THE CONCENTRATION PRINCIPLE FOR DIRAC OPERATORS

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ABSTRACT

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The symbol map σ of an elliptic operator carries essential topological and geometrical information about the underlying manifold. We investigate this connection by studying Dirac operators with a perturbation term. These operators have the form $D_s = D + s\mathcal{A} : \Gamma(E) \to \Gamma(F)$ over a Riemannian manifold (X, g) for special bundle maps $\mathcal{A} : E \to F$ and their behavior as $s \to \infty$ is interesting. We start with a simple algebraic criterion on the pair $(\sigma, \mathcal{A})$ that insures that solutions of $D_s\psi = 0$ localize as $s \to \infty$ around the singular set $Z_{\mathcal{A}}$ of $\mathcal{A}$. Under certain assumptions of $\mathcal{A}$, $Z_{\mathcal{A}}$ is a union of submanifolds, and this gives a new localization formula for the index of D as a sum over contributions from the components of $Z_{\mathcal{A}}$. We give numerous examples.

To my parents Petros and Pavlina

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Introduction

Many problems in geometric analysis involve studying solutions of first-order elliptic systems $D\psi = 0$ for operators

$$D : \Gamma(E) \to \Gamma(F)$$

between bundles over a Riemannian manifold (X, g) . Given such an operator D, one can look for zeroth-order perturbation A such that all finite-energy solutions of the equation

$$D_s\psi \;=\; (D + s\mathcal{A})\psi \;=\; 0 \tag{0.0.1}$$

concentrate along submanifolds Z_ℓ as $s \to \infty$ in a precise sense. There are several examples of this in the literature, the most well-known occurring in Witten's approach to Morse Theory. The aim of this thesis is to find a general setting for such concentration phenomenon.

We start with a simple criterion that insures concentration : $D + s\mathcal{A}$ localizes if the composition $A^* \circ \sigma_D(\alpha)$ of the adjoint of $\mathcal{A}$ with the symbol of D is self-adjoint (Definition (1.0.1)). It is natural to regard D as a Dirac operator; its symbol is then

Clifford multiplication c and the concentration condition becomes

$$\mathcal{A}^* \circ c(u) = c(u)^* \circ \mathcal{A} \qquad \text{for every } u \in T^*X.$$

This simple algebraic condition implies the analytic fact that solutions concentrate(in the precise sense of Proposition 1.0.4).

This thesis has two main results. The first is a Spectral Decomposition Theorem for operators that satisfy the concentration condition and three transversality conditions. It shows that:

- The eigenvectors corresponding to the low eigenvalues of $D_s^* D_s$ concentrate near the singular set of the perturbation bundle map $\mathcal{A}$ as $s \to \infty$.
- The eigenvalues of $D_s^* D_s$ corresponding to the eigensections that do not concentrate grow as at least $O(s)$ as $s \to \infty$.
- The perturbation $\mathcal{A}$ determines submanifolds Z_ℓ and each determines an associated decomposition of the normal bundle to the Z_ℓ giving a specific asymptotic formula for the solutions of (0.0.1) for large s.

Our second main result is an Index Localization Theorem (Theorem 1), which follows from the Spectral Decomposition Theorem. It describes how the Atiyah-Singer index formula for the index of D decomposes as a sum of local indices associated with specific operators on bundles over the submanifolds Z_ℓ.

I. Prokhorenkov and K. Richardson [PR] previously found the concentration condition (1.0.1), but found few examples because they assumed $\mathcal{A}$ to be complex-linear and found concentration only at points. Their list of examples does *not* include most of the' examples given in Chapter 2. The reason is that they assumed that A is self-adjoint and complex-linear, while in many of our examples A is conjugate-linear or has no complex structure and in some examples concentration occurs along submanifolds i.e. the zero set of a spinor. Thus it is essential to study (0.0.1) as a *real* operator.

