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Regularity and Stability for Periodic Solutions to
Nonlinear Klein-Gordon and Schrodinger Equations

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Xinming Zhao

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**Regularity and Stability for Periodic Solutions to Nonlinear
Klein-Gordon and Schrödinger Equations**

By

Xinming Zhao

A DISSERTATION

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ABSTRACT

Regularity and Stability for Periodic Solutions to Nonlinear Klein-Gordon and Schrödinger Equations

By

Xinming Zhao

In this dissertation, we first prove the regularity of the periodic solutions of nonlinear Klein-Gordon Equation on the compact manifold S^3 .

The L^p - L^q estimates for periodic solutions of linear wave equations have been developed and utilized to establish the regularity for even dimensions. It turns out to be very difficult to handle the regularity for odd dimensions due to the structure of the kernel of the wave operator. Yet dimension three is the most meaningful case in physical sense because $S^1 \times S^3$ is conformally equivalent to $\mathbb{R}^1 \times \mathbb{R}^3$. We consider three dimensions and look for symmetric periodic solutions. The original equation is reduced into a one dimensional equation with singularity. We prove that under some conditions on the nonlinearity, the smoothness of the periodic solutions is one degree higher than that of nonlinear interaction.

We then study the orbital stability for standing waves of least energy to nonlinear Klein-Gordon(NLKG) and nonlinear Schrödinger(NLS) equations.

Many authors have studied the orbital stability of standing waves for these equations, but their attention has been principally focused on space domain $\mathbb{R}^n$. We extend stability results to bounded domains and compact manifolds. Our method also applies on $\mathbb{R}^n$. The mass term and nonlinear interaction term we are considering depend on space variable as well as on solution. The following results are obtained: (A) If the equations have positive energy, then standing waves of any frequency of NLKG or NLS are orbitally stable. (B) If the equations have indefinite energy, a very sharp condition is obtained on relation between the least energy and the frequency

of standing waves. We apply this condition to bounded domains, compact manifolds and whole space $\mathbb{R}^n$; in each case, we produce orbital stable standing waves.

Finally, for instability, we prove that for a certain class of nonlinearity, the steady state of least energy(ground state) of NLKG are unstable in a very strong sense: there is a region whose boundary ground states lie on, such that every solution starting from this region will blow up in finite time.

To my mother
To my wife Rujie and my daughters Victoria and Alice

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Introduction

In this dissertation, we devote half of the effort to the regularity of nonlinear wave equations and another half to the stability and instability of stationary states and standing waves of nonlinear Klein-Gordon and Schrödinger equations.

Regularity of Nonlinear Wave Equations

We first investigate the regularity and the existence of symmetric and periodic solutions of the semi-linear wave equation on S^3 :

$$u_{tt} - \Delta u + u + F(t, x, u) = 0 \quad \text{in } S^1 \times S^3, \quad (0.1)$$

where Δ is the Laplace-Beltrami operator on S^3 . Under various conditions on function F , the existence results for semi-linear wave equation on $S^1 \times S^n$ are obtained by many authors, e.g., Benci and Fortunato [1], Brezis and Nirenberg [6, 3], Rabinowitz [37, 38] and Zhou [56, 57], among others. The regularity results in the case of $n=1$ are obtained by Brezis and Nirenberg [6] for asymptotically linear F and by Rabinowitz [37] for super-linear F . Jerison, Sogge and Zhou [57, 21] studied and proved the regularity results for $n = 2$, $n = 4$ and $n = 6$. The case of $n = 3$ is the most interesting and meaningful in physics, since $S^1 \times S^3$ is conformally equivalent to $\mathbb{R}^1 \times \mathbb{R}^3$. It was pointed out in [57, 21] that the kernel of operator

$$\square_n = \frac{\partial^2}{\partial t^2} - \Delta + \left(\frac{n-1}{2}\right)^2 \quad \text{on } S^1 \times S^n$$

plays an important role in investigating the regularity of the solution of the wave equation. When n is even, the kernel of the operator $\square_n$ is $\{0\}$, which makes it easy to handle the regularity. In this case, we can apply the $L^p - L^q$ estimates developed in [57], [21] which are generalizations of estimates for $\mathbb{R} \times \mathbb{R}^n$ in [34] and [52]. If

n is odd, the kernel is infinite-dimensional, those $L^p - L^q$ estimates only apply to the component orthogonal to the kernel of $\square$. The component in the kernel is very difficult to control or estimate. Hence it is usually hard to obtain the regularity if n is odd. Currently, little is known of the regularity for the case of odd $n > 1$.

In this paper, we will seek a periodic solution of a symmetric semi-linear wave equation on S^3 . The key observation is that the problem reduces to a standard one-dimensional semi-linear wave equation with singularity, and the kernel for the one dimensional wave is very well understood. Some techniques developed by Brezis, Nirenberg [6, 3] and Rabinowitz [37, 38] can be modified to work for this case. Using standard spherical coordinates, i.e., if $x = (x_1, x_2, x_3, x_4) \in S^3 \subset \mathbb{R}^4$,

$$\begin{cases} x_1 = \sin \rho \sin \phi \cos \theta, \\ x_2 = \sin \rho \sin \phi \sin \theta, \\ x_3 = \sin \rho \cos \phi, \\ x_4 = \cos \rho, \end{cases} \quad (0.2)$$

where $\rho \in [0, \pi]$, $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$, we can rewrite equation (0.1) in the form

$$u_{tt} - \frac{1}{\sin^2 \rho} [(\sin^2 \rho u_\rho)_\rho + \Delta_2 u] + u + F(t, \rho, \phi, \theta, u) = 0, \quad (0.3)$$

where $\Delta_2 u = \frac{1}{\sin \phi} [(\sin \phi u_\phi)_\phi + \frac{1}{\sin \phi} u_{\theta\theta}]$ is the Laplacian of u on S^2 .

If the function F is independent of ϕ and θ , i.e., $F = F(t, \rho, u)$, it is natural to search for solutions of (0.3) which are also independent of ϕ and θ . Such solutions satisfy the equation

$$u_{tt} - u_{\rho\rho} - 2 \cot \rho u_\rho + u + F(t, \rho, u) = 0, \quad (0.4)$$

and will be called **symmetric** solutions. If $u = u(t, \rho) \in C^2(S^1 \times S^3)$ is a solution of (0.4), it is immediate that $v(t, \rho) = u(t, \rho) \sin \rho$ satisfies the equation

$$\begin{cases} v_{tt} - v_{\rho\rho} + F(t, \rho, v/\sin \rho) \sin \rho = 0 & \text{in } S^1 \times [0, \pi], \\ v(t, 0) = v(t, \pi) = 0, \end{cases} \quad (0.5)$$

which is a standard one-dimensional semi-linear wave equation with singularities at $\rho=0$ and $\rho=\pi$. Conversely, we will show that $u = v/\sin \rho$ is also a weak solution of Equation (0.4) for any weak solution v of Equation (0.5).



The procedure for establishing the regularity is the following: We first prove the existence of an L^∞ symmetric solution under the condition that F is monotone in u (i.e. $F_u \geq \alpha > 0$) and asymptotically linear ($F_u \leq \beta$). We then show that every bounded symmetric solution is actually a classical solution provided that F is sufficiently smooth.

Stability and Instability of Standing Waves and Stationary States

In this part, we study the stability and instability of standing waves and stationary states of nonlinear Klein-Gordon equation(NLKG)

$$\begin{cases} u_{tt} - \Delta u + u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \quad u_t(0, x) = V(x), \end{cases} \quad (0.6)$$

and nonlinear Schrödinger equation(NLS)

$$\begin{cases} iu_t - \Delta u + u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \end{cases} \quad (0.7)$$

where Ω is $\mathbb{R}^n$, a bounded domain in $\mathbb{R}^n$ or n -dimensional compact manifold. Here, u is a complex function of $(x, t) \in \Omega \times \mathbb{R}^+$, Δ denotes the Laplace operator with respect to space variable $x \in \Omega$, f is a continuous function of the form $f(x, u) = g(x, |u|)u$ where $g : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$.

Equation (0.6) arises in particle physics ([28, 32]). Special cases of (0.7) include Hartree-type and Pekar-Chaoquard equations [9] which arises in various domains of physics, e.g. in the study of propagation of laser beams([24, 53]) and quantum mechanics([20, 30]).

Of special importance are the “solitary waves” solutions of Equations (0.6) and (0.7). These solutions include time dependent periodic solutions(standing waves) of the form $e^{i\omega t}\phi(x)$ and time independent solutions(stationary states or steady states) corresponding to $\omega = 0$.

The search for both types of solutions leads to the consideration of nonlinear elliptic equation of the form

$$\begin{cases} -\Delta u + g(x, u) = 0, \\ u = 0 \text{ on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \end{cases} \quad (0.8)$$

It is well-known (see, e.g., [10], [50], [39] and [40]) that, there exist infinitely many solutions of (0.8) apart from the trivial solution $u \equiv 0$, among them a positive, classical solution u_0 which has minimal energy among all nontrivial solutions. By minimal we mean the functional

$$J(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + G(x, u) dx,$$

where $G(x, u) = \int_0^u f(x, s) ds$ achieves its minimum at u_0 . We call such a solution “ground state”.

The stability of both stationary states and standing waves of equations (0.6) and (0.7) are physically very important and naturally have been extensively studied. We can not expect the conventional stability of the stationary state due to the nature of NLKG. Indeed, it has been proved that solutions of the NLKG (0.6) will blow up in finite time under some conditions on nonlinearity f : For $\Omega = \mathbb{R}^n$, John [11] and Glassey [12, 13], among others, studied the case $f = -|u|^p$. They obtained finite time blow-up results of classical solutions of equation (0.6) for $p < p_0(n) \equiv (n + 1 + \sqrt{n^2 + 10n - 7}) / (2(n - 1))$, which is less than $1 + 4/(n - 2)$. For a bounded domain and a compact manifold Ω , Payne and Sattinger [36] and Sternberg [48] proved that any weak solution of equation (0.6) starting from some neighborhood of stationary state will blow up in the L^2 -norm in finite time.

Blow up results for solutions of NLS (0.7) have also been obtained by Berestycki and Cazenave [2] who showed that for $\Omega = \mathbb{R}^n$, under certain conditions on f , solutions of NLS starting from some region near a standing wave will blow up in finite time in L^2 -norm. Under much more relaxed conditions on f , Shatah [43] proved for $\Omega = \mathbb{R}^n$ that any solution of equation (0.6) starting from some neighborhood of stationary ground state may not necessarily blow up in finite time, but its L^2 -norm will approaches infinity as t goes to infinity.

An interesting instability result for solution of equation (0.6) was obtained by Keller [26]. He showed that for $\Omega = \mathbb{R}^n$ some stationary state u_0 can be perturbed

into a time-dependent solution of $u_{tt} + \alpha u_t - \Delta u + f(u) = 0$ ($\alpha < 0$) that remains bounded in energy norm for $t > 0$. If f satisfies some growth conditions at infinity, then solutions tend to zero as $t \rightarrow \infty$. In particular, u_0 is not stable.

A more common type of instability of solutions of equation (0.6) and (0.7) (see [45]) is that no matter how close a solution may initially be to a ground state, the solution will eventually leave any prescribed neighborhood of the ground state. Such solutions may not blow up in finite time, nor go to infinity as t goes to infinity, nor approach zero. In terms of techniques used for obtaining this type of instability, there are two disparate types of instability results in the literature. The approach developed by Strauss-Shatah [45] gives an instability criterion coming from the variational structure of the problem; Jones' approach [22] produced a complementary criterion related to the difference between the number of negative eigenvalues of two selfadjoint operators using quite different techniques. Grillakis [14] tried to combine these approaches into one single framework.

Due to the nature of NLKG and NLS, stability in the strict sense can not be established for the standing waves of NLKG and NLS. However a concept of orbital stability has been proposed (for a precise definition, see the later chapters). A number of authors (e.g. Grillakis, Shatah, Strauss [18], [19], [44], [51], Cazenave and Lions [8], [9] and Weinstein [55]) have studied the orbital stability of standing waves of NLKG and NLS.

It appears that almost all existing stability, instability and finite-time blow-up results have been established for the situation where $\Omega = \mathbb{R}^n$.

What we shall do in this dissertation is to extend orbital stability results to the case where Ω is a bounded domain in $\mathbb{R}^n$ or a compact manifold and carry out the proofs for stability in two chapters according to the behavior of nonlinearity f . Chapter two deals with the equations with positive energy, while chapter three treats the case in which the equations have indefinite energy. Our approach to proving orbital stability of standing waves does not need the scaling property which is essential in previous arguments for $\Omega = \mathbb{R}^n$, and consequently our approach applies on more general domains Ω . We also prove the orbital stability for dimension $n = 2$ on $\mathbb{R}^n$,

which was left unresolved in [44] and [45]. We also present a unified approach to get the results of finite-time blow-up of weak solutions of NLKG for all cases of the space domain Ω that we are considering. A very important feature of our method is that the mass m and the nonlinearity f we are considering may depend on the space variable x as well as on the solution u .

Chapter 1

Regularity of Periodic Solutions of Nonlinear Wave Equations

1.1 Preliminaries and Notations

Let $W = S^1 \times S^3$, $\square = \frac{\partial^2}{\partial t^2} - \Delta_3 + 1$. We consider the solutions of the following nonlinear wave equation on S^3

$$\square u(t, x) + F(t, x, u) = 0 \quad \text{in } W. \quad (1.1)$$

It is well-known [35] that the eigenvalues of $\square$ are

$$\lambda(j, l) = l^2 - j^2, \quad j = 0, 1, 2, \dots, \quad l = 1, 2, \dots,$$

and the eigenfunction associated with $\lambda(j, l)$ are

$$e_{ji}(t)s_{lm}(x), \quad i = 1, 2, \quad m = 1, 2, \dots, l^2,$$

where $\{s_{lm}\}_{m=1}^{l^2}$ are orthonormal spherical harmonics of degree $l - 1$ on S^3 , and

$$e_{j1}(t) = \frac{1}{\sqrt{\pi}} \cos jt, \quad e_{j2}(t) = \frac{1}{\sqrt{\pi}} \sin jt.$$

It is also known that these eigenfunction constitute an orthonormal basis of the real Hilbert space $L^2(W)$. Thus for any $u \in L^2(W)$, u can be expanded as

$$u(t, x) = \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} \sum_{i=1}^2 \sum_{m=1}^{l^2} u_{jilm} e_{ji}(t) s_{lm}(x),$$

and

$$\|u\|_{L^2}^2 = \sum_{j,i,l,m} |u_{jilm}|^2,$$

where $\|u\|_{L^2}^2 = (u, u) = \int_W u^2 d\omega dt$, and u_{jilm} is the Fourier coefficient with respect to this basis in $L^2(W)$, and $d\omega$ is the standard measure on S^3 .

For $k \geq 0$ we define the Hilbert space

$$H^k(W) = \left\{ u = \sum_{j,i,l,m} u_{jilm} e_{ji} s_{lm} \mid \|u\|_k^2 = \sum_{j,i,l,m} (1 + j^2 + l^2)^k |u_{jilm}|^2 < \infty \right\}.$$

From this definition, we see that for $H^0(W) = L^2(W)$ and $\|u\|_0 = \|u\|_{L^2}$. We use the subscript s to denote any space of functions on W which are independent of θ and ϕ . For any $u, w \in L^2(W)$, the usual inner product

$$(u, w) = \int_W u w d\omega dt.$$

is given in spherical coordinates by

$$(u, w) = \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^{2\pi} u w \sin^2 \rho \sin \phi d\theta d\phi d\rho dt.$$

If $u, w \in L_s^2(S^1 \times S^3)$, this simplifies to

$$(u, w) = 4\pi \int_0^{2\pi} \int_0^\pi u w \sin^2 \rho d\rho dt.$$

The restriction of $\square$ to $C_s^2(W)$, is the operator

$$\square_s = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2} - 2 \cot \rho \frac{\partial}{\partial \rho} + 1.$$

From now on, we assume that F is a symmetric function, i.e.,

$$F(t, x, u) = F(t, \rho, u).$$

A symmetric solution of Equation (1.1) satisfies the equation

$$\square_s u(t, \rho) + F(t, \rho, u) = 0 \quad \text{on } W. \tag{1.2}$$

It is easy to show that $e_{ji}(t) \frac{\sin l\rho}{\sin \rho}$ is an eigenfunction of $\square_s$ associated with $\lambda(j, l)$, and that the set

$$\left\{ \frac{1}{\sqrt{2\pi}} e_{ji}(t) \frac{\sin l\rho}{\sin \rho}, \quad j = 0, 1, \dots, \quad l = 1, 2, \dots, \quad i = 1, 2 \right\}$$

is an orthonormal basis of $L_s^2(W)$. That is, for any $u(t, \rho) \in H_s^k(W)$, we have

$$u(t, \rho) = \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} \sum_{i=1}^2 u_{jil} \frac{1}{\sqrt{2\pi}} e_{ji}(t) \frac{\sin l\rho}{\sin \rho}$$

with

$$\|u\|_k^2 = \sum_{j,i,l} (1 + j^2 + l^2)^k |u_{jil}|^2.$$

Let $T = S^1 \times [0, \pi]$. Consider the set S of all $C^\infty(T)$ functions vanishing near $\rho = 0$ and $\rho = \pi$. For any $v(t, \rho) \in S$, we have

$$v(t, \rho) = \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} \sum_{i=1}^2 v_{jil} \sqrt{\frac{2}{\pi}} e_{ji}(t) \sin l\rho,$$

where v_{jil} is the Fourier coefficient of v with respect to the orthonormal basis

$$\left\{ \sqrt{\frac{2}{\pi}} e_{ji}(t) \sin l\rho, \quad j = 0, 1, \dots, \quad l = 1, 2, \dots, \quad i = 1, 2 \right\}.$$

Let $H_0^k(T)$ be the completion of S under the norm

$$\|v\|_k = \sqrt{\sum_{j,i,l} (1 + l^2 + j^2)^k |v_{jil}|^2}.$$

Remark 1.1 We use $\|\cdot\|$ to denote norms associated with $T = S^1 \times [0, \pi]$, and $\|\cdot\|$ to denote norms associated with domain $W = S^1 \times S^3$ or S^1 (the distinction will be clear in the context). $u \in H_s^k(W)$ if and only if $v = u \sin \rho \in H_0^k(T)$.

With these definitions, $\|u\|_k^2 = 4\pi \|v\|_k^2$. and hence the identity

$$\square_s u(t, \rho) = \frac{1}{\sin \rho} \square_1 v(t, \rho) \quad \text{for } v(t, \rho) = u(t, \rho) \sin \rho,$$

where $\square_1 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2}$. This identity immediately implies the following

Lemma 1.1. $u \in H_s^k(W)$ is a weak solution of (1.2) if and only if $v = u \sin \rho \in H_0^k(T)$ is a weak solution of

$$\begin{cases} \square_1 v(t, \rho) + F(t, \rho, v/\sin \rho) \sin \rho = 0 & \text{in } T, \\ v(t, 0) = v(t, \pi) = 0. \end{cases} \quad (1.3)$$

In order to apply results from functional analysis, we extend the domain of $\square$ to

$$D(\square) = \left\{ u \in L^2(W) \mid \sum_{j,i,l,m} |\lambda(j, l)|^2 |u_{jilm}|^2 < \infty \right\}.$$

For $u = \sum u_{jilm} e_{ji} s_{lm} \in D(\square)$, we define

$$\square u = \sum_{j,i,l,m} \lambda(j, l) u_{jilm} e_{ji} s_{lm}.$$

This definition coincides with the classical one if $u \in C^2(W) \subset D(\square)$. Under this definition, $\square$ is a self-adjoint operator from $D(\square) \subset L^2(W)$ to $L^2(W)$ with kernel

$$Ker(\square) = \left\{ u \mid u(t, x) = \sum_{l=1}^{\infty} \sum_{i=1}^2 \sum_{m=1}^{l^2} u_{lilm} e_{li}(t) s_{lm}(x) \right\},$$

and the range $R(\square) = Ker(\square)^\perp$.

Similarly, we can extend the domain of operators $\square_s$ and $\square_1$ whose kernels play an important role in the proof of the regularity of the periodic solutions (cf. Brezis and Nirenberg in [6] and Rabinowitz in [37]). When this is done, the kernel of $\square_1$ is

$$N \equiv Ker(\square_1) = \left\{ p(t + \rho) - p(t - \rho) \mid p \in L^2(S^1), [p] \equiv \int_{S^1} p(s) ds = 0 \right\},$$

or in terms of Fourier series expansion

$$N = \left\{ v \mid v(t, \rho) = \sum_{l=1}^{\infty} \sum_{i=1}^2 v_{li} \sqrt{\frac{2}{\pi}} e_{li}(t) \sin l\rho \right\}.$$

Similarly the kernel of $\square_s$ is

$$K \equiv Ker(\square_s) = \{ v / \sin \rho \mid v \in N \},$$

or in Fourier series expansion

$$K = \left\{ u \mid u(t, \rho) = \sum_{l=1}^{\infty} \sum_{i=1}^2 u_{li} \frac{1}{\sqrt{2\pi}} e_{li}(t) \frac{\sin l\rho}{\sin \rho} \right\}.$$

We will prove the existence of an L^2 periodic solution, we make the following hypothesis on F .

