as follows. In Fig. 2 we see that the server makes service cycles in order of increasing station indices, moving over to station 1 after station N has been visited. On each cycle made with respect to station j, the server constructs a (binary) vector, called a *service vector*, that records individual station events. On completing the cycle, the server instantaneously (i.e., in zero time) transfers the vector over to the observer and begins another cycle. We make the assumption that this transfer may take place at any time prior to or precisely the scan instant at the reference station, but only after the server has left the preceding station's departure point. Additionally, we require that the transfer be made at the same physical point for every cycle. Let Θ be the set of all N-bit binary vectors. The observer can be viewed as a randomly moving particle, with each vector transfer corresponding to a transition by the particle between states of a Markov renewal process. In this case, the state space S^* will be the 2^N states of the set Θ.

Let $T_1, T_2, \ldots, T_n, \ldots$ be the instants in time at which the observer acquires the service vector from the server. Assume that $T_0 = 0$, and $(T_{n+1} - T_n) > 0$, for all n, $n \in I^+$, $T_n \in R^+$. Define the service vector $Z = <z_j, \ldots, z_N, z_1, \ldots, z_{j-1}>$ as

$$z_i = \begin{cases} 1 & \text{if a station } i \text{ customer is served in the current cycle} \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in S$. If $j = 1$, we take $(j-1)$ to denote station N. At time T_n, the observer receives a service vector $Z_n = <z_j^{(n)}, \ldots, z_N^{(n)}, z_1^{(n)}, \ldots, z_{j-1}^{(n)}>$ associated with the n^{th} transition of the process.

The sequence of states $Z_0, Z_1, \ldots, Z_n, \ldots$ forms a Markov chain. When time spent by the observer in any state (i.e., a cycle-time) is taken into consideration, then since this random time is generally not exponentially distributed, $\{Z_n, T_n\}$ will be a Markov renewal process. Due to the nature of the transitions, i.e., different service vectors define different service patterns, the time spent by the server in a state before a transition is made will be a function of the current state and the next state.

3.2 Asymmetric Systems With General Distributions

In making a transition from a state $Z_n = z$ to a state $Z_{n+1} = z'$, the random time that elapses is precisely defined by the vectors $z = <z_j, \ldots, z_{j-1}>$ and $z' = <z_j', \ldots, z'_{j-1}>$. If the limiting distribution of the related semi-Markov process can be determined, the cycle-time distribution for station j easily follows. In order to deal with the Markov renewal process, we need its kernel of transitions **Q**.

In order to obtain the kernel, our approach requires the probability transition matrix **P** corresponding to the Markov chain $\{Z_n\}$ embedded at the instants of vector transfer from server to observer. For each possible transition $z \to z'$, with $z,z' \in \Theta$, we require the steady-state probability $p(z,z')$ of making the transition.

In the situation where the buffer capacity at each station is exactly one, the stochastic process governing packet queue lengths can be placed in a simple framework. If the server can encounter at most one customer waiting for service at any queue, the queueing process (seen by the server) at any station takes on a regenerative form, with future events that are probabilistic replicas and independent of past events at this station. In essence, this becomes a slightly modified version of the M/G/1/K queueing problem whose solution and transition matrix for the embedded chain are well known. In our case, the assumption of unrestricted buffer capacities makes events associated with the server's current scan instant at a station depend on past events at the station. The lack of a regenerative or easily identified semi-regenerative structure for this process requires another approach in obtaining **P**.

In order to obtain a particular transition probability, it is necessary to examine the time involved in making the transition from the starting state into the target state. Define a shift operator **M** that maps a vector two-tuple from the set $\Theta x \Theta$ into the same set such that

$$\mathbf{M}\{(x_1,..,x_N),(x_1',..,x_N')\} = \{(x_2,..,x_N,x_1'),(x_2',..,x_N',x_1\} \qquad (1)$$

The mapping views a vector two-tuple as a $2N$-bit computer word and does an end-around left shift on the word in such a way that all bits move to the left by one position, while the leftmost bit falls off the left edge and is placed in the rightmost (vacated) bit position. Applying **M** recursively for a total of $2N$ times will yield the original two-tuple. If we define the starting state z and final state z' of a particular transition as a two-tuple in $\Theta x \Theta$, then with the starting vector two-tuple and $(N-1)$ repeated applications of **M**, we can obtain N vector two-tuples corresponding to the given transition. Let the N components of a transition $z \to z'$ (for scan-cycles at station j) be given by the vector pairs $<x_j,y_j>,..., <x_N,y_N>, <x_1,y_1>,.. <x_{j-1},y_{j-1}>$. Observe that the vectors $<x_j>$ and $<y_j>$ are, respectively, the first and second N bits of the $2N$ bit computer word representing $[z,z']$. We fix our attention on each vector of the form $<x_i>$, $i \in \mathbf{S}$, and let $\delta_i(k)$ represent the k^{th} entry (i.e., either 0 or 1) of this vector. Physically, the entries of $<x_i>$ indicate exactly which stations required service (and which stations did not) on station i's scan-cycle during the transition $z \to z'$, $i \in \mathbf{S}$. Given such information, it is a simple matter to compute the length of this particular cycle as a sum of independent random variables.

The probability of a transition from z to z' is obtained as a product of N probabilities, one from each station. For each m, $1 \leq m \leq N$, the m^{th} station on the path of the server after a vector transfer to the observer is denoted by $\gamma(m,j) = (j+m-1) \bmod N$. For each transition $z \to z'$ recorded by our observer at station j, and each m, $1 \leq m \leq N$, the time taken by the server to make a complete cycle with respect to station $\gamma(m,j)$ is given by

$$T_{\gamma(m,j)}(z,z') = \sum_{k=1}^{N} \{ J_{\gamma(m,j)}(k) + Y_{\gamma(k,j)} \} \qquad (2)$$

where

$$J_{\gamma(m,j)}(k) = \delta_{\gamma(m,j)}(k)\, X_{\gamma(m,j)} + (1 - \delta_{\gamma(m,j)}(k))\, V_{\gamma(m,j)} \qquad (3)$$

Since the random variables X_k, V_k and Y_k are independent for all k, Eq.(2) essentially describes the random cycle-time associated with station $\gamma(m,j)$ (for transition $z \to z'$ at the observer's point) as a linear combination of independent random variables. The distributions of arbitrary linear combinations of independent random variables can be obtained by the usual techniques [Spri79]. In the case of exponential random variables, and also gamma random variables with a restricted class of parameter types, exact results and computational forms are well known [AlOb82, Math83]. Given arbitrary distributions for each X_k , V_k, and Y_k, the distributions of the random times described by Eq.(2). can, in principle, always be obtained. In practice, unless the distributions B_k, S_k, and U_k are extremely complicated, the distribution of $T_{\gamma(m,j)}(z,z')$ can be determined either in exact form or a form suitable for computation. For the remainder of this chapter, let the distribution of the random variable $T_{\gamma(m,j)}(z,z')$ in Eq.(2) be denoted by $F_{\gamma(m,j)}(.,z,z')$, for $m = 1,2,\ldots,N$, transition $z \to z'$, and each (fixed) reference station j. Observe that a strictly asymmetric system will possess 2^N possible distributions of this type.

In order to make the vector transition $z \to z'$, the server must encounter individual station transitions of the form $z_{\gamma(m,j)} \rightarrow z'_{\gamma(m,j)}$ during station j's scan-cycle. For each m, $1 \leq m \leq N$, denote the probability of transition $z_{\gamma(m,j)} \rightarrow z'_{\gamma(m,j)}$ by $\xi_{\gamma(m,j)}(z,z')$. This is the probability that the server encounters $z_{\gamma(m,j)}$ customer(s) (i.e., either 0 customers or 1 customer) on the first visit and $z'_{\gamma(m,j)}$ customer(s) on the second visit to station $\gamma(m,j)$. Let $p_k(n,z,z')$ denote the probability that n customers arrive at station k during a random time $T_k(z,z')$, where k takes on integer values between (and including) $\gamma(1,j)$ and $\gamma(N,j)$. Note that this corresponds to the probability of n arrivals at station k during the time the server was away from this station, i.e., during a reference station's scan-cycle transition. If the arrival process at station k is Poisson, with positive, constant intensity λ_k, $k \in \mathbf{S}$, this probability can be obtained as

$$p_k(n,z,z') = \int_0^\infty \frac{\exp(-\lambda_k t)(\lambda_k t)^n}{n\,!} \, dF_k(t,z,z'). \tag{4}$$

In the case of a general customer inter-arrival distribution for a given station, the probability defined in Eq.(4) can always be obtained by substituting the appropriate distribution in the integral to describe the probability of n customer arrivals. Each scan-cycle transition probability $p(z,z')$ can be obtained as

$$p(z,z') = \prod_{m \in \mathbf{S}} \xi_{\gamma(m,j)}(z,z') \tag{5}$$

where the approch taken in obtaining each component of the product in Eq.(5) is explained as follows. For transition $z \to z'$, and $k = \gamma(1,j),\ldots,\gamma(N,j)$, let events A_k and B_k denote the outcomes of the binary random variables z_k and z'_k, respectively, for given j. We use the basic probabilistic idea that

$$P(\, B \wedge A\,) \;=\; P(\, B \mid A\,)\;\; P(\, A\,) \qquad (6)$$

to obtain each probability $\xi_k(z,z')$. The event A_k tells us if a customer was served at station k on the server's last visit to this station, during transition $z \to z'$. If a customer was not served at this station (i.e., $A_k = 0$), the cycle-time takes on a value that is smaller than c, $c \in \mathcal{R}^+$, with a probability that is strictly less than the corresponding probability for a cycle-time incorporating a customer service at this station. In other words, for a given vector transition, cycle-times obtained by excluding a service-time contribution from a particular station are stochastically less than cycle-times that include such a contribution.

We are interested in computing the conditional probability $P(\, B_k \mid A_k\,)$ for each station $k \in \mathbf{S}$. Observe that knowing A_k by itself is not sufficient to compute this probability. If $A_k = 0$, then B_k is strictly a function of the arrival process at station k and the random time $T_k(z,z')$. This can be computed with the aid of Eq.(5). If $A_k = 1$, then the situation becomes complicated, since we have no way of knowing exactly how many customers are queued for service at station k. In fact, this distribution of customer queue length is one of the many distributions we seek. If we strictly seek the conditional probability that $B_k = 1$ given that $A_k = 1$, then indeed we are in trouble, since we must require the queue length distribution for this. Alternately, if we seek the conditional probability that $B_k = 0$ given that $A_k = 1$, then we can utilize the additional knowledge that this conditional probability makes sense only if the number of customers queued for service at station k when the server arrived there was precisely one. Additionally, no customers must have arrived in the random time $T_k(z,z')$. The latter probability is obtained from Eq.(5), but the probability q_k that station k's queue length is exactly one (given that $A_k = 1$ or that it was found nonempty) still remains to be computed.

An example is presented in the following section to demonstrate a method for computing each q_k. Other methods are also described. For now, assume that we know how q_k may be computed for each $k \in \mathbf{S}$. Then with the aid of Eq.(5), we can write down an expression for computing $\xi_k(z,z')$, for each $k \in \mathbf{S}$, as

$$\xi_k(z,z') = \begin{cases} p_k(0,z,z') & z_k=0,\ z_k'=0 \\ [1 - p_k(0,z,z')] & z_k=0,\ z_k'=1 \\ p_k(0,z,z')q_k & z_k=1,\ z_k'=0 \\ [1 - p_k(0,z,z')q_k] & z_k=1,\ z_k'=1 \end{cases} \tag{7}$$

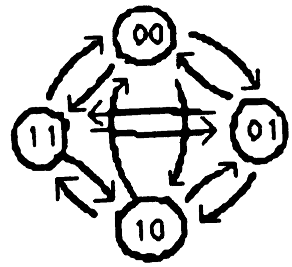
At this stage, we have the necessary material to enable us to use results from Markov renewal theory, if we interested in renewal analysis. We are interested in the distribution of time that the token spends in scan-cycles, as seen by the observer at the reference station. So, we first focus our attention on P. In particular, we are interested in the limiting distribution of the Markov chain whose transitions are governed by P. An application of standard techniques [Klei75] will yield the limiting distribution $\{\pi_z, z \in \Theta\}$, where π_z is the steady-state probability that the vector transferred to the observer by the server is $z \in \Theta$. Note that this distribution ignores the time spent by the particle in the various states of the related semi-Markov process. The length of time spent by the observer in any state $z = \langle z_j, \ldots z_N, z_1, \ldots, z_{j-1} \rangle$ of Θ is given by the random variable

$$C(z) = \sum_{i \in S} [z_{\gamma(i,j)} X_i + (1 - z_{\gamma(i,j)}) V_i + Y_i] \quad (8)$$

with law $F_{C(z)}(.)$, easily obtained as the distribution of a linear combination of independent random variables.

In order to obtain the limiting density of the semi-Markov process $\{Z(t)\}$, we compute the probability of each state z as the product of π_z and $E(C(z))$ normalized by the sum of all such product probabilities. Let $\{\phi_z, z \in \Theta\}$ be the probability density obtained in this manner. Each random service vector z describes a service cycle (made with respect to the observer's position at the reference station) of length $C(z)$, and occuring with steady-state probability ϕ_z. The cycle-time random variable as seen by the observer will have a distribution given by

$$F_C(t) = \sum_{z \in \Theta} \pi_z F_{C(z)}(t) \quad (9)$$



$\lambda_1 = 0.0032$ $\lambda_2 = 0.003492$

$E(X_1) = 198$ $E(X_2) = 100$

$E(Y_1) = 1$ $E(X_2) = 2$

$V_1 = 0$ $V_2 = 0$

Mean Cycle-time = 174.4186

Probability station 1 is empty = 0.44186

Probability station 2 is empty = 0.39093

Probability transition matrix

	(00)	(01)	(10)	(11)
(00)	0.9801	0.0103	0.0056	0.0040
(01)	0.0491	0.7012	0.0097	0.2400
(10)	0.0478	0.0005	0.5568	0.3949
(11)	0.0024	0.0342	0.0373	0.9261

Limiting service vector is (0.3604,0.0717,0.0497,0.5181)

Fig.3 Markov model for two station example

The time-considered distribution of cycle-time may be obtained by replacing π_i by ϕ_i for each state i in Eq.(9).

3.3 Application Of The SV Method

In this section we apply the methods just presented to demonstrate how the cycle-time distribution may be obtained. For convenience, we assume negligible switching times on a strictly asymmetric, two station system. Let the arrival rates for stations 1 and 2 be denoted by λ_1 and λ_2, respectively. Let the mean service times and mean walk times be given by $1/\beta_k$ and $1/a_k$, respectively, for $k = 1,2$. A graphic description of this scenario is given in Fig. 3, with the system observer positioned at a point on the path of the server from station 2 to 1. Thus, station 1 is taken to be our reference station.

Assume that the system is operating at steady-state. There are four possible service vectors (corresponding to service cycles) on our two station system, and these are (00), (01), (10) and (11). The i^{th} entry of each vector tells if station i did or did not require a customer's service on the cycle represented by the vector, for $i = 1,2$. For example, if the observer obtains a vector 10 from the server, this is taken to mean that station 1 had one customer served, but station 2 had no customer served in the most recent cycle. For a given vector (ij), let x_{ij} be the probability that no customers arrive at station 1 during a service cycle represented by vector (ij). The corresponding probability for station 2 is denoted by y_{ij}.

We are interested in constructing a 4×4 transition matrix P representing transition probabilities for the four different kinds of cycles seen by the observer. Assume that two vectors consecutively transferred to the observer are *(00)* and *(01)*. This means that station *1* had no arrivals during its server cycle *(00)*. One application of the shift operator in Eq.(1) will allow us to determine events at station *2*. Thus, the transition is possible only if station *2* had one or more customer arrivals during its own server cycle *(00)*. We are interested in computing the probabilities that the server encounters no customers at each station given that each station saw the cycles *(00)*, *(01)*, *(10)*, and *(11)*, respectively. Note that computing the complementary probabilities directly would be extremely difficult (if not impossible), since we have no information on queue lengths. Let these probabilities be denoted by $P_k[\ 0 \mid ij\ ]$, for $i,j \in \{0,1\}$, and $k = 1,2$. Recalling that q_k is the probability that exactly one customer is queued for service at station k, given that it is not empty, the conditional probabilities for $k = 1,2$ are easily determined as

$$
\begin{aligned}
a_{11} &= P_1[\ 0 \mid 00\ ] = x_{00} & b_{11} &= P_2[\ 0 \mid 00\ ] = y_{00} \\
a_{21} &= P_1[\ 0 \mid 01\ ] = x_{01} & b_{21} &= P_2[\ 0 \mid 01\ ] = y_{01} \\
a_{31} &= P_1[\ 0 \mid 10\ ] = x_{10}\, q_1 & b_{31} &= P_2[\ 0 \mid 10\ ] = y_{10}\, q_2 \\
a_{41} &= P_1[\ 0 \mid 11\ ] = x_{11}\, q_1 & b_{41} &= P_2[\ 0 \mid 11\ ] = y_{11}\, q_2
\end{aligned}
\qquad (10)
$$

Let the complementary probabilities for stations *1* and *2* be denoted by $a_{n,2}$ and $b_{n,2}$, respectively, for $n = 1,2,3,4$. It now remains to apply Eq.(5) in order to obtain **P**. Once this is done, the transition matrix corresponding to the embedded Markov chain of server cycles may be written as

	00	01	10	11
00	$a_{11}*b_{11}$	$a_{11}*b_{12}$	$a_{12}*b_{21}$	$a_{12}*b_{22}$
01	$a_{21}*b_{31}$	$a_{21}*b_{32}$	$a_{22}*b_{41}$	$a_{22}*b_{42}$
10	$a_{31}*b_{11}$	$a_{31}*b_{12}$	$a_{32}*b_{21}$	$a_{32}*b_{22}$
11	$a_{41}*b_{31}$	$a_{41}*b_{32}$	$a_{42}*b_{41}$	$a_{42}*b_{42}$

(11)

At this stage the matrix in P described by Eq.(11) is easily computed if the two conditional probabilities q_1 and q_2 are known. We now describe a method that allows us to compute them explicitly. Since P describes a Markov chain for which each state is aperiodic, recurrent, and nonnull, the Markov chain of vector transitions must be ergodic. Treating the four limiting state probabilities π_{00}, π_{01}, π_{10}, and π_{11} as well as the conditional probabilities q_1 and q_2 as our six unknowns, we proceed to determine the matrix in Eq.(11). With Foster's criteria [Fell66] it can be shown that the Markov chain's ergodicity implies that the four limiting state probabilities satisfy a system of four linearly independent equations (one of them being the conservation equation, in which all four probabilities add to one). These four equations still involve the unknowns q_1 and q_2 . To determine these, we make use of the mean cycle-time (see Eq.(5) of chapter V) and mean conditional cycle-times (see Eq.(1) of chapter VII). With these additional equations, q_1 and q_2 are easily computed. Thus, the matrix and the invariant vector of limiting probabilities are both determined. By multiplying the q_i's by their respective conditions (i.e., by the probability that station i is not empty at its scan instants, obtained from Eq.(4) of chapter V), the unconditional probability that station i has exactly one customer queued at its scan instants can also be determined.

Consider the following numerical example. Let $\lambda_1 = 0.0032$ and $\lambda_2 = 0.003492$ be the mean arrival rates for stations 1 and 2, respectively, assuming Poisson arrivals. Also, assume that the service times and walk times for both stations are exponentially distributed random variables. Let the mean service times for stations 1 and 2 be $E(X_1) = 1/\beta_1 = 198$ and $E(X_2) = 1/\beta_2 = 100$, respectively. Let the server's mean walking time from station 1 to station 2 and vice-versa be given by $1/\alpha_1 = 1$ and $1/\alpha_2 = 2$, respectively. It is now a simple matter to compute x_{ij} and y_{ij} for $i,j \in \{0,1\}$. Once this is done, a system of six independent equations involving π_{ij} and q_i , for $i,j \in \{0,1\}$, can be set up to enable us to solve.

In this particular example, we chose to obtain q_1 and q_2 from the mean cycle-time equation in an iterative fashion. Note that an iteration is not really necessary since these probabilities may be determined explicitly. Using an iterative criterion that selected both q_i subject to a permissible difference of 10^{-15} between the iterated mean cycle-time and the exact mean cycle-time, we obtained $q_1 = 0.0870366$ and $q_2 = 0.0805674$. Consequently, the limiting state probabilities of the embedded Markov chain are obtained as $\pi_{00} = 0.3604$, $\pi_{01} = 0.0717$, $\pi_{10} = 0.0497$, and $\pi_{11} = 0.5181$.

Let the random cycle-lengths corresponding to the four service vectors be F_{00}, F_{01}, F_{10}, and F_{11}, respectively. Note that each F_{ij}, for $i,j \in \{0,1\}$, is a generalized Erlangian distribution composed of at

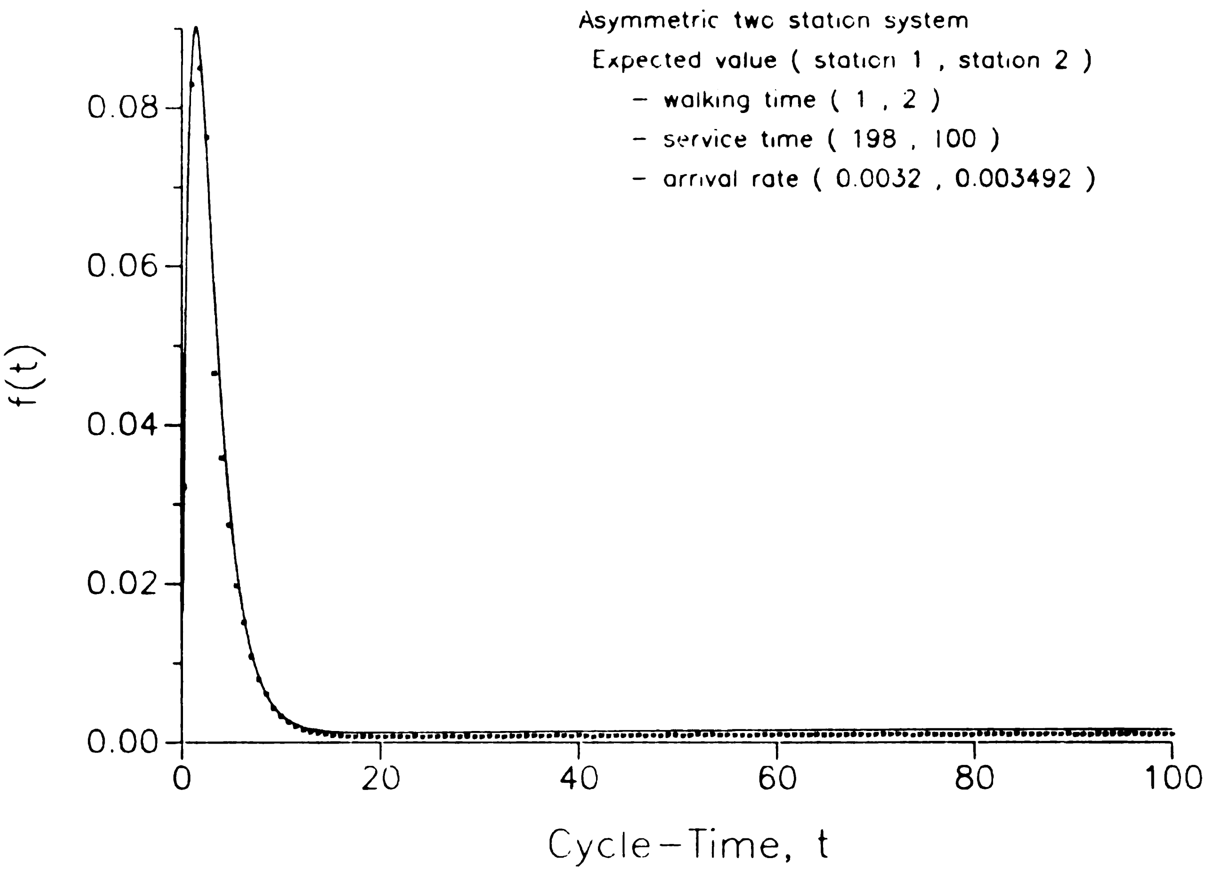


Fig.4a SV cycle-time density for two station example

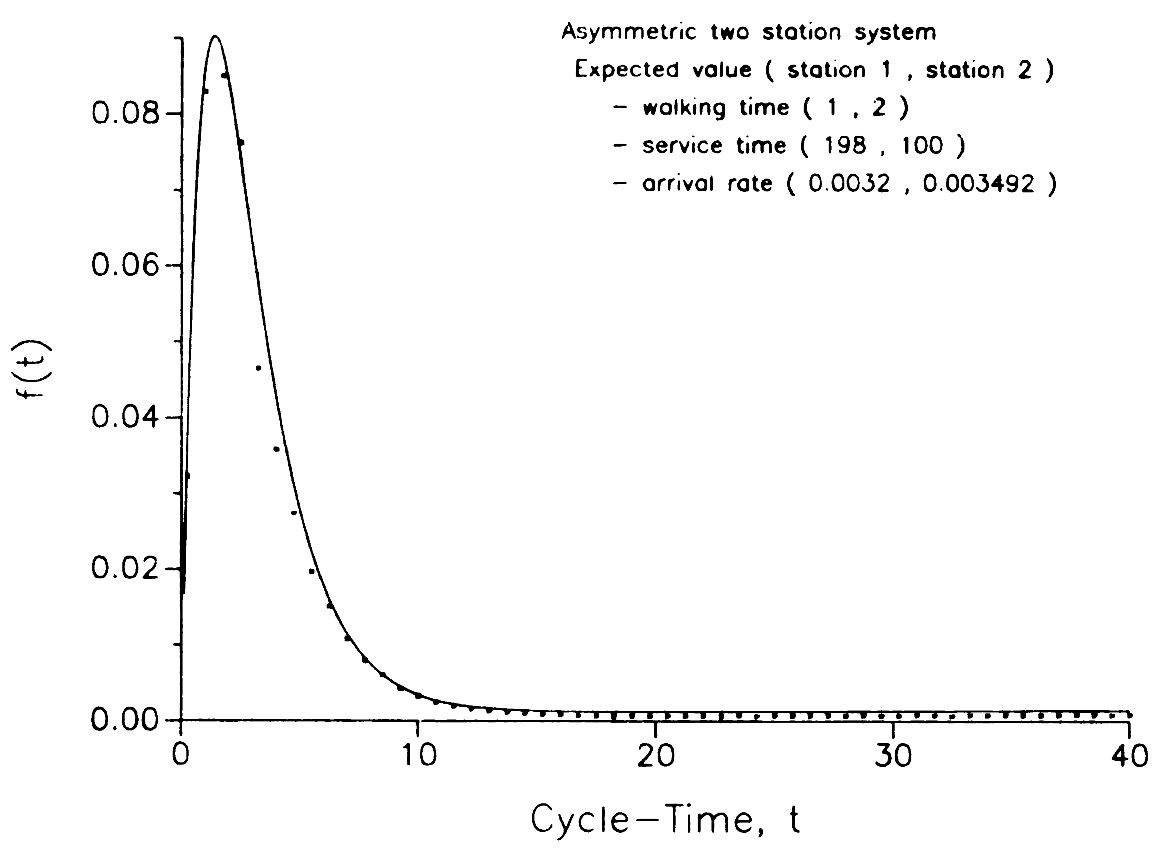


Fig.4b Enlarged view of peak in SV density

most four distributions of independent, exponentially distributed random variables. Thus, the (embedded) cycle-time distribution witnessed by the observer at the instants of vector transfers is given by

$$F_C(c) = \pi_{00}\, F_{00}(c) + \pi_{01}\, F_{01}(c) + \pi_{10}\, F_{10}(c) + \pi_{11}\, F_{11}(c) \qquad (12)$$

The method for computing q_i described above is one in which a system of equations must be solved. In general for an N station system possessing ergodicity, the transition matrix (and the conservation equation) will yield 2^N equations for the 2^N limiting state probabilities associated with the service vectors of the system. In order for the invariant vector to be determined, it is essential that the N probabilities q_1, ..., q_N first be determined. Thus, an N station system will have $2^N + N$ unknowns. Having already accounted for the 2^N equations obtained from the transition matrix, the N mean conditional cycle-times (Eq.(1), chapter VII) fully determine the system. Figure 4a depicts the analytic versus simulated cycle-time densities for this moderately loaded system. Figure 4b is an enlarged view of the extremely peaked behaviour of the density for small cycle-times.

An alternate method is one in which all the limiting state probabilities are expressed in terms of the unknowns q_i, $i = 1,\ldots,N$. An expression can now be set up involving the limiting state probabilites and the mean unconditional cycle-time. Equation (9) tells us what $E(C)$ should be in terms of the limiting state probabilities, while on the other hand Eq.(5) of chapter V tells us what the exact value of $E(C)$ is. Denote the estimate of mean cycle-time obtained from EQ.(9) as C_q. The problem then reduces to choosing a point $(q_1, q_2, .., q_N)$ in the N-dimensional unit cube such that the (nonlinear) constraint $|E(C) - C_q|$ is minimized. Still another method, involving entropy maximization, can also be used. This method is used in chapter VIII to solve a similar problem.

3.4 Invariance Of Cycle-Times

We obtain the result that for a general asymmetric system, the distribution of the observer's scan cycle-time is independent of the index of the station from which the observer makes the measurement. At first glance, this result appears surprising since two different stations must possess different transition probabilities corresponding to the same vector transition. But, note that the cycle-time random variable can be defined as the time taken by the token to make one complete traversal of the (logical) ring. This random time must possess the same distribution regardless of the reference point where the measurement is made. This is an interesting property, in that the cycle-time distribution really is unique.

Let A_1, A_2, ..., A_N be the N binary valued random variables associated with service events at the respective stations during any steady-state cycle. Define $p_i = P(A_i = 1)$ to be the unconditional probability that a customer is queued at station i, waiting for the server during an arbitrary steady-state cycle, and let $P(A_i = 0)$ to be the probability that station i is empty. Observe that these N random variables are *not* independent. However, given the limiting probabilities for the 2^N vectors, we can proceed to solve for the probabilities p_i as follows. We make use of the identity

$$P(A_1 \vee A_2 \vee \ldots A_N) = \sum_{i=1}^{N} P(A_i) - \sum_{i<j=2}^{N} P(A_i \wedge A_j) + \sum_{i<j<r=3}^{N} P(A_i \wedge A_j \wedge A_r) + \ldots. + (-1)^{N-1} P(A_1 \wedge A_2 \wedge \ldots \wedge A_N) \tag{13}$$

Equation (13) expresses the joint distribution of the N random variables A_i, $i = 1,\ldots,N$, in term of $2^N - 1$ unknowns. Since the 2^N limiting state probabilities in $\{\pi_i\}$ are really joint probabiities, Eq.(13) can be used to solve for the joint distribution of an arbitrary combination of the A_i or even the marginals, $P(A_i) = p_i$, $i = 1,\ldots,N$. The following results are proved in the appendix.

