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Seiberg-Witten Invariants of Rational Blow-downs and Geography Problems of Irreducible 4-Manifolds

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SEIBERG-WITTEN INVARIANTS OF RATIONAL BLOW-DOWNS AND GEOGRAPHY PROBLEMS OF IRREDUCIBLE 4-MANIFOLDS

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ABSTRACT

SEIBERG-WITTEN INVARIANTS OF RATIONAL BLOW-DOWNS AND GEOGRAPHY PROBLEMS OF IRREDUCIBLE 4-MANIFOLDS

BY

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One of the main problems in Seiberg-Witten theory is to find (SW)-basic classes and its invariants for a given smooth 4-manifold. Rational blow-down procedure introduced by R. Fintushel and R. Stern is one way to compute these invariants for some smooth 4-manifolds.

This thesis consists of two parts. First, we find (SW)-basic classes and Seiberg-Witten invariants for rational blow-down 4-manifolds by using index computations. (R. Fintushel and R. Stern did $q = 1$ case.)

Secondly, we investigate the geography problems (in particular, the existence problem and the uniqueness problem) for simply connected smooth irreducible 4-manifolds. By taking fiber sums along an embedded surface of square 0 and by the rational blow-down procedure, we construct many new examples of irreducible 4-manifolds. Furthermore, we prove "All but finitely many lattice points (a, b) lying in between $c_1^2 = 0$ and $c_1^2 = 8\chi$ (non-positive signature region) can be realized as (χ, c_1^2) of a simply connected irreducible 4-manifold which has infinitely many distinct smooth structures."

To my parents.

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Contents

Chapter 1

Introduction

As gauge theory (Donaldson theory and Seiberg-Witten theory) is developed, the fundamental problem in this area is to find its invariants for a given smooth 4-manifold.

In 1993, R. Fintushel and R. Stern introduced a surgical procedure, called rational blow-down, to compute the Donaldson series for simply connected regular elliptic surfaces with multiple fibers of relatively prime orders. 'Rational blow-down' means that if a smooth 4-manifold X contains a certain configuration C_p of transversally intersecting 2-spheres whose boundary is $L(p^2, 1-p)$, then one can construct a new smooth 4-manifold X_p from X by replacing C_p with a rational ball B_p.

In fact, A. Casson and J. Harer ([CH]) showed that for any pair of relatively prime integers p and q, $L(p^2, 1-pq)$ bounds a rational ball $B_{p,q}$. Hence one can extend this rational blow-down procedure to the general case, that is, whenever a smooth 4-manifold X contains a certain configuration $C_{p,q}$ of transversally intersecting 2-spheres whose boundary is $L(p^2, 1-pq)$, one can always construct a new smooth 4-manifold $X_{p,q}$ by replacing $C_{p,q}$ with a rational ball $B_{p,q}$.

For the $q = 1$ case, R. Fintushel and R. Stern initially computed the Donaldson series of $X_p = X_{p,1}$ from the Donaldson series of X, and later they computed the

Seiberg-Witten invariants of X_p ([FS3]). In Chapter 3 of this paper we extend these results to the general case. Explicitly, we prove the following theorem by using index computations:

Theorem 1.0.1 *Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$. If L is a characteristic line bundle on X such that $SW_X(L) \neq 0$, $(L|_{C_{p,q}})^2 = -b_2(C_{p,q})$ and $c_1(L|_{L(p^2,1-pq)}) = mp \in \mathbf{Z}_{p^2} \cong H^2(L(p^2,1-pq);\mathbf{Z})$ with $m \equiv (p-1) \pmod 2$, then L induces a characteristic line bundle $\overline{L}$ on $X_{p,q}$ such that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$.*

Furthermore, we prove the following theorem:

Theorem 1.0.2 *If a simply connected smooth 4-manifold X contains a configuration $C_{p,q}$ satisfying condition $(*)$ below, then the SW-invariants of $X_{p,q}$ are completely determined by those of X. That is, for any characteristic line bundle $\overline{L}$ on $X_{p,q}$ with $SW_{X_{p,q}}(\overline{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_{p,q}}(\overline{L})$.*

The condition $(*)$ in the theorem above is following:

$$(*)\ \{\partial(\sum_{i=1}^{k} \epsilon_i e_i|_{B_{p,q}}) : \epsilon_i = \pm 1, \forall i\} = \{mp : -(p-1) \leq m \leq (p-1) \text{ and } m \equiv (p-1) \pmod 2\}$$

All known configurations $C_{p,q}$ satisfy this condition.

As applications, we explore the geography problems for simply connected smooth irreducible 4-manifolds. Namely, the existence problem: which pairs $(\chi = \frac{b^+ + 1}{2}, c_1^2 = 3\sigma + 2e)$ of lattice points are realized by simply connected smooth irreducible 4-manifolds and the uniqueness problem: are there infinitely many distinct irreducible smooth 4-manifolds which are all homeomorphic at each lattice point? In Chapter 4 we give partial answers for these problems. That is, we find many new examples of such 4-manifolds in the wedge between $c_1^2 = 0$ and $c_1^2 = 2\chi - 6$. Actually we construct new examples which cover all lattice points in this region (The examples lying in the wedge were first found by R. Fintushel and R. Stern, and later by R. Gompf and A. Stipsicz.). Note that any such 4-manifold in this region cannot admit a complex structure with either orientation. We also investigate uniqueness problem, i.e. the problem of finding infinitely many diffeomorphism types for a given pair (χ, c_1^2) lying in between $c_1^2 = 0$ and $c_1^2 \leq 9\chi$. As a result of Theorem 1.0.2, we can compute Seiberg-Witten invariants of $X(p)$, where $X(p)$ is the result of p-surgery in the cusp neighborhood of a cusp fiber in X. Under a mild condition on X making $X(p)$ simply connected, such $X(p)$ is not diffeomorphic, but is homeomorphic to X. In fact, for infinitely many (χ, c_1^2), we can find irreducible 4-manifolds containing a cusp fiber satisfying the mild condition, for example, by taking fiber sums. The main result we prove in this paper is

Theorem 1.0.3 *All but finitely many lattice points (a, b) lying in between $c_1^2 = 0$ and $c_1^2 = 8\chi$ (non-positive signature region) can be realized as (χ, c_1^2) of a simply connected irreducible 4-manifold which has infinitely many distinct smooth structures.*

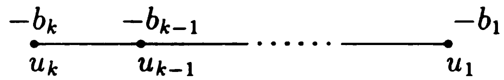
Chapter 2

The Topology of Rational Blow-downs

In this chapter we describe topological aspects and several examples of rationally blowdown 4-manifolds.

2.1 Topological Properties

For any relatively prime integers p and q with $1 \leq q < p$, we define a configuration $C_{p,q}$ as a smooth 4-manifold obtained by plumbing disk bundles over 2-sphere instructed by the following linear diagram



where $\frac{p^2}{pq-1} = [b_k, b_{k-1}, \ldots, b_1]$ is a unique continued linear fraction with all $b_i \geq 2$, and each vertex u_i represents a disk bundle over 2-sphere whose Euler number is $-b_i$. Then the configuration $C_{p,q}$ has the following properties:

1. It is a simply connected smooth 4-manifold whose boundary is lens space $L(p^2, 1-pq)$.

2. $H_2(C_{p,q};\mathbf{Z}) \cong \bigoplus_{i=1}^k \mathbf{Z}$ has generators $\{u_i : 1 \leq i \leq k\}$ which can be represented by embedded 2-spheres, that is, each u_i is represented by zero-section S_i^2 of the disk bundle u_i over S^2. (We use u_i for both a generator and the corresponding disk bundle.)

3. The plumbing matrix for $C_{p,q}$ with respect to the basis $\{u_i : 1 \leq i \leq k\}$ is given by the symmetric $k \times k$ matrix

$$P = \begin{pmatrix} -b_1 & 1 & 0 & & & \\ 1 & -b_2 & 1 & & 0 & \\ 0 & 1 & -b_3 & & & \\ & & & \ddots & & \\ & 0 & & & -b_{k-1} & 1 \\ & & & & 1 & -b_k \end{pmatrix}$$

so that $C_{p,q}$ is negative definite.

4. The intersection form on $H^2(C_{p,q};\mathbf{Z})$ with respect to the dual basis $\{\gamma_i : 1 \leq i \leq k\}$ (i.e. $<\gamma_i\ ,\ u_j>=\delta_{ij}$) is given by

$$Q := (\gamma_i \cdot \gamma_j) = P^{-1}$$

Proof : Note that the intersection form Q on $H^2(C_{p,q};\mathbf{Z})$ is defined by

$$\gamma_i \cdot \gamma_j := \frac{1}{p^2} < \gamma_i\ ,\ PD\gamma_j' >$$

where $\gamma_j' \in H^2(C_{p,q}, \partial C_{p,q};\mathbf{Z})$ is determined by $\jmath^*(\gamma_j') = p^2 \cdot \gamma_j$ in the sequence

$$0 \longrightarrow H^2(C_{p,q}, \partial C_{p,q};\mathbf{Z}) \xrightarrow{\jmath^*} H^2(C_{p,q};\mathbf{Z}) \xrightarrow{\partial} H^2(\partial C_{p,q};\mathbf{Z}) \longrightarrow 0$$

Since $\jmath^* = P$, we have

$$\begin{aligned} \gamma_i \cdot \gamma_j &:= \frac{1}{p^2} < \gamma_i\,, PD\gamma_j' > = \frac{1}{p^2} < \gamma_i\,, P^{-1}(p^2 \cdot PD\gamma_j) > = < \gamma_i\,, P^{-1}(PD\gamma_j) > \\ &= (P^{-1})_{ij}. \quad \square \end{aligned}$$

Lemma 2.1.1 *The inclusion induced homomorphism* $\partial : H^2(C_{p,q};\mathbf{Z}) \longrightarrow H^2(\partial C_{p,q};\mathbf{Z}) \cong \mathbf{Z}_{p^2}$ *is given by* $\partial(\gamma_i) = n_i$, *where* n_i *is a number satisfying*

$$\begin{pmatrix} * \\ n_i \end{pmatrix} := \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ b_2 & 1 \end{pmatrix}\cdots\begin{pmatrix} -1 & 0 \\ b_{i-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Proof : By Poincaré duality , it suffices to show $\partial : H_2(C_{p,q}, \partial C_{p,q};\mathbf{Z}) \to H_1(\partial C_{p,q};\mathbf{Z})$ is given by $\partial(PD\gamma_i) = n_i$. For each i, choose a fiber D_i^2 of a disk bundle u_i over S^2 so that $D_i^2 \cdot S_j^2 = \delta_{ij}$. Then D_i^2 is a representative for $PD(\gamma_i) \in H_2(C_{p,q}, \partial C_{p,q};\mathbf{Z})$. Since

$$\begin{aligned} \partial C_{p,q} &= D^+ \times S_k^1 \cup_{A_k} \partial D^- \times S_k^1 \cup_B \partial D^+ \times S_{k-1}^1 \cup_{A_{k-1}} \cdots \cup_{A_1} D^- \times S_1^1 \\ &= D^+ \times S_k^1 \cup_A D^- \times S_1^1 \end{aligned}$$

where $S_i^1 := \partial D_i^2$ and $A := A_k B A_{k-1} \cdots A_1$ with $A_i := \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}$ and $B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$\begin{aligned} \partial(PD\gamma_i) &= \partial(D_i^2) \\ &= \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} -1 & 0 \\ b_{i-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ n_i \end{pmatrix} \end{aligned}$$

which is homologous to $\begin{pmatrix} 0 \\ n_i \end{pmatrix}$ in $H_1(\partial C_{p,q};\mathbf{Z})$. Hence, by choosing $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a generator of $H_1(\partial C_{p,q};\mathbf{Z})$, we have $\partial(PD\gamma_i) = n_i$. □

Lemma 2.1.2 *The lens space* $L(p^2, 1-pq) = \partial C_{p,q}$ *bounds a rational ball* $B_{p,q}$ *with* $\pi_1(B_{p,q}) = \mathbf{Z}_p$, *and the inclusion induced homomorphism*

$$\iota^* : H^2(B_{p,q};\mathbf{Z}) \cong \mathbf{Z}_p \longrightarrow H^2(L(p^2,1-pq);\mathbf{Z}) \cong \mathbf{Z}_{p^2}$$

can be given by $n \longmapsto np$.

