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GENUS ONE ENUMERATIVE INVARIANTS IN P^n

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GENUS ONE ENUMERATIVE INVARIANTS IN $\mathbf{P}^n$

By

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ABSTRACT

GENUS ONE ENUMERATIVE INVARIANTS IN $\mathbf{P}^n$

By

Eleny-Nicoleta Ionel

In this thesis we prove recursive formulas for τ_d, the number of degree d elliptic curves with fixed j invariant in $\mathbf{P}^n$. We use analysis to relate the classical invariant τ_d to the genus one perturbed invariant $RT_{1,d}$ defined recently by Ruan and Tian (the later invariant can be computed inductively).

Our approach is to start with $RT_{1,d}$ and consider a sequence of perturbations converging to zero. In the limit we get not only holomorphic tori (i.e. τ_d), but also bubble trees with ghost base. We use Taubes Obstruction Bundle method to compute the contribution of the ghost base stratum to the difference between the two invariants.

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0 Introduction.

A classical problem in enumerative algebraic geometry is to compute the number of degree d, genus g holomorphic curves in $\mathbf{P}^n$ that pass through a certain number of constraints (points, lines, etc).

Let σ_d denote the number of degree d rational curves ($g = 0$) through appropriate constraints. For example $\sigma_1(pt, pt) = 1$ (since 2 points determine a line). The first nontrivial cases were computed around 1875 when Schubert, Halphen, Chasles et al. found σ_2 for $\mathbf{P}^2$ and $\mathbf{P}^3$. Later, more low degree examples were computed in $\mathbf{P}^2$ and P^3, but the progress was slow. Then in 1993 Kontsevich [K] predicted, based on ideas of Witten, that the number σ_d of degree d rational curves in $\mathbf{P}^2$ through $3d-1$ points satisfies the following recursive relation:

$$\sigma_d = \sum_{d_1+d_2=d} \left[\binom{3d-1}{3d_1-1} d_1^2 d_2^2 - \binom{3d-1}{3d_1-2} d_1^3 d_2 \right] \sigma_{d_1} \sigma_{d_2}$$

where $d_i \neq 0$, and $\sigma_1 = 1$. Ruan-Tian ([RT], 1994) extended these formulas for σ_d in any $\mathbf{P}^n$.

When genus $g = 1$, the classical problem splits into two totally different problems: one can count (i) elliptic curves with a fixed complex structure, or (ii) elliptic curves with unspecified complex structure (each satisfying the appropriate number of constraints). This paper gives recursive formulas which completely solve the first of these.

Thus our goal is to compute the number τ_d of degree d elliptic curves in $\mathbf{P}^n$ with fixed j invariant. This problem is considerable harder than the $g = 0$ case. The simplest nontrivial case – τ_3 in $\mathbf{P}^2$ – was determined by Alluffi (1991) using classical methods. Recently Pandharipande [Pan] has made more significant progress, using the Kontsevich moduli space of stable curves to give recursive formulas for τ_d in $\mathbf{P}^2$.

We will approach this problem from a different direction, using analysis. This approach is based on the ideas introduced by Gromov to study symplectic topology. If (Σ, j) is a fixed Riemann surface, let

$$\{\, f : \Sigma \to \mathbf{P}^n \mid \overline{\partial}_J f = 0,\ [f] = d \cdot l \in H_2(\mathbf{P}^n, \mathbf{Z}) \,\}/Aut(\Sigma, j)$$

be the moduli space of degree d holomorphic maps $f : \Sigma \to \mathbf{P}^n$, modulo the automorphisms of (Σ, j). Each constraint, such as the requirement that the image of f passes through a specified point, defines a subset of this moduli space.

Imposing enough constraints gives a 0-dimensional "cutdown moduli" space $\mathcal{M}_d$. To see whether or not it consists of *finitely many* points, one looks at its bubble tree compactification $\overline{\mathcal{M}}_d$ [PW]. If the constraints are cut transversely, then all the boundary strata of $\overline{\mathcal{M}}_d$ are at least codimension 1, and thus empty. Unfortunately, transversality fails at multiply-covered maps or at constant maps (called *ghosts*), so $\overline{\mathcal{M}}_d$ is not a manifold.

This was a real problem until 1994, when Ruan and Tian considered the moduli space $\mathcal{M}_\nu$ of solutions of the perturbed equation:

$$\overline{\partial}_J f = \nu(x, f(x))$$

and used marked points instead of moding out by $Aut(\Sigma, j)$. For a generic perturbation ν the moduli space $\mathcal{M}_\nu$ is smooth and compact, so it consists of finitely many points that, counted with sign, give an invariant $RT_{d,g}$ (independent of ν).

In $\mathbf{P}^n$, the genus 0 perturbed invariant $RT_{d,0}$ is equal to the enumerative invariant σ_d. The perturbed invariants satisfy a degeneration formula that gives not only recursive formulas to compute the enumerative invariant σ_d in $\mathbf{P}^n$, but also expresses the higher genus perturbed invariants in terms of the genus zero invariants [RT]. For convenience, these formulas are included in the Appendix.

Unfortunately, when $g = 1$, the perturbed invariant $RT_{d,1}$ does not equal the enumerative invariant τ_d. For example, for $d = 2$ curves in $\mathbf{P}^2$ the Ruan-Tian invariant is $RT_{2,1} = 2$ (cf. (A.2)), while $\tau_2 = 0$ (there are no degree 2 elliptic curves in $\mathbf{P}^2$).

Thus while the Ruan-Tian invariants are readily computable, they differ from the enumerative invariants τ_d. One should seek a formula for the difference between the two invariants. For that, we take the obvious approach:

Start with the genus 1 perturbed invariant $RT_{d,g}$ and consider a sequence of generic perturbations $\nu \to 0$. A sequence of (J,ν)-holomorphic maps converges either to a holomorphic torus or to a bubble tree whose base is a constant map (ghost base). Proposition 1.21 shows that the contribution of the $(J,0)$-holomorphic tori is a multiple of τ_d.

We show that the only other contribution comes from bubble trees with ghost base such that the bubble point is equal to the marked point $x_1 \in T^2$. To compute this contribution, we use the Taubes "Obstruction Bundle" method. Proposition 1.7 identifies the moduli space of (J,ν)-holomorphic maps that are close to a bubble tree with the zero set of a specific section of the obstruction bundle. Studying the leading order term of this section, we are able to compute the corresponding contribution (Proposition 1.26). Adding both contributions, yields our main analytic result:

Theorem 0.1 *Consider the genus 1 enumerative invariant $\tau_d(\beta_1, \dots, \beta_k)$ in $\mathbf{P}^n$. Let $\mathcal{U}_d$ be the $n-1$ dimensional moduli space of 1-marked rational curves of degree d in $\mathbf{P}^n$ passing through $\beta_1, \dots, \beta_k$. Let $L \to \mathcal{U}_d$ be the relative tangent bundle, and denote by $\widetilde{L} \to \widetilde{\mathcal{U}}_d$ its blow up defined by (1.17). Then:*

$$n_j\ \tau_d(\beta_1, \dots, \beta_k) = RT_{d,1}(\beta_1 \mid \beta_2, \dots, \beta_k) - \sum_{i=0}^{n-1} \binom{n+1}{i+2} \mathrm{ev}^*(H^{n-i-1}) c_1^i(\widetilde{L}^*)$$

where H^i is a codimension i hyperplane in $\mathbf{P}^n$, ev $: \mathcal{U}_d \to \mathbf{P}^n$ *is the evaluation map corresponding to the special marked point and n_j is the order of the group of automorphisms of the complex structure j that fix a point.*

Theorem 0.1 becomes completely explicit provided we can compute the top power intersections $\mathrm{ev}^*(H^{n-i-1})c_1^i(\widetilde{L}^*)$. We do this in the second part of the paper, in several steps. For simplicity of notation, let

$$x = c_1(L^*) \in H^2(\mathcal{U}_d, \mathbf{Z}), \qquad \tilde{x} = c_1(\widetilde{L}^*) \in H^2(\widetilde{\mathcal{U}}_d, \mathbf{Z}) \ , \quad \text{and} \quad y = \mathrm{ev}^*(H) \qquad (0.1)$$

where $y \in H^2(\mathcal{U}_d, \mathbf{Z})$ or $y \in H^2(\widetilde{\mathcal{U}}_d, \mathbf{Z})$ depending on the context. In this notation, Theorem 0.1 becomes:

$$n_j \tau_d(\,\cdot\,) = \sum_{i_1+i_2=n} \sigma_d(H^{i_1}, H^{i_2}, \,\cdot\,) + \sum_{i=0}^{n-1} \binom{n+1}{i+2} \tilde{x}^i y^{n-1-i} \cdot [\widetilde{\mathcal{U}}_d] \tag{0.2}$$

Proposition 2.2 gives recursive formulas relating $\tilde{x}^i y^j$ to $x^i y^j$ and Proposition 2.5 gives recursive formulas for $x^i y^j$ in terms of the enumerative invariant σ_d. Finally, the recursive formulas for σ_d are known (see [RT], [K]), so the right hand side of (0.2) can be recursively computed.

In the end, we give applications of these formulas. We explicitly work out the formulas expressing the number of degree d elliptic curves passing through generic constraints in $\mathbf{P}^2$ and $\mathbf{P}^3$ in terms of the rational enumerative invariant σ_d. For example:

Proposition 0.2 *For $j \neq 0, 1728$, the number $\tau_d = \tau_d(p^a, l^b)$ of elliptic curves in $\mathbf{P}^3$ with fixed j invariant and passing through a points and b lines (such that $2a + b = 4d - 1$) is given by:*

$$\tau_d(\cdot) = \frac{(d-1)(d-2)}{d} \sigma_d(l, \,\cdot) - \frac{1}{d} \sum_{d_1+d_2=d} d_2(2d_1d_2 - d)\sigma_{d_1}(l, \,\cdot)\sigma_{d_2}(\cdot) \tag{0.3}$$

where $\sigma_d(l, \,\cdot) = \sigma_d(l, p^a, l^b)$ is the number of degree d rational curves in $\mathbf{P}^3$ passing through same conditions as τ_d plus one more line. The sum above is over all decompositions into a degree d_1 and a degree d_2 component, $d_i \neq 0$, and all possible ways of distributing the constraints p^a, l^b on the two components.

Using a computer program, one computes then specific examples, e.g.

$$\tau_3(l^{11}) = 6 \cdot 25920 \quad \text{and} \quad \tau_5(p, l^{17}) = 6 \cdot 15856790593536.$$

when $j \neq 0, 1728$. To get τ_d for $j = 0$ or $j = 1728$ one simply divides the τ_d computed for a generic j by 2 or 3 respectively.

1 Analysis

1.1 Setup

Let τ_d be the genus one degree d *enumerative invariant* (with fixed j invariant) and σ_d be the genus zero degree d enumerative invariant in $\mathbf{P}^n$. Using analytic methods, we will compute τ_d by relating it to the perturbed invariant $RT_{d,g}$ introduced by Ruan and Tian [RT]. The later is defined as follows.

Let (Σ, j) be a genus g Riemann surface with a fixed complex structure and ν an inhomogenous term. A (J,ν)-*holomorphic* map is a solution $f : \Sigma \to \mathbf{P}^n$ of the equation

$$\overline{\partial}_J f(x) = \nu(x, f(x)). \tag{1.1}$$

For $2g + l \geq 3$, let $x_1, \ldots, x_l$ be fixed marked points on Σ, and $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_k$ be various codimension submanifolds in $\mathbf{P}^n$, such that

$$\text{index } \overline{\partial}_J = (n+1)d - n(g-1) = \sum_{i=1}^{l}(n - |\alpha_i|) + \sum_{i=1}^{k}(n - 1 - |\beta_i|)$$

For a generic ν, the invariant

$$RT_{d,g}(\alpha_1, \ldots, \alpha_l \mid \beta_1, \ldots, \beta_k)$$

counts the number of (J,ν)-holomorphic degree d maps $f : \Sigma \to \mathbf{P}^n$ that pass through $\beta_1, \ldots, \beta_k$ with $f(x_i) \in \alpha_i$ for $i = 1, \ldots, l$.

The first part of this paper is devoted to the proof of Theorem 0.1.

Outline of the Proof of Theorem 0.1. The proof is done in several steps. The basic idea is to start with the genus 1 perturbed invariant

$$RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l) \tag{1.2}$$

and take a sequence of generic perturbations $\nu \to 0$. Denote by $\mathcal{M}_{d,1,t\nu}$ the moduli space of $(J, t\nu)$-holomorphic maps satisfying the constraints in (1.2), and let

$$\mathcal{M}^{\nu} = \bigcup_{t \geq 0} \mathcal{M}_{d,1,t\nu}. \tag{1.3}$$

As $t \to 0$, a sequence of $(J, t\nu)$-holomorphic maps converges to a $(J, 0)$-holomorphic torus or to a bubble tree ([PW]). Let $\overline{\mathcal{M}}^{\nu}$ denote the bubble tree compactification of $\mathcal{M}^{\nu}$ (for details on bubble tree compactifications, see [P]).

Proposition 1.21 shows that the number of $(J, t\nu)$-holomorphic maps converging to a J-holomorphic torus is equal to

$$n_j \tau_d(\beta_1, \ldots, \beta_k)$$

where $n_j = |Aut_{x_1}(j)|$ is the order of the group of automorphisms of the complex structure j that fix the point $x_1$namely,

$$n_j = \begin{cases} 2 & \text{if } j \neq 0, 1728 \\ 4 & \text{if } j = 0 \\ 6 & \text{if } j = 1728 \end{cases} \tag{1.4}$$

These multiplicities occur because if f is a J-holomorphic map, then so is $f \circ \phi$ for any $\phi \in Aut_{x_1}(j)$, but they get perturbed to distinct $(J, t\nu)$-holomorphic maps.

As $t \to 0$, there are also a certain number of solutions converging to bubble trees. Because the moduli space of $(J, 0)$-holomorphic tori passing through $\beta_1, \ldots, \beta_k$ is 0 dimensional, the only bubble trees which occur have with a multiply-covered or a ghost base (for these transversality fails, so dimensions jump up).

A careful dimension count shows that the multiply-covered base strata are still codimension at least one for genus $g = 1$ maps in $\mathbf{P}^n$. (This is *not* true for $g \geq 2$.) But at a ghost base bubble tree the dimension jumps up by n so these strata are $n - 1$ dimensional. There are actually 2 such pieces, corresponding to bubble tree where (i) the bubble point is at the marked point x_1 and (ii) the bubble point is somewhere else. To make this precise, a digression is necessary to set up some notation.

Let

$$\mathcal{M}_d^0 = \{\ (f, y_1, \ldots, y_k) \mid f : S^2 \to \mathbf{P}^n \text{ degree d holomorphic, } f(y_j) \in \beta_j\ \} \tag{1.5}$$

be the moduli space of bubble maps, and $\mathcal{M}_d = \mathcal{M}_d^0/G$ be the corresponding moduli space of curves, where $G = \mathbf{PSL}(2, \mathbf{C})$. Introduce one special marked point $y \in S^2$ and let

$$\mathcal{U}_d = \{\ [f, y, y_1, \ldots, y_k]\ \mid\ [f, y_1, \ldots, y_k] \in \mathcal{M}_d\} \tag{1.6}$$

be the moduli space of *1-marked curves* and

$$\mathrm{ev} : \mathcal{U}_d \to \mathbf{P}^n, \quad \mathrm{ev}([f, y, y_1, \ldots, y_k]) = f(y). \tag{1.7}$$

be the corresponding evaluation map. We will use $f(y)$ to record the image of the ghost base

For generic constraints $\beta_1, \ldots, \beta_k$ the bubble tree compactification of $\mathcal{U}_d$ is a smooth manifold that comes with a natural stratification, depending on the possible splittings into bubble trees and how the degree d and the constraints $\beta_1, \ldots, \beta_k$ distribute on each bubble.

With this, the two "pieces" of the boundary of $\overline{\mathcal{M}}^\nu$ are:

$$\{x_1\} \times \overline{\mathcal{U}}_d \qquad \text{and} \qquad T^2 \times \mathrm{ev}^*(\beta_1) \tag{1.8}$$

The first factor records the bubble point, while the image of the ghost base is encoded in the second factor. For generic constraints each piece, as well as their intersection, is a smooth manifold, again stratified.

To see which bubble trees with ghost base appear as a limit of perturbed tori, we use the Taubes Obstruction Bundle. This construction must be performed on the link of each strata. We do this first on the top statum of $\{x_1\} \times \overline{\mathcal{U}}_d$, which consists of bubble trees with ghost base and a single bubble.

First we construct in Section 1.2 a set of approximate maps by gluing in the bubble. The "gluing data" $[f, y, v]$ consists of a nonvanishing vector v tangent to the bubble at the bubble point y. Proposition 1.4 shows that the obstruction bundle is then diffeomorphic to $\mathrm{ev}^*(T\mathbf{P}^n)$.