Lemma 3.1:

Let $\mathbf{P}_j$ be the embedded probability transition matrix for scan-cycles obtained with respect to station j, $j \in \mathbf{S}$. Then, the probability transition matrix $\mathbf{P}_k$ for station $k = (j) \bmod N + 1$ can be obtained as

$$\mathbf{T}^j(\mathbf{P}_j) = \mathbf{P}_k$$

where $\mathbf{T}$ is a transformation performed on each entry of the matrix $\mathbf{P}_j$ and the superscript j indicates that the transformation is performed on the j^{th} multiplicand of each matrix element.

Lemma 3.2:

Let $\mathbf{P}_j(\mathrm{s})$ denote the embedded probability transition matrix for scan-cycles obtained with respect to station j, $j \in \mathbf{S}$. Let $\mathbf{P}_j(\mathrm{d})$ denote the corresponding probability for dep-cycles. Then, for each $j \in \mathbf{S}$,

$$\mathbf{P}_k(\mathrm{s}) = \mathbf{P}_j(\mathrm{d})$$

where $k = j \bmod N + 1$.

Theorem 3.3:

Let $F_j(c)$ and $F_k(c)$ be cycle-time distributions measured with respect to (scan or departure points of) reference stations j and k, $j \neq k$, $j,k \in \mathbf{S}$. Then,

$$F_j(c) = F_k(c) = F(c)$$

3.4 Summary

In this chapter, we have shown that service-cycles on token-passing systems can be uniquely associated with binary service vectors. Using a fixed station as a reference point, the consecutive service vectors seen at the reference station form consecutive states of a Markov chain, and the corresponding service-cycle lengths behave as renewal times in a Markov renewal process. Given the four input distributions to a token-passing configuration (i.e., arrival, service, walk, and switching) it is possible to completely formulate the behaviour of the cycle-time process in a Markov renewal framework, consequently allowing for the derivation of a (unique) cycle-time distribution.

A result of some importance has not been stressed in this chapter. This is the fact that the distribution of cycle-time can be reduced to a function of the mean cycle-time. The only other parameters involved are the N probabilities that each of the N stations has exactly one customer queued for service. There are methods for determining these probabilities given the mean cycle-times. Additional results include invariance properties for the cycle-time random variable, insensitivity to reference station, and insensitivity to type of cycle.

The algorithm required to compute the distribution of cycle-time as defined in Eq.(9) is an exponential algorithm. This will always be the case for an asymmetric system since we must deal with the 2^N states of an N-bit binary vector. For symmetric and partially symmetric systems the complexity will generally be less than exponential. In the symmetric case, the expression in Eq.(9) can be put into a binomial form. Note that for asymmetric systems the matrices involved will be of the order 2^N, while for symmetric systems the matrices will be of the order N. The results obtained in the last section can be proved easily for symmetric systems using the notion of *exchangeable* random variables [JoKo77]. Only in the case of pure symmetry will certain key random variables be exchangeable. Symmetric systems are by far more simple to deal with than asymmetric systems. For example, Eq. (9) requires the computation of only N Erlang (*not* generalized Erlangian) distributions, if we are working with exponential random variables. In the asymmetric case, this will require 2^N generalized Erlangian distributions. It is possible to reduce this to N distributions by performing certain approximations. But this necessarily introduces some degree of error in the result. In conclusion, the results of this chapter are very new and as a consequence there is considerable scope for various extensions and applications of renewal theory, reversibility, numerical analysis, and several specialized results.

CHAPTER IV

PERFORMANCE MEASUREMENTS USING SERVICE VECTORS

If we allow the arrival process at any one station, say reference station j, to be Poisson, then the queueing process at this station can be placed in the framework of a queue with semi-Markovian service times. In order to do this we require the ideas developed in chapter III on cycle-time distributions. The critical pieces of information are the probability transition matrix P with entries given by Eq.(3.5), and the distribution functions $F_{C(z)}(.)$, for $z\in\Theta$. There is a difference between the queueing process at the reference station and the classical M/SM/1 queue [Neut66, Cinl67]. In the latter system arriving customers are assumed to belong to a finite class of customer types, and the service time of the n^{th} customer is a function of the types of customers $(n-1)$ and n. So far the situation is the same with our queueing process at station j. In the classical M/SM/1 queue, an idle period of the server begins when the server finds the queue empty. In the queueing process at station j, there really is *no* idle period. When the queue is found empty, the server begins another cycle (with respect to station j), in order to make an attempt to service each of the N stations once again.

A reference station customer who is positioned at the head of the queue upon arrival does not find an idle server (or initiate a busy period) as would be the case if the queueing system was M/SM/1. Instead, this customer must wait for a random time that is the forward recurrence time of the cycle-time corresponding to the particular cycle in progress when the customer arrives. Due to this difference, we call our queueing process an *M/SM/1 queue with vacationing server*. With the aid of the invariance properties for cycle-times, i.e., the insensitivity of cycle-times to scan points and to departure points, we can freely move the observer from scan points to departure points (in a consistent manner) and vice-versa, to help with the analysis. The reference station index we use must remain fixed for the duration of the analysis, since we obtain results for only one queue at a time.

In section 4.1 we embed the queueing process viewed by the observer at station j in the M/SM/1 queue with vacationing server framework. The busy periods and the vacation periods of the server are defined in section 4.1. In order to describe the distributions of the busy and vacation periods it is convenient to resort to distributions of phase type, or PH-distributions (see [Neut81]). This is done in section 4.2, along with a description of the different service times required by customers of different types. We make the assumption of exponentially distributed service, walk, and switching times at each station. This is done for convenience and is not a necessary assumption.

In section 4.3 we define a three dimensional (countable state) Markov renewal matrix that describes time-considered transitions for queue length, successive customer types, and corresponding service times for station j customers. Also, a stability criterion for the queueing process at station j is stated. The distributions for packet queue length and packet queueing delay are obtained in section 4.4 and section 4.5, respectively. In section 4.6 we apply the methods of this chapter to the two station example presented in chapter III, and in section 4.7 we introduce a random variable to represent the system's throughput, or token utilization.

4.1 Busy Periods, Vacation Periods, And Service Times

Assume that an observer is positioned at the reference station's scan point and is recording service vectors for scan-cycles. Each vector is of the form $z = <z_j, \ldots , z_{j-1}>$, where $(j-1)$ is meant to indicate a station i such that $j = i \bmod N + 1$. Define $v_k(z)$ to be the k^{th} entry of such a service vector, and let $\Theta_1 = \{z \in \Theta | v_1(z)=1\}$ be the set of all service vectors whose corresponding cycles include service of a reference station customer. Define its complement in Θ to be $\Theta_0 = \Theta \backslash \Theta_1$. For each (binary) service vector z recorded by the observer, $z \in \Theta$, let $d(z)$ be its decimal representation. Observe that $0 \leq d(z) \leq 2^N-1$. Let T_0 be the instant at which the server completes some cycle and hands over a service vector z to the observer. The observer instantaneously identifies the most recent cycle-time as either the service time of a customer of type $d(z)$ if $z \in \Theta_1$, or a vacation-time from station j if $z \in \Theta_0$. Since it is possible for one or more such vacation times to occur consecutively, a vaction period is the sum of consecutively occuring vacation-times.

Consider a sequence of random pairs $\{Z_k, T_k\}$, where T_k is the instant at which the observer hands over vector Z_k to the observer. This corresponds to the completion of the k^{th} cycle defined with respect to scan-cycles at station j. Without loss of generality we take $T_0 = 0$, and Z_k to be defined only for $k \geq 1$. We define a *busy period* of the server to be the random time b defined by

$$b = T_{e-1} - T_{n-1} \tag{1a}$$

where

(1) $e > n$, for $e, n \in I^+$,

(2) $Z_e \in \Theta_0$, $Z_n \in \Theta_1$, $Z_{n-1} \in \Theta_0$, and

(3) $Z_m \in \Theta_1$, $n \leq m < e$.

A *vacation period* of the server is the random time v defined by

$$v = T_{n-1} - T_{e-1} \tag{1b}$$

where

(1) $n > e$, for $e, n \in I^+$,

(2) $Z_e \in \Theta_0$, $Z_n \in \Theta_1$, $Z_{e-1} \in \Theta_1$, and

(3) $Z_m \in \Theta_0$, $e \leq m < n$.

Let (Z_n, T_n) be the state of the two dimensional stochastic process at the end of the n^{th} cycle and let $Z_n = z_n \in \Theta_1$. The random service time of the customer who entered service at time T_n depends on the vector that the server gives to the observer at the instant T_{n+1}. At this instant, the observer identifies the customer to be of type $d(z)$, and the service time distribution of this customer is thus found to be $F_{C(z)}(.)$. This distribution was defined for an arbitrary vector z in Eq.(3.2) and in terms of cycle-times in Eq.(3.8).

4.2 Distributions Of Busy And Vacation Periods Of The Token

In order to cast our M/SM/1 queue with vacationing server into an M/SM/1 framework in the sense of Neuts [Neut66] we are required to make certain modifications. We permute the rows and columns of the transition matrix **P** in order to obtain a new matrix $\mathbf{P}^*$. Let the columns of **P** (in increasing order) be associated with the elements $c_1,\ldots,c_n$, for $n = 2^N$, each $c_i \in \Theta$. Apply a (non-unique) permutation ω on the vector $(d(c_1),\ldots,d(c_n))$ to obtain the matrix $\mathbf{P}^*$ as

$$\mathbf{P}^* = \begin{vmatrix} \mathbf{P}_0 & \mathbf{P}_{01} \\ \mathbf{P}_{10} & \mathbf{P}_1 \end{vmatrix} \tag{2}$$

The intention of doing a permutation ω on rows and columns of **P** is to obtain $\mathbf{P}_0$ and $\mathbf{P}_1$ as probability matrices describing transitions for Markov chains in Θ_0 and Θ_1, respectively. $\mathbf{P}_{01}$ is the transition matrix describing probabilities of transitions from states of Θ_0 to states of Θ_1, and its dual matrix $\mathbf{P}_{10}$ defines probabilities of transitions from Θ_1 to Θ_0. We relabel the rows (and columns) of $\mathbf{P}^*$ as $1,2,\ldots,n$. Note that this corresponds to a one-to-one mapping $\mathbf{r}_0:\Theta_0 \to \mathbf{L}$, where $\mathbf{L} = \{1,2,\ldots,n\}$. If we restrict our attention to the set $\mathbf{L}_0 = \{1,2,\ldots,m\}$, $m = 2^{N-1}$, and the elements of Θ that are mapped onto $\mathbf{L}_0$, we obtain a Markov chain on the integers $\{1,2,\ldots,m\}$ with transition matrix $\mathbf{P}_0$.

Applying a similar mapping $\mathbf{r}_1:\Theta \to \mathbf{L}_1$, where $\mathbf{L}_1 = \mathbf{L}\backslash\mathbf{L}_0$. The matrix $\mathbf{P}_1$ can thus be considered as the transition matrix of a Markov chain on the integers $\{m+1,\ldots,n\}$. Since each element x in $\mathbf{L}_1$ can be uniquely associated with an element y in $\mathbf{L}_0$ by the linear transformation $y = x - m$, we may conveniently consider $\mathbf{P}_1$ to describe the transitions of a Markov chain on the integers in $\mathbf{L}_0$.

When dealing with our relabelled states, if we wish to recover an element $z \in \Theta$ associated with some integer x of $\mathbf{L}_0$, we do the following. If the transition is in $\mathbf{P}_0$, apply the inverse permuation ω^{-1} on vector $(1,\ldots,m)$ and identify the element z such that $\omega(.,.,d(z),.) = (.,.,x,.)$. If the transition is in $\mathbf{P}_1$, first obtain the vector whose entries are translated by the positive quantity m and then apply the inverse permutation to discover z as before. Note that the relabelling procedure described is not necessary if we begin with a probability transition matrix with row (and column) indices already in ascending order.

Let $\mathbf{N}^0$ and $\mathbf{E}^0$ be the column vectors whose i^{th} entries are given by

$$\mathbf{N}^0(i) = \sum_{1 \le k \le m} \mathbf{P}_{01}(i,k) \tag{3}$$

$$\mathbf{E}^0(i) = \sum_{1 \le k \le m} \mathbf{P}_{10}(i,k) \tag{4}$$

Define Markov chains on the states $\{1,2,\ldots,m+1\}$ with transition matrices obtained as

$$\mathbf{N} = \begin{vmatrix} \mathbf{P}_0 & \mathbf{N}^0 \\ \mathbf{0} & 1 \end{vmatrix} \quad \text{and} \quad \mathbf{E} = \begin{vmatrix} \mathbf{P}_1 & \mathbf{E}^0 \\ \mathbf{0} & 1 \end{vmatrix} \tag{5}$$

where $\mathbf{P}_0$ *and* $\mathbf{P}_1$ are the substochastic matrices defined earlier, satisfying the property that $(\mathbf{I} - \mathbf{P}_0)$ and $(\mathbf{I} - \mathbf{P}_1)$ are nonsingular matrices, where I denotes the identity matrix.

We motivate our approach to obtaining the token's busy and vacation period distributions by recalling the definitions in Eqs.(1a) and (1b). A busy period b is initiated at the first instant that the server transfers a vector of Θ_1 to the observer and terminates at the first instant that the server transfers a vector of Θ_0 to the observer. Similarly, the vacation period v of the token (again, defined with respect to the reference station) is the dual period starting with a vector transfer involving Θ_0 and ending with a vector transfer involving Θ_1. Since a busy period begins when a vacation period ends and vice-versa, we see that the random points in time at which vector transfers are made correspond to points in an alternating renewal process. We are interested in the distributions of times between alternating renewals.

Let $F_V(.)$ denote the distribution function of a token-vacation period. A cycle that generates a service vector $z \in \Theta_0$ is understood to be a cycle of the server's vacation period and is called a *vacation cycle*. A token-vacation period is comprised of a random number of vacation cycle-times. Recall that for an asymmetric system, vacation cycle-times are not i.i.d random variables. Each vacation cycle-time will be a sum of independent random variables, and will have a distribution given by a finite convolution. Since each vacation type cycle occurs with a certain steady state probability π_z^j, $z \in \Theta\ U$, there is a natural way to formulate the distribution of a random vacation cycle-time as a finite mixture of finite convolutions. Since a token-vacation period is comprised of a random number of such vacation cycle-times, the distribution of a token-vacation period is a compound distribution, given by repeated convolutions of the mixture. Because of the repeated convolutions required, this is generally not a convenient form for direct numerical computation. An alternative approach to describing token-vacation periods is to resort to distributions of *phase-type* [Neut81].

Consider an $(m+1)$-state continuous-time Markov process with m transient states and a single absorbing state. Denote its infinitesimal generator to be the matrix $\mathbf{Q}'$, given by

$$\mathbf{Q}' = \begin{vmatrix} \mathbf{T} & \mathbf{R}^0 \\ \mathbf{0} & 0 \end{vmatrix} \qquad (6)$$

where $\mathbf{T}$ is a nonsingular square matrix of order m with all diagonal elements negative and off-diagonal elements nonnegative. Define the m-vector $\mathbf{e}$ to have all entries as 1. The m-vector $\mathbf{R}^0$ has only nonnegative entries and is equal to $-\mathbf{Te}$. Define $\nu = (\nu_1,\ldots,\nu_m)$ and let (ν, ν_{m+1}) be the vector of initial probabilities, satisfying $\nu\mathbf{e} + \nu_{m+1} = 1$, for $0 \le \nu_{m+1} < 1$. The probability distribution $F(.)$ of the time till absorption in the state $(m+1)$ is given by

$$F(x) = 1 - \nu \exp(\mathbf{T}x)\mathbf{e} \qquad \text{for } x \ge 0. \qquad (7)$$

The distribution $F(.)$ is said to be of *phase-type* (i.e., F is a PH-distribution). The pair $(\nu,\mathbf{T})$ is called a *representation* of $F(.)$. We will always assume that $\nu_{m+1} = 0$, so that $F(.)$ does not have a jump at 0. Under the assumption that the representation guarantees each state a positive probability of being visited prior to absorption, the

Markov chain with generator $\mathbf{T} + \mathbf{R}^{0}\nu$ is irreducible. The moments of $F(.)$ about the origin all exist and are given by

$$\mu(k) \quad = \quad (-1)^{k}(k!)\nu\mathbf{T}^{-k}\mathbf{e} \qquad \text{for } k \geq 1. \tag{8}$$

In the following discussion we present a scheme that allows us to view $F_V(.)$ as a PH-distribution. The same argument can be used to obtain the phase representation of the token-busy period distribution, but we do not pursue that distribution any further. Assume that each of the N stations utilizes exponentially distributed random variables for service, switch, and walk times, with parameters given by $1/\beta_i$, $1/\gamma_i$, and $1/a_i$, respectively, for each station i, $i \in \mathbf{S}$. This assumption is made for analytic convenience, since generalized Erlangian distributions lend themselves very easily to phase representations. For any vector $z \in \Theta_0$, the distribution $F_{d(z)}(.)$ is an Erlangian distribution with a phase representation described as follows. Let $\mathbf{a}_{d(z)}$ be a $2N$-bit vector with all entries as zero except the very first entry, which is a 1. Then the pair $(\mathbf{a}_{d(z)}, \mathbf{T})$ is said to be a representation of $F_{d(z)}$, with entry (i,k) of the order $2N$ square matrix $\mathbf{T}$ given by

$$\mathbf{T}_{ik} \quad = \quad \begin{array}{l} -a_i,\ i \text{ odd-valued},\ i = k \\ a_i,\ i \text{ odd-valued},\ k = (i+1) \\ \upsilon_i(z)\ \beta_i \ + \ (1 - \upsilon_i(z))\ \gamma_i,\ i \text{ even-valued},\ i = k \\ -\upsilon_i(z)\ \beta_i \ - \ (1 - \upsilon_i(z))\ \gamma_i,\ i \text{ even-valued},\ k = (i+1) \end{array} \tag{9}$$

Since the process of observer transitions in the set Θ_0 is not a Markov process (i.e., it is *semi-Markov*) we cannot apply Eq.(7)

directly. Instead, it is necessary to define a semi-Markov point process that is a very special case of the "versatile Markovian point process" introduced by Neuts [Neut79]. The number of generalized Erlangian distributions (or PH-distributions) required to completely specify the different vacation cycle-times in Θ_0 is $m = 2^{N-1}$. Each vacation cycle-time is thus comprised of $2N$ phases of a generalized Erlangian distribution, N phases given by the service and switching times, and N phases given by the walk times. We define a point process with events governed by epochs of transitions of the Markov chain $\mathbf{N}$ defined in Eq.(5). Note that each state of this order $(m+1)$ chain has a positive probability of being visited before absorption. If the Markov chain has made a transition to the state i, $1 \le i \le m$, the next transition is to state k, with probability p_{ik}, and the time between transitions has a PH-distribution $F_i(.)$ of order $2N$.

Define the vector $(\pi_1, \pi_2, \ldots , \pi_n)$ to be the invariant vector of the probability transition matrix in Eq.(2), for $n = 2^N$. Recall that π_i is the probability that in steady state, the observer sees the server in state $z \in \Theta$, with $i = d(z) + 1$. We are only interested in those vectors z, $z \in \Theta_0$. Define the vector $\nu = (\nu_1, \ldots , \nu_m)$ to be a set of (normalized) vectors corresponding to probabilities between vectors in Θ_0, given that we are restricted to this set. In other words, $\nu_i = [\pi_i/(1 - \pi_{m+1} - \ldots - \pi_N)]$, for $1 \le i \le m$.

Let S(t) and π(t) denote the state of the Markov chain $\mathbf{N}$ at time t, and the phase of the Markov chain $T_{S(t)}$ at time t, respectively. Assume that the current time instant is t and the last event occured at time τ, at which time the Markov chain $\mathbf{N}$ made a transition to the state

$S(\tau) = k$. Recall that the initial vector chosen for the Markov chain $\mathbf{T}_k$ is of order $2N$, given by $\mathbf{a}_k = (1,0,\ldots..)$ for all k, $1 \le k \le m$. In the interval $(\tau, t]$, the Markov chain $\mathbf{T}_k$ triggered through zero, one, or more than one transition, without entering its absorbing state. At time t, $S(t) = k$, and the Markov chain $\mathbf{T}_k$ is in phase $\pi(t)$. It is necessary to make the assumption that for each $t > 0$, the intervals of time between events is conditionally independent, given the path function of the Markov chain $\mathbf{N}$. Under this assumption $\{S(t), \pi(t)\}$ is a continuous-time Markov process.

Given the generator of the process $\{S(t), \pi(t)\}$, it is now possible to describe the distribution $F_V(.)$ as a PH-distribution. Let $\mathbf{e}'$ be a $2N$-bit column vector with each entry equal to 1. Define an order $m*2N$ square block-partitioned matrix $\mathbf{A}^*$ with block-entry (i,k) as $\mathbf{A}^*(i,k) = p_{ik}\, \mathbf{e}'\mathbf{a}_k$, where the vector product denotes the product of a column vector by a row vector. Define also an order $m*2N$ square block-diagonal matrix $\mathbf{T}^*$ with diagonal block-entry i as $\mathbf{T}_i$. The infinitesimal generator of the Markov process $\{S(t), \pi(t)\}$ is given by $\mathbf{T}' = \mathbf{T}^*(\mathbf{I} - \mathbf{A}^*)$, where $\mathbf{I}$ is the order $m*2N$ identity matrix. Thus, the token's vacation-period distribution is a PH-distribution, with the representation $(\nu, \mathbf{T}')$. A similar procedure using the transition matrix $\mathbf{E}$ will yield the token's busy-period distribution as a PH-distribution.

4.3 The Countable State Markov Renewal Matrix

Let **D** be a square matrix of order m with entry (i,k) as the service-time distribution of customer type i (or cycle-time distribution for vector of type i, $i \in \Theta_1$), for each k, $1 \le k \le m$. Thus each row of the matrix is the same. Given the current customer's service-time, the type of the next customer and service time of the current customer are both defined through **D**. Given the m customer types, with corresponding service vectors coming from Θ_1, we require a probability transition matrix describing the probability transitions between the various customer types. Consider the matrix $\mathbf{P}_1$ defined in section 4.2. Define the row sum of the i^{th} row of this matrix to be s_i, for $1 \le i \le m$. Define a new matrix $\mathbf{E}^*$ as

$$\mathbf{E}^*(i,k) \quad = \quad \mathbf{P}_1(i,k)/s_i \tag{10}$$

for $1 \le i,k \le m$.

Let **A** be a square matrix of order m with entry (i,k) given by $\mathbf{A}(i,k) = \mathbf{D}(i,k)\ \mathbf{E}^*(i,k)$. It can be shown that **A** is aperiodic and irreducible and satisfies the property that $\mathbf{A}(+\infty) = \mathbf{E}^*$.

Denote the row sum distributions of $\mathbf{A}(.)$ by $\mathbf{H}_i(.)$, $i = 1,2, \ldots, m$. Let η_i be the mean service time of customer type i, and let $\eta = (\eta_1,\ldots,\eta_m)$. Observe that η_i is the (finite) mean of the distribution $\mathbf{H}_i(.)$. If $\zeta = (\zeta_1,\ldots,\zeta_m)$ is the limiting invariant vector of $\mathbf{E}^*$, the traffic intensity at station j is given by

$$\rho \quad = \quad \rho_j \quad = \quad \lambda\zeta\eta \tag{11}$$

where we must assume that $\rho < 1$ for well defined steady state distributions to exist (see [Neut77]). The triple defining queue lengths, customer types at departure times, and various customer service times or cycle-times, leads to the classical definition [Neut66] for an embedded Markov renewal sequence. The matrix of transition functions for this sequence is given by

$$\mathbf{Q}(x) = \begin{vmatrix} B_0(x) & B_1(x) & B_2(x) & B_3(x) & \cdots \\ A_0(x) & A_1(x) & A_2(x) & A_3(x) & \cdots \\ 0 & A_0(x) & A_1(x) & A_2(x) & \cdots \\ 0 & 0 & A_0(x) & A_1(x) & \cdots \\ 0 & 0 & 0 & A_0(x) & \cdots \\ . & . & . & . & . \end{vmatrix} \qquad (12)$$

where $$A_k(x) = \int_0^x P(k,t)\, dA(t), \qquad \text{for } k \geq 0, \qquad (13)$$

and $P(k,t)$ is the generating function of the Poisson arrival process at station j, describing the probability that k customers arrive during $(0,t]$. Using "*" to denote convolution, define

$$B_k(x) = F_V(x) * A_k(x), \text{ for } k \geq 0. \qquad (14)$$

4.4 Distribution of Packet Queue Length

Define $\mathbf{x} = \{x(i,k), i \geq 0, 1 \leq k \leq m\}$ to be the invariant probability vector of the matrix $\mathbf{Q}(\infty)$. This vector defines the stationary joint probability for packet queue length and packet type, at the *scan instants* of reference station j. The stationary infinite vector $\mathbf{x}$

consists of m-vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots$ etc. Similarly, define $\mathbf{y} = \{y(i,k), i \geq 0, 1 \leq k \leq m\}$ to be the joint probability for packet queue length and packet type at an arbitrary instant in time for a stationary queue. This infinite vector consists of m-vectors $\mathbf{y}_0, \mathbf{y}_1, \ldots$ etc. The k^{th} entries in $\mathbf{x}_0$ and $\mathbf{y}_0$ describe the embedded and arbitrary-time probabilities that station j has no packets queued, and that the next type of packet to enter service is associated with cycle vector type k. It remains to solve the equation $\mathbf{xQ} = \mathbf{x}$. The solution may be written as

$$\mathbf{x}_i = \mathbf{x}_0 B_i(\infty) + \sum_{k=1}^{i+1} \mathbf{x}_k A_{i-k+1}(\infty), \qquad \text{for } i \geq 0 \tag{15}$$

The unique solution to the nonlinear matrix equation comprised of substochastic matrices

$$G = \sum_{k=0}^{\infty} A_k G^k \tag{16}$$

is given by (see [Neut77]) a matrix G which is both positive and stochastic. Let $\mathbf{g}$ be the invariant probability vector of G. Let $\mathbf{X}(t)$ be the probability generating function of the distribution $\mathbf{x}_0, \mathbf{x}_1, \ldots$ in terms of complex variable z, for $|z| \leq 1$. Denote the Laplace-Stieltjes transform of the Markov renewal matrix A defined in section 4.3 as $\underline{A}(s)$. By using the system in Eq.(15), we obtain

$$\mathbf{X}(z) = (1 - \rho)(z-1)\mathbf{g}\underline{A}(\lambda - \lambda z)[zI - \underline{A}(\lambda - \lambda z)]^{-1} \tag{17}$$

If $\mathbf{Y}(z)$ is the probability generating function for the components of $\mathbf{y}$, for $|z|<1$, then in a similar fashion ([Neut77]) we obtain

$$\mathbf{Y}(z) = (1-\rho)\mathbf{g} + z(1-\rho)\mathbf{g}[\underline{\mathbf{A}}(\lambda - \lambda z) - z\mathbf{I}]^{-1}[\mathbf{I} - \mathbf{H}(\lambda - \lambda z)] \quad (18)$$

Analagous to the M/G/1 situation, we find that

$$\mathbf{x}_0 = (1-\rho)\mathbf{g} = \mathbf{y}_0 \quad (19)$$

By using a recursive method, Neuts [Neut77] obtains derivatives (at 0^+) of the Perron-Frobenius eigenvalues of the matrix $\underline{\mathbf{A}}(s)$ and conseqently (complicated) expressions for the moments of the queue length distributions. In particular, the restricted mean packet queue length at station j is obtained as

$$\begin{aligned} L = {} & \lambda \mathbf{g}(\mathbf{I} - \mathbf{E}^* + \Pi)^{-1}\eta \\ & + \{ -2\rho^2 + \lambda^2[\zeta\eta^{(2)} - 2\zeta\underline{\mathbf{A}}'(0^+)(\mathbf{I} - \mathbf{P} + \Pi)^{-1}\eta]\}/2(1-\rho) \end{aligned} \quad (20)$$

where Π is the matrix having all m rows equal to ζ, and $\eta^{(2)}$ is the vector whose entries are the second moments of the m different cycle-times (service-times) from Θ_1.

4.5 Distribution of Packet Queueing Delay

Following Neuts [Neut77], the virtual delay of a packet can be obtained as follows. The Laplace-Stieltjes transform $\mathbf{W}(s)$ for the virtual delay distribution is given by

$$\mathbf{W}(s) = s\mathbf{x}_0[(s-\lambda)\mathbf{I} + \lambda\underline{\mathbf{A}}(s)]^{-1}\mathbf{e} \quad (21)$$

where $\mathbf{e}$ is an m-bit unit vector, and $\mathbf{x}_0 = (1-\rho)\mathbf{g}$. Define a vector $\mathbf{U}(s)$ (to enables us to compute the moments of virtual delay) as

$$\mathbf{U}(s)\{(s-\lambda)\mathbf{I} + \lambda\underline{\mathbf{A}}(s)\} = (1-\rho)s\mathbf{g} \tag{22}$$

from which the mean packet delay W is obtained (in accordance with the classical formula due to Little) as

$$\begin{aligned} W &= -\mathbf{U}'(0^{+})\mathbf{e}. \\ &= (L - \rho)/\lambda \end{aligned} \tag{23}$$

The interpretation of the transform in Eq.(21) is as follows. Let $\mathbf{W}_k(x)$ be the stationary probability that a virtual packet arriving at time $t = 0$ waits for a length of time not exceeding x and initiates a service-vector that is mapped into element k of the set $\mathbf{L}_0$. Then the k^{th} entry in $\mathbf{U}(s)$ is the Laplace-Stieltjes transform of the corresponding entry in $\mathbf{W}(s)$. An inversion performed on Eq.(21) will yield a Volterra system of integral equations

$$\mathbf{W}(t) = (1-\rho)\mathbf{g} + \lambda \int_0^t \mathbf{W}(y)[\mathbf{I} - \underline{\mathbf{A}}(t-y)]dy, \quad \text{for } t \geq 0. \tag{24}$$

Note that the system in Eq.(24) is computationally attractive once $\mathbf{g}$ has been obtained. In implementing a numerical solution, a check on the accuracy of the implementation is easily made by observing the property $\mathbf{W}(\infty) = \boldsymbol{\zeta}$ for $s \to 0$ in Eq.(21). Computationally useful formulas for the moments of delay time can be derived [Neut85].