Proof : The first part was proved by Casson and Harer ([CH]). For the second part, since Mayer-Vietoris sequence for $X \equiv C_{p,q} \cup_L \overline{B_{p,q}}$ which is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$

$$0 \longrightarrow H_2(C_{p,q};\mathbf{Z}) \oplus H_2(B_{p,q};\mathbf{Z}) \longrightarrow H_2(\sharp k\overline{\mathbf{CP}}^2;\mathbf{Z}) \longrightarrow \cdots$$

implies $H_2(B_{p,q};\mathbf{Z})$ is torsion free, by Poincaré duality, $H^2(B_{p,q},\partial B_{p,q};\mathbf{Z}) \cong H_2(B_{p,q})$ $= 0$. On the other hand, since the exact sequence for $(B_{p,q},\partial B_{p,q})$ also implies that

$$\iota^* : H^2(B_{p,q};\mathbf{Z}) \cong \mathbf{Z}_p \longrightarrow H^2(\partial B_{p,q};\mathbf{Z}) \cong \mathbf{Z}_{p^2}$$

is injective, $\iota^*(1) = lp$ for some l with $\gcd(l,p) = 1$. Hence, by re-choosing a generator of $H^2(\partial B_{p,q};\mathbf{Z}) \cong \mathbf{Z}_{p^2}$, we may assume that $\iota^*(1) = p$, so that $\iota^*(n) = np$. □

Lemma 2.1.3 *$B_{p,q}$ is spin if p is odd, and $B_{p,q}$ is not spin if p is even.*

Proof : If p is odd, then $H_1(B_{p,q}) \cong \mathbf{Z}_p$ implies $H^2(B_{p,q};\mathbf{Z}_2) \cong Ext(H_1(B_{p,q});\mathbf{Z}_2)$ $= 0$. Assume p is even and $B_{p,q}$ is spin. Then the index of Dirac operator on $B_{p,q}$ should be an integer. But the index computation on $B_{p,q}$ (Proposition 3.2.2 and its remark) shows that it is not an integer—a contradiction! □

Now we define the rational blow-down procedure: Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$ for some relatively prime integers p and q. We construct a new smooth 4-manifold $X_{p,q}$, called the **rational blow-down of** X, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$(Fig 2.1). We call this procedure a '**(generalized) rational blow-down**'. Note that this procedure is well defined, i.e. $X_{p,q}$ is uniquely constructed (up to diffeomorphism) from X because each diffeomorphism of $\partial B_{p,q} = L(p^2, 1-pq)$ extends over the rational ball $B_{p,q}$ by the same argument as Corollary 2.2 in [FS3].

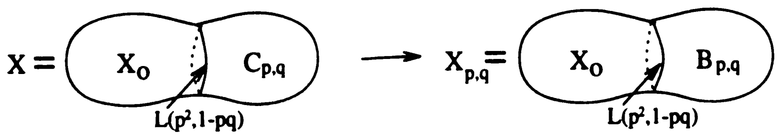


Figure 2.1:

Lemma 2.1.4 *$b^+(X_{p,q}) = b^+(X)$ and $c_1^2(X_{p,q}) = c_1^2(X) + k$, where $k = b_2(C_{p,q})$.*

Proof : Since $C_{p,q}$ is negative definite, $b^+(X_{p,q}) = b^+(X)$ and

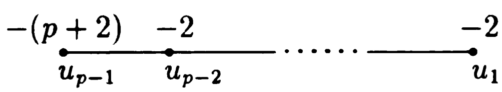
$$\begin{aligned} c_1^2(X_{p,q}) &= 3\sigma(X_{p,q}) + 2e(X_{p,q}) \\ &= 3(\sigma(X) + k) + 2(e(X) - k) \\ &= c_1^2(X) + k. \end{aligned}$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X. □

2.2 Examples

Here are several configurations $C_{p,q}$ that will be used later.

Case $q = 1$: This case is studied in [FS3], whose configuration $C_{p,1}$ is



Fintushel and Stern used this configuration to show that the rational blow-down of $E(n)\sharp(p{-}1)\overline{\mathbf{CP}}^2$ is diffeomorphic to $E(n;p)$, p-log transform on $E(n)$, and to compute the Donaldson and Seiberg-Witten invariants of simply connected elliptic surfaces with multiple fibers. Here $E(n)$ is a simply connected elliptic surface with no multiple fibers and holomorphic Euler characteristic n, and 'p-log transform on $E(n)$' is the result of removing tubular neighborhood of torus fiber in $E(n)$, say $T^2 \times D^2$, and regluing it by a diffeomorphism

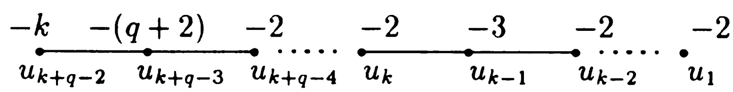
$$\varphi : T^2 \times \partial D^2 \longrightarrow T^2 \times \partial D^2$$

such that the absolute value of the degree of the map

$$\text{proj}_{\partial D^2} \circ \varphi : pt \times \partial D^2 \longrightarrow \partial D^2$$

is p. Note that 'p-log transform on $E(n)$' is well defined, i.e. $E(n;p)$ is uniquely determined up to diffeomorphism by the fact that if $\text{proj}_{\partial D^2} \circ \varphi$ and $\text{proj}_{\partial D^2} \circ \varphi'$ have the same degree up to sign, then the resulting two manifolds are diffeomorphic ([G1, Proposition 2.1]).

Case $p = kq-1 (k, q \geq 2)$: We assume $q \geq 3$ ($q = 2$ case is also obtained in a similar way). The configuration $C_{p,q}$ is given by



which can be embedded in $\sharp(k+q-2)\overline{\mathbf{CP}}^2$ by choosing

$$u_i := \begin{cases} e_{k+q-2-i} - e_{k+q-1-i} & \text{i=1,...,k-2} \\ e_{q-2} - e_{q-1} - e_q & \text{i=k-1} \\ e_{k+q-3-i} - e_{k+q-2-i} & \text{i=k,...,k+q-4} \\ -2e_1 - e_2 - \cdots - e_{q-1} & \text{i=k+q-3} \\ e_{q-1} - e_q - \cdots - e_{k+q-2} & \text{i=k+q-2} \end{cases}$$

where each e_i $(1 \leq i \leq k+q-2)$ is the exceptional divisor in $\sharp(k+q-2)\overline{\mathbf{CP}}^2$. Furthermore, by using Lemma 2.1.1, we get its boundary values

$$\partial\gamma_i = \begin{cases} i & \text{i=1,...,k-1} \\ (i+2-k)k - i & \text{i=k,...,k+q-3} \\ pq-1 & \text{i=k+q-2} \end{cases} \tag{2.1}$$

which implies that $C_{kq-1,q}$ satisfies the condition (∗) mentioned in the introduction.

Theorem 2.2.1 *For any integers k and q* $(k, q \geq 2)$, *there is an embedding* $C_{kq-1,q} \subset E(n)\sharp(k+q-2)\overline{\mathbf{CP}}^2$ *such that the rational blow-down is diffeomorphic to* $E(n; kq-1)$.

Proof : Consider the homology class f of the fiber in $E(n)$ which can be represented by an immersed 2-sphere with one positive double point and self-intersection 0 (a nodal fiber). Blow up this double point so that $f - 2e_1$ (e_1 is the exceptional divisor) is represented by an embedded sphere. Since e_1 intersects $f - 2e_1$ at two positive points, blow up one of these points again. By continuing in this way, we get a configuration $C_{kq-1,q}$ in $E(n)\sharp(k+q-2)\overline{\mathbf{CP}}^2$. We draw the case $q \geq 3$ (Fig 2.2) ($q = 2$ case is similar). The claim that the rational blow-down of $E(n)\sharp(k+q-2)\overline{\mathbf{CP}}^2$ is diffeomorphic to $E(n; kq-1)$ can be proved by Kirby calculus on the neighborhood of a cusp fiber as the same way as Theorem 3.1 in [FS3]. □

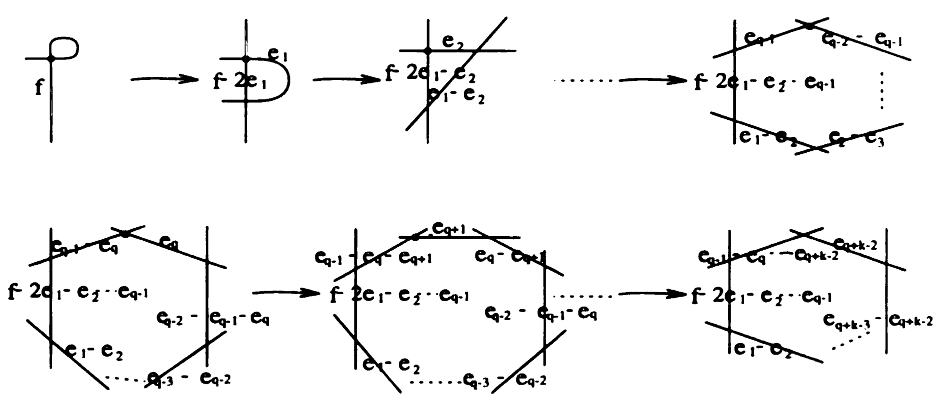


Figure 2.2:

Here are a few remarks on this theorem:

1. The theorem above implies that there are many ways to obtain $E(n;p)$, p-log

transform on $E(n)$, from $E(n)$ via a rational blow-down procedure; so one can choose an 'economical' way to get $E(n;p)$. For example, $E(n,11)$ is diffeomorphic to the rational blow-down of $C_{11,1} \subset E(n)\sharp 10\overline{\mathbf{CP}}^2$, of $C_{11,2} \subset E(n)\sharp 6\overline{\mathbf{CP}}^2$, and of $C_{11,3} \subset E(n)\sharp 5\overline{\mathbf{CP}}^2$.

2. One expects that for any relative prime integers p and q, there is an embedding $C_{p,q}$ in $E(n)\sharp k\overline{\mathbf{CP}}^2$, for some $k \in \mathbf{Z}$, such that the rational blow-down is diffeomorphic to $E(n;p)$.

3. The key ingredient in the proof of the theorem is to find such a configuration $C_{kq-1,q}$. We chose u_i exactly the same u_i embedded in $\sharp(k+q-2)\overline{\mathbf{CP}}^2$ except

$$u_{k+q-3} = f - 2e_1 - e_2 \cdots - e_{q-1} \quad (u_{k-1} = f - 2e_1 - e_2, \ \text{ if } \ q = 2)$$

4. One can extend the 'logarithmic transform' procedure to any 4-manifold which contains a cusp neighborhood. A *cusp* in a 4-manifold means a PL embedded 2-sphere of self-intersection 0 with a single non-locally flat point whose neighborhood is the cone on the right-hand trefoil knot, and we define a *cusp neighborhood* in a 4-manifold to be a manifold N obtained by performing 0-framed surgery on the trefoil knot in the boundary of the 4-ball. Note that since the trefoil knot is a fibered knot with a genus 1 fiber, N is fibered by tori with one singular fiber which is a cusp. Hence one can perform 'p-log transform' on a regular torus fiber in N exactly the same way as in $E(n)$, so that the theorem above is also true for any smooth 4-manifold containing a cusp neighborhood.