(F₁) $F(t, x, u)$ is nondecreasing in u for $(t, x) \in W$;

(F₂) There exist positive constants η_1 and η_2 such that

$$|F(t, x, u)| \geq \eta_1 |u| - \eta_2 \quad \forall (t, x) \in W;$$

(F₃) There exist positive constants $\gamma < 3$ and ν such that

$$|F(t, x, u)| \leq \gamma |u| + \nu.$$

Then we have the following existence result.

Theorem 1.1. *If $F \in C(W \times \mathbb{R})$ satisfies (F₁) - (F₃), then the wave equation (1.1) possesses a solution $u \in L^2(W)$. Furthermore, if F is symmetric, then equation (1.2) possesses a symmetric solution.*

Proof. If we let $A = \square$, $H = L^2(W)$, $\delta = 3$, $B = F$, then this theorem is a direct consequence of the following result due to Brezis and Nirenberg [7]. $\square$

Theorem 1.2 (Brezis and Nirenberg). *Let H be a real Hilbert space, let $A : D(A) \subset H \rightarrow H$ be a closed linear operator with dense domain and closed range. Assume*

(1) $N(A) = N(A^*)$;

(2) $A^{-1} : R(A) \rightarrow R(A)$ is compact.

Denote by δ the largest positive constant such that

$$(Au, u) \geq -(1/\delta) \|Au\|^2 \quad \forall u \in D(A)$$

Assume $B : H \rightarrow H$ is a nonlinear monotone demicontinuous (i.e., mapping strongly convergent sequence in H to weakly convergent sequence in H) operator and satisfies the condition that there exists a positive constant $\gamma < \delta$ and a constant $C(w)$ depending only on w such that

$$(Bu - Bw, u) \geq (1/\gamma) \|Bu\|^2 - C(w) \quad \forall u, w \in H.$$

Then

$$R(A + B) \simeq R(A) + \text{convex hull of } R(B),$$

where $\simeq$ means the sets on both sides have the same interior and same closure. Furthermore if $\|Bu\| \rightarrow \infty$ as $\|u\| \rightarrow \infty$, then $A + B$ is onto.

1.2 Boundedness of Solutions

The following stronger hypotheses on F are needed to make all L^2 solutions bounded.

(F_{3t}) there exist positive constants μ_1 and μ_2 such that

$$|F_t| \leq \mu_1 |u| + \mu_2 \quad \forall (t, x, u) \in W \times \mathbb{R};$$

(F₄) there exists positive constant α such that $\alpha \leq F_u \quad \forall (t, x, u) \in W \times \mathbb{R};$

(F₅) there exists a positive constant β such that $F_u \leq \beta \quad \forall (t, x, u) \in W \times \mathbb{R}.$

Theorem 1.3. *Assume that $F \in C^1(W \times \mathbb{R})$ satisfies (F₃)-(F₅) and (F_{3t}). Let u be any L^2 symmetric solution. Then there exists a constant C independent of u , F , ν , α , β , μ_1 and μ_2 such that*

$$\|u\|_{L^\infty(W)} \leq C \left[\frac{\mu_1 + \beta + 1}{\alpha^2} \left(\|F\|_{L^2(W)} + \nu \right) + \frac{\mu_2}{\alpha} \right].$$

First, to prove the theorem, we need an estimate on solutions of the linear one-dimensional wave equation

$$\begin{cases} \square_1 w(t, \rho) + g(t, \rho) = 0 & \text{in } T, \\ w(t, 0) = w(t, \pi) = 0. \end{cases} \quad (1.4)$$

Lemma 1.2. *Given $g \in L^q(T) \cap N^\perp$ for $1 \leq q \leq \infty$, there exists a unique solution $w \in C^{0,\alpha}(T) \cap N^\perp$ of (1.4) such that for $\alpha = 1 - 1/q$,*

$$\|w\|_{C^{0,\alpha}(T)} \leq C \|g\|_{L^q(T)}, \quad (1.5)$$

where $C^{0,\alpha}(T)$ is a Hölder space with $C^{0,0}(T) = C(T)$ and $C^{0,1}(T)$ being the space of Lipschitz continuous functions. Moreover we have an explicit representation for w

$$w(t, \rho) = \psi(t, \rho) + p(t + \rho) - p(t - \rho) \quad (1.6)$$

with

$$\psi(t, \rho) = \frac{1}{2} \int_{\rho}^{\pi} dx \int_{t+\rho-x}^{t-\rho+x} g(\tau, x) d\tau + a \frac{(\pi - \rho)}{\pi},$$

$$a = -\frac{1}{2} \int_0^{\pi} dx \int_{t-x}^{t+x} g(\tau, x) d\tau \quad (a \text{ is a constant}),$$

$$p(y) = \frac{1}{2\pi} \int_0^{\pi} [\psi(y - s, s) - \psi(y + s, s)] ds.$$

Remark 1.2. The explicit representation was given by Lovicarová in [33]. With such a representation, estimate (1.5) can be easily verified.

Let $u(t, \rho)$ be an L^2 solution of Equation (1.2). We can write $u = u_1 + u_2$ with $u_1 \in K$, $u_2 \in K^{\perp}$. Let $v = u \sin \rho = u_1 \sin \rho + u_2 \sin \rho \equiv v_1 + v_2$, then v satisfies Equation (1.3) with $v_1 \in N$, $v_2 \in N^{\perp}$. Let $p \in L^2(S^1)$ be the function such that $v_1 = p(t + \rho) - p(t - \rho)$ and $[p] \equiv \int_{S^1} p(s) ds = 0$. It is easy to check that

$$p(y) = \frac{1}{2\pi} \int_0^{\pi} [v_1(y - s, s) - v_1(y + s, s)] ds. \quad (1.7)$$

One can easily verify that for $f, g \in L^2(S^1)$,

$$\int_T f(t + \rho) g(t - \rho) d\rho dt = \frac{1}{2} [f][g], \quad (1.8)$$

from which it follows that

$$|||v_1|||_{L^2}^2 = 2\pi \|p\|_{L^2}^2. \quad (1.9)$$

From the Fourier series expansion, it is easy to see that $\|D_t^k u_1\|_{L^2}$ is equivalent to $\|u_1\|_k$, and $|||D_t^k v_1|||_{L^2}$ is equivalent to $|||v_1|||_k$. In this section, C will be used to denote various constants independent of u , F , ν , α , β , μ_1 and μ_2 . For the simplicity of notations, we denote $\|F\|_{L^2(W)} + \nu$ by B_F .

We will carry out the proof of Theorem 1.3 in several lemmas. As we pointed out earlier, it is relatively easier to estimate u_2 and v_2 . For u_2 , one uses Lemma 1.2 to see that $v_2 \in L^\infty(T)$ from $F(t, \rho, u) \sin \rho \in L^2(T)$, and $v_2 \in C^{0,1}$ from $v_2 \in L^\infty(T)$ and $v_1 \in L^\infty$ (Lemma 1.3). Consequently $u_2 \in L^\infty$. The estimate for u_1 is more complicated. We first use the fact that $F(t, \rho, u) \in K^\perp$ or $F(t, \rho, u) \sin \rho \in N^\perp$. By carefully choosing $\xi \in N$ in following equation

$$\int_T F(t, x, u) \sin \rho \xi(t, \rho) dt d\rho = 0,$$

we will obtain the boundedness of p in Lemma 1.3. Then Lemma 1.4 uses the test function $\xi \in N$, which is an approximation to $D_t^2 v_1$, to obtain $D_t v_1 \in L^2$. Lemma 1.5 uses the fact

$$\int_T D_t [F(t, x, u(t, \rho)) \sin \rho] \xi(t, \rho) dt d\rho = 0$$

for any $\xi \in N$ and shows that $p' \in L^\infty$. Finally from Lemma 1.6 we conclude that $\|u_1\|_{L^\infty} \leq C \|p'\|_{L^\infty}$, and hence that u_1 is bounded.

Define a function $q \in C(\mathbb{R})$ as follows. For positive number M ,

$$q(s) = \begin{cases} s + M, & \text{if } s \leq -M, \\ 0, & \text{if } |s| < M, \\ s - M, & \text{otherwise.} \end{cases} \quad (1.10)$$

Lemma 1.3. *There exists a constant C such that*

$$\|v_2\|_{L^\infty} \leq C B_F, \quad \|p\|_{L^\infty} \leq C \frac{B_F}{\alpha}.$$

Proof. By (F_3) , $u \in L_s^2(W)$ implies $F(t, \rho, u) \in L_s^2(W)$, which in turn implies that $F \sin \rho \in L^2(T)$. Therefore by Lemma 1.2, $v_2 \in C^{0,1/2}$. In particular,

$$\|v_2\|_{L^\infty} \leq C \|F \sin \rho\|_{L^2} \leq C \|F\|_{L^2} \leq C B_F.$$

Since v is a solution of (1.3), we have

$$\int_T F(t, \rho, v / \sin \rho) \xi \sin \rho = 0 \quad \text{for all } \xi \in N, \quad (1.11)$$

or

$$\int_T (F(t, \rho, u) - F(t, \rho, u_2)) \xi \sin \rho = - \int_T F(t, \rho, u_2) \xi \sin \rho. \quad (1.12)$$

Write $v_1 = p(t + \rho) - p(t - \rho) \equiv v_1^+ - v_1^-$, then $\xi = q(v_1^+) - q(v_1^-) \in N$ by construction.

Note that

$$(F(t, \rho, u) - F(t, \rho, u_2)) \sin \rho = F_u(t, \rho, u^*) v_1, \quad (1.13)$$

where u^* is between u and u_2 . The above equality, the monotonicity of q , $sq(s) \geq M |q(s)|$, (F_4) , and (1.8) lead to the estimate

$$\begin{aligned} & \int_T (F(t, \rho, u) - F(t, \rho, u_2)) \xi \sin \rho \\ &= \int_T F_u(t, \rho, u^*) (v_1^+ - v_1^-) (q(v_1^+) - q(v_1^-)) \\ &\geq \alpha \int_T (v_1^+ q(v_1^+) + v_1^- q(v_1^-) - v_1^+ q(v_1^-) - v_1^- q(v_1^+)) \\ &\geq \alpha M \int_T (|q(v_1^+)| + |q(v_1^-)|). \end{aligned} \quad (1.14)$$

On the other hand, (F_3) implies that

$$\left| \int_T F(t, \rho, u_2) \xi \sin \rho \right| \leq C B_F \int_T (|q(v_1^+)| + |q(v_1^-)|). \quad (1.15)$$

Combining (1.12), (1.14) and (1.15) gives

$$\alpha M \int_T (|q(v_1^+)| + |q(v_1^-)|) \leq C B_F \int_T (|q(v_1^+)| + |q(v_1^-)|). \quad (1.16)$$

Whenever $M < \|p\|_{L^\infty}$, the integral term in (1.16) is nonzero and we can divide by it, and therefore $M \leq 2C B_F / \alpha$. Since M is an arbitrary number less than $\|p\|_{L^\infty}$, we have $\|p\|_{L^\infty} \leq C B_F / \alpha$. Thus the proof is completed. $\square$

Lemma 1.4. *There exists a constant C such that*

$$\|v_2\|_{C^{0,1}} \leq C \frac{B_F}{\alpha}, \quad \|u_2\|_{L^\infty} \leq C \frac{B_F}{\alpha},$$

and

$$\|v_{1t}\|_{L^2} \leq C \left(\frac{\beta}{\alpha^2} B_F + \frac{\mu_1}{\alpha} \|u\|_{L^2} + \frac{\mu_2}{\alpha} \right). \quad (1.17)$$

Proof. From preceding lemma and (F₃) we know that $|||F \sin \rho|||_{L^\infty} \leq C \frac{B_F}{\alpha}$. Lemma 1.2 implies that

$$|||v_2|||_{C^{0,1}} \leq C |||F \sin \rho|||_{L^\infty} \leq C \frac{B_F}{\alpha}, \quad (1.18)$$

from which we get

$$||u_2||_{L^\infty} = |||v_2 / \sin \rho|||_{L^\infty} \leq C |||v_2|||_{C^{0,1}} \leq C \frac{B_F}{\alpha}, \quad (1.19)$$

where the boundary conditions $v_2(t, 0) = v_2(t, \pi) = 0$ have been used. As for v_1 , let $z^h = (z(t+h, \rho) - z(t, \rho))/h$, for $z \in L^2(T)$ and $h \in \mathbb{R}$. Since v is a solution of (1.3), we have

$$\int_T F(t, \rho, v / \sin \rho) \xi \sin \rho = 0 \quad \text{for all } \xi \in N. \quad (1.20)$$

In the above equation, set $\xi = (v_1^h)^{-h}$ which clearly belongs to N , we get

$$\int_T (F \sin \rho)^h v_1^h = 0. \quad (1.21)$$

A direct computation gives

$$(F \sin \rho)^h = F_t(t^*, \rho, u(t + \rho)) \sin \rho + F_u(t, \rho, u^*)(v_1^h + v_2^h), \quad (1.22)$$

where u^* is between $u(t, \rho)$ and $u(t+h, \rho)$, and t^* is between t and $t+h$. Substituting (1.22) into (1.21) and then using (F_{3t}), (F₄) and (F₅) yield that

$$\begin{aligned} |||v_1^h|||_{L^2} &\leq \frac{1}{\alpha} (\beta |||v_2^h|||_{L^2} + |||F_t \sin \rho|||_{L^2}) \\ &\leq \frac{C}{\alpha} (\beta |||v_2|||_{C^{0,1}} + \mu_1 |||v|||_{L^2} + \mu_2), \end{aligned}$$

from which the inequality (1.17) follows. $\square$

Remark 1.3. In the proof of the previous lemma, we used the well-known fact that if $z \in L^2(T)$ is periodic in t , then $z_t \in L^2(T)$ if and only if there is a constant δ independent of h such that $||z^h||_0 \leq \delta \quad \forall h \in \mathbb{R}$. The test function $\xi = (v_1^h)^{-h}$ is an approximation to $D_t^2 v_1$. We did not use $D_t^2 v_1$ itself since we don't know whether $D_t^2 v_1 \in L^2(T)$.

Lemma 1.5. *There exists a constant C such that*

$$\|p'\|_{L^\infty} \leq C \left(\frac{\mu_1 + \beta}{\alpha^2} B_F + \frac{\mu_2}{\alpha} \right). \quad (1.23)$$

Proof. It is easy to see that $F \sin \rho \in N^\perp$ implies

$$D_t(F \sin \rho) = F_t \sin \rho + F_u v_t \in N^\perp.$$

Actually one can easily verify that

$$D_t^k : H_0^k(T) \cap N^\perp \rightarrow N^\perp; \quad D_t^k : H_0^k(T) \cap N \rightarrow N. \quad (1.24)$$

Consequently, we have

$$\int_T (F_t \sin \rho + F_u(v_{1t} + v_{2t})) \xi = 0 \quad \forall \xi \in N, \quad (1.25)$$

or equivalently

$$-\int_T (F_t \sin \rho + F_u v_{2t}) \xi = \int_T F_u v_{1t} \xi \quad \forall \xi \in N. \quad (1.26)$$

Using Lemmas 1.3 and 1.4, (F_{3t}) and (F_5) , we have

$$|||(F_t \sin \rho + F_u v_{2t})|||_{L^\infty} \leq \mu_1 |||v|||_{L^\infty} + \mu_2 + \beta |||v_{2t}|||_{L^\infty} \leq C \left(\frac{\mu_1 + \beta}{\alpha} B_F + \mu_2 \right).$$

Write $v_{1t} = p'(t + \rho) - p'(t - \rho) \equiv v_{1t}^+ - v_{1t}^-$ with $p \in H^1(S^1)$ by (1.7). Take $\xi = q(v_{1t}^+) - q(v_{1t}^-)$ in (1.26), where the function q is defined by (1.10).

The right hand side of (1.26) gives

$$\begin{aligned} \int_T F_u v_{1t} \xi &= \int_T F_u(t, \rho, u^*)(v_{1t}^+ - v_{1t}^-)(q(v_{1t}^+) - q(v_{1t}^-)) \\ &\geq \alpha \int_T (v_{1t}^+ q(v_{1t}^+) + v_{1t}^- q(v_{1t}^-) - v_{1t}^+ q(v_{1t}^-) - v_{1t}^- q(v_{1t}^+)) \\ &\geq \alpha M \int_T (|q(v_{1t}^+)| + |q(v_{1t}^-)|). \end{aligned}$$

Hence

$$\alpha M \int_T |q(v_{1t}^+)| + |q(v_{1t}^-)| \leq C \left(\mu_2 + \frac{\mu_1 + \beta}{\alpha} B_F \right) \int_T |q(v_{1t}^+)| + |q(v_{1t}^-)|, \quad (1.27)$$

where M is any constant less than $\|p'\|_{L^\infty}$, which implies (1.23). $\square$

So far, we have shown that $u_2 \in L^\infty$, $p' \in L^\infty$. The following lemma completes the proof of Theorem 1.3.

Lemma 1.6. *For any non-negative integer k ,*

$$\|D_t^k u_1\|_{L_s^\infty(W)} \leq \|p^{(k+1)}\|_{L^\infty(S^1)}.$$

Proof. Fix $\sigma \in (0, \pi/4)$, let $T_1 = S^1 \times [0, \sigma]$, $T_2 = S^1 \times [\pi - \sigma, \pi]$, $T_3 = T \setminus (T_1 \cup T_2)$.

For $(t, \rho) \in T_1$, using

$$D_t^k v_1(t, \rho) = p^{(k)}(t + \rho) - p^{(k)}(t - \rho) = \int_{t-\rho}^{t+\rho} p^{(k+1)}(s) ds,$$

we get

$$\|D_t^k v_1 / \sin \rho\|_{L^\infty(T_1)} \leq \|p^{(k+1)}\|_{L^\infty} \sup_{(0 \leq \rho \leq \sigma)} |2\rho / \sin \rho| \leq C \|p^{(k+1)}\|_{L^\infty}.$$

Similarly on T_2 , using $p(t - \rho) = p(t - \rho + 2\pi)$, we have

$$\|D_t^k v_1 / \sin \rho\|_{L^\infty(T_2)} \leq C \|p^{(k+1)}\|_{L^\infty}.$$

Finally on T_3 , we have

$$\|D_t^k v_1 / \sin \rho\|_{L^\infty(T_3)} \leq \frac{1}{\sin \sigma} \|D_t^k v_1\|_{L^\infty} \leq C \|p^{(k+1)}\|_{L^\infty}.$$

From these three inequalities, we get $D_t^k u_1 \in L^\infty(W)$, and the proof of lemma is completed. $\square$

1.3 Regularity of Solution

In this section, we will show that every bounded weak solution of (1.2) is a classical solution if F is sufficiently regular. We state our main result first.

Theorem 1.4. *Let $k \geq 2$. Suppose that $F \in C_s^k(W \times \mathbb{R})$ satisfies (F_4) and that $u = u_1 + u_2 \in K \oplus K^\perp$ is a bounded weak solution of (1.2), then $u_1 \in H_s^k \cap C^{k-2, \mu} \cap K$ and $u_2 \in H_s^{k+1} \cap C^{k-1, \mu} \cap K^\perp$ for any $\mu \in [0, 1/2)$. In particular, if F is C^∞ , so is u .*

Remark 1.4. *As long as $u \in L^\infty$, the conditions (F_3) , (F_{3t}) and (F_5) are no longer needed for the regularity of the solution. The condition (F_4) can not be relaxed to a monotonicity condition. Indeed, let $F(t, \rho, s) = F(s) \in C^\infty$ be such that F is monotone increasing on $\mathbb{R}$ and $F(s) = 0$ for $s \in [-1, 1]$, then any L^∞ function $\phi(t, \rho) \in K$ with $\|\phi\|_{L^\infty} < 1$ is a solution of (1.2).*

Throughout this section, let $u = u_1 + u_2 \in K \oplus K^\perp$ be a bounded solution given in Theorem 1.4. Let $v = u \sin \rho = v_1 + v_2$ with $v_1 \in N$ and $v_2 \in N^\perp$. We know that v is a solution of (1.3) and $v_1 = p(t + \rho) - p(t - \rho)$ for some $p \in L^2(S^1)$ with $[p] = 0$. The main idea is to differentiate the equation k times with respect to t to obtain

$$\square D_t^k u = D_t^k F(t, \rho, u(t, \rho)).$$

For u_2 or $v_2 = u_2 \sin \rho$ one uses Lemma 3.1 and Lemma 4.1 below to get $D_t^k v_2 \in C^{0,1}(T)$ and $u_2 \in H_s^{k+1}$. For u_1 or $v_1 = u_1 \sin \rho$ one can consider the identity

$$\int_T D_t^{k-1} [F(t, \rho, u(t, \rho)) \sin \rho] D_t^{k+1} v_1(t, \rho) dt d\rho = 0,$$

which produces a good term $F_u |D_t^k v_1|^2$ and yields $D_t^k v_1 \in L^2(T)$. Again, $D_t^{k+1} v_1$ must be replaced by suitable approximations in the detailed argument. Consequently $u_1 \in H_s^k$. The facts $u_1 \in C^{k-2, \mu}$ and $u_2 \in C^{k-1, \mu}$ follow from some sharp estimates on spherical harmonics (Lemmas 4.2-4.4). It should be pointed out that Lemmas 4.5-4.9 are independent of Lemmas 4.2-4.4. In particular, one can conclude that u is a C^∞ solution when $F \in C^\infty$ without using Lemmas 4.2-4.4.