4.6 An Application Of The SV Method

Consider the two-station example presented in section 3.3. The matrix $\mathbf{E}^*$ describing customer transitions at station 1 is obtained from Eq.(10). Recall that $\mathbf{g}$ is the invariant probability vector of a stochastic matrix G that is the solution to the nonlinear equation in (16). In Theorem 6 of [Neut74], it is shown that the sequence of matrices G_k defined by

$$G_0 = A_0, \quad G_{k+1} = \sum_{n=0}^{\infty} A_n G_k{}^n, \qquad k \geq 0,$$

is non-increasing and converges to the minimal non-negative solution G of the equation defined in Eq.(16). The matrix G may be computed by the method of successive substitutions. For n restricted to a maximum value of 150, and 30 iterations, we obtain G and $\mathbf{g}$ for station 1 as

$$G = \begin{vmatrix} .523 & .477 \\ .051 & .949 \end{vmatrix} \quad \text{and} \quad \mathbf{g} = \begin{vmatrix} .096 \\ .904 \end{vmatrix}$$

Using the limiting vector $\zeta = (0.085, 0.915)$, $\lambda = 0.0032$, and $\eta = (201, 301)$, the traffic intensity (for station 1) is obtained from Eq.(11) as $\rho = 0.9359$. Finally, on using Eq.(20), we obtain the conditional mean queue length at station 1 as $L = 9.9123$, and the restricted mean waiting time for this station's customers as $W = 2805.13$. It is possible to compute the unconditional moments for the queue length and waiting time distribution with some additional effort, but we do not try for maximum generality in this direction.

4.7 Distribution Of Channel Throughput

Since the stationary cycle-time random variable is independent of station indices, without any loss of generality we can assume that the reference index $j = 1$. For a given service vector $z = \langle z_1, z_2, \ldots, z_N \rangle$ and corresponding service-cycle $C(z)$, define the random time $V(z)$ to be the time that the channel is actually being used for station transmissions (and not for overhead). That is,

$$V(z) = z_1X_1 + z_2X_2 + \ldots\ldots + z_NX_N \qquad (25)$$

For each $z \in \Theta$, define the ratio random variable $U(z) = V(z)/C(z)$. A method of obtaining an approximate distribution for server utilization is given as follows. The random utilization of the server corresponding to a service vector z is simply $U(z)$, and the overall server utilization is the random variable U^* given by

$$U^* = \sum_{z \in \Theta} \pi_z \, U(z) \qquad (26)$$

The expression in Eq.(26) describes the channel utilization of the system at the time instants of vector transfer from server to observer. Replacing π_z by ϕ_z in Eq.(26), the corresponding result is obtained for arbitrary time instants. For a data rate of R Mbps, the channel throughput of a token-passing system is obtained as $S^* = RU^*$. Observe that the random variable U^* is defined in the range $[0,1)$.

For each z, the distribution functions of $V(z)$ and $C(z)$ can be obtained by the methods indicated in section 3.2. Given these distributions, the next step is to obtain the distribution of the quotient random variable $U(z)$. This is where a problem arises, forcing us to make an assumption in order to obtain an approximate distribution. Note that $U(z)$ is a ratio of dependent random variables. If we make an assumption of independence, we can proceed as follows. One method is to resort to the Mellin transform which, for a given function $f(x)$, is defined as

$$M_f(s) = \int_0^{\infty} f(x)\, x^{s-1}\, dx \tag{27}$$

Treating $1/C(z)$ as a random variable, we apply transform techniques to obtain the Mellin transform of the product $(V(z))(1/C(z))$ (for distribution functions denoted by their respective random variables) as

$$M_U(s) = M_V\, M_C(2-s) \tag{28}$$

and the inverse transform, or the distribution of each $U(z)$ as

$$F_U(t) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{c-ib}^{c+ib} t^{-s}\, M_U(s)\, ds \tag{29}$$

The complex inversion integral shown above falls in one of four classes [Spri79] each of which determines the transformed function uniquely. In this case, if at least one pole exists, the integrand of the inversion integral can be expressed as a Laurent series, with a unique expansion. The inversion integral may then be evaluated by the method of residues. Since poles can be shown to exist in our case, the next step is to evaluate the integral over the Bromwich path

$(c-i\infty, c+i\infty)$. In any event, we assume that the distribution for U may be obtained by resorting to Bromwich contours and consequently the residue theorem [Spri79]. At this time, we are not interested in the an explicit form for the distribution, but only a method to show that it can be obtained. The approach will necessarily depend on the form of the Mellin transform convolution and any convenient algebraic manipulation suggested by the function in determing its series representation.

An approximate expression for mean system throughput can be obtained by computing $E(C(z))$ and $E(V(z))$, for each vector z, $z \in \Theta$. The latter expectation may be obtained by computing the mean service-time associated with service vector z, and the former expectation is obtained from $E(V(z))$ simply by adding mean switching times for stations not served during this cycle, as well as the sum of all the mean walk times. Thus, both $E(C(z))$ and $E(V(z))$ are known constants, for a given vector z. The mean utilization of the token on a cycle generating service vector z is then obtained as $E(U(z)) = E(V(z))/E(C(z))$. If we take utilization and throughput to mean the same, then approximate mean system throughput is simply given as $\Sigma\ \pi_z\ E(U(z))$, where the summation is over all vectors z, $z \in \Theta$.

It is also possible to obtain an exact distribution for server utilization. We briefly explain how this can be done in terms of our two station example. Since there are four possible service vectors, there are four random cycle-times generated by these vectors. Define the random variable $Y = (Y_1 + Y_2)$ to represent total (walking-time) overhead. Then, the four cycle-times are Y, $U = (Y + X_1)$, $V = (Y + X_2)$, and $W = (Y + X_1 + X_2)$. During three of the four random cycle-times, the server spends some portion of the cycle doing useful work, i.e., serving customers. The three ratio random variables corresponding to useful server utilization times can be defined as

$$R_1 = X_1/(Y + X_1),\ R_2 = X_2/(Y + X_2),\ \text{and}\ R_3 = (X_1 + X_2)/(Y + X_1 + X_2)$$

The server utilization during the zero vector generated cycle-time Y is nil. The vectors $(Y_1/U,\ Y_2/U,\ R_1)$, $(Y_1/V,\ Y_2/V,\ R_2)$, and $(Y_1/W,\ Y_2/W,\ X_1/W,\ X_2/W)$ can each be shown to have a generalized Dirichlet distribution [Spri79]. In oder to use the form in Eq.(26) to define utilization, we must obtain the marginal distributions for the random variables R_1, R_2, and R_3. In the case of R_3 it will be necessary to compute the sum of two dependent random variables once the joint distribution is obtained. Since this is a fairly straightforward matter, the exact distribution of server utilization or system throughput follows.

4.8 Summary

In this chapter, applications of the SV method in obtaining performance measures of the MQCS/s-QED = 1 queueing system are presented. An observer positioned at an arbitrary reference station pictures the queueing system at that station (in isolation) as an M/SM/1 queue with a vacationing server. Unlike the usual server-vacation queueing models, the vacation-periods in this model are random sums of random variables that are conditionally independent but *not* i.i.d. For an appropriate treatment of vacation times, the methods of Neuts [Neut81] can be applied. This involves an application of PH-distributions and some detailed, interesting results obtained by Neuts on the Perron-Frobenius eigenvalue of a Laplace-Stieltjes transform matrix. The latter matrix is obtained from a consideration of dependent cycle-times (chapter III).

The main results of this chapter include generating functions for packet queue length densities and delay times, and first moments of these distributions. Using [Neut77], additional moments may also be obtained. The only necessary assumption is that of Poisson arrivals. Arbitrary distributions may be used for service, walk, and switching on an asymmetric system (as in chapter III). We take a novel approach in viewing the system's channel utilization as a random variable defined on [0,1). An expression is obtained for this random variable, and it is shown how its distribution may be computed. The use of such a random variable is clear. A distributional form for utilization will yield considerably more information on channel behaviour than the usual mean value approach. It is especially simple to compute the mean system throughput using this method. Additional results in this chapter are (exact) mean queue lengths and waiting times for each station, and distributions for token-busy and token-vacation periods with respect to a given station. In 4.6 the two station example presented in Chapter III is further used in order to demonstrate how the SV method may be applied.

CHAPTER V

CYCLE-TIME DISTRIBUTIONS VIA SERVICE PROBABILITIES

In the following analysis, another method of obtaining cycle-time distributions is presented. This is based on a vector of probabilities $(p_1, p_2, \ldots, p_N)$, where p_i is the probability that at steady-state, the server encounters at least one customer queued at station i at its scan instants. The essence of the approach is to decompose C into a sum of independent random variables. The idea was originally used by Hashida and Ohara [HaOh72], and later by Kuehn [Kueh79]. Using negligible switching times, Hashida and Ohara expressed the Laplace-Stieltjes transform for $F_C(.)$ as a product of walk time distribution and service time distribution transforms. This requires a major assumption, i.e., at steady state the probabilities of service events at stations i and j are independent, for $i \neq j$, $i,j \in \mathbf{S}$. This assumption is difficult to justify, especially since dependence between the various stations' queueing patterns can be shown to exist for certain choices of parameters. Intuitively, the longer the period of time spent by the server at any one queue, the higher the probability that the server finds a customer at the next queue.

We use the same approach as in [HaOh72] and [Kueh79] to define C, except that we have included the additional switching time random variables. We assume exponential distributions for service, switching, and walk distributions, in order to obtain an explicit form for the distribution $F_C(.)$. In general, it is possible to work with any distributions, and this can be done for symmetric as well as asymmetric systems. Interestingly enough, dependence between queues does not affect the limiting distribution of a stationary random cycle-time when system load is either very high, or very low. But the effects of dependence can be observed for loads that are not extreme.

From the results of chapter III we know that a sequence of random cycle-times can be shown to possess an m^{th} order Markov dependence, where m varies with the parameters of the system. To determine m as a function of system parameters is not easy. We content ourselves with approximating m by studying sequences of cycle-times, with distributions that are obtained under the independence assumption. In section 5.1, the server's behaviour at the instants of transition between stations and walks is modelled as a Markov chain. In section 5.2, the transition matrix for this chain is acquired via the probabilities p_i, $i = 1,\ldots,N$. The distribution of cycle-time is obtained for asymmetric and symmetric systems in sections 5.3 and 5.4, respectively. A simple result on the existence of limiting distributions is presented in section 5.5, and the property of cycle-time dependence is discussed in section 5.6.

5.1 The Markov Chain Of Server Transitions

Consider the behaviour of the process as the server moves from a walk to a station and from a station to a walk. Due to the nature of server transitions, the sequence of states visited clearly forms a Markov chain. If each station, walk, and switching-action is considered a state, the system will consist of $3N$ states in total. By the process of *lumping* [KeSn60], states of switching can be combined with corresponding stations to reduce the state space to $2N$ states. We are interested in a Markov renewal process $\{Z_n, T_n\}$, where Z_n is defined over the finite set $\mathbf{S}^* = \mathbf{S} \cup \mathbf{W}$. For each $n \in \mathrm{I}^+$, T_n is the (positive) random time spent by the process in the specific state of $\mathbf{S}^*$ defined by Z_n. The kernel of the process is a $2N \times 2N$ matrix $\mathbf{Q}$ (identified with the distributions characterizing the sojourn times of the token in the various states of $\mathbf{S}^*$) defined as follows :

$$Q_{ij}(t) = \begin{cases} p_i U_i(t) & i=w_k \in \mathbf{W},\ j=k \in \mathbf{S} \\ q_i U_i(t) & i=w_k \in \mathbf{W},\ j=w_{k+1} \in \mathbf{W} \\ p_i B_i(t) + q_i S_i(t) & i=k \in \mathbf{S},\ j=w_{k+1} \in \mathbf{W} \end{cases} \quad (1)$$

where $p_i + q_i = 1$ for all $i \in \mathbf{S}^*$. For the interpretation of $\mathbf{Q}$, the ordering of states (aligned with rows and columns) is taken to be $\{w_1, 1, w_2, 2, \ldots, w_N, N\}$.

The functions associated with the Markov renewal process (see section 2.2) can now be interpreted in terms of our model. The basic dynamic particle of our system is the token. Its behaviour in moving among the various states of $\mathbf{S}^*$, as given by Z_n, is governed by a monodesmic semi-Markov process which has a kernel $\mathbf{Q}$. The matrix $\mathbf{P}$ is the matrix of transition probabilities of the underlying Markov chain, which is aperiodic, recurrent, and nonnull. $\mathbf{P}$ describes the transition probabilities of the token from state i to state j. On leaving state w_i, p_i is the probability that the next state visited is i, and q_i is the probability that the next state visited is w_{i+1}. It is shown in section 5.2 that under the independence assumption, the chain is homogeneous in time. It follows that p_i is the probability that the token encounters at least one waiting packet at station i, and q_i is the probability that the token finds queue i empty and switches to the walk before station $(i) \bmod N + 1$. From each state j, $j \in \mathbf{S}$, the token moves to state w_{j+1} with probability 1. The function $H_{ij}(t)$ is the conditional transition time distribution for the time to make a state transition from i to j. For each $i \in \mathbf{S}^*$, the distribution of the sojourn time of the token in state i can be defined by

$$
\begin{aligned}
h_i(t) &= \sum_{j \in \mathbf{S}^*} Q_{ij}(t) \\
&= Pr(\, T_{n+1} - T_n \le t \mid Z_0, \ldots\ldots, Z_n \,) \\
&= Pr(\, T_{n+1} - T_n \le t \mid Z_n = i \,). \qquad (2)
\end{aligned}
$$

We are interested in determining the limiting density of $f_C(.)$, if it exists. Since stationarity can be shown to hold (see section 5.6), the cycle-time C can be expressed as a finite sum of independent random variables with distributions $B_i(t)$, $U_i(t)$, and $S_i(t)$, $i \in \mathbf{S}$. With the aid of the independence assumption, C reduces to a sum of independent random variables.

If the other stations are not in line of sight, the observer at station j sees a single server queueing system. Each customer from this queue keeps the server occupied for a random time C. Thus, to the observer, this system resembles a GI/G/1 queue [Klei75]. In reality, there are two important differences. In a GI/G/1 queueing system, customer service times are i.i.d. In our system, the observer clearly sees the dependence between consecutive server cycles, i.e., customer service times will be positively correlated. The second difference arises with respect to the waiting time of a customer who arrives when the queue is empty. In a GI/G/1 queue, this customer finds the server either serving the previous customer, or idle. In the queue at station j, the server may either be serving the previous customer at station j, or be in some other state of $\mathbf{S}^*$, but never idle. Hence, while a customer at station j records a service time of X_j, the observer records a service time of C for that customer. The key to determining $f_C(.)$ lies in determining the probability p_i that at least one customer is found waiting for service at the scan instant at station i, for all $i \in \mathbf{S}$.

5.2. Service Probabilities For Poisson Arrivals

We assume that the arrival processes are all Poisson with positive and constant rates, and the queue length distribution at the reference station is stationary. Using standard M/G/1 methods [Klei75], the geometric transform of the packet queue length distribution at this station is given by

$$G(z) = \frac{r_{0j}\ (z-1)\ L[\lambda_j(1-z)]}{z - L[\lambda_j(1-z)]} \qquad (3)$$

where λ_j is the rate of the Poisson arrival process at the reference station j, $L[s]$ is the Laplace-Stieltjes transform of the service time density expressed in terms of the complex variable s, and r_{0j} is the probability that the token finds no packet queued at the reference station at its scan instants.

The initial condition r_{0j} is evaluated from the geometric transform property $G(z=1) = 1$. This yields $1 = r_{0j}L[0]/(1 + \lambda_j L'[0])$. Since the transform of the service time density evaluated at $s = 0$ is unity and its derivative is the mean value, it follows that

$$r_{0j} = 1 - \lambda_j E(C) \tag{4}$$

If the expected length of a cycle in the stationary state and the Poisson arrival parameter λ_j of station j are known, Eq.(4) can be applied to any station. The probability r_{0i} that the token encounters no packet at a station i during any visit to the station can be computed. Thus, r_{0i} defines the parameter q_i specified in Eq.(1). In this way the token's holding time at station i is obtained as a mixture of the station's service and switching distributions, the mixing density being Bernoulli with parameter p_i, $i \in S$. The expectation $E(C)$ in Eq.(4) can be obtained by investigating the flow balance of the system in steady-state. When the system is in the steady state, the mean number or customers arriving at any station is equal to the mean number of customers served at that station. In fact, the mean number of customers served at station j during a cycle is identical to the probability that the server encounters at least one customer at queue j at this station's scan instants [Kueh79]. For each station j, this probability is simply $\lambda_j E(C)$. From this we obtain

$$E(C) = \sum_{j \in S} \{ E(Y_j) + (\lambda_j E(C))E(X_j) + (1 - \lambda_j E(C))E(V_j) \}$$

and consequently the mean cycle time as

$$E(C) = \frac{\sum_{j \in S} [\, E(Y_j) + E(V_j) \,]}{(\, 1 - \sum_{i \in S} \lambda_i E(X_i) + \sum_{i \in S} \lambda_i E(V_i) \,)} \quad . \tag{5}$$

5.3 Asymmetric Systems With Exponential Distributions

In this section, $f_C(.)$ is derived for an asymmetric model, i.e., one with all distributions having different parameters. The arrival processes are assumed to be Poisson(λ_j), and the B_j's, S_j's, and U_j's are assumed to be exponential, with means $1/\mu_{j0}$, $1/\mu_{j1}$, and $1/a_j$, respectively. Since a cycle is defined in terms of contributions from *all* stations, the random length of a cycle will remain the same regardless of the index of the station from which the observer measures it. An observer positioned at any station will record the random variable C decomposed in terms of its various sojourn times as

$$C = \sum_{j \in S} X_j^{'} + \sum_{j \in S} Y_j = X^* + Y^* \tag{6}$$

where the starred terms denote the respective sums. The random variable Y_j has density $a_j \, exp(-a_j t)$, and the random variable $X_j^{'}$ has a density that is a mixture of the densities of X_j and V_j, given by $p_j \mu_{j0} \, exp(-\mu_{j0} t) + q_j \mu_{j1} \, exp(-\mu_{j1} t)$. C can be viewed as the sum of N hyperexponential random variables and a generalized Erlangian random variable.

Let the Laplace-Stieltjes transforms of the densities of the random variables X^* and Y^* be given by $L[f_{X^*}]$ and $L[f_{Y^*}]$, respectively. For all $j \in S$ we have

$$L[f_{X_j}'] = \frac{a_{j0}}{(s + \mu_{j0})} + \frac{a_{j1}}{(s + \mu_{j1})} \tag{7}$$

with $a_{j0} = p_j \mu_{j0}$, $a_{j1} = q_j \mu_{j1}$, and

$$L[f_{Y_j}] = \frac{\alpha_j}{(s + \alpha_j)} . \tag{8}$$

The transform of f_C is obtained as

$$L[f_C] = L[f_{X^*}] \cdot L[f_{Y^*}] = \prod_{j \in S} L[f_{X_j}'] \cdot L[f_{Y_j}] \tag{9}$$

Let Θ be the set of all N digit binary numbers representing the non-negative integers in the range $[0, 2^N - 1]$. An element $k \in \Theta$ is an N-bit binary vector of the form $[k(1), k(2), \ldots., k(N)]$. In terms of our new notation, we have

$$L[f_{X^*}] = \sum_{k \in \Theta} \prod_{i \in S} \frac{a_{i\,k(i)}}{(s + \mu_{i\,k(i)})} \tag{10}$$

and
$$L[f_{Y^*}] = \sum_{j \in S} \frac{\beta_j}{(s + \alpha_j)} \tag{11}$$

where $\beta_j = \left(\prod_{\substack{i \in S \\ i \neq j}} \frac{\alpha_i}{(\alpha_i - \alpha_j)} \right) \alpha_j$.

$L[f_C]$ can now be obtained from Eqs.(9), (10) and (11). Note that $L[f_C]$ contains $2^N N$ terms, where each term has the form

$$D_{kj}^*(s) = \prod_{i \in S} \frac{a_{i\,k(i)}\,\beta_j}{(s + \mu_{i\,k(i)})(s + a_j)}, \quad \text{for } j \in S \text{ and } k \in \Theta. \tag{12}$$

Using partial fraction expansion, the resulting expression consists of terms $(s + \mu)$ and $(s + a)$, that are convergent for $\mathrm{Re}(s) > -\mu$ and $\mathrm{Re}(s) > -a$, respectively. Upon inverting the transform in Eq.(12), we obtain

$$D_{kj}(t) = \prod_{i \in S} a_{i\,k(i)}\beta_j \left\{ \frac{\exp(-a_j t)}{\prod_{m \in S} (\mu_{m\,k(m)} - a_j)} \right.$$

$$\left. + \sum_{n \in S} \frac{\exp(-\mu_{n\,k(n)} t)}{\prod_{\substack{m \in S \\ m \neq n}} (\mu_{m\,k(m)} - \mu_{n\,k(n)})(a_j - \mu_{n\,k(n)})} \right\}. \tag{13}$$

The cycle time density thus can be obtained as

$$f_C(c) = \sum_{j \in S} \sum_{k \in \Theta} D_{kj}(c)\ . \tag{14}$$

The computational complexity of the asymmetric density can be obtained as follows. Let the time taken for each addition be t_s, and the time taken for each multiplication be t_p. Consider the expression for $D_{kj}(c)$ given in Eq.(13). The second (summation) term in the sum requires a time of $N^2 t_p + (N-1)t_s$, and the first term in the sum requires a time of Nt_p. The sum itself requires a time of t_s, and the product term involving the $a_{i\ k(i)}$'s and the β_j's requires an effort of $(N+1)t_p$. The time required for any D_{kj} is $N^2[t_p + 2N + 1] + Nt_s$. For a given value of $c \in \mathbb{R}^+$, the effort required to compute $f_C(c)$ is $2^N N\{N^2[t_p + 2N + 1] + Nt_s\} + 2^N(N-1)t_s$. This requires an algorithm of exponential complexity and is inefficient for large N. In fact, due to the presence of the summation over the set Θ in Eq.(14), any algorithm for $f_C(.)$ will always be an exponential algorithm.

5.4 Symmetric Systems With Exponential Distributions

In the event that $\alpha_j = \alpha$, $\mu_{j0} = \mu_0$, $\mu_{j1} = \mu_1$, and $\lambda_j = \lambda$, for all $j \in S$, the computation can be shown to be tractable. In this case we have $a_0 = p\mu_0$ and $a_1 = q\mu_1$. The transforms for f_X* and f_Y* become

$$L[f_X*] = \sum_{j=0}^{N} \binom{N}{j} \left[\frac{a_0}{s + \mu_0} \right]^{N-j} \left[\frac{a_1}{s + \mu_1} \right]^{j} \tag{15}$$

and

$$L[f_Y*] = \left[\frac{\alpha}{s + \alpha} \right]^{N} \tag{16}$$

The transform $L[f_C]$ can be obtained from Eqs.(15) and (16). A direct inversion by partial fraction expansion will involve repeated differentiation in the computation of the coefficients of the fractions. To be precise, each term in the inverse will require the computation of $2N$ coefficients, where the N^{th} coefficient involves the N^{th} derivative of an expression of the form $(s + \zeta)^{-m}.(s + \kappa)^{-n}$, where $m \le N$, and

$n \leq N$. This requires an overall effort of $O(2^N)$ and is clearly an undesirable scheme. As an alternative, we represent the transform of the density of X^* as

$$L[f_{X^*}] = \sum_{j \in S} \frac{d_j}{(s + \mu_0)^j} + \sum_{j \in S} \frac{e_j}{(s + \mu_1)^j} \tag{17}$$

where

$$d_j = \frac{1}{(\mu_0 - \mu_1)^{N-j}} \sum_{n=1}^{N-j} \binom{N}{n} \binom{N-j-1}{n-1} (-1)^n (a_0)^{N-n} (a_1)^n \tag{18}$$

$$e_j = \frac{1}{(\mu_1 - \mu_0)^{N-j}} \sum_{n=1}^{N-j} \binom{N}{n} \binom{N-j-1}{n-1} (-1)^n (a_1)^{N-n} (a_0)^n \tag{19}$$

for $j = 1, 2, 3, \ldots\ldots, N-1$, with $d_N = a_0^{\,N}$ and $e_N = a_1^{\,N}$.

The transform of the density of C can then be expressed as

$$L[f_C] = \sum_{j \in S} \frac{d_j a^N}{(s + \mu_0)^j (s + a)^N} + \sum_{j \in S} \frac{e_j a^N}{(s + \mu_1)^j (s + a)^N} \tag{20}$$

where the computation of the coefficients of the partial fraction expansion becomes considerably simpler. Each of the $2N$ terms in the above transform is expressed as a fraction

$$\frac{x(s)}{y(s)} = \frac{1}{(s + \mu)^m (s + a)^N}$$

$$= \sum_{j=0}^{m-1} \xi_j \frac{1}{(s+\mu)^{m-j}} + g(s)$$

for some $m \in S$. The coefficient ξ_k is given by

$$\xi_k = \frac{1}{k!} \frac{d^k}{ds} \left[\frac{(s+\mu)^m x(s)}{y(s)} \right] \Bigg|_{s=-a}$$

$$= \frac{1}{k!} \frac{d^k}{ds} \left[\frac{1}{(s+a)^N} \right] \Bigg|_{s=-a}$$

where since $m \le N$ we get

$$\xi_k = \frac{(-1)^k \; \Gamma(N+k)}{k! \; \Gamma(N) \; (a-\mu)^{N+k}}$$

for $k = 0,1,2,\ldots,m-1$. The coefficients ζ_k, $k = 0,1,2,\ldots,N-1$, for the $(s+a)$'s are computed in an identical fashion. An arbitrary term from the first summation in Eq.(20), say

$$D_j^*(s) = \frac{d_j a^N}{(s+\mu_0)^j (s+a)^N} \tag{21}$$

can be seen to invert to

$$D_j(t) = d_j a^N \left\{ \sum_{k=0}^{j-1} \frac{\xi_k \, t^{j-k-1} \exp(-\mu_0 t)}{\Gamma(j-k)} + \sum_{k=0}^{N-1} \frac{\zeta_k \, t^{N-k-1} \exp(-at)}{\Gamma(N-k)} \right\} \tag{22}$$

and $E_j(t)$ can be obtained in the same manner, with coefficients ξ_k' and ζ_k', from the term in the second summation. Hence, we obtain the density for C in the symmetric case as

$$f_C(c) \quad = \sum_{j \in S} D_j(c) \; + \sum_{j \in S} E_j(c) \; . \tag{23}$$

The complexity of an algorithm using Eq.(23) can be determined by investigation of the function D_j. Consider the quotient immediately after the first summation in D_j. The partial fraction coefficient ξ_k requires an effort of $t_p(N+3k-1)$, the power of t requires $t_p(j-k-2)$, the gamma function requires $t_p(j-k-1)$, and obtaining the quotient with the final products requires $3t_p$. For a fixed value of k, the quotient term requires a time of $t_p(N+2j+k-1)$. Varying k from 0 to $(j-1)$ to obtain the first summation requires an overall effort of $t_p[5j^2/2 + j(N - 3/2)]$, for each $j \in S$. In like fashion, the second summation can be seen to require an overall effort of $t_p[7N^2/2 - 3N/2]$, for each $j \in S$. All products involved in the computation of a single D_j require a total effort of $t_p[5j^2 + j(N - 3/2) + 7N^2/2 - N/2 + 1]$. All sums involved in the same computation require a total effort of $t_s(N+j-1)$. The total time required to compute the first summation in Eq.(23) is

$$t_p[5(N+1)^2N^2/4 + N(N+1)(N-3/2)/2 + 7N^3/2 - N^2/2 + N] \; + \; t_s[N(N-1)/2].$$

By symmetry, E_j can be shown to require the same effort. Thus, we find that Eq.(23) yields an algorithm of (fourth-order) polynomial complexity.

5.5 Stationary Cycle-Time Distributions

Assume that the observer at station j witnesses the operation of the entire system starting from time $t = 0$. At first, the observer will notice that the probabilities p_i, $i \in S$, follow a transient path, fluctuating with changes in the system. After a time equal to the *relaxation* time [Giff78] of the system, these probabilities will attain constant values. For the queueing process at any station i to be stable, it is necessary (but not sufficient) that the limiting value of p_i be strictly less than one. The stability of the queueing processes is not a necessary condition for the stability of the cycle-time random variable C. That is, if the arrival rate at some station k is so large that $\lambda_k E(C)$ is greater than one, we take the convention that $p_k = 1$. This will ensure that in steady state, the server stops at station k with probability one during each cycle, in order to serve a customer. Viewed in this fashion, it is easy to see that C will always be stable, i.e., its expectation will be finite since it is the sum of independent random variables, each of which has finite expectation.

Let us suppose that the observer positioned at the reference station begins to record the successive cycle times starting from time $t = 0$, where without loss of generality we will assume that the token arrives at the reference station at time $t = 0$. The observer sees a sequence $C_1, C_2, .., C_n, ..$ of *transient* cycle times, where C_i is the random time taken by the token to complete the i^{th} cycle, $i \in I^+$. If $p_j^{(i)}$ is the probability that queue j is not empty at station j's scan instant during the i^{th} cycle, then clearly the length of cycle

C_i depends on the probabilities $p_j^{(i)}$, $j \in \mathbf{S}$. The distribution of C_i is obtained as

$$F_{Ci} = [F_1^{(i)} * F_2^{(i)} * \ldots\ldots * F_N^{(i)}] * G$$

where "*" is used to denote the convolution operation. $F_j^{(i)}$ is the holding time distribution of the token in state j, $j \in \mathbf{S}$, during the i^{th} cycle. G is a generalized Erlangian distribution constructed from the walk distributions. We now state a theorem regarding the distribution of cycle times as seen by the observer in steady state. The proof can be found in the appendix.