Chapter 3

Seiberg-Witten Theory of Rational Blow-downs of 4-Manifolds

In this chapter we compute the Seiberg-Witten invariants of rational blow-downs of 4-manifolds.

3.1 Basics of Seiberg-Witten Invariants

We start by recalling the basics of Seiberg-Witten invariants introduced by Seiberg and Witten (cf. [W],[KM]).

Let X be an oriented, closed Riemannian 4-manifold, and let L be a characteristic line bundle on X, i.e. $c_1(L)$ is an integral lift of $w_2(X)$. This determines a $Spin^c$-structure on X. We denote the associated $U(2)$-bundles by $W^\pm := S^\pm \otimes L^{1/2}$, where $S^\pm$ is a (locally defined) spinor bundle on X. (One may choose a $Spin^c$-structure first, and associated $U(2)$-bundles $W^\pm$ on X. Then $L := \det(W^+) \cong \det(W^-)$ is the associated characteristic line bundle on X.) For simplicity we assume that $H^2(X;\mathbf{Z})$ has no 2-torsion so that the set $Spin^c(X)$ of $Spin^c$-structures on X is identified with the set of characteristic line bundles on X.

Note that Clifford multiplication $c : T^*X \to \mathrm{Hom}(W^+, W^-)$ leads to an isomorphism

$$\rho : \Lambda^+ \otimes \mathbf{C} \longrightarrow sl(W^+)$$

taking Λ^+ to $su(W^+)$, and the Levi-Civita connection on TX together with a unitary connection A on L induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$. This connection, followed by Clifford multiplication, induces a $Spin^c$-Dirac operator D_A : $\Gamma(W^+) \to \Gamma(W^-)$. The Seiberg-Witten equations ([W]) are the following pair of equations for a unitary connection A of L and a section Ψ of $\Gamma(W^+)$:

$$\begin{cases} D_A\Psi &= 0 \\ \rho(F_A^+) &= i(\Psi \otimes \Psi^*)_0 \end{cases} \tag{3.1}$$

where F_A^+ is the self-dual part of the curvature of A and $(\Psi \otimes \Psi^*)_0$ is the trace-free part of $(\Psi \otimes \Psi^*)$ which is interpreted as an endomorphism of W^+.

The gauge group $\mathcal{G} := Aut(L) \cong Map(X, S^1)$ acts on the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ by

$$g \cdot (A, \Psi) = (g \cdot A \cdot g^{-1}, g \cdot \Psi)$$

In particular, if $b_1(X) = 0$, then the gauge group $\mathcal{G}$ is homotopy equivalent to S^1 so that the quotient

$$\mathcal{B}_X^*(L) := \mathcal{A}_X(L) \times (\Gamma(W^+) - 0)/S^1$$

is homotopy equivalent to $\mathbf{CP}^\infty$. Since the set of solutions is invariant under the action, it induces an orbit space, called the *(Seiberg-Witten) moduli space*, denoted by $M_X(L)$, whose formal dimension is

$$\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2e(X))$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X.

Definition A solution (A, Ψ) of the Seiberg-Witten equation (3.1) is called *irreducible* (*reducible*) if $\Psi \not\equiv 0$ ($\Psi \equiv 0$).

Note that if $b^+(X) > 0$ and $M_X(L) \neq \phi$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions, so that it is a compact, smooth manifold of the given dimension. Furthermore the moduli space $M_X(L)$ is orientable and its orientation is determined by a choice of orientation on $\det(H^0(X;\mathbf{R}) \oplus H^1(X;\mathbf{R}) \oplus H^2_+(X;\mathbf{R}))$.

Definition The *Seiberg-Witten invariant* for X with $b_1(X) = 0$ is a function $SW_X : Spin^c(X) \to \mathbf{Z}$ defined by

$$SW_X(L) = \begin{cases} 0 & \text{if } \dim M_X(L) < 0 \text{ or odd} \\ \sum_{(A,\Psi)\in M_X(L)} sign(A,\Psi) & \text{if } \dim M_X(L) = 0 \\ < \beta^{d_L}, [M_X(L)] > & \text{if } \dim M_X(L) := 2d_L > 0 \text{ and even} \end{cases}$$

where $sign(A, \Psi)$ is ± 1 whose sign is determined by an orientation on $M_X(L)$, and β is a generator of $H^2(\mathcal{B}^*_X(L);\mathbf{Z}) \cong H^2(\mathbf{CP}^\infty;\mathbf{Z})$. For convenience, we denote the Seiberg-Witten invariant for X by $SW_X = \sum_L SW_X(L) \cdot e^L$.

Note that if $b^+(X) > 1$, the Seiberg-Witten invariant $SW_X = \sum SW_X(L) \cdot e^L$ is a diffeomorphism invariant, i.e. SW_X does not depend on the choice of generic metric on X and generic perturbation of the Seiberg-Witten equation. Furthermore, only finitely many $Spin^c$-structures on X have a non-zero Seiberg-Witten invariant.

Definition Let X be an oriented, smooth 4-manifold with $b_1 = 0$ and $b^+ > 1$. We say a cohomology class $c_1(L) \in H^2(X;\mathbf{Z})$ is a *Seiberg-Witten basic class* (for brevity, *SW-basic class*) for X if $SW_X(L) \neq 0$.

Definition An oriented, smooth 4-manifold X is called *Seiberg-Witten simple type* (for brevity, *SW-simple type*) if $SW_X(L) = 0$, for all L satisfying $\dim M_X(L) > 0$.

Next we describe a (Seiberg-Witten) gluing theory for computing Seiberg-Witten invariants of a smooth 4-manifold $X = X_+ \cup_Y X_-$ which is separated into two pieces X_+, X_- by an embedded 3-manifold Y. Let (X_R, g_R) be the Riemannian manifold obtained from X by cutting along Y and inserting a cylinder $[-R, R] \times Y$ on which g_R is a product metric. As in Donaldson theory, if the moduli space $M_{X_R}(L)$ is non-empty for all sufficiently large R, then by stretching neck along Y in X (i.e. $R \to \infty$) each solution $(A, \Psi) \in M_X(L)$ is split into three relative solutions

$$((A_+, \Psi_+), (A_0, \Psi_0), (A_-, \Psi_-)) \in M_{X_+}(L|_{X_+}) \times M_{R \times Y}(L|_{R \times Y}) \times M_{X_-}(L|_{X_-}),$$

and conversely any such three relative solutions $(A_+, \Psi_+), (A_0, \Psi_0)$ and (A_-, Ψ_-) induce a global solution $(A_+, \Psi_+)\sharp_{g_1}(A_0, \Psi_0)\sharp_{g_2}(A_-, \Psi_-) \in M_X(L)$, where g_1 and g_2 are gluing parameters. (In general, there is an obstruction to construct a global solution from relative solutions [D].) In particular, if the embedded 3-manifold Y in X has a positive scalar curvature metric (e.g. $Y = S^3, L(p^2, 1-pq)$), then any such solution $(A_0, \Psi_0) \in M_{R \times Y}(L|_{R \times Y})$ is reducible. I.e.

$$\begin{aligned} M_{R \times Y}(L|_{R \times Y}) &= \{(A_0, 0) : A_0 \text{ is an ASD U(1)}-\text{connection on } Y\} \\ &\cong H^1(Y; \mathbf{R})/H^1(Y; \mathbf{Z}) \end{aligned}$$

For example, if $Y = S^3$ or $L(p^2, 1-pq)$, then $M_{R \times Y}(L|_{R \times Y})$ is a single reducible solution. Furthermore, since L is a $U(1)$-bundle, gluing parameters are S^1. In summary, we have

Proposition 3.1.1 *If a smooth 4-manifold X is split into two pieces X_+ and X_- by an embedded 3-manifold $Y = S^3$ or $L(p^2, 1-pq)$, then each solution $(A, \Psi) \in M_X(L)$ can be obtained from two relative solutions $((A_+, \Psi_+), (A_-, \Psi_-)) \in M_{X_+}(L|_{X_+}) \times M_{X_-}(L|_{X_-})$ and*

$$\dim M_X(L) = \dim M_{X_+}(L|_{X_+}) + \dim M_{X_-}(L|_{X_-}) + 1$$

where $M_{X_i}(L|_{X_i})$ is the set of solutions (modulo gauge group) which converge asymptotically to a reducible solution in $M_Y(L|_Y)$.

Note that if $\dim M_{X_-}(L|_{X_-}) < 0$, then $M_{X_-}(L|_{X_-})$ consists of reducible solutions.

3.2 Index Computations

The technical part in the rest of this chapter is to show that $\dim M_{B_{p,q}}(L|_{B_{p,q}}) = -1$ and $\dim M_{C_{p,q}}(L|_{C_{p,q}}) \leq -1$, so that both $M_{B_{p,q}}(L|_{B_{p,q}})$ and $M_{C_{p,q}}(L|_{C_{p,q}})$ consist of a single reducible solution. Before doing this, as a warm-up, we can get a well-known blow-up formula ([FS2]) for Seiberg-Witten invariants by using index computations.

Proposition 3.2.1 *If X is a SW-simple type 4-manifold, then the blow-up $\tilde{X} \equiv X \sharp \overline{\mathbf{CP}^2}$ is also of SW-simple type, and the Seiberg-Witten invariants of $\tilde{X} \equiv X \sharp \overline{\mathbf{CP}}^2$ are*

$$SW_{\tilde{X}} = SW_X \cdot (e^E + e^{-E})$$

where E is the exceptional divisor of $\overline{\mathbf{CP}}^2$.

Proof : Note that a characteristic line bundle on $\tilde{X} \equiv X \sharp \overline{\mathbf{CP}}^2$ is of the form $L + (2k+1)E$, where L is a characteristic line bundle on X and $k \in \mathbf{Z}$. (We identify the exceptional divisor E with its corresponding line bundle on $\overline{\mathbf{CP}}^2$.) Suppose $\tilde{L} :=$

$L+(2k+1)E$ is a characteristic line bundle on $\tilde{X}$ such that $SW_{\tilde{X}}(\tilde{L}) \neq 0$. When splitting apart $\tilde{X}$ along S^3, Proposition 3.1.1 implies that any solution in $M_{\tilde{X}}(\tilde{L})$ can be obtained from two relative solutions which are identified with two (absolute) solutions in $M_X(L) \times M_{\overline{\mathbf{CP}}^2}((2k+1)E)$ (Since stretching neck along S^3 corresponds to choosing a sequence of metric so that the neck is pinched down to a point, the last statement follows from a simple removable singularities argument.). But since

$$\begin{aligned}
\dim M_{\overline{\mathbf{CP}}^2}((2k+1)E) &= 2 \cdot \mathrm{ind} D_A|_{\overline{\mathbf{CP}}^2} + \mathrm{ind}(d^+ + d^*)|_{\overline{\mathbf{CP}}^2} \\
&= 2 \cdot (e^{\frac{(2k+1)E}{2}} \cdot \hat{A}(\overline{\mathbf{CP}}^2)) \cdot [\overline{\mathbf{CP}}^2] + (h^1 - h^0 - h^+)(\overline{\mathbf{CP}}^2) \\
&= 2 \cdot \int_{\overline{\mathbf{CP}}^2} (\frac{((2k+1)E)^2}{8} - \frac{p_1}{24}) \; - \; 1 \\
&= 2 \cdot \frac{-(4k^2 + 4k)}{8} - 1 \\
&\leq -1.
\end{aligned}$$

(In case $Y = S^3$, $\mathrm{ind} D_A$ has no boundary terms.) Thus $M_{\overline{\mathbf{CP}}^2}((2k+1)E)$ consists of a single reducible solution, and $M_{\tilde{X}}(\tilde{L})$ can be identified with $M_X(L)$. Furthermore, since

$$\begin{aligned}
\dim M_{\tilde{X}}(\tilde{L}) &= \frac{1}{4}\{(c_1(L) + (2k+1)E)^2 - (3\sigma(\tilde{X}) + 2e(\tilde{X}))\} \\
&= \frac{1}{4}\{c_1(L)^2 - (3\sigma(X) + 2e(X))\} - (k^2 + k) \\
&= \dim M_X(L) - (k^2 + k),
\end{aligned}$$

the SW-simple type condition on X and $SW_{\tilde{X}}(\tilde{L}) \neq 0$ imply that $\dim M_{\tilde{X}}(\tilde{L}) = 0$ and $k = 0$ or -1. Hence $\tilde{X}$ is also of SW-simple type and $SW_X(L) = SW_{\tilde{X}}(L+E) = SW_{\tilde{X}}(L-E)$.□

In order to compute $\mathrm{ind} D_A$ on $B_{p,q}$ and $C_{p,q}$, we need the following two elementary trigonometric computations.