In Section 1.4 we correct the approximate maps to make then $(J, t\nu)$-holomorphic by pushing them in a direction normal to the kernel of the linearized equation. Those approximate maps that can be corrected to solutions of the equation (1.1) are then identified with the zero set of a section ψ_t of the obstruction bundle. Proposition 1.7 shows that actually all the solutions of the equation (1.1) are obtained this way, i.e. the end of the moduli space of $(J, t\nu)$-holomorphic maps is diffeomorphic to the zero set of the section ψ_t.

To understand the zero set of ψ_t it is enough to look at the leading order term of its expansion as $t \to 0$. By Proposition 1.45 this has the form $df_y(v) + t\bar{\nu}$ where $\bar{\nu}$ is the projection of ν on the obstruction bundle.

The construction described above extends naturally to all the other boundary strata. Each bubble comes with "gluing data " $[f_i, y_i, v_i]$, consisting of a vector v_i tangent to the bubble at the bubble point y_i. But the leading order term of the section ψ_t depends only on the vectors tangent to the *first level of nontrivial bubbles.*

More precisely, let $\mathcal{Z}_h \subset \overline{\mathcal{U}}_d$ denote the collection of bubble trees for which the image $u = f(y)$ of the ghost base lies on h nontrivial bubbles. Geometrically, the image of a bubble tree in $\mathcal{Z}_h$ has h components $C_1, \ldots, C_h$ that meet at u. Let $\mathcal{W}|_{\mathcal{Z}_h} \to \mathcal{Z}_h$ be the bundle whose fiber is $T_uC_1 \oplus \cdots \oplus T_uC_h$. The leading order term of ψ_t on $\mathcal{Z}_h$ is a section of $\mathcal{W}$, equal to

$$a(f, y, v]) + t\bar{\nu} \overset{def}{=} df_1(y_1)(v_1) + \ldots + df_h(y_h)(v_h) + t\bar{\nu}$$

where $([f_i, y_i, v_i])_{i=1}^h$ is the gluing data corresponding to the bubbles C_i, $i = 1, \ldots, h$.

Unfortunately $\mathcal{W} \to \overline{\mathcal{U}}_d$ is not a vector bundle. But if we blow up each strata $\mathcal{Z}_h$ starting with the bottom one, then the total space of $\mathcal{W}$ is the same as the total space of $\tilde{L}$, the blow-up of the relative tangent sheaf $L \to \mathcal{U}_d$. The leading order term of ψ_t descends as a map $a + t\bar{\nu} : \tilde{L} \to \mathrm{ev}^*(T\mathbf{P}^n)$. Moreover, $\bar{\nu}$ doesn't vanish on

$\mathrm{Im}(\mathcal{M}) = ev_*(\overline{\mathcal{U}}_d)$ so it induces a splitting on the restriction

$$T\mathbf{P}^n/\mathrm{Im}(\mathcal{M}) = \mathbf{C}\langle\bar{\nu}\rangle \oplus E.$$

Finally, we put all these pieces together in Proposition 1.26 to prove that the number of (J,ν)-holomorphic maps converging as $\nu \to 0$ to the boundary strata $\{x_1\} \times \overline{\mathcal{U}}_d$ is given by the Euler class $c_{n-1}(\mathrm{ev}^*(E) \otimes \tilde{L}^*)$.

In Section 1.8 we show that the other boundary strata $T^2 \times \mathrm{ev}^*(\beta_1)$ gives trivial contribution, concluding the proof of the Theorem 0.1.

1.2 The Approximate gluing map

Let $\mathcal{U}_d$ be the moduli space of 1-marked rational curves of degree d passing through the conditions $\beta_1, \ldots, \beta_k$. In this section we construct a set of approximate maps starting from $\{x_1\} \times \overline{\mathcal{U}}_d$, the first boundary strata in (1.8). We will use a:

Cutoff function. In what follows, fix a smooth cutoff function β such that $\beta(r) = 0$ for $r \leq 1$ and $\beta(r) = 1$ for $r \geq 2$. Let $\beta_\lambda(r) = \beta(r/\sqrt{\lambda})$. Then β_λ has the following properties:

$$|\beta_\lambda| \leq 1\ ,\ \ |d\beta_\lambda| \leq 2/\sqrt{\lambda} \text{ and } d\beta_\lambda \text{ is supported in } \sqrt{\lambda} \leq r \leq 2\sqrt{\lambda}$$

The definition of the approximate gluing map on the top stratum. Let $\mathcal{N}$ denote the top stratum of $\{x_1\} \times \overline{\mathcal{U}}_d$. First we need to choose a canonical representative of each bubble curve $[f, y] \in \mathcal{N}$ (recall that $f(y)$ is the image of the ghost base). Using the $G = \mathbf{PSL}(2, \mathbf{C})$ action, we can assume that y is the North pole and f is centered on the vertical axis, which leaves a $\mathbf{C}^* \cong S^1 \times \mathbf{R}_+$ indeterminancy. To break it off, include as gluing data a unit vector tangent to the domain S^2 of the bubble at the bubble point y. The frame bundle

$$Fr = \{\ [f, y, u] \mid [f, y] \in \mathcal{U}_d,\ u \in T_yS^2,\ |u| = 1\} \tag{1.9}$$

models the link of $\mathcal{N}$. The notation $[f, y, u]$ means the equivalence class under the action of G given by:

$$g \cdot (f, y, u) = (f \circ g^{-1},\ g(y),\ g(u))$$

where the compact piece $SO(3) \subset G$ acts on the unit frame u by rotations and the noncompact part acts trivially.

Fix a nonzero vector u_1 tangent to the torus at x_1. This determines an identification $T_{x_1}(T^2) \cong \mathbf{C}$ such that $u_1 = 1$, giving local coordinates on the torus at $x_1 = 0$. Similarly, let u_0 be a unit vector tangent to the sphere S^2 at the north pole and consider the identification

$$(T_{x_1}T^2, u_1) \cong (T_N S^2, u_0) \tag{1.10}$$

that induces natural coordinates on the sphere via the stereographical projection (such that $N = 0,\ u_0 = 1$). These choices of local coordinates on the domain of the bubble tree will be used for the rest of the paper. Fix also a metric on $\mathbf{P}^n$ such that we can use normal coordinates up to radius 1.

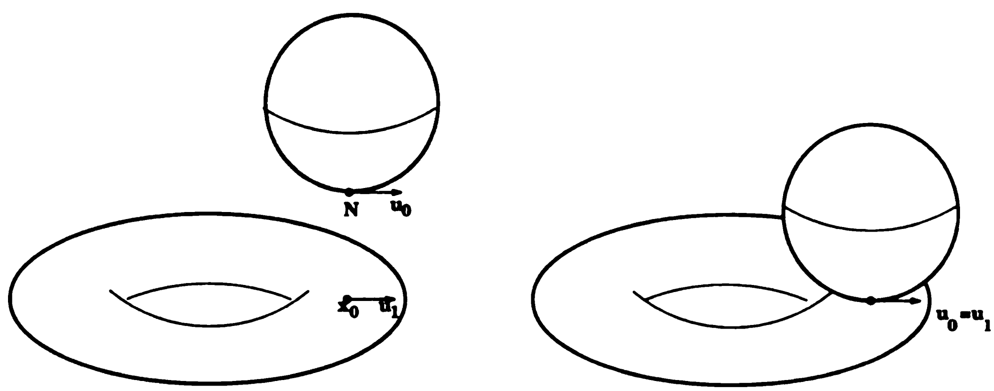

Figure 1. The domain of the bubble tree.

To glue, one needs to make sure that only a small part of the energy of f is concentrated in a neighbourhood of y. The convention in [PW] is to rescale f until ε_0 of its energy is distributed in H_y, the hemisphere centered at y.

But since the constructions in the next couple of sections involve quite a few estimates, we prefer to do a different rescaling, that will simplify the analysis. Choose a representative of $[f, y, v]$ such that

$$y = 0, \quad u = 1, \quad f \text{ centered on the vertical axis} \tag{1.11}$$

Since on the top strata $[f, y]$ cannot be a ghost, such representative is uniquely determined up to a rescaling factor $r \in \mathbf{R}_+$. We will choose this rescaling factor such that moreover

$$\max\{ \, |\nabla^2 f(z)|, \ |z| \leq 1\} \leq 2 \tag{1.12}$$

Note that if the degree of f is not 1, then imposing the extra condition

$$\max\{ \, |\nabla^2 f(z)|, \ |z| \leq 1\} = 2 \tag{1.13}$$

determines uniquely the representative. To see this, choose some representative $\tilde{f}$ as in (1.11) and look for a map $f(z) = \tilde{f}(rz)$ satisfying also (1.13). The uniqueness comes from the fact that the map $s(r) = \max\{ \, |\nabla^2 \tilde{f}(z)|, \ |z| \leq r\} - 2/r^2$ is decreasing.

If the degree of f is 1, (i.e. the image curve is a line), then we could replace (1.13) by say $|df(0)| = 1$ and still have (1.12) satisfied.

Finally, the *approximate gluing map*

$$\gamma_\epsilon : Fr \times (0, \varepsilon) \to \mathcal{M}aps(T^2, X)$$
$$\gamma_\epsilon(\, [f, y, u], \ \lambda) = f_\lambda \tag{1.14}$$

is constructed as follows: Choose the unique representative of $[f, y, u]$ satisfying (1.11) and (1.13). The approximate map f_λ is obtained by gluing to the constant map $f(y)$ defined on T^2 the bubble map f rescaled by a factor of λ inside a disk $D(0, \sqrt{\lambda}) \subset T^2$,

$$f_\lambda(z) = \beta_\lambda(|z|) f\left(\frac{\lambda}{z}\right)$$

where the multiplication is done in normal coordinates at $f(0)$.

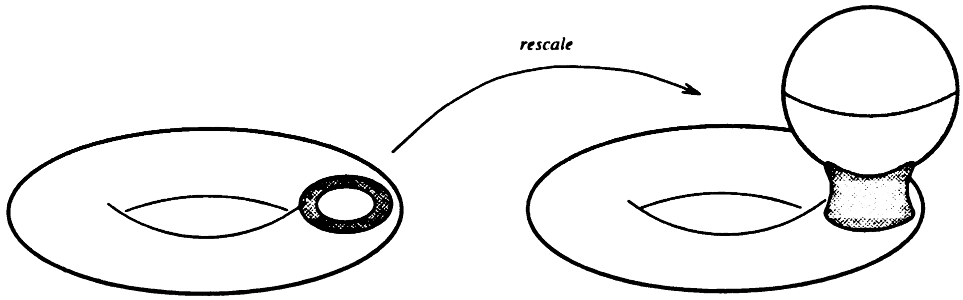

Figure 2. The rescaled domain of the approximate map.

In what follows, we will denote by $Gl = Fr \times (0, \varepsilon)$ the set of gluing data.

Weighted Norms. On the domain of f_λ we will use the rescaled metric $g_\lambda = \theta_\lambda^{-2} dz d\bar{z}$, where

$$\theta_\lambda(z) = (1 - \beta_\lambda(z))(\lambda + \lambda^{-1}|z|^2) + \beta_\lambda(z)$$

Define

$$\|\xi\|_{1,p,\lambda} = \left(\int |\xi|^p \theta_\lambda^{-2} + |\nabla\xi|^p \theta_\lambda^{p-2}\right)^{1/p} \text{ for } \xi \text{ vector field along } f_\lambda \text{ and}$$

$$\|\eta\|_{p,\lambda} = \left(\int |\eta|^p \theta_\lambda^{p-2}\right)^{1/p} \text{ for } \eta \text{ 1-form along } f_\lambda$$

The weighted norm of a vector field or 1-form on f_λ equals its usual norm off $B(0, 2\sqrt{\lambda})$ and on $B(0, \sqrt{\lambda})$ it is equal with the norm of its pulled back on S^2 via a rescaling of factor λ. The usual Sobolev embeddings hold for this weighted norms with constants independent of λ.

Lemma 1.1 *There exists $\varepsilon_0 > 0$ and constants $C > 0$ such that for any $p \geq 1$ and $\lambda \leq \varepsilon_0$:*

$$\|df_\lambda\|_{p,\lambda} \leq C \text{ and } \|\bar{\partial}_J f_\lambda\|_{p,\lambda} \leq C\lambda^{1/p} \tag{1.15}$$

Moreover on the annulus A: $\{\sqrt{\lambda} \leq |z| \leq 2\sqrt{\lambda}\}$ we have the following expansion:

$$\bar{\partial}_J f_\lambda = \frac{\sqrt{\lambda}}{|z|} d\beta \cdot df(y)(u) + O(\lambda) \tag{1.16}$$

The estimates are uniform on $Gl \to \mathcal{N}$.

Proof. Let B be the disk $|z| \le \sqrt{\lambda}$. Note that df_λ vanishes for $|z| \ge 2\sqrt{\lambda}$ and by the definition of the weighted norm on B,

$$\|df_\lambda\|_{p,\lambda,B} = \|df\|_{p,B}$$

But (1.12) implies that

$$\max\{\, |df(z)|,\ |z| \le 1\} \le 2 \tag{1.17}$$

In the same time, $\overline{\partial}_J f_\lambda = 0$ outside A. Hence we need only to consider what happens in A. But on A

$$|\overline{\partial}_J f_\lambda| \ \le\ C|df_\lambda| \ \le\ C(|d\beta_\lambda|\,|f| + |\beta_\lambda|\,|df|)\,\frac{\lambda}{|z|^2} \ \le\ C\frac{1}{\sqrt{\lambda}}\sup_B |f| + C \ \le\ C$$

since $\sup\limits_B |f(z)| \ \le\ \sqrt{\lambda}\ \sup\limits_B |df| \ \le\ 2\sqrt{\lambda}$ in normal coordinates on $\mathbf{P}^n$ at $f(y)$. This concludes the first part of the proof. For the second part, notice that on A

$$\overline{\partial}_J f_\lambda = \overline{\partial}_J \beta_\lambda \cdot f + \beta_\lambda \cdot \overline{\partial}_J f = \frac{1}{\sqrt{\lambda}}\, d\beta\, \frac{z}{|z|} \cdot f\left(\frac{\lambda}{z}\right)$$

since f is holomorphic. But using (1.12) in normal coordinates on $\mathbf{P}^n$ at $f(y)$ and $y = 0$, we get $|f(z) - f(0) - df(0)(z)| \le 2|z|^2$ so

$$f\left(\frac{\lambda}{z}\right) = \frac{\lambda}{z} \cdot df_y(u) + O(\lambda) \quad \text{on} \quad A$$

Substituting this in the formula for $\overline{\partial}_J f_\lambda$ we obtain (1.16). □

Extending the approximate gluing map. The approximate gluing map extends naturally to the bubble tree compactification $\overline{\mathcal{U}}_d$ of the moduli space of 1-marked curves. For simplicity, let $\mathcal{N}$ denote some boundary stratum modeled on a bubble tree B and corresponding to a certain distribution of the degree $d = d_1 + \ldots + d_m$ on the bubbles. If $[f_i, y_i]$, $i = 1, \ldots, m$ are the bubble curves corresponding to the bubble map $f : B \to \mathbf{P}^n$, then the gluing data Gl is a collection of unit vectors tangent to each sphere in the domain at the corresponding bubble point together with gluing parameters:

$$Gl = \{ \ (\ [f_i, y_i, u_i], \lambda_i \ )_{i=1}^m \mid u_i \in T_{y_i} S^2, \ |u_i| \neq 0, \lambda_i \leq \varepsilon \} \tag{1.18}$$

Note that as long as f_i is not a constant map, then we can choose a unique reresentative of $[f_i, y_i, u_i]$ as in (1.11), (1.13). Then Lemma 1.1 extends naturally to $\mathcal{N}$ to give

Lemma 1.2 *With the notations above, let f_λ be an approximate gluing map, and $A_1, \ldots A_m$ be the corresponding annuli of radii λ_i in which the cutoff functions are supported. Then for ε small enough, there exists a constant C such that:*

$$\|df_\lambda\|_{p,\lambda} \leq \quad C \ , \quad \|\overline{\partial}_J f_\lambda\|_{p,\lambda} \leq C\lambda^{1/p}$$

Moreover, $\overline{\partial}_J f_\lambda = 0$ except on the annuli A_i that correspond to nontrivial bubbles, where

$$\overline{\partial}_J f_\lambda = -\frac{\sqrt{\lambda_i}}{|z|} \, d\beta \cdot df_i(y_i)(u_i) + O(\lambda_i) \tag{1.19}$$

The estimates above are uniform on $Gl \to \mathcal{N}$.

We will see later that most of the important information is encoded in the first level of nontrivial bubbles.