Theorem 5.1:

If the distribution of queue length at each station j, $j \in \mathbf{S}$ is stationary, then the random cycle time C posseses a stationary distribution given by

$$F_C = \lim_{i \to \infty} [F_1^{(i)} * F_2^{(i)} * \ldots\ldots * F_N^{(i)}] * G . \qquad (24)$$

The effects of varying system parameters on the cycle-time distributions obtained via the independence assumption are easily demonstrated. It is intuitively true that at very high loads, cycle-times are asymptotically independent. In Fig. 5a is shown a comparison of simulation versus analytic results for the cycle-time densities for asymmetric sets of two, five, and eight station systems, respectively. In Fig. 5b, similar results are shown for asymmetric systems with three, six, and ten station systems, respectively. Observe, that the independence assumption performs well at high loads.

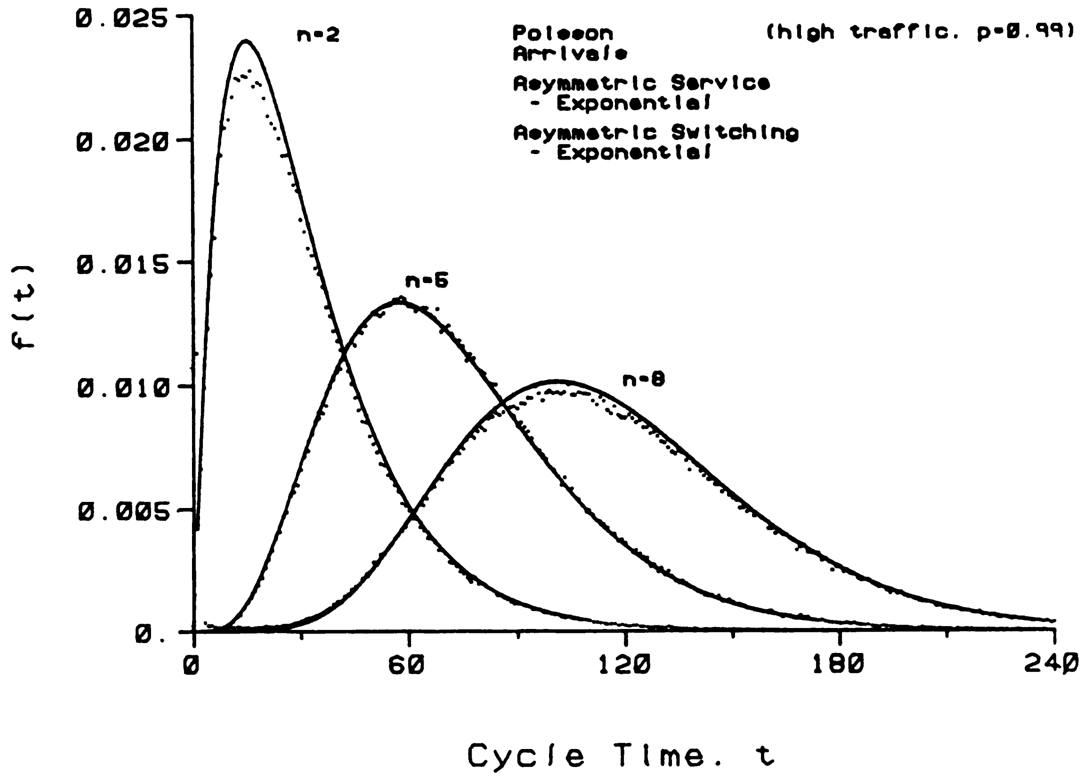


Fig.5a SP density for two, five, and eight station systems

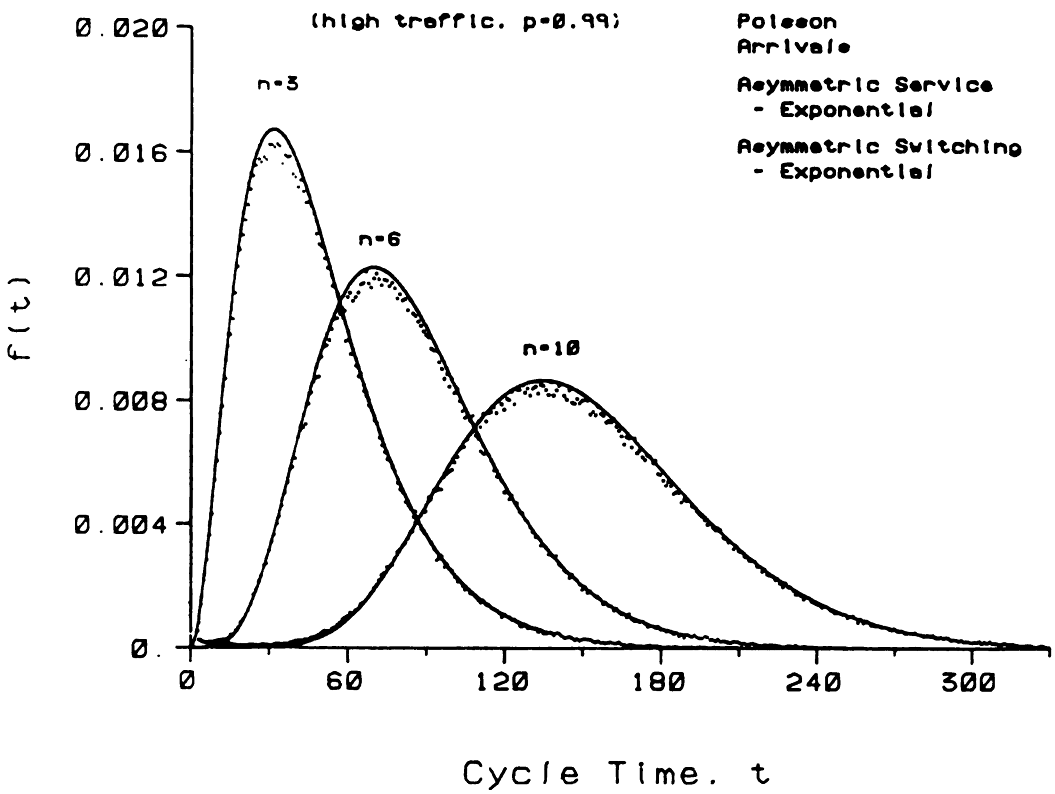


Fig.5b SP density for three, six, and ten station systems

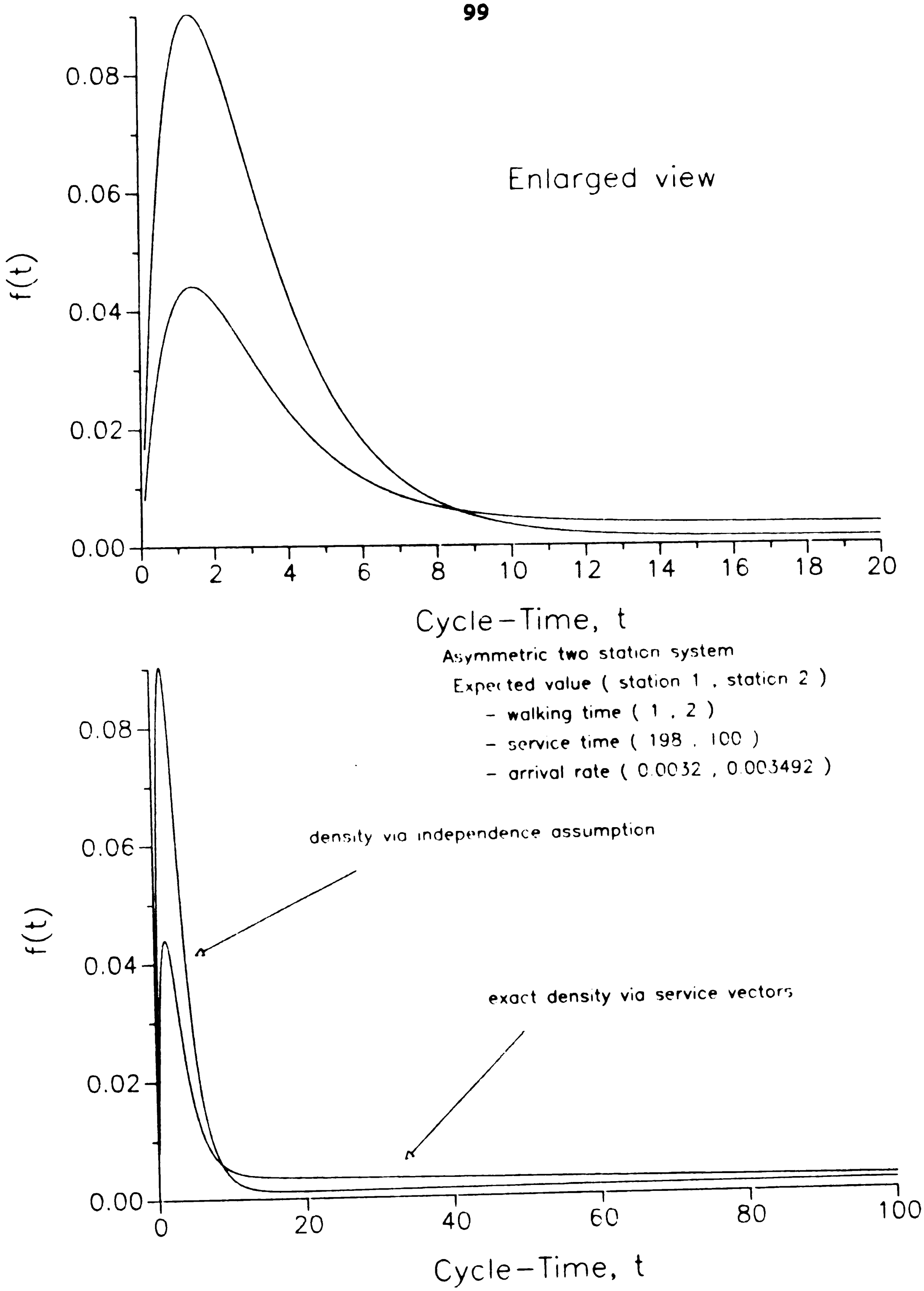


Fig.5c SP density degradation for moderate loads

Similarly, it can be shown that independence also does well at low loads. Unfortunately, for system loads that are neither extremely high, nor extremely low, the independence assumption appears to perform rather poorly, as can be seen in Fig. 5c.

5.6 Serial Dependence Of Cycle-Times

If we recall the distribution of cycle-time obtained in chapter III, we see that Eq.(3.9) defines a *strictly stationary* distribution. Since the system regenerates itself whenever all the queues become empty, the time points corresponding to empty-queues also correspond to regeneration epochs in a regenerative process. By hypothesis, the first two moments of all system distributions are finite, and consequently $E(C)$ is finite. It is easy to prove that under this condition, the mean time between epochs of regeneration is finite. From Theorems 10-4 and 10-5 of [HeSo82] it is clear that the stationary version of this regenerative process is stationary, and the regenerative process itself is asymptotically stationary. Consequently, the cycle-time process must be asymptotically stationary. Thus, Eq.(3.9) defines the strictly stationary distribution that the cycle-time process converges to asymptotically. Equation (3.9) is strictly stationary simply because we chose as the initial distribution of the semi-Markov process what would have been the limiting distribution from an arbitrary initial state.

The fact that we have identified a regenerative process is important in that a study of the sample paths of the process in any one interval of the regenerative process will help us completely understand the system's behaviour. A method of pursuing this will be via the busy period distribution of chapter III. In this section, we are interested in studying the effects of the independence assumption, and consequently leave the idea just mentioned aside. If we work with the cycle-time distributions defined in sections 5.3 and 5.4, and assuming an arbitrary starting random variable C, it is difficult to prove (without using asymptotic stationarity) the stationarity of the cycle-time process. Indeed, starting with a regeneration epoch, it can be argued that the ensuing sequence of cycle-times (which in general will be dependent) is *weakly stationary*. It would be interesting to investigate the effects of the first cycle-time random variable (obtained under independence) on the following random variables to determine the behaviour of deteriorating dependence.

An arbitrary cycle seen by the observer at the reference station is found to have a random length with density $f_C(.)$. If the arrival rates at all stations are sufficiently low, then successive cycle lengths are approximately independent. The queueing situation at each station will closely approximate a GI/G/1 queue with uncorrelated service times. For example, when arrivals at each queue occur sparingly enough so that an average of at most one customer per queue is awaiting service at a scan instant, and the time between this customer's beginning of service and the next customer's arrival is an independently random time. Again, approximate independence holds also for sufficiently high arrival rates. In this case, a customer is always found waiting for service at each queue at its scan instants, ensuring that the cycle time random variable (which now will be the sum of N

random walk times and N service times) takes the same form for each cycle, and is independent of the previous cycle. But in general, even in steady state, successive cycle time lengths are not independent. Consequently, any study of the system utilizing $f_C(.)$ must take into account this dependence.

Consider an infinite sequence of consecutive cycle lengths $..C_{-n},..,C_{-1},C_0,C_1,..,C_n,..$, denoted by $\{C_n\}$, that are observed at the reference station when the system is in the steady state. If cycles C_i and C_j of the sequence are independent, the autocorrelation function for terms at lag $m = |j-i|$ in the sequence will be zero. A nonzero autocorrelation for a given lag m implies a degree of dependence between cycles that are m positions apart. Given the parameters of the system, it is not a simple task to establish the value of m for which the autocorrelation goes to zero. Conversely, if this value of m is determined in an empirical fashion, it is not correct to assume that cycles of lag m are independent. Thus, we need to develop an analytic approach towards identifying independent cycles, making appropriate assumptions along the way, if necessary.

The discrete parameter stochastic process $\{C_n\}$ possesses the property that the joint distribution function for (C_i,C_{i+1}) is different from the joint distribution function for (C_{i+1},C_{i+2}). By our assumption that the service, walk and switching distributions possess finite first and second moments, $\{C_n\}$ is a *second order* process. Further, if it can be shown that the process $\{C_n'\}$ defined by $C_n' = C_{n+k}$ has the same mean and covariance functions as the $\{C_n\}$ process, for every fixed number $k \in I^+$, then $\{C_n\}$ is *second order stationary* or *weakly stationary*. For

$i,j,n \in I^+$, let $E(C_n)$ and $r_{ij} = Cov(C_i,C_j)$ be the mean and autocovariance function of the $\{C_n\}$ process, respectively. We resort to an equivalent but more functional definition of weak stationarity, i.e., $E(C_n)$ is independent of n and r_{ij} depends only on the difference between i and j, to prove weak stationarity. In order to emphasize the importance of lags, and not the index of the process, we define $r_j = r_{0j}$ as the autocovariance function of the process with some cycle C_0 chosen arbitrarily from $\{C_n\}$ to be the reference cycle. From the symmetry property of autocovariance functions, it can be shown [HoPS72] that $r_{-j} = r_j$, $j \in I^+$, provided that there are a large number of cycles that occur prior to C_0 in $\{C_n\}$. Since $V(C_n) = Cov(C_n,C_n)$, the common variance of the random variables in $\{C_n\}$ is given by r_0. Using Schwarz's inequality, it can be shown [HoPS72] that since $r_0 > 0$, the correlation between C_n and C_{n+k} can be given independently of n by r_k/r_0.

From Theorem 5.1 it follows that $E(C_n) = E(C)$ is independent of n, with $E(C)$ being given by Eq.(5). To prove weak stationarity of $\{C_n\}$, it is left to show that r_{ij} depends only on $|j-i|$ and not the particular values of i and j. We now present an argument to demonstrate the dependence between consecutive cycles of the sequence beginning at C_0 (measured with respect to the reference station). This approach treats the symmetric and asymmetric situations simultaneously, with the appropriate density used for f_C throughout the rest of this discussion. Since the arrival process at station j is Poisson, $\lambda_j C_0$ is the probability that an arrival event at queue j occurs during the token's C_0 cycle. This corresponds to the probability that at the start of the

C_1 cycle, at least one customer is seen awaiting service at station j given that a cycle of length C_0 has just occurred and no further information is available about the queue status at its C_0 scan instant. Since all stations on the network see the same cycle time random variable, corresponding probabilities can be generated for each station by utilizing its arrival parameter. Thus, a larger value of C_0 ensures a higher probability that customers will be awaiting service at each station during the C_1 cycle than a probability generated by a smaller value of C_0. Since the probability that at least one customer awaits service at a station is equal to the mean number of customers served at the station, we can conclude that if C_0 is large, then C_1 will also have a tendency to be large rather than small. Large cycles will tend to follow large cycles with a high probability. By the same vein, small cycles will tend to follow small cycles with a high probability. The notion of 'high probability' simply describes a probability strictly greater than one-half. By exactly how much this probability exceeds one-half is a function of the parameters of the system. If $E(C)$ is used as a means for discriminating between large and small cycles, then since consecutive cycles tend to group on the same side of the mean, the covariances between neighbouring terms in $\{C_n\}$ will have a strong tendency to be positive.

Let f_i correspond to the probability density function of the random variable $C_i \in \{C_n\}$. By $f_{i/j}$ is meant the conditional density, and by $f_{i,j}$ is meant the joint density of the random variables C_i and C_j. The same notation is generalized to three or more random variables. We take the convention that $f_0 = f_C$. Define $\mathbf{p}^{m} = (p_1^{(m)},\ldots,p_N^{(m)})$ to be the vector of probabilities associated with the N-station configuration during cycle C_i. Here, $p_j^{(m)} = \lambda_j \sum_{j=0}^{m} C_k/m$ is the probability that at least one customer awaits service at station j during at the server's C_m scan instant, $C_m \in \{C_n\}$, and $j \in \mathbf{S}$. By using the mean of the random sample $C_0, \ldots, C_{m-1}$, we obtain queue state probabilities that use some history of the cycle time process.

The density $f_C(c)$ can be also be expressed as a function of probabilities, i.e., as $f_C[c,\mathbf{p}]$, where $\mathbf{p}$ is the vector $(p_1, \ldots, p_N)$, with $\mathbf{q} = \mathbf{1} - \mathbf{p}$. The conditional density of C_1 given C_0, may be obtained simply by replacing the term p_j by $p_j^{(1)}$ in f_C, for all $j \in \mathbf{S}$. That is

$$f_{1/0} = f_C[c,\mathbf{p}^{\mathbf{1}}] \quad , \tag{25}$$

and with the aid of Eq.(25), the joint density of (C_0,C_1) is obtained as

$$f_{0,1} = f_{1/0} \cdot f_0 \tag{26}$$

The general form of the conditional density function for C_m given $C_0, \ldots, C_{m-1}$ can be obtained by extending the idea outlined above, i.e.,

$$f_{m/0,1,\ldots,m-1} = f_C[c,\mathbf{p}^{\mathbf{m}}] \tag{27}$$

and the joint density of $(C_0, \ldots, C_m)$ as

$$f_{0,1,\ldots,m} = f_{m/0,1,\ldots,m-1} \cdot f_{0,1,\ldots,m-1} \tag{28}$$

If we compute the marginal distribution of C_0 by treating this cycle as the first cycle after the system is found empty (i.e., after a regeneration epoch, when it will be easy to compute the probabilities p_i) we will have described a way to express the conditional density and joint density functions of random variables from $\{C_n\}$. By the manner in which the joint densities are defined, it is clear that the covariances between pairs of terms in $\{C_n\}$ depend only on their distances from each other. Thus, weak stationarity follows. Given any pair of random variables in the sequence $C_0, C_1, \ldots, C_m, \ldots,$ their joint and marginal distribution functions may be obtained by repeated integration. These distributions incorporate the local dependence properties of the second order process. The intention of the discussion was to demonstrate how the process may be viewed as weakly stationary, since strict stationarity is not easily proved without looking for the asymptotic distribution.

Let the *maximal dependent set* of random variables in the sequence $C_0, C_1, \ldots, C_n, \ldots$ be defined as $\{C_k | k \geq 0, Cov(C_0, C_k) \geq \delta\}$, where δ is some specified tolerance level. Terms in the sequence whose dependence on C_0 is too small (smaller than δ) are excluded from the set. Let m be the largest positive integer satisfying $Cov(C_0, C_m) \geq \delta$. Then, m_δ is called the *memory of the sequence* with respect to δ. Henceforth, we suppress the subscript and understand the memory to be defined in terms of δ. The number m will vary systematically with the parameters of the process. Generally, higher arrival rates generate sequences with longer memories, and lower arrival rates give rise to shorter memory sequences. Also, choice of δ will affect the memory length considerably. The relationship between the memory of the process and the parameters of the process is a subject for further research, as also the covariance structure of $\{C_n\}$. Observe that if δ is made arbitrarily small, the size of the maximal set grows large. By passing to the limit, it is trivial that the marginal density for each C_m sufficiently far (in the limit) from C_0 enjoys independence of C_0. This is a consequence of the fact that the sample mean is a consistent estimate of the true mean of the cycle times.

By assuming that terms in $\{C_n\}$ are independent, the variance of the cycle time process obtained from the density f_C underestimates the "true" (unknown) cycle time variance. The independence assumption made by Hashida and Ohara [HaOh72] disregards the positive correlation between successive random variables in the sequence. Kuehn [Kueh79] uses a form of conditioning that accounts for two kinds of cycles. One kind involves a server cycle where a reference station customer is served, and the other kind involves a server cycle where the reference station queue is empty. The net effect is to increase the cycle time variance of the process. This form of conditioning uses information pertaining only to the current cycle and not previous cycles. To incorporate such effects, one needs to consider sequences of dependent cycles and find methods to describe such dependence. The joint density functions obtained above were developed with the intention of demonstrating one such method. Given the $(m+1)$ dependent random variables $C_0, C_1, \ldots, C_m$, the marginal density function for each random variable and joint density functions corresponding to every pair of random variables in the maximal dependent set are obtained. From Eq.(26) it can be seen that the procedure is computationally simple

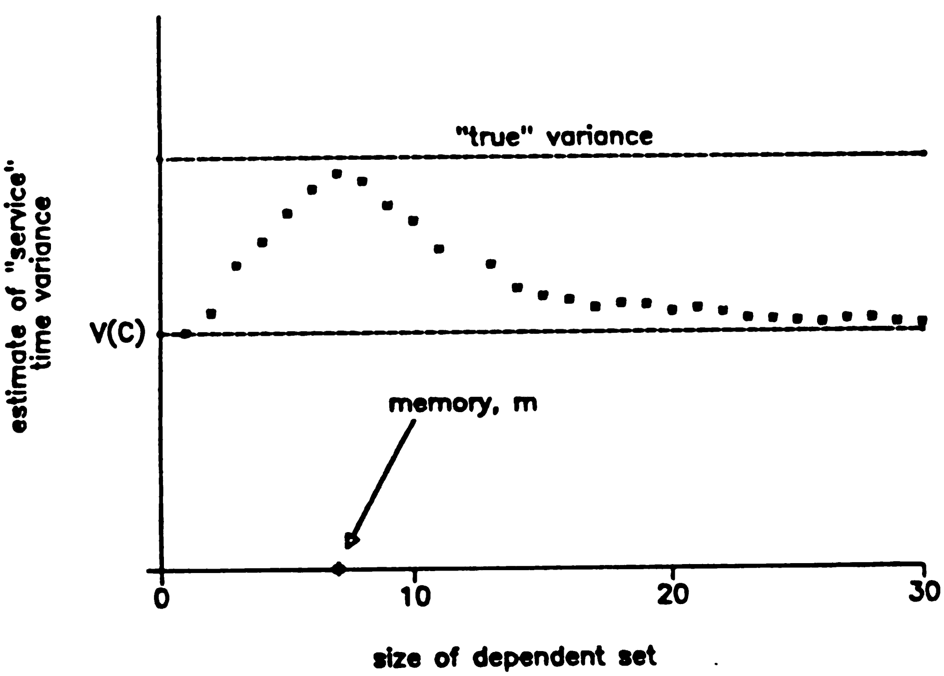


Fig.6 Variance Estimate

since the form of the joint densities is merely the product of the individual marginals with the appropriate probabilities substituted. The variance-covariance matrix is obtained with the aid of the joint and marginal densities. Let σ_m^2 represent the maximum "true" variance of the cycle time process. Then an estimate of variance may be obtained as

$$\hat{\sigma}_m^2 = \sum_{i=0}^{m} V(C_i)/(m+1) + 2 \sum_{i<j}\sum Cov(C_i, C_j)/m(m+1) \tag{29}$$

The variance estimate obtained in this fashion can be seen to increase the cycle-time variance by considering the history of cycles that form a chain of dependence. This increase is due to the positivity of the covariance terms. The picture in Fig. 6 describes how the study of dependence is made to increase the estimate. Observe that $V(C)$ takes the shape of a lower bound for $\hat{\sigma}_m^2$. An upper bound may be achieved simply by choosing the marginal density in the maximal set that defines a C_i the sum of whose variance and covariance with C_0 is the largest. The repeated addition of covariance terms from successive entries in the stationary sequence causes $\hat{\sigma}_m^2$ to increase to a point beyond which either all covariances are actually zero, or all are assumed be negligible (in terms of δ). Thus, the estimate obtained as the maximum of this averaging process increases the underestimate $V(C)$ by accounting for dependence.

As a first approximation to mean waiting time, the variance estimate used above can be used as service-time variance for the approximate M/G/1 queue at station j. Since this estimate uses more information that Kuehn's estimate (and certainly yields a larger variance for "service-time"), it is expected that the mean waiting times obtained in this way will be more accurate. It must be noted that though the procedure for computing the covariance matrix is computationally simple, the complexity grows with $(m+1)^2$ for a process having memory m.

5.7 Summary

The essence of this chapter is a method for obtaining the cycle-time distribution function under the assumption that, at steady state, the emptiness (or non-emptiness) of one station's buffer does not affect the emptiness (or non-emptiness) of any other station's buffer. Additionally, it is required that the arrival process be Poisson. We have used exponential random variables for service, walk, and switching times. This was done for convenience, and is not really necessary. Computationally, the method is simple for both symmetric, as well as asymmetric systems. Unfortunately, for asymmetric systems, the complexity of the computational algorithm grows exponentially with N. For symmetric systems, by resorting to a binomial form for the coefficients, the computational effort can be reduced to polynomial time.

The independence assumption is shown to perform very well under extreme system loads. When system load is moderate, the independence assumption is poor. It is easy to see that the approximation will surely degrade with increasing values of N and increasing variance in service and switching times. In a simple result, the distribution of a stationary cycle-time random variable is obtained as the limit of a sequence of finite convolutions. It is important to note that this can only be done under a decomposition rule, which in our case is the independence assumption.

From our previous work in chapter III, we already have a stationary distribution for the cycle-time random variable. By assuming independence, we obtain another form for the limiting density of a stationary (by hypothesis) cycle-time random variable. In this case, it is difficult to prove strict stationarity without resorting to regenerative theory and asymptotic stationarity. Consequently, we introduce an argument that allows us to view a sequence of cycle-times as weakly stationary. If the first random variable in the sequence occurs immediately after a regeneration epoch, then it is interesting to examine the effects of deteriorating dependence. We basically use this idea to suggest that a number of such dependent cycles may be completely characterized (by obtaining their joint distribution), and further used to investigate properties of the system. One such property is the memory of the cycle-time process. Of course, this can be done directly with the methods of chapter III, but we must stress that we are interested in examining the effects of independence here. Using SV methods, an estimate of memory is easily obtained by taking the ratio of the mean busy period (i.e., mean time between regeneration epochs) to the mean cycle-time. The SP methods require the computation of a covariance matrix. Once the memory is established, we can move on to the next phase of the analysis, which is to approximate the "service-time" variance for station j customers with a function of the variances of a number of cycle-time random variables. This idea is discussed further in the next chapter.

CHAPTER VI

PERFORMANCE MEASUREMENTS USING SERVICE PROBABILIIES

In section 5.1, we mentioned two important differences between the queueing system at station j and an M/G/1 system. If we view the queueing process at station j as a single server queue with Poisson arrivals, the dependence between consecutive cycle-times leads to dependence between the "service times" of consecutive customers from station j. The reason for this dependence was explained in section 5.6. Additionally, we were able to show that consecutive service times are positively correlated. By assuming that cycle-times are i.i.d, Hashida and Ohara [HaOh72] ignore the positive covariance between neighbouring cycles. This amounts to a loss of information and consequently, as claimed by Kuehn [Kueh79], underestimates for mean waiting times. Kuehn attempts to account for dependence by considering some history of a current cycle. Assume that C' and C'' are random cycle-times corresponding to cycles in which a customer from station j is served, and no customer from station j is served, respectively. Kuehn treats a station j customer's service time as a random variable whose distribution is a mixture of the distributions of C' and C''. The mixing density is taken to be Bernoulli, with parameter p_j. With this approach, Kuehn obtains improved estimates for mean waiting time and queue length. Unfortunately, just as in the Hashida-Ohara model, the accuracy of the cycle-time variance decreases with increasing N or increasing variance in the random variables X_i, $i \in S$.

The Hashida/Ohara and Kuehn models view the multiqueueing system through the Laplace-Stieltjes transform of the cycle-time distribution. Since we have gone a step further by actually obtaining the distribution of cycle-time, we can obtain more information about the system than in either [HaOh72] or [Kueh79]. Using the service probability approach, these authors obtained approximate expressions for mean customer queue lengths and waiting times, whereas we can aim for approximate moments of higher order, and even approximate distributions. Using the explicit form for the distribution of a stationary cycle-time, it is possible to develop several models to examine the effects of covariance. In particular, we choose to look at two such models with the intention of obtaining approximate results for the MQCS/s-QED=1 problem. Both models reduce the MQCS queueing scheme into single server queueing schemes, one assuming an M/G/1 scheme, and the other assuming an M/G(r+1)/1 scheme, where Gr denotes the distribution of a moving average random variable of order r.

The first model assumes the existence of a quasi-service time distribution (where the term "quasi" is meant to indicate an approximating random process) that can be used to model the "service time" distribution at station j The M/G/1 approximating model is introduced in section 6.1. In section 6.2, methods for obtaining the covariance matrix of a set of dependent cycles are given. The idea is to first establish covariance properties for the $(m+1)$ random variables in a maximal dependent set. This can be done by using either the conditional distributions (if they are known) or by simply approximating the covariance between cycles at lag k with some continuously decreasing function of k. Such a function is not always easy to obtain, but can be developed in certain special cases under approximating assumptions. In any case, we require a variance covariance matrix with the aid of which we can proceed to apply a principle component analytic approach. In section 6.3, methods for obtaining (approximate) marginal distributions are discussed, and in section 6.4, we briefly describe how the quasi-service time random variable may be obtained. Approximate expressions for packet queue length and waiting time distributions are obtained in sections 6.5 and 6.6, respectively. An approximate form of the system's mean throughput is obtained in section 6.7.