Lemma 3.2.1 *For relatively prime integers p and q, and* $z = e^{\frac{2\pi i}{p^2}}$

$$\sum_{k=1}^{p^2-1} \frac{z^{tpk}}{(z^k-1)(z^{(pq-1)k}-1)} = \sum_{k=1}^{p^2-1} \frac{1}{(z^k-1)(z^{(pq-1)k}-1)} \quad , \text{ for all } t \in \mathbf{Z}$$

Proof : There exist integers r and s satisfying $rp + sq = 1$; so $z^{tpk} = z^{stpqk}$.

Thus it suffices to show

$$\sum_{k=1}^{p^2-1} \frac{z^{tpqk}-1}{(z^k-1)(z^{(pq-1)k}-1)} = 0 \quad , \text{ for all } t \in \mathbf{Z}$$

Given $t \in \mathbf{Z}$ and setting $w = z^{pq-1}$,

$$\begin{aligned}
\sum_{k=1}^{p^2-1} \frac{z^{(t+1)pqk} - z^{tpqk}}{(z^k-1)(z^{(pq-1)k}-1)} &= \sum_{k=1}^{p^2-1} \frac{z^{tpqk}\{(z^k-1)(w^k-1)\} + z^{tpqk}\{(w^k-1)+(z^k-1)\}}{(z^k-1)(w^k-1)} \\
&= \sum_{k=1}^{p^2-1}\{z^{tpqk} + \frac{2}{(z^k-1)}\} + \sum_{k=1}^{p^2-1}\{\frac{(z^{tpqk}-1)}{(z^k-1)} + \frac{(w^{-(pq+1)tpqk}-1)}{(w^k-1)}\} \\
&= \sum_{k=1}^{p^2-1}\{z^{tpqk} + \frac{2}{(z^k-1)}\} + \sum_{k=1}^{p^2-1}\{\frac{(z^{tpqk}-1)}{(z^k-1)} - \frac{(w^{tpqk}-1)}{w^{tpqk}(w^k-1)}\} \\
&= \sum_{k=1}^{p^2-1}\{z^{tpqk} + \frac{2}{(z^k-1)}\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1}\{z^{lk} - (w^{-1})^{(tpq-l)k}\} \\
&= \sum_{k=1}^{p^2-1}\{z^{tpqk} + \frac{2}{(z^k-1)}\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1} z^{lk} - \sum_{l=1}^{tpq}\sum_{k=1}^{p^2-1}(w^{-1})^{lk} \\
&= \sum_{k=1}^{p^2-1}\{z^{tpqk} + \frac{2}{(z^k-1)}\} + \sum_{l=0}^{tpq-1}\sum_{k=1}^{p^2-1} z^{lk} - \sum_{l=1}^{tpq}\sum_{k=1}^{p^2-1} z^{lk} \\
&= \sum_{k=1}^{p^2-1} \frac{2}{(z^k-1)} + (p^2-1) \\
&= 0
\end{aligned}$$

Hence the lemma follows from induction on t. □

Lemma 3.2.2 *For relatively prime integers p and q, and* $z = e^{\frac{2\pi i}{p^2}}$

$$s(1-pq,p^2) = \sum_{k=1}^{p^2-1} \cot(\frac{\pi k}{p^2}) \cdot \cot(\frac{\pi k(1-pq)}{p^2}) \; = \; \frac{2}{3}(1-p^2)$$

$$\text{equivalently,} \quad \sum_{k=1}^{p^2-1} \frac{1}{(z^k-1)(z^{(pq-1)k}-1)} \; = \; \frac{1}{12}(p^2-1)$$

Note that this lemma can also be proved by using different method ([HZ]).

Proof : An easy computation shows that

$$s(1-pq,p^2) = (1-p^2) + \sum_{k=1}^{p^2-1} \frac{4}{(z^k-1)(z^{(pq-1)k}-1)}$$

Note that for $0 \le t \le p-1$ and $w = z^p$,

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{w^{tk}-1}{(w^k-1)(w^{-k}-1)} &= \sum_{l=0}^{t-1}\sum_{k=1}^{p-1} \frac{w^{lk}}{(w^{-k}-1)} \\
&= \sum_{k=1}^{p-1} \frac{-t}{(w^k-1)} - \sum_{l=1}^{t}\sum_{k=1}^{p-1} \frac{(w^{lk}-1)}{(w^k-1)} \\
&= \frac{t(p-1)}{2} - \sum_{l=1}^{t}((p-1)-(l-1)) \\
&= \frac{t^2-tp}{2}
\end{aligned}$$

(The third equality follows from the fact that $\sum_{k=1}^{p-1} w^{lk} = -1$, for $1 \le l \le p-1$).

Hence by using the equality $\sum_{t=0}^{p-1} w^{tk} = 0$ for $1 \le k \le p-1$,

$$\begin{aligned}
0 &= \sum_{t=1}^{p-1}\sum_{k=1}^{p-1} \frac{w^{tk}}{(w^k-1)(w^{-k}-1)} + \sum_{k=1}^{p-1} \frac{1}{(w^k-1)(w^{-k}-1)} \\
&= \sum_{t=1}^{p-1} \frac{(t^2-tp)}{2} + \sum_{k=1}^{p-1} \frac{p}{(w^k-1)(w^{-k}-1)}
\end{aligned}$$

so that
$$\frac{p}{12}(p^2-1) = \sum_{k=1}^{p-1} \frac{p}{(w^k-1)(w^{-k}-1)}$$

Finally by using the fact that $\sum_{l=0}^{p-1} z^{lpqk} = 0$ if $k \neq tp$ and $\sum_{l=0}^{p-1} z^{lpqk} = p$ if $k = tp$, and by Lemma 3.2.1, we have

$$\begin{aligned}\sum_{k=1}^{p^2-1} \frac{p}{(z^k-1)(z^{(pq-1)k}-1)} &= \sum_{l=0}^{p-1}\sum_{k=1}^{p^2-1} \frac{z^{lpk}}{(z^k-1)(z^{(pq-1)k}-1)} \\ &= \sum_{t=1}^{p-1} \frac{p}{(z^{tp}-1)(z^{(pq-1)tp}-1)} \\ &= \sum_{t=1}^{p-1} \frac{p}{(w^t-1)(w^{-t}-1)} \\ &= \frac{p}{12}(p^2-1) \qquad \square\end{aligned}$$

Proposition 3.2.2 *For any characteristic line bundle L_B on $B_{p,q}$ with a cylindrical end*

$$B^+_{p,q} = B_{p,q} \cup L(p^2, 1-pq) \times [1,\infty)$$

$dim M_{B^+_{p,q}}(L_B) = -1$; so the moduli space $M_{B^+_{p,q}}(L_B)$ consists of a single reducible solution.

Proof : It suffices to show that $\mathrm{ind}(D_A|_{B^+_{p,q}}) = 0$ because

$$\begin{aligned}\dim M_{B^+_{p,q}}(L_B) &= 2\cdot \mathrm{ind}(D_A|_{B^+_{p,q}}) + \mathrm{ind}(d^+ + d^*)|_{B^+_{p,q}} \\ &= 2\cdot \mathrm{ind}(D_A|_{B^+_{p,q}}) + (b^1 - b^0 - b^+)(B^+_{p,q}) \\ &= 2\cdot \mathrm{ind}(D_A|_{B^+_{p,q}}) \ - 1\end{aligned}$$

where A is a $U(1)$-connection on $L_B \to B^+_{p,q}$. Now compute

$$\begin{aligned}\mathrm{ind}(D_A|_{B^+_{p,q}}) &= (e^{\frac{c_1(L_B)}{2}}\cdot \hat{A}(B^+_{p,q}))\cdot[B^+_{p,q}] \\ &= \int_{B^+_{p,q}} (\frac{c_1(L_B)^2}{8} - \frac{p_1}{24}) \ - \ (\frac{h+\eta(0)}{2})\end{aligned}$$

Since L_B is a flat connection on $B^+_{p,q}$ the first term $\frac{c_1(L_B)^2}{8} = 0$, and the second term can be computed by using Proposition 2.12 in [APS]

$$0 = \sigma(B^+_{p,q}) = \int_{B^+_{p,q}} (\frac{p_1}{3}) + \frac{1}{p^2}\sum_{k=1}^{p^2-1} \cot(\frac{\pi k}{p^2}) \cdot \cot(\frac{\pi k(1-pq)}{p^2})$$

Hence, by Lemma 3.2.2,

$$\int_{B^+_{p,q}} (\frac{p_1}{24}) = \frac{-1}{8p^2} \cdot s(1-pq, p^2) = \frac{1}{12p^2}(p^2-1)$$

The boundary term, $\frac{h+\eta(0)}{2}$, can also be computed by using Atiyah-Singer fixed point theorem ([Sh, §19]) for a $Spin^c$-Dirac operator D_A on $D^4/\mathbf{Z}_{p^2} \cong$ cone on $L(p^2, 1-pq)$:

$$\begin{aligned}
\frac{h+\eta(0)}{2} &= \frac{-1}{p^2} \sum_{g\in \mathbf{Z}_{p^2}-\{0\}} Spin(g, D^4) \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{(e^{\pi ki/p^2} - e^{-\pi ki/p^2})(e^{(1-pq)\pi ki/p^2} - e^{-(1-pq)\pi ki/p^2}) \cdot e^{mp\cdot\pi ki/p^2}}{(1-e^{\pi ki/p^2})(1-e^{-\pi ki/p^2})(1-e^{(1-pq)\pi ki/p^2})(1-e^{-(1-pq)\pi ki/p^2})} \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{e^{mp\cdot\pi ki/p^2}}{(e^{\pi ki/p^2} - e^{-\pi ki/p^2})(e^{(1-pq)\pi ki/p^2} - e^{-(1-pq)\pi ki/p^2})}
\end{aligned}$$

where $c_1(L_B|_{L(p^2,1-pq)}) = mp \in H^2(L(p^2, 1-pq); \mathbf{Z}) \cong \mathbf{Z}_{p^2}$ (Lemma 2.1.2). Since L_B is a characteristic line bundle, we can always choose an integer m so that $m+q$ is even. (If p and $m+q$ are odd, choose $m+p+q \equiv m+q \pmod p$. If p is even, then m and q are odd.) By setting $z := e^{2\pi i/p^2}$ and $t := (m+q)/2 \in \mathbf{Z}$, we have

$$\begin{aligned}
\frac{h+\eta(0)}{2} &= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{e^{\pi(m+q)ki/p}}{(e^{2\pi ki/p^2}-1)(e^{2\pi(pq-1)ki/p^2}-1)} \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{z^{tpk}}{(z^k-1)(z^{(pq-1)k}-1)} \\
&= \frac{-1}{p^2}\sum_{k=1}^{p^2-1} \frac{1}{(z^k-1)(z^{(pq-1)k}-1)} \qquad \text{(by lemma 3.2.1)} \\
&= \frac{1}{12p^2}(1-p^2) \qquad \text{(by lemma 3.2.2)}
\end{aligned}$$

Combining these computations we get $\operatorname{ind}(D_A|_{B^+_{p,q}}) = 0$. □

Remark : In the proof of Proposition 3.2.2 above, if both p and m are even (in particular $m=0$), a similar computation shows that $\operatorname{ind} D_A$ on $B_{p,q}$ is not an integer. This contradiction means that $B_{p,q}$ is not spin for p even (cf. Lemma 2.1.3).