The proof of Theorem 1.4 will be carried out in several lemmas. To begin with, some results on the solutions of linear wave equation

$$\square u(t, x) = g(t, x) \quad \text{in } W \tag{1.28}$$

are required.

Lemma 1.7. *Given $g \in R(\square) \cap H^k$, $k \geq 0$, then there exists a unique $u \in R(\square) \cap H^{k+1}$ satisfying (1.28) and there exists a constant C such that*

$$\|u\|_{k+1} \leq C \|g\|_k.$$

Proof. Recall that

$$R(\square) = \left\{ u \mid u(t, x) = \sum_{l \neq j} u_{jilm} e_{ji}(t) s_{lm}(x) \right\},$$

and that $\{e_{ji}(t) s_{lm}(x)\}_{l \neq j}$ is an orthonormal basis of $R(\square)$.

The uniqueness follows from $Ker(\square)^\perp = R(\square)$.

To show the existence, let $g(t, x) = \sum_{j \neq l} g_{jilm} e_{ji}(t) s_{lm}(x)$. Then clearly

$$u(t, x) = \sum_{j \neq l} \frac{g_{jilm}}{\lambda(j, l)} e_{ji}(t) s_{lm}(x) \in R(\square)$$

is a solution of (1.28). Moreover

$$\begin{aligned} \|u\|_{k+1}^2 &= \sum_{j \neq l} (1 + j^2 + l^2)^{k+1} \left| \frac{g_{jilm}}{\lambda(j, l)} \right|^2 \\ &= \sum_{j \neq l} (1 + j^2 + l^2)^k |g_{jilm}|^2 \frac{1 + j^2 + l^2}{(l + j)^2 (l - j)^2} \\ &\leq C \|g\|_k^2, \end{aligned}$$

which completes the proof of the lemma. $\square$

Actually we can get a sharper estimate of the solution of (1.28) by using Sogge's 'the best estimate on the spherical harmonics' [46].

Lemma 1.8 (Sogge). *Let $H(x)$ be a spherical harmonics on S^3 of degree $l \geq 1$, then there exists a constant C independent of H , p and l such that*

$$\|H\|_{L^p(S^3)} \leq C l^{(3/p'-2)} \|H\|_{L^2(S^3)} \quad \forall p \geq 4,$$

where p' is the exponent conjugate of p , i.e., $1/p + 1/p' = 1$.

First, we use this result to get a sharper L^p estimate for functions in K . Note that $K = Ker(\square_s) \subset Ker(\square)$.

Lemma 1.9. *The identity map*

$$I : H^{k+1}(W) \cap Ker(\square) \hookrightarrow H^{k,r}(W)$$

is a continuous embedding for any $1 \leq r < 8$, where $H^{k,r}(W)$ is the standard Sobolev space on W consisting of all functions which together with their derivatives of order up to k are in $L^r(W)$.

Remark 1.5. *The standard Sobolev embedding theorem only concludes that embedding*

$$H^{k+1}(W) \hookrightarrow H^{k,r}(W)$$

is continuous for $1 \leq r \leq 4$. Lemma 1.9 is obviously an improvement when restricted to $\text{Ker}(\square)$.

Proof. It suffices to show that for $4 \leq r < 8$, $u \in L^r$ if $u \in H^1 \cap \text{Ker}(\square)$. Expand u in Fourier series

$$u(t, x) = \sum_l \sum_i \sum_m u_{lilm} s_{lm}(x) e_{li}(t)$$

with

$$\|u\|_1^2 = \sum_l \sum_i (1 + 2l^2) \sum_m |u_{lilm}|^2 \geq 2 \sum_l \sum_i l^2 a_{li}^2,$$

where $a_{li}^2 = \sum_m |u_{lilm}|^2$. Set $H_{li}(x) = \sum_m s_{lm}(x) u_{lilm}$, then $\|H_{li}\|_{L^2}^2 = a_{li}^2$. Using interpolation theory, one easily proves that for $r > 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$,

$$\|\sum_l \sum_i b_{li} e_{li}(t)\|_{L^r(S^1)} \leq C \left(\sum_l \sum_i b_{li}^{r'} \right)^{1/r'}.$$

Using this inequality, Hölder inequality and Sogge's lemma, we obtain that for $4 \leq r < 8$

$$\begin{aligned} \|u\|_{L^r} &= \|\sum_{l,i} H_{li} e_{li}(t)\|_{L^r} \\ &= \left\{ \int_{S^3} \left[\left(\int_{S^1} \left| \sum_{l,i} H_{li} e_{li}(t) \right|^r dt \right)^{1/r} \right]^r dx \right\}^{1/r} \\ &\leq C \left\{ \int_{S^3} \left(\sum_{l,i} |H_{li}|^{r'} \right)^{r/r'} dx \right\}^{1/r} \\ &\leq C \left(\sum_{l,i} \left(\int_{S^3} |H_{li}|^r dx \right)^{r'/r} \right)^{1/r'} \\ &\leq C \left(\sum_{l,i} l^{3-2r'} \|H_{li}\|_{L^2}^{r'} \right)^{1/r'} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\sum_{l,i} l^{3-2r'} a_{li}^{r'} \right)^{1/r'} \\
&\leq C \left(\sum_{l,i} l^2 a_{li}^2 \right)^{1/2} \left(\sum_i \sum_l l^{\frac{6-6r'}{2-r'}} \right)^{\frac{2-r'}{2r'}} \\
&= C_0 \|u\|_1,
\end{aligned}$$

where $C_0 = C \left(\sum l^{\frac{6-6r'}{2-r'}} \right)^{\frac{2-r'}{2r'}} < \infty$ since $r' > 8/7$ and $\frac{6-6r'}{2-r'} < -1$. Hence the proof is completed. $\square$

Lemma 1.10. *Suppose $g \in R(\square) \cap H^k$, and $u \in K^\perp$ solves (1.28), then for $1 \leq r < 8$, there exists a positive constant $C(r)$ such that*

$$\|u\|_{H^{k,r}(W)} \leq C(r) \|g\|_k.$$

Proof. The proof is similar to that of Lemma 4 in [57], so we omit it. $\square$

Lemma 1.11. *Let M be 4-dimensional compact manifold and $f \in C^k$, then for any $u \in H^k(M) \cap L^\infty(M)$, $f(x, u(x)) \in H^k(M)$.*

Proof. It suffices to consider the case $f(x, u(x)) = f(u(x))$. Set $h(x) = f(u(x))$. It is well-known that if M is an n -dimensional manifold, $f \in C^k(M \times \mathbb{R})$ and $k > n/2$, then $u \in H^k(M)$ implies $f(x, u(x)) \in H^k(M)$. Therefore we only need prove for $k = 1$ and 2.

For $k = 1$, the proof easily follows from

$$|\nabla h|^2 = |f'(u)|^2 |\nabla u|^2.$$

For $k = 2$, we only need to check that $\Delta f(u) \in L^2(M)$. Since

$$\Delta f(u) = f''(u) |\nabla u|^2 + f'(u) \Delta u. \quad (1.29)$$

The standard embedding theorem asserts that $H^1(M) \hookrightarrow L^4(M)$. Therefore $|\nabla u|^2 \in L^2(M)$, which combining with (1.29) shows that $\nabla f(u) \in L^2(M)$. $\square$

Now we come back to establish Theorem 1.4.

Lemma 1.12. $p' \in L^\infty(S^1)$, $D_t v_2 \in C^{0,1}(T)$ and $u_2 \in H_s^2(W)$.

Proof. First, $u \in L^\infty(W)$ implies $v \in L^\infty(T)$ and $F(t, \rho, u) \sin \rho \in L^\infty(T)$. Since v is a solution of (1.3), Lemma 1.2 asserts that $v_2 \in C^{0,1}(T)$. Therefore $v_1 = v - v_2 \in L^\infty$. Exactly the same arguments as in the proofs of Lemma 1.5 yield $p' \in L^\infty(S^1)$, which implies that v_t is bounded. Next, since the operators $\square_1$ and D_t commute, we see that $D_t v_2$ is a periodic solution of (1.3) with nonlinear term $F \sin \rho$ replaced by $D_t(F \sin \rho) = F_t \sin \rho + F_u v_t$. From the boundedness of v_t and Lemma 1.2, it follows that $D_t v_2 \in C^{0,1}$. Finally, the fact that $p' \in L^\infty$ ($v_1 \in H_0^1(T)$) and $v_2 \in C^{0,1}$ implies that $u \in H_s^1(W)$. Lemma 1.11 asserts that $F(t, \rho, u) \in H_s^1(W)$. Therefore $u_2 \in H_s^2(W)$ by Lemma 1.7, and the proof is completed. $\square$

Lemma 1.13. $p'' \in L^2(S^1)$.

Proof. Boundary conditions and $v_{2t} \in C^{0,1}(T)$ yield that

$$|||v_{2t}/\sin \rho|||_{L^\infty} \leq C |||v_{2t}|||_{C^{0,1}} < \infty. \quad (1.30)$$

Let $\xi = -(v_{1t}^h)^{-h}$ in

$$\int_T D_t(F \sin \rho) \xi = 0 \quad \forall \xi \in N.$$

We obtain

$$\int_T (D_t(F \sin \rho))^h v_{1t}^h = 0. \quad (1.31)$$

The mean value theorem implies

$$\begin{aligned} & (D_t(F \sin \rho))^h - F_u v_{1t}^h \\ &= F_{tt} \sin \rho + F_{tu} v^h + F_u v_{2t}^h + F_{ut} v_t + F_{uu} v_t v^h / \sin \rho, \end{aligned} \quad (1.32)$$

where on the right side, all v 's are evaluated at some point in $v_R \equiv [t, t+h] \times \{\rho\}$, and all F 's at some point in $F_R \equiv v_R \times [u(t, \rho), u(t+h, \rho)]$. Hence by (F₄), (1.31) and (1.32) we have

$$\begin{aligned} |||v_{1t}^h|||_{L^2} &\leq (1/\alpha) [\|F_{tt}\|_{L^\infty} + \|F_{ut}\|_{L^\infty} |||v^h|||_{L^2} + \|F_{ut}\|_{L^\infty} |||v_t|||_{L^2} + \\ &\quad \|F_u\|_{L^\infty} |||v_{2t}^h|||_{L^2} + \|F_{uu}\|_{L^\infty} |||v_t|||_{L^\infty} |||v^h/\sin \rho|||_{L^2}]. \end{aligned} \quad (1.33)$$

Lemma 1.12 implies that the first four terms in bracket of (1.33) are bounded by a constant independent of h , and so are $\|F_{uu}\|_{L^\infty}$ and $\|v_t\|_{L^\infty}$ in the fifth term.

As for $v^h/\sin \rho$ in the fifth term, write $v^h = v_1^h + v_2^h$. By (1.30) we have

$$\|v_2^h/\sin \rho\|_{L^\infty} \leq \|D_t v_2/\sin \rho\|_{L^\infty} < \infty. \quad (1.34)$$

Let $\sigma \in (0, \pi/4]$, let $T_1 = S^1 \times [0, \sigma]$, $T_2 = S^1 \times [\pi - \sigma, \pi]$, and $T_3 = T \setminus (T_1 \cup T_2)$.

Then

$$\left\| \left\| \frac{v_1^h}{\sin \rho} \right\| \right\|_{L^2} = \left(\int_T \left(\frac{v_1^h}{\sin \rho} \right)^2 dt d\rho \right)^{\frac{1}{2}} \leq \left(\int_{T_1} \right)^{\frac{1}{2}} + \left(\int_{T_2} \right)^{\frac{1}{2}} + \left(\int_{T_3} \right)^{\frac{1}{2}}. \quad (1.35)$$

On T_1 , we have $0 \leq \rho \leq \sigma \leq \pi/4$, $v_1 = 2\rho \int_0^1 p'(t - \rho + 2s\rho) ds$. Therefore we get

$$\begin{aligned} (v_1^h/\sin \rho)^2 &= (2\rho/\sin \rho)^2 \left(\int_0^1 p'^h(t - \rho + 2s\rho) ds \right)^2 \\ &\leq C \int_0^1 (p'^h(t - \rho + 2s\rho))^2 ds, \end{aligned} \quad (1.36)$$

where constant C is independent of σ , from which it follows that

$$\begin{aligned} \left[\int_{T_1} (v_1^h/\sin \rho)^2 \right]^{\frac{1}{2}} &\leq C \left[\int_0^\sigma \int_0^{2\pi} \int_0^1 (p'^h(t - \rho + 2s\rho))^2 ds dt d\rho \right]^{\frac{1}{2}} \\ &= \left[\int_0^\sigma \int_0^1 \int_0^{2\pi} (p'^h(t + (2s-1)\rho))^2 dt ds d\rho \right]^{\frac{1}{2}} \\ &= C \left(\int_0^\sigma \|p'^h\|_{L^2(S^1)}^2 d\rho \right)^{\frac{1}{2}} = C\sqrt{\sigma} \|p'^h\|_{L^2}. \end{aligned} \quad (1.37)$$

Similarly on T_2 , using

$$v_1 = p(t + \rho) - p(t - \rho + 2\pi) = 2(\rho - \pi) \int_0^1 p'(t - \rho + 2s(\rho - \pi)) ds,$$

we obtain

$$\left(\int_{T_2} \right)^{\frac{1}{2}} \leq C\sqrt{\sigma} \|p'^h\|_{L^2}. \quad (1.38)$$

On T_3 , we estimate that

$$\left(\int_{T_3}\right)^{\frac{1}{2}} \leq \frac{1}{\sin \sigma} |||v_1^h|||_{L^2} \leq \frac{C}{\sin \sigma} |||v_{1t}|||_{L^2}. \quad (1.39)$$

Writing $v_{1t}^h = p'^h(t + \rho) - p'^h(t - \rho)$ and using (1.8) and $[p'] = 0$, we get

$$|||v_{1t}^h|||_{L^2}^2 = 2\pi \|p'^h\|_{L^2}^2. \quad (1.40)$$

Then it follows from (1.33)-(1.40) that for some constants C_1, C_2 and C_3 independent of h and σ ,

$$\|p'^h\|_{L^2} \leq C_1 + C_2/\sin \sigma + C_3\sqrt{\sigma} \|p'^h\|_{L^2}.$$

Choose σ in the above inequality so that $C_3\sqrt{\sigma} = \frac{1}{2}$, we get $\|p'^h\|_{L^2} \leq 2(C_1 + C_2/\sin \sigma)$. Therefore Remark 1.3 concludes that $\|p''\|_{L^2} < \infty$, which completes the proof. $\square$

Lemma 1.14. $u_2 \in H_s^3(W)$, $p'' \in L^\infty(S^1)$, and $D_t^2 v_2 \in C^{0,1}(T)$.

Proof. Lemmas 1.11 - 1.13 imply that $F \in H_s^2(W)$; therefore we have $u_2 \in H_s^3(W)$ by Lemma 1.7. Also $[D_t^2, \square_1] = 0$ and $F \sin \rho \in H_0^2(T)$ imply that

$$\int_T D_t^2(F \sin \rho) \xi = 0 \quad \forall \xi \in N.$$

We write $v_{1tt} = p''(t + \rho) - p''(t - \rho)$, let $q^\pm = q(p''(t \pm \rho))$, where q is defined in (1.10). Take $\xi = q^+ - q^- \in N$. Similar to the proof of Lemma 1.5, in order to show $p'' \in L^\infty(S^1)$, it suffices to prove that

$$D_t^2(F \sin \rho) - F_u v_{1tt} = F_u v_{2tt} + F_{tt} \sin \rho + 2F_{tu} v_t + F_{uu} v_t^2 / \sin \rho. \quad (1.41)$$

is in $L^\infty(T)$. In fact, the first three terms of (1.41) are in $L^\infty(t)$ by Lemma 1.12. That the fourth term is in $L^\infty(T)$ follows from Lemma 1.13 the following estimates.

By Lemma 1.12, we have

$$|||v_{2t}^2 / \sin \rho|||_{L^\infty} \leq |||v_{2t}|||_{L^\infty} |||v_{2t} / \sin \rho|||_{L^\infty} < \infty, \quad (1.42)$$

$$|||v_{1t} v_{2t} / \sin \rho|||_{L^\infty} \leq |||v_{1t}|||_{L^\infty} |||v_{2t} / \sin \rho|||_{L^\infty} < \infty. \quad (1.43)$$

Fix $\sigma \in (0, \pi/4]$. On T_1 , using Schwartz inequality and Lemma 1.13, we obtain

$$\begin{aligned} \|v_{1t}^2 / \sin \rho\|_{L^\infty(T_1)} &= \left\| \frac{1}{\sin \rho} \left(\int_{t-\rho}^{t+\rho} p''(s) ds \right)^2 \right\|_{L^\infty(T_1)} \\ &\leq \sup_{(t,\rho) \in T_1} \left\{ \frac{2\rho}{\sin \rho} \int_{t-\rho}^{t+\rho} (p''(s))^2 ds \right\} < C \|p''\|_{L^2}^2. \end{aligned} \quad (1.44)$$

Similarly on T_2 we get

$$\|v_{1t}^2 / \sin \rho\|_{L^\infty(T_2)} < \infty. \quad (1.45)$$

On T_3 we can directly estimate

$$\|v_{1t}^2 / \sin \rho\|_{L^\infty(T_3)} \leq C \|v_{1t}^2\|_{L^\infty}. \quad (1.46)$$

Now $p'' \in L^\infty$ and (4.15) tell us that $D_t^2(F \sin \rho) \in L^\infty(T)$, which implies that $v_{2tt} \in C^{0,1}(T)$ since $w = v_{2tt}$ is the solution of $\square_1 w = D_t^2(F \sin \rho)$ in $R(\square_1)$. $\square$

In general, we have the following regularity result

Lemma 1.15. *If $F \in C^k(W \times \mathbb{R})$ ($k \geq 1$) is symmetric, then $u_2 \in H_s^{k+1}(W)$, $p^{(k)} \in L^\infty(S^1)$, and $D_t^k v_2 \in C^{0,1}(T)$.*

Proof. This holds for $k = 1$ and 2 by Lemma 1.12 and Lemma 1.14. We proceed by induction. Assume the lemma is true for $k=j \geq 2$, i.e., $p^{(j)} \in L^\infty(S^1)$, $D_t^i v_2 \in C^{0,1}(T)$ for $i = 1, \dots, j$, and $u_2 \in H_s^{j+1}(W)$.

For $k = j+1$, we first show $p^{(j+1)} \in L^2(S^1)$ and $u_2 \in H_s^{j+2}(W)$. By (1.7) it suffices to show $D_t^{j+1} v_1 \in L^2(T)$. By the induction hypothesis,

$$\int_T D_t^j(F(t, \rho, u) \sin \rho) \xi = 0 \quad \forall \xi \in N. \quad (1.47)$$

Taking $\xi = ((D_t^j v_1)^h)^{-h} \in N$, we get

$$\int_T (D_t^j(F \sin \rho))^h (D_t^j v_1)^h = 0. \quad (1.48)$$

Expanding the j th derivative and using mean value theorem, we obtain

$$\begin{aligned} &[D_t^j(F \sin \rho)]^h - F_u(D_t^j v_1)^h \\ &= F_u(D_t^j v_2)^h + \sum_{m,r} C_{mr} F_{t^m u^r} \frac{D_t^{i_1} v}{\sin \rho} \dots \frac{D_t^{i_r} v}{\sin \rho} \sin \rho, \end{aligned} \quad (1.49)$$

where C_{mr} 's are positive constants, and $i_1 + \dots + i_r + m = j + 1$ with

$$1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq j.$$

Note that on the right hand side of (1.49), all v 's are evaluated at some point in $v_R = [t, t+h] \times \{\rho\}$ and F 's at some point in $F_R = v_R \times [u(t, \rho), u(t+h, \rho)]$. Denoting the second term in the right hand side of (1.49) by $I(F, v, h)$, substituting (1.49) into (1.48) and using (F₄) lead to

$$|||(D_t^j v_1)^h|||_{L^2} \leq (1/\alpha) (||F||_{L^\infty} |||(D_t^j v_2)^h|||_{L^2} + |||I(F, v, h)|||_{L^2}). \quad (1.50)$$

In the expression of $I(F, v, h)$, consider a typical term

$$F_{t^m u^r} \frac{D_t^{i_1} v}{\sin \rho} \dots \frac{D_t^{i_r} v}{\sin \rho} \sin \rho.$$

By hypothesis

$$|||F_{t^m u^r}|||_{L^\infty} \leq C. \quad (1.51)$$

If $r = 0$, then $m = j + 1$ and we have no derivative on v .