The second model, due to Pearce [Pear67], assumes that successive customer service times at a reference station (in actuality, the successive cycle-times) can be represented as moving average type service times. Without resorting to an i.i.d service time random variable, it is possible to obtain queueing distributions. In defining the moving average, we are allowed considerable flexibility in choosing the weights of the moving average. Since the cycle-time process is positively correlated, the only restriction is that the weights be positive. This model, based on $M/G(r+1)/1$ queueing systems, is presented in section 6.8, and the analysis is reviewed in section 6.9.

6.1 The M/G/1 Approximating Model

In this approach, we use the notion of conditioning discussed in section 5.6. Since we are interested in modelling the queue at station

j as an M/G/1 queue, we require a service time distribution. Consequently, we assume that the "service time" process seen by the observer at the reference station j can be represented by a quasi-service time random variable S_q. If we can determine the distribution of S_q, then M/G/1 theory can be applied to obtain descriptive results for the queueing system at station j. The problem we face is that instead of a single service time random variable, what we really have is a sequence of $(m+1)$ serially dependent cycle-time random variables. In the following analysis, we demonstrate a method that enables us to construct S_q as the first principal component of the $(m+1)$ dependent sequence. In essence, we try to reduce a random variable in $(m+1)$ dimensions through a filtering process to a single dimension.

6.2 Covariance Matrix Obtaining Methods

Given a stationary random cycle-time C_0 whose distribution is either of asymmetric or symmetric form (as obtained in chapter V), the first step is to construct a maximal dependent set of cycle-time random variables. This is to enable us to determine the order of the required covariance matrix. One obvious way to do this is to use the approach outlined in section 5.6. This yields $(m+1)$ correlated random variables and their corresponding distributions. Thus, the covariance matrix can be obtained directly with the aid of the marginal distributions. Another method involves the case where the marginal distributions are unknown. It is often convenient to first obtain an approximating form for the covariance function of the cycle-time process, and then use this to obtain the marginal distributions. Two methods for doing this will be described.

The first method attempts to approximate the covariance function of the cycle-time process by some continuously decreasing function of cycle lags k. For example, if the server viewed the contents of the queue at station j at the scan instants of this queue, the server would see a queueing process with an arrival rate that is a function of the cycle-time. In this sense, the arrival process at station j can be approximated by Poisson process whose rate is a function of a stationary, continuous-time stochastic process, or a process known as a doubly stochastic Poisson process. We are specifically interested in the covariance function which relates to how dependent the rate at one instant of time is to the rate at another instant of time.

Let $p_j(c)$ be the packet rate intensity function at station j. For a large class of doubly stochastic Poisson processes, the integral of the arrival rate covariance function is directly related to the limiting variance-to-mean ratio of the number of intervals during a time interval [CoLe66]. If $n(c)$ is the number of arrivals in a time interval of length c, then

$$\lim_{c \to \infty} \frac{Var[n(c)]}{E[n(c)]} = 1 + \frac{2 \int_0^\infty r(t)\, dt}{\kappa} \tag{1}$$

where κ is the mean of $p_j(c)$. If we use the second order process of section 5.6, then κ is simply p_j, or $\lambda_j E(C)$. The covariance integral in

Eq.(1) is useful in defining a time constant for the random process. One easy way to define a time constant is given by the integral

$$\tau_C = (1/v) \int_0^\infty r(t)\, dt \qquad (2)$$

where the constant τ_C is such that an exponential covariance function approximation of the form

$$r_e(t) = v \exp(-t/\tau_C) \qquad (3)$$

is generally a good approximation to time function $r(t)$ under a wide range of conditions [Heff80]. We require that $v = Var(C)$, where the variance is computed from the densities given in section 5.6 (for symmetric and asymmetric cases, respectively), in order that the condition $r(0) = Var(C)$ is satisfied. Heff [Heff80] shows that the covariance function approximation works very well for the correlated Poisson arrival process. If we proceed along these lines to determine the explicit form of the covariance function, then since we obtained the doubly stochastic view of the arrival process via the dependent cycle-times, we can assume that the correlation function obtained is really that of the cycle-time process. In this fashion, we can determine the correlation between pairs of cycles. Using a threshold δ (see section 5.6) to define a maximal dependent set of cycles, the next step is to determine the distributions of the conditional cycle-times in the set.

The second method is based on the concept of moving averages [Cono81]. We begin with the random variable C_0 (whose density is given by either Eq.(5.14) or Eq.(5.23) and demonstrate how a sequence of dependent random variables can be constructed. Let X_n, $n=1,2,\ldots$ be a sequence of i.i.d random variables with mean zero and variance one. A law for this sequence can be obtained from the law of C_0 by appropriate scaling. Define the sequence

$$X_n^{(1)} = z_1 X_n + z_2 X_{n+1}, \tag{4}$$

where $z_1 + z_2 = 1$, and each z_1 is strictly positive. This defines a moving average of order two. The superscript 1 in Eq.(4) is used to denote moving averages made up of level one iterations and is used to differentiate between moving averages defined in terms of moving averages , i.e., a level two iteration is a moving average obtained from the level one terms defined in Eq.(4). Applying a forward shift operator $EX_n = X_{n+1}$, a little algebraic manipulation will yield covariance terms

$$Cov(X_n^{(1)}, X_n^{(1)}) = z_1^2 + z_2^2$$

and $$Cov(X_n^{(1)}, X_{n+1}^{(1)}) = z_1 \cdot z_2$$

Using the zero order and first order covariances, a covariance matrix of the cycle-time process with memory two is obtained. This procedure can be generalized to higher order covariances for level one and order two moving average sequences. Consequently, this will lead to larger covariance matrices with sizes that may more accurately reflect the memory of the cycle-time process. Let $\rho_k^{(1)}$ be the correlation of two cycle-times at lag k in an order two, level one, moving average sequence. Using the normal approximation, it can be shown [Cono81] that

$$\rho_k^{(1)} \sim exp\{-k^2/4z_1z_2\} \quad (5)$$

If we choose, we can generalize our solution to level one moving averages of order three and higher. In any case, we can always obtain either an exact correlation [Cono81] or the approximation given in Eq.(5), and as a consequence arrive at a covariance matrix for the approximating process.

6.3 Marginal Distributions Of Dependent Cycle-Times

In section 5.6, a method for obtaining the distribution of each cycle in a maximal dependent set of cycles was outlined. If we begin with a covariance matrix, say obtained via methods described in section 6.2, we have yet another method to obtain the marginal distributions of cycles in a maximal dependent set. The problem is to determine marginal distribution functions such that the covariance matrix defined by these distributions agrees with the given covariance matrix. In general, we seek forms for the marginal densities $f_{C1},\ldots,f_{Cm}$ given the density f_{C0} of C_0. By hypothesis, each C_i is a sum of independent random variables $X_j^{(i)'}$ and Y_j, $j \in S$. If such is the case, then dependencies between the C_i must appear only via the probabilities p_j and q_j in the distribution of X_j' (see section 5.3).

For simplicity, we concentrate only on correlations for random variable pairs (C_i,C_{i+1}), $i = 0,..,m-1$. The method uses f_{C0} to determine f_{C1}, and the latter density to obtain f_{C2} and so on, until the density of f_{Cm} is obtained. Hence, we need only examine how f_{C1} is obtained from f_{C0}, where f_{C0} is the (known) cycle-time density of a steady-state cycle. Define $X_j^{(1)'}$ to be the random time spent by the server at station j, in either switching past or servicing this station during cycle C_1. By convention, $X_j^{(0)'}$ is taken to be X_j'. Clearly, C_1's dependence on C_0 follows from the dependence of $X_j^{(1)'}$ on C_0.

Let c_{ij} be the correlation between cycles C_i and C_j and let f_{Xj}^{1} be the marginal density of $X_j^{(1)'}$. To arrive at f_{C1} we must first determine the density f_{Xj}^{1}, $j \in S$. Using the methods developed in chapter V for sums of independent mixtures, f_{C0} can be obtained. Let $\{\phi_j, j \in S^*\}$ be the limiting density of the semi-Markov process $\{Z(t)\}$ associated with $\{Z_n, T_n\}$ defined in chapter V. An expression for this distribution will be given in section 6.7. Define the correlation between $X_j^{(1)'}$ and C_0 as $[c_{01}\phi_j/(\phi_1+\ldots+\phi_N)]$. In defining this correlation, our motivation lies in the fact that $X_j^{(1)'}$ plays a part in making C_1 dependent on C_0, and this is reflected by a function of j. Ignoring the random walk times (since they are independent), we define this function to be the probability ϕ_j normalized by a term that is the limiting probability for the server being in any station. In essence, this defines the contribution of station j to the correlation.

The key to determining f_{Xj}^{1} lies in determining the probability $p_j^{(1)}$. For notational ease we let p denote this probability, with $q = 1- p$. The covariance between random variables $X_j^{(1)'}$ and C_0 can be expressed as

$$Cov[X_j^{(1)'}, C_0] = E[X_j^{(1)'}, C_0] - E[X_j^{(1)'}] E[C_0] \qquad (6)$$

Since the term on the left of Eq.(6) is known, we obtain an equation in the two unknowns p and q. With the knowledge that $p + q = 1$, the probability p can be determined uniquely. Note that each j will yield different values for p, wih each (p,q) combination determining the density $f_{Xj}^{(1)'}$ for one value of j. Given this density for each j, $j \in S$, the density f_{C1} can be obtained by the methods described in chapter V. Continuing in this trend, we can obtain the density functions for $C_1, \ldots, C_m$.

6.4 Principle Component Analysis For Quasi-Service Time

In this section, we are interested in defining the quasi-service time random variable S_q. Clearly, such a random variable will be a function of the dependent sequence $C_0, \ldots, C_m$, and additionally, must possess the maximum variability exhibited by the sequence. In particular, the first principle component of the maximal dependent sequence is the function that will yield the desired property. In other words, we seek the normalized linear combination (i.e., with sum of squares of the coefficients equal to one) that possesses maximum variance. In order to obtain a positive covariance between terms C_i, we must impose the restriction that coefficients are strictly positive.

Following Anderson [Ande58], let $\mathbf{X}'$ be a vector of $(m+1)$ components $C_0, \ldots, C_m$. For analytic convenience we concentrate on the random variable $\mathbf{X}$ with mean vector $\mathbf{0}$ and obtained from $\mathbf{X}'$ by appropriate scaling. Let Σ be the covariance matrix of $\mathbf{X}$ and let β be an $(m+1)$

component column vector such that $\beta'\beta = 1$. By a theorem of Anderson (see theorem 11.2.1 in [Ande58]), there exists an orthogonal linear transformation $\mathbf{U} = \mathbf{B}'\mathbf{X}$ such that the covariance matrix of $\mathbf{U}$ is $E\,\mathbf{U}\mathbf{U}' = \Lambda$, and Λ is a diagonal matrix whose entries are the eigenvalues of the transformation. The matrix $\mathbf{B}$ is the matrix of eigenvectors associated with the transform. We take the first component of $\mathbf{U}$ to be $S_q = \mathbf{B}^{(1)}\mathbf{X}$, where $\mathbf{B}^{(1)}$ is the first column of $\mathbf{B}$ and the superscript is used to denote the first eigenvector, or the eigenvector corresponding to the first eigenvalue. Observe that the largest eigenvalue is actually the variance of S_q.

Having determined the linear combination that defines S_q, the next step is to obtain the distribution of this linear combination. Since we have the marginal distributions, standard methods can be applied in determining the distribution for S_q. Note that a way of doing this is to first obtain the joint density of $C_0,\ldots,C_m$. Fortunately, by the manner in which we have described their dependence, this can be obtained by simply taking the product of the marginal density functions. That is, they are dependent only via probabilities $p_j^{(i)}$, for $j \in \mathbf{S}$, and $i = 0,\ldots,m$. Thus the joint distribution will also be a mixture of exponentials. In taking the Jacobian of the required transformation to obtain S_q, since we deal with linear combinations, we will obtain only constants. Since the constants do not change the form of the distribution, we can obtain the distribution of S_q as another mixture distribution.

6.5 Distribution Of Packet Queue Length

In this section and the next we briefly present the use of the quasi-service time distribution in deriving approximate distributions for queue length and waiting time for packets in a stationary asymmetric system. We denote S_q by S, and consequently the density of S_q by f_S. In order for the solution to hold it is necessary that the system be completely stable [ReNi84]. For high asymmetric loads it is possible that $(1 - r_{0j}) > 1$ so that it can no longer be interpreted as a probability. If the other queues are stable, then the system is only partially stable since queue j is unstable. In such cases it is necessary to set $(1 - r_{0j})$ to 1. This means that a customer is present with probability 1 at each scan instant of this unstable queue.

Since we know that f_S is a mixture density, we can determine unique E_{km} such that f_S takes the form of Eq.(5.14), with D_{km} replaced by E_{km}. Let $L_j(z)$ be the geometric transform for the number of arrivals at reference station j (as seen by the observer) during a random time S. Let r_{nj} be the distribution of queue length at station j, $n \in I^+$. In order to derive the distribution of queue length it is necessary to exploit the geometric transform of the queue length distribution at station j (see Eq.(5.3)). If we can represent part of this transform as a ratio of two power series, the result is a single power series. By evaluating the low order coefficients of the new series, the terms r_{nj} may be recovered [AbSt64], for $n \in I^+$. We may write $L_j(z)$ as

$$L_j(z) = \sum_{m \in S} \sum_{k \in \theta} \sum_{n=0}^{\infty} \{ E_{km}{}^{(n)} \} \; z^n \qquad (7)$$

where $$E_{km}^{(n)} = \int_0^{\infty} \frac{(\lambda_j t)^n \exp(-\lambda_j t)}{n!} \; E_{km} \, dt \tag{8}$$

with E_{km} defined earlier. Observe that $L_j(z)$ is of the form $L_j(z) = \sum_0^{\infty} \omega_n z^n$ where ω_n describes the probability of n packet arrivals at station j during a random "service time" S. Using the above equation, we can write the geometric transform for the length of station j's packet queue as

$$G(z) = \frac{r_{0j}(z-1) \sum_{n=0}^{\infty} \omega_n z^n}{z - \sum_{n=0}^{\infty} \omega_n z^n} \tag{9}$$

from which, after some algebraic manipulation we obtain

$$G(z) = \frac{r_{0j}\left[\, 1 + \sum_{n=1}^{\infty} \left(\frac{\omega_n - \omega_{n-1}}{\omega_0}\right) z^n \,\right]}{1 + \left(\frac{\omega_1 - 1}{\omega_0}\right) z \quad \sum_{n=2}^{\infty} \left(\frac{\omega_n}{\omega_0}\right) z^n} \tag{10}$$

The expression for $G(z)$ shown above is clearly of the form $(1 + \sum_1^{\infty} a_i z^i)/(1 + \sum_1^{\infty} b_i z^i)$ which simply reduces to $\sum_0^{\infty} d_i z^i$. Thus d_i can be determined recursively as

$$d_i = \begin{cases} a_i - \sum_{j=1}^{i} b_j d_{i-j} & i = 1,2,3... \\ 1 & i = 0 \end{cases} \tag{11}$$

With the aid of d_i it is an easy matter to compute the distribution of queue length upto any desired value of n , $n \in I^+$. For example, to obtain r_{1j} we compute $d_1 = 1/\omega_0 - 1$, and then $r_{1j} = r_{0j}d_1$. In order to compute r_{2j} we first compute $d_2 = (1 - \omega_0 - \omega_1)/\omega_0^2$, and then obtain $r_{2j} = r_{0j}d_2$. The higher order probabilities are computed in a similar fashion. For a stable system, the distribution of queue length will be unimodal and will possess the property that $r_{nj} = 0$, $n \geq k$, for some fixed $k \in I^+$. Thus, the compuation will necessarily involve only a finite number of terms.

There are other methods available for determining the distribution of queue length using $L_j(z)$. For example, if this transform is extremely complicated, then an approximation may be desirable. Often, $L_j(z)$ can be viewed as a rational function of the form $Q(z)/R(z)$, with these functions being polynomials without common factors. The algebra of partial fractions will yield [Fell68] a simpler (exact) decomposition for $L_j(z)$. If $R(z)$ has distinct roots, the decomposition is exact. Methods for determining the zeros [RuTa77, CoBo72, Henr64] via algorithmic techniques are known. When the roots can only be found appriximately due to limitations of solving polynomials for zeros [CoBo72, Henr64], such as when the degree of the polynomial is large, the transform of the queue length distribution may be approximated [Henr64, Fell68, Doug63, CrCG79]. Once a simpler form for the transform is obtained, it may be inverted by one of several methods [Jage78, Jage84].

6.6 Distribution Of Packet Delay

Let $F^*(s)$ be the Laplace-Stieltjes transform for F_S, where s is a complex argument. We formulate the delay characteristics of an arbitrary packet at station j as that of a customer in an M/G/1 system, with service time distribution $F_S(.)$. We define the delay of a packet as the time it spends in its queue before its transmission begins. Let $\hat{f}_S(.)$ be the residual service time density for a transmission from station j (i.e., a packet in service). The Laplace-Stieltjes transform of this density is given by

$$\hat{F}^*(s) = \frac{1 - F^*(s)}{s\,E(S)} \tag{12}$$

Using this, we can express the transform of the packet delay distribution as

$$W_j^*(s) = \frac{1 - \lambda_j E(S)}{1 - \lambda_j E(S)\hat{F}^*(s)} \tag{13}$$

Inverting this last transform to obtain the delay density function, we obtain

$$W_j(t) = \sum_{k=0}^{\infty} (1 - \lambda_j E(S))[\lambda_j E(S)]^k\, \hat{f}_{(k)}(t) \tag{14}$$

where $\hat{f}_{(k)}(t)$ is the k-fold convolution of the residual service time density. For a stable queueing system at station j, $[\lambda_j E(S)]^k$ will approach zero as k increases. This will enable us to approximate $W_j(t)$

by a finite number of terms, with accuracy increasing as the number of terms increases.

Since $E(S^r)$ is seen to exist for $r = 1,2....,$ an application of Takaċs recurrence theorem [Taka62] will allow us to evaluate the moments of packet delay distribution. Let L_j be the random variable representing mean queueing time for packets queued at station j. Then,

$$E(L_j^{\,r}) = \frac{\lambda_j}{(1 - \rho_j^{\,'})} \sum_{i=1}^{r} \binom{r}{i} \frac{E(C^{i+1})E(L^{r-i})}{(i + 1)} \qquad r = 1,2..... \qquad (15)$$

with $E(L^0) = 1$, and $\rho_j^{\,'} = \lambda_j E(S)$ as the mean contribution of station j to the token's cycle time.

Let M_j represent the random time that station j must wait from the instant that it begins a transmission to the instant that it next gains access to the channel (assuming that it does require a free token on the next pass). By hypothesis, since at least $(r+1)$ moments of the quasi-service time distribution, and at least r moments of the packet delay distribution exist, the moments of station j's response time can be obtained as

$$E(M_j^{\,r}) = \sum_{i=0}^{r} \binom{r}{i} E(S^r)\, E(L_j^{\,k-r}) \qquad r = 1,2..... \qquad (16)$$

6.7 Channel Utilization

The fraction of time that each station keeps the token occupied (detains the token during packet transmission) is the station's channel utilization. The total utilization (summed over all stations) yields the channel utilization of the system. This quantity can also be interpreted as the throughput of the system. In order to obtain an expression for utilization, it is necessary to obtain the stationary distributions associated with the semi-Markov process. Let $\{Z(t)\}$ be the continuous time semi-Markov process associated with $\{Z_n, T_n\}$. The embedded chain $\{Z_n\}$ observed at the instants of state transitions behaves like a Markov chain. $\{Z_n\}$ is aperiodic, positive recurrent and irreducible and can be shown [HoPS72] to possess a stationary distribution $\Pi = (\pi_{W1}, \pi_1, \ldots, \pi_{WN}, \pi_N)$. The probability of finding the token in state i after the process has been operating for an arbitrarily long time is π_i, $i \in \mathbf{S}^*$.

The process $\{Z(t)\}$ can be shown to exhibit a unique limiting behaviour within the chain $\{Z_n\}$. The equilibrium distribution of $\{Z(t)\}$ is given by $\Phi = (\phi_{W1}, \phi_1, \ldots, \phi_{WN}, \phi_N)$, where ϕ_j is the limiting interval transition probability of observing the token in state $j \in \mathbf{S}^*$. This probability is different from that obtained from the chain due to the consideration of the holding time distributions in the various states of $\mathbf{S}^*$. With $q_k = r_{0k}$ and $p_k = (1 - q_k)$ we obtain

$$\phi_j = \begin{cases} \dfrac{\pi_j \; E(Y_j)}{\sum_{k \in \mathbf{S}} \pi_k \; \{p_k E(X_k) + q_k E(V_k)\} + \sum_{k \in \mathbf{W}} \pi_k E(Y_k)} & j \in \mathbf{W} \\[2ex] \dfrac{\pi_j \; \{p_j E(X_j) + q_j \; E(V_j)\}}{\sum_{k \in \mathbf{S}} \pi_k \; \{p_k E(X_k) + q_k E(V_k)\} + \sum_{k \in \mathbf{W}} \pi_k E(Y_k)} & j \in \mathbf{S} \end{cases} \qquad (17)$$

Let u_j be the mean channel utilization by station j alone. Then u_j can be given by

$$u_j \quad = \quad \begin{cases} \phi_j \, E(Y_j) & j \in \mathbf{W} \\ \phi_j \, \{(1-r_{0j})E(X_j) \quad + \quad r_{0j}E(V_j) & j \in \mathbf{S} \end{cases} \tag{18}$$

and the approximate mean system throughput is obtained as

$$U^{*} \quad = \quad \sum_{j \in \mathbf{S}} u_j \tag{19}$$

6.8 The M/G(r+1)/1 Approach

Another interesting queueing model applicable to the MQCS/s-QED=1 problem can be obtained from analyzing systems in which service times of adjacent or near adjacent customers (in a single server queue) are correlated. If we can determine the stationary distributions for such a queue, then we essentially have a closed-form solution to the problem. Unfortunately, our solution must depend on the correlation of the service times. In all the analysis we have carried out from section 5.5 through the present, we have only been able to propose models for describing the correlation function for a given process. In the following discussion, we do not require the i.i.d service time assumption that is often made. Observe that while our closed-form solution skips the i.i.d assumption, the one remaining difference between the queueing system at station j and an M/G(r+1)/1 queueing system still remains to be addressed (see section 5.1). The letter r in the queueing notation above is used in conjunction with G to denote the distribution of a moving average random variable of order r.

As in the past, the queue at station j is viewed as a single server queueing system. The service time of the n^{th} customer is given by

$$S_n = g_0(C_{n+r}) + g_1(C_{n+r-1}) + \ldots + g_r(C_n), \qquad n \geq 0,$$

with
$$\sum_{i=1}^{r} \inf \; g_i \geq 0,$$

where $\{C_n\}$ is a sequence of i.i.d random variables with density function given by Eq.(5.14) or Eq.(5.23). The constraint on the g_i's is to ensure nonnegative service times. We are particularly interested in the situation where the g_i's are positive constants. Thus, S_n becomes a linear combination of independent random variables,

$$S_n = b_0C_{n+r} + b_1C_{n+r-1} + \ldots + b_rC_n , \qquad n \geq 0.$$

An analysis of such queueing systems was pioneered by Loynes [Loyn62a, Loyn62b], and Pearce [Pear66, Pear67]. Pearce was interested in queues with moving average service times of order r (i.e., M/G(r+1)/1 systems) where the moving averages guaranteed service times that were correlated. However, in choosing the particular values of b_i, no guidelines are available. Since we are interested in modelling positive correlation, we require that the b_i's be positive. Consider the $(m+1)$ dependent random variables $S_0,\ldots,S_m$. Application of the PCA method described in section 6.4 will yield coefficients a_i, $i = 0,..,m$, such that the corresponding linear combination of the S_n's possesses maximum variability. In theory, given the a_i, it is possible that each b_i can be determined from a system of linear equations that involve b_i's and a_i's. Thus, each b_i will be defined in terms of the a_i's. In practice, since principle component analysis is difficult for sequences of random variables with incompletely specified distributions, this approach is difficult. Instead, an iterative method can be used to determine a set of positive constants b_i that maximizes the variability of a set of S_i's via principle component analysis.

6.9 Queue Length Distributions Via M/G(r+1)/1 Systems

In this section, we focus on the packet-queue length distribution at station j assuming that this station's packet transmission lengths (as seen by the observer) can be represented by moving averages of order two. For convenience, we take the sum of the b_i's to be unity. Using lower case letters to denote actual realizations of random variables, the $(n+r)$ tuples $(c_0,..,c_{n+r-1})$, $(C_0,..,C_{n+r-1})$ are compactly represented as $(c^{(n+r-1)})$, $(C^{(n+r-1)})$, with the cycle time distribution $f_C(.)$ taken as the common distribution of the C_i's.

Let $P_k(c^{(n+r-1)})$, $k \geq 0$, be the probability that $(C^{(n+r-1)}$ is equal to $(c^{(n+r-1)})$ and the server finds k customers queued at station j (the reference station) after serving customer n. This corresponds to the instant that the token returns to station j for the $(n+1)^{th}$ time. The generating function of this distribution is given by

$$P(c^{(n+r-1)};z) = \sum_{i=0}^{\infty} P_i(c^{(n+r-1)})z^i, \qquad |z|\leq 1 \tag{20}$$

and its integral transform $P^*(s^{(r)};z;n)$ is given by

$$P^*(s^{(r)};z;n) = E[P(U^{(n+r-1)};z)\exp(-s_rC_{n+r-1} - -s_{r-1}C_{n+r-2} \cdots s_1C_n],$$

with $|z|\le 1$, Re $s_i \ge 0$, $1\le i\le r$.

For the limiting distribution, we obtain the form

$$P(w_1,\ldots,w_r;z) = \lim\ E\,P(C_0,\ldots,C_{n-1},c_n,\ldots,c_{n+r-1};z),$$

where $w_1,\ldots,w_r$ is a realization of $u_n,\ldots,u_{n+r-1}$, and its integral transform

$$P^*(s^{(r)};z) = E\ P(W^{(r)};z)\exp(-s_rW_r - \ldots - s_1W_1), \qquad |z|\le 1,\ \text{Re } s_i\ge 0,$$

where f_C is used as the density of the i.i.d random variables W_i. Define $\psi = E[\exp(-\lambda_j sC)]$, for Re $s\ge 0$. When $r = 1$, for $|z|\le 1$, Re $s_1\ge 0$, we obtain

$$zP^*(s_1;z)=-z(1 - z)\psi\{(1 - z)b_0 + s_1/\lambda_j\}P^*{}_0\{(1 - z)\lambda_jb_1\}[z - \psi(1 - z)]^{-1}.$$

After some algebraic manipulation of the integral transform of the limiting distribution, dividing by z and taking the limit of the resulting expression as $z \to 0$, we arrive at

$$P^*(s_1;z) = (1 - z)\psi\{(1 - z)b_0 + s_1/\lambda_j\}\psi(1 - b_1z)[\psi(b_0)]^{-1}.$$

$$(1 - \lambda_j \int_0^\infty c\ dF_C(c))[\psi(1 - z)],^{-1}, \qquad |z|\le\ 1,\ \text{Re } s_1\ge 0.$$

Finally, the generating function of packet-queue length distribution can be found by letting $s_1 = 0$, or

$$P(z) = (1 - z)\psi\{(1 - z)b_0\}\psi(1 - b_1 z)[\psi(b_0)]^{-1}.$$
$$(1 - \lambda_j \int_0^\infty c \, dF_C(c))[\psi(1 - z) - z]^{-1}, \qquad |z| \le 1.$$

In the case of $r = 2$, the limiting packet queue length distribution's generating function is obtained by similar methods to be

$$\begin{aligned}P(z) = & - z^{-1}(1 - z)\psi\{(1 - z)b_0\}[1 + \psi'(0)][\psi(b_0)]^{-1} \\ & \times [\{1 + b_0\psi'(1)\}\{\psi(1)\}^{-1}\psi(b_0 + b_1) - b_0\psi'(b_0 + b_1)]^{-1} \\ & \times [\{1+b_0\psi'(1)\}\{\psi(1)\}^{-1}\psi(b_0+b_1+(1-z)b_2\} - b_0\psi'(b_0 + b_1 + (1 - z)b_2\}] \\ & \times [\{z - \psi(1 - z)\}^{-1}\psi\{(1 - z)(b_0 + b_1)\}\psi\{b_0 + (1 - z)(b_1 + b_2)\} \\ & + \psi\{b_0 + (1 - z)b_1\}], \qquad |z|\le 1.\end{aligned}$$

6.10 Summary

This chapter uses the cycle-time distributions developed in chapter V to develop two kinds of queueing models. One kind assumes i.i.d service times for queues, and the other kind attempts to incorporate the dependence between service times. The i.i.d assumption leads to an M/G/1 type approximation that typically yields underestimates for steady-state queueing and delay distributions. The other models proposed attempt to take into account the correlations between pairs of dependent cycles. The intention here was to show that once an explicit form for the cycle-time distribution is obtained, several approaches (all giving special attention to dependence) in modelling these queues are possible.

CHAPTER VII

STABILITY AND FAIRNESS IN TOKEN-PASSING SCHEMES

If the mean queue length at station j, $j \in S$ is finite, for finite λ_j and positive service, walk, and switching times, then the queueing process at station j is called a *stable* process. Assuming that the queueing process at station j has a stationary distribution, it can be shown that the queueing process is stable. Observe that while stability is a consequence of stationarity, the converse need not hold. A stable queue is not necessarily stationary since a mean queue length may be finite even though its distribution may vary with time. Let $\rho_j = \lambda_j E(X_j)$ be the mean channel utilization by customers at station j. To the observer at station j, ρ_j takes the form of a *traffic intensity* at queue j. Unlike standard GI/G/1 queueing situations, no simple condition exists involving ρ_j alone that can ensure stable queueing conditions at station j [Kueh79].