1.3 The Obstruction Bundle

In order to see which of the approximate maps can be corrected to solutions of the equation $\overline{\partial}_J f = \nu$ we need first to understand the behaviour of the linearization of this equation over the space of approximate solutions.

Recall that transversality fails at a bubble tree with ghost base, so the linearization at such bubble tree is not onto. The cause of that is the ghost base. Thus we start by analysing the ghost maps:

Consider the moduli space of holomorphic maps $f : T^2 \to \mathbf{P}^n$ representing $0 \in H_2(\mathbf{P}^n)$. Obviously, the only such maps are the constant ones (ghosts). If D_u is the linearization of the section $\overline{\partial}_J : \mathcal{M}aps(T^2, \mathbf{P}^n) \to \Lambda^{0,1}$ at $f : T^2 \to \mathbf{P}^n$, $f(x) = u$ a constant map, then

$$\text{index } \mathrm{D}_u \;=\; \dim \mathrm{KerD}_u - \dim \mathrm{CokerD}_u = c_1(0) + n(1-1) = 0$$

and

$$\mathrm{CokerD}_u = H^1(T^2, f^*T\mathbf{P}^n) \cong T_u\mathbf{P}^n \quad \text{(canonically)}$$

since $f^*(T\mathbf{P}^n)$ is a trivial bundle, so the elements $\omega \in H^1(T^2, f^*T\mathbf{P}^n)$ are constant on the torus, i.e. have the form $\omega = Xdz$ for some $X \in T_u\mathbf{P}^n$.

Now if $f : B \to \mathbf{P}^n$ is a bubble tree map whose base is a ghost torus $u = f(y) \in \mathbf{P}^n$, let D_f be the linearization at f of the section $\overline{\partial}_J : \mathcal{M}aps(B, \mathbf{P}^n) \to \Lambda^{0,1}$. Then

$$\text{index } \mathrm{D}_f \;=\; \dim \mathrm{KerD}_f - \dim \mathrm{CokerD}_f = -1$$

To describe CokerD_f we will use the following:

Definition 1.3 *If $f : B \to \mathbf{P}^n$ is as above, let*

$B_1 \subset B$ consist of the domains of all the ghost bubbles with image $f(y)$,

$B_2 = B - B_1$ and

$\tilde{B} \subset B$ denote the first level of bubbles that are not in B_1.

Then CokerD_f is n dimensional, consisting of 1-forms ω such that

$$\omega = \begin{cases} Xdz & \text{on } B_1 \\ 0 & \text{on } B_2 \end{cases}$$

for some $X \in T_u\mathbf{P}^n$. In particular, there is a natural isomorphism

$$\begin{array}{ccc} \mathrm{CokerD} & \cong & \mathrm{ev}^*(TP^n) \\ & \searrow \quad \swarrow & \\ & \overline{\mathcal{U}}_d & \end{array} \tag{1.20}$$

where $\mathrm{ev} : \overline{\mathcal{U}}_d \to \mathbf{P}^n$ is the evaluation map. Since the moduli space of bubble trees $\overline{\mathcal{U}_d}$ is compact, there exists a constant $E > 0$ such that $D_f D_f^*$ has a zero eigenvalue with multiplicity n, and all the other eigenvalues are greater than $2E$.

When f_λ is an approximate map, let D_λ be the linearization of

$$\bar{\partial}_J : \mathcal{M}aps(T^2, \mathbf{P}^n) \to \Lambda^{0,1}$$

at f_λ and D_λ^* its L^2-adjoint with respect to the metric g_λ on T^2. Then D_λ is not uniformly invertible. More precisely,

Lemma 1.4 *For $\lambda > 0$ small, the operator $\Delta_\lambda = D_\lambda D_\lambda^*$ has exactly n eigenvalues of order $\sqrt{\lambda}$ and all the others are greater than E. Moreover, over the set of gluing data Gl, the span of low eigenvalues*

$$\begin{array}{ccc} \Lambda^{0,1}_{low}(f_\lambda{}^* T\mathbf{P}^n) & \hookrightarrow & \Lambda^{0,1}_{low} \\ & & \downarrow \\ & & Gl \end{array}$$

is a n-dimensional vector bundle (called the Taubes obstruction bundle*), naturally isomorphic to the bundle*

$$\mathrm{ev}^*(T\mathbf{P}^n) \to Gl$$

where $\mathrm{ev} : Gl \to \mathbf{P}^n$ *is the evaluation map.*

Proof. The proof is more or less the same as the one Taubes used for the similar result in the context of Donaldson theory, [T1]. For each gluing data in Gl, by cutting and pasting eigenvectors we show that the eigenvalues of $\Delta_\lambda = D_\lambda D_\lambda^*$ are $O(\sqrt{\lambda})$ close to those of $\Delta_u = D_u D_u^*$, where u is the point map in the base of the bubble tree.

Take for example the top stratum of $\overline{\mathcal{U}}_d$. Choose $\{\omega_i,\ i = 1, n\}$ a local orthonormal base of CokerD $\cong \mathrm{ev}^*(T\mathbf{P}^n)$ and define

$$\omega^i_\lambda(z) = \beta\left(\frac{z}{2\sqrt{\lambda}}\right)\omega^i(z) \qquad (1.21)$$

A straightforward computation shows that:

$$\|D_\lambda^*\omega_\lambda\|_{2,\lambda} \leq \lambda^{1/4}\|\omega_\lambda\|_{2,\lambda} \tag{1.22}$$

$$\langle\overline{\omega}_\lambda^i, \overline{\omega}_\lambda^j\rangle_{2,\lambda} = \delta_{ij} + O(\lambda) \tag{1.23}$$

The Gramm-Schmidt orthonormalization procedure then provides n eigenvectors $\overline{\omega}_\lambda^i$ for Δ_λ with eigenvalues $O(\sqrt{\lambda})$ such that

$$\overline{\omega}_\lambda^i = \omega_\lambda^i + O(\lambda)$$

The construction above extends naturally to the other substrata of $\bar{\mathcal{U}}_d$. Note that for example when B_1 has other components besides T^2 then $\overline{\omega}_\lambda$ is equal to ω not only on the ghost base, but on all B_1 and is extended with 0 starting from the first level of nontrivial bubbles.

An adaptation of Taubes argument from [T1] shows that there are at most n low eigenvalues of Δ_λ. Therefore there is a well defined splitting

$$\Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n) = \Lambda^{0,1}_{low}(f_\lambda{}^*T\mathbf{P}^n) \oplus \Lambda^{0,1}_E(f_\lambda{}^*T\mathbf{P}^n)$$

The definition (1.21) combined with (1.20) provides the isomorphism $\Lambda^{0,1}_{low} \cong \mathrm{ev}^*(T\mathbf{P}^n)$, concluding the proof. □

The partial right inverse of D_λ. The restriction of $D_\lambda D_\lambda^*$ to $\Lambda^{0,1}_E$ is invertible (since all its eigenvalues are at least E). Define P_λ to be the composition of the L^2-othogonal projection $\Lambda^{0,1} \to \Lambda^{0,1}_E$ with the operator $D_\lambda^*(D_\lambda D_\lambda^*)^{-1}$ on $\Lambda^{0,1}_E$. Then

$$P_\lambda : \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n) \to \Lambda^0(f_\lambda{}^*T\mathbf{P}^n) \tag{1.24}$$

is the *partial right inverse of* D_λ and satisfies the uniform estimate:

$$\|P_\lambda\eta\|_{1,p,\lambda} \leq E^{-1}\|\eta\|_{p,\lambda} \tag{1.25}$$

We will denote by $\pi_-^{f_\lambda} : \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n) \to \Lambda^{0,1}_{low}(f_\lambda{}^*T\mathbf{P}^n)$ the projection onto the fiber of the obstruction bundle.

1.4 The Gluing map

The next step is to correct the approximate gluing map to take values in the moduli space $\mathcal{M}_{t\nu}$ of solutions to the equation

$$\overline{\partial}_J f(x) = t \cdot \nu(x, f(x)) \tag{1.26}$$

where ν is generic and fixed and t is a small parameter.

If f_λ is an approximate map, use the exponential map to write any nearby map in the form $f = \exp_{f_\lambda}(\xi)$, for some correction $\xi \in \Lambda^0(f_\lambda{}^*T\mathbf{P}^n)$. Let D_λ be the linearization of the $\overline{\partial}_J$-section at f_λ so

$$\overline{\partial}_J f = \overline{\partial}_J f_\lambda + D_\lambda(\xi) + Q_\lambda(\xi) \tag{1.27}$$

where Q_λ is quadratic in ξ. Similarly,

$$\nu(x, f(x)) = \nu(x, f_\lambda(x)) + d\nu(\xi) + \tilde{Q}_\lambda(\xi)$$

so equation (1.26) can be rewritten as:

$$D_\lambda(\xi) + N_\lambda(\xi, t) = t\nu(x, f_\lambda(x)) - \overline{\partial}_J f_\lambda \tag{1.28}$$

where $N_\lambda(\xi, t) = Q_\lambda(\xi) - t d\nu(\xi) - t\tilde{Q}_\lambda(\xi)$ is quadratic in (ξ, t).

The kernel of D_λ models the tangent directions to the space of approximate maps, so is natural to look for a correction in the normal direction. More precisely, we will consider the solutions of (1.28) of the form

$$f = \exp_{f_\lambda}(P_\lambda \eta) \quad \text{where} \quad \pi_-(\eta) = 0 \tag{1.29}$$

Since $D_\lambda(P_\lambda(\eta)) = \eta$ for such η, then equation (1.28) becomes

$$\eta + N_{\lambda,t}(P_\lambda \eta) = t\nu - \overline{\partial}_J f_\lambda \tag{1.30}$$

The existence of a solution of (1.30) is a standard aplication of the Banach fixed point theorem combined with the estimates in the previous sections.

Lemma 1.5 *There exists a constant $\delta > 0$ (independent of λ, t) such that for t small enough and for any $\alpha \in \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n)$ so that $\|\alpha\|_{p,\lambda} < \delta/2$ the equation:*

$$\eta + N_{\lambda,t}(P_\lambda \eta) = \alpha$$

has a unique small solution $\eta \in \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n)$ with $\|\eta\|_{p,\lambda} < \delta$. Moreover,

$$\|\eta\|_{p,\lambda} < 2\|\alpha\|_{p,\lambda}$$

and if α is C^∞, so in η.

Proof. Apply the contraction principle to the operator

$$T_\lambda : \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n) \to \Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n)$$

$$T_\lambda \eta = \alpha - N_{\lambda,t}(P_\lambda \eta).$$

defined on a small ball centered at 0 in the Banach space $\Lambda^{0,1}(f_\lambda{}^*T\mathbf{P}^n)$ with the weighted Sobolev norm L^p_λ. To prove that T is a contraction we note that:

$$\|T_\lambda \eta_1 - T_\lambda \eta_2\|_{p,\lambda} = \|N_{\lambda,t}(P_\lambda \eta_1) - N_{\lambda,t}(P_\lambda \eta_2)\|_{p,\lambda}$$

and use some estimates of Floer. He proved in [F] that for the quadratic part Q of (1.27), there exists a constant C depending only on $\|df\|_{p,\lambda}$ such that:

$$\|Q_f(\xi_1) - Q_f(\xi_2)\|_{p,\lambda} \leq C\,(\,\|\xi_1\|_{1,p,\lambda} + \|\xi_2\|_{1,p,\lambda})\|\xi_1 - \xi_2\|_{1,p,\lambda} \tag{1.31}$$

$$\|Q_f(\xi)\|_{p,\lambda} \leq C\,\|\xi\|_{\infty,\lambda} \cdot \|\xi\|_{1,p,\lambda}. \tag{1.32}$$

(Floer's estimates are for the usual Sobolev norm, but the same proof goes through for the weighted norms.) Since $\|df\|_{p,\lambda}$ is uniformly bounded by Lemma 1.2, the same constant C works for all $f_\lambda \in Im(\gamma_\epsilon)$. Moreover, for t very small the same estimates hold for the nonlinear part $N_{\lambda,t}$. Hence by (1.31):

$$\begin{aligned} \|T_\lambda \eta_1 - T_\lambda \eta_2\|_{p,\lambda} &\leq C\,(\,\|P_\lambda \eta_1\|_{1,p,\lambda} + \|P_\lambda \eta_2\|_{1,p,\lambda})\|P_\lambda(\eta_1 - \eta_2)\|_{1,p,\lambda} \\ &\leq C/E^2\,(\,\|\eta_1\|_{p,\lambda} + \|\eta_2\|_{p,\lambda}) \cdot \|\eta_1 - \eta_2\|_{p,\lambda}. \end{aligned}$$

Choosing $\delta < E^2/(4C)$ this implies

$$\|T_\lambda\eta_1 - T_\lambda\eta_2\|_{p,\lambda} \;\leq\; 1/2\ \|\eta_1 - \eta_2\|_{p,\lambda}$$

for any $\eta_1, \eta_2 \in B(0,\delta)$. Moreover, since $\|T_\lambda(0)\|_{p,\lambda} \;\leq\; \delta/2$ then $T_\lambda : B(0,\delta) \to B(0,\delta)$ is a contraction. Therefore T_λ has a unique fixed point η in the ball such that moreover

$$\|\eta\|_{p,\lambda} \;\leq\; \|T_\lambda\eta - T_\lambda(0)\|_{p,\lambda} + \|T_\lambda(0)\|_{p,\lambda} \;\leq 1/2\ \|\eta\|_{p,\lambda} + \|T_\lambda(0)\|_{p,\lambda}$$

so $\|\eta\|_{p,\lambda} \leq 2\ \|T_\lambda(0)\|_{p,\lambda} = 2\|\alpha\|_{p,\lambda}$. Elliptic regularity implies that η is smooth when α is. $\square$

Corollary 1.6 *For t, λ small enough, equation (1.30) has a unique small solution $\|\eta\|_{p,\lambda} \;\leq\; \delta$. Moreover,*

$$\|\eta\|_{p,\lambda} \;\leq\; C(t|\nu| + \lambda^{\frac{1}{p}}).$$

Proof. Follows immediately from Lemmas 1.2 and 1.4 and the estimate

$$\|\alpha\|_{p,\lambda} = \|\ t\nu - \overline{\partial}_J f_\lambda\|_{p,\lambda} \;\leq\; t|\nu| + C\lambda^{\frac{1}{p}}. \qquad \square$$

The gluing map. Let Gl be the set of gluing data. The *gluing map* is defined by

$$\bar{\gamma}_\epsilon : Gl \to \mathcal{M}aps(T^2, X)$$

$$\bar{\gamma}_\epsilon([f,y,u],\lambda) = \bar{f}_\lambda = \exp_{f_\lambda}(P_\lambda\eta)$$

where $\eta = \eta(f,y,u,\lambda)$ is the unique solution to the equation (1.30) given by Corrolary 1.6.

By construction, $\bar{\gamma}_\epsilon$ is a local diffeomorphism onto its image. Moreover, if $\pi_-^{f_\lambda}(\eta) = 0$ then $\bar{f}_\lambda$ is actually a solution of (1.26).

The obstruction to gluing. The section

$$\psi_t : Gl \to \Lambda^{0,1}_{low}(f_\lambda{}^*T\mathbf{P}^n) \quad \text{given by}$$

$$\psi_t(f,y,u,\lambda) = \pi_-^{f_\lambda}(\eta) = \pi_-^{f_\lambda}(t\nu - \overline{\partial}_J f_\lambda) - \pi_-^{f_\lambda}(N_{\lambda,t}(P_\lambda\eta))$$

will be called the ***obstruction to gluing.*** Let $Z_t = \psi_t^{-1}(0)$ be the zero set of this section. By applying the gluing construction to bubble trees in Z_t we obtain a subset of the moduli space $\mathcal{M}^{t\nu}$.

1.5 Completion of the construction

We have seen in the previous section that applying the gluing construction to the bubble trees in the zero set Z_t we will get elements of the moduli space $\mathcal{M}_{d,1,t\nu}$. It is not clear yet why all the elements of this moduli space close enough to the boundary stratum $\mathcal{N}$ can be obtained by the gluing procedure. The purpose of this section is to clarify this issue.

Recall the construction of the gluing map: Starting with a bubble tree we glue in the bubble to obtain an approximate map f_λ. Then we correct f_λ by pushing it in a direction normal to the kernel of D_λ in order to get an element of the moduli space $\mathcal{M}^{t\nu}$. The key fact here is that the kernel of the linearization D_λ models the tangent space to the approximate maps, and therefore, at least in the linear model, it is enough to look for solutions only in a normal direction. For the construction to be complete though, we need to show that the same thing is true for the nonlinear problem.

More precisely, we will show that for t small, all the elements of the moduli space $\mathcal{M}_{d,1,t\nu}$ close to the boundary stratum $\mathcal{N}$ can be reached starting with an approximate map and going out in a normal direction. The proof of the following Theorem is an adaptation of the proof for the same kind of result in the context of Donaldson theory [DK]. It is pretty technical and we include it just for continuity.