Corollary 3.2.1 *For any characteristic line bundle L_C on $C^+_{p,q} = C_{p,q} \cup L(p^2, 1-pq) \times [1,\infty)$, $\dim M_{C^+_{p,q}}(L_C)$ is odd and ≤ -1; so the moduli space $M_{C^+_{p,q}}(L_C)$ consists of a single reducible solution.*

Proof : Since $\operatorname{ind}(d^+ + d^*|_{C^+_{p,q}}) = (b^1 - b^0 - b^+)(C^+_{p,q}) = -1$, as the same way in the proof above, it suffices to show that $\operatorname{ind}(D_A|_{C^+_{p,q}}) \leq 0$. Since $X = C^+_{p,q} \cup_L \overline{B^+_{p,q}}$ is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$ with $k = b_2(C_{p,q})$, for any characteristic line bundle L on X, $c_1(L)^2 \leq -k$ and

$$\operatorname{ind}(D_A|_{C^+_{p,q}}) + \operatorname{ind}(D_A|_{\overline{B^+_{p,q}}}) = \operatorname{ind}(D_A|_X) = \int_X \frac{(c_1(L)^2 + k)}{8} \leq 0$$

Hence $\operatorname{ind}(D_A|_{C^+_{p,q}}) \leq -\operatorname{ind}(D_A|_{B^+_{p,q}}) = 0$. □

3.3 Main Technical Theorems

Lemma 3.3.1 *Let X be a smooth 4-manifold containing a configuration $C_{p,q}$, that is, $X = X_0 \cup_{L(p^2,1-pq)} C_{p,q}$, and let $X_{p,q}$ be its rational blow-down. Then a line bundle L on $X_{p,q}$ is characteristic if and only if both $L|_{X_0}$ on X_0 and $L|_{B_{p,q}}$ on $B_{p,q}$ are characteristic.*

Proof : Since $H^1(B_{p,q}; \mathbf{Z}_2) \to H^1(L(p^2, 1-pq); \mathbf{Z}_2)$ is surjective, $i^* \oplus j^* : H^2(X_{p,q}; \mathbf{Z}_2) \to H^2(X_0; \mathbf{Z}_2) \oplus H^2(B_{p,q}; \mathbf{Z}_2)$ is injective. Hence the proof follows from the following commutative diagram

$$\begin{array}{ccccc} & 0 & \longrightarrow & H^2(X_{p,q};\mathbf{Z}) & \longrightarrow H^2(X_0;\mathbf{Z}) \oplus H^2(B_{p,q};\mathbf{Z}) \\ & & & \downarrow & \downarrow \\ H^1(L(p^2,1-pq);\mathbf{Z}_2) & \longrightarrow & & H^2(X_{p,q};\mathbf{Z}_2) & \stackrel{i^*\oplus j^*}{\longrightarrow} H^2(X_0;\mathbf{Z}_2) \oplus H^2(B_{p,q};\mathbf{Z}_2) \end{array} \qquad \square$$

Theorem 3.3.1 *Suppose X is a smooth 4-manifold which contains a configuration $C_{p,q}$. If L is a characteristic line bundle on X such that $SW_X(L) \neq 0$, $(L|_{C_{p,q}})^2 = -b_2(C_{p,q})$ and $c_1(L|_{L(p^2,1-pq)}) = mp \in \mathbf{Z}_{p^2} \cong H^2(L(p^2,1-pq);\mathbf{Z})$ with $m \equiv (p-1) \pmod 2$, then L induces a characteristic line bundle $\overline{L}$ on $X_{p,q}$ such that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$.*

Proof : Lemma 2.1.2 and the condition $c_1(L|_{L(p^2,1-pq)}) = mp$ with $m \equiv (p-1)$ (mod 2) imply that the characteristic line bundle $L|_{X_0}$ on X_0 extends uniquely to a characteristic line bundle $\overline{L}$ on $X_{p,q}$. First we study the solutions of Seiberg-Witten equations on X for L by pulling apart $X = X_0 \cup_{L(p^2,1-pq)} C_{p,q}$ along $L(p^2,1-pq)$. Then Proposition 3.1.1 and Corollary 3.2.1 imply that each solution in $M_X(L)$ can be obtained by gluing a solution $(A_{X_0}, \Psi_{X_0}) \in M_{X_0}(L|_{X_0})$ with a unique reducible solution $(A_{C_{p,q}}, 0) = M_{C_{p,q}}(L|_{C_{p,q}})$. But, not every solution in $M_{X_0}(L|_{X_0})$ produces a global solution in $M_X(L)$. Explicitly, using Corollary 3.2.1, the inequality

$$2d_L = \dim M_X(L) = \dim M_{X_0}(L|_{X_0}) + \dim M_{C_{p,q}}(L|_{C_{p,q}}) + 1 \leq \dim M_{X_0}(L|_{X_0}) = 2d_{L|_{X_0}}$$

implies that there is an obstruction bundle ξ of rank $d_{L|_{X_0}} - d_L$ associated to the basepoint fibration over $M_{X_0}(L|_{X_0})$ such that the zero set of a generic section of ξ is homologous to $M_X(L)$ in $\mathcal{B}^*_X(L)$ (Theorem 4.53 in [D], or [FS2, §4]). Hence

$$SW_X(L) = <\beta^{d_L}, [M_X(L)]> = <\beta^{d_L}, \beta^{d_{L|_{X_0}} - d_L} \cap [M_{X_0}(L|_{X_0})]> = <\beta^{d_{L|_{X_0}}}, [M_{X_0}(L|_{X_0})]>$$

where β is a generator of $H^2(\mathcal{B}^*_X(L);\mathbf{Z})$. Similarly, since $\dim M_{B_{p,q}}(\overline{L}|_{B_{p,q}}) = -1$ by Proposition 3.2.2, the same argument as above shows

$$SW_{X_{p,q}}(\overline{L}) = < \beta^{d_{L|X_0}}, [M_{X_0}(L|_{X_0})] >$$

so that $SW_{X_{p,q}}(\overline{L}) = SW_X(L)$. □

Corollary 3.3.1 *If two characteristic line bundles L and L' on X satisfying the hypothesis in Theorem 3.3.1 induce the same characteristic line bundle $\overline{L}$ on $X_{p,q}$, then $SW_X(L) = SW_X(L')$.*

Freedman's classification of simply connected topological 4-manifolds implies that $X \equiv C_{p,q} \cup_L \overline{B_{p,q}}$ is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$ with $k = b_2(C_{p,q})$. Each generator e_i of $H^2(X;\mathbf{Z})$ when restricted to $B_{p,q}$ has the boundary value $\partial(e_i|_{B_{p,q}}) = mp \in H^2(L(p^2, 1-pq);\mathbf{Z})$ for some m. We impose the following condition $(*)$ on $C_{p,q}$:

$$(*)\ \{\partial(\sum_{i=1}^{k} \epsilon_i e_i|_{B_{p,q}}) : \epsilon_i = \pm 1, \forall i\} = \{mp : -(p-1) \leq m \leq (p-1) \text{ and } m \equiv (p-1) \pmod 2\}$$

All known configurations $C_{p,q}$ satisfy the condition $(*)$ above. (One expects that all relatively prime integers (p,q) satisfy the condition $(*)$.) Under this assumption, we prove

Lemma 3.3.2 *Suppose X is a simply connected smooth 4-manifold which contains a configuration $C_{p,q}$ satisfying the condition $(*)$, and let $X_{p,q}$ be its rational blow-down. If $\overline{L}$ is a characteristic line bundle on $X_{p,q}$, there exists a characteristic line bundle L on X such that $L|_{X_0} = \overline{L}|_{X_0}$ and $c_1(L|_{C_{p,q}})^2 = -k$, where $k = b_2(C_{p,q})$.*

Proof : The condition $(*)$ on $C_{p,q}$ implies that there exists $\epsilon_i = \pm 1$, for $1 \leq i \leq k$, such that $\partial(\sum_{i=1}^{k} \epsilon_i e_i|_{B_{p,q}}) = mp = \partial c_1(\overline{L}|_{B_{p,q}})$. Since the corresponding line bundle, denoted by the same notation $\sum_{i=1}^{k}\epsilon_i e_i$, is characteristic on $C_{p,q}\cup_L \overline{B_{p,q}}$ which is homeomorphic to $\sharp k\overline{\mathbf{CP}}^2$, its restriction $\sum_{i=1}^{k}\epsilon_i e_i|_{C_{p,q}}$ is also characteristic on $C_{p,q}$ and $(\sum_{i=1}^{k}\epsilon_i e_i|_{C_{p,q}})^2 = (\sum_{i=1}^{k}\epsilon_i e_i)^2 - (\sum_{i=1}^{k}\epsilon_i e_i|_{\overline{B_{p,q}}})^2 = (\sum_{i=1}^{k}\epsilon_i e_i)^2 = -k$. Now define a line bundle L on X by

$$L = \begin{cases} \overline{L}|_{X_0} & \text{on } X_0 \\ \sum_{i=1}^{k}\epsilon_i e_i|_{C_{p,q}} & \text{on } C_{p,q} \end{cases}$$

Then L has the desired properties except (possibly) characteristic, that is, if p is odd, then L is automatically a characteristic line bundle on X, so we are done. If p is even, we can change L (see below) so that L is characteristic on X satisfying the same properties.

Suppose p is even.

$$\begin{array}{ccccc} & 0 & \longrightarrow & H^2(X;\mathbf{Z}) & \longrightarrow H^2(X_0;\mathbf{Z})\oplus H^2(C_{p,q};\mathbf{Z}) \\ & & & \downarrow h_* & \qquad\qquad \downarrow \\ H^1(L(p^2,1-pq);\mathbf{Z}_2) & \xrightarrow{\delta} & & H^2(X;\mathbf{Z}_2) & \xrightarrow{i^*\oplus j^*} H^2(X_0;\mathbf{Z}_2)\oplus H^2(C_{p,q};\mathbf{Z}_2) \end{array}$$

Since X is simply connected, $H_1(X_0;\mathbf{Z})\cong \mathbf{Z}_t$ for some t dividing p^2. If t is even, then $i^*\oplus j^* : H^2(X;\mathbf{Z}_2)\to H^2(X_0;\mathbf{Z}_2)\oplus H^2(C_{p,q};\mathbf{Z}_2)$ is injective so that L is characteristic. If t is odd, then $i^*\oplus j^*$ is not injective, and in this case $h_*(c_1(L)) = w_2(X)$ or $w_2(X)+\delta(1)$.