In Eq.(4.11) it is shown that the traffic intensity at a single station can be computed via the SV method provided the limiting distribution for the vectors is known. Additionally, it is known [Neut77] that steady-state queueing distributions at a given station will be stationary only if the traffic intensity at the station is less that *1*. In fact, only under this condition will the matrix G defined in Eq.(4.16) be positive and stochastic. In section 7.1, a stability condition is given in a form that is related to the SP method. In section 7.2 we describe how a system's degree of stability is related to the distance between the vector of mean arrival rates and a critical point in an N-dimensional space. Simple descriptive measures of system stability are presented. In section 7.3 we introduce the issue of *fairness* to describe how fair the operating protocol is to the N stations. The intention is to describe fairness *within* an operating protocol, and not between protocols. With some additional effort, the ideas presented here can be extended to compare fairness between different protocols.

For a given set of system parameters, a flexible measure that describes how fairly service is distributed across the system is presented in section 7.4. The token-intervisit time for a station during its active cycles (i.e., those cycles in which this station makes a transmision) is the key statistic used in the fairness measure. The flexibility arises in our ability to choose an arbitrary linear combination of the moments of this random variable to define fairness in our own context. In section 7.5, we present an application of rarefactions to show how an approximate first passage time distribution may be computed. Since the approximation is based on a renewal assumption for the cycle-time process, the approximation will work well only for high and low loads.

7.1 Stability Of MQCS Schemes

Assume that the arrival, walk, service, and switching distributions at all queues are stationary. Treating each queue j, $j \in S$, as a GI/G/1 queueing scheme, stability criteria for individual queues and for the whole system can be developed. Given that all queues in the set $S\backslash\{j\}$ are stable (which means that they have sufficiently small arrival rates), queue j arrives at its maximum contribution to the cycle time as ρ_j approaches 1. If the arrival rates of the other (stable) queues remain unchanged, the longest cycles seen from station j correspond to ones in which a customer from queue j is served with probability 1. These cycles give rise to a mean cycle time of

$$E_j(C) = \frac{\sum_{k\in S} E(Y_k) + \sum_{\substack{k\in S\\ k\neq j}} E(V_k) + E(X_j)}{1 - \sum_{\substack{k\in S\\ k\neq j}} \rho_k + \sum_{\substack{k\in S\\ k\neq j}} \lambda_k E(V_k)}, \qquad (1)$$

where $E_j(C)$ is the mean cycle time conditioned on the event that a customer from queue j is served during each cycle with probability 1. Observe that for Poisson arrivals, the successive (random) queue lengths seen by the server at station j form an embedded Markov chain. This fact was already made use of in Eq.(5.3) to obtain the probability that the server finds station j empty at any of its scan instants (i.e., Eq.(5.4)). Since the queue at the reference station has the *same* embedded Markov chain as an M/G/1 queue, a limiting queue length probability vector with positive entries will exist if, and only if, this probability vector is the invariant vector of the transition matrix describing the chain. It is easy to prove ergodicity for the chain, and consequently, the existence of the invariant vector. But such a vector (with positive entries) can exist only if Eq.(5.4) holds. Using station

j's conditional cycle-time as a service time random variable, and noting that the first two moments of this random variable are finite, by hypothesis, we arrive at the following. The reciprocal of $E_j(C)$ behaves as an upper bound for values of λ_j that generate stable queueing conditions at station j. In other words, queue j is stable provided that

$$\lambda_j \quad < \quad \frac{1}{E_j(C)} \quad = \lambda_j^* , \qquad (2)$$

where λ_j^* is the *critical arrival rate* for queue j customers. If the arrival rate at queue j equals or exceeds this value, then queue j grows in unbounded fashion, or is said to *saturate*. This critical value for queue j defines its *stability boundary*.

In the case of QEDs other than s-QED = 1, corresponding stability criteria may be constructed. In particular, consider the service scheme r-QED = n_j. In this discipline, the server attends to the queue at station j upto n_j times in succession before moving on to station $(j+1)$. The conditional cycle-time mean at station j can be obtained from Eq.(1) provided that the term $E(X_j)$ is replaced by $n_jE(X_j)$. In order to obtain the *critical arrival rate* for this discipline, the quantity 1 given in the fraction in Eq.(2) must be replaced by n_j.

It can be shown [Kueh79] that conditions of stability at queue j are sensitive to the arrival rates, service times, and switching times of all the other queues, as well as the sum of all the walk times. Regardless of the various customer arrival rates in the system, the server's mean cycle time is always stable since it remains bounded above by the sum of all the mean walk times and mean service times. This characteristic of token-passing guarantees that all stations receive server attention within a random time that has a stable mean.

7.2 Stability Index: A Measure of System Stability

If the condition in inequality (2) is satisfied by all N queueing processes then the entire system is stable. When modelling symmetric systems, the stability boundary is the same for all queues. Investigation of the condition in inequality (2) will show that λ_j is forced towards zero as N is increased asymptotically. Keeping all other parameters fixed, increasing the number of stations in a symmetric model requires that the arrival rates be decreased in order that condition (2) is satisfied and stability maintained. It would appear that asymmetric models are more realistic since it will often be the case that some queues fail to satisfy the stability condition. A typical situation is one where some station is doing large file transfers (generating large volumes of traffic). Such heavily loaded stations will contribute one packet to every token cycle until their packet traffic decreases. Thus it is necessary to investigate conditions of *partial stability*, where *some* queueing processes are stable. In such an asymmetric system, the lightly loaded stations witness increased token-passing overhead for the duration of time that some other stations are heavily loaded. If the arrival parameters do not vary with time, we can define a simple *stability index* to capture the relationship between *loading* (offered system traffic) and *throughput* (channel utilization). Let the vector of critical arrival rates for a stable system be denoted by $\Lambda^* = (\lambda_1{}^*,\ldots,\lambda_N{}^*)$. A simple index of stability may be defined as

$$I(\Lambda^*) \quad = \quad \frac{|\{\ j \mid \ \lambda_j \geq \lambda_j{}^* \}|}{N} \tag{3}$$

which really is the fraction of unstable queues in the system. Different configurations of unstable queues can give rise to the same value for the stability index. In order to understand the relation between the stability index, throughput, and the notion of partial stability, one must study the effects of varying arrival rates on the N-station system.

Consider the following argument presented in terms of an asymmetric system. From inequality (2) it can be seen that the boundary of total system stability is defined by the sides of an N-dimensional cube in $\mathbb{R}^N$, whose corners are vectors with entries all zero, all nonzero, or have all but one entry zero. These respectively correspond to the zero vector, the vector Λ^*, and vectors of the form $\Lambda_k{}^* = (0,..,\lambda_k{}^*,...0)$ that describe the stability boundary for station k alone. When the vector of arrival rates lies in the interior of the cube, the system is totally stable. Replacing the inequality in (2) by an equality, we obtain a system of N equations in the variables λ_k, $k \in S$. The critical arrival rate vector Λ^* can then be determined as the solution to this linear system. Let $u_k = (X_k - V_k)$ and $s_k = [\sum_i E(Y_i) + \sum_{i \neq k} E(V_i)] + E(X_k)$. The individual station stability boundaries are obtained as

$$\lambda_j^* = \frac{\prod_{\substack{k \in S \\ k \neq j}} (s_k - u_k)}{\{ \prod_{k \in S} s_k + \prod_{k \in S} u_k \} - \{ \sum_{k \in S} \prod_{\substack{i \in S \\ i \neq k}} s_k u_i \}} , \quad j \in S. \tag{4}$$

Let the (variable) vector of arrival rates be given by $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. The regions of stability as given by inequality (2) are bounded by planes in an N-dimensional space, with Λ^* as a point common to all planes. Thus Λ^* defines 2^N different regions, only one of which is bounded on all sides. When Λ lies inside this region (which is the cube just described), we have total stability. Here $I(\Lambda^*)$ is zero and does not say much about system throughput. If $\Lambda \geq \Lambda^*$, then Λ lies in an infinite region where all queues are unstable causing total instability, with $I(\Lambda^*) = 1$. In this case, the throughput attains its maximum possible value. But for these two regions, all other regions describe situations of partial system stability. This happens when *some* but *not all* coordinates of Λ are strictly less than corresponding coordinates in Λ^*, causing $I(\Lambda^*)$ to take on values between *0* and *1*. In this region, the system throughput will be a nondecreasing function of the stability index.

When Λ lies within the cube that ensures total stability, $I(\Lambda^*)$ does not yield any information regarding system throughput. Here, a stability index that is more sensitive to changes in Λ can be defined. This is given by

$$I^S(\Lambda^*) = || \Lambda - \Lambda^* || , \tag{5}$$

where $||\cdot||$ is used to denote the Euclidean norm in N dimensions. The system throughput in the absolutely stable region can be seen to be a nondecreasing function of the inverse of $I^S(\Lambda^*)$. A phenomena involving stability occurs here that appears to characterize the multiqueueing situation and is unlike conditions in typical queueing systems. If Λ lies in the absolutely stable region and all coordinates are simultaneously increased, the queues that are first to become unstable are precisely those with the highest arrival rates, and they attain saturation in the order of decreasing arrival rates, *independently* of all other parameters [Kueh79].

7.3 Issues of Fairness

The *fairness* of a distributed protocol is an important measure of network performance since systems operating under global optimality are not necessarily fair [GeSt80] in terms of individual usage of the resource(s). In local-area networks, since protocols are generally designed to function far from saturation, individual station delay is an important parameter in the determination of protocol fairness. Unlike long-haul and contention based networks, individual station channel utilizations no longer maintain great importance in fairness determination, especially for the one-packet at a time QED. This is due to the fact that active stations are periodically visited by the server (token), independent of their buffer status or desire to transmit. This is not to imply that utilization cannot be taken as a factor in the resolution of fairness; only that a station's channel utilization by itself does not reflect this station's ease of channel accessibility. For example, if the station under consideration transmitted packets with a mean length much smaller than the mean packet lengths of all the other stations, and if its mean packet arrival rate is not greater than that of any other station, then this station will have a smaller channel utilization value than any other station. But this only means that this station is less demanding of the channel than the other stations, and does not imply that the protocol is necessarily unfair.

The issue of fairness in the context of local area networks was first introduced by Marsan and Gerla [MaGe82]. These authors define a fairness measure in terms of mean delay and mean throughput for individual stations. It is a well known fact that ordered access schemes are generally superior to contention-based or random schemes at conditions of high load. But under light load, the contention schemes perform better. The performance keywords here are mean throughput (at high loads) and mean delay (at low load) for individual stations. Define a particular station's accessibility to the channel (in the mean) as a ratio of its throughput at high load to its delay at low load. For convenience, we call this the mean throughput-delay ratio, or TDR. Fairness in [MaGe82] is defined as the smallest ratio of TDR statistics generated by pairs of stations, for every pair of stations on the network. Additionally, a protocol is defined to operate in an optimally fair fashion if the fairness statistic is unity, i.e., every TDR is equal to every other TDR.

In the next subsection, we present an alternate approach to determine fairness. This method is based on the random time between a station's access of the transmission medium and its next chance to access the medium. A definition of fairness is obtained in terms of distributions. Our approach is motivated by the fact that mean values are limited in their scope and do not yield any information about variations in resource usage. That is, the variation in the utilization of the channel (by a station) has important consequences on how the channel usage varies between the other stations. Since the total capacity of the channel is bounded, a highly variable or highly skewed transmission time by a single station can have strange affects on delays experienced by other stations. It is fairly easy to build examples where the fairness measure proposed in [MaGe82] cannot capture such information.

7.4 A Measure of Fairness

In token-passing schemes, the cycles that are important to a particular active station are those cycles in which the station makes a transmission. Let D_j be the random length of a stationary cycle conditioned on the event that a transmission from station j is made. The notion of stationary here is taken to mean a random cycle-time with law defining the density in Eq.(5.14) for the case of asymmetric systems, and Eq.(5.23) for symmetric systems. Define the order statistics $R_1 = max\ \{D_1,\dots,D_N\}$ and $R_2 = min\ \{D_1,\dots,D_N\}$, corresponding to the largest and smallest such cycles, respectively. Let $\omega_i = E[R_2^{\,i}]/E[R_1^{\,i}]$ and $\Omega_i = a_i\omega_i$, for $a_i \in \mathcal{R}^+$ and $i=0,1,2\dots$. For convenience, we limit ourselves to a (generally small) finite number of weights, with a sum normalized to unity. For each integer k, $k \geq 0$, we define vectors $\Omega^{(k)} = (\Omega_0, \Omega_1, \ldots, \Omega_k)$ and $\mathbf{a}^{(k)} = (a_0, a_1, \ldots, a_k)$, and a corresponding fairness measure as $X(k) = ||\Omega^{(k)} - \mathbf{a}^{(k)}||_k$, where $||.||_k$ is used to represent the euclidean norm in k dimensions. The protocol is said to exhibit *perfect fairness in the* k^{th} *degree* if $X(k) = 0$, for arbitrary choice of the weights a_i. Note that fairness in the first degree is equivalent to the condition that $E(D_j) = d$, $d \in \mathfrak{R}^+$, for all $j \in S$.

The measure introduced above can be motivated by the following argument. Focusing our attention on the conditional cycle times, we see that if every station's conditional cycle time converges to the same random variable in distribution, then every station must have the same fair chance of accessing the transmission medium. In other words, not only is their "waiting time" fair in the mean, but it is also fair in terms of moments upto order k. For a given value of k, we can choose a weight vector $\mathbf{a}$ that appropriately describes our level of interest in each of the k moments. Thus if we are equally interested in the mean and variance, we set $a_1 = a_2 = 0.5$, and $a_i = 0$ for all i, $2 \leq i \leq k$. The further away the value of $(\Omega_1 + \Omega_2)$ from unity (i.e., the sum of the weights), the less fair will be the protocol.

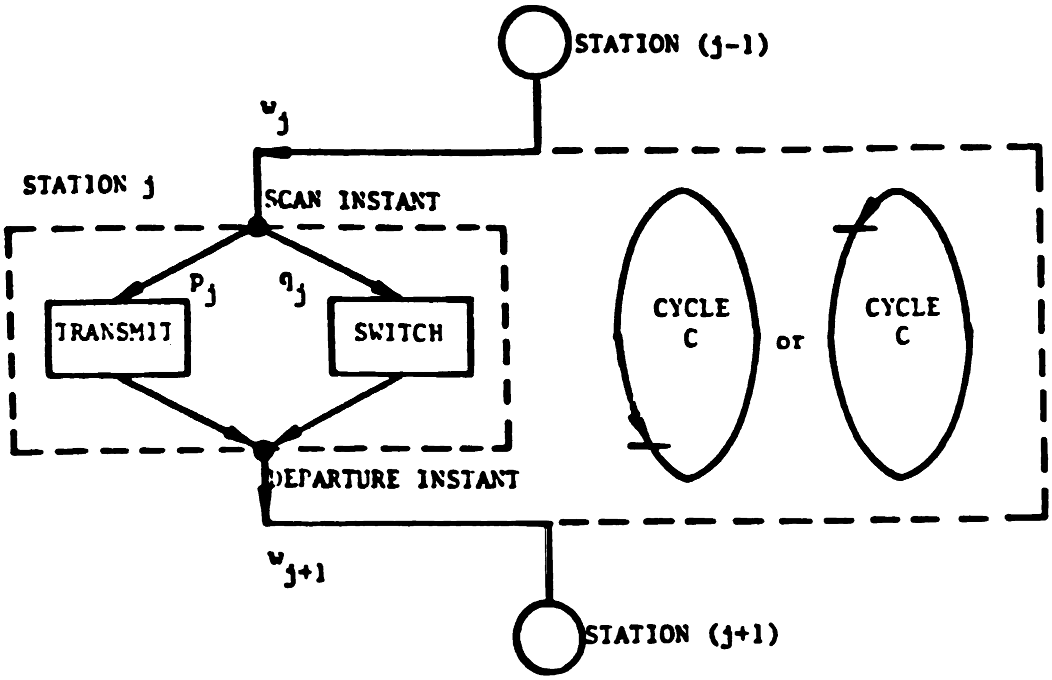
We can view the vector $\mathbf{a}^{(k)}$ as the centre of a unit ball in a k-dimensional space. The measure $X(k)$ lies within the ball or on the boundary (for a totally unfair protocol). The distance of $X(k)$ from $\mathbf{a}^{(k)}$ determines exactly how far the protocol operates from perfect fairness in the context of the first k moments of conditional cycle times.

There is an interesting problem that presents itself at this time. Note that a fairness measure is really a random function of the parameters of the system. That is, it changes according to variations in p_j, μ_{j0}, μ_{j1}, a_j, $j \in \mathbf{S}$, and N. Keeping some parameters fixed while varying others will allow us to study the distribution of a general fairness statistic X^* as a function of the changing parameters. Consequently, this will allow us to determine those parameters that have the maximum effect (where we must define the kind of effect we are interested in) on the fairness of the protocol.

Consider a measure $X(k)$ for a given system to be defined in terms of the cycle-time density given in Eq.(5.14). To obtain the density of the conditional cycle-time random variable D_j, simply replace the probability p_j wherever it appears in Eq.(5.13) by 1. This can be done for each station on the system. By definition, the random variables D_j, $j \in \mathbf{S}$, are conditionally independent. The distributions of the order statistics R_1 and R_2 can be determined in a fairly straightforward fashion using the N distributions of conditional cycle-times $D_1, \ldots, D_N$. Let the distribution of D_j be denoted by F_{Dj}, $j \in \mathbf{S}$. In order to obtain the distribution F_{R1} of the random variable R_1 we proceed as follows. By definition,

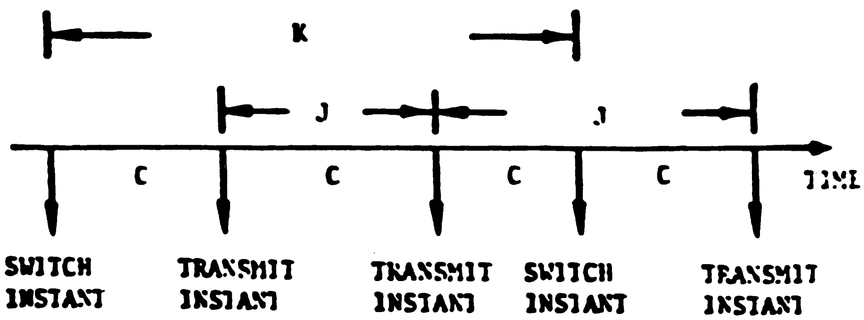
$$\begin{aligned} F_{R1}(r) &= P(R_1 \le r) \\ &= P(D_1 \le r; \ldots\ldots, D_N \le r) \qquad (6) \end{aligned}$$

The largest of all the D_j's is less than or equal to r if and only if all the D_j's are less than or equal to r. By the assumption of independence of conditional cycle-times, this gives us



(a) A cycle involves either a transmit phase or a switch phase.

Fig.7a Transmit phases and Switch phases



(b) Relationship between cycle and first passage times.

Fig.7b Cycle-times and Passage Times

$$F_{R1}(r) = F_{D1}(r) \cdot F_{D2}(r) \ldots\ldots F_{DN}(r) \tag{7}$$

and by using a similar reasoning, we obtain

$$F_{R2}(r) = 1 - (1 - F_{D1}(r)).(1 - F_{D2}(r))\ldots\ldots(1 - F_{DN}(r)). \tag{8}$$

Given the distributions of R_1 and R_2, the next step is to obtain the distribution of the quotient random variable $R = R_2/R_1$. Since we are dealing with a quotient random variable, and not with ratios of moments as before, the fairness statistic X^* is a function of random variables. In general, determining the distribution of R is not a straightforward matter since we are now dealing with dependent random variables (i.e., order statistics are dependent). In order to find the distribution of the product of dependent random variables the two-dimensional Mellin transform [Spri79] can be used.

7.5 Approximate Distributions For Token-Intervisit Times

Consider the scene involving reference station j in Fig. 7a. On arrival at this station, the token may go into either a transmit phase or a switch phase depending on whether a customer is present or not. We are interested in successive token departure instants from station j. The token's departure instant from the reference station after a transmit phase is called a T-instant, and the departure instant after a switch phase is called an S-instant. If each type of instant is considered to be an event, the random times of the process $\{Z(t)\}$ from any T-instant to the next T-instant, and from any S-instant to the next

S-instant are the *first passage times* of the process between the respective events.

Let the random variables J and K represent the first passage times for the transmit events and switch events (Fig. 7b), with distribution functions F_J and F_K, respectively. Let $J_1, J_2, \ldots, J_n, \ldots$ and $K_1, K_2, \ldots, K_m, \ldots$ be i.i.d (independent and identically distributed) sequences of random variables from the distributions F_J and F_K, respectively. Given the parameters of the system the joint densities of successive cycle time random variables can be developed and examined for serial dependence, as has already been outlined in the previous chapter. This dependence is small when the various traffic intensities are either very small or very large [Kueh79]. If the dependence is known to be small, then the cycle times (intervals between successive departure instants) seen at station j can be approximated by a renewal process $\{S_n, n \geq 1; F_C(.)\}$.

Suppose that we delete each point S_n of the renewal process with probability q_j. Next, expand the time scale by the factor $1/p_j$. The deletion of each point is done independently of the other points, and independent of the process $\{S_n\}$. By this *rarefaction* procedure [Rény56] we obtain a new point process $Rp_j = \{S_n^*\}$. The notation Rp_j is used to denote a rarefaction procedure with respect to scale $1/p_j$. Each term in the rarefied sequence $\{S_n^*\}$ is a scaled random sum of intervals of the process $\{S_n\}$, where each interval has distribution $F_C(.)$. Clearly, the random number of intervals used in the sum is geometrically distributed with parameter p_j. Thus $S_1^* = p_j S_X$, where X is the first index that is not deleted. S_X is actually the first passage time of the process

$\{Z(t)\}$ from a transmit event to a transmit event at station j. Alternately, S_X is precisely the random variable J, and S_1^* is a scaled version of this passage time. Consequently, F_J is the distribution of a geometric sum of random variables C with density f_C. Let F_J^* be the interval distribution of the rarefied process $\{S_n^*\}$. Then, except for the change in scale, F_J^* and F_J are the same distribution. We now state a theorem [Rény56] that enables us to determine the interval distribution F_J^*.

Theorem 7.1:

Given a renewal process $\{S_n, n\geq 1; F_C(c)\}$, the rarefied process $Rp_j\{S_n\} = \{S_n^*\}$ is also a renewal process. The interval distribution F_J^* of the new process is given by

$$F_J^*(c) = p_j \sum_{i=1}^{\infty} F_C^{(*i)}(c/p_j) q_j^{\,i-1} \tag{9}$$

where $0 < q_j < 1$, $p_j = 1 - q_j$, and $F_C^{(*i)}$ is the i-fold convolution of the interval distribution F_C.

Corollary 7.2:

For renewal processes $\{S_n\}$ and $\{S_n^*\}$, where the latter process is a rarefaction of the former, if $E(S_k) < +\infty$, then $E(S_k) = E(S_k^*)$ for $k \geq 1$. If $E(S_k) = +\infty$, then $E(S_k^*) = +\infty$ for $k \geq 1$.

Note that except for a scale change, Theorem 7.1 gives us the distribution of the random variable J. Replacing p_j by q_j and vice-versa in the above argument yields a rarefied process whose interval distribution F_K^* is a scaled version of the distribution of F_K. Thus, we obtain the distributions of the first passage time random variables J and K.

Using the fact that J and K are sums of random variables obtained by geometric compounding, we can apply Wald's theorem [Taka62] to obtain the means as $E(J) = E(C)/p_j$ and $E(K) = E(C)/q_j$, and their variances as

$$V(J) = (1/p_j)\, V(C) + (q_j/p_j^2)\, (E(C))^2$$
$$V(K) = (1/q_j)\, V(C) + (p_j/q_j^2)\, (E(C))^2 \qquad (10)$$

The mean passage times obtained in this manner are exact, independent of the renewal assumption. This means that the expressions for mean passage times are valid under all traffic conditions in a stationary system. If the traffic conditions are very low or very high, $V(C)$ is a good approximation to the "true" cycle time variance, and the expressions for passage time variances can be expected to perform well as approximations. If the arrival rates at all queues are zero or all queues are unstable, the renewal assumption is asymptotically exact. Since successive cycle times are now independent, $V(C)$ obtained from f_C is exact, and so the variances for the first passage times are also exact. For other traffic conditions, the serial dependency of cycle times must be taken into consideration. The approximation improves with the accuracy in the estimate of cycle-time memory.

When $p_j = q_j$, we find that J and K reduce to random variables with the same distribution. When $p_j \neq q_j$, J and K will have different distributions. Using this line of thought, we try to explain the behaviour of Kuehn's model.

Let random variables S and T represent the lengths of cycles involving switch phases and transmit phases at the reference station, respectively. In the stationary state, the instants at which packets from station j are transmitted can be viewed as renewal points. At these instants, the queueing process at this station can be treated as an embedded Markov chain. Using renewal theory arguments, Kuehn obtains approximate expressions for the mean number of customers waiting at station j at the server's scan and departure instant, and the mean customer waiting time (service time excluded). We observe that independence between queueing processes at the various stations follows indirectly from the assumption that S and T are each i.i.d. random variables. Intuitively, we would suspect that the switch random variables (i.e., like S) and the transmit random variables (i.e., like T) are dependent, both within types, as well as between types. Given the occurrence of a *T-cycle*, it is more likely that the following cycle is a *T-cycle* rather than an *S-cycle*. That is, the longer it takes the server to return to the reference queue, the greater is the probability of finding a packet awaiting service at this queue. Thus, assuming that S and T are independent, or assuming that sequences of switching cycles and sequences of transmit cycles are i.i.d in themeslves gives sufficient cause for concern regarding the validity of the independence assumption. Note that the manner in which S and T are defined ensures that $E(T) > E(S)$.

If the packet traffic offered by station j is very small, then

p_j is close to 0. In this case $E(S)$ is very close to $E(T)$, since packet contributions from station j are scarce. So the approximation can be expected to work well for station j. If the offered packet traffic at station j is very high, then p_j is close to 1, and station j makes frequent packet contributions to the cycle time. Now we have scarce *S-cycles* and conditioning on the distribution of these cycles does not offer much to the end result. So the approximation can be expected to work well again. When the offered traffic is moderate, then p_j is closer to q_j than before. Our ability to discriminate between the two types of cycles begins to get weaker as p_j approaches q_j. Consequently, the variance of the cycle time begins to depend more heavily on the actual arrivals at station j. At this stage, the approximation begins to degrade. Indeed, this was demonstrated both experimentally and analytically in Fig. 5c.

7.3 Summary

In this chapter, we proposed a new and very general definition of fairness for token-passing protocols utilizing an S-QED = 1 scheme. It is possible to obtain corresponding measures of fairness for other QEDs by generalizing this approach. The intention of resorting to distributions in our definition is motivated by the fact that mean values tend to neglect valuable information in a statistical sense. Consequently, it follows that the approach to determining the distributional measures is computationally more complicated than the approach suggested by Marsan and Gerla [MaGe82].

The fairness measure defined in section 7.4 is a function of several parameters, and in this sense is really a statistic. An interesting problem that immediately presents itself is the distribution of this statistic. In section 7.4 we presented a method to determine its distribution, and reserve the resolution of an explicit form for future work. Once the distribution is obtained, other insightful questions may be asked. In particular, we may be interested in the effect of the parameter N on the distribution of the fairness statistic, or the limiting and asymptotic forms of this distribution.

The discussion on stability is a generalized form of Kuehn's [Kueh79] work on multiqueues. The proof for stability is obtained via the embedded Markov chain of queue length states. We presented two simple measures of stability, one in which the system's arrival rate vector fell outside an N-dimensional cube defining a stable region, and one in which the vector fell within the cube. Additionally, we discussed a possible interpretation of stability in relation to system throughput. There is considerable scope for future work in the realm of system stability, especially for more general arrival processes and time-varying rates.

The last section of this chapter makes use of the fact that the cycle-time process closely approximates a renewal process for very high and very low loads. Clearly, the approximation improves as the load is further increased, or decreased. By treating consecutive service events and switching events as renewal points in their respective renewal processes, an application of rarefactions easily yields the interval distributions for these processes from the original (approximate) renewal cycle-time process. The two intervals can be treated as first-passage times between service events and switching events, respectively.

CHAPTER VIII

ADAPTIVE TOKEN-PASSING SCHEMES

Token-passing on large asymmetrically loaded systems can cause unnecessary delay for some stations (i.e., those that are more active) when the system load averaged over all stations is low. If a station's delay periods coincide with fruitless token-solicitations at inactive stations, a reduction in delay will follow a reduction in the number of such solicitations. This can be done by scheduling alternate token-paths at frequent intervals, in a way that the scheduled paths bypass any inactive station visits. Scheduling is done during a combined transmission and information gathering cycle (complete network traversal) of the token, with dynamically assessed priorities assigned to the active stations. An adaptive token-passing protocol (ATP) that achieves this has been presented in [ReHu85]. The ATP method is introduced as an upward compatible enhancement to the non-adaptive token-passing (NATP) bus protocol proposed by the Standards Committee [IEEE84b]. Since ATP requires that the token-path be modified to accomodate the heavily loaded stations more frequently than in NATP, an adaptive system behaviour is required in ATP. Hence, the corresponding queueing model is called the *multi-queue and adaptive server* (MQAS) model.

Following in the spirit of chapter five, the method of analysis used is based on service probabilities. Thus, since the underlying assumption is one of independence, the results of this chapter are approximations. A key part of the analysis used in determining cycle-time distributions involves the queue length density function values for one-customer queues, for each queue on the system. In this chapter, these probabilities are determined by M/G/1 approximation methods (i.e., service time represented by cycle-times, quasi-service times, moving averages etc.). This introduces a second approximation into the analysis. If such an approximation is not good enough, then the exact probabilities can be used. Recall that a method for computing these probabilities exactly was introduced in chapter III.

The main results in this section are the distributions of cycle-times in the MQAS model or equivalently, in the ATP protocol. These distributions are important in performance considerations. With the aid of such distributions, other important distributions (i.e., station intervisit-time distributions, packet queue length distributions etc.) can be obtained. The derivation of these distributions, various important performance measures, and criteria to aid in comparing NATP and ATP protocols are a part of an ongoing research programme.