Theorem 1.7 *The end of the moduli space $\mathcal{M}_{d,1,t\nu}$ close to the boundary strata $\mathcal{N}$ is diffeomorphic to the zero set of the section ψ_t. More precisely, for δ and t small* *enough, there exists an isomorphism*

$$\mathcal{M}_{d,1,t\nu} \cap \mathbf{U}_\delta \cong \psi_t^{-1}(0) \qquad \textit{where} \tag{1.33}$$

$$\mathbf{U}_\delta = \{f : T^2 \to X \mid \exists f_\lambda \ s.t.\ f = \exp_{f_\lambda}(\xi),\ \|\xi\|_{1,2,\lambda} \le \delta \ and \ \|\overline{\partial}_J f\|_{2,\lambda} \le \delta^{3/2}\} \tag{1.34}$$

and $f_\lambda \in \mathrm{Im}\gamma_\varepsilon$ *is some approximate map.*

Proof. The proof consists of 2 steps. First, Lemma 1.8 shows that $\mathbf{U}_\delta$ is actually a neighborhood of $\mathcal{N}$ in the bubble tree convergence topology. Second, recall that in constructing the section ψ_t we were looking for solutions of the equation (1.26) that have the form

$$f = \exp_{f_\lambda}(P_\lambda \eta) \quad \text{for some} \ \|\eta\|_{2,\lambda} \le \delta \tag{1.35}$$

To prove the Theorem it is enough to show that for t small, all the solutions of the equation (1.26) can be written in the form (1.35). This is a consequence of Proposition 1.9.

Lemma 1.8 $\mathbf{U}_\delta \cap \overline{\mathcal{M}^\nu}$ *is a neighborhood of* $\mathcal{N}$ *in the bubble tree convergence topology. More precisely, for any* $(J, t\nu)$*-holomorphic map* f *close to the boundary strata* $\mathcal{N}$ *there exists an approximate map* f_λ *such that* f *can be written in the form*

$$f = \exp_{f_\lambda}(\xi) \quad \textit{for some} \ \|\xi\|_{1,2,\lambda} \le \delta$$

Proof. By contradiction, assume there exists a sequence f_n of $(J, t_n\nu)$-holomorphic maps for $t_n \to 0$ such that f_n do not have the required property. By the bubble tree convergence Theorem ([PW]) there exists a bubble tree f such that $f_n \to f$ uniform on compacts. Moreover, after rescaling the functions f_n by some λ_n, this becomes a $L^{1,2}$-convergence. But this is equivalent to saying that f_n is $L^{1,2,\lambda_n}$ close to f. In particular, for λ small enough, f_n is $L^{1,2,\lambda_n}$ close to f_{λ_n}, which contradicts the assumption. $\square$

Proposition 1.9 *For small enough* δ, t *any map in* $\mathbf{U}_\delta$ *can be represented in the form*

$$f = \exp_{f_\lambda}(P_\lambda \eta) \quad \textit{for some} \ f_\lambda \in \mathrm{Im}\gamma_\varepsilon,\ \|\eta\|_{2,\lambda} < \delta \ \ \textit{and} \ \ \pi_-^{f_\lambda}(\eta) = 0$$

Proof. We will use the continuation method. The key fact is that a neighborhood of f_λ in Imγ_ϵ is modeled by Λ^0_{low} and that P_λ spans the normal directions to Imγ_ϵ.

Let $f \in \mathbf{U}_\delta$. By definition, there is $f_\lambda \in$ Imγ_ϵ such that $f = \exp_{f_\lambda} \xi$, where $\|\xi\|_{1,2,\lambda} < \delta$. Consider the path $f_s = \exp_{f_\lambda}(s\xi)$. Let

$$S = \{s \in [0,1] \mid \exists f_{\lambda_s} \text{ and } \|\eta_s\|_{p,\lambda_s} < \delta \text{ such that } f_s = \exp_{f_{\lambda_s}}(P_{\lambda_s}\eta_s)\}. \tag{1.36}$$

Note that by definition $f = f_\lambda = \exp_{f_\lambda}(0)$ so $0 \in S$. We will show that S is both open and closed and since it is nonempty, $1 \in S$.

S is closed. The only open condition in the definition of S is $\|\eta_s\|_{p,\lambda_s} < \delta$. But since

$$\begin{aligned}
\overline{\partial}_J f_s &= \overline{\partial}_J f_{\lambda_s} + D_{\lambda_s}(P_{\lambda_s}\eta_s) + N_\lambda(P_{\lambda_s}\eta_s) \qquad \text{then} \\
\eta_s &= \overline{\partial}_J f_s - \overline{\partial}_J f_{\lambda_s} - N_{\lambda_s}(P_{\lambda_s}\eta_s) \qquad \text{so} \\
\|\eta_s\|_{2,\lambda} &\leq \|\overline{\partial}_J f_s\|_{2,\lambda} + \|\overline{\partial}_J f_{\lambda_s}\|_{2,\lambda} + \frac{C}{E^2}\|\eta_s\|^2_{2,\lambda} \\
&\leq \|\overline{\partial}_J f_s\|_{2,\lambda} + C\sqrt{\lambda} + C\|\eta_s\|^2_{2,\lambda}
\end{aligned} \tag{1.37}$$

We need to estimate $\|\overline{\partial}_J f_s\|_{2,\lambda}$. Since

$$\overline{\partial}_J f_s = \overline{\partial}_J f_\lambda + sD_\lambda(\xi) + N_\lambda(s\xi) \quad \text{and} \quad \overline{\partial}_J f_1 = \overline{\partial}_J f_\lambda + D_\lambda(\xi) + N_\lambda(\xi) \quad \text{then}$$

$$\overline{\partial}_J f_s = s\overline{\partial}_J f_1 + (1-s)\overline{\partial}_J f_\lambda + N_\lambda(s\xi) - sN_\lambda(\xi)$$

The estimate (1.32) gives $\|N_\lambda(\xi)\|_{2,\lambda} \leq C\ \|\xi\|^2_{1,2,\lambda}$ so

$$\|\overline{\partial}_J f_s\|_{2,\lambda} \leq \|\overline{\partial}_J f_1\|_{2,\lambda} + \|\overline{\partial}_J f_\lambda\|_{2,\lambda} + 2\,C\ \|\xi\|^2_{1,2,\lambda} \leq \sqrt{\lambda} + \delta^{3/2} + C\ \delta^2$$

Therefore for $\lambda << \delta$,

$$\|\overline{\partial}_J f_s\|_{2,\lambda} \leq 2\,C\ \delta^{3/2} \tag{1.38}$$

Using (1.38) in (1.37) we get

$$\|\eta_s\|_{2,\lambda} \leq 2\,C\ \delta^{3/2} + C\sqrt{\lambda} + C\|\eta_s\|^2_{2,\lambda}$$

For small $\lambda \le \delta^3$, the constraint $\|\eta_s\|_{2,\lambda} < \delta$ implies $\|\eta_s\|_{2,\lambda} < \delta/2$ so it is a closed condition too.

S is open. Assume that $s_0 \in S$, i.e. there exists an approximate map f_{λ_0} such that $f_{s_0} = \exp_{f_{\lambda_0}}(P_{\lambda_0}(\eta_0))$. We will show that $s \in S$ for s sufficiently close to s_0. For that we need to find an approximate map f_{λ_s} and an $\eta_s \in \Lambda^0_E$ such that:

$$f_s = \exp_{f_\lambda}(s\xi) = \exp_{f_{\lambda_s}}(P_{\lambda_s}\eta_s) \tag{1.39}$$

It is enough to prove that the linearization of the equation (1.39) is onto at s_0. First we prove that:

Lemma 1.10 *A small neighborhood $\mathcal{N}_\delta$ of f_λ in $\mathrm{Im}\gamma_\epsilon$ is modelled by Λ^0_{low}. More precisely, there is a well defined map $g : \Lambda^0_{low} \to \Lambda^{0,1}_E$ such that any approximate map $f \in \mathrm{Im}\gamma_\epsilon$ has the form $f = \exp_{f_\lambda}(\zeta + P_\lambda g(\zeta))$ for some $\zeta \in \Lambda^0_{low}$, $\|\zeta\|_{1,2,\lambda} \le \delta$.*

Proof. The first statement is an immediate consequence of the way we constructed the approximate maps. For the second part, notice that any $f \in \mathrm{Im}\gamma_\epsilon$ close to f_λ can be written in the form $f = \exp_{f_\lambda}(\chi)$, with χ small. Let $\chi = \zeta + P_\lambda\eta$ be the orthogonal decomposition of χ in $\Lambda^0_{low} \oplus \Lambda^0_E$, where $\eta \in \Lambda^{0,1}_E$ (recall that $P_\lambda : \Lambda^{0,1}_E \to \Lambda^0_E$ is an isomorphism). Using the same techniques as in Section 1.4 we can prove that for any $\zeta \in \Lambda^0_{low}$ there exists a unique solution $\eta = g(\zeta)$ to the equation

$$\eta + N_\lambda(P_\lambda\eta) = \overline{\partial}_J f$$

which concludes the proof of Lemma. $\square$

Since the notations are becoming cumbersome, we will illustrate for simplicity the case $s_0 = 0$. The general case follows similarly. Using Lemma 1.10 we can regard the equation (1.39) as an equation in $(\zeta, \eta) \in \Lambda^0_{low} \oplus \Lambda^{0,1}_E$. More precisely, for a fixed s small, we need to find $\zeta \in \Lambda^0_{low}$ and $\eta \in \Lambda^0_E$ such that the approximate map $f = \exp_{f_\lambda}(\zeta + P_\lambda g(\zeta))$ solves the equation:

$$\exp_f(P_f\eta) = \exp_{f_\lambda}(s\xi) \tag{1.40}$$

The linearization of the equation (1.40) at $(0, \eta)$ is $\mathbf{D} : \Lambda^0_{low} \oplus \Lambda^{0,1}_E \to \Lambda^0$,

$$\mathbf{D}_{(0,\eta)}(\mathbf{z}, \mathbf{n}) = \mathbf{z} + P_\lambda \nabla g(\mathbf{z}) + P_\lambda \mathbf{n} + \Pi(\mathbf{z}, \eta)$$

where $\Pi(\mathbf{z}, \eta)$ is the derivative of $P_\lambda \eta$ with respect to f_λ.

Our goal is to show that the operator $\mathbf{D}_{(0,\eta)}$ is an isomorphism in some appropriate norms on $\Lambda^0_{low} \oplus \Lambda^{0,1}_E$ and Λ^0.

Definition 1.11 *On $\Lambda^0_{low} \oplus \Lambda^{0,1}_E$ and Λ^0 define the following norms:*

$$\|\,(\mathbf{z}, \mathbf{n})\,\|_{B_1} = \|\mathbf{z}\|_{1,2,\lambda} + \|\mathbf{n} + \nabla g(\mathbf{z})\|_{2,\lambda} \qquad \textit{for any } (\mathbf{z}, \mathbf{n}) \in \Lambda^0_{low} \oplus \Lambda^{0,1}_E$$

$$\|\xi\|_{B_2} = \|D_\lambda \xi\|_{2,\lambda} \qquad \textit{for any } \xi \in \Lambda^0$$

Consider the operator $\mathbf{T} : \Lambda^0_{low} \oplus \Lambda^{0,1}_E \to \Lambda^0$ given by $T(\mathbf{z}, \mathbf{n}) = z + P_\lambda(\mathbf{n} + \nabla g(\mathbf{z}))$. Then $\mathbf{T}$ is continuous, since

$$\|\mathbf{T}(\mathbf{z}, \mathbf{n})\|_{B_2} = \|D_\lambda \mathbf{z} + \mathbf{n} + \nabla g(\mathbf{z})\|_{2,\lambda} \leq \|D_\lambda \mathbf{z}\|_{2,\lambda} + \|\mathbf{n} + \nabla g(\mathbf{z})\|_{2,\lambda}$$

$$\leq C\lambda^{1/4}\|\mathbf{z}\|_{1,2,\lambda} + \|\mathbf{n} + \nabla g(\mathbf{z})\|_{2,\lambda} \leq \|(\mathbf{z}, \mathbf{n})\|_{B_1}$$

for λ small enough. Recall that the low eigenvalues of D_λ are of order $\lambda^{1/4}$, and thus $\|D_\lambda \mathbf{z}\|_{2,\lambda} \leq \lambda^{1/4}\|\mathbf{z}\|_{1,2,\lambda}$ on Λ^0_{low}.

Lemma 1.12 *For λ, δ small enough $\mathbf{T}$ is invertible, with the operator norm of the inverse uniformly bounded $\|\mathbf{T}^{-1}\| \leq C_T$ (independent of λ, δ).*

Proof. Let $\alpha = \mathbf{z} + P_\lambda(\mathbf{n} + \nabla g(\mathbf{z}))$. We need to estimate $\|\mathbf{z}\|_{1,2,\lambda}$ and $\|\mathbf{n} + \nabla g(\mathbf{z})\|_{p,\lambda}$ in terms of $\|\alpha\|_{B_2}$. Since $D_\lambda \alpha = D_\lambda \mathbf{z} + \mathbf{n} + \nabla g(\mathbf{z})$ then

$$\begin{aligned}\|\mathbf{n}+\nabla g(\mathbf{z})\|_{2,\lambda} &\leq \|\alpha\|_{B_2}+\|D_\lambda \mathbf{z}\|_{2,\lambda} \leq \|\alpha\|_{B_2}+C\lambda^{1/4}\|\zeta\|_{1,2,\lambda} \\ &\leq \|\alpha\|_{B_2}+C\lambda^{1/4}\,\|\alpha-P_\lambda(\mathbf{n}+\nabla g(\mathbf{z}))\|_{1,2,\lambda} \\ &\leq \|\alpha\|_{B_2}+C\lambda^{1/4}\,\|\alpha\|_{B_2}+C\lambda^{1/4}\,\|\mathbf{n}+\nabla g(\mathbf{z})\|_{2,\lambda}\end{aligned}$$

So for λ small we get the uniform estimate $\|\mathbf{n}+\nabla g(\mathbf{z})\|_{2,\lambda} \leq C_1\|\alpha\|_{B_2}$. Then

$$\|\mathbf{z}\|_{1,2,\lambda}=\|\alpha-P_\lambda(\mathbf{n}+\nabla g(\mathbf{z}))\|_{1,2,\lambda} \leq \|\alpha\|_{B_2}+C\|\mathbf{n}\|_{2,\lambda} \leq C_2\|\alpha\|_{B_2}$$

thus $\|(\mathbf{z},\mathbf{n})\|_{B_1} \leq C_T\|\mathbf{T}(\mathbf{z},\mathbf{n})\|_{B_2}$. This implies that $\mathbf{T}$ is injective. By construction, index$(\mathbf{T})=0$ so $\mathbf{T}$ is invertible, with $\|\mathbf{T}^{-1}\| \leq C_T$ (independent of λ,δ). □

Lemma 1.13 *For* $\mathbf{z}$ *small,* $\|\Pi(\mathbf{z},\eta)\|_{B_2} \leq C\|\eta\|_{2,\lambda}\|(\mathbf{z},0)\|_{B_1}$.