If $r = 1$, then since $i_1 \leq j$, we have

$$|||F_{t^m u} D_t^{i_1} v|||_{L^2} \leq ||F_{t^m u}||_{L^\infty} |||D_t^{i_1} v|||_{L^\infty} < \infty. \quad (1.52)$$

If $r \geq 2$, then it is easy to check that $i_{r-1} \leq j - 1$. Therefore by Lemma 1.6 and induction hypothesis we have

$$\frac{D_t^{i_s} v}{\sin \rho} = \frac{D_t^{i_s} v_1}{\sin \rho} + \frac{D_t^{i_s} v_2}{\sin \rho} \in L^\infty(T) \quad \text{for } 1 \leq s \leq r - 1, \quad (1.53)$$

and

$$\frac{D_t^{i_r} v}{\sin \rho} \sin \rho \in L^\infty(T) \quad (\text{since } i_r \leq j). \quad (1.54)$$

Summarizing these estimates, we obtain

$$|||I(F, u, h)|||_{L^2} \leq C |||I(F, u, h)|||_{L^\infty} < \infty. \quad (1.55)$$

Remark 1.3, (1.50) and (1.55) yield that $D_t^{j+1} v_1$ is in $L^2(T)$. The induction hypothesis and $v_1 \in H^{j+1}$ imply that $u = u_1 + u_2 \in H_s^{j+1}(W)$. Therefore applying Lemma 1.11 and Lemma 1.7, we conclude that $u_2 \in H_s^{j+2}(W)$.

Next, we show $p^{(j+1)} \in L^\infty(S^1)$. Let $q^\pm = q(p^{(j+1)}(t \pm \rho))$, then $\xi = q^+ - q^- \in N$, where the function q is given in (1.10). Set

$$I(F, u) = D_t^{j+1}(F \sin \rho) - F_u D_t^{j+1} v_1. \quad (1.56)$$

The same argument as we used for $I(F, u, h)$ shows that

$$|||I(F, u)|||_{L^\infty} \leq C. \quad (1.57)$$

Using (1.56), $\xi = q^+ - q^- \in N$ and $D_t^{j+1}(F \sin \rho) \in N^\perp$ we get

$$\int_T F_u(p^{(j+1)}(t + \rho) - p^{(j+1)}(t - \rho))(q^+ - q^-) = \int_T I(F, u)(q^+ - q^-).$$

Then following the same procedure as in the proof of Lemma 1.5, we obtain

$$p^{(j+1)} \in L^\infty(S^1). \quad (1.58)$$

Finally, (1.56)- (1.58) imply that $D_t^{j+1}(F \sin \rho) \in L^\infty(T)$. Since $D_t^{j+1} v_2$ solves equation (1.3) with $F \sin \rho$ replaced by the bounded function $D_t^{j+1}(F \sin \rho)$, Lemma 1.2 implies that $D_t^{j+1} v_2 \in C^{0,1}(T)$. $\square$

Conclusion of the proof of Theorem 1.4. Lemma 1.15 shows that $u = u_1 + u_2 \in (H_s^k \cap K) \oplus (H_s^{k+1} \cap K^\perp)$. Furthermore, Lemmas 1.9 and 1.10 imply that $u_1 + u_2 \in W^{k-1,r}(W) \cap K \oplus W^{k,r}(W) \cap K^\perp$ for $r \in [1, 8)$. The standard Sobolev embedding theorem shows that $u_1 + u_2 \in (C^{k-2,\mu}(W) \cap K) \oplus (C^{k-1,\mu}(W) \cap K^\perp)$ for $\mu \in [0, \frac{1}{2})$. $\square$

Remark 1.6. *The regularity we established strongly relies on two hypothesis:*

- (1) *there exists a positive constant α such that $F_u \geq \alpha$;*
- (2) *F is symmetric, i.e, F is independent of ϕ and θ , when S^3 is expressed in spherical coordinates, which allows us to apply some results from one-dimensional wave equation;*

We strongly believe that the condition (1) can be relaxed to the condition that F is strictly increasing in u . We also hope that our results can be extended to non-symmetric F .

1.4 Small Forcing Problem

As in the one-dimensional vibration problem [6], we can deal with the small forcing problem. Consider the equation

$$\square u(t, x) + g(u) + f(t, x) = 0 \quad \text{in } S^1 \times S^3. \quad (1.59)$$

Again we assume f is symmetric, i.e., $f(t, x) = f(t, \rho)$, and look for symmetric solutions of

$$\square_s u(t, \rho) + g(u) + f(t, \rho) = 0. \quad (1.60)$$

We assume that $g(0) = 0$, $\|f\|_{L^\infty}$ is small, and that for some positive number L , $g \in C^1([-L, +L])$ satisfies

$$|g'(s)| \geq \alpha \quad \text{and} \quad |g(s)| \leq \gamma |s| \quad \forall s \in [-L, +L], \quad (1.61)$$

where α and γ are positive constants specified in (F₃) and (F₄) respectively.

Theorem 1.5. *Assume g has the above properties. Then there is a $\delta > 0$ such that for each $f \in L^\infty(W)$ with $\|f\|_{L^\infty} + \|f_t\|_{L^\infty} \leq \delta$, there exists a weak symmetric solution of (1.60) with $\|u\|_{L^\infty} \leq L$. Furthermore, if $\alpha \leq g'(s) \leq \gamma$ for all s in $[-L, +L]$, then such solution of (1.60) is unique.*

Remark 1.7. *Theorem 1.5 and Theorem 1.4 guarantee the existence, uniqueness, and smoothness of the small solution for the equations*

$$\square_s u + u^3 + \alpha u + f(t, \rho) = 0 \quad \text{in } W,$$

$$\square_s u + \sin u + f(t, \rho) = 0 \quad \text{in } W,$$

provided that $0 < \alpha < 3$ and f is small enough.

Proof of Theorem 1.5. For the uniqueness, assume u and $\bar{u}$ are two solutions of (1.60) with $\|u\|_{L^\infty} \leq L$, and $\|\bar{u}\|_{L^\infty} \leq L$, then

$$\int_W \square(u - \bar{u})(u - \bar{u}) + \int_W (g(u) - g(\bar{u}))(u - \bar{u}) = 0. \quad (1.62)$$

From the Fourier series expansion, we have for any $u \in D(\square)$

$$3(\square u, u) + \|\square u\|_{L^2}^2 \geq 0. \quad (1.63)$$

It follows from the above two inequalities and assumption on g that

$$\begin{aligned} (1/\gamma) \int_W |g(u) - g(\bar{u})|^2 &\leq \int_W (g(u) - g(\bar{u}))(u - \bar{u}) \\ &\leq (1/3) \int_W |\square(u - \bar{u})|^2 = (1/3) \int_W |g(u) - g(\bar{u})|^2. \end{aligned}$$

Hence $|g(u) - g(\bar{u})| = 0$, therefore $u = \bar{u}$.

Now let us turn to the existence. We can construct an extension $\bar{g}$ of g satisfying

1. $\bar{g} \in C^1(\mathbb{R})$ and $\bar{g}(s) = g(s)$ for all $s \in [-L, L]$;
2. There exist $3 > \gamma' \geq \gamma$, $0 < \alpha' \leq \alpha$ and $\beta > 0$ such that

$$\beta \geq \bar{g}'(s) \geq \alpha' \quad \text{and} \quad |\bar{g}(s)| \leq \gamma' |s| \quad \forall s \in \mathbb{R}.$$

Let $F(t, \rho, u) = \bar{g}(u) + f(t, \rho)$, then $F_t = f_t(t, \rho)$, $F_u = \bar{g}'$ and

$$|F| \leq \gamma' |u| + \nu \quad |F_t| \leq \mu_2 \quad \text{with } \nu = \|f\|_{L^\infty} \quad \text{and} \quad \mu_2 = \|f_t\|_{L^\infty}. \quad (1.64)$$

Therefore F satisfies (F_3) , (F_{3t}) with $\mu_1 = 0$, (F_4) and (F_5) . Theorems 1.1 and 1.3 imply that there exists a solution of (1.60) such that

$$\|u\|_{L^\infty} \leq C \left[\frac{\beta + 1}{(\alpha')^2} (\|F\|_{L^2} + \nu) + \frac{\|f_t\|_{L^\infty}}{\alpha'} \right]. \quad (1.65)$$

Note that $\|F\|_{L^2}$ depends on u . To get $\|u\|_{L^\infty} \leq L$, we need to estimate $\|F\|_{L^2}$ in terms of $\|f\|_{L^\infty}$. Since $\bar{g}' \geq \alpha' > 0$, the Implicit Function Theorem implies that there exists a $u_0(t, \rho) \in L^\infty(W)$ such that

$$F(t, \rho, u_0(t, \rho)) = \bar{g}(u_0(t, \rho)) + f(t, \rho) = 0,$$

and

$$\|u_0\|_{L^\infty} \leq \frac{1}{\alpha'} \|f\|_{L^\infty}. \quad (1.66)$$

We write $u = u_1 + u_2 \in (L^\infty \cap K) \oplus (L^\infty \cap K^\perp)$. Then u_2 satisfies

$$\square_s u_2 + F = 0.$$

Taking L^2 scalar product with u , we find that

$$(\square_s u_2, u_2) + (F, u) = 0. \quad (1.67)$$

The Fourier series expansion gives

$$3(\square u_2, u_2) + \|\square u_2\|_{L^2}^2 \geq 0. \quad (1.68)$$

From the definition of $u_0(t, \rho)$, we have

$$F(t, \rho, u)(u - u_0) = |F(t, \rho, u)| |u - u_0| \geq |F(t, \rho, u)| |u| - |F(t, \rho, u)| |u_0|,$$

from which we obtain

$$|F(t, \rho, u)| |u| \leq F(t, \rho, u)u + 2 |F(t, \rho, u)| |u_0|. \quad (1.69)$$

Estimates (1.67)-(1.69) together with $\|\square_s u_2\|_{L^2}^2 = \|F\|_{L^2}^2$ imply that

$$\int_W |F| |u| \leq (1/3) \|F\|_{L^2}^2 + 2 \int_W |F(t, \rho, u)| |u_0|. \quad (1.70)$$

Equations (1.64) and (1.70) yield

$$(1/\gamma') \int_W |F| (|F| - \nu) \leq (1/3) \|F\|_{L^2}^2 + 2 \int_W |F| |u_0|,$$

which gives us

$$\|F\|_{L^2} \leq \frac{3\gamma'}{3 - \gamma'} \left(\frac{\nu}{\gamma'} + 2 \|u_0\|_{L^\infty} \right) \sqrt{8\pi^3}. \quad (1.71)$$

Together, (1.65), (1.66), and (1.71) show that there exists a constant C depending only on γ' , α' and β such that

$$\|u\|_{L^\infty} \leq C (\|f\|_{L^\infty} + \|f_t\|_{L^\infty}). \quad (1.72)$$

We can, therefore choose δ so small that $\|f\|_{L^\infty} + \|f_t\|_{L^\infty} \leq \delta$ implies $\|u\|_{L^\infty} \leq L$.

Chapter 2

Orbital Stability with Positive Energy

2.1 Problem and Notations

In this chapter, we assume that Ω is a bounded domain in $\mathbb{R}^n$ or a n -dimensional compact manifold. We shall prove a sharp stability theorem for a wide variety of nonlinearity f for ground state standing waves of NLKG

$$\begin{cases} u_{tt} - \Delta u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \quad u_t(0, x) = V(x), \end{cases} \quad (2.1)$$

and NLS

$$\begin{cases} iu_t - \Delta u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \end{cases} \quad (2.2)$$

By a standing wave we mean a solution of the form $u(x, t) = e^{i\omega t}\phi(x)$ with ω a real parameter, called frequency. The nonlinear interaction f we consider here is very general, and has the form $f(x, u) = g(x, |u|)u$ and satisfies the following conditions:

(F) There exist constants $l > 2$ and $C > 0$ such that for all $x \in \Omega$

$$|F(x, s)| \leq cs^l \quad \forall \quad s \geq 0, \quad (2.3)$$

where $F(x, \phi) = F(x, |\phi|) = \int_0^{|\phi|} f(x, s) ds$.

(FG) For l specified in **(F)**, one of the following is true

1. $F(x, s)/s^2 \rightarrow \infty$ as $s \rightarrow \infty$ uniformly on Ω if $l < 2n/(n-2)$
2. $g(x, s)/s^{l-2} \rightarrow A$ for some $A > 0$ as $s \rightarrow \infty$ uniformly on Ω if $l \geq 2n/(n-2)$.

We will carry out the details for stability of NLKG (2.1). The investigation of orbital stability for NLS is very similar, and is outlined at the end of this chapter.

Note that the search for standing wave of NLKG leads us to following nonlinear elliptic equation

$$\begin{cases} -\Delta\phi - \omega^2\phi + f(x, \phi) = 0, \\ \phi = 0 \quad \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset. \end{cases} \quad (2.4)$$

Any solution of (2.4) is a critical point of energy functional

$$J_\omega(\phi) = \frac{1}{2} \int_\Omega |\nabla\phi|^2 - \omega^2 |\phi|^2 + \int F(x\phi).$$

Note that by **(FG)** the functional J_ω is bounded from below. Since we do not need the explicit use of dependencies of ϕ and J on frequency ω , we suppress the subscript ω for notational brevity.

Next we introduce some notations which will be used throughout the rest of this dissertation. $H = H^1(\Omega) \cap L^l(\Omega)$ if Ω is a n -dimensional compact manifold, and $H = H_0^1(\Omega) \cap L^l(\Omega)$ if Ω is a bounded domain in $\mathbb{R}^n$ and $L^p = L^p(\Omega)$. As usual, we use

$$\begin{aligned} (u, v) &= \int_\Omega u \bar{v} dx, \\ \|u\|_p &= \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}, \\ \|u\|_H^2 &= \int_\Omega (|\nabla u|^2 + |u|^2). \end{aligned}$$

to denote L^2 inner product, L^p norm and H norm in space variable. Next we introduce the notion of weak solution [41] [49] of NLKG (2.1). A weak solution of (2.1) is a function $u(x, t)$ defined a.e. in Ω for each real t such that

(A) u (resp. u_t) is a weakly continuous function of t with values in H (resp. L^2).

(B) f is locally integrable function on $\Omega \times \mathbb{R}$.

(C) $u_{tt} - \Delta u + f(x, u) = 0$ in the sense of distribution.

We now are ready to specify our problem. From assumption (FG), J is bounded from below for any frequency ω , so it makes sense to define the minimization problem to search for the least energy solution.

$$d = \inf_{\phi \in H} J(\phi) \quad (2.5)$$

In next section, we show that d is actually achieved at some $\phi_0(x) \geq 0$. Any minimizer of (2.5) is called “ground state”, and corresponding $e^{i\omega t}\phi(x)$ called “ground state with frequency ω ” or ground state for short. For fixed frequency ω , define S to be the set of all minimizer of minimization problem (2.5), i.e., $S = \{\phi \in H \mid J(\phi) = d\}$.

Now we are in a position to state main result in this chapter.

Theorem 2.1 *For any fixed frequency ω , the standing waves of NLKG (2.1) with frequency ω is orbitally stable in the following sense: for any given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that any weak solution u of (2.1) with initial data satisfying*

$$\inf_{\phi \in S} (\|U - \phi\|_H + \|V - i\omega\phi\|_2) < \delta$$

has the property

$$\inf_{\phi \in S} (\|u(t) - \phi\|_H + \|u_t(t) - i\omega\phi\|_2) < \epsilon \quad \forall t \geq 0$$

2.2 Existence of Ground State

In this section, we prove the existence, positiveness and regularity of ground state, and give some characterization of ground state set S .

Lemma 2.1 *Every minimizing sequence in H of problem (2.5) has a convergent subsequence in H . In particular, d is achieved at some ϕ . Moreover the minimizer can be chosen nonnegative.*

Proof. Let $\{\phi_k\}$ be a minimizing sequence. We divide the proof into two cases

Case 1: $l < 2n/(n-2)$. By **(FG)**, there exist constants C_1 and C_2 that

$$J(\phi_k) \geq C_1 \|\phi_k\|_H^2 - C_2 \quad \forall \phi \in H$$

from which it follows that there exists a ϕ_0 and a subsequence of $\{\phi_k\}$, still denoted by $\{\phi_k\}$, such that $\phi_k \rightharpoonup \phi_0$ weakly in H and $\phi_k \rightarrow \phi_0$ a.e. in Ω and strongly in L^p for any $1 \leq p < 2n/(n-2)$. Thus from **(F)** and weak lower semicontinuity of $\|\cdot\|_H$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int F(x, \phi_k) &= \int F(x, \phi_0), \\ \lim_{k \rightarrow \infty} \int |\phi_k|^2 &= \int |\phi_0|^2, \\ \liminf_{k \rightarrow \infty} \int |\nabla \phi_k|^2 &\geq \int |\nabla \phi_0|^2. \end{aligned} \quad (2.6)$$

It is easy to see that if a strict inequality held in (2.6) then we would be led to following contradiction:

$$d \leq J(\phi_0) < \liminf_{k \rightarrow \infty} J(\phi_k) = d$$

Therefore we have $\phi_k \rightarrow \phi_0$ strongly in $H^1(\Omega)$, thus in H by Sobolev embedding theorem.

Case 2: $l \geq 2n/(n-2)$. **(FG)** implies that there exist some constants C_1, C_2 and C_3 such that

$$J(\phi_k) \geq C_1 \|\phi_k\|_H^2 + C_2 \|\phi_k\|_l^l - C_3$$

which implies that there exist a ϕ_0 and a subsequence, also denoted by $\{\phi_k\} \in H$ such that

$$\begin{aligned} \phi_k &\rightharpoonup \phi_0 \quad \text{weakly in } H^1(\Omega), \\ \phi_k &\rightharpoonup \phi_0 \quad \text{weakly in } L^l(\Omega), \\ \phi_k &\rightarrow \phi_0 \quad \text{strongly in } L^p(\Omega) \quad \forall p \in [1, 2n/(n-2)), \\ \phi_k &\rightarrow \phi_0 \quad \text{a.e. on } \Omega. \end{aligned}$$

Assumption **(FG)**, Fatou's lemma, boundedness of Ω and weak lower semicontinuity of $\|\cdot\|_H$ yield that

$$\liminf_{k \rightarrow \infty} \int F(x, \phi_k) \geq \int F(x, \phi_0), \quad (2.7)$$

$$\liminf_{k \rightarrow \infty} |\nabla \phi_k|_2^2 \geq |\nabla \phi_0|_2^2. \quad (2.8)$$

A strict inequality in either (2.7) or (2.8) would lead to contradiction

$$d \leq J(\phi_0) < \liminf_{k \rightarrow \infty} J(\phi_k) = d.$$

Therefore equalities hold both in (2.7) and (2.8). An equality in (2.8) implies $\phi_k \rightarrow \phi_0$ strongly in $H^1(\Omega)$. It now remains to show that $\phi_k \rightarrow \phi_0$ strongly in $L^l(\Omega)$. Assumption **(FG)** implies that there exists some constant C such that

$$F(x, \phi_k) - \frac{A}{2} |\phi_k|^l + C \geq 0,$$

which combined with Fatou's Lemma gives that

$$\liminf_{k \rightarrow \infty} \int \left(F(x, \phi_k) - \frac{A}{2} |\phi_k|^l \right) \geq \int \left(F(x, \phi_0) - \frac{A}{2} |\phi_0|^l \right).$$

Hence we get

$$\begin{aligned} \int F(x, \phi_0) &= \liminf_{k \rightarrow \infty} \int F(x, \phi_k) \\ &\geq \liminf_{k \rightarrow \infty} \int \left(F(x, \phi_k) - \frac{A}{2} |\phi_k|^l \right) + \liminf_{k \rightarrow \infty} \int \frac{A}{2} |\phi_k|^l \\ &\geq \int \left(F(x, \phi_0) - \frac{A}{2} |\phi_0|^l \right) + \int \frac{A}{2} |\phi_0|^l \\ &= \int F(x, \phi_0). \end{aligned}$$

From this expression, all the inequalities are forced into equalities, therefore we obtain

$$\liminf_{k \rightarrow \infty} \int |\phi_k|^l = \int |\phi_0|^l,$$

which implies $\phi_k \rightarrow \phi_0$ strongly in $L^l(\Omega)$ by a theorem in [4].

The existence of a non-negative minimizer follows from the fact that

$$J(|\phi|) \leq J(\phi) \quad \forall \phi \in H.$$

Lemma 2.2 *Assume that f is Lipschitz continuous on $\Omega \times \mathbb{R}$, then every real minimizer of problem (2.5) is a classical solution of equation of (2.4).*

Proof. Case 1: $l < 2n/(n-2)$. The lemma is a direct consequence of standard elliptic theory.