A brief description of the organization of this chapter is as follows. In sections 8.1 through 8.3 we describe the operation of the ATP scheme in terms of the MQAS model. Since the system uses two kinds of service cycles, the distributions involving both cycle-types are introduced in section 8.1, the scheduling method and the two cycle-types are introduced in section 8.2, and the distributions involving the second cycle-type are given in section 8.3. The semi-Markov model for server behaviour is described in section 8.4 and the distributions for

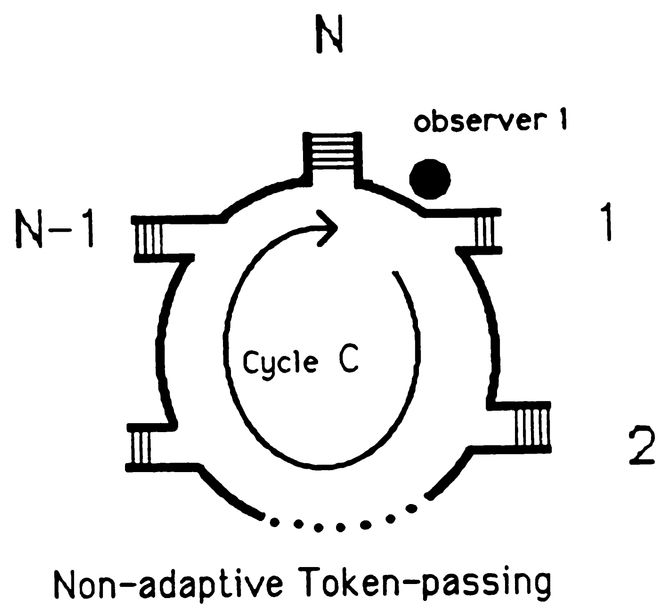


Fig.8a MQCS abstraction for NATP

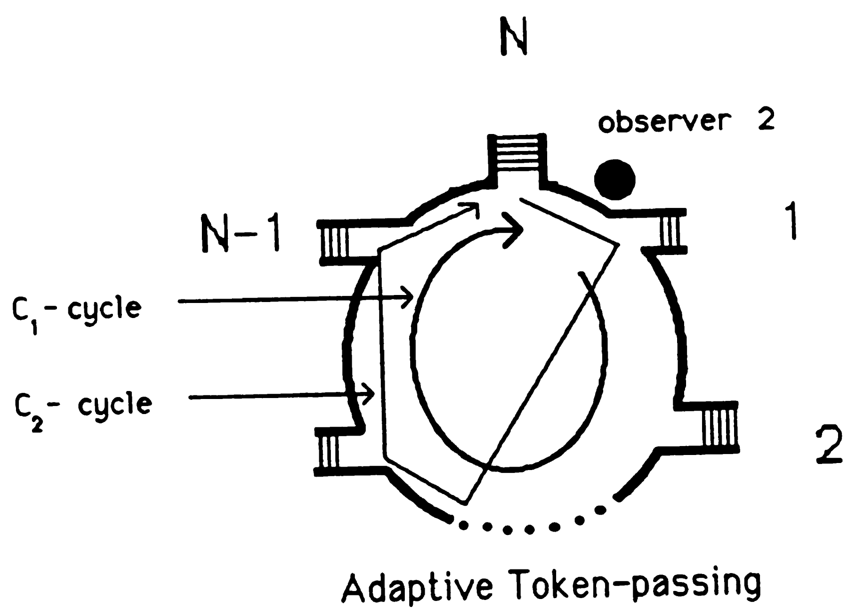


Fig.8b MQAS abstraction for ATP

both cycle-times are derived in section 8.5. The maximum entropy approach used in computing steady-state probabilities for nonempty and one-customer queues (i.e., probabilities that are require to determine the cycle-time distributions) is outlined in section 8.6 and the existence of stationary limiting distributions (under independence) is shown in section 8.7.

8.1 Description Of Model Distributions

The ATP bus is modelled by a system of N independent and infinite buffers chained together to form a logical ring by sections of varying cable lengths. Figures 8a and 8b depict similar abstractions of the MQCS and MQAS models. It must be noted that the server's behaviour as described by this model is only one of a host of possible adaptive schemes. Packet arrivals at station j are generated by some process with interarrival distribution given by $A_j(t) = Pr(I_j \leq t)$, where I_j is the interarrival time random variable at station j, $j \in \mathbf{S} = \{1, 2, \ldots, N\}$. Let us label the walk between station $(j-1)$ and station j as w_j, $j \in \mathbf{S}$. In our notation $(j-1)$ indicates station j's predecessor and $(j+1)$ indicates station j's successor on the path of the token. The station index immediately after j is obtained by computing $(j) \bmod N + 1$. The time spent by the server in w_j is given by Y_j, where Y_j has distribution $P(Y_j \leq t) = U_j(t)$, for all $w_j \in \mathbf{W}$.

8.2 Cycle-Times For NATP And ATP

For as long as it remains part of a logical ring on a token-passing bus, the station that initiates a logical ring formation is defined to be a *reference station,* and similarly, the station that precedes this station on the logical ring is called the *predecessor station* [ReHu85]. If these stations desire to retire, they must first transfer their respective identities to qualified stations on the ring. For the rest of our discussion, we take $j \in S$ to be an index in which we have a special interest. In the case of NATP, j is arbitrary; since certain quantities are defined with respect to station j, it is called a reference station. In ATP, our definitions require that j be the single *predecessor station* on the bus. Note that this restriction is merely due to the fact that ATP cycles are defined with respect to the *predecessor station*. The analysis applies regardless of the particular station defined to be the *predecessor*.

In this analysis we consider unrestricted queues, with at most one customer served per station at each of its scan instants. Let the NATP cycles (defined with respect to station j) be called *C-cycles*, with C denoting the random length of a stationary cycle. Let the density and distribution functions of this random variable be given by $f_C(.)$ and $F_C(.)$ respectively.

The behaviour of the token in the ATP scheme involves the repetition of a pair of cycles. During the first cycle (which is similar in appearance to a C-cycle and defined with respect to the *predecessor station*), the server walks, switches, or services stations just as in NATP. The only difference is that all stations in S whose customer queue sizes exceed one at their respective scan instants, are labelled by the server as *active* stations. Thus, when the server returns to the *predecessor station* (and the first cycle is complete), a set **A** of momentarily active stations on the bus will have been determined. The status of the active stations is made known to the other stations on the bus during the first cycle via broadcasting. The second cycle begins at the *predecessor Station* and in this cycle only stations in **A** are served, in the same order that they were visited in the first cycle. In this way, stations that are known to have a backlog of packets are given priority, with information that is accurate upto a cycle-time. As soon as the last active station (if the first cycle does find any active) has completed its transmission, the *predecessor station* immediately transmits a token *control_frame* to the *reference station*, from where the first cycle begins all over again. If the *predecessor station* detects the condition $\mathbf{A} = \phi$, by performing a counter scan [ReHu85], it immediately transmits a token *control_frame* to the *reference station* in order to initiate a C_1-cycle. Thus, steady-state operation of the ATP bus involves the repetition of the first and second cycles, in alternating fashion.

Let the first cycle be called a C_1-*cycle* and the second cycle be called a C_2-*cycle*. Correspondingly, let C_1 and C_2 be the random lengths of these cycles. Note that if $\mathbf{A} = \mathbf{S}$, then the C_2-cycle is the same as the C_1-cycle, since every station is labelled active. If $\mathbf{A} = \phi$, then no station is found active, and the C_2-cycle is of length *0*. The situations $\mathbf{A} = \mathbf{S}$ and $\mathbf{A} = \phi$ occur under heavy traffic and light traffic conditions, respectively. We follow a convention that assumes the C_2-cycle always occurs after a C_1-cycle, even if the latter cycle determines no momentarily active stations and consequently generates a null C_2-cycle. In this event, its length is taken to be zero.

8.3 Service, Switch, And Scheduled-Walk Distributions

If station i is found empty during the C_1-cycle, the server takes a random time V_i to switch past station i to walk W_{i+1}, $i \in \mathbf{S}$, where V_i has distribution $P(V_i \leq t) = S_i(t)$. If a customer is served at station i during the C_1-cycle, then the customer's service time is a random variable X_i with distribution $P(X_i \leq t) = B_i(t)$. During the C_2-cycle, since only the active stations are visited, there are no switching distributions involved. Define a *scheduled-walk* to be the server's walk between stations in the C_2-cycle. In order to be able to visit only the active stations, the server's path in the C_2-*cycle* is scheduled in advance with the aid of station counters [ReHu85]. Thus a scheduled-walk is a walk between active stations, or between an active station and the *predecessor station*, but not necessarily between adjacent stations.

For each station $i \in \mathbf{A}$ that is visited during a C_2-cycle, the server spends a random time Y_i' in the scheduled-walk to reach this station, and a random time X_i to serve a customer at this station. The random variable Y_i' has distribution $P(Y_i' \leq t) = U_i'(t)$. Each set $\mathbf{A}$ that is generated by a C_1-cycle (and consequently defines a subsequent C_2-cycle) is defined to be a subset of the set $\mathbf{S}' = \{1',\ldots,N'\}$ of stations that potentially require transmission of a packet in each C_2-cycle. The time spent by the server in any state $i \in \mathbf{A}$ is given by the random sum $(Y_i' + X_i)$. In other words, this is precisely the time taken by the server to walk from the previous active station (or *predecessor station*) to station i, plus the time taken to serve a station i customer. For each $i \in S'$, we take the random sum $(Y_i' + X_i)$ to have the distribution $B_i'(t)$. For analytic convenience, we assume that all distributions described in sections 8.1 through 8.3 possess finite first and second moments.

8.4 The Markov Chain Of Server Transitions

The analysis of the ATP bus is similar to that of the NATP bus, where the MQCS model is used. Throughout the analysis, we use the MQCS model as a background model in order to obtain the distributions of interest in the MQAS model. To be precise, we seek representations for the distributions of the random times spent by the token in the first and second cycles, respectively. In order to arrive at these, we require certain probability vectors, and in order to obtain the vectors, we must examine how MQAS can be obtained as a modification of MQCS. In this section, we are only concerned with *asymmetric* systems.

Consider the scene depicted in Figs. 8a and 8b, where observer 1 is stationed at the reference queue of the NATP bus, and observer 2 is stationed at the reference queue of the ATP bus. Assume that both systems are operating under identical conditions (i.e., the respective arrival, service, switching, walk, and scheduled-walk distributions are the same, for corresponding stations and walks). Also, assume that the systems are operating under steady-state conditions, and all queue length distributions are stationary.

Observer 1 sees a single server queueing system at station j. Recall that the difference between this system and a GI/G/1 system was discussed in section 5.1. In the ATP system, observer 2 also sees an approximate GI/G/1 queueing system. But in this case, the "service distribution" is more complicated. For each station $i \in \mathbf{S}$, let T_i^* be the random time between two successive (server) scan instants at station i. We call this the *trip-time* of the server with respect to station i. Note that an instance of T_i^* could either be C_1 or C_2. If F_{Tj^*}, F_{C1}, and F_{C2} are the distributions of T_j^*, C_1, and C_2, respectively, then observer 2 sees the trip-time as a compound distribution involving both C_1 and C_2. The distribution F_{Ti^*} , $i \in \mathbf{S}$, is useful in the investigation of system performance measures such as queue length distributions, packet delay distributions etc.

Let Z_n be a random variable taking values in the finite set $\mathbf{S}^*$, where $\mathbf{S}^*$ is a union of sets $\mathbf{S}$, $\mathbf{S}'$, and $\mathbf{W}$, with n an index from the set of non-negative integers $\mathbf{I}^+$. Let T_n be another random variable such that for each $Z_n \in \mathbf{S}^*$, T_n takes values in the non-negative real numbers $\mathbb{R}^+$. The sequence of pairs $\{(Z_n,T_n)\}$ is a Markov renewal process whose limiting behaviour is our main interest. Note that the assumptions made in chapter II carry over to the present analysis as well.

The transition functions of the ATP server are considerably different from those of the NATP model's server. In this case, $Q_{ij}(t)$ is an element of a $3N \times 3N$ matrix $\mathbf{Q}$, i.e., the *semi-Markov kernel* of the process. The server's behaviour over states of $\mathbf{S}^*$ can be viewed as a semi-Markov process. Let p_{0j}, p_{1j}, and q_j be the probabilities that the server encounters zero, one, and more than one customer at station j's scan instants of the C_1-cycle, respectively, for all $j \in \mathbf{S}$. These probabilities are the mixing densities used to obtain the holding time distributions of the process in terms of the distributions specified earlier. The scene depicted in Fig. 9 describes the transitions of the token over the state space $\mathbf{S}^* = \{1,2,3\} \cup \{w_1,w_2,w_3\} \cup \{1',2',3'\}$. In general, the kernel $\mathbf{Q}$ is defined as

$$Q_{ij}(t) = \begin{cases} (1 - p_{0i})U_i(t) & w_k \in \mathbf{W},\ j=k \in \mathbf{S} \\ p_{0i}U_i(t) & i=w_k \in \mathbf{W}\setminus\{w_1\},\ j=w_{k+1} \in \mathbf{W} \\ (1 - p_{0i})B_i(t) + p_{0i}S_i(t) & i=k \in \mathbf{S}\setminus\{N\},\ j=w_{k+1} \in \mathbf{W} \\ q_k \prod_{m=1}^{N} (1 - q_m)B_i(t) & i=N \in \mathbf{S},\ j=k' \in \mathbf{S}',\ j \neq w_1 \\ \prod_{m=1}^{N} (1 - q_m)B_i(t) & i=N \in \mathbf{S},\ j=w_1 \in \mathbf{W} \\ q_k \prod_{m=i+1}^{k-1} (1 - q_m)B_i'(t) & i=k' \in \mathbf{S}',\ j=r' \in \mathbf{S}',\ j \neq w_1 \\ \prod_{m=i+1}^{N} (1 - q_m)B_i'(t) & i=k' \in \mathbf{S}'\setminus\{N'\},\ j=w_1 \in \mathbf{W} \\ B_i' & i=N',\ j=w_1 \\ p_{0i}q_k \prod_{m=1}^{N} (1 - q_m)U_i(t) & i=w_N,\ j=k' \in \mathbf{S}' \\ p_{0i} \prod_{m=1}^{N} (1 - q_m)U_i(t) & i=w_N,\ j=w_1 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

for all $i,j \in \mathbf{S}^*$. For the interpretation of $\mathbf{Q}$, the ordering of states (aligned with rows and columns) is taken to be

$\{w_1,1,w_2,2,\ldots,w_N,N,1',2',\ldots,N'\}$. Define **P** to be the matrix with entries $p_{ij} = Q_{ij}(\infty)$, for all $i,j\in\mathbf{S}^*$.

The model can be interpreted as follows. The basic dynamic particle of our system is the server, or token. Its behaviour in moving among the various states of $\mathbf{S}^*$, as given by Z_n, is governed by a monodesmic semi-Markov process which is homogeneous in time and has a kernel **Q**. The matrix **P** is the matrix of transition probabilities of the underlying Markov chain. **P** describes the transition probabilities of the token from state i to state j. On leaving state $w_i\in\mathbf{W}\backslash\{w_N\}$, p_{0i} is the probability that the token encounters no customer at station i. It follows that for each C_1-cycle, p_{0i} is the probability that the token encounters an empty buffer at station i, and $(1 - p_{0i})$ is the probability that the token finds at least one waiting packet at station i's buffer, for all $i\in\mathbf{S}$. Note that p_{1i} is the probability that exactly one customer is seen waiting at station i, for all $i\in\mathbf{S}$. From state w_N, the token visits state N with probability $(1 - p_{0N})$, and with probability p_{0N} it either moves to some state in the set $\mathbf{S}'$, or to state w_1. From states w_N or N, the states visited in $\mathbf{S}'$ depend entirely on the probabilities q_i, for all $i\in\mathbf{S}$, where q_i is the probability that the token encounters *more* that one waiting customer at the buffer of station i during each C_1-cycle. From states $k\in\{N'\}\cup\mathbf{S}\backslash\{N\}$, the token moves to state w_{k+1} with probability 1.

The trip-time at station j, T_j^*, can also be defined as the *first passage time* of the process from a state in the set $\{j,j'\}$ to a state in the same set. This corresponds to the random time between token reappearances at the *predecessor station*. The random time C_1 is the time between the instant at which state w_1 is entered and either (i) the instant at which state w_N is left (if no customer is found waiting for service at station N), or (ii) the instant at which state N is left (if a customer at station N is served). This corresponds to the random time spent by the server in visiting (and possibly serving customers in) the various states of **S**. The random variable C_2 is the time spent by the server in the states of different subsets **A** of **S**$'$, for each such subset of active states generated by the preceding C_1-cycle. Under the independence assumption, C_1 can be expressed as a finite sum of random variables with distributions $B_i(.)$, $U_i(.)$, $S_i(.)$, and C_2 can be expressed as a finite sum of random variables with distributions $B_i'(t)$, $i \in \mathbf{S}$. C_1 and C_2 will possess limiting densities $f_{C1}(.)$ and $f_{C2}(.)$ that we would like to determine. Note that since the C_2-cycle depends on probabilities associated with the C_1-cycle, the random variable C_2 depends on C_1.

8.5 Cycle-Time Distributions for Asymmetric Systems

In the following discussion, we demonstrate how $F_{C1}(.)$ and $F_{C2}(.)$ may be determined with the aid of the distribution $F_C(.)$. The arrival process at each station $i \in \mathbf{S}$ is assumed to be Poisson(λ_i), and B_i, S_i, U_i, and U_i' are assumed to be exponential, with means $1/\mu_{i0}$, $1/\mu_{i1}$, $1/a_i$, and $1/a_i'$, respectively. Since a cycle is defined in terms of contributions from *all* stations, the random length of a cycle will remain the same regardless of the index of the station from which the observer measures it. In this paper, we restrict our attention to asymmetric cycle-time distributions for the ATP system. By using the symmetric form of the cycle-time distribution for C, it is simple to apply our results to the symmetric ATP distributions. Exponential distributions are used mainly for convenience and required only for interarrival times.

The Distribution Of AN NATP C-cycle

Let p_{0i}^* be the probability that in the NATP process the server encounters no customer at queue i, for all $i \in \mathbf{S}$, in a stationary system (and recall that p_{0i} is the corresponding probability in the ATP process). Also, let $a_{i0} = (1 - p_{0i}^*)\mu_{i0}$ and $a_{i1} = p_{0i}^*\mu_{i1}$, for all $i \in \mathbf{S}$. Let Θ be the set of all N digit binary numbers representing the non-negative integers in the range $[0, 2^N-1]$. An element $k \in \Theta$ is an N-bit binary vector of the form $[k(1), k(2), \ldots., k(N)]$. The asymmetric and symmetric cycle time densities are given by Eqs.(5.14) and (5.23), respectively, with p_{0i}^* used in place of p_{0i}. The latter term is now used to denote a measure with respect to the C_1-cycle.

The Distribution Of An ATP C_1-cycle

The expectation $E(T_j*)$ is a maximum when all stations in the set $\mathbf{S} \backslash \{j\}$ take part in a C_2-cycle while station j sees only C_1-cycles. That is, observer 2 never gets to see the server in the C_2-cycle due to very low traffic at station j. In this case, $E(T_j^*)$ takes on the value $[E(C_1) + E(C_2)]$. But in general,

$$E(T_j^*) \leq [E(C_1) + E(C_2)]$$

or $$E(T_j^*) \leq [E(C) + E(C)] \leq 2E(C) \tag{2}$$

The inequality in Eq.(2) merely says that $E(C)$ takes the shape of an upper bound for $E(C_1)$ and $E(C_2)$. The inequality $E(C_1) \leq E(C)$ leads us to conclude that

$$\frac{(1 - p_{0j})}{\lambda_j} \leq \frac{(1 - p_{0j}^*)}{\lambda_j} \tag{3}$$

By our assumption that both systems (NATP and ATP) are operating under identical conditions, inequality (3) yields

$$(1 - p_{0j}) \leq (1 - p_{0j}^*)$$

or $$p_{0j}^* \leq p_{0j} \tag{4}$$

From inequality (4), the effect of the C_2-cycle in ATP can be interpreted as follows. The server has a greater probability of encountering an empty station buffer (at each station) in the C_1-cycle of the ATP bus than in the C-cycle of the NATP bus, simply due to the server's adaptive behaviour. The important point here is that $F_{C1}(.)$ may be obtained in asymmetric and symmetric forms from Eqs.(5.14) and (5.23), respectively, provided that p_{0i}^* is replaced by p_{0i} for all $i \in S$. Thus, the probabilities p_{0i} for all $i \in S$ must first be obtained in order to obtain $F_{C1}(.)$.

The Distribution Of An ATP C_2-cycle

Given that the server visits state i, $i \in S'$, the random time

$(X_i + Y_i')$ spent by the server in this state has distribution $B_i'(t)$. Since this is the sum of independent random variables, B_i' may be obtained as $B_i * U_i'$ for all $i \in \mathbf{S}$, where "*" is used to denote the convolution operation. Note that in general, it is not necessary to differentiate between index sets $\mathbf{S}$ and $\mathbf{S}'$ since a specific index in each set really refers to the same station. However, we do differentiate between these sets in instances where the kind of cycle being considered (i.e., first or second) is important. The distribution $B_i'(t)$ is a generalized Erlangian distribution with a density given by

$$b_i'(t) = \mu_{0i} a_i' \left[\frac{\exp(-\mu_{0i} t)}{(a_i' - \mu_{0i})} + \frac{\exp(-a_i' t)}{(\mu_{0i} - a_i')} \right] \tag{5}$$

for all $i \in \mathbf{S}$. Note that Eq. (5) is given in asymmetric form, but will work for the symmetric case if the subscript i is suppressed.

If the system is in steady-state operation, q_i is the probability that the server visits state i, $i \in \mathbf{S}'$, during the C_2-cycle. The distribution of time spent by the server in state $i \in \mathbf{S}'$ under conditions of stationarity is given by

$$F_{2i}(t) = q_i B_i'(t) + (1 - q_i)\delta_0(t) \tag{6}$$

where $\delta_0(t)$ is the dirac delta function, defined with

$$\delta_0(t) = \begin{cases} 0 & t \neq 0,\ t \in \mathbb{R}^+ \\ \infty & t = 0 \end{cases}$$

and $\int_{\mathbb{R}^+} \delta_0(t) = 1$. Note that this agrees with the transition matrix $\mathbf{P}$ obtained from $\mathbf{Q}$ in Eq. (3). That is,

$$\lim_{t\to\infty} F_{2i}(t) = q_i \qquad \forall i \in S' . \tag{7}$$

The distribution of a C_2-cycle is thus a sum of independent random variables with the distributions $F_{21}, F_{22}, \ldots, F_{2N}$. Let $a_{i0}'' = q_i a_i' \mu_{i0}$ and $a_{i1}'' = (1 - q_i)$ for $i \in S'$. The Laplace-Stieltjes transform for the density of the random variable C_2 is given by

$$L[f_{C2}] = \sum_{k\in\Theta} \left[\prod_{i\in S} a_{i\,k(i)}'' \right] \left[\prod_{n\in G_1} \frac{1}{(s + \mu_{n\,k(n)})(s + a_n')} \right] \left[\prod_{m\in G_2} 1 \right] \tag{8}$$

with $G_1 = \{x|k(x)=0\}$, $G_2 = \{x|k(x)=1\}$, and the set Θ defined earlier. The plane of convergence for terms $1/(s + \mu_{i\,k(i)})(s + a_i')$ is the region defined by $Re(s) > -min\,(a_i', \mu_{i\,k(i)})$, $\forall i \in S$. Inverting the expression in Eq.(8) we obtain $f_{C2}(.)$ as

$$\sum_{k\in\Theta} \left\{ \prod_{i\in S} a_{i\,k(i)}'' \left[\sum_{n\in G_1} \frac{exp(-\mu_n c)}{(a_n' - \mu_{n\,k(n)})} + \frac{exp(-a_n' c)}{(\mu_{n\,k(n)} - a_n')} \right] \right\} \tag{9}$$

Thus, $F_{C2}(.)$ is completely determined provided that p_{0i} and p_{1i} are given, for all $i \in S$. From this and the previous subsection, we see that obtaining the probabilities p_{0i} and p_{1i}, $\forall i \in S$, is crucial in determining the distributions F_{C1} and F_{C2}. In the next subsection we examine how these probabilities may be obtained.

8.6 Computation Of Probabilities For The First Cycle

A sufficient condition for the existence of a stationary cycle-time distribution $F_C(.)$ in the NATP process is the stationarity of each station's queue length distribution. This is also the case with the existence of stationary distributions $F_{C1}(.)$ and $F_{C2}(.)$ of the ATP process. A necessary condition for $F_C(.)$ to be stationary is the existence of a time-invariant value for p_{0i}^*, $\forall i \in S$. For $F_{C1}(.)$ and $F_{C2}(.)$ the necessary condition is the existence of time-invariant values for p_{0i} and p_{1i}, $\forall i \in S$. This is summarized in the following lemma.

Lemma 8.1:

If the queue length distribution at each queue j, $j \in S$, is stationary, then the NATP system possesses a stationary cycle-time distribution F_C and the ATP system possesses stationary cycle-time distributions F_{C1} and F_{C2}. The expectations $E(C)$, $E(C_1)$, and $E(C_2)$ are stable even in the absence of stationarity for the respective distributions. Also, there exist unique probabilities p_{0j}^* (for the NATP system), p_{0j}, and p_{1j} (for the ATP system) such that

$$p_{0j} = 1 - \lambda_j E(C_1)$$

$$p_{1j} = \lambda_j \{E(C_1) - E(C_2)\}$$

$$p_{0j} \geq p_{0j}^*$$

and $$p_{1j} = p_{0j}(1 - \nu_{0j})/\nu_{0j}$$

for $$\nu_{0j} = \sum_{m \in S} \sum_{k \in \Theta} \int_0^{\infty} (\lambda_j t) \exp(-\lambda_j t)\, D_{km}(t)\, dt$$

with $D_{km}(t)$ given by

$$D_{km}(t) = \prod_{i \in S} a_{i\ k(i)}{}' \beta_m \left\{ \frac{\exp(-a_m t)}{\prod_{r \in S} (\mu_{r\ k(r)} - a_m)} \right.$$

$$+ \sum_{n \in S} \frac{\exp(-\mu_{n\ k(n)} t)}{\prod_{\substack{r \in S \\ r \neq n}} (\mu_{r\ k(r)} - \mu_{n\ k(n)})(a_m - \mu_{n\ k(n)})} \}.$$

with $a_{i0}' = (1 - p_{0i})\mu_{i0}$ and $a_{i1}' = p_{0i}\mu_{i1}$, $\forall i \in S$.

Let $\{Z(t)\}$ be the continuous time semi-Markov process associated with $\{Z_n, T_n\}$. The embedded chain $\{Z_n\}$ observed at the instants of state transitions behaves like a Markov chain. $\{Z_n\}$ is aperiodic, positive recurrent and irreducible and can be shown [Klei75] to possess a stationary distribution $\Pi = (\pi_{W1}, \pi_1, \ldots, \pi_{WN}, \pi_N, \pi_1', \ldots, \pi_N')$. The probability of finding the token in state i after the process has been operating for an arbitrarily long time is π_i, $i \in S^*$.

The process $\{Z(t)\}$ can be shown to exhibit a unique limiting behaviour within the chain $\{Z_n\}$. The equilibrium distribution of $\{Z(t)\}$ is given by $\Phi = (\phi_{W1}, \phi_1, \ldots, \phi_{WN}, \phi_N, \phi_1', \ldots, \phi_N')$, where ϕ_j is the limiting interval transition probability of observing the token in state $j \in S^*$. This probability is different from that obtained from the chain due to the consideration of the holding time distributions in the various states of S^*. The equilibrium distribution of the server over states of S^* is obtained as

$$\phi_j = \begin{cases} \dfrac{\pi_j \{(1 - p_{0j})E(X_j) + p_{0j}E(Y_j)\}}{\Delta} & j \in S \\ \pi_j E(Y_j)/\Delta & j \in W \\ \dfrac{\pi_j \{q_j E(X_j) + q_j E(Y_j')\}}{\Delta} & j \in S' \end{cases} \qquad (10)$$

with

$$\Delta = \sum_{k \in S} \pi_k \{E(Y_k) \; + \; [p_{0k} E(X_k) + (1 - p_{0k}) E(V_k)] \; + \; q_k [E(X_k) + E(Y_k')]\}.$$

Let $\{\phi_j(t) = P[Z(t) = j]\}$ be the server's time-considered state distribution over S^*, and suppose that the initial distribution $\{\phi_j(0)\}$ may not be the equilibrium distribution. By obtaining the equilibrium distribution Π of the chain $\{Z_n\}$, the limiting probabilities $\phi_j = lim \;\; \phi_j(t)$, for $j \in S^*$, may be obtained. In order to identify conditions under which this equilibrium distribution is obtained, we proceed as follows. Define the function

$$H(t) = \sum_{j \in S^*} \phi_j \; h[\frac{\phi_j(t)}{\phi_j}] \qquad (11)$$

where $h(y)$ is a strictly concave function of y, $y \in R^+$. Clearly, $H(t)$ is a function of the server's position (over states in S^*) at time t, i.e., $\phi_j(t)$, $j \in S^*$. If the initial distribution is the equilibrium distribution, then $H(t)$ takes on a constant value. Otherwise, $H(t)$ increases monotonically to this constant value, as shown in the next lemma [Kell79].

Lemma 8.2:

If the initial distribution $\{\phi_j(0)\}$ is not the equilibrium distribution of the process $\{Z(t)\}$, then the function $H(t)$, $t > 0$, is strictly increasing.

Theorem 8.3:

The equilibrium distribution $\{\phi_j\}$ of the semi-Markov process $\{Z(t)\}$ is the distribution obtained by maximizing the function

$$H^*(t) = \sum_{j\in S^*} \phi_j(t) \ln[\frac{\phi_j(t)}{\phi_j}] ,$$

where $H^*(t)$ corresponds to the *entropy* of the distribution $\{\phi_j(t)\}$, with respect to the equilibrium distribution.

In Theorem 8.3 we see that the monotonic increase of $H^*(t)$ is a consequence of the convergence of transient distribution $\{\phi_j(t)\}$ to the equilibrium distribution $\{\phi_j\}$. Additionally, this theorem gives conditions under which the equilibrium probabilities p_{0j} and p_{1j} of Lemma 8.1 can be determined, $\forall j\in S$. The method involves systematic increments to probability p_{0j}^*, $\forall j\in S$, until probability sets $H_1 = \{p_{0j}, j\in S\}$ and $H_2 = \{p_{1j}, j\in S\}$ that maximize the entropy $H^*(t)$ are obtained.