Proof. By differentiating the relation $D_fP_f\eta=\eta$ with respect to f at f_λ we get

$$\partial D_f(P_\lambda\eta)(\mathbf{z})+D_f(\Pi(\mathbf{z},\eta))=0 \qquad \text{so}$$

$$\|\ D_\lambda(\Pi(\mathbf{z},\eta))\ \|_{2,\lambda}=\|\ \partial D_f(P_\lambda\eta)(\mathbf{z})\ \|_{2,\lambda}.$$

Using the expansion of

$$D_f\xi=\frac{1}{2}\left(\nabla\xi+J(f)\circ\nabla\xi\circ j\right)+\frac{1}{8}N_f(\partial_Jf,\xi)$$

(cf. [MS]) then

$$\|\ \partial D_f(P_\lambda\eta)(\mathbf{z})\ \|_{2,\lambda} \leq C\|\mathbf{z}\|_{\infty,\lambda}\|P_\lambda\eta\|_{1,2,\lambda}$$

uniformly in a neighborhood of f_λ. Therefore

$$\begin{aligned}\|\Pi(\mathbf{z},\eta)\|_{B_2} &= \|D_f(\Pi(\mathbf{z},\eta))\ \|_{2,\lambda} \leq C\|\mathbf{z}\|_{\infty,\lambda}\|P_\lambda\eta\|_{1,2,\lambda} \\ &\leq C\|\mathbf{z}\|_{1,2,\lambda}\|\eta\|_{2,\lambda}=C\|\mathbf{z}\|_{B_2}\|\eta\|_{2,\lambda}.\end{aligned}$$

□

If we choose δ small enough then for $\|\eta\|_{2,\lambda} < \delta$, $\|\Pi(\mathbf{z},\eta)\|_{B_2} \leq C_T/2\ \|(\mathbf{z},\mathbf{n})\|_{B_2}$ where C_T is the constant in Lemma 1.12 so $\mathbf{D}_{(0,\eta)}(\mathbf{z},\mathbf{n}) = \mathbf{T}(\mathbf{z},\mathbf{n}) + \Pi(\mathbf{z},\mathbf{n})$ is still invertible. This concludes the proof of Proposition 1.9. □

1.6 The leading order term of the obstruction ψ_t for t small

Next step is to identify the leading order term of the section ψ_t as $t \to 0$. Let $\mathcal{N}$ denote some stratum of $\overline{\mathcal{U}}_d$ and $Gl \to \mathcal{N}$ denote the gluing data as in (1.18). For the sake of the gluing construction, the gluing data has to be defined on the domain of the bubble tree. But we will see in a moment that the important information is encoded in the image curves. Introduce first some notation: If $u_i \in T_{y_i}S^2$ is a unit frame and λ_i is the gluing parameter, let

$$v_i = \lambda_i \cdot u_i \in T_{y_i}S^2, \ \ (v_i \neq 0) \qquad \text{denote the gluing data.}$$

Definition 1.14 *For any $[f,y,v] \in Gl$, such that $f : B \to \mathbf{P}^n$ is an element of $\mathcal{N}$, let $([f_i,y_i,v_i])_{i=1}^m$ be the bubble maps together with the gluing data and let u be the image of the ghost base (so $u = f_j(y_j)$ for all $j \in \tilde{B}$). Set*

$$a([f,y,v]) \ = \ \sum_{i=1}^{n}\sum_{j\in\tilde{B}}\langle\, df_j(y_j)(v_j)\ ,\ X_i\,\rangle\omega_i \tag{1.41}$$

$$\bar{\nu}(x) \ = \ \sum_{i=1}^{n}\int_{T^2}\langle\, \nu(z,u)\ ,\ \omega_i(z)\,\rangle\omega_i \tag{1.42}$$

where $\{\omega_i = X_i dz,\ i = 1,n\}$ is an orthonormal base of $H^1(T^2, u^*T\mathbf{P}^n)$, $X_i \in T_u\mathbf{P}^n$ and $\tilde{B}$ is as in (1.3).

Note that a depends only on the gluing data on the first level $\tilde{B}$ of essential bubbles, and $\bar{\nu}$ depends only on the image of the ghost base. Then

Lemma 1.15 *Using the notation above, let f_λ be an approximate gluing map. Then for t and $|\lambda| = \sqrt{\lambda_1^2 + \ldots \lambda_i^2}$ small enough,*

$$\begin{aligned}
\pi_-^{f_\lambda}(\nu) &= \bar{\nu}(u) + O(|\lambda|) && (1.43)\\
\pi_-^{f_\lambda}(\overline{\partial}_J f_\lambda) &= a([f,y,v]) + O(|\lambda|^{3/2}). && (1.44)
\end{aligned}$$

and the section ψ_t has the form

$$\psi_t([f,y,v]) = t\bar{\nu}(u) + a([f,y,v]) + O(|\lambda|^{5/4} + t\sqrt{|\lambda|} + t^2). \tag{1.45}$$

The estimates above are uniform on $\mathcal{N}$.

Proof. For the first 2 relations, it is enough to check them on components. Assume for simplicity that $\tilde{B}$ consists of a single bubble $[f,y,v]$. If $\omega = Xdz$ is an element of the base for $H^{0,1}$, let ϖ_λ be the element of the local orthonormal frame for $\Lambda^{0,1}_{low}(f_\lambda{}^*T\mathbf{P}^n)$ provided by Lemma 1.4. Then

$$|\langle \nu, \omega_\lambda - \varpi_\lambda\rangle| \le \|\nu\|_\infty \|\omega_\lambda - \varpi_\lambda\|_{2,\lambda} \le C\lambda \quad \text{so}$$

$$\langle \nu, \varpi_\lambda\rangle = \langle \nu, \omega_\lambda\rangle + O(\lambda)$$

On the other hand, using the definition of ω_λ

$$\begin{aligned}
\langle \nu, \omega_\lambda\rangle &= \int_{|z|>\sqrt{\lambda}} \langle \nu(z, f(y)), \omega\rangle = \int_{T^2} \langle \nu(z,f(y)),\omega\rangle + O(\lambda) \quad \text{so}\\
\langle \nu, \varpi_\lambda\rangle &= \int_{T^2} \langle \nu(z,f(y)),\omega\rangle + O(\lambda)
\end{aligned}$$

which gives (1.43). Similarly,

$$|\langle \overline{\partial}_J f_\lambda, \omega_\lambda - \varpi_\lambda\rangle| \le \|\overline{\partial}_J f_\lambda\|_{2,\lambda}\|\omega_\lambda - \varpi_\lambda\|_{2,\lambda} \le C\lambda^{1/2}\lambda \le C\lambda^{3/2}$$

and using the estimate (1.16) and the definition of ω_λ we get

$$\langle \overline{\partial}_J f_\lambda, \omega_\lambda\rangle = \int_{\sqrt{\lambda} \le |z| \le 2\sqrt{\lambda}} \frac{\sqrt{\lambda}}{|z|}\, d\beta \langle df(y)(u), X\rangle + O(\lambda^2) = \lambda\langle df(y)(u), X\rangle + O(\lambda^2)$$

Combine the previous 2 relations we get

$$\langle \overline{\partial}_J f_\lambda, \overline{\omega}_\lambda \rangle = \langle df(y)(\lambda u), X\rangle + O(\lambda^{3/2}) = \langle df(y)(v), X\rangle + O(\lambda^{3/2})$$

which implies (1.44).

The general case when B has more bubbles follows in a similar maner using the relation (1.19) and the fact that ω is 0 pass the first level of nontrivial bubbles.

Finally, the relation (1.45) is a consequence of (1.15) provided we have an estimate of the the quadratic part. For that use (1.32) to get

$$\begin{aligned} \langle\, N_\lambda(P_\lambda \eta)\,,\, \overline{\omega}_\lambda \rangle \;&\le\; \|N_\lambda(P_\lambda \eta)\,\|_{4/3,\lambda}\, \|\overline{\omega}_\lambda\|_{4,\lambda} \;\le\; C\|\eta\|_{2,\lambda}\, \|\eta\|_{4/3,\lambda}\, \|\omega\|_4 \\ &\le\; O(|\lambda|^{1/2} + t)\, O(|\lambda|^{3/4} + t). \end{aligned}$$

Thus the quadratic part is $O(|\lambda|^{5/4} + t\sqrt{|\lambda|} + t^2)$. □

The definition of $\widetilde{L} \to \widetilde{\mathcal{U}}_d$. From this point on, since we are going to look at the leading order term, it will become easier if we forget part of the gluing data. We have already observed that the map a depends only on the gluing data on the first level $\widetilde{B}$ of essential bubbles. Moreover, if we denote by

$$w = \sum_{j\in\widetilde{B}} df_j(y_j)(v_j) \in T_u\mathbf{P}^n \tag{1.46}$$

then the map a and the linear part $\widetilde{\psi}_t$ of ψ_t become respectively

$$a(w) = \sum_{i=1}^{n} \langle w, X_i\rangle \omega_i \tag{1.47}$$

$$\widetilde{\psi}_t(w) = t\bar{\nu}(u) + a(w) \tag{1.48}$$

Introduce a space $\mathcal{W}$ together with a projection $\pi : \mathcal{W} \to \overline{\mathcal{U}}_d$ such that the fiber of π at a 1-marked curve (possibly with more components) is the span of the tangent planes to all the image bubbles that meet at the marked point. By definition $w \in \mathcal{W}$

so (1.46) defines a projection $p : Gl \to \mathcal{W}$. Note though that $\pi : \mathcal{W} \to \overline{\mathcal{U}}_d$ is not a vector bundle, and that $\mathcal{W}$ it is equal to the *relative tangent bundle* $L \to \mathcal{U}_d$ on the top strata of $\overline{\mathcal{U}}_d$.

Here is a more precise description of $\mathcal{W}$. Stratify $\overline{\mathcal{U}}_d$ by letting $\mathcal{Z}_h$ be the union of all boundary strata such that the image of the marked point is on h nontrivial bubbles, i.e.

$$\mathcal{Z}_h = \{ f : B \to \mathbf{P}^n \mid \tilde{B} \text{ has } h \text{ elements } \} \tag{1.49}$$

Note that $\overline{\mathcal{Z}}_2 \supset \overline{\mathcal{Z}}_3 \supset \ldots$ and each $\overline{\mathcal{Z}}_h$ is a smooth variety with normal crossings. For transversality arguments we need to use the moduli space $\hat{\mathcal{Z}}_h$ obtained from $\mathcal{Z}_h$ by collapsing all the ghost bubbles up to the first level of essential bubbles. The natural projection

$$q : \mathcal{Z}_h \to \hat{\mathcal{Z}}_h$$

has fiber $\mathcal{U}_{0,h} = \mathcal{M}_{0,h+1}$, the moduli space of $h+1$ marked points on the sphere. Moreover,

$$\dim \mathcal{Z}_h = n - h - 1 \qquad \text{and} \qquad \dim \hat{\mathcal{Z}}_h = n - 2h + 1 \tag{1.50}$$

In particular, $\mathcal{Z}_h \neq \emptyset$ only for $h \leq [\frac{n+1}{2}]$.

Let L_i be the pullback of the relative tangent bundle to the i'th factor of $\hat{\mathcal{Z}}_h$. When the constraints $\beta_1, \ldots, \beta_k$ are in generic position, the fibers of $L_1, \ldots, L_h$ over a point in $\hat{\mathcal{Z}}_h$ are linearly independent subspaces of $\mathbf{P}^n$. This is because linear dependence imposes $n + 1 - h$ conditions, and $\hat{\mathcal{Z}}_h$ is only $n - 2h + 1$ dimensional. So on $\mathcal{Z}_h$

$$\mathcal{W}|_{\mathcal{Z}_h} = q^*(L_1 \oplus \ldots \oplus L_h) \tag{1.51}$$

Remark 1.16 Since not all the gluing parameters can be zero, a dimension count argument similar to the one above shows that w defined by (1.46) is an element of $\mathcal{W} - \{0\}$, the space nonzero vectors in $\mathcal{W}$, thus $p : Gl \to \mathcal{W} - \{0\}$.

Note that $\mathcal{W}|_{\mathcal{Z}_h}$ is nothing but the normal bundle of $\mathcal{Z}_h$ in $\overline{\mathcal{U}}_d$, for any $2 \leq h \leq [\frac{n+1}{2}]$. This observation allows us to get a line bundle out of $\mathcal{W}$ as follows:

Definition 1.17 *Let $N = [\frac{n+1}{2}]$. Blow up $\overline{\mathcal{U}}_d$ along $\mathcal{Z}_N$ (the bottom strata), then blow up the proper transform of $\mathcal{Z}_{N-1}$ and so on, all the way up to blowing up the proper transform of $\mathcal{Z}_2$ and denote by*

$$\rho : \widetilde{\mathcal{U}}_d \to \mathcal{U}_d$$

the resulting manifold. Similarly, after the first blow up, extend L over the exceptional divisor $\widetilde{\mathcal{Z}}_N$ as the universal line bundle over $\mathbf{P}(N_{\mathcal{Z}_N})$, the projectivization of the normal bundle of $\mathcal{Z}_N$, and so on. Let $\widetilde{L} \to \widetilde{\mathcal{U}}_d$ denote the blow up of L constructed above.

By definition, the total space of $\widetilde{L} \to \widetilde{\mathcal{U}}_d$ is the same as $\rho^*(\mathcal{W})$. From now on, we will make this identification.

Note that both the map a and the linear part $\widetilde{\psi}_t$ of ψ_t pull back to $\widetilde{L} - \{0\}$ as

$$a(w) = \sum_{i=1}^{n} \langle w, X_i \rangle X_i \tag{1.52}$$

$$\widetilde{\psi}_t(w) = t\bar{\nu}(\pi(w)) + a(w) \tag{1.53}$$

where $\pi : \widetilde{L} \to \mathbf{P}^n$ is the composition $\widetilde{L} \to \widetilde{\mathcal{U}}_d \xrightarrow{ev} \mathbf{P}^n$. For simplicity of notation, we have also denoted by ev : $\widetilde{\mathcal{U}}_d \to \mathbf{P}^n$ the composition $\widetilde{\mathcal{U}}_d \xrightarrow{\rho} \mathcal{U}_d \xrightarrow{ev} \mathbf{P}^n$. Note that by definition, a is a linear map but $\widetilde{\psi}_t$ is not, and we have the following diagramm:

$$\begin{array}{ccccc}
\widetilde{L} - \{0\} & \xrightarrow{a,\,\widetilde{\psi}_t} & \mathrm{ev}^*(T\mathbf{P}^n) & & T\mathbf{P}^n \\
\downarrow & & \downarrow & & \downarrow\uparrow \bar{\nu} \\
\widetilde{\mathcal{U}}_d & = & \widetilde{\mathcal{U}}_d & \xrightarrow{\mathrm{ev}} & \mathbf{P}^n
\end{array}$$

Proposition 1.18 *As $t \to 0$ the zero set of the section ψ_t is homotopic to the zero set of its leading order term*

$$\widetilde{\psi}_t : \widetilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$$

Proof. In generic conditions and for t small enough the zero sets of both sections

$$\psi_t : Gl \to \mathrm{ev}^*(T\mathbf{P}^n) \quad \text{and} \quad \rho^* p_*(\psi_t) : \tilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$$

consist of points lying on the top stratum of $\mathcal{U}_d$ and $\tilde{\mathcal{U}}_d$ respectively. But on the top stratum, the projection $pr : Gl \to \tilde{L} - \{0\}$ is an isomorphism, thus the two zero sets are diffeomorphic for t small. Note that (1.45) gives

$$p_*(\psi_t(w)) = t\bar{\nu}(u) + a(w) + O(|w|^{5/4} + t\sqrt{|w|} + t^2)$$

Finally, Lemma 1.19 gives that $w = O(t)$ on the zero set of ψ_t, so

$$p_*(\psi_t(w)) = t\bar{\nu}(u) + a(w) + O(t^{5/4})$$

giving the desired homotopy as $t \to 0$. □

Lemma 1.19 *The linear map $a : \tilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$ defined in (1.52) has no zeros when the constraints $\beta_1, \ldots, \beta_l$ are in a generic position, thus there exists $C > 0$ such that*

$$|a(w)| \geq C|w| \tag{1.54}$$

Moreover, there exists a uniform constant C on $\tilde{L} - \{0\}$ such that the zero set of ψ_t is contained in $|w| \leq Ct$.

Proof. First part is a standard transversality argument and dimension count. Note that a induces a map

$$\begin{aligned} a \otimes id : \tilde{L} \otimes \tilde{L}^* &\to \mathrm{ev}^*(T\mathbf{P}^n) \otimes \tilde{L}^* \qquad \text{i.e.} \\ a \otimes id : \overline{\mathcal{U}}_d \times \mathbf{C} &\to \mathrm{ev}^*(T\mathbf{P}^n) \otimes \tilde{L}^* \end{aligned}$$

Because of the $\mathbf{C}^*$-equivariance of a, the zero set of $a : \tilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$ is the same as the zero set of the section

$$\begin{aligned} \tilde{a} : \tilde{\mathcal{U}}_d &\to ev^*(T\mathbf{P}^n) \otimes \tilde{L}^* \\ \tilde{a}(x) &= (a \otimes id)(x, 1) \end{aligned}$$

If the constraints $\beta_1, \ldots, \beta_k$ are in generic position, then $\tilde{a}$ is transverse to the zero set of $\mathrm{ev}^*(T\mathbf{P}^n)$. But the base $\tilde{\mathcal{U}}_d$ is only $n-1$ dimensional, while the fiber is n dimensional, so generically $\tilde{a}$ and thus a has no zeros.