Case 2: $l \geq 2n/(n-2)$. We use some kind of bootstrap argument. Let ϕ be a real minimizer of (2.5), then we have

$$(\nabla \phi, \nabla v) - \omega^2(\phi, v) + (f(\cdot, \phi), v) = 0 \quad \forall v \in H. \quad (2.9)$$

For $\sigma > 0$, set

$$v_\sigma = \begin{cases} \phi & \text{if } |\phi| \leq \sigma, \\ -\sigma & \text{if } \phi < -\sigma, \\ \sigma & \text{if } \phi > \sigma \end{cases}$$

Then for $q = l - 2 > 0$, $v = |v_\sigma|^q v_\sigma \in H$ and $\nabla v = (q+1)|v_\sigma|^{q-1} \nabla v_\sigma$. Substituting them into (2.9) yields that

$$\begin{aligned} & (q+1) \int_{\Omega_\sigma} |\phi|^q |\nabla \phi|^2 - \omega^2 \int_{\Omega_\sigma} |\phi|^{q+2} - \omega^2 \int_{\Omega \setminus \Omega_\sigma} \sigma^{q+1} |\phi| \\ & + \int_{\Omega_\sigma} |\phi|^{q+2} g(x, |\phi|) + \int_{\Omega \setminus \Omega_\sigma} \sigma^{q+1} |\phi| g(x, |\phi|) = 0, \end{aligned} \quad (2.10)$$

where

$$\Omega_\sigma = \{x \in \Omega \mid |\phi(x)| \leq \sigma\}$$

Our choice of q and assumption **(F)** imply that each term in (2.10) is well defined. Choose σ large enough so that the last term to the left hand side of equality sign in (2.10) becomes positive due to assumption **(FG)**. Therefore the following estimate holds

$$\int_{\Omega_\sigma} |\phi|^{q+2} g(x, |\phi|) \leq \omega^2 \int_{\Omega} |\phi|^{q+2} = \omega^2 \|\phi\|_{q+2}^{q+2}. \quad (2.11)$$

On the other hand, using **(FG)**, there exist positive constants C_1 and C_2 independent of σ such that

$$\int_{\Omega_\sigma} |\phi|^{q+2} g(x, |\phi|) \geq C_1 \int_{\Omega_\sigma} |\phi|^{l+q} - C_2 \int_{\Omega_\sigma} |\phi|^{q+2}. \quad (2.12)$$

Combining (2.11) and (2.12) yields for some positive constant C_3

$$\int_{\Omega_\sigma} |\phi|^{l+q} \leq C_3 \|\phi\|_{q+2}^{q+2},$$

which by arbitrariness of σ implies that $\phi \in L^{l+q}$. Thus the regularity of ϕ is increased by an order of $l - 2$. Repeating the above procedure, we are able to improve the regularity of ϕ so that $\phi \in L^p$ for all $1 \leq p < \infty$. Therefore, by elliptic theory, we have $\phi \in H^{2,p}(\Omega)$ for all $1 \leq p < \infty$, which implies $\phi \in C^{1,\alpha}(\Omega)$ by embedding theorem. From the Lipschitz continuity of f , we have $f \in C^\alpha(\Omega)$. Hence $\phi \in C^{2,\alpha}(\Omega)$ by Schauder theory, and the proof is completed.

To conclude this section, we give some characterization of the set of ground states. The proof for the following lemma follows the same idea as in [9].

Lemma 2.3 *For any $\phi(x) \in S$, there exists a non-negative function $\phi_0(x) \in S$ such that $\phi(x) = e^{i\theta} \phi_0(x)$.*

Proof. Let $\phi \in S$ and $\phi = \phi^1 + i\phi^2$ where $\phi^1, \phi^2 \in H$ are real-valued, then $\hat{\phi} = |\phi^1| + i|\phi^2|$ is still in S , and this yields that

$$\begin{cases} -\Delta \phi^j - \omega^2 \phi^j + g(x, |\phi|) \phi^j = 0, \\ -\Delta |\phi^j| - \omega^2 |\phi^j| + g(x, |\phi|) |\phi^j| = 0. \end{cases}$$

where $j = 1$ and 2 . This shows that ω^2 is the first eigenvalue of the operator $-\Delta - g(x, |\phi|)$ and $\phi^1, \phi^2, |\phi^1|$ and $|\phi^2|$ are all multiples of the positive normalized eigenfunction of $-\Delta - g(x, |\phi|)$ and the proof is finished.

2.3 Orbital Stability of Standing Waves

In this section, we will prove Theorem 2.1. Let $u(t)$ be a weak solution of NLKG (2.1) with initial data U and V .

Define the total energy and charge as

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \left(\int |u_t(t)|^2 + \int |\nabla u(t)|^2 \right) + \int F(x, u(t)) dx, \\ Q(u(t)) &= \text{Im}(u_t(t), u(t)). \end{aligned}$$

It has been shown [41, 49] that energy inequality holds for weak solutions of NLKG (2.1) for the nonlinearity f we are considering, i.e., $E(u(t_1)) \leq E(u(t_2))$ for all $t_2 \geq t_1 \geq 0$. From the arguments in [41, 49], it is easy to show that the charge identity holds for weak solutions of NLKG (2.1), i.e. $Q(u(t)) \equiv Q(u(0))$ for all $t \geq 0$.

Proof of Theorem 2.1: Suppose that there exists a frequency ω for which standing waves are not orbitally stable. Then there exist $\epsilon_0 > 0$, sequences of $\{t^k\}$, $\{U^k\}$, $\{V^k\}$ and $\{u^k\}$ such that

$$\inf_{\phi \in S} (\|U^k - \phi\|_H + \|V^k - i\omega\phi\|_2) \longrightarrow 0 \quad (2.13)$$

and

$$\inf_{\phi \in S} (\|u^k(t^k) - \phi\|_H + \|u_t^k(t^k) - i\omega\phi\|_2) \geq \epsilon_0 \quad (2.14)$$

A direct computation shows that

$$E(u(t)) - \omega Q(u(t)) = \frac{1}{2} \int |u_t(t) - i\omega u(t)|^2 + J(u(t)), \quad (2.15)$$

$$E(u(0)) - \omega Q(u(0)) = \frac{1}{2} \int |V - i\omega U|^2 + J(U). \quad (2.16)$$

By Lemma 2.1, S is a compact set in H , thus we may assume that $U^k \longrightarrow \phi_0$ in H , $V^k \longrightarrow i\omega\phi_0$ in L^2 for some $\phi_0 \in S$. Hence using (2.16) we obtain

$$E(u^k(0)) - \omega Q(u^k(0)) \longrightarrow d. \quad (2.17)$$

On the other hand, (2.15), energy inequality, charge identity and (2.17) give rise to

$$J(u^k(t^k)) \leq E(u^k(t^k)) - \omega Q(u^k(t^k)) \leq E(u^k(0)) - \omega Q(u^k(0)) \longrightarrow d, \quad (2.18)$$

which implies that $\{u^k(t^k)\}$ is a minimizing sequence of problem (2.5). Lemma 2.1 implies that there exists a $\hat{\phi} \in S$ such that

$$u^k(t^k) \longrightarrow \hat{\phi} \text{ strongly in } H, \quad (2.19)$$

$$J(u^k(t^k)) \longrightarrow d. \quad (2.20)$$

Using (2.20) and (2.15), we have

$$u_t^k(t^k) \longrightarrow i\omega\hat{\phi} \text{ strongly in } L^2. \quad (2.21)$$

Combination of (2.19) and (2.21) is a contradiction to (2.14), and the proof is completed.

Remark 2.1 *First if J has unique positive minimizer ϕ for some frequency ω , then, by Lemma 2.3, the ground set can be characterized as*

$$S = \{e^{i\theta}\phi \mid \theta \in \mathbb{R}\},$$

and we have “real” orbital stability: For every given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\inf_{\theta \in \mathbb{R}} (\|U - e^{i\theta}\phi\|_H + \|V - i\omega e^{i\theta}\phi\|_2) \leq \delta$$

implies

$$\inf_{\theta \in \mathbb{R}} (\|u(t) - e^{i\theta}\phi\|_H + \|u_t(t) - i\omega e^{i\theta}\phi\|_2) \leq \epsilon \quad \forall t \geq 0.$$

*For $\Omega = \mathbb{R}^n$, the minimization problem (2.5) usually is either **NOT** defined or only has trivial solution. So minimization is taken over some hypersurface in H . In this case, the NLKG or NLS may not possess standing waves for all frequency ω , even they have standing waves for some frequency, the standing waves may not necessarily be orbitally stable [9, 18, 19, 44, 51].*

2.4 Outline for Schrödinger Equation

In this section, we illustrate the proof for the orbital stability for standing waves of NLS. The associated nonlinear elliptic equation used to seek the standing waves is

$$\begin{cases} -\Delta\phi - \omega\phi + f(x, \phi) = 0, \\ \phi = 0 \quad \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \end{cases} \quad (2.22)$$

By same argument, we see that the nonlinear elliptic equation (2.22) has non-negative ground state $\phi_\omega \geq 0$ for all frequency ω . Let

$$S = \{\phi \in H \mid J(\phi) = d\}$$

where

$$d = \inf_{\phi \in H} J(\phi) \equiv \frac{1}{2} \int (|\nabla\phi|^2 - \omega|\phi|^2) + \int F(x, \phi).$$

We have the stability theorem.

Theorem 2.2 *The standing waves of NLS of frequency ω are orbitally stable in the following sense: given $\epsilon > 0$, there exists a $\delta > 0$ such that if*

$$\inf_{\phi \in S} \|U - \phi\|_H < \delta,$$

then

$$\inf_{\phi \in S} \|u(t) - \phi\|_H < \epsilon \quad \forall t \geq 0.$$

The proof for this theorem is similar to that for stability of standing waves of NLKG.

Here we can use

$$\begin{aligned} \tilde{Q}(t) &= \int |u(t)|^2, \\ \tilde{E}(u(t)) &= \frac{1}{2} \int |\nabla u|^2 + \int F(\cdot, u) \end{aligned}$$

to replace E and Q for NLKG. The details are omitted.

Chapter 3

Orbital Stability with Indefinite Energy

3.1 Introduction

In this chapter, let Ω be $\mathbb{R}^n$ or a bounded domain or a compact manifold. We shall give a sharp condition for orbital stability of standing waves of nonlinear Klein-Gordon equation(NLKG)

$$\begin{cases} u_{tt} - \Delta u + m(x)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \quad u_t(0, x) = V(x), \end{cases} \quad (3.1)$$

and nonlinear Schrödinger equation(NLS)

$$\begin{cases} iu_t - \Delta u + m(x)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \end{cases} \quad (3.2)$$

where m is a real bounded function such that the lower bound λ_1 of the spectrum of the operator $-\Delta + m$ is positive. In this chapter, the nonlinear interaction f is very different from that in last chapter, consequently new techniques must be employed. Again $f(x, u) = g(x, |u|)u$ and is imposed on the following conditions:

- (f₁) $f \in C^1(\Omega \times \mathbb{R})$ and $f'_u(x, s)$ and $f(x, s) \rightarrow 0$ uniformly as $s \rightarrow 0$.
- (f₂) There exist constants $2 < l < 2n/(n-1)$ and C such that $|f'_u(x, s)| \leq Cs^{l-2}$ for large $s > 0$ and for all $x \in \Omega$.
- (f₃) $f'_u(x, s) < 0$ for a.e. $x \in \Omega$ and all $s > 0$ and there exists a constant $\theta > 1$ such that $sf'_u(x, s) \leq \theta f(x, s)$ for all $x \in \Omega$ and all $s > 0$.

Remark 3.1 *It follows from the assumptions (f₁)-(f₃) that the following statements are true:*

1. $F(x, s) = \int_0^s f(x, \tau) d\tau \leq 0$ for all $x \in \Omega$ and all $s > 0$.
2. $sf(x, s) \leq 0$ for all $x \in \Omega$ and all $s > 0$;
3. For any given $x \in \Omega$;

$$F(x, s) - \frac{1}{\theta + 1} sf(x, s)$$

is a non-decreasing non-negative function of s on $(0, \infty)$.

4. For any non-negative $v \in H$,

$$\int v(x) f(x, v(x)) = 0 \iff v(x) = 0.$$

We will carry out the detailed proof for NLKG. The proof for NLS is similar and will be outlined at the end of this chapter.

As in last chapter, search for standing waves of NLKG (3.1) leads to the following nonlinear elliptic equation

$$\begin{cases} -\Delta \phi - (m(x) - \omega^2) \phi + f(x, \phi) = 0, \\ \phi = 0 \text{ on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset. \end{cases} \quad (3.3)$$

If we define

$$d(\omega) = \frac{1}{2} \int |\nabla \phi_\omega|^2 + \frac{1}{2} \int (m(x) - \omega^2) |\phi_\omega|^2 + \int F(x, \phi_\omega)$$

where ϕ_ω is a least energy solution among all solutions of (3.3), then we have the following main result:

Theorem 3.1 *If $d''(\omega_0) > 0$ and $\omega_0^2 < \lambda_1$, then the standing waves of frequency ω_0 are orbitally stable.*

It should be pointed out that although our main result is similar to that in [44], our result applies to both $\Omega = \mathbb{R}^n$ and $\Omega = \text{compact manifold or bounded domain}$, and our method allows nonlinearity f to depend on space variable x as well as on u .

3.2 Least Energy Solution

In this section, we shall prove the existence of a positive least energy solution of (3.3).

Since we include the case $\Omega = \mathbb{R}^n$, we need to redefine Hilbert spaces H and L^2 as follows

$$H = \begin{cases} H_r^1(\mathbb{R}^n) & \text{if } \Omega = \mathbb{R}^n, \\ H_0^1(\Omega) & \text{if } \Omega \text{ is a bounded domain,} \\ H^1(\Omega) & \text{if } \Omega \text{ is a compact manifold.} \end{cases}$$

$$L^2 = \begin{cases} L_r^2(\mathbb{R}^n) & \text{if } \Omega = \mathbb{R}^n, \\ L^2(\Omega) & \text{if } \Omega \text{ is a bounded domain or a compact manifold.} \end{cases}$$

where subscript r indicates that the corresponding function space consists of only radially symmetric functions. When $\Omega = \mathbb{R}^n$, we also assume that $m(x) = m(|x|)$ and $f(x, u) = f(|x|, u)$.

It is easy to see that every solution $\phi \in H$ of equation (3.3) is a critical point of energy functional

$$J_\omega(\phi) = \frac{1}{2} \int (|\nabla \phi|^2 + (m(x) - \omega^2) |\phi|^2) + \int F(x, \phi).$$

It is also easy to verify that every solution of (3.3) satisfies the functional identity

$$K_\omega(\phi) \equiv \int (|\nabla \phi|^2 + (m(x) - \omega^2) |\phi|^2) + \int |\phi| f(x, |\phi|) = 0.$$

Therefore, it is natural to search the nontrivial least energy solution by solving the following minimization problem.

$$d(\omega) = \inf_{\phi \in M_\omega} J_\omega(\phi). \quad (3.4)$$

where surface $M_\omega = \{\phi \in H \mid K_\omega(\phi) = 0, \phi \neq 0\}$.

Indeed we will show that for every $\omega^2 < \lambda_1$, $d(\omega)$ is achieved at some nontrivial ϕ and all minimizer of (3.4) are least energy solutions of equation (3.3).

First we define functional

$$\begin{aligned} I_\omega(\phi) &= J_\omega(\phi) - \frac{1}{\theta + 1} K_\omega(\phi) \\ &= \frac{\theta - 1}{2(\theta + 1)} \int (|\nabla \phi|^2 + (m(x) - \omega^2) |\phi|^2) \\ &\quad + \int \left(F(x, |\phi|) - \frac{1}{\theta + 1} |\phi| f(x, |\phi|) \right), \end{aligned}$$

and set

$$M_\omega^- = \{\phi \in H \mid K_\omega(\phi) \leq 0, \phi \neq 0\}.$$

Next we give several lemmas to lay foundation for existence theorem of ground states. The first lemma is about equivalent H -norm

Lemma 3.1 *Let $\mu < \lambda_1$, define*

$$B(\mu) = \inf_{v \in H} \left\{ \int (|\nabla v|^2 + (m(x) - \mu) |v|^2), \|v\|_H = 1 \right\},$$

then $B(\mu)$ is a positive decreasing function of μ .

Proof. $B(\mu)$ is a decreasing function since the integral is a decreasing function of μ .

For positiveness, we prove by contradiction. For $\mu < \lambda_1$, suppose that there exists a sequence $\{v_k\}$ such that

$$\|v_k\|_H^2 = \int (|\nabla v_k|^2 + |v_k|^2) = 1, \quad (3.5)$$

$$(\lambda_1 - \mu) \int |v_k|^2 \leq \int (|\nabla v_k|^2 + (m(x) - \mu) |v_k|^2) \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (3.6)$$

From (3.6), $|v_k|_2 \longrightarrow 0$ as $k \longrightarrow \infty$. By boundedness of m and by second part of (3.6), we obtain

$$\int |\nabla v_k|^2 \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (3.7)$$

Therefore a combination of (3.7) and $|v_k|_2 \longrightarrow 0$ leads to a contradiction of (3.5), which completes the proof.

Next let us prove that minimization problem (3.4) is equivalent to a very useful minimization problem.

Lemma 3.2 *For any $\omega^2 < \lambda_1$, M_ω and M_ω^- are non-empty, and*

$$d(\omega) = \inf_{\phi \in M_\omega^-} I_\omega(\phi).$$

Furthermore, $I_\omega(\phi) > d(\omega)$ if $K_\omega(\phi) < 0$.

Proof. It easily follows from assumptions (f₁) and (f₃) imply that M_ω^- is non-empty for all ω . The non-emptiness of M_ω is a consequence of the following arguments. Consider any function $v \in H$ such that $K_\omega(v) < 0$. Let $v_\alpha(x) = \alpha v(x)$, then

$$K_\omega(v_\alpha) = \frac{\alpha^2}{2} \int (|\nabla v|^2 + (m - \omega^2) |v|^2) + \alpha \int |v| f(x, \alpha |v|).$$

Now for $\alpha = 1$, $K_\omega(v_1) = K_\omega(v) < 0$ and for α close to zero $K_\omega(v_\alpha) > 0$. Therefore there exists an $\alpha_0 \in (0, 1)$ such that $K_\omega(v_{\alpha_0}) = 0$. Remark 3.1 and definition of I_ω imply that

$$\forall v \in H, I_\omega(sv) \text{ is an increasing function of } s \text{ on } (0, \infty),$$

which yields

$$d(\omega) \leq I_\omega(v_{\alpha_0}) = I_\omega(\alpha_0 v) < I_\omega(v).$$

Hence we get

$$d(\omega) \leq \inf_{\phi \in M_\omega^-} I_\omega(\phi).$$

But by definition

$$d(\omega) = \inf_{\phi \in M_\omega} J_\omega(\phi) = \inf_{\phi \in M_\omega} I_\omega(\phi) \geq \inf_{\phi \in M_\omega^-} I_\omega(\phi).$$

which concludes our proof.

Lemma 3.3 *For $\omega^2 < \lambda_1$, M_ω is a C^1 -hypersurface in H , and both M_ω and M_ω^- are bounded away from zero.*

Proof. (f₁)-(f₃) imply that K_ω is a C^1 -functional in H which in turn implies that M_ω is C^1 hypersurface.

For any small $\epsilon > 0$, from (f₁) and (f₂) there exists a $C(\epsilon) > 0$ such that

$$|\phi| f(x, |\phi|) \geq -\epsilon |\phi|^2 - C(\epsilon) |\phi|^l. \quad (3.8)$$

Using (3.8), Lemma 3.1, Sobolev embedding theorem and L^p interpolation theorem, we have that for $\epsilon < \lambda_1 - \omega^2$

$$\begin{aligned} K_\omega(\phi) &\geq \frac{1}{2} \int (|\nabla \phi|^2 + (m - \omega^2 - \epsilon) |\phi|^2) - C(\epsilon) \int |\phi|^l \\ &\geq \frac{B(\omega^2 + \epsilon)}{2} \|\phi\|_H^2 - C(\epsilon) \int |\phi|^l \\ &\geq C_1 \|\phi\|_H^2 - C_2 \|\phi\|_H^l. \end{aligned}$$

which implies that M_ω and M_ω^- are bounded away from zero, and the proof is completed.

Remark 3.2 For $\Omega = \mathbb{R}^n$, if the mass term m and nonlinear interaction term f are independent of space variable x , then any nontrivial solution $v \in H$ of nonlinear elliptic equation (3.3) also lies on another C^1 hypersurface

$$M'_\omega = \{\phi \in H \mid K'_\omega(\phi) = 0, \phi \neq 0\},$$

where

$$K'_\omega(u) = \frac{n-2}{2} \int |\nabla u|^2 + n \int \left[\frac{1}{2}(m - \omega^2) |u|^2 + F(u) \right].$$

To prove this, we need to use the scaling property of function in $H^1(\mathbb{R}^n)$. Let $u \in H_r^1(\mathbb{R}^n)$ be a solution of (3.3). Put $u_\mu(x) = u(x/\mu)$, then

$$\begin{aligned} J_\omega(u_\mu) &= \frac{1}{2} \int |\nabla u_\mu|^2 + \frac{1}{2} \int (m - \omega^2) |u_\mu|^2 + \int F(u_\mu) \\ &= \frac{\mu^{n-2}}{2} \int |\nabla u|^2 + \frac{\mu^n}{2} \int (m - \omega^2) |u|^2 + \mu^n \int F(u). \end{aligned}$$

Since u is a solution, $d(J_\omega(u_\mu))/d\mu = 0$ at $\mu = 1$. An easy computation shows that

$$\frac{d(J_\omega(u_\mu))}{d\mu} \Big|_{\mu=1} = \frac{n-2}{2} \int |\nabla u|^2 + n \int \left[\frac{1}{2}(m - \omega^2) |u|^2 + F(u) \right].$$

Note that for $n = 1$ and $n = 2$, M'_ω is not bounded away from zero, and the minimization problem can not be defined.