Since $C_{p,q}$ satisfies the condition $(*)$, there exists $\delta_i=\pm 1$ satisfying $\sum_{i=1}^{k}\delta_i e_i|_{C_{p,q}} = (p-m)p$. Then setting $\gamma_i \equiv \frac{\epsilon_i+\delta_i}{2}$ we have

1) $\partial(\sum_{i=1}^{k}\gamma_i e_i|_{C_{p,q}}) = (\frac{p}{2})p \neq 0$
2) $\partial(\sum_{i=1}^{k}(\epsilon_i - 2\gamma_i)e_i|_{C_{p,q}}) = \partial(\sum_{i=1}^{k}\epsilon_i e_i|_{C_{p,q}}) = mp$
3) $\sum_{i=1}^{k}(\epsilon_i - 2\gamma_i)e_i|_{C_{p,q}} = \sum_{i=1}^{k}\epsilon_i' e_i|_{C_{p,q}}$, for some $\epsilon_i' = \pm 1$.

Hence there exists a bundle L' on X such that $L'|_{C_{p,q}} = \sum_{i=1}^{k}(\epsilon_i - 2\gamma_i)e_i|_{C_{p,q}}$ and $L'|_{X_0} = L|_{X_0}$. Then we claim either L or L' is characteristic: Suppose neither L nor L' is characteristic, i.e. $h_*(c_1(L)) = h_*(c_1(L')) = w_2(X) + \delta(1)$. Then $h_*(L - L') = 0$, so that there exists an element $\alpha \in H^2(X;\mathbf{Z})$ satisfying $2\alpha = L - L'$. Since both $H^2(X_0;\mathbf{Z})$ and $H^2(C_{p,q};\mathbf{Z})$ are 2-torsion free,

$$2(\alpha|_{X_0}, \alpha|_{C_{p,q}}) = (i^* \oplus j^*)(2\alpha) = (i^* \oplus j^*)(L - L') = 2(0, \sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}})$$

implies $\alpha|_{X_0} = 0$ and $\alpha|_{C_{p,q}} = \sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}$ which contradicts $\partial(\sum_{i=1}^{k} \gamma_i e_i|_{C_{p,q}}) = (\frac{p}{2})p \neq 0$. □

Finally, by using the same argument as in the proof of Theorem 3.3.1 with the characteristic line bundle L on X constructed in the Lemma 3.3.2 above, we get our main technical theorem.

Theorem 3.3.2 *If a simply connected smooth 4-manifold X contains a configuration $C_{p,q}$ satisfying the condition $(*)$, then the Seiberg-Witten invariants of $X_{p,q}$ are completely determined by those of X. That is, for any characteristic line bundle $\overline{L}$ on $X_{p,q}$ with $SW_{X_{p,q}}(\overline{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_{p,q}}(\overline{L})$. Furthermore, if X is of SW- simple type, then $X_{p,q}$ is also of SW-simple type.*

Chapter 4

The Geography of Irreducible 4-Manifolds

In this chapter we apply the result of the previous section to several examples of rational blow-downs and explore geography problems for simply connected smooth irreducible 4-manifolds (Fig 4.1). The geography problems we are interested in studying are twofold, that is, which lattice points in the $(\frac{b^+ + 1}{2}, 3\sigma + 2e)$-plane are 'populated' by simply connected smooth irreducible 4-manifolds (the existence problem) and if so, are there infinitely many distinct smooth 4-manifolds which are all homeomorphic (the uniqueness problem)? These coordinates are chosen because of their relation to complex surfaces where holomorphic Euler characteristic $\chi = \frac{1}{12}(c_1^2 + c_2) = \frac{b^+ + 1}{2}$ and the chern number $c_1^2 = 3\sigma + 2e$. The geography problem for surfaces of general type has been studied extensively by algebraic surface theorists (see remarks below), and for topologists, the problems are to find constructions of new 4-manifolds and to be able to compute invariants (such as Donaldson invariants and Seiberg-Witten invariants) which can show that the result is an irreducible 4-manifold. Note that a smooth 4-manifold X is called *irreducible* if X is not a connected sum of other manifolds except for a homotopy sphere, i.e. if $X = X_1 \sharp X_2$ implies that one of X_i is a homotopy sphere. One of the most powerful applications of gauge theory to

4-dimensional topology related to geography problems is that both Donaldson invariants and Seiberg-Witten invariants for a connected sum manifold $X = X_1 \sharp X_2$ with $b^+(X_i) > 0$ $(i = 1,2)$ vanish. Hence $SW_X \neq 0$ (or $D_X \neq 0$) implies that X is irreducible unless X is a blow-up manifold.

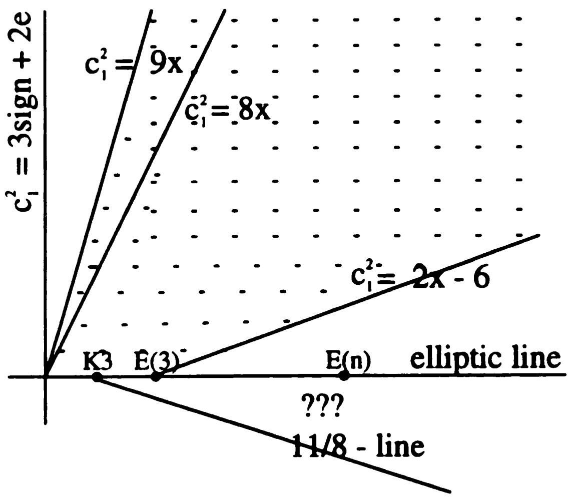


Figure 4.1:

Here are a few remarks on Figure 4.1 below:

1. The simply connected minimal complex surfaces of general type live in the dotted region determined by the "Noether line", $c_1^2 = 2\chi - 6$ $(5\sigma + 3e + 12 \geq 0)$, and the "Bogomolov-Miyaoka-Yau line", $c_1^2 = 9\chi$ $(3\sigma \leq e)$. A surface of signature $= 0$ has $c_1^2 = 8\chi$, so any surface of negative signature lies in the region $c_1^2 < 8\chi$, and any lattice point lying in this region and above $c_1^2 = 2\chi - 6$ can be realized as (χ, c_1^2) of a minimal surface which is a hyperelliptic fibration ([P]).

2. Moishezon and Teicher constructed infinitely many simply connected minimal surfaces of positive signature (equivalently, lying in between $c_1^2 = 8\chi$ and

$c_1^2 = 9\chi$). Xiao and Chen also constructed other minimal surfaces of positive signature which are hyperelliptic fibrations ([C]).

3. Any irreducible 4-manifold in the wedge between "elliptic line", $c_1^2 = 0$, and "Noether line" cannot admit a complex structure with either orientation because it violates Noether inequality or B-M-Y inequality. The examples lying in this wedge were first found by Fintushel and Stern ([FS3]). Actually, they found examples realizing all lattice points below the Noether line. We also construct other examples lying in this wedge (see Example 2 and Theorem 4.2.5).

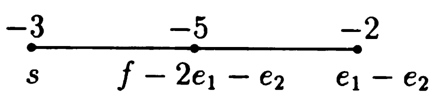
4. There are no known irreducible 4-manifolds lying in elliptic line below, $c_1^2 < 0$, and there is a conjecture that every smooth spin 4-manifold satisfies $\frac{b_2}{|\sigma|} \geq \frac{11}{8}$. Note that the rational blow-down procedure moves a manifold vertically upward and blowing up procedure moves a manifold vertically downward in Figure 4.1.

4.1 Examples

We compute the Seiberg-Witten invariants of a manifold constructed from $E(n)$ via blowing up and rationally blowing down.

Example 1 Consider a 4-manifold $X \equiv E(3)\sharp 2\overline{\mathbf{CP}}^2$ constructed by the following blowing up process (Fig 4.2):

Then we get a configuration $C_{5,2} \subset X$



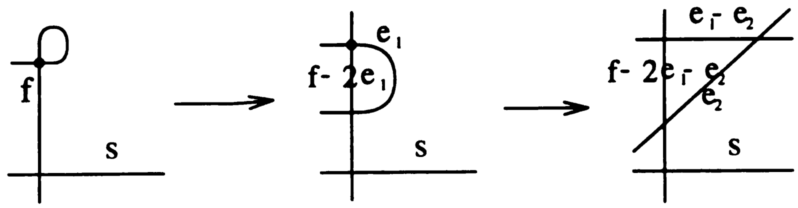


Figure 4.2:

where s is a section in $E(3)$ and e_i $(i = 1, 2)$ is the exceptional divisor in $\overline{\mathbf{CP}}^2$. Since SW-basic classes in $E(3)$ are $\pm f$, up to sign the SW-basic classes of X are of the form

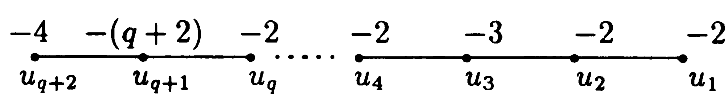
$$L = f + \epsilon_1 e_1 + \epsilon_2 e_2 \qquad (\epsilon_i = \pm 1)$$

By using boundary values (cf. equation (2.1)), compute $L|_{C_{5,2}}$ and $\partial(L|_{C_{5,2}})$

$$\begin{aligned} L|_{C_{5,2}} &= (L \cdot u_1)\gamma_1 + (L \cdot u_2)\gamma_2 + (L \cdot u_3)\gamma_3 \\ &= (\epsilon_2 - \epsilon_1)\gamma_1 + (2\epsilon_1 + \epsilon_2)\gamma_2 + \gamma_3 \\ \partial(L|_{C_{5,2}}) &= (\epsilon_2 - \epsilon_1) + 2(2\epsilon_1 + \epsilon_2) + 9 \\ &= 3(\epsilon_1 + \epsilon_2) + 9 \end{aligned}$$

Then $\partial(L|_{C_{5,2}})$ is a multiple of $p = 5$ if and only if $\epsilon_1 = \epsilon_2 = 1$. Hence by Theorem 3.3.1, only $L = f + e_1 + e_2$ descends to a SW-basic class $\overline{L}$ of $X_{5,2}$, and by Theorem 3.3.2, $\overline{L}$ is the only SW-basic class of $X_{5,2}$. Since $c_1(\overline{L})^2 = c_1(L)^2 - c_1(L|_{C_{5,2}})^2 = -2 + 3 = 1$, $X_{5,2}$ is a SW-simple type 4-manifold with $c_1^2 = 1$ which has one basic class $\overline{L} = \overline{f + e_1 + e_2}$ (up to sign) and its Seiberg-Witten invariant is $SW_{X_{5,2}}(\overline{L}) = SW_X(L) = 1$.

Next, let us consider a configuration $C_{4q-1,q}$



whose boundary values (cf. equation (2.1)) are given by

$$\partial\gamma_i = \begin{cases} i & i = 1,2 \\ 4i-9 & i = 3,\dots,q+1 \\ (4q-1)q-1 & i = q+2 \end{cases}$$

Then we have

Proposition 4.1.1 *Suppose X is a simply connected smooth 4-manifold containing a configuration $C_{p,q}$ $(p = 4q-1)$. If each u_i satisfies $|L\cdot u_i| + u_i^2 \leq -2$, for each basic class L in X, then Seiberg-Witten invariants of $X_{p,q}$ are given by*

$$SW_{X_{p,q}}(\overline{L}) = \begin{cases} SW_X(L) & \text{if } L\cdot u_3 = \epsilon, \quad L\cdot u_{q+1} = \epsilon q \quad \text{and} \quad L\cdot u_{q+2} = 2\epsilon \ (\epsilon = \pm 1) \\ 0 & \text{otherwise} \end{cases}$$

Remark : The hypothesis, $|L\cdot u_i| + u_i^2 \leq -2$, in Proposition 4.1.1 above comes from the adjunction inequality in [FS2]. Our assumption is that the u_i are generic in the sense that they do not fall into the special case of Theorem 1.3 in [FS2].