For the second part, note that on the zero set of $p_*(\psi_t)$

$$\begin{aligned} 0 = p_*(\psi_t) &= a(w) + t\bar{\nu}(u) + O(|w|^{5/4} + |w|^{1/2}\, t + t^2) \quad \text{so} \\ a(w) &= -t\bar{\nu}(u) - O(|w|^{5/4} + |w|^{1/2}\, t + t^2) \end{aligned}$$

which combined with (1.54) gives

$$C|w| \le |a(w)| \;\le\; t|\bar{\nu}(u)| + \tilde{C}(|w|^{5/4} + |w|^{1/2}\, t + t^2) \quad \text{i.e.}$$

$$|w|(C - \tilde{C}|w|^{1/4}) \;\le\; Ct$$

For t and w small, the left hand side is positive, completing the proof. □

1.7 The enumerative invariant τ_d

Next step is to find the zero set of the leading order term of ψ_t. As a warm-up we will discuss first the limit case $t = 0$. The constructions described in the previous sections apply equally in this case, giving:

Proposition 1.20 *Let $\mathcal{N}$ be a ghost base boundary stratum of $\overline{\mathcal{U}}_d$. Then the moduli space of J-holomorphic tori close to $\mathcal{N}$ is isomorphic to the zero set of a section in the obstruction bundle over the space of gluing data*

$$\psi([f_i, y_i, v_i]_{i=1}^m) = a([f_i, y_i, v_i]]_{i=1}^m) + O(|\lambda|^{5/4})$$

where a is defined by (1.41). Moreover, for generic constraints $\beta_1, \ldots, \beta_l$, the number of J-holomorphic tori that define the enumerative invariant

$$\tau_d(\beta_1, \ldots, \beta_l)$$

is finite, and the moduli space of these holomorphic tori is at a positive distance from the ghost base boundary strata of the bubble tree compactification.

Proof. For the second part, note that ψ and $\lambda^{-1}\psi$ have the same zero set, so as $\lambda \to 0$ the limit of the end of the moduli space of J-holomorphic tori is modeled by the zero set of the section a. But we have seen that generically a has no zeros, and thus there are no J-holomorphic tori in the neighborhood of that boundary stratum. □

Now we are finally ready to evaluate the contribution from the interior.

Proposition 1.21 *For t small, the number of $(J, t\nu)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l)$ and are close to some $(J, 0)$-holomorphic torus is equal to*

$$n_j \tau_d(\beta_1, \ldots, \beta_l)$$

where $n_j = |Aut_{x_1}(j)|$ is the order of the group of automorphisms of the complex structure j that fix the point x_1.

Proof. Recall that $RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l)$ counts the number of solutions of the equation

$$\overline{\partial}_J f(x) = \nu(x, f(x))$$

such that $f(x_1) \in \beta_1$ and f passes through $\beta_2, \ldots, \beta_l$.

A generic path of perturbations converging to 0 provides a cobordism $\mathcal{M}^\nu$ to the solutions of the equation

$$\overline{\partial}_J f(x) = 0$$

such that $f(x_1) \in \beta_1$ and f passes through $\beta_2, \ldots, \beta_l$. A $(J, 0)$-holomorphic torus $f : T^2 \to \mathbf{P}^n$ is a smooth point of this cobordism, i.e. all the intersections are transversal and the cokernel $H^{0,1}(T^2, f^*(T\mathbf{P}^n))$ vanishes (since $f^*(T\mathbf{P}^n)$ is positive for the standard complex structure).

But the invariant $\tau_d(\beta_1, \ldots, \beta_l)$ counts the number of such solutions mod the automorphism group of j. Imposing the condition $f(x_1) \in \beta_1$ reduces the stabilizer to just $Aut_{x_1}(j)$. □

Remark 1.22 Note that the pertubed invariant counts the number of (J,ν)- holomorphic maps with sign. This sign is determined by the spectral flow of the linearization D_f to $\overline{\partial}_J$. In the limit, when $\nu = 0$, we have $D_f = \overline{\partial}_J$ thus all $(J,0)$-tori have a positive sign. This agrees with the way they were counted classically to obtain τ_d.

Lemma 1.23 *For generic ν the section $\bar{\nu} : \mathrm{ev}_*(\overline{\mathcal{U}}_d) \to T\mathbf{P}^n$ defined by (1.42) has no zeros.*

Proof. For generic ν, the section $\bar{\nu}$ is transverse to the zero section. But the fiber of $T\mathbf{P}^n$ is n dimensional, and the base $\mathrm{Im}(\mathcal{M}) = \mathrm{ev}_*(\overline{\mathcal{U}}_d)$ is only $n-1$ dimensional, so $\bar{\nu}$ has no zeros generically. $\Box$

Remark 1.24 The zeros $u \in \mathbf{P}^n$ of $\bar{\nu}$ give the location of the point maps u that can be perturbed away to get genus one (J,ν)-holomorphic maps representing $0 \in H_2(\mathbf{P}^n)$. Since index=0 then generically $\bar{\nu}$ has finitely many zeros. But $\mathrm{Im}(\mathcal{M})$ is a codimension 1 subvariety in $\mathbf{P}^n$ that doesn't depend on ν. Then we can choose ν generic so that its zeros do not lie in $\mathrm{Im}(\mathcal{M})$, and thus $\bar{\nu}(f(y)) \neq 0$ for any $[f,y] \in \overline{\mathcal{U}}_d$.

Moreover, Lemma 1.18 showed that as $t \to 0$ the zero set Z_t of ψ_t is homotopic to the zero set Z_0 of the map

$$\psi_0 : \widetilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$$

$$\psi_0(w) = \bar{\nu}(\pi(w)) + a(w)$$

where a, $\bar{\nu}$ are defined in (1.52), (1.42) and $\pi : \widetilde{L} \to \mathbf{P}^n$ is the composition $\widetilde{L} \to \widetilde{\mathcal{U}}_d \xrightarrow{\mathrm{ev}} \mathbf{P}^n$. We have also made a change of variables $w \to w/t$.

We are ready now to identify the zero set Z_0. Since $\bar{\nu}(u) \neq 0$ on $\mathrm{Im}(\mathcal{M})$ then it induces a splitting of the obstruction bundle:

$$T\mathbf{P}^n|_{\mathrm{Im}(\mathcal{M})} = \mathbf{C} < \bar{\nu} > \oplus E \tag{1.55}$$

where E is an $n-1$ dimensional bundle, so

$$\mathrm{ev}^*(T\mathbf{P}^n) = \mathbf{C} < \bar{\nu} > \oplus\, \mathrm{ev}^* E \tag{1.56}$$

Lemma 1.25 *The number of zeros (counted with multiplicity) of ψ_0 is equal to*

$$c_{n-1}(\mathrm{ev}^*(E) \otimes \tilde{L}^*)$$

Proof. Using (1.56) map $\psi_0 : \tilde{L} - \{0\} \to \mathrm{ev}^*(T\mathbf{P}^n)$ splits as

$$\psi_1(w) = \bar{\nu}(\pi(w)) + a_1(w) \tag{1.57}$$

$$\psi_2(w) = a_2(w) \tag{1.58}$$

where a_i denotes the corresponding projection of $a(w)$. The map $a_2 : \tilde{L} - \{0\} \to \mathrm{ev}^*(E)$ is $\mathbf{C}^*$-equivariant, so tensored with the identity on $\tilde{L}^*$ induces a $\mathbf{C}^*$-equivariant map

$$\bar{a}_2 : \tilde{\mathcal{U}}_d \times \mathbf{C}^* \to \mathrm{ev}^*(E) \otimes \tilde{L}^*$$

that has the same zero set as a_2. Let

$$\tilde{a}_2 : \tilde{\mathcal{U}}_d \to \mathrm{ev}^*(E) \otimes \tilde{L}^* \quad \text{given by} \quad \tilde{a}_2(x) = \bar{a}_2(x, 1)$$

Then the zero set of a_2 is equal to $Z(\tilde{a}_2) \times \mathbf{C}^*$. To find the zero set of ψ_0, for any $(x, v) \in Z(\tilde{a}_2) \times \mathbf{C}^*$ solve the equation

$$0 = \psi_1(x, v) = \bar{\nu}(x) + a_1(x, v) = \bar{\nu}(x) + v \cdot a_1(x, 1)$$

Note that $a_1 \neq 0$ on $Z(a_2)$ since a has no zeros so for any $x \in Z(\tilde{a}_2)$ there exists a unique $v \in \mathbf{C}^*$ such that

$$-\bar{\nu}(x) = v \cdot a_1(x, 1)$$

This implies that there exists an isomorphism between the zero set of ψ_0 and the zero set of $\tilde{a}_2$. To complete the proof, note that for generic ν the section $\tilde{a}_2$ is transversal to the zero section of $\mathrm{ev}^*(E) \otimes \tilde{\mathrm{L}}^*$, so its zero set is given by the Euler class of $\mathrm{ev}^*(E) \otimes \tilde{\mathrm{L}}^*$.
$\square$

Finally, we can compute the boundary contribution

Proposition 1.26 *For t small, the number of $(J, t\nu)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $\{x_1\} \times \overline{\mathcal{U}}_d$ is equal to*

$$\sum_{i=0}^{n-1} \binom{n+1}{i+2} \mathrm{ev}^*(H^{n-1-i}) \cdot c_1^i(\tilde{L}^*)$$

where $\tilde{L}$ is the blow up of the relative tangent bundle L defined in (1.17).

Proof. As we have seen previously, the moduli space of $(J, t\nu)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $\{x_1\} \times \overline{\mathcal{U}}$ is diffeomorphic to the zero set of the section ψ_0. Using Lemma 1.25, the later is equal to

$$c_{n-1}(\mathrm{ev}^*(E) \otimes \tilde{L}^*) = \sum_{i=0}^{n-1} \mathrm{ev}^*(c_{n-i-1}(E)\ ) \cdot c_1^i(\tilde{L}^*)$$

But by definition $c_i(E) = c_i(T\mathbf{P}^n) = \binom{n+1}{i} H^i$, completing the proof. □

1.8 The other contribution

In the previous sections we have described in great length the gluing construction corresponding to the strata $\{x_1\} \times \mathcal{U}_d$, that consists of a ghost base and a bubble at the marked point x_1. Finally, it is the time to sketch the gluing construction corresponding to other boundary stratum $T^2 \times \mathrm{ev}^*(\beta_1)$ and to explain why it does not give any contribution.

Proposition 1.27 *For t small, the number of $(J, t\nu)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $T^2 \times \mathrm{ev}^*(\beta_1)$ is equal to 0.*

Proof. Construct first the space of approximate maps. The only difference from the gluing construction decribed in Section 1.2 is that we need to allow the bubble point $x \in T^2$ to vary. Since the tangent bundle of the torus is trivial, choose an isomorphism

$$TT^2 \cong T^2 \times \mathbf{C}$$

which gives an identification $T_xT^2 \cong \mathbf{C}$ for all $x \in T^2$ (providing local coordinates on T^2). The set of gluing data will then be modeled on:

$$T^2 \times Fr \times (0, \varepsilon)$$

where

$$Fr = \{ \, [f, y, u] \mid [f, y] \in \mathrm{ev}^*(\beta_1), \; u \in T_yS^2 \; |u| = 1\}$$

is the restriction of the frame bundle over $\mathcal{U}_d$ defined by (1.9).

To glue, use the unit frame $u \in TyS^2$ to identify $T_xT^2 \cong T_yS^2$ which will induce natural coordinates on the sphere via the stereographic projection.

Then all the constructions decribed in Sections 1.2-1.7 extend to this case. Since the holomorphic 1-form $\omega \in H^{0,1}(T^2, \mathbf{C})$ is constant along the torus, then the isomorphism between the obstruction bundle and $\mathrm{ev}^*(T\mathbf{P}^n)$ is independent of the bubble point, so

$$\begin{array}{ccc} H^{0,1} \cong \rho^*\mathrm{ev}^*(T\mathbf{P}^n) & & \mathrm{ev}^*(T\mathbf{P}^n) \\ \searrow \quad \swarrow & & \downarrow \\ T^2 \times \mathrm{ev}^*(\beta_1) & \xrightarrow{\rho} & \mathrm{ev}^*(\beta_1) \end{array}$$

Moreover, the linear part of the section ψ_t that models the end of the moduli space is also independent of the bubble point. But a dimension count shows that the zero set of a T^2-equivariant section in the obstruction bundle must be empty generically.

□

2 Applications

In this second part of the paper we explain how one can compute the top power intersections $c_1^i(\widetilde{L}^*)\mathrm{ev}^*(H^{n-1-i})$ involved in Theorem 0.1. The programm is simple: first we find recursive formulas for the top intersections $c_1^i(L^*)\mathrm{ev}^*(H^{n-1-i})$ (see Proposition 2.2), where L is the relative tangent bundle of $\mathcal{U}_d$, and object well known to the algebraic geometers. Next we exploit the fact that $\widetilde{L}$ is a blow up of L to compute its coresponding top intersections recursively in Proposition 2.5.

Unfortunately, the notation becomes quickly pretty complicated if we insist on keeping track of all the information, so we chose to indicate at each step only the new changes, leaving out the data that stays the same.

Notations. If $\beta_0, \dots, \beta_k$ are various codimension constraints let

$$\mathcal{U}_d(\beta_0\ ;\ \beta_1, \dots, \beta_k) = \mathrm{ev}^*(\beta_0)\, [\, \mathcal{U}_d(\ ; \beta_1, \dots, \beta_k)\,]$$

denote the moduli space of 1-marked cuves in $\mathbf{P}^n$ passing through $\beta_0, \dots, \beta_k$, such that the special marked point is on β_0 and let

$$\mathcal{M}_d(\beta_0, \beta_1, \dots, \beta_k)$$

denote the corresponding moduli space of curves (in which we forget the special marked point).

In particular, let $\mathcal{U}_d = \mathcal{U}_d(\ ; \beta_1, \dots, \beta_k)$ be the moduli space of 1-marked curves that appears in Theorem 0.1. If $i, j \geq 0$ are such that $i + j = \dim \mathcal{U}_d$ then let

$$\phi_d(i, j \mid \beta_1, \dots, \beta_k) = c_1^i(L^*)\ \mathrm{ev}^*(H^j)\ [\, \mathcal{U}_d\,] \tag{2.1}$$

denote the top intersection. Moreover, if $\widetilde{\mathcal{U}}_d$ is the blow-up $\mathcal{U}_d$ as in (1.17), let

$$x = c_1(L^*) \in H^2(\mathcal{U}_d, \mathbf{Z}), \qquad \tilde{x} = c_1(\widetilde{L}^*) \in H^2(\widetilde{\mathcal{U}}_d, \mathbf{Z}) \quad \text{and} \quad y = \mathrm{ev}^*(H) \tag{2.2}$$

where $y \in H^2(\mathcal{U}_d, \mathbf{Z})$ or $y \in H^2(\widetilde{\mathcal{U}}_d, \mathbf{Z})$ depending on the context. Note that

$$\phi_d(i, j \mid \cdot\,) = x^i y^j \, [\,\mathcal{U}_d\,] = x^i \, [\mathcal{U}_d(H^j; \cdot)] \tag{2.3}$$

Remark 2.1 Using the notation above and the degeneration formula (A.2), Theorem 0.1 becomes:

$$n_j \tau_d(\,\cdot\,) = \sum_{i_1+i_2=n} \sigma_d(H^{i_1}, H^{i_2}, \,\cdot\,) + \sum_{i=0}^{n-1} \binom{n+1}{i+2} \tilde{x}^i y^{n-1-i} \cdot [\widetilde{\mathcal{U}}_d] \tag{2.4}$$

2.1 Recursive formulas for $c_1^i(L^*)\mathrm{ev}^*(H^j)$

Let $\mathcal{U}_d$ be some r-dimensional moduli space of 1-marked curves of degree d through some constraints $\beta_1, \ldots, \beta_k$ (not necessarily the same as in Theorem 0.1) and let $L \to \mathcal{U}_d$ be its relative tangent sheaf. In this section we give recursive formulas for

$$\phi_d(i, j \mid \cdot) = c_1^i(L^*)\mathrm{ev}^*(H^j)[\mathcal{U}_d]$$

where the constaints $\beta_1, \ldots, \beta_k$ are dropped from the notation.

Proposition 2.2 *For every r-dimensional moduli space $\mathcal{U}_d$ of any degree $d \geq 1$, there are the following recursive relations:*

$$\begin{aligned}
\phi_d(0, j \mid \cdot) &= \sigma_d(H^j, \,\cdot) && (2.5) \\
\phi_d(i+1, j \mid \cdot) &= -\frac{2}{d}\phi_d(i, j+1 \mid \cdot\,) + \frac{1}{d^2}\phi_d(i, j \mid H^2, \,\cdot\,) && (2.6) \\
&+ \sum_{\substack{d_1+d_2=d \\ i_1+i_2=n}} \frac{d_2^2}{d^2} \, \phi_{d_1}(i, j \,|H^{i_1} \,, \,\cdot\,) \cdot \sigma_{d_2}(H^{i_2}, \,\cdot\,) && \\
&+ (-1)^{i+1} \sum_{\substack{d_1+d_2=d \\ i_1+i_2=n+j}} \frac{d_2^2}{d^2}\phi_{d_1}(i-1, i_1 \mid \cdot\,) \cdot \sigma_{d_2}(H^{i_2}, \,\cdot\,) && (2.7)
\end{aligned}$$

for any $i \geq 0$, where the sums above are over all possible distributions of the constraints $\beta_1, \ldots, \beta_k$ on the two factors and $d_1, d_2 \neq 0$. When $i = 0$, the last term in (2.6) is missing.