Now we are ready to present our existence theorem for ground states.

Theorem 3.2 Let $\omega^2 \in (0, \lambda_1)$. Then,

1. $d(\omega)$ is positive;
2. Every minimizing sequence of problem (3.4) possesses a convergent subsequence.
In particular, $d(\omega)$ is attained at some ϕ_ω ;
3. This minimizer ϕ_ω can be chosen positive;
4. Every minimizer of problem (3.4) is a solution of equation (3.3) and is called the ground state.

Proof. Let $\{\phi_k\}$ be a minimizing sequence in M_ω for problem (3.4). Remark 3.1 and Lemma 3.1 imply that there exists a constant $C(\omega, \theta) > 0$ such that for all $\phi \in M_\omega$

$$C(\omega, \theta) \|\phi\|_H^2 \leq \frac{\theta - 1}{2(\theta + 1)} \int |\nabla \phi|^2 + (m - \omega^2) |\phi|^2 \leq I_\omega(\phi) = J_\omega(\phi), \quad (3.9)$$

which implies that ϕ_k is bounded in H . Thus by Sobolev embedding theorem (if $\Omega = \mathbb{R}^n$ we need corresponding embedding theorem developed in [50]), there exist a $\phi_0 \in H$ and a subsequence, still denoted by $\{\phi_k\}$, such that

$$\begin{aligned} \phi_k &\rightharpoonup \phi_0 \text{ weakly in } H, \\ \phi_k &\longrightarrow \phi_0 \text{ strongly in } L^p(\Omega), \\ \phi_k &\longrightarrow \phi_0 \text{ a.e. on } \Omega. \end{aligned}$$

where $2 < p < 2n/(n - 2)$ if $\Omega = \mathbb{R}^n$, $1 < p < 2n/(n - 2)$ otherwise.

Next we want to get strong convergence of sequence $\{\phi_k\}$. To that end, let $0 < \sigma = \frac{1}{2}(\lambda_1 - \omega^2)$ and rewrite $K_\omega(\phi)$ as follows:

$$K_\omega(\phi) = S(\phi) + P(\phi) \quad (3.10)$$

with

$$S(\phi) = \int |\nabla \phi|^2 + \int (m - \omega^2 - \sigma) |\phi|^2,$$

and

$$P(\phi) = \sigma \int |\phi|^2 + \int |\phi| f(x, |\phi|).$$

By Lemma 3.1, $\sqrt{S(\phi)}$ makes an equivalent norm on Hilbert space H , therefore after selecting another subsequence of $\{\phi_k\}$, we get that $\phi_k \rightharpoonup \phi_0$ weakly in H under the new norm, and by weak lower semicontinuity of the norm $\sqrt{S(\cdot)}$, we have

$$\liminf_{k \rightarrow \infty} \int |\nabla \phi_k|^2 + \int (m - \omega^2 - \sigma) |\phi_k|^2 \geq \int |\nabla \phi_0|^2 + \int (m - \omega^2 - \sigma) |\phi_0|^2. \quad (3.11)$$

Without loss of generality, we assume the existences of $\lim \int |\phi_k|^2$ and $\lim P(\phi_k)$. In (3.8), choose $0 < \epsilon < \sigma$, we have that for some positive constants C_1 and C_2

$$P(\phi_k) \geq C_1 \int |\phi_k|^2 - C_2 \int |\phi_k|^l,$$

which by Fatou's lemma implies that

$$\lim_{k \rightarrow \infty} \left[P(\phi_k) + C_2 \int |\phi_k|^l \right] \geq P(\phi_0) + C_2 \int |\phi_0|^l.$$

Since $\phi_k \rightharpoonup \phi_0$ strongly in L^p for $2 < p < 2n/(n-2)$, we immediately obtain

$$\lim_{k \rightarrow \infty} P(\phi_k) \geq P(\phi_0). \quad (3.12)$$

(3.11), (3.12) and Remark 3.1 yield that

$$I_\omega(\phi_0) \leq \liminf_{k \rightarrow \infty} I_\omega(\phi_k) = d(\omega), \quad (3.13)$$

$$K_\omega(\phi_0) \leq \liminf_{k \rightarrow \infty} K_\omega(\phi_k) = 0, \quad (3.14)$$

A strict inequality in (3.11) would imply a strict inequality in both (3.13) and (3.14) which in turn would imply $\phi_0 \neq 0$, and thus by Lemma 3.2 would generate a contradiction

$$d(\omega) < I_\omega(\phi_0) < d(\omega).$$

Therefore we obtain the strong convergence of $\{\phi_k\}$ to ϕ_0 under the equivalent norm which implies the strong convergence under the original norm $\|\cdot\|$.

Lemma 3.3 and (3.9) show that $\phi_0 \neq 0$ and $d(\omega) > 0$.

The existence of positive minimizer ϕ_ω follows from $J_\omega(|\phi|) \leq J_\omega(\phi)$, $K_\omega(|\phi|) \leq K_\omega(\phi)$, Lemma 3.2 and strong maximal principle.

Finally to show that ϕ_0 is a solution of equation (3.3), we have by Lagrange multiplier method

$$\delta J_\omega(\phi_0) = \lambda \delta K_\omega(\phi_0),$$

or

$$\begin{aligned} & -\Delta\phi_0 + (m(x) - \omega^2)\phi_0 + f(x, \phi_0) \\ &= \lambda [-2\Delta\phi_0 + 2(m - \omega^2)\phi_0 + f(x, \phi_0) + \phi_0 f'(x, |\phi_0|)]. \end{aligned}$$

Taking inner product with ϕ_0 on both sides and using $K_\omega(\phi_0) = 0$ lead to

$$0 = \lambda \int (|\nabla\phi_0|^2 + (m(x) - \omega^2)|\phi_0|^2 + |\phi_0|^2 f'(x, |\phi_0|)).$$

Using $K_\omega(\phi_0) = 0$ again, we obtain

$$0 = \lambda \int |\phi_0| (f(x, |\phi_0|) - |\phi_0|^2 f'(x, |\phi_0|)). \quad (3.15)$$

From (f₃), it follows that

$$\begin{aligned} & \int |\phi_0| (f(x, |\phi_0|) - |\phi_0|^2 f'(x, |\phi_0|)) \\ & \geq (1 - \theta) \int |\phi_0| f(x, |\phi_0|) \\ & = (\theta - 1) \int [|\nabla\phi_0|^2 + (m - \omega^2)|\phi_0|^2] > 0, \end{aligned} \quad (3.16)$$

which implies that $\lambda = 0$ and the proof is completed.

Corollary 3.1 *Every minimizing sequence of the minimization problem*

$$\inf_{\phi \in M_\omega^-} I_\omega(\phi), \quad (3.17)$$

has a subsequence converging to a $\phi_\omega \in M_\omega$. In particular, ϕ_ω is also a minimizer of minimization problem (3.4).

We conclude this section with the definition of the set of ground states

$$S_\omega = \{\phi \in M_\omega \mid J_\omega(\phi) = d(\omega)\}.$$

3.3 Standing Wave as a Function of Frequency

In this section, we prove that standing waves are smooth functions of frequency.

Lemma 3.4 $d(\omega)$ and $\|\phi_\omega\|_H$ are uniformly bounded for ω^2 on compact subsets of $(0, \lambda_1)$.

Proof. The uniform boundedness of $d(\omega)$ follows from the fact that given $\omega_0 \in (0, \lambda_1)$, there exists $\phi_0 \in M_\omega^-$, hence there exists an $\epsilon > 0$ such that

$$K_\omega(\phi_0) < 0 \text{ for } \omega \in (\omega_0 - \epsilon, \omega_0 + \epsilon),$$

from which and Lemma 3.2 it follows that

$$d(\omega) \leq I_\omega(\phi_0) \leq C \text{ for } \omega \in (\omega_0 - \epsilon, \omega_0 + \epsilon),$$

By Remark 3.1 and Lemma 3.1, we have

$$\begin{aligned} d(\omega) &= J_\omega(\phi_\omega) = I_\omega(\phi_\omega) \geq \frac{\theta - 1}{2(\theta + 1)} \int [|\nabla \phi_\omega|^2 + (m - \omega^2)|\phi_\omega|^2] \\ &\geq \frac{B(\omega^2)(\theta - 1)}{2(\theta + 1)} \|\phi_\omega\|_H^2, \end{aligned} \quad (3.18)$$

which implies the uniform boundedness of $\|\phi_\omega\|_H$.

Lemma 3.5 $d(\omega)$ is a decreasing and continuous function of ω for $\omega \in (0, \sqrt{\lambda_1})$.

Proof. Let $0 < \omega_1 < \omega_2 < \sqrt{\lambda_1}$ and $d(\omega_1) = J_{\omega_1}(\phi_{\omega_1})$, then

$$K_{\omega_2}(\phi_{\omega_1}) = K_{\omega_1}(\phi_{\omega_1}) - \frac{1}{2}(\omega_2^2 - \omega_1^2) \int |\phi_{\omega_1}|^2 < 0.$$

Therefore by Lemma 3.2 we have

$$d(\omega_2) \leq I_{\omega_2}(\phi_{\omega_1}) < I_{\omega_1}(\phi_{\omega_1}) = d(\omega_1).$$

which concludes the proof for monotonicity of d .

Next to show continuity, let $\omega_0 \in (0, \sqrt{\lambda_1})$, and we will show d is left and right continuous at ω_0 .

For left continuity, let $0 < \omega < \omega_0$ and $v_\alpha(x) = \alpha\phi_{\omega_0}(x)$, then

$$g(\omega, \alpha) \equiv K_\omega(v_\alpha)$$

is a smooth function of α and ω . Moreover, we have

$$g(\omega_0, 1) = 0,$$

and by (3.16)

$$g'_\alpha(\omega_0, 1) = \int (|\phi_{\omega_0}|^2 f'(x, \phi_{\omega_0}) - |\phi_{\omega_0}| f(x, \phi_{\omega_0})) < 0. \quad (3.19)$$

Therefore by implicit function theorem, there exists a neighborhood of ω_0 and a C^1 function $\alpha = \alpha(\omega)$ in this neighborhood such that $\alpha(\omega_0) = 1$ and $g(\omega, \alpha(\omega)) = 0$. Hence we have

$$\begin{aligned} d(\omega_0) &\leq d(\omega) \leq I_\omega(\alpha(\omega)\phi_{\omega_0}) \\ &= \alpha^2(\omega) \frac{\theta - 1}{2(\theta + 1)} (\omega_0^2 - \omega^2) \int |\phi_{\omega_0}|^2 + I_{\omega_0}(\alpha(\omega)\phi_{\omega_0}). \end{aligned}$$

Let $\omega \rightarrow \omega_0$, then $\alpha(\omega) \rightarrow 1$ and $I_{\omega_0}(\alpha(\omega)\phi_{\omega_0}) \rightarrow d(\omega_0)$. Hence $d(\omega) \rightarrow d(\omega_0)$ as $\omega \rightarrow \omega_0$, which concludes the proof for left continuity.

For right continuity, select ω_2 such that $\omega_0 < \omega_2 < \sqrt{\lambda_1}$ and let $\omega_0 < \omega \leq \omega_2$. To show that

$$\lim_{\omega \rightarrow \omega_0^+} d(\omega) = d(\omega_0).$$

It suffices to find a function $\alpha(\omega)$ such that

$$K_{\omega_0}(\alpha(\omega)\phi_\omega) = 0 \text{ and } \alpha(\omega) \rightarrow 1 \text{ as } \omega \rightarrow \omega_0^+. \quad (3.20)$$

In fact,

$$\begin{aligned} d(\omega_0) &= J_{\omega_0}(\phi_{\omega_0}) \leq J_{\omega_0}(\alpha(\omega)\phi_\omega) \\ &= J_\omega(\alpha(\omega)\phi_\omega) + \frac{\alpha^2(\omega)}{2} (\omega^2 - \omega_0^2) \int \phi_\omega^2 \\ &= d(\omega) + [J_\omega(\alpha(\omega)\phi_\omega) - J_\omega(\phi_\omega)] + \frac{\alpha^2(\omega)}{2} (\omega^2 - \omega_0^2) \int \phi_\omega^2. \end{aligned}$$

Hence $d(\omega) \rightarrow d(\omega_0)$ as $\omega \rightarrow \omega_0^+$, since $\|\phi_\omega\|_H$ is uniformly bounded on $[\omega_0, \omega_2]$.

Now we come back to find $\alpha(\omega)$ satisfying (3.20). Set

$$\begin{aligned} g(\omega, \alpha) &= \frac{K_{\omega_0}(\alpha\phi_\omega)}{\alpha^2} \\ &= \int |\nabla\phi_\omega|^2 + (m - \omega_0^2)\phi_\omega^2 + \frac{1}{\alpha} \int \phi_\omega f(\cdot, \alpha\phi_\omega), \end{aligned}$$

where $\omega \in [\omega_0, \omega_2]$ and $\alpha > 0$.

Note that we can write

$$g(\omega, \alpha) = h(\omega, \alpha) + Q(\omega)$$

with

$$\begin{aligned} h(\omega, \alpha) &= \int |\nabla\phi_\omega|^2 + (m - \omega^2)\phi_\omega^2 + \frac{1}{\alpha} \int \phi_\omega f(\cdot, \alpha\phi_\omega), \\ Q(\omega) &= (\omega^2 - \omega_0^2) \int \phi_\omega^2. \end{aligned}$$

It is easy to see that $h(\omega, 1) = 0$, $Q(\omega) > 0$ for $\omega > \omega_0$ and that $Q(\omega) \rightarrow 0$ as $\omega \rightarrow \omega_0$ by uniform boundedness of $\|\phi_\omega\|_H$. We can find the derivative of h with respect to α as follows:

$$h'_\alpha(\omega, \alpha) = \frac{1}{\alpha^3} \int [(\alpha\phi_\omega)^2 f'_u(\cdot, \alpha\phi_\omega) - \alpha\phi_\omega f(\cdot, \alpha\phi_\omega)].$$

Hence $h(\omega, \alpha) = h_\alpha(\omega, \beta)(\alpha - 1)$ for some β between α and 1. Using (f₃), $K_\omega(\phi_\omega) = 0$, Remark 3.1 and Theorem 3.2 we have

$$h_\alpha(\omega, \beta) \leq \frac{\theta - 1}{\beta^3} \int \beta\phi_\omega f(x, \beta\phi_\omega) < 0. \quad (3.21)$$

Therefore we can find an $\alpha = \alpha(\omega)$ satisfying

$$g(\omega, \alpha) = (\alpha - 1)h_\alpha(\omega, \beta) + Q(\omega) = 0,$$

i.e.,

$$\alpha = 1 - \frac{Q(\omega)}{h_\alpha(\omega, \beta)} > 1. \quad (3.22)$$

Note that by Remark 3.1, we have

$$d(\omega) = J_\omega(\phi_\omega) \leq \frac{1}{2} \int (|\nabla\phi_\omega|^2 + (m - \omega^2)\phi_\omega^2). \quad (3.23)$$

Using (3.21), (3.22), Remark 3.1, $K_\omega(\phi_\omega) = 0$ and (3.23) we arrive at

$$\begin{aligned}
 h_\alpha(\omega, \beta) &\leq \frac{\theta-1}{\beta^3} \int \beta \phi_\omega f(x, \beta \phi_\omega) \\
 &\leq \frac{\theta-1}{\alpha^3} \int \phi_\omega f(x, \phi_\omega) \\
 &= -\frac{\theta-1}{\alpha^3} \int (|\nabla \phi_\omega|^2 + (m - \omega^2) \phi_\omega^2) \\
 &\leq \frac{2(1-\theta)d(\omega)}{\alpha^3} \leq \frac{2(1-\theta)d(\omega_2)}{\alpha^3}.
 \end{aligned} \tag{3.24}$$

Since $Q(\omega) \rightarrow 0$ as $\omega \rightarrow \omega_0$, from (3.22) and (3.24) it follows that to show that $\alpha(\omega) \rightarrow 1$ as $\omega \rightarrow \omega_0^+$, we only need to show that α is uniformly bounded on $[\omega_0, \omega_2]$. Suppose that this is not true, by uniform boundedness of ϕ_ω on $[\omega_0, \omega_2]$, there exist a sequence $\{\omega_k\} \in [\omega_0, \omega_2]$ such that

$$\begin{aligned}
 \omega_k &\rightarrow \tilde{\omega} \in [\omega_0, \omega_2], \\
 \alpha &= \alpha(\omega_k) \rightarrow \infty, \\
 \phi_k &= \phi_{\omega_k} \rightharpoonup \tilde{\phi} \in H \text{ weakly,}
 \end{aligned}$$

and

$$\frac{1}{\alpha_k^2} K_{\omega_0}(\alpha_k \phi_k) = \int |\nabla \phi_k|^2 + (m - \omega_0) \phi_k^2 + \frac{1}{\alpha_k} \int \phi_k f(x, \alpha_k \phi_k) = 0. \tag{3.25}$$

Notice on one hand, by uniform boundedness of $\|\phi_k\|_H$, we have

$$\limsup_{k \rightarrow \infty} \int [|\nabla \phi_k|^2 + (m - \omega_0^2) \phi_k^2] < \infty, \tag{3.26}$$

On the other hand, we define for $s \geq 1$

$$G(s) = \frac{1}{s} \int \phi_\omega f(x, s \phi_\omega), \tag{3.27}$$

then

$$\begin{aligned}
 G'(s) &= \frac{1}{s^2} \int [s \phi_\omega^2 f'(x, s \phi_\omega) - \phi_\omega f(x, s \phi_\omega)] \\
 &\leq \frac{1}{s^2} \int (\theta - 1) \phi_\omega f(x, s \phi_\omega) \quad (\text{by } f_2) \\
 &= \frac{\theta - 1}{s} G(s).
 \end{aligned}$$

Therefore we have

$$\frac{G'(s)}{G(s)} \geq \frac{\theta - 1}{s} \quad \text{since } G(s) < 0 \quad \text{if } s \geq 1,$$

from which it follows that

$$G(s) \leq G(1)S^{\theta-1}. \quad (3.28)$$

Next let us estimate $G(1)$. By $K_\omega(\phi_\omega) = 0$ we have

$$\begin{aligned} G(1) &= \int \phi_\omega f(x, \phi_\omega) \\ &= - \int (|\phi_\omega|^2 + (m - \omega^2)\phi_\omega^2) \\ &\leq -2d(\omega_2) < 0. \end{aligned} \quad (3.29)$$

Therefore (3.27) (3.28) and (3.29) yield

$$\liminf_{k \rightarrow \infty} \frac{1}{\alpha_k} \int \phi_k f(x, \alpha_k \phi_k) = -\infty.$$

which combined with (3.26) is a contradiction to (3.25), hence the proof is concluded.

If we assume the minimization problem (3.4) has a unique positive solution, then we have the following theorem regarding the continuity of ϕ_ω on ω .

Theorem 3.3 *For ω near ω_0 , let ϕ_ω be the unique positive solution of problem (3.4). Suppose that zero is not an eigenvalue of the linearized operator $\mathcal{L}_0 = -\Delta + m - \omega_0^2 + f'(\cdot, \phi_{\omega_0})$ at ϕ_{ω_0} acting on L^2 (real valued). If f is a C^1 function, then $\omega \rightarrow \phi_\omega$ is a C^1 mapping from a small neighborhood of ω_0 into H .*

We carry out the proof of this theorem in two lemmas following the same procedures as in [45].

Lemma 3.6 $\omega \rightarrow \phi_\omega$ is continuous with values in H .

Proof. From Lemma 3.4 and 3.5, $d(\omega) = I_\omega(\phi_\omega)$ is continuous in ω and $\|\phi_\omega\|_H$ is a bounded function of ω . Let $\{\omega_k\}$ be a sequence tending to ω_0 . Then $\{\phi_{\omega_k}\}$ is bounded in H . A subsequence may be chosen converging weakly in H to some v . Note $v \geq 0$ since each ϕ_{ω_k} is positive and $\phi_{\omega_k} \rightarrow v$ a.e. on Ω . Now

$$0 = K_\omega(\phi_\omega) = \int (|\nabla \phi_\omega|^2 + (m - \omega^2)\phi_\omega^2 + \phi_\omega f(\cdot, \phi_\omega)), \quad (3.30)$$

Letting $\omega = \omega_k \rightarrow \omega_0$, by uniform boundedness of ϕ_{ω_k} , continuity of K and d and lower semicontinuity of weak limits, we have

$$K_{\omega_0}(v) \leq 0,$$

and

$$I_{\omega_0}(v) \leq \liminf_{k \rightarrow \infty} I_{\omega_k}(\phi_{\omega_k}) = d(\omega_0).$$

By similar arguments in the proof of Theorem 3.2, we have $K_{\omega_0}(v) = 0$, $I_{\omega_0}(v) = d(\omega_0)$ and $\phi_{\omega_k} \rightarrow v$ strongly in H . Then by uniqueness, $v = \phi_{\omega_0}$, which completes the proof.