Proof : The condition $|L\cdot u_i| + u_i^2 \leq -2$ implies $L\cdot u_i = 0$ $(i = 1,2,4,\dots,q)$, so that

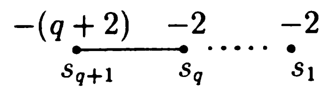
$$\begin{aligned} L|_{C_{p,q}} &= (L\cdot u_3)\gamma_3 + (L\cdot u_{q+1})\gamma_{q+1} + (L\cdot u_{q+2})\gamma_{q+2} \\ \partial(L|_{C_{p,q}}) &= 3(L\cdot u_3) + (4q-5)(L\cdot u_{q+1}) + (pq-1)(L\cdot u_{q+2}) \\ &\equiv 3(L\cdot u_3) - 4(L\cdot u_{q+1}) - (L\cdot u_{q+2}) \qquad (\text{mod } p) \end{aligned}$$

Since $L|_{C_{p,q}}$ is characteristic, the condition $\partial(L|_{C_{p,q}}) \equiv 0 \pmod p$ in Theorem 3.3.1 implies that only basic class $\overline{L}$ in $X_{p,q}$ comes from L of X satisfying

$$L\cdot u_3 = \epsilon, \quad L\cdot u_{q+1} = \epsilon q \quad \text{and} \quad L\cdot u_{q+2} = 2\epsilon \qquad (\epsilon = \pm 1)$$

The rest of the proof follows from Theorem 3.3.2. □

Example 2 Let $X \equiv E(q+2)\sharp 2\overline{\mathbf{CP}}^2 (1 \leq q \leq 8)$ be a manifold constructed as follows: Consider the following configuration in $E(q+2)$



where s_{q+1} is a section and $f \cdot s_i = 0$, for $i = 1, \cdots, q$. (One can choose such a configuration by using $\tilde{E}_8$-fiber $\subset E(2)\sharp_f E(q) \cong E(q+2)$.) By blowing up the double point of a nodal fiber f in $E(q+2)$ and another point in s_3, we have a configuration $C_{4q-1,q} \subset X$ such that

$$u_{q+2} = f - 2e_1, \ u_3 = s_3 - e_2 \text{ and } u_i = s_i, \ i \neq 3, q+2$$

Since the SW-basic classes of X have the form

$$L = kf + \epsilon_1 e_1 + \epsilon_2 e_2 \quad (|k| \leq q, \ k \equiv q \pmod 2 \ \text{ and } \ \epsilon_i = \pm 1)$$

this example satisfies the hypothesis of the Proposition 4.1.1 above. It follows that $X_{p,q}$ has one basic class $\overline{L} = \overline{qf + e_1 + e_2}$ (up to sign) with $c_1(\overline{L})^2 = q$. Hence $X_{p,q}$ is a SW-simple type smooth 4-manifold lying in $c_1^2 = \chi - 2$ which has one basic class and cannot admit a complex structure. Note that for $q > 8$, if one can find such a configuration in $E(q+2)$ (It seems to be possible), then the same argument holds.

Example 3 (p-log transform) As we see in [FS3] (or Theorem 2.2.1), $E(n;p)$ is obtained by blowing up and rational blow-down from $E(n)$, so that Seiberg-Witten invariant of $E(n;p)$ can be computed explicitly as the same way as in Example 1:

Theorem 4.1.1 ([FS3]) *The Seiberg-Witten invariants of $E(n;p)$ are*

$$SW_{E(n;p)} = SW_{E(n)} \cdot (e^{(p-1)f_p} + e^{(p-3)f_p} + \cdots + e^{-(p-1)f_p})$$

where f_p is a multiple fiber obtained by p-log transform on $E(n)$.

Furthermore, by extending the notion of 'p-log transform' to any smooth 4-manifold containing a cusp neighborhood, we extend this result

Corollary 4.1.1 *Let $X(p)$ be the result of p-log transform in the neighborhood of a cusp, say f, in a SW-simple type irreducible 4-manifold X. Then the Seiberg-Witten invariants of $X(p)$ are*

$$SW_{X(p)} = SW_X \cdot (e^{(p-1)f_p} + e^{(p-3)f_p} + \cdots + e^{-(p-1)f_p})$$

where f_p is a multiple fiber in $X(p)$ obtained by p-log transform on X.

Proof : It suffices to show that $f \cdot L = 0$ for each basic class L of X. Since genus$(f) = 1$ and $f^2 = 0$, this is implied by the adjunction inequality

$$f^2 + |f \cdot L| \leq 2 \cdot \text{genus}(f) - 2 \ . \quad \square$$

4.2 Applications of Seiberg-Witten Theory to Geography Problems

Corollary 4.1.1 allows us to partially answer our uniqueness question above. Before going on, we quote a well-known theorem on $X(p)$, the p-log transform of X.

Theorem 4.2.1 ([FS1]) *Let X be a simply connected 4-manifold containing a cusp neighborhood N whose complement Z has $\pi_1(Z) = \mathbf{Z}_q, q \geq 1$, and $\pi_1(\partial N) \to \pi_1(Z)$*

is surjective. Let $\Xi(X)$ be the class of 4-manifolds $\{X(p) : p, q$ coprime, $p \neq 0\}$. Then

(a) Each $X(p) \in \Xi(X)$ is simply connected.

(b) If X is not spin or q is even, then all the manifolds in $\Xi(X)$ are homeomorphic.

(c) If X is spin and q is odd, then the manifolds $X(p)$ and $X(p')$ in $\Xi(X)$ are homeomorphic if and only if $p \equiv p' \pmod 2$.

Note that any smooth 4-manifold containing a Brieskorn manifold $B(p,q,r)$ with $(p,q,r) \geq (2,3,7)$ contains a cusp neighborhood satisfying the hypothesis of Theorem 4.2.1 ([FS1]). Hence we can apply Theorem 4.2.1 and Corollary 4.1.1 to show that such a manifold has infinitely many distinct smooth structures.

Now we construct more irreducible 4-manifolds which have infinitely many diffeomorphism types, but all are homeomorphic. First we define another topological surgery, called *fiber sum.*

Definition Let X and Y be closed, oriented smooth 4-manifolds containing a smoothly embedded surface Σ of genus $g \geq 1$. Suppose Σ represents a homology class of infinite order and of square zero, so that there exists a tubular neighborhood, say $D^2 \times \Sigma$, in X and Y. Let $X_0 = X \setminus D^2 \times \Sigma$ and $Y_0 = Y \setminus D^2 \times \Sigma$, and let $N = S^1 \times \Sigma = \partial D^2 \times \Sigma$ be the common boundary of $D^2 \times \Sigma$. By choosing an orientation–reversing, fiber–preserving diffeomorphism

$$\varphi : D^2 \times \Sigma \longrightarrow D^2 \times \Sigma$$

and gluing X_0 to Y_0 along their boundary by the map $\varphi| : N \longrightarrow N$, we define a new oriented smooth 4-manifold $X\sharp_\Sigma Y$, called the **fiber sum** of X and Y along Σ. Note that there is an induced embedding of Σ into $X\sharp_\Sigma Y$ well-defined up to isotopy which represents a homology class of infinite order and of square zero.

Lemma 4.2.1 *$c_1^2(X\sharp_\Sigma Y) = c_1^2(X) + c_1^2(Y) + 8(g-1)$ and $\chi(X\sharp_\Sigma Y) = \chi(X) + \chi(Y) + (g-1)$, where $\chi = \frac{b^+ + 1}{2}$ and $g = genus(\Sigma)$.*

Proof : These follow from the fact

$$\begin{aligned} e(X\sharp_\Sigma Y) &= e(X) - (2-2g) + e(Y) - (2-2g) \\ &= e(X) + e(Y) + 4(g-1) \\ \sigma(X\sharp_\Sigma Y) &= \sigma(X) + \sigma(Y) \qquad \square \end{aligned}$$

We quote a product formula for Seiberg-Witten invariants of a fiber sum manifold which provides an important tool for studying geography problems for irreducible 4-manifolds.

Theorem 4.2.2 ([MST]) *Let X and Y be closed,oriented smooth 4-manifolds containing a smoothly embedded surface Σ of genus $g > 1$. Suppose Σ represents a homology class of infinite order and of square zero, and $b^+(X), b^+(Y) \geq 1$. If there are characteristic classes $l_1 \in H^2(X;\mathbf{Z})$ and $l_2 \in H^2(X;\mathbf{Z})$ with $< l_1, [\Sigma] >=< l_2, [\Sigma] >= 2g-2$ and with $SW_X(l_1) \neq 0$ and $SW_Y(l_2) \neq 0$, then there exists a characteristic class $k \in H^2(X\sharp_\Sigma Y;\mathbf{Z})$ with $k|_N = \mathrm{proj}^*(k_0)$ for $k_0 \in H^2(\Sigma;\mathbf{Z})$ satisfying $< k_0, [\Sigma] >= 2g-2$ for which $SW_{X\sharp_\Sigma Y}(k) \neq 0$.*

In case genus(Σ) = 1 (i.e. Σ = torus), they also proved

Theorem 4.2.3 ([MS]) *Suppose X and Y contain a cusp neighborhood of a cusp fiber f. Then SW-basic classes of $X\sharp_f Y$ are given by*

$$\{K_X + K_Y + n\cdot f \ : \ K_X(K_Y) \text{ is a basic class of } X(Y) \text{ and } n = 0, \pm 2\}$$

Theorem 4.2.4 *The fiber sum of two minimal symplectic 4-manifolds with $b^+ \geq 2$ along a symplectic (or lagrangian) surface is also minimal symplectic.*

Proof : Since the fiber sum along a symplectic (or lagrangian) surface is also symplectic ([G2, Corollary 1.7]), it suffices to show its minimality (a fiber sum of two minimal symplectic 4-manifolds does not contain an embedded -1-sphere) which can be proved by W. Lorek's argument. Here is a sketch of an alternative argument: Suppose E is an embedded -1-sphere in a symplectic manifold $X\sharp_\Sigma Y$. Since there is a symplectically embedded -1-sphere representing the same homology class as E, we may assume that E is symplectically embedded. As the radius of a tubular neighborhood $\Sigma \times D^2$ of Σ goes to zero, E goes to a limit surface $C_X \amalg C_{\Sigma\times S^2} \amalg C_Y$ in the compactification space $X \amalg \Sigma \times S^2 \amalg Y$. Since the genus of a limit surface cannot increase and genus$(E) = 0$, each piece of $C_X \amalg C_{\Sigma\times S^2} \amalg C_Y$ should be S^2. Furthermore, $C_{\Sigma\times S^2} = S^2 \subset \Sigma \times S^2$ has square 0. Hence, since an essential sphere S^2 of non-negative square cannot be embedded in a symplectic 4-manifold with $b^+ \geq 2$, $E^2 = -1 = C_X^2 + C_Y^2$ implies that either $C_X^2 = -1$ or $C_Y^2 = -1$ which contradicts that X and Y are minimal. □

These theorems enable us to partially solve our existence question.