Algorithm to obtain the sets H_1 and H_2

step 1: Compute the mean NATP cycle time i.e.,

$$E(C) = [\sum_{k\in S} E(Y_k) + E(V_k)]/(1 - \rho_0 + \gamma_0), \text{ where } \rho_0 = \sum_{k\in S} \lambda_k E(X_k)$$

and $\gamma_0 = \sum_{k\in S} \lambda_k E(V_k)$.

step 2: Set $\delta = 10^{-n}$ where $(n-1)$ is the number of digits of accuracy required, and set $H_{old} = -\infty$.

step 3: Set $p_{0j} = p_{0j} + \delta$, and compute p_{1j}, $\forall j\in S$.

step 4: Compute Π and then Φ. Using Φ, compute the entropy H^*.

step 5: If $H^* < H_{old}$, then H_{old} is maximimum; stop. Otherwise, set

$H_{old} = H^*$ and go to *step 3*.

The algorithm uses p_{0j}^*, $j \in \mathbf{S}$, as prior probabilities in order to obtain the sets $\mathbf{H}_1$ and $\mathbf{H}_2$. By theorem 8.3, since the probabilities in these sets maximize the entropy H^*, these can be used as equilibrium probabilities for empty and one-customer buffers. The convergence of the solution is dependedent on the value of n chosen in *step 2;* in general, it is rapid. Once these probability sets are obtained, $F_{C1}(.)$ and $F_{C2}(.)$ are completely determined for the ATP process.

8.7 Stationary Distributions

As in the case of $F_C(.)$, the distributions $F_{C1}(.)$ and $F_{C2}(.)$ can be obtained as stationary distributions, given that each queue length distribution is stationary. Let us suppose that observer 2 begins to record consecutive cycle-time lengths starting at time $t = 0$, where without loss of generality we will assume that the token arrives at the reference station at time $t = 0$. Thus, observer 2 sees a sequence of cycle-time pairs $(C_{11},C_{21}),\ldots(C_{1n},C_{2n}),\ldots$ where C_{in} is the random time taken by the server to complete the C_i-cycle $(i=1,2)$, in the n^{th} pair of cycles. Let $p_{0j}^{(n)}$ and $p_{1j}^{(n)}$ be the probabilities corresponding to the empty buffer and one-customer buffer, respectively, at station j's scan instants during the C_{1n}-cycle, for all $j \in \mathbf{S}$. Clearly, the length of cycle C_{1n} depends on probabilities $p_{0j}^{(n)}$ and $p_{1j}^{(n)}$. The distribution of cycle C_{1n} can be represented as

$$F_{C1}^{(n)} = [F_{11}^{(n)} * F_{12}^{(n)} \ldots * F_{1N}^{(n)}] * G$$

where $F_{1i}^{(n)}$ is used to denote the holding-time distribution of the token in state $i \in S$ during cycle C_{1n}. If $F_{2i}^{(n)}$ is the holding-time distribution of the token in state $i \in S'$ during cycle C_{2n}, then the distribution of C_{2n} can be represented as

$$F_{C2}^{(n)} = [F_{21}^{(n)} * F_{22}^{(n)} \ldots * F_{2N}^{(n)}].$$

The stationary distributions observed for C_1 and C_2, when the ATP system is operating at steady-state, can be obtained from the limiting values of the above time-dependent distributions.

Theorem 8.4:

If the distribution of queue length at each queue in S is stationary, the random cycle times C_1 and C_2 each possess stationary distributions, given by

$$F_{C1} = \lim_{n \to \infty} F_{C1}^{(n)}$$

and

$$F_{C2} = \lim_{n \to \infty} F_{C2}^{(n)}$$

8.8 Summary

A well known disadvantage of token-passing on a bus is the delay suffered by a heavily loaded station on a lightly loaded network. If the network is large, and the fraction of stations contributing to most of the network traffic is small, then these stations will experience a considerable amount of delay in waiting for channel access. In this study, we reviewed a scheme presented earlier [ReHu85], where such stations are given priority over the inactive (or relatively inactive) stations, with the intention of reducing their delays. The protocol that achieves this is called the adaptive token-passing (ATP) protocol. The cycle-time distributions of the the token in the two ATP cycles are derived in this section order to (i) be utilized in subsequent work for deriving performance measures of the ATP scheme, and to (ii) compare it to the standard nonadaptive token-passing (NATP) scheme. For this presentation, we restricted our attention to the situation where at most one customer is served per station at its scan instants.

The cycle-time distributions are obtained for an asymmetric system under exponential assumptions. It is possible to obtain these results for symmetric systems by either using the asymmetric results themselves, or by resorting to simpler expressions that are characteristic of symmetry. The analysis shows how adaptive token-passing can be formulated as a semi-Markov process, with cycle-time distributions obtained as stationary distributions (under certain conditions) when the system is operating at steady-state. Certain resulting approximate distributions are currently being studied as candidates for approximate models of performance. Note that it is possible to apply the results of chapters III and IV to obtain exact results for the ATP system. This is reserved for future work.

CHAPTER IX

CONCLUSIONS AND FUTURE RESEARCH

In concluding the thesis, we briefly summarize the contents of each chapter and discuss the consequences of the main results. The goal of the project was a detailed study of token-passing systems with a view towards results that would be of interest to researchers in the area of performance modelling. We were specifically interested in modelling asymmetric token-passing systems in which at most one packet is transmitted by a station that is both ready to transmit and is in possession of the free token. This service discipline has been analyzed only in an approximate fashion in the past. The usual approximations involve independence assumptions [Heym83, Kueh79], borrowed-models such applications of exhaustive service models or Erlang-loss models [Buxw81, Heym83, Kaye72] and applications of gated service models [FeAm85], or light and heavy traffic approximations [Stuc83]. In our analysis, we are require to make no assumptions other than Poisson arrivals, general distributions with finite first and second moments for service, switching, and token-passing times, and the existence of a limiting or asymptotically stationary behaviour of the system at steady state. The last assumption can be derived for arbitrary distributions possessing finite first and second moments. Consequently, for a system with given parameters, the existence (or non-existence) of well defined steady state distributions can be established as a very first step.

Ideally, a performance model should yield results that are directly amenable to practical use, either for a quantitative or a qualitative measurement of the real world system being modelled. Unfortunately, and as is often the case, the results in this dissertation do not lend themselves to algorithmic coding and testing with any ease. Most, if not all of the results, involve a number of nontrivial steps before the final measure is established. The overall process is usually a trifle tedious and time-consuming, and with increasing time also increases the possibility of human error. In this regard, much of the work is largely analytic. It is open to application in a variety of queueing phenomena that utilize service disciplines resembling the round-robin service scheme. Due to limitations in resources and time some of the experimental work was left aside in favour of the analysis. The experimental and exploratory work in the neighbourhood of this dissertation area is part of an ongoing research programme.

In chapter I we introduced the basic research problem in a simple form. In the interest of clarity, we also introduced the notions of service-disciplines (or schemes) and QEDs (queue emptying disciplines) to describe the server's motion through the queueing network, and the number of customer's served at each queue, respectively. Note that the term service-discipline is not to be confused with the meaning of the term (i.e., FIFO, LIFO, random order) in usual queueing terminology. In the "standard" queueing literature, servers are stationary and customer's arrive, receive service, and depart. In our systems of multi-queues, it is the server who moves, from one queue to another.

Hence, the term service-discipline is meant to capture server behaviour. Within queues, we always take service to be FIFO, though this can be changed.

The term QED is introduced as an aid to classification of multiqueueing systems. Apparently, prior to this, such queues were treated on an individual basis. QEDs and service-disciplines link multiqueueing systems together in a manner that facilitates analysis. The literature review of closely related models (in terms of our notation) is intended to demonstrate the similarities between members of the class of multiqueueing problems, only a few of which have ever been addressed. A brief review of Markov renewal processes is presented in chapter II, mainly to introduce notation. We also take the opportunity to present a formal definition of the problem.

In chapter III we introduce a key random variable, i.e., the cycle-time random variable, and proceed to outline an exact method to determine its stationary distribution. The uses of the cycle-time random variable should be clear. It is important in that it describes the length of time between server reappearances at any station. Thus, we can compute the probability that any station has to wait for upto t units of time for service. In the past, this was known only approximately, and the approximation was known to be poor for systems with many stations. Note that our Markov methods of analysis allow for a transient analysis even though we choose to stay with steady-state results.

The bad side of this approach is that the size of the required Markov matrix grows exponentially with the number of station's on the system being modelled. A method for avoiding this drawback is currently under investigation. On a brighter note, it is shown that one Markov matrix is all that is required to describe cycle-transitions for all stations on the system. Given one station's transition matrix, applying a simple procedure will yield another station's matrix. Consequently, we also prove some invariance properties and demonstrate insensitivity of cycle-times.

In chapter IV we take a novel approach to modelling N-station multiqueues. The method of analysis is based on work by Neuts [Neut66, Neut77] and Cinlar [Cinl67] on semi-Markovian, single server queues. The semi-Markovian approach yields the following elegant solution. We focus our attention on station j and pretend this is a single-server queueing system. The cycle that begins with service of a particular station j customer is taken to be the "service time" of this customer. There are $m = 2^{N-1}$ different types of customers (differentiated through their service vectors), and these arrive at station j according to a Poisson process. A transition matrix obtained via the Markov transition matrix of chapter III describes transition probabilities between successive customer types. Viewed in this fashion, we have a modified M/SM/1 queue. The modification is due to customers who arrive to find an empty queue.

The notion of *server vacation periods* is introduced to reduce this queueing scheme into an M/SM/1 queue, using PH-distributions (i.e., phase-type distributions) [Neut81]. We use PH-distributions to model busy periods and vacation periods of the token. The final stage involves M/SM/1 queueing theory to obtain an algorithmic solution for steady-state distributions of packet queueing characteristics at the reference station. Thus, performance measures can be obtained for each station on the network. The measures obtained in chapter IV are restricted to the queue at station j due to conditioning. With some effort, the actual queue-length measures (i.e., moments of the distribution) may be obtained. An interesting application of this approach is the determination of the distribution of throughput, with throughput considered as a random variable defined on [0,1). Just as in chapter III, a major problem here is exponential growth of the transition matrix with increasing N.

The approximate methods discussed in chapters V through VIII chronologically precede the material in chapters III and IV. Still, these methods have certain advantages over the exact methods. For example, their ease of use and the computational convenience in dealing with square matrices of size N instead of size 2^N. Loosely speaking, the ideas are generalizations of previous work in the area of multiqueueing, notably work by Kuehn [Kueh79], and Hashida and Ohara [HaOh72]. In chapter V, we use exponential distributions to compute asymmetric and symmetric cycle-time distributions. We introduce switching time random variables to avoid working with impulse functions. In our experimental work we found that empirical distributions computed with simulated output did not differ from a uniformly random sample subset of the simulation output. Additionally, we found that for high and low loads, the cycle-time distribution obtained via independence assumptions performs remarkably well. For moderate loads, the independence assumption performs poorly. We can only conclude that this is due to serial cycle-time depencencies that arise in a wide range of moderate loads. Again the problem of computational complexity arises. With regard to cycle-time densities, asymmetric systems require computational algorithms of exponential complexity, while symmetric systems require computational algorithms of polynomial complexity.

In chapter VI, we introduce approximate methods to address the problem of serial dependence. Just as in chapter IV, this chapter deals with methods of obtaining queueing distributions. Since the i.i.d cycle-time assumption of Hashida and Ohara neglected covariance information, and the methods of Kuehn (mixing two kinds of cycle-times) which took into account two non i.d cycle-times proved to be superior to the Hashida/Ohara results, we decided to attack the covariance problem. Two methods for covariance function approximation, and a method for determining marginal cycle-time distributions (based on the original cycle-time distributions) are outlined. In choosing a linear combination of dependent cycle-times to construct a quasi-service time random variable, we demonstrate an application of principle component analysis. The quasi-service time random variable so constructed is guaranteed to possess maximum variability. Thus, the quasi-service time random variable may either be used in the Hashida/Ohara sense, or in the Kuehn sense. In either case, due to consideration of covariance, we are certain to obtain better results.

In chapter VII, we discuss the stability of token-passing systems. Two notions of stability are presented. In the SV queueing sense, the stability criterion (as stated in Chapter IV) is a function of customer transition probabilities, mean arrival rates, and mean service times.

In the SP queueing sense, the stability criterion is a function of mean arrival rates and mean conditional cycle-times. Interestingly enough, both criteria are based on embedded Markov chains, where the SV chain is obtained via conditioning and the SP chain is obtained via the independence hypothesis. A study of the stability criterion in N dimensions allows us to define simple measures of system stability. The use of conditional cycle-times is carried one step further in defining a flexible fairness measure. The flexibility arises in our ability to change the weights involved in the measure, thereby giving the particular aspect of the measure that we are more interested in a correspondingly heavier weight. In this sense, we can change the measure to suit our purpose. Another result of this section is one that is very useful for conditions of extreme load. This is based on work by Reńyi [Reńy56] on rarefactions and determines the random time between two consecutive transmissions by a given station.

In chapter VIII, we introduce a complex service discipline with the intention of deriving cycle-time distributions. This discipline is applicable to token-passing bus networks. Consider an asymmetric system with N stations, N very large (i.e., of the order 10^2). Examine the effects obtained if a few stations are continuously active and generate a large proportion of the total load on the network. For example, 5% of the stations on the system generate 90% of the total (transmission) load, while the remaining 95% are relatively inactive. This example illustrates the well-known disadvantage of token-passing for either high asymmetric station loads or low average loads, with the token spending a good deal of time in fruitless token-passes [Stal84].

In order to analyze the complex service scheme, we require the distributions of the two cycle-time random variables involved. With the aid of these, the server's interarrival time distribution may be derived. As an approximate solution, we use the entropy approach [Kell79] in obtaining the limiting probabilities of certain queue states. This finally allows for the determination of the distribution of the server's interarrival time, T_j, at station j. The conditions under which this simple adaptive token-passing scheme performs better than a corresponding standard token-passing scheme are simple to derive.

If we keep a given system fixed and change only the QED we obtain a new queueing problem. Similarly, if we change only the service discipline, we obtain yet another queueing problem. Thus, there is a rich class of strongly linked queueing problems that has not surfaced in the literature in any structured form. In this regard, there is more than ample scope for future research in this area. It may well turn out that a single method of analysis is applicable to a host of QEDs simultaneously. Such a result is not known at the present time.

Some of the problems open for future research can be described as:

(1) Generalizing solutions to accomodate related QEDs such as s-QED = n, $n > 1$, finite queue capacity, *relaxed* disciplines, more *complex* disciplines etc.

(2) Multiqueueing configurations with more than one server and a variety of QEDs.

(3) Multiqueueing systems with time-varying arrival rates or randomly varying arrival parameters.

(4) Multiqueueing systems with arbitrary distributions for arrival, service, walk, and switching.

(5) Asymptotic analysis of multiqueueing problems.

(6) Distribution of the Fairness statistic.

(7) Computational algorithms.

The research areas outlined by no means exhaust the variety of interesting multiqueueing problems available. The topic in (7) is important for practical applications but is probably also extremely difficult. In fact, all of the topics mentioned above are challenging research problems. From our work, it is clear that one inherent problem of asymmetric multiqueues is the high degree of computational complexity. In this regard, the topic in (5) will provide valuable insight into the effects of increasing N. With an understanding of asymptotic behaviour, we can attempt to work around the problem of computation.

APPENDIX

APPENDIX

Proof for Lemma 3.1:

Without loss of generality, we first work with stations 1 and 2. Consider any transition of the form $z \rightarrow z'$ in P_1, where $z = <z_1, .., z_N>$ and $z' = <z_1', .., z_N'>$. By Eq.(5) of chapter III, the probability of this transition, i.e., $p(z,z')$, can be written as a product of conditional probabilities, one from each station. Denote these probabilities as $\xi_i(z_i' | z_i, .. , z_{i-1}')$ where $i = 1,..,N$ and $z_0' = N$.

For ease of notation we write station i's conditional probability simply as ξ_i, unless specific mention of the condition is required. At steady state, given that an observer at station 1's scan point sees a transition $<z_1, .. , z_N> \rightarrow <z_1', .. , z_N'>$ occur, another observer at station 2's scan point must see a transition of the form $<z_2, .. , z_N, z_1'> \rightarrow <z_2', .. , z_N', X>$. The X is meant to indicate that the original transition does not yield sufficient information to determine this entry. Recalling Eq.(5), we see that $(N-1)$ out of the N conditional probabilities that go to make the probability $p(z,z')$ in P_1 will also be present in the partial (due to absence of a specific X) transition probability seen from station 2. Since z_1 is merely a descriptor for events at station 1, substitute z_1 in place of X for station 2's transition. Then, station 2's transition tuple is obtained as the first and second set of N bits in $<M(z,z')>$, where M is defined in Eq.(1) of chapter III.

Let $M(z,z')$ denote a transition from a starting state (first N

bits) to a target state (second N bits), seen from station 2's scan point. Let $p(\mathbf{M}(z,z'))$ denote the probability of this transition in $\mathbf{P}_2$ as seen by the observer at station 2. Then this probability can be obtained as

$$p(\mathbf{M}(z,z')) = [\ \xi_2\xi_3\cdots\xi_N\ ]*[\xi_1(z')/\xi_1(z)]$$

where all the probabilities on the right hand side are defined. In particular, $\xi_1(z)$ describes the conditional probability $\xi_1(z_1|z)$ and $\xi_1(z')$ describes the conditional probability $\xi_1(z_1'|z')$. Repeating this procedure for every transition pair (z,z') in $\mathbf{P}_1$ will yield the entire transition probability matrix $\mathbf{P}_2$. Observe that the rows (and columns) of $\mathbf{P}_2$ will not be in the same order as the rows (and columns) of $\mathbf{P}_1$.

In general, to obtain $\mathbf{P}_k$ from $\mathbf{P}_j$, $k = j \bmod N + 1$, assuming that $p(z,z')$ describes a transition probability in $\mathbf{P}_j$, the transformation $\mathbf{T}^j$ applied to each entry of $\mathbf{P}_j$ is formally described as

$$\mathbf{T}^j(z,z') = [p(\mathbf{M}(z,z'))]\ [\xi_j(z_j|z_j',\ldots,z_{j-1})/\xi_j(z_j'|z_j,\ldots,z'_{j-1})].$$

Extending the above idea, it is easy to show that the probability matrix viewed with respect to station k's scan-cycle can be obtained for any station k. Observe that N repetitions of the transformation on $\mathbf{P}_1$ will yield $\mathbf{P}_1$ with rows and columns transposed, and $2N$ repitions will yield the original $\mathbf{P}_1$.

Proof for Lemma 3.2:

If the observer is positioned to witness scan-cycles at station j, then the service vector obtained from the server is of the form $<z_j, z_k, \ldots, z_{j-1}>$. Similarly, for scan-cycles seen at station k, the service vector obtained is $<z_k, \ldots, z_j>$. If the observer is positioned to witness dep-cycles at station j, then the first and last entries of the service vector must correspond to service events at stations k and j, respectively. Hence, this service vector must be of the form $<z_k, \ldots, z_j>$, and this is precisely the vector seen at or before station k's scan point. Furthermore, since the probability transitions for service vectors are not dependent on scan instants (as in [Kueh79]), we have that for all $j \in S$,

$$P_k(s) = P_j(d).$$

Proof for theorem3.3:

Let $\{\Pi^j\}$ be the invariant vector obtained from the Markov transition matrix P_j. The elements of this vector are 2^N probabilities corresponding to limiting state probabilities for the 2^N different service vectors. Thus, the invariant vector is a vector of probabilities corresponding to states that are vectors (i.e., service vectors) in themselves. For each $z \in \Theta$, let π^j_z be the steady-state probability that the observer at station j receives vector z from the server.

Let $z = \langle z_j,..,z_N,z_1,..,z_{j-1}\rangle$, and define an operator $\mathbf{R}$ on elements of Θ such that

$$\mathbf{R}(z_j,..,z_N,..,z_{j-1}) = (z_{j+1},..,z_N,..,z_j).$$

In other words, $\mathbf{R}$ treats each element of Θ as an N-bit computer word and performs an end-around left shift. The bit that falls off the left edge is made to occupy the (vacated) rightmost bit position.

From Lemma 3.1, we see that the transition matrix for any station will yield an invariant probability vector which, when suitably transformed, becomes the invariant vector for a neighbouring station. For each service vector z seen on a scan-cycle at station j, $j \in \mathbf{S}$, it is easy to see that

$$\pi^j{}_z = \pi^k{}_{\mathbf{R}(z)}$$

where $k = j \bmod N + 1$. Thus, given the invariant service vector distribution with respect to scan-cycles for any one station, it is possible to obtain the invariant service vector distribution with respect to scan-cycles for any other station.

If station j sees a scan-cycle vector z (in the steady-state of the system) with probability $\pi^j{}_z$, $z \in \Theta$, then station k sees a different vector $\mathbf{R}(z)$ with precisely the same probability. Since the vectors z and $\mathbf{R}(z)$ both generate the same cycle-time random variable (see Eq.(8) in chapter III), the mixture distribution obtained in Eq.(9) will be the same for both stations. That is, station j and station k see the same cycle-time random variable. By replacing indices, this can be shown to hold for every pair of stations j and k, for $j,k \in \mathbf{S}$.

Proof for Theorem 5.1:

Consider a sequence of transient cycles witnessed by our observer at station j from the instant the system begins to operate. Without loss of generality we assume that the system begins at $T_0 = 0$, with $Z_0 = j$, and that cycles are measured with respect to scan instants at station j. Let C_i be the i^{th} cycle from the start of the process, $i \in I^+$. Let $p_j^{(i)}$ be the probability that the server finds at least one customer awaiting service at station j at its C_i scan instant, and $q_j^{(i)} = 1 - p_j^{(i)}$.

Let e_m be the mean of the first m terms in the cycle sequence. Since e_m is a consistent estimate of the true cycle mean $E(C)$, we have

$$\lim_{m \to \infty} P[|e_m - E(C)| \le \epsilon] = 1, \quad \text{for all } \epsilon > 0.$$

Next, consider the observer's view of the system after it has been operating for an arbitrarily long time. By hypothesis, all queueing distributions are stationary and consequently stable. Since the observer sees an M/G/1 queueing system at station j, it follows that what e_m converges to is precisely the mean $E(C)$ of the G distribution. From M/G/1 theory we have that $\lim_{i \to \infty} q_j^{(i)} = q_j$ always exists and is independent of a customer's virtual waiting time at the instant $t = 0$ [Taka62]. That is,

$$\lim_{k \to \infty} (1/k) \sum_{i=0}^{k} q_j^{(i)} = q_j$$

and $p_j = 1 - q_j$. Consequently, for each $j \in S$ we obtain weak convergence of the distributions $F_j^{(k)}$, i.e.,

$$\lim_{k \to \infty} F_j^{(k)} = p_j B_j + q_j S_j = F_j, \qquad j \in S.$$

Given the weak convergence, we can interchange the operations of convolution and limit to obtain the theorem.

Proof for Corollary 7.1:

If $E(S_1) < \infty$, then

$$E(S_1^*) = \sum_{k=1}^{\infty} E(S_1^* \mid S_1^* = p_j S_k) \; P(S_1^* = p_j S_k)$$

$$= \sum_{k=1}^{\infty} E(S_k) \, q_j^{k-1} p_j = E(S_1) \, p_j^2 \sum_{k=1}^{\infty} k q_j^{k-1},$$

which is equal to $E(S_1)$. On the other hand, since $E(S_1^*) \geq p_j E(S_1)$, we have that $E(S_1^*) = \infty$ if $E(S_1) = \infty$. That the result holds for all integers $k > 1$ follows from the fact that both processes are renewal processes.

Proof for Lemma 8.1:

By assumption of Poisson arrivals, the queueing process at station j is an M/G/1 queue. The queue length states of queue j viewed at its discrete set J of scan instants in the C_1-cycle forms an embedded Markov chain $\{L(t), t \in J\}$. Define $p_{nj} = P[L(t) = n]$, for all $t \in J$, $n \in I^+$, $j \in S$. By hypothesis, $\{L(t)\}$ is stationary. From M/G/1 theory [Klei75], it follows that $p_{0j} = 1 - \lambda_j E(C_1)$, for all $j \in S$. Similarly, the M/G/1 queue formed by customers at station j with service distribution F_{C2} yields

$$p_{0j} + p_{1j} = 1 - \lambda_j E(C_2), \text{ for all } j \in S. \text{ Consequently,}$$

$$p_{1j} = \lambda_j \{E(C_1) - E(C_2)\} \text{ for all } j \in S.$$

The inequality $p_{0j}^* \geq p_0$ is obtained in the analysis [Eq.(14), Chapter III].

The fact that expectations $E(C_1)$ and $E(C_2)$ are finite follows from our assumption that all distributions in section II possess finite first moments. Hence C_1 and C_2 are random variables with stable means. The corresponding proof for the NATP case can be found in [ReNi84]. Finally, the proof that p_{1j} can be obtained as a function of p_{0j} and ν_{0j} may be found in section 6.5. This is acquired as a consequence of obtaining the queue length distribution (as a performance measure).

Proof for Lemma 8.2:

Consider the semi-Markov process $\{Z(t)\}$. For some fixed $\delta \geq 0$, define

$$\begin{aligned} p_{jk}' &= P(Z(t+\delta) = k \mid Z(t) = j) \\ &= P(Z_{n+1} = k, T_{n+1} - T_n < \delta \mid Z_n = j) \\ &= P(Z_{n+1} = k \mid Z_n = i) P(T_{n+1} - T_n < \delta \mid Z_{n+1} = k, Z_n = j) \\ &= p_{jk} H_{jk}(\delta), \text{ for all } j,k \in S^*, \end{aligned}$$

where $H_{ij}(t)$ is the conditional transition time distribution function defined in Eq.(4), and $\{Z_n\}$ is the Markov chain of queue length states at station j obtained via embedding.

Let $\phi_j(t) = P(Z(t)=j)$. It follows that

$$\phi_k(t + \delta) = \sum_{j \in S^*} \phi_j(t) p'_{jk}$$

$$= \sum_{j \in S^*} \phi_j(t) p_{jk} H_{jk}(\delta)$$

$$\text{and } \phi_k = \lim_{t \to \infty} \phi_k(t) = \lim_{t \to \infty} \sum_{j \in S^*} \phi_j(t) p'_{jk}$$

$$= \sum_{j \in S^*} \phi_j p_{jk} H_{jk}(\delta).$$

Let $b_{kj} = \phi_j p_{jk} / \phi_k$. Consequently,

$$b_{kj} > 0, \text{ and } \sum_j b_{kj} = 1,$$

$$\phi_k(t + \delta)/\phi_k = \sum_{j \in S^*} \phi_j(t) p'_{jk} / \phi_k$$

$$= \sum_{j \in S^*} \phi_j(t) H_{jk}(\delta)/\phi_j$$

$$\geq \sum_{j \in S^*} b_{kj} \phi_j(t)/\phi_j$$

We thus obtain,

$$H(t + \delta) = \sum_{j \in S^*} \phi_k h[\phi_j(t + \delta)/\phi_k]$$

$$\geq \sum_{j \in S^*} \phi_k h[\sum_{k \in S^*} b_{kj} \phi_j(t)/\phi_j].$$

Since $h(y)$ is a strictly increasing, concave function of y,

$$H(t + \delta) > \sum_{j \in S^*} \sum_{k \in S^*} \phi_k b_{kj} h[\phi_j(t)/\phi_j]$$

$$= \sum_{j\in S^*} \sum_{k\in S^*} \phi_j p_{jk} h[\phi_j(t)/\phi_j]$$

$$= H(t).$$

Thus, $H(t)$ is a strictly increasing function of t, achieving a maximum as

$$H = \lim_{t\to\infty} H(t) = \lim_{t\to\infty} \sum_{j\in S^*} \sum_{k\in S^*} \phi_j p_{jk} h[\phi_j(t)/\phi_j]$$

$$= \sum_{j\in S^*} \sum_{k\in S^*} \phi_j p_{jk} h(1).$$

Proof for Theorem 8.3:

Let the concave function $h(y)$ [see Lemma 8.2] be defined as $h(y) = (-y)\ ln(y)$.

By Lemma 8.2,

$$H(t) = - \sum_{j\in S^*} \phi_j(t)\ ln[\phi_j(t)/\phi_j]$$

is a strictly increasing function, and takes on its maximum at

$$H^*(t) = \lim_{t\to\infty} - \sum_{j\in S^*} \phi_j ln(1)$$

$$= 0.$$

The monotonic increase of $H^*(t)$ is a consequence of the convergence of the transient distribution $\{\phi_j(t)\}$, $t\in\mathbb{R}^+$, to the equilibrium distribution Φ of the process $\{Z(t)\}$.

Proof for Theorem 8.4:

Let $p_{0j}^{(i)}$, $p_{1j}^{(i)}$, and $q_j^{(i)}$ be the probabilities that the server encounters zero, one, or more than one waiting customer at station j, respectively, at the scan instant of this station during the C_{1i} cycle, $\forall j \in S$, $\forall i \in I^+$. The token's holding-time distribution at station j has the representation

$$F_{1j}^{(i)} = [1 - p_{0j}^{(i)}]B_j + p_{0j}^{(i)}S_j,$$

and at station j' has the representation

$$F_{2j}^{(i)} = q_j^{(i)}B_j', \qquad \forall j \in S,\ i \in I^+.$$

By hypothesis, the queue length distribution at each queue in S is stationary. It follows that after an infinite number of cycles, the ATP system must reach a steady state, in which

$$\lim_{i\to\infty} p_{0k}^{(i)} = p_{0k},$$

$$\lim_{i\to\infty} p_{1k}^{(i)} = p_{1k},$$

$$\text{and} \quad \lim_{i\to\infty} q_k^{(i)} = q_k, \ \forall k \in S,$$

where p_{0k}, p_{1k}, and q_k are the stationary probabilities for queue $k \in S$.

We thus obtain

$$\lim_{i\to\infty} F_{C1}^{(i)} = F_{11} * F_{12} * \ldots * F_{1N} * G, \text{ and}$$

$$\lim_{i\to\infty} F_{C2}^{(i)} = F_{21} * F_{22} * \ldots * F_{2N}, \qquad \forall j \in S.$$

Hence, C_{1i} and C_{2i} converge to C_1 and C_2 in distribution.

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