Proof. The first relation follows by definition, and provides the initial step of the recursion. The second one requires more work. In what follows, we will identify a cohomology class like $c_1(L)$ with a divisor representing it. Then:

Lemma 2.3 *On $\mathcal{U}_d$, we have the following relation:*

$$c_1(L^*) = \frac{1}{d^2}\mathcal{H} - \frac{2}{d}\mathrm{ev}^*(H) + \frac{1}{d^2}\sum_{d_1+d_2=d} d_2^2\mathcal{M}_{d_1,d_2} \tag{2.8}$$

where $\mathcal{H}$ denotes the extra condition that the curve passes through H^2, and $\mathcal{M}_{d_1,d_2}$ denotes the boundary stratum corresponding to the splittings in a degree d_1 1-marked curve and a degree d_2 curve, for $d_i \neq 0$ (for all possible distributions of the constraints $\beta_1, \ldots, \beta_k$ on the two components).

Proof. Fix 2 hyperplanes in generic position in $\mathbf{P}^n$. Each curve in $\mathcal{U}_d$ intersects a hyperplane in d points. Then the moduli space $Y = \mathrm{ev}_{k+1}^*(H) \cap \mathrm{ev}_{k+2}^*(H)$ of 1-marked curves passing through $\beta_1, \ldots, \beta_k, H, H$ is a d^2 fold cover of $\mathcal{U}_d$:

$$\pi : Y \to \mathcal{U}_d, \qquad [f, y_1, \ldots, y_k, a, b\,; y] \to [f, y_1, \ldots, y_k\,; y]$$

Define the section

$$s([f, y_1, \cdots, y_k, a, b\,; y]) = \frac{(a-b)dy}{(y-a)(y-b)}$$

Then s is a section in the relative cotangent bundle L^*, and it extends to the compactification $\overline{\mathcal{U}_d}$. As a and b are getting closer together, the section s converges to 0. Thus its zero set is the sum of the divisors $\{a = b\}$ and $\mathcal{M}(y\,;\, a, b)$, where $\mathcal{M}(y\,;\, a, b)$ is the sum of all boundary strata corresponding to splittings into a degree d_1 1-marked bubble and a degree d_2 bubble containing a, b for $d = d_1 + d_2$. Note that $d_i \neq 0$. The infinity divisor is $\{y = a\} + \{y = b\}$. Thus

$$\pi^*(c_1(L^*)) = \{a = b\} + \mathcal{M}(y\,;\, a, b) - \{y = a\} - \{y = b\}$$

Note that

$$d^2 c_1(L^*) = \pi_* \pi^*(c_1(L^*))$$

When projecting down to $\overline{\mathcal{U}}_d$, the divisor $\{a = b\}$ becomes $\mathcal{H}$, and the divisors $\{y = a\}$, $\{y = b\}$ become each $d \cdot \mathrm{ev}^*(H)$. The rest amounts to summing over all codimension 1 boundary strata. The boundary strata $\mathcal{M}_{d_1,d_2}$ appears with coefficient d_2^2 in $\pi_*(\mathcal{M}(y, a\ ;\ b))$. Combining all the pieces together completes the proof of Lemma. $\Box$

Remark 2.4 We could have chosen any 2 marked points out of the already existent ones, and then express $c_1(L)$ in terms of them. But then this expression would not look independent of choice. Nevertheless, with some work, one can actually see that all these divisors are homotopic. We have chosen to introduce 2 new marked points to avoid this issue.

Relation (2.8) provides the basic relation for proving (2.6):

$$c_1^{i+1}(L^*) = -\frac{2}{d}c_1^i(L^*) \cdot \mathrm{ev}^*(H) + \frac{1}{d^2}c_1^i(L^*) \cdot \mathcal{H} + \sum_{d_1+d_2=d} \frac{d_2^2}{d^2}\, c_1^i(L^*) \cdot \mathcal{M}_{d_1,d_2}$$

so taking a cup product with $\mathrm{ev}^*(H^j)$ we get:

$$\begin{aligned} \phi_d(i+1, j \mid \cdot) &= -\frac{2}{d}\phi_d(i, j+1 \mid \cdot) + \frac{1}{d^2}\phi_d(i, j \mid H^2,\ \cdot) \\ &+ \sum_{d_1+d_2=d} \frac{d_2^2}{d^2}\, c_1^i(L^*)\mathrm{ev}^*(H^j) \cdot \mathcal{M}_{d_1,d_2} \end{aligned} \tag{2.9}$$

Next, we need to understand the restriction of L to the boundary stratum $\mathcal{M}_{d_1,d_2}$. Let

$$p : \mathcal{M}_{d_1,d_2} \to \mathcal{U}_{d_1}$$

be the projection on the first component (the one that contains the special marked point y). If A, B are the 2 special points of $\mathcal{M}_{d_1,d_2}$ (where the 2 components meet), and

$$\mathrm{ev}_A \times \mathrm{ev}_B : \mathcal{U}_{d_1} \times \mathcal{M}_{d_2} \to \mathbf{P}^n \times \mathbf{P}^n$$

be the corresponding evaluation map. Then by definition

$$\mathcal{M}_{d_1,d_2} = (\mathrm{ev}_A \times \mathrm{ev}_B)^*([\Delta]) \tag{2.10}$$

where Δ is the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$. Moreover, it is known that as divisors,

$$c_1(L^*)/\mathcal{M}_{d_1,d_2} = p^* c_1(L_A^*) + \{y = A\} \tag{2.11}$$

where $L_A = L_{|\mathcal{U}_{d_1}}$ is the relative tangent bundle of $\mathcal{U}_{d_1}$. Next step is to find

$$c_1^i(L^*)/\mathcal{M}_{d_1,d_2} = \sum_{l=0}^{i} \binom{i}{l} p^* c_1^{i-l}(L_A^*) \cdot (\{y = A\})^l \tag{2.12}$$

For the self intersection of the divisor $\{y = A\}$ note that its normal bundle N inside $\mathcal{M}_{d_1,d_2}$ is nothing but $p^*(L_A)/\{y = A\}$, so for $l > 0$,

$$(\{y = A\})^l = c_1(N)^{l-1} = (-1)^{l-1} p^* c_1^{l-1}(L_A^*) \cdot [\{y = A\}]$$

Substituting in (2.12) and after some algebraic manipulations we get:

$$c_1^i(L^*) \cdot [\mathcal{M}_{d_1,d_2}] = p^* c_1^i(L_A^*) + (-1)^{i-1} p^* c_1^{i-1}(L_A^*) \cdot [\{y = A\}] \tag{2.13}$$

Compute the intersections above inside $\mathcal{U}_{d_1} \times \mathcal{M}_{d_2}$. Then (2.10) combined with the relation $[\Delta] = \sum\limits_{i_1+i_2=n} H^{i_1} \times H^{i_2}$ gives

$$\begin{aligned} \mathrm{ev}^*(H^j) \cdot [\mathcal{M}_{d_1,d_2}] &= \sum_{i_1+i_2=n} \mathcal{U}_{d_1}(H^j \,;\, H^{i_1}, \,\cdot\,) \times \mathcal{M}_{d_2}(H^{i_2}, \,\cdot\,) \\ \mathrm{ev}^*(H^j) \cdot [\{y = y_A\}] &= \sum_{i_1+i_2=n+j} \mathcal{U}_{d_1}(H^{i_1} \,;\, \cdot\,) \times \mathcal{M}_{d_2}(H^{i_2}, \,\cdot\,) \end{aligned}$$

where we sum over all possible distributions of the constraints on the two components. The relations above imply

$$\begin{aligned} \mathrm{ev}^*(H^j) \cdot p^* c_1^i(L_A^*) \cdot [\mathcal{M}_{d_1,d_2}] &= \sum_{i_1+i_2=n} (c_1^i(L_A) \cdot \mathcal{U}_{d_1}(H^j \,;\, H^{i_1}, \,\cdot\,)\,) \times \mathcal{M}_{d_2}(H^{i_2}, \,\cdot\,) \\ &= \sum_{i_1+i_2=n} \phi_{d_1}(i, j \mid H^{i_1}, \,\cdot\,) \cdot \sigma_{d_2}(H^{i_2}, \,\cdot\,) \qquad (2.14) \\ \mathrm{ev}^*(H^j) \cdot p^* c_1^{i-1}(L_A^*) \cdot [\{y = y_A\}] &= \sum_{i_1+i_2=n+j} (c_1^{i-1}(L_A) \cdot \mathcal{U}_{d_1}(H^{i_1} \,;\, \cdot\,)\,) \times \mathcal{M}_{d_2}(H^{i_2}, \,\cdot\,) \\ &= \sum_{i_1+i_2=n+j} \phi_{d_1}(i-1, i_1 \mid \cdot\,) \cdot \sigma_{d_2}(H^{i_2}, \,\cdot\,) \qquad (2.15) \end{aligned}$$

Substituting these relations in (2.9) using (2.13) we get (2.6), which concludes the proof of Proposition 2.2. □

2.2 Recursive formulas for $c_1^i(\tilde{L}^*) \cdot \mathrm{ev}^*(H^j)$

Let $\mathcal{U}_d = \mathcal{U}_d(\ ;\beta_1, \ldots, \beta_k)$ the some r-dimensional moduli space of 1-marked curves and $\widetilde{\mathcal{U}}_d$ be its repeated blow-up along $\mathcal{Z}_{[\frac{n+1}{2}]}, \ldots, \mathcal{Z}_1$ as in (1.17). Next we give recursive formulas for computing the top intersections $\tilde{x}^i y^j$:

Proposition 2.5 *If $\mathcal{U}_d$ is some r-dimensional cutdown moduli space of 1-marked curves then for $i + j = r$, the top intersection*

$$\tilde{x}^i y^j[\widetilde{\mathcal{U}}_d] \quad = \quad x^i y^j[\mathcal{U}_d] - \sum_{\substack{d_1+d_2=d \\ j_1+j_2=n+j}} \left[\frac{(\tilde{x}_1 + \tilde{x}_2 - 1)^i}{(1-\tilde{x}_1)(1-\tilde{x}_2)}\right]_{i-2} y_1^{j_1} y_2^{j_2}[\widetilde{\mathcal{U}}_{d_1} \times \widetilde{\mathcal{U}}_{d_2}] \tag{2.16}$$

where $\tilde{x}_i = c_1(\tilde{L}^*|_{\widetilde{\mathcal{U}}_{d_i}})$ and $y_i = \mathrm{ev}^*(H)$ on $\widetilde{\mathcal{U}}_{d_i}$, with $d_i \neq 0$ for $i = 1, 2$.

Proof. Recall the construction of $\widetilde{\mathcal{U}}_d$: starting with $\mathcal{U}_d$, we first blow up along $\mathcal{Z}_{[\frac{n+1}{2}]}$, then we blow up the proper transform of $\mathcal{Z}_{[\frac{n-1}{2}]}$ and so on, up to blowing up the proper transform of $\mathcal{Z}_2$. Since $\tilde{L}$ extends as the blow up of L then

$$c_1(\tilde{L}) = c_1(L) - \sum_{h=2}^{[\frac{n+1}{2}]} \tilde{\mathcal{Z}}_h \tag{2.17}$$

where $\tilde{\mathcal{Z}}_h$ is the exceptional divisor corresponding to $\mathcal{Z}_h$.

For simplicity, let $N = [\frac{n+1}{2}]$ and $\tilde{L}^{(h)}$ be the partial blow up of L along $\mathcal{Z}_N, \ldots, \mathcal{Z}_h$ only. Denote by

$$\tilde{x}(h) = -c_1(\tilde{L}^{(h)}) \tag{2.18}$$

Recall that an element of $\mathcal{Z}_h$ has h components having a point in common, the image of the ghost base. Then:

Lemma 2.6 *Using the notations above,*

$$\tilde{x}^i(h)y^j[\tilde{\mathcal{U}}_d^{(h)}] = \tilde{x}^i(h+1)y^j[\tilde{\mathcal{U}}_d^{(h+1)}] - \left[\frac{(\tilde{x}_1 + \ldots + \tilde{x}_h - 1)^i}{(1-\tilde{x}_1)\cdot \ldots \cdot (1-\tilde{x}_h)}\right]_{i-h} \pi^* y^j [pr\mathcal{Z}_h] \tag{2.19}$$

where $\tilde{x}_i$ *and* y_i *are* $c_1(\tilde{L}^*)$ *and* $\mathrm{ev}^*(H)$ *corresponding to the i'th component of each element of* $pr\mathcal{Z}_h$*, the proper transform of* $\mathcal{Z}_h$*.*

Proof. Let $\pi : \tilde{\mathcal{U}}_d^{(h)} \to \tilde{\mathcal{U}}_d^{(h+1)}$ be the blow up of $\tilde{\mathcal{U}}_d^{(h+1)}$ along $pr\mathcal{Z}_h$, and $\tilde{\mathcal{Z}}_h$ be that exceptional divisor. Then

$$c_1(\tilde{L}^{(h)}) = \pi^* c_1(L^{(h+1)}) - [\tilde{\mathcal{Z}}_h] \qquad \text{i.e.} \qquad x(h) = \pi^* x(h+1) + [\tilde{\mathcal{Z}}_h] \qquad \text{so}$$

$$\tilde{x}^i(h)y^j[\tilde{\mathcal{U}}_d^{(h)}] = \pi^*(\ \tilde{x}^i(h+1)y^j\ ) + \sum_{l=1}^{i}\binom{i}{l}\pi^*(\ \tilde{x}^{i-l}(h+1)y^j\ )[\tilde{\mathcal{Z}}_h]^l \tag{2.20}$$

Thus we need to understand terms like

$$\pi^*(\alpha)[\tilde{\mathcal{Z}}_h]^l \qquad \text{for} \qquad \alpha = \tilde{x}^{i-l}(h+1)y^j, \ \ l \geq 1$$

But this kind of intersections were computed by Fulton [Ful]. Here is a brief sketch of the argument: Let N be the normal bundle of the proper transform of $\mathcal{Z}_h$ and $\xi = \mathcal{O}_N(1)$ be the hyperplane class in $\mathbf{P}(N)$. Recall that $\mathcal{Z}_h$ is codimension h. Combining the relation

$$\tilde{\mathcal{Z}}_h^l = c_1^{l-1}(\mathcal{O}_N(-1))$$

with the definition of the Segree class of the normal bundle ([Ful] p 47)

$$s_i(N)\cdot\alpha = \pi_*(c_1^{h-1+l}(\mathcal{O}_N(1))\cdot\pi^*(\alpha))$$

one gets

$$\pi_*(\pi^*(\alpha)\cdot[\tilde{\mathcal{Z}}_h]^l) = (-1)^{l-1}\alpha\cdot s_{i-h}(N)$$

In our case we look at top intersections (i.e. $|\alpha| + l = r$) so for $l \geq 1$

$$\pi^*(\alpha) \cdot [\tilde{\mathcal{Z}}_h]^l = (-1)^{l-1} \alpha \cdot s_{i-h}(N)[pr\mathcal{Z}_h] \tag{2.21}$$

as integers. To complete the proof, we need to find the Segree class of N.