Lemma 3.7 *In a neighborhood in H of ϕ_{ω_0} , all solutions of (3.3) lie on a C^1 curve.*

Proof. Write (3.3) as

$$-\Delta\phi + m(x)\phi + \tau\phi + f(x, \phi) = 0, \quad (3.31)$$

where $\tau = -\omega^2$. Let $\tau_0 = -\omega_0^2$, $\phi_0 = \phi_{\omega_0}$ and let

$$\mathcal{L}(\tau, v) = v + (m - \Delta + \tau)^{-1}f(\cdot, v), \quad \tau > -\lambda_1, \quad v \in H. \quad (3.32)$$

Then $\mathcal{L}(\tau, v) \in H$ since $v \in H \subset L^{2n/(n-2)}$, $f(\cdot, v) \in L^{2n/(n+2)}$ by (f_1) and (f_2) , and therefore $(\tau + m - \Delta)^{-1}f(\cdot, v) \in H$ by elliptic theory. In fact, $\mathcal{L}(\tau, v)$ is a C^1 operator from $(-\lambda_1, \infty) \times H$ into H . Note $\mathcal{L}(\tau_0, \phi_0) = 0$. Now the operator $\mathcal{L}_0 = -\Delta + m - \omega_0^2 + f'(\cdot, \phi_{\omega_0})$ is invertible by assumption. It follows that the compact operator $(\tau_0 + m - \Delta)^{-\frac{1}{2}}f'(\cdot, \phi_0)(\tau_0 + m - \Delta)^{-\frac{1}{2}}$ on L^2 does not have -1 in its spectrum. Hence

$$\frac{\partial \mathcal{L}}{\partial v}(\tau_0, \phi_0) = I + (\tau_0 + m - \Delta)^{-1}f'(\cdot, \phi_0),$$

acting from H to H , is invertible. By implicit function theorem, the solutions of $\mathcal{L}(\tau, v) = 0$ in a neighborhood of (τ_0, ϕ_0) form a C^1 curve in $(-\lambda_1, \infty) \times H$.

With these preparations, we can go on to find the derivative of d , which concludes this section.

Lemma 3.8 *Under the conditions specified in Theorem 3.3, we have*

$$d'(\omega) = -\omega \int |\phi_\omega|^2.$$

Proof. From

$$d(\omega) = J_\omega(\phi_\omega) = \frac{1}{2} \int (|\nabla \phi_\omega|^2 + (m - \omega^2) |\phi_\omega|^2) + \int F(x, \phi_\omega),$$

we have

$$d'(\omega) = \int (-\Delta \phi_\omega + (m - \omega^2) \phi_\omega + f(x, \phi_\omega)) \frac{\partial \phi_\omega}{\partial \omega} - \omega \int |\phi_\omega|^2. \quad (3.33)$$

The first integral in (3.33) is zero since ϕ_ω is a solution of equation (3.3), and we proved the lemma.

3.4 Stability of Standing Waves

We consider NLKG

$$\begin{cases} u_{tt} - \Delta u + m(x)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x) \in H, \quad u_t(0, x) = V(x) \in L^2. \end{cases} \quad (3.34)$$

For $\Omega = \mathbb{R}^n$ it is shown in [15, 16] that strong solutions $u(\cdot) \in C([0, T], H)$, $u_t(\cdot) \in ([0, T], L^2)$ exist for nonlinear interaction we are considering. For other cases of Ω , it is shown in [41, 50] that weak solutions exist and for these solutions energy inequality holds. In this dissertation, we will only consider the weak solutions of NLKG or NLS. The proof of stability for strong solution is relatively easier.

Let

$$E_\omega(u, v) = \frac{1}{2} \int |v|^2 + J_\omega(u).$$

Define

$$R_\omega = \{(u, v) \in H \oplus L^2 \mid E_\omega(u, v) < d(\omega)\}.$$

Next we introduce two invariant sets which plays a very important role in the establishment of stability.

$$R_\omega^1 = \{(u, v) \in R_\omega \mid K_\omega(u) > 0\} \cup \{(0, v) \in R_\omega\},$$

$$R_\omega^2 = \{(u, v) \in R_\omega \mid K_\omega(u) < 0\}.$$

It is easy to prove that we have the following equivalent expressions

$$R_\omega^1 = \{(u, v) \in R_\omega \mid I_\omega(u) < d(\omega)\},$$

$$R_\omega^2 = \{(u, v) \in R_\omega \mid I_\omega(u) > d(\omega)\}.$$

Lemma 3.9 R_ω^1 and R_ω^2 are invariant regions under the solution flow of the following modulated equation

$$\begin{cases} u_{tt} + 2i\omega u_t - \Delta u + (m(x) - \omega^2)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x) \in H, \quad u_t(0, x) = V(x) \in L^2. \end{cases} \quad (3.35)$$

Proof. Let $(U, V) \in R_\omega^1$ and assume that there exists a τ such that $(u(\tau), u_t(\tau)) \notin R_\omega^1$. Then $u(\tau) \neq 0$ and $K_\omega(u(\tau)) \leq 0$, i.e., $u(\tau) \in M_\omega^-$. Let

$$s = \inf \{0 \leq t \leq \tau \mid (u(t), u_t(t)) \notin R_\omega^1\}, \quad (3.36)$$

then $K_\omega(u(t)) \geq 0$ for all $0 < t < s$. Let $\{s_k\}$ be the minimizing sequence for problem (3.36), then arguing similarly as in the proof of Theorem 3.2 we have

$$K_\omega(u(s)) \leq \liminf_{k \rightarrow \infty} K_\omega(u(s_k)) \leq 0.$$

Note $u(s) = 0$ would imply that $K_\omega(u(s)) = 0$ which in turn, would imply $u(s_k) \rightarrow u(s)$ strongly in H . Then Lemma 3.3 and $u(s_k) \in M_\omega^-$ would imply $u(s) \neq 0$ which contradicts the original assumption. Hence we have

$$K_\omega(u(s)) \leq 0 \text{ and } u(s) \neq 0. \quad (3.37)$$

On the other hand

$$\begin{aligned} I_\omega(u(s)) &= \liminf_{t \rightarrow s^-} I_\omega(u(t)) \\ &\leq \liminf_{t \rightarrow s^-} \left(I_\omega(u(t)) + \frac{1}{\theta + 1} K_\omega(u(t)) \right) \\ &\leq \liminf_{t \rightarrow s^-} E_\omega(u(t), u_t(t)) < d(\omega), \end{aligned}$$

which, in view of inequality (3.37), contradicts Lemma 3.2 and completes the proof for the invariance of R_ω^1 .

To show the invariance of R_ω^2 , we just need to switch the roles of I_ω and K_ω . Let $(U, V) \in R_\omega^2$ and assume that there exists a τ such that $(u(\tau), u_t(\tau)) \notin R_\omega^2$, i.e., $I_\omega(u(\tau)) \leq d(\omega)$. Let

$$s = \inf \{0 \leq t \leq \tau \mid (u(t), u_t(t)) \notin R_\omega^2\}, \quad (3.38)$$

then by weak lower semicontinuity $I_\omega(u(s)) \leq d(\omega)$ and $I_\omega(u(t)) > d(\omega)$ for all $0 < t < s$. On the other hand

$$\begin{aligned} K_\omega(u(s)) &= \liminf_{t \rightarrow s^-} (\theta + 1) [J_\omega(u(t)) - I_\omega(u(t))] \\ &\leq \liminf_{t \rightarrow s^-} (\theta + 1) [E_\omega(u(t), u_t(t)) - d(\omega)] \\ &\leq (\theta + 1) [E_\omega(U, V) - d(\omega)] < 0, \end{aligned}$$

which, in view of $I_\omega(u(s)) \leq d(\omega)$, contradicts Lemma 3.2

Lemma 3.10 Assume $d''(\omega_0) > 0$. Then there exists an $M(\omega_0) \geq 0$ such that for every $M > M(\omega_0)$ there exists a $\delta = \delta(M)$ such that if $u(t)$ is a weak solution of NLKG equation (3.34) with initial data satisfying

$$\|U - \phi_{\omega_0}\|_H + \|V - i\omega_0\phi_{\omega_0}\|_2 < \delta,$$

then

$$d(\omega_+) \leq I_{\omega_\pm}(u(t)) \leq d(\omega_-) \quad \forall t > 0, \quad (3.39)$$

and

$$\frac{1}{2} \int |u_t(t) - i\omega_\pm u(t)|^2 + J_\pm(u(t)) < d(\omega_\pm) \quad \forall t > 0, \quad (3.40)$$

where $\omega_\pm = \omega_0 \pm 1/M$.

Proof. Set $v_\pm(t) = e^{-i\omega_\pm t} u(t)$. Then $v_\pm$ satisfies

$$\begin{cases} v_{\pm tt} + 2i\omega_\pm v_{\pm t} - \Delta v_\pm + (m(x) - \omega_\pm^2)v_\pm + f(x, v_\pm) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_\pm = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ v_\pm(0, x) = U(x), \quad v_{\pm t}(0, x) = V(x) - i\omega_\pm U. \end{cases} \quad (3.41)$$

Note

$$J_{\omega_{\pm}}(u) = J_{\omega_{\pm}}(v_{\pm}),$$

$$\int |v_{\pm t}|^2 = \int |u_t(t) - i\omega_{\pm}u(t)|^2.$$

The energy inequality of modulated equation (3.41) becomes

$$\frac{1}{2} \int |v_{\pm t}(t)|^2 + J_{\omega_{\pm}}(u(t)) \leq \frac{1}{2} \int |V - i\omega_{\pm}U|^2 + J_{\omega_{\pm}}(U). \quad (3.42)$$

To show (3.39) and (3.40), by invariance of $R_{\omega_{\pm}}^1$ and $R_{\omega_{\pm}}^2$ under solution flow of modulated equation (3.41) and by energy inequality (3.42), it is sufficient to prove that

$$d(\omega_+) < I_{\omega_{\pm}}(U) < d(\omega_-), \quad (3.43)$$

and

$$E_{\omega_{\pm}}(U, V - i\omega_{\pm}V) < d(\omega_{\pm}). \quad (3.44)$$

We first prove (3.43). Note

$$I_{\omega_{\pm}}(U) = I_{\omega_{\pm}}(\phi_{\omega_0}) + O(\delta),$$

therefore δ can be chosen if

$$d(\omega_+) < I_{\omega_{\pm}}(\phi_{\omega_0}) < d(\omega_-).$$

Set $a = \frac{\theta-1}{2(\theta+1)} < \frac{1}{2}$. It is obvious to see that

$$I_{\omega_+}(\phi_{\omega_0}) = I_{\omega_0}(\phi_{\omega_0}) + a(\omega_0^2 - \omega_+^2) \int |\phi_{\omega_0}|^2$$

$$< d(\omega_0) < d(\omega_-),$$

and

$$I_{\omega_-}(\phi_{\omega_0}) = I_{\omega_0}(\phi_{\omega_0}) + a(\omega_0^2 - \omega_-^2) \int |\phi_{\omega_0}|^2$$

$$> d(\omega_0) > d(\omega_+).$$

Note

$$K_{\omega_+}(\phi_{\omega_0}) = K_{\omega_0}(\phi_{\omega_0}) + (\omega_0^2 - \omega_+^2) \int |\phi_{\omega_0}|^2 < 0,$$

hence Lemma 3.2 implies that

$$d(\omega_+) < I_{\omega_+}(\phi_{\omega_0}).$$

To see $d(\omega_-) > I_{\omega_-}(\phi_{\omega_0})$, we use $d''(\omega_0) > 0$ and $d'(\omega_0) = -\omega_0 \int |\phi_{\omega_0}|^2$ to get

$$\begin{aligned} I_{\omega_-}(\phi_{\omega_0}) &= I_{\omega_0}(\phi_{\omega_0}) + a(\omega_0^2 - \omega_-^2) \int |\phi_{\omega_0}|^2 \\ &= d(\omega_0) + a \frac{\omega_0 + \omega_-}{\omega_0} (\omega_- - \omega_0) d'(\omega_0) \\ &< d(\omega_0) + (\omega_- - \omega_0) d'(\omega_0) < d(\omega_-). \end{aligned}$$

Now we turn our attention to (3.44). It is easy to see that

$$J_{\omega_{\pm}}(U) = J_{\omega_{\pm}}(\phi_{\omega_0}) + O(\delta) = J_{\omega_0}(\phi_{\omega_0}) + \frac{\omega_0^2 - \omega_{\pm}^2}{2} \int |\phi_{\omega_0}|^2 + O(\delta), \quad (3.45)$$

and

$$\begin{aligned} \|V - i\omega_{\pm}U\|_2 &\leq \|V - i\omega_0\phi_{\omega_0}\|_2 + \|\omega_0\phi_{\omega_0} - \omega_{\pm}\phi_{\omega_0}\|_2 + \|\omega_{\pm}\phi_{\omega_0} - \omega_{\pm}U\|_2 \\ &= |\omega_0 - \omega_{\pm}| \|\phi_{\omega_0}\|_2 + O(\delta). \end{aligned} \quad (3.46)$$

Using (3.45), (3.46) and $d''(\omega_0) > 0$, and choosing δ small enough, we obtain

$$E_{\omega_{\pm}}(U, V - i\omega_{\pm}U) \leq d(\omega_0) + (\omega_{\pm} - \omega_0) d'(\omega_0) + O(\delta) < d(\omega_{\pm}).$$

which concludes our proof for the lemma.

Finally we can present our main result.

Theorem 3.4 *If $d''(\omega_0) > 0$, then the ground state standing waves of frequency ω are orbitally stable in the following sense: for every given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$\inf_{\phi \in S_{\omega_0}} (\|U - \phi\|_H + \|V - i\omega_0\phi\|_2) \leq \delta$$

implies

$$\inf_{\phi \in S_{\omega_0}} (\|u(t) - \phi\|_H + \|u_t(t) - i\omega_0\phi\|_2) \leq \epsilon \quad \text{for all } t \geq 0.$$

Proof. Suppose that standing waves of frequency ω_0 are not orbitally stable. Then there exist $\{(U_k, V_k)\}$, $\{t_k\}$, and weak solutions $\{u^k(t)\}$ and $\epsilon_0 > 0$ such that

$$\inf_{\phi \in S_{\omega_0}} (\|U_k - \phi\|_H + \|V_k - i\omega_0\phi\|_2) \longrightarrow 0, \quad (3.47)$$

and

$$\inf_{\phi \in S_{\omega_0}} (\|u^k(t_k) - \phi\|_H + \|u_t^k(t_k)\|_2) \geq \epsilon_0. \quad (3.48)$$

Since S_{ω_0} is compact in H , without loss of generality, we may assume that

$$(U_k, V_k) \longrightarrow (v, i\omega_0 v) \text{ for some } v \in S_{\omega_0}.$$

From Lemma 3.10, there a subsequence of $\{k\}$ such that

$$d(\omega_0 + 1/k) \leq I_{\omega_{\pm}}(u^k(t_k)) \leq d(\omega_0 - 1/k), \quad (3.49)$$

and

$$\|u_t^k(t_k) - i\omega_{\pm} u^k(t_k)\|_2^2 + J_{\omega_{\pm}}(u^k(t_k)) < d(\omega_0 + 1/k), \quad (3.50)$$

where $\omega_{\pm} = \omega_0 \pm 1/k$. (3.49) and (3.9) imply that $u^k(t_k)$ is bounded in H , therefore by continuity and (3.49) again

$$I_{\omega_0}(u^k(t_k)) \longrightarrow d(\omega_0). \quad (3.51)$$

From (3.50) it follows that there exists another subsequence of $\{k\}$ such that

$$J_{\omega_0}(u^k(t_k)) \longrightarrow d \leq d(\omega_0) \text{ for some } d. \quad (3.52)$$

Hence (3.52) and (3.51) yield

$$\liminf_{k \rightarrow \infty} K_{\omega_0}(u^k(t_k)) = (\theta + 1) \liminf_{k \rightarrow \infty} (J_{\omega_0}(u^k(t_k)) - I_{\omega_0}(u^k(t_k))) \leq 0. \quad (3.53)$$

(3.51) and (3.53) imply that $\{u^k(t_k)\}$ is a minimizing sequence of problem (3.17), therefore by Corollary 3.1 there exist a sequence of $\{k\}$ and a $\phi \in S_{\omega_0}$ such that

$$u^k(t_k) \longrightarrow \phi \text{ as } k \longrightarrow \infty,$$

which together with (3.50) implies that

$$u_t^k(t_k) \longrightarrow i\omega_0 \phi \text{ in } L^2.$$

Therefore we get a contradiction to (3.48).

3.5 Nonlinear Schrödinger Equation

We recall the nonlinear Schrödinger equation

$$\begin{cases} iu_t - \Delta u + m(x)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \end{cases}$$

The proof of stability of standing waves of NLS will be similar to that of stability of standing waves of NLKG which we presented in the previous sections. So we will state the relevant lemmas and theorem without proof.

The associated nonlinear elliptic equation resulted from searching the standing waves of the form $e^{i\omega t}\phi(x)$ is

$$\begin{cases} -\Delta\phi - (m(x) - \omega)\phi + f(x, \phi) = 0, \\ \phi = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \end{cases} \quad (3.54)$$

The modulated equation is

$$\begin{cases} iu_t - \Delta u + (m(x) - \omega)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \end{cases} \quad (3.55)$$

The energy for modulated equation (3.55) becomes the energy for (3.54):

$$E_\omega(u) = J_\omega(u).$$

The corresponding J , K and I functionals are

$$\begin{aligned} J_\omega(\phi) &= \frac{1}{2} \int (|\nabla\phi|^2 + (m(x) - \omega)|\phi|^2) + \int F(x, \phi), \\ K_\omega(\phi) &= \int (|\nabla\phi|^2 + (m(x) - \omega)|\phi|^2) + \int f(x, |\phi|)|\phi|, \\ I_\omega(\phi) &= \frac{\theta - 1}{2(\theta + 1)} \int (|\nabla\phi|^2 + (m(x) - \omega)|\phi|^2) \\ &\quad + \int \left(F(x, |\phi|) - \frac{1}{\theta + 1} f(x, |\phi|)|\phi| \right). \end{aligned}$$

The minimization problem is

$$d(\omega) = \inf_{\phi \in M_\omega} J_\omega(\phi),$$

where $M_\omega = \{\phi \in H \mid K_\omega(\phi) = 0, \phi \neq 0\}$.

The invariant regions R_ω^1 and R_ω^2 in H are defined as

$$\begin{aligned} R_\omega &= \{u \in H \mid E_\omega(u) < d(\omega)\}, \\ R_\omega^1 &= \{u \in R_\omega \mid K_\omega(u) > 0\} \cup \{0\}, \\ R_\omega^2 &= \{(u, v) \in R_\omega \mid K_\omega(u) < 0\}. \end{aligned}$$

The range for frequency ω is $\omega < \lambda_1$ and the derivative of d over this range is

$$d'(\omega) = - \int |\phi_\omega|^2$$

Finally the stability theorem is

Theorem 3.5 *If $d''(\omega_0) > 0$, then the ground state standing waves of frequency ω are orbitally stable in the following sense: for every given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$\inf_{\phi \in S_{\omega_0}} \|U - \phi\|_H \leq \delta$$

implies

$$\inf_{\phi \in S_{\omega_0}} \|u(t) - \phi\|_H \leq \epsilon \quad \text{for all } t \geq 0$$

where $S_{\omega_0} = \{\phi \in M_{\omega_0} \mid J_{\omega_0}(\phi) = d(\omega_0)\}$.

3.6 Applications

In this section, we consider several cases of nonlinearity f or domain Ω where we have orbitally stable standing waves.

Theorem 3.6 *If the lower bound λ_1 of spectrum of operator $-\Delta + m$ is a positive eigenvalue (This is certainly true if the underlying domain Ω is a bounded domain in $\mathbb{R}^n$ or a compact manifold. It is also true if $m(x)$ is a potential, and the operator $-\Delta + m$ has discrete spectrum to the left of a continuous spectrum), then*

1. *The NLKG have orbitally stable standing waves for $\omega^2 \in (0, \lambda_1)$.*
2. *The NLS have orbitally stable standing waves for $\omega \in (-\infty, \lambda_1)$.*

Proof. By Theorem 3.4 or Theorem 3.5, it suffices to show that there exists a ω_0 such that $d''(\omega_0) > 0$. Again we only give the proof for NLKG. The proof for NLS is very similar.