Theorem 4.2.5 *Every lattice point in the wedge between the elliptic line ($c_1^2 = 0$) and Noether line ($c_1^2 = 2\chi - 6$) is realized as (χ, c_1^2) of a simply connected smooth irreducible 4-manifold. Furthermore, each of these manifolds has infinitely many diffeomorphism types, but all are homeomorphic.*

Proof : Consider a torus fiber sum $X(k,n) \equiv E(k)\sharp_f E(n-k)$ obtained by choosing a cusp f in a cusp neighborhood N in $B(2,3,6k-1) \subset E(k)$ and in $B(2,3,6n-6k-1) \subset$

$E(n-k)$.

$$X(k,n) = N_k \cup_{\Sigma(2,3,6k-1)} B(2,3,6k-1)\sharp_f B(2,3,6n-6k-1) \cup_{\Sigma(2,3,6n-6k-1)} N_{n-k}$$

where N_k and N_{n-k} are a neighborhood of a singular fiber and a section in $E(k)$ and $E(n-k)$ respectively. Then $X(k,n)$ and $E(n)$ have the same $(\chi, c_1^2) = (n, 0)$, but they are not homeomorphic. Since $E(k) \setminus N$ contains two disjoint configurations $C_{k,1}$, so does $X(k,n)$. Furthermore, $X(k,n)$ also contains -4-sphere, a configuration $C_{4,1}$, which intersects f at one point. Hence, by rationally blowing down these configurations, we can fill every lattice point in the wedge. Explicitly, if $1 \leq c_1^2 \leq n-5$, rationally blow down one $C_{k,1}$ for $4 \leq k \leq n-2$. If $n-4 \leq c_1^2 \leq 2n-9$ and c_1^2 is even (odd), rationally blow down two $C_{k,1}$ (two $C_{k,1}$ and one $C_{4,1}$). Finally, if $c_1^2 = 2n-8(2n-7)$, rationally blow down two $C_{n-2,1}$ and two $C_{4,1}$ (two $C_{n-2,1}$ and three $C_{4,1}$). Irreducibility of these rational blow-down manifolds follows from Theorem 4.2.3 and the fact that such manifolds cannot be blow-up manifolds. (Otherwise, there exist SW-basic classes $\overline{K_1}, \overline{K_2}$ satisfying $(\overline{K_1} - \overline{K_2})^2 = -4$ such that $\overline{K_i}|_{X_0} (i = 1,2)$ extends to a basic class $K_i (i = 1,2)$ for $X(k,n)$. Since $(K_1 - K_2)^2 = 0$ (Theorem 4.2.3) and $C_{k,1}$ is negative definite,

$$\begin{aligned} 0 &= (K_1|_{X_0} - K_2|_{X_0})^2 + (K_1|_{C_{k,1}} - K_2|_{C_{k,1}})^2 \\ &= (\overline{K_1} - \overline{K_2})^2 + (K_1|_{C_{k,1}} - K_2|_{C_{k,1}})^2 \\ &\leq (\overline{K_1} - \overline{K_2})^2 = -4 \end{aligned}$$

which is a contradiction.) Second statement follows from Corollary 4.1.1 and Theorem 4.2.1 because all such rational blow-down manifolds still contain a nicely embedded cusp neighborhood. □

Here are a few remarks on this theorem:

1. Fintushel and Stern showed this theorem by using twisted fiber sums. Gompf ([G2]) and Stipsicz ([St]) also showed that any lattice point with c_1^2 = even in the wedge can be realized as (χ, c_1^2) of symplectic manifolds. Stipsicz' examples are $X\sharp_f E(n)$, where X is a Horikawa surface with $c_1^2 = 2\chi - 6$.

2. In fact, we can show that every lattice point in the wedge is realized as (χ, c_1^2) of symplectic manifolds by using symplectically embedded -4-spheres in $E(4)$ and by a slight modification of Stipsicz' examples. That is, first consider a torus fiber sum manifold $Y \equiv X\sharp_{f'}E(4)$, where f' is an embedded torus in the cusp neighborhood contained in a Milnor fiber $B(2,3,23) \subset E(4)$ and X is a Horikawa surface as above. Since such an embedded torus f' is lagrangian ([St]), by Gompf's argument ([G2]), Y is a symplectic manifold and still contains a symplectically embedded -4-sphere (a configuration $C_{4,1}$). Hence the rational blow-down manifold $Y_{4,1}$ of Y is again a symplectic irreducible 4-manifold. Now, as the same way in Stipsicz' examples, construct torus fiber sum manifolds $Y_{4,1}\sharp_f E(n)$. Then any lattice point with c_1^2 = odd in the wedge is realized by these symplectic manifolds. (It is also known that Fintushel and Stern's twisted fiber sum examples covering the wedge are all symplectic.)

3. Each of these manifolds constructed in the proof above has more than one SW-basic classes. Actually one can construct infinitely many irreducible 4-manifolds which have up to sign two (three, four, $\cdots$) SW-basic classes by using slightly different examples.

4. Let Ω be the set of all lattice points in the wedge between $c_1^2=0$ and $c_1^2 \leq 2\chi{-}6$. If a simply connected irreducible 4-manifold X contains a cusp neighborhood, then by torus fiber sum of X with manifolds constructed above, each lattice

point in $(\chi(X), c_1^2(X)) + \Omega$ is represented by a simply connected 4-manifold which has infinitely many diffeomorphism types. All these manifolds seem likely to be minimal (so that they are irreducible). If one choose a simply connected irreducible symplectic manifold X which contains a symplectic (or lagrangian) torus in a cusp neighborhood, then, by taking a torus fiber sum of X with symplectic manifolds constructed in Remark 2 above, we get a family of desired irreducible manifolds. (Irreducibility follows from either Taubes' result ([T]) and Theorem 4.2.4, or Theorem 4.2.3 and Theorem 4.2.4.)

Let us consider a Milnor fiber $B(p,q,pq-1) = \{(z_1,z_2,z_3) \in \mathbf{C}^3 : z_1^p + z_2^q + z_3^{pq-1} = \epsilon, \text{ for } \epsilon > 0\}$ which has a natural compactification (by adding a complex curve at infinity) as a complete intersection in a weighted homogeneous space. Note that the singularities of this compactification can be resolved to obtain a simply connected algebraic surface $X(p,q,pq-1)$.

Example 4 The Milnor fiber $B(2,2n+1,4n+1)$ is contained in the simply connected algebraic surface $X(2,2n+1,4n+1)$ which is diffeomorphic to $B(2,2n+1,4n+1) \cup_{\Sigma(2,2n+1,4n+1)} T(2,2n+1)$. $T(2,2n+1)$ is the manifold obtained from $+1$ surgery on the $(2,2n+1)$ torus knot, and it contains an obvious surface T of genus n and square $+1$. The canonical class K_X of $X(2,2n+1,4n+1)$ is represented by a multiple of T. Let $X' = X(2,2n+1,4n+1)\sharp\overline{\mathbf{CP}}^2$ be the manifold obtained by blowing up at a point in T, so that

$$X' \cong B(2,2n+1,4n+1) \cup_{\Sigma(2,2n+1,4n+1)} C(2,2n+1)$$

where $C(2,2n+1)$ is the blow up of $T(2,2n+1)$. In X' is an embedded surface Σ representing $T - e$, and Σ has genus n and self-intersection 0. Since Σ is symplectically embedded, by taking a fiber sum of X' with itself along Σ (Fig 4.3), we get a simply connected symplectic 4-manifold $Z \equiv X' \sharp_\Sigma X'$ which satisfies the hypothesis of Theorem 4.2.2. Furthermore, Z is irreducible if n is odd. Note that the irreducibility of Z follows from Theorem 4.2.2 and the fact that Z is spin because each part of the following Figure 4.3 is spin. (The middle part, $C(2,2n+1)\sharp_\Sigma C(2,2n+1)$, is embedded in the elliptic surface

$$E(n+1) \cong Q \cup_{\Sigma(2,2n+1,4n+1)} C(2,2n+1)\sharp_\Sigma C(2,2n+1) \cup_{\Sigma(2,2n+1,4n+1)} Q,$$

where Q is the canonical resolution of singularity of $z_1^2 + z_2^{2n+1} + z_3^{4n+1} = 0$ in $\mathbf{C}^3$.)

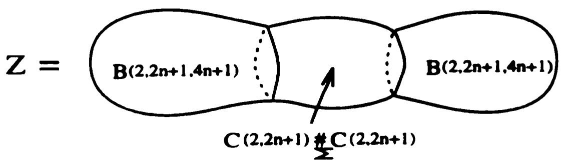


Figure 4.3:

Finally, by using the symplectic manifold Z constructed above, we prove the main result of this paper:

Theorem 4.2.6 *There is a linear function $y = f(x)$ such that any lattice point (a,b) satisfying $b \le f(a)$ in the first quadrant can be realized as (χ, c_1^2) of a simply connected symplectic 4-manifold which has infinitely many distinct irreducible smooth structures. In particular, all lattice points (a,b) except at most finitely many in between $c_1^2 = 0$ and $c_1^2 = 8\chi$ (non-positive signature region) satisfy $b \le f(a)$.*

Proof : Choose a simply connected minimal surface, say Y, of positive signature which is a hyperelliptic fibration whose genus is odd. (One may choose a minimal surface constructed by Xiao and Chen ([C]) for Y). Let Σ be a fiber of Y and let g = genus(Σ). Take any irreducible symplectic 4-manifold Z which contains a symplectically embedded surface Σ satisfying $\Sigma^2 = 0$ and $<\Sigma, K_Z> = 2g-2$, and also contains a symplectic (or lagrangian) torus f in a cusp neighborhood N satisfying $N \cap \Sigma = \phi$, where K_Z is a (SW)-basic class of Z (Such an irreducible 4-manifold exists! – see Example 4). Note that since

$$c_1^2(Y\sharp_\Sigma Z) - 8\chi(Y\sharp_\Sigma Z) = [c_1^2(Y) - 8\chi(Y)] + [c_1^2(Z) - 8\chi(Z)] \text{ and } [c_1^2(Y) - 8\chi(Y)] > 0$$

there exists an integer $k > 0$ such that $X \equiv \overbrace{Y\sharp_\Sigma \cdots \sharp_\Sigma Y}^{k}\sharp_\Sigma Z$ has a positive signature. Let Ω be the set of all lattice points in the wedge between $c_1^2 = 0$ and $c_1^2 \leq 2\chi - 6$. Then, by taking a torus fiber sum of X with manifolds constructed in Remark 2 (Theorem 4.2.5 below), each lattice point in $(\chi(X), c_1^2(X)) + \Omega$ is represented by a simply connected irreducible symplectic 4-manifold which has infinitely many diffeomorphism types. The same is true for $X\sharp_f X$, $X\sharp_f X\sharp_f X, \ldots$ So define $y = f(x)$ by

$$f(x) = c_1^2(X)/\chi(X) \cdot [x - c_1^2(X)/2 - \chi(X) - 3] + 2c_1^2(X)$$

Then each lattice point (a, b) satisfying $b \leq f(a)$ in the first quadrant is realized as (χ, c_1^2) of a simply connected irreducible symplectic 4-manifold $\overbrace{X\sharp_f X\sharp_f \cdots \sharp_f X}^{n}\sharp_f W$, for some $n \in \mathbf{Z}$ and a manifold W constructed in Remark 2 (Theorem 4.2.5 below). Note that the irreducibility of such manifolds follows from either Taubes' result in [T] (A symplectic 4-manifold with $b^+ \geq 2$ has a non-zero Seiberg-Witten invariant) and Theorem 4.2.4, or Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4. The second statement follows from the fact that the slope of $f(x), c_1^2(X)/\chi(X)$, is greater than 8. □

Remarks 1. In the proof above, the reason we use a symplectic manifold Z is to make sure that all involved manifolds are minimal. Hence if one can prove that all involved manifolds are minimal by using different argument, then one can drop the symplectic condition of Z.

2. Note that there are still many lattice points in the region $y > f(x)$ which are realized as (χ, c_1^2) of $\overbrace{X\sharp_f X\sharp_f \cdots \sharp_f X}^{n}\sharp_f W$, for some $W \in \Omega$ and $n \in \mathbf{Z}$. Furthermore, we do not claim that the $f(x)$ constructed in the proof above is the best choice. In fact, one may choose better $y = f(x)$ having the same property by choosing other surface X required in the proof.

We close this paper by suggesting the following problem:

Problem *For each pair of integers (χ, c_1^2) between elliptic line ($c_1^2 = 0$) and Bogomolov-Miyaoka-Yau line ($c_1^2 = 9\chi$), are there infinitely many diffeomorphism types which are all homeomorphic?*

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