Note that $\mathcal{Z}_h$ has components indexed by the different distributions of the degree on the h bubbles:

$$\mathcal{Z}_h = \bigcup_{d_1+\ldots+d_h=d} \mathcal{Z}_{d_1,\ldots,d_h}$$

where $d_i \neq 0$ for $i = 1, \ldots, n$. When we blow up, the components of $pr\mathcal{Z}_h$ become disjoint, and the component $pr\mathcal{Z}_{d_1,\ldots,d_h}$ can be viewed as a subset of

$$(\mathcal{M}_{0,h+1} \times \tilde{\mathcal{U}}_{d_1} \times \ldots \times \tilde{\mathcal{U}}_{d_h})/S_h \tag{2.22}$$

where $\mathcal{M}_{0,h+1}$ is the moduli space of $h+1$ marked points on a sphere, and the symetric group S_h acts freely on the h bubbles by permuting them (giving the same bubble tree). Consider the projections:

$$\begin{array}{ccc}
\mathcal{M}_{0,h+1} \times \tilde{\mathcal{U}}_{d_1} \times \ldots \times \tilde{\mathcal{U}}_{d_h} & \xrightarrow{\tilde{p}_i} & \tilde{\mathcal{U}}_{d_i} \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{M}_{0,h+1} \times \mathcal{U}_{d_1} \times \ldots \times \mathcal{U}_{d_h} & \xrightarrow{p_i} & \mathcal{U}_{d_i}
\end{array}$$

be the projections on the i'th factor, and L_i be the relative tangent bundle of the i'th factor. The normal bundle of $\mathcal{Z}_h$ is isomorphic to

$$p_1^* L_1 \oplus \ldots \oplus p_h^* L_h$$

so the normal bundle of the proper transform of $\mathcal{Z}_h$ inside $\tilde{\mathcal{U}}_d^{(h+1)}$ is

$$N = \tilde{p}_1^* \tilde{L}_1 \oplus \ldots \oplus \tilde{p}_h^* \tilde{L}_h,$$

where $\tilde{L}_i$ is the full blow up of L_i as in (1.17). Thus:

$$s(N) = \frac{1}{(1-\tilde{x}_1) \cdot \ldots \cdot (1-\tilde{x}_h)} \tag{2.23}$$

where $\tilde{x}_i$ is the pull back via $\tilde{p}_i$ of the first Chern class of the relative cotangent bundle $\tilde{L}^*$ of $\tilde{\mathcal{U}}_{d_i}$. Putting together (2.20), (2.21) and (2.23) we get:

$$\begin{aligned}
\tilde{x}^i(h)y^j[\tilde{\mathcal{U}}_d^{(h)}] &= \tilde{x}^i(h+1)y^j[\tilde{\mathcal{U}}_d^{(h+1)}] \\
&+ \sum_{l=h}^{i}\binom{i}{l}(-1)^{l-1}\left[\frac{1}{(1-\tilde{x}_1)\cdot\ldots\cdot(1-\tilde{x}_h)}\right]_{l-h}\pi^*(\,\tilde{x}^{i-l}(h-1)y^j\,)[pr\tilde{\mathcal{Z}}_h] \\
&= \tilde{x}^i(h+1)y^j[\tilde{\mathcal{U}}_d^{(h+1)}] - \left[\frac{(\pi^*\tilde{x}(h+1)-1)^i}{(1-\tilde{x}_1)\cdot\ldots\cdot(1-\tilde{x}_h)}\right]_{i-h}\pi^*y^j[pr\tilde{\mathcal{Z}}_h]
\end{aligned}$$

But

$$\tilde{x}(h+1)[pr\mathcal{Z}_h] = \tilde{x}_1 + \ldots + \tilde{x}_h$$

which gives (2.19). □

Next, note that all the cohomology classes involved in the last term of (2.19) are pulled back from $\hat{\mathcal{Z}}_h$, the result of collapsing the ghost bubbles in $pr\mathcal{Z}_h$ to a point. But note that dim $\hat{\mathcal{Z}}_h <$ dim $\mathcal{Z}_h$ unless $h = 2$ (the difference is dim $\mathcal{M}_{0,h+1}$) so a top intersection on $\mathcal{Z}_h$ of classes pulled back from $\hat{\mathcal{Z}}_h$ vanishes unless $h = 2$. This, combined with (2.19) and the fact that $\tilde{L}^{(h)} = L$ when $h = N+1$, implies:

$$\tilde{x}^i y^j[\tilde{\mathcal{U}}_d] = x^i y^j[\mathcal{U}_d] - \left[\frac{(\tilde{x}_1+\tilde{x}_2-1)^i}{(1-\tilde{x}_1)(1-\tilde{x}_h)}\right]_{i-2}\pi^*y^j[pr\mathcal{Z}_2] \tag{2.24}$$

But $pr\mathcal{Z}_2 = \hat{\mathcal{Z}}_2 = ev_0^*(\Delta)$ for Δ is the diagonal of $\mathbf{P}^n$ so

$$[pr\mathcal{Z}_2] = \frac{1}{2}\sum_{\substack{d_1+d_2=d\\ i_1+i_2=n}}[\,\tilde{\mathcal{U}}_{d_1}(H^{i_1};\cdot)\times\tilde{\mathcal{U}}_{d_2}(H^{i_2};\cdot)\,] = \sum_{\substack{d_1+d_2=d\\ i_1+i_2=n}} y_1^{i_1}y_2^{i_2}[\tilde{\mathcal{U}}_{d_1}\times\tilde{\mathcal{U}}_{d_2}] \tag{2.25}$$

The factor of $1/2$ in front of the sum comes from action of the symmetric group S_2 on the 2 bubbles (yielding the same bubble tree). After distributing y on the 2 components (2.24) becomes (2.16). □

2.3 Applications to $\mathbf{P}^n$, $n \leq 3$

In this section we apply the inductive algorithms described in the previous section to compute the elliptic enumerative invariant τ_d in $\mathbf{P}^2$ and $\mathbf{P}^3$.

Proposition 2.7 *The number $\tau_d(p^{3d-1})$ of degree d elliptic curves in $\mathbf{P}^2$ with fixed j invariant and passing though $3d-1$ points is*

$$\tau_d(p^{3d-1}) = \frac{2}{n_j}\binom{d-1}{2}\sigma_d(p^{3d-1}) \tag{2.26}$$

where σ_d is the number of rational curves through $3d$ points. and n_j is the order of the group of automorphisms of the complex structure j fixing a point.

Proof. For $n = 2$, relation (0.2) gives:

$$n_j\tau_d(p^{3d-1}) = \sigma_d(l, l, p^{3d-1}) - 3ev^*(H) - c_1(\widetilde{L}^*) \tag{2.27}$$

where $L \to \mathcal{U}_d$ is the relative tangent bundle over the moduli space of 1-marked rational curves of degree d passing through $3d-1$ generic points and $\widetilde{L}$ is its blow up as in (1.17).

The moduli space $\mathcal{M}_d$ of unmarked curves is $n - 2 = 0$ dimensional, consisting of $\sigma_d(p^{3d-1})$ curves. In particular, we do not need to blow up to do the intersection theory, i.e. $\widetilde{L} = L$. Using (2.2) (or easier by inspection)

$$c_1(L^*) = -\frac{2}{d}\sigma_d(l, p^{3d-1}) = -2\sigma_d(p^{3d-1})$$

$$ev^*(H) = \sigma_d(l, p^{3d-1}) = d\sigma_d(p^{3d-1}) \quad \text{and} \quad \sigma_d(l, l, p^{3d-1}) = d^2\sigma_d(p^{3d-1})$$

So pluging them back in (2.27) we obtain

$$\tau_d(p^{3d-1}) = \frac{1}{n_j}(d^2 - 3d + 2)\,\sigma_d$$

which gives (2.7). □

In particular,

$$\tau_d(p^{3d-1}) = \begin{cases} \binom{d-1}{2}\sigma_d & \text{if } j \neq 0, 1728 \\ \frac{1}{2}\binom{d-1}{2}\sigma_d & \text{if } j = 0 \\ \frac{1}{3}\binom{d-1}{2}\sigma_d & \text{if } j = 1728 \end{cases} \tag{2.28}$$

This formula was recently obtained by Panharipande [Pan] using different methods.

Similarly, we can prove that:

Proposition 2.8 *The number $\tau_d = \tau_d(p^a, l^b)$ of elliptic curves in $\mathbf{P}^3$ with fixed j invariant and passing through a points and b lines (such that $2a + b = 4d - 1$) is given by:*

$$\tau_d = \frac{2(d-1)(d-2)}{dn_j}\sigma_d(l) - \frac{2}{dn_j}\sum_{d_1+d_2=d} d_2(2d_1d_2 - d)\sigma_{d_1}(l)\sigma_{d_2} \tag{2.29}$$

where $\sigma_d(l) = \sigma_d(p^a, l^b,\ l)$ is the number of degree rational curves in $\mathbf{P}^3$ passing through same conditions as τ_d plus one more line. By the term $\sigma_{d_1}(l)\sigma_{d_2}$ we understand the sum over all decompositions into a degree d_1 and a degree d_2 bubble such that the constraints are distributed in all possible ways on the bubble, and $d_i \neq 0$.

Proof. When $n = 3$, Theorem 0.1 becomes:

$$n_j\tau_d(p^a, l^b) = \sum_{i_1+i_2=3} \sigma_d(H^{i_1}, H^{i_2}, p^a, l^b) - 6\mathrm{ev}^*(H^2) - 4\mathrm{ev}^*(H)c_1(\widetilde{L}^*) - c_1^2(\widetilde{L}^*) \tag{2.30}$$

The moduli space $\mathcal{M}_d$ of degree d unmarked curves passing through a points and b lines is $n - 2 = 1$ dimensional, with a finite number of bubble trees in the boundary. Then Proposition 2.8 is a consequence of (2.30) and the following Lemma after some simple algebraic manipulations:

Lemma 2.9 *We have the following relations*

$$\sum_{i_1+i_2=3} \sigma_d(H^{i_1}, H^{i_2}, p^a, l^b) = 2d \cdot \sigma_d(p^a, l^{b+1}) \tag{2.31}$$

$$\mathrm{ev}^*(H^2) = \sigma_d(p^a, l^{b+1}) \tag{2.32}$$

$$\mathrm{ev}^*(H) \cdot c_1(\tilde{L}^*) = \mathrm{ev}^*(H) \cdot c_1(L^*) = -\frac{1}{d}\sigma_d(l) + \frac{1}{d} \sum_{d_1+d_2=d} d_1 d_2^2 \sigma_{d_1}(l)\sigma_{d_2} \tag{2.33}$$

$$c_1(L^*)^2 = -\sum_{d_1+d_2=d} d_2 \sigma_{d_1}(l)\sigma_{d_2} \tag{2.34}$$

$$c_1(\tilde{L}^*)^2 = -2\sum_{d_1+d_2=d} d_2 \sigma_{d_1}(l)\sigma_{d_2} \tag{2.35}$$

Proof. Relations (2.31) and (2.32) follow immediately by definition. The remaining relations involve more work. To prove (2.33), use (2.8) to get

$$\begin{aligned}
\mathrm{ev}^*(H) \cdot c_1(L^*) &= \frac{1}{d^2}\mathrm{ev}^*(H)[\mathcal{U}_d(\ ; H^2)] - \frac{2}{d}\sigma_d(H^2) + \frac{1}{d^2} \sum_{\substack{i_1+i_2=3 \\ d_1+d_2=d}} d_2^2\, \sigma_{d_1}(H, H^{i_1})\sigma_{d_2}(H^{i_2}) \\
&= -\frac{1}{d}\sigma_d(l) + \frac{1}{d^2} \sum_{d_1+d_2=d} d_2^2 (\ \sigma_{d_1}(H,l)\sigma_{d_2}(H) + \sigma_{d_1}(H,H)\sigma_{d_2}(l)\ ) \\
&= -\frac{1}{d}\sigma_d(l) + \frac{1}{d^2} \sum_{d_1+d_2=d} d_2^2 \Big(\ d_1 d_2 \sigma_{d_1}(l)\sigma_{d_2} + d_1^2 \sigma_{d_1}\sigma_{d_2}(l)\ \Big)
\end{aligned}$$

which gives (2.33) after switching the indices in the second sum.

For simplicity we have recorded only the new constraints, whereas the old ones get distributed in all possible ways. The equality between $\mathrm{ev}^*(H) \cdot c_1(L^*)$ and $\mathrm{ev}^*(H) \cdot c_1(\tilde{L}^*)$ is a consequence of (2.16).

Next, we need to evaluate $c_1^2(L^*)$. We use (2.8)

$$\begin{aligned}
c_1^2(L^*) &= -\frac{2}{d}c_1(L^*) \cdot \mathrm{ev}^*(H) \cdot [\mathcal{U}_d] + \sum_{\substack{d_1+d_2=d \\ i_1+i_2=3}} \frac{d_2^2}{d^2}\, c_1(L^*) \cdot [\, \mathcal{U}_{d_1}(\ ; H^{i_1}, \cdot)\,] \cdot \sigma_{d_2}(H^{i_2}, \cdot) \\
&+ \frac{1}{d^2}c_1(L^*) \cdot [\, \mathcal{U}_d(\ ; H^2, \cdot)\,] - \sum_{\substack{d_1+d_2=d \\ i_1+i_2=3}} \frac{d_2^2}{d^2}\, [\mathcal{U}_{d_1}(H^{i_1}; \cdot)] \cdot \sigma_{d_2}(H^{i_2}, \cdot) \qquad (2.36)
\end{aligned}$$

The first term is given by (2.33). Using (2.8) again, we get:

$$c_1(L^*)\cdot[\,\mathcal{U}_d(\,;H^2,\cdot)\,] = -2\sigma_d(l)$$

$$c_1(L^*)\cdot[\,\mathcal{U}_{d_1}(\,;H^{i_1},\cdot)\,] = -\frac{2}{d_1}\sigma_{d_1}(H,H^{i_1},\cdot) = -2\sigma_{d_1}(H^{i_1},\cdot)$$

Plugging these relations in (2.36), expanding the sums and then combining the terms together, we get (2.34).

Finally, to get (2.35), use (2.16)

$$\begin{aligned}
\tilde{x}^2 &= x^2 - \frac{1}{2}\sum_{\substack{d_1+d_2=d\\ i_1+i_2=3}}\left[\frac{(\tilde{x}_1+\tilde{x}_2-1)^2}{(1-\tilde{x}_1)(1-\tilde{x}_2)}\right]_0 [\mathcal{U}_{d_1}(H^{i_1};\cdot)\times\mathcal{U}_{d_2}(H^{i_2};\cdot)] \\
&= x^2 - \frac{1}{2}\sum_{\substack{d_1+d_2=d\\ i_1+i_2=3}}\sigma_{d_1}(H^{i_1},\cdot)\cdot\sigma_{d_2}(H^{i_2},\cdot) = x^2 - \sum_{d_1+d_2=d} d_1\sigma_{d_1}(l,\cdot)\cdot\sigma_{d_2}(\cdot)
\end{aligned}$$

after expanding the sum and then combining terms together. □

If we distribute the constraints in Proposition 2.8 in all possible ways, formula (2.29) becomes:

$$\begin{aligned}
\tau_d(p^a,l^b) &= \frac{2(d-1)(d-2)}{n_j d}\sigma_d(p^a,l^{b+1}) \qquad (2.37)\\
&+ \frac{2}{n_j d}\sum_{d_1=1}^{d}\sum_{a_1=0}^{a} d_2(2d_1d_2-d)\,\sigma_{d_1}(p^{a_1},l^{b_1+1})\cdot\sigma_{d_2}(p^{a_2},l^{b_2})\binom{a}{a_1}\binom{b}{b_1}
\end{aligned}$$

Example. Using a computer program based on (2.37) and the recursive formulas (A.3) for σ_d, one recovers for example that in $\mathbf{P}^3$ all the degree 2 elliptic invariants are 0 (fact known for a very long time) but also one gets new examples, like:

	$\tau_3(l^{11})$	$\tau_5(p,l^{17})$
$j \neq 0, 1728$	$6\cdot 25920$	$6\cdot 15856790593536$
$j = 0$	$3\cdot 25920$	$3\cdot 15856790593536$
$j = 1728$	$2\cdot 25920$	$2\cdot 15856790593536$

APPENDIX

3 Appendix

If we let $\sigma_d(j_1, j_2, \ldots, j_k) = \sigma_d(H^{j_1}, H^{j_2}, \ldots, H^{j_k})$ be the genus 0 enumerative invariant in $\mathbf{P}^n$, then Ruan-Tian proved that the genus zero perturbed invariant and the genus zero enumerative invariant are equal in $\mathbf{P}^n$, i.e.

$$\sigma_d(H^{j_1}, H^{j_2}, \ldots, H^{j_k}) = RT_{d,0}(H^{j_1}, H^{j_2}, H^{j_3} | H^{j_4}, \ldots, H^{j_k}) \tag{A.1}$$

Consequences of Ruan-Tian degeneration formula are:

$$RT_{d,1}(\beta_1 \mid \beta_2, \ldots, \beta_l) = \sum_{i_1+i_2=n} \sigma_d(H^{i_1}, H^{i_2}, \beta_1, \ldots, \beta_l) \tag{A.2}$$

and that σ_d in $\mathbf{P}^n$ satisfies the following recursive formula: for $j_1 \geq j_2 \geq \ldots \geq j_k \geq 2$,

$$\begin{aligned} \sigma_d(j_1, j_2, j_3) &= -\sigma_d(j_1, j_2+1, j_3-1) + d\sigma_d(j_1+j_3-1, j_2) - d\sigma_d(j_1+j_2, j_3-1) \\ &+ \sum_{d_1=1}^{d-1} \sum_{i=0}^{n} (\, \sigma_{d_1}(j_1, j_2, i)\sigma_{d_2}(j_3-1, n-i) - \sigma_{d_1}(j_1, j_3-1, i)\sigma_{d_2}(j_2, n-i)\,) \end{aligned} \tag{A.3}$$

where $\sigma_d(j_1, j_2, j_3) = \sigma_d(j_1, j_2, j_3, j_4, \ldots, j_k)$ and the conditions $H^{j_4}, \ldots, H^{j_k}$ are distributed in the right hand side in all possible ways. Note that $\sigma_1(pt, pt) = 1$ gives the initial step of the recursion.

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