Suppose not, then $d''(\omega) \leq 0$ for all $\omega \in (0, \sqrt{\lambda_1})$ which implies that

$$d'(\omega) = -\omega \int |\phi_\omega|^2 \text{ is decreasing for } \omega \in (0, \lambda_1),$$

which yields that there exists a positive constant C independent of ω such that

$$\int |\phi_\omega|^2 \geq C \text{ for } \omega \in (\epsilon, \sqrt{\lambda_1})$$

for any constant $0 < \epsilon < \sqrt{\lambda_1}$. By definition of $d(\omega)$, we have

$$d(\omega) = I_\omega(\phi_\omega) \geq \frac{\theta - 1}{2(\theta + 1)} \int (|\nabla \phi_\omega|^2 + (m - \omega^2) |\phi_\omega|^2) = A_1(\theta, C)(\lambda_1 - \omega^2), \quad (3.56)$$

where $A_1 = A_1(\theta, C)$ is a positive constant independent of ω . Next we estimate an upper bound for $d(\omega)$. Let $v(x)$ be the first eigenvalue of operator $-\Delta + m$, and $v_\delta(x) = \delta v(x)$. We can find $\delta = \delta(\omega)$ so that

$$K_\omega(v_\delta) = \delta^2 \int (|\nabla v|^2 + (m - \omega^2) |v|^2) + \delta \int v f(x, \delta v) = 0,$$

or

$$(\lambda_1 - \omega^2) \int v^2 = -\frac{1}{\delta} \int v f(x, \delta v),$$

which implies from Remark 3.1 that

$$\delta = \delta(\omega) \longrightarrow 0 \text{ as } \omega^2 \longrightarrow \lambda_1. \quad (3.57)$$

Using (f₃) and alternative expression for $d(\omega)$ we get

$$\begin{aligned} d(\omega) &\leq J_\omega(\delta v) \leq \frac{\delta^2}{2} \int (|\nabla v|^2 + (m - \omega^2) |v|^2) \\ &= \frac{\lambda_1 - \omega^2}{2} \delta^2 \int |v|^2 = A_2(\theta, v) \delta^2 (\lambda_1 - \omega^2), \end{aligned} \quad (3.58)$$

where $A_2(\theta, v)$ is a positive constant independent of ω . Combining (3.56) and (3.58) gives

$$0 < \frac{A_1}{A_2} \leq \delta^2,$$

a contradiction to (3.57), and the theorem is proved.

The second application we consider is for the case $\Omega = \mathbb{R}^n$ with $n \geq 2$. We investigate the stability of standing waves for NLKG in the following form.

$$u_{tt} - \Delta u + u - |u|^{p-1} u = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}, \quad (3.59)$$

which corresponds to $m(x) \equiv 1$ and $f(x, u) = -|u|^{p-1} u$, and for NLS in the form

$$iu_t - \Delta u - |u|^{p-1} u = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}, \quad (3.60)$$

which corresponds to $m(x) \equiv 0$ and $f(x, u) = -|u|^{p-1} u$.

Theorem 3.7 *The NLKG (3.59) and NLS (3.60) have orbitally standing waves for $1 < p < 1 + 4/n$.*

Proof. We consider the NLKG first. Due to the scaling property of the solutions in $\mathbb{R}^n$, we can find an explicit expression for $d(\omega)$. Let ϕ_ω be the positive radial symmetric solution satisfying

$$-\Delta \phi_\omega + (1 - \omega^2) \phi_\omega - \phi_\omega^p = 0.$$

Put $v(x) = (1/\delta) \phi_\omega(x/\beta)$, then

$$-\delta \beta^2 \Delta v + (1 - \omega^2) \delta v - \delta^p v^p = 0.$$

which is transformed to

$$-\Delta v + v - v^p = 0$$

if we select $\beta^2 = \delta^{p-1} = 1 - \omega^2$. Therefore we have

$$\begin{aligned} d(0) &= J_0(v) = \frac{1}{2} \int (|\nabla v|^2 + v^2) - \frac{1}{p+1} \int v^{p+1} \\ &= \frac{\delta^{-2}}{2} \int (\beta^{n-2} |\nabla \phi_\omega(x/\beta)|^2 + \phi_\omega^2(x/\beta)) - \frac{\delta^{-p-1}}{p+1} \int \phi_\omega^{p+1}(x/\beta) \\ &= \beta^{n-2} \delta^{-2} J_\omega(\phi_\omega) = \beta^{n-2} \delta^{-2} d(\omega), \end{aligned}$$

which implies that

$$d(\omega) = \delta^2 \beta^{2-n} d(0) = (1 - \omega^2)^\alpha d(0),$$

where

$$\alpha = \frac{4 + (p-1)(2-n)}{2(p-1)}.$$

Taking the second derivative we find

$$d''(\omega) = 2\alpha [-1 + (2\alpha - 1)\omega^2] (1 - \omega^2)^{\alpha-2} d(0),$$

which shows that if $\omega^2 < 1$, then

$$\{\omega \mid d''(\omega) > 0\} = \{\omega \mid 0 < \frac{1}{2\alpha - 1} < \omega^2 < 1\}.$$

The set in right hand side is nonempty if $1 < p < 1 + 4/n$.

As for NLS

$$d(\omega) = (-\omega)^\alpha d(\omega).$$

and

$$d''(\omega) = \alpha(\alpha - 1)(-\omega)^{\alpha-2}$$

which is positive for all $\omega < 0$ and $1 < p < 1 + 4/n$.

Remark 3.3 • *The same stability result for standing waves of NLKG (3.59) was obtained by Shatah in [43], but the approach in [43] can not handle the case $n = 1$ or $n = 2$ due to the usage of a different functional K_ω which is not well defined for $n = 1$ and $n = 2$.*

- *Similar orbital stability result for standing waves of NLS (3.60) was obtained by Cazenave and Lions in [9], their method is very different from the one developed here. In [9], the frequency ω can not be prescribed to find corresponding ground state. Instead, they solve minimization problem*

$$\inf \left\{ \int |\nabla u|^2 - \frac{1}{p+1} \int |u|^{p+1} \mid \|u\|_2 = \mu, u \in H^1(\mathbb{R}^n) \right\}$$

where $\mu > 0$ to find a ground state, then use Lagrange multiplier to find corresponding frequency.

Chapter 4

Finite Time Blow Up for Nonlinear Klein-Gordon Equation

4.1 Introduction

In this chapter, Ω is $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$ or n -dimensional compact manifold. We will only consider the following nonlinear Klein-Gordon equation:

$$\begin{cases} u_{tt} - \Delta u + m(x)u + f(x, u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{if } \partial\Omega \neq \emptyset, \\ u(0, x) = U(x), \quad u_t(0, x) = V(x), \end{cases} \quad (4.1)$$

where condition on $m(x)$ is the same as defined in last chapter, i.e., $m(x)$ is a real bounded function and if $\Omega = \mathbb{R}^n$, $m(x)$ is assumed to be radially symmetric, i.e., $m(x) = m(|x|)$. We also assume that the lower bound λ_1 of spectrum of operator $-\Delta + m$ is positive. We still assume $f(x, u) = g(x, |u|)u$. We prove that for a class of nonlinearity f the steady states of the least energy, i.e., ground states are unstable in a very strong sense: there is a region on boundary of which ground states lie, such that that every solution of NLKG (4.1) starting from this region will blow up in finite time. This type of instability basically means the nonexistence of global solutions for some initial data and some nonlinearities. Keller's work [27] represents one of the earliest results in this direction. Since then, a number of authors (e.g.,

Berestycki and Cazenave [2], John [11], Glassey [12], [13], Levein [29], Payne and Sattinger [36], Sternberg [48] and Tsutsumi [54]) have investigated the conditions on which the solutions of (4.1) will blow-up in finite time. The second kind of instability is that the solution of (4.1) starting near ground state may exist globally but will approach infinity in L^2 norm as time t approaches infinity. Shatah's work [43] and Keller's work [25] represent the work in this direction. The third type of instability is that the solution of (4.1) may exist globally and may not approach infinity as time t approaches infinity, nevertheless the ground state is unstable. Shatah and Strauss' paper [45] and Keller's paper [25] represent the work in this direction.

In this chapter, we shall deal with first type of instability. We first prove the existence of the ground state, then establish more properties of weak solutions. Finally we prove that every solution of NLKG (4.1) starting from some region with ground state on its boundary will blow up in finite time.

4.2 Steady State and Weak Solution

In this section, we prove the existence of steady state of the least energy, i.e, ground state, and establish more properties of weak solutions of NLKG (4.1). For the existence of ground state, we mainly state the relevant results since the conditions imposed on nonlinearity f in this chapter will be the same as in those in last chapter except that we do not assume that $f_u < 0$ which was used to prove the continuity of $d(\omega)$, therefore the proofs for some related lemmas and theorems (for instance, the theorem of the existence of ground states) in this chapter are also similar and will be omitted. Now let us go to the existence of ground state. The steady state of NLKG (4.1) satisfies the nonlinear elliptic equation

$$\begin{cases} -\Delta\phi + m(x)\phi + f(x, \phi) = 0, \\ \phi = 0 \text{ on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset, \end{cases} \quad (4.2)$$

We know that every solution $u \in H$ of (4.2) is a critical point of potential energy functional

$$J(v) = \frac{1}{2} \int (|\nabla v|^2 + m(x)|v|^2) + \int F(x, v),$$

where $F(x, v) = F(x, |v|) = \int_0^{|v|} f(x, s) ds$. It is also easy to check that every solution of (4.2) satisfies

$$K(v) \equiv \int (|\nabla v|^2 + m(x) |v|^2) + \int |v| f(x, |v|) = 0.$$

Therefore every solution of (4.2) lies on hypersurface

$$M = \{v \in H \mid K(v) = 0 \text{ and } v \neq 0\}.$$

To find the steady state of the least energy, it is natural to turn to minimization problem

$$d = \inf_{v \in M} J(v). \quad (4.3)$$

We make following assumptions on f to ensure J to attain its minimum.

(H1) $f \in C^1(\Omega \times \mathbb{R})$ and $f'_u(x, s), f(x, s) \rightarrow 0$ uniformly in Ω as $s \rightarrow 0$.

(H2) There exist constants $2 < l < 2n/(n-1)$ and C such that $|f'_u(x, s)| \leq Cs^{l-2}$ for large $s > 0$ and for all x in Ω .

(H3) There exists a constant $\theta > 1$ such that $sf'_u(x, s) \leq \theta f(x, s)$ for all $x \in \Omega$ and all $s > 0$.

Note that the following functions satisfy the above assumptions (H1)-(H3).

$$f(x, s) = -s^p \text{ with } 1 < p < 1 + 4/(n-2),$$

$$f(x, s) = -s^p + s^q \text{ with } 1 < q < p < 1 + 4/(n-2).$$

Introduce the I functional and the region M^- in H

$$I(v) = \frac{\theta - 1}{2(\theta + 1)} \int (|\nabla v|^2 + m(x) |v|^2) + \int \left(F(x, v) - \frac{1}{\theta + 1} |v| f(x, |v|) \right),$$

$$M^- = \{v \in H \mid K(v) \leq 0 \text{ and } v \neq 0\}.$$

Lemma 4.1 *M is a C^1 -hypersurface in H , and both M and M^- are bounded away from zero.*

Lemma 4.2

$$d = \inf_{v \in M} I(v) = \inf_{v \in M^-} I(v).$$

Theorem 4.1 *Let f satisfy conditions (H1), (H2) and (H3), then*

1. *d is positive;*
2. *Every minimizing sequence of problem (4.3) posses a convergent subsequence.
In particular, d is attained at some ϕ ;*
3. *This minimizer ϕ can be chosen positive;*
4. *Every minimizer of problem (4.3) is a solution of Equation (4.2) and is called the ground state.*

We define the set of ground states

$$S = \{\phi \in M \mid J(\phi) = d\}.$$

Next we present some properties about the weak solutions of NLKG (4.1) (see [36], [49] and [54] for details). Suppose that u is a weak solution of (4.1) on $\Omega \times [0, T)$, then the following statements hold

- (1) $u(t)$ is weakly continuous from $[0, T)$ to H , $u_t(t)$ is weakly continuous from $[0, T)$ to L^2 . So $\|u(t)\|_H$ and $\|u_t(t)\|_2$ are bounded on compact subsets of $[0, T)$.
- (2) There exists a weakly continuous mapping from $[0, T)$ to L^2 denoted by u_t , such that

$$(u, \phi) \big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (u_t, \phi) dt,$$

for $0 \leq t_1 \leq t_2$ and $\phi \in L^2$ where $(u, v) = \int_{\Omega} u \bar{v} dx$ is standard inner product on complex valued $L^2(\Omega)$.

- (3) For any $\psi : [0, T) \longrightarrow H$ satisfying (1) and (2), we have

$$(u_t, \psi) \big|_{t_1}^{t_2} = \int_{t_1}^{t_2} [(u_t, \psi_t) - (\nabla u, \nabla \psi) - (m(\cdot)u, \psi) - (f(\cdot, u), \psi)] dt \quad (4.4)$$

It follows from (2) that $(u(t), \phi)$ is Lipschitz continuous for any ϕ in L^2 , i.e., $u(t)$ is weakly Lipschitz continuous in t . In (4.4), if we put $\psi = u$, we have

$$(u_t, u) \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} [\|u_t\|_2^2 - \|\nabla u\|_2^2 - (m(\cdot)u, u) - (f(\cdot, u), u)] dt. \quad (4.5)$$

Let $u(t)$ be weak solution of NLKG (4.1), define

$$N(t) = (u(t), u(t)).$$

We follow the approaches in [36] to get some smoothness of $N(t)$.

Lemma 4.3 *$N''(t)$ exists a.e. in $[0, T)$, and $N'(t)$ is Lipschitz continuous there.*

Proof. Let $Q(t, s) = (u(t), u(s))$. Since $u(t)$ is weakly absolutely continuous (by (2)), and $u_t(t)$ is weakly continuous

$$N'(t) = \left(\frac{\partial}{\partial t} Q(t, s) + \frac{\partial}{\partial s} Q(t, s) \right) \Big|_{s=t} = 2\text{Re}(u_t, u). \quad (4.6)$$

From special form of our nonlinearity $f(x, u) = g(x, |u|)u$ and (4.5) we conclude that (u_t, u) is real, so (4.6) and (4.5) yield

$$N'(t_2) - N'(t_1) = 2 \int_{t_1}^{t_2} [\|u_t\|_2^2 - \|\nabla u\|_2^2 - (m(\cdot)u, u) - (f(\cdot, u), u)] dt \quad (4.7)$$

for all $0 \leq t_1 < t_2 < T$. Since each term in the integrand is bounded on compact subsets of $[0, T)$, we see that N' is Lipschitz continuous on such sets. Therefore, N'' exists a.e. in $[0, T)$, and

$$N''(t) = 2 [\|u_t\|_2^2 - \|\nabla u\|_2^2 - (m(\cdot)u, u) - (f(\cdot, u), u)]. \quad (4.8)$$

This concludes the proof for the lemma.

4.3 Finite Time Blow Up

In this section, with preparation made in last section, we shall prove the finite time blow up of solutions starting from a region that has ground states on its boundary.

First we define total energy(E) as the sum of potential energy($J(u)$) and kinetic energy($\frac{1}{2}\|v\|_2^2$)

$$E(u, v) = J(u) + \frac{1}{2}\|v\|_2^2.$$

For weak solutions $u(t)$ of NLKG (4.1), we have energy inequality

$$E(u(t_1), u_t(t_1)) \geq E(u(t_2), u_t(t_2)) \quad \forall 0 \leq t_1 \leq t_2.$$

Next define the NLKG solution-invariant sets

$$\begin{aligned} R_1 &= \{(u, v) \in R \mid K(u) > 0\} \cup \{(0, v) \in R\}, \\ R_2 &= \{(u, v) \in R \mid K(u) < 0\}, \end{aligned}$$

where

$$R = \{(u, v) \in H \oplus L^2 \mid E(u, v) < d\}.$$

By definition of d , K and I , R_1 and R_2 have the following alternative definitions

$$\begin{aligned} R_1 &= \{(u, v) \in R \mid I(u) < d\} \cup \{(0, v) \in R\}, \\ R_2 &= \{(u, v) \in R \mid I(u) > d\}, \end{aligned}$$

Lemma 4.4 *R_1 and R_2 are invariant regions under the solution flow of NLKG (4.1).*

Proof. The proof is similar to the proof of similar invariance lemma in last chapter, so we skip it.

Now we are ready to present our main result.

Theorem 4.2 *Let u be a weak solution of Equation (4.1) with initial data $(U, V) \in R_2$, let $[0, T)$ be the existence interval. Then T must be finite.*

Proof. Suppose that $T = \infty$. Note $K(u) = (\theta + 1)(J(u) - I(u))$. From invariance of R_2 and energy inequality, we have for all $0 < t < \infty$

$$K(u) \leq (\theta + 1)(E(u, u_t) - d) \tag{4.9}$$

$$\leq (\theta + 1)(E(U, V) - d) \equiv -\epsilon < 0. \tag{4.10}$$

Let $N(t) = (u(t), u(t))$, recall from last section that

$$N''(t) = 2 [\|u_t\|_2^2 - \|\nabla u\|_2^2 - (m(\cdot)u, u) - (f(\cdot, u), u)] \quad (4.11)$$

Therefore by (4.10) and (4.11) we have

$$N''(t) = 2[\|u_t\|^2 - K(u)] \geq 2\epsilon > 0$$

which implies that N' is strictly increasing for $t \in [0, \infty)$, and there exists some $t_1 \geq 0$ such that

$$N'(t) \geq N'(t_1) > 0 \quad \forall t \geq t_1 \quad (4.12)$$

From (H1) and (H3) we get

$$(\theta + 1) \int F(x, u) \geq (f(\cdot, u), u).$$

Energy inequality implies that

$$\int F(x, u) \leq E(U, V) - \frac{1}{2} (\|\nabla u\|_2^2 + (mu, u) + \|u_t\|_2^2).$$

Hence we arrive at

$$\begin{aligned} -(f(\cdot, u), u) &\geq -(\theta + 1) \int F(x, u) \\ &\geq \frac{\theta + 1}{2} (\|\nabla u\|_2^2 + (mu, u) + \|u_t\|_2^2) - (\theta + 1)E(U, V), \end{aligned}$$

which yields by (4.11)

$$N''(t) \geq (\theta + 3)\|u_t\|_2^2 + (\theta - 1)\lambda_1\|u\|_2^2 - 2(\theta + 1)E(U, V).$$

Note that $N(t)$ is strictly increasing for $t > t_1$ from (4.12), therefore there exists a $t_2 \geq t_1$ such that

$$N''(t) > (\theta + 3)\|u_t\|_2^2.$$

Hence for $t > t_2$, we have that

$$NN'' - \frac{\theta + 3}{4} (N')^2 > (\theta + 3)\|u\|_2^2\|u_t\|_2^2 - (\theta + 3)(u, u_t)^2 \geq 0,$$

which leads to

$$(N^{-\alpha})'' = -\frac{\alpha}{N^{\alpha+2}} \left[NN'' - (\alpha + 1)(N')^2 \right] < 0 \quad \text{for } t > t_2,$$

where $\alpha = (\theta - 1)/4 > 0$, and so $M^{-\alpha}$ is concave and decreasing for $t \geq t_2$. Therefore there exists a T_0 such that $N^{-\alpha} \rightarrow 0$ as $t \rightarrow T_0$ which implies $N \rightarrow \infty$ as $t \rightarrow T_0$. Thus we get contradiction to $T = \infty$.

Finally we need to show that R_2 is not empty. Choose initial data $U(x) = \tau\phi(x)$ with $\phi \in S$, and choose $V(x) \equiv 0$. Then $E(U, V) = J(U) = J(\tau\phi)$ and $K(U) = K(\tau\phi)$. It suffices to find τ_0 so that

$$E(U, V) < d \text{ and } K(U) < 0.$$

The following arguments complete our proof. Let $h(\tau) = J(\tau\phi)$ and $g(\tau) = K(\tau\phi)$. Then

$$h(1) = J(\phi) = d, \quad g(1) = K(\phi) = 0,$$

and by (H3) and $K(\phi) = 0$ we have

$$\begin{aligned} g'(1) &= \int (|\nabla\phi|^2 + m(x)|\phi|^2 + |\phi|^2 f'(x, |\phi|)) \\ &\leq (\theta - 1) \int |\phi| f(x, |\phi|) \\ &= (1 - \theta) \int (|\nabla\phi|^2 + m(x)|\phi|^2) < 0, \end{aligned}$$

and

$$h'(\tau) = \frac{1}{\tau}g(\tau) \text{ and } h''(1) = g'(1) < 0.$$

Remark 4.1 *Interesting enough, if a weak solution u starts from invariance region R_1 , then the solution exists globally. To see this, energy inequality implies that*

$$E(u(t), u_t(t)) = \frac{1}{2}\|u_t\|_2^2 + I(u) + \frac{1}{\theta + 1}K(u) \leq E(U, V),$$

and since R_1 is invariant, then $K(u(t)) > 0$ for all t . Therefore

$$\frac{1}{2}\|u_t\|_2^2 + I(u) \leq E(U, V).$$

So $\|u_t\|_2$ and $\|u\|_H$ are uniformly bounded, and this implies the global existence of the solution.

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List of References

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