This thesis has six chapters. Chapter 1 introduces the Concentration Condition, describes some elementary consequences, and states the main results, which are proved in later sections. An important part of the story is the vector bundles $\mathcal{K}_\ell$ and $\hat{\mathcal{K}}_\ell$ over the sub manifolds Z_ℓ that are introduced in Chapter 1 and described in detail later.

Chapter 2 presents many examples, some already known, and some new. The point here is to search for real linear perturbations on reducible Clifford bundles. The first example is probably classical, but it already illustrates the idea that concentration occurs when a complex operator is perturbed by a conjugate-linear operator. Example 2 stems from an observation of Taubes. Taubes used a concentrating family of operators to give an interesting new proof of the classical Riemann-Roch Theorem for line bundles on complex curves; our Example 2 extends this to higher-rank bundles over curves. Example 3 is Witten's Morse Theory operator, which we show fits into the general setup of Chapter 1.

The next set of examples is more interesting because the singular set $z_{\mathcal{A}}$ arises as the zero set of a spinor on X. In these examples, D acts on "spinor-form pairs ", meaning sections of a subbundle of $W \oplus \Lambda^*(T^*X)$ where W is a bundle of spinors, a Spin^c bundle and $\Lambda^* T^* X$ the bundle of forms. These examples are most natural in low dimensions, especially in dimension four. In particular, Examples 5 and 6 are linearizations of operators that occur in Seiberg-Witten theory.

In Chapter 3 we examine the geometric structure of $D + s\mathcal{A}$ near the singular set $Z_{\mathcal{A}}$ of $\mathcal{A}$. We first examine the perturbation term $\mathcal{A}$, regarding it as a section of a bundle of linear maps, and requiring it to be transverse to certain subvarieties, where the linear maps jump in rank. This transversality condition, allows us to write down a Taylor series expansion of $\mathcal{A}$ in the normal directions along each connected component Z_ℓ of $Z_{\mathcal{A}}$, and this gives a similar expansion of the coefficients of the operator D.

The technical analysis needed to prove the Spectral Separation Theorem is done in Chapters 4 and 5. We use a maximum principle argument to obtain decay estimates on the concentrating eigensections, and we use the Taylor expansions from Chapter 3 to decompose D into the sum of a "vertical" operator D^V that acts on sections along each fiber of the normal bundle, and a "horizontal" operator D^H that acts on sections of the bundles $\mathcal{K}_\ell$ on each component Z_ℓ of $Z_{\mathcal{A}}$. We then define a space of approximate eigensections, and work through estimates needed to show that these approximate eigensections are uniquely corrected to true eigensections.

Chapter 6 heads in a new direction. It describes a simple "non-linearization" procedure that associates to each operator (0.0.1) a non-linear elliptic system whose linearization is (0.0.1). We show how this non-linearization procedure produces, quite automatically, two famous non-linear elliptic systems: the Vortex equations in dimension two and the Seiberg-Witten equations in dimension four. In these cases, concentration phenomena are well-known as seen, for example, in the work Bradlow, Garcia-Prada, and Taubes. One of the interesting consequences of this thesis is that the concentration that occurs in these examples, which many believe to be a reflection of nonlinearity, appears instead to be a result of an algebraic interplay between first and zeroth order terms that is already present in the linearized equations.

Finally, we provide an appendix with the structure of the subvarieties that produce the singular set of $\mathcal{A}$ in Chapter 3. A good reference is [K]. The extensive study of their topology is crucial in the study of the localizing Dirac type operators.

Chapter 1

The Concentration Principle

The section describes some very general conditions in which one has a family D_s of first order elliptic operators whose low eigenvectors concentrate around submanifolds Z_ℓ as $s \to \infty$. At the end of the section, we state the main theorem of this thesis. We also state one important corollary, which shows how the index of D decomposes as a sum of indices of operators on the submanifolds Z_ℓ.

Let (X, g) be a closed Riemannian manifold and $E,\ F$ be real vector bundles over X. Suppose that

- $D : \Gamma(E) \to \Gamma(F)$ is a first order elliptic differential operator with its symbol σ.
- $\mathcal{A} : E \to F$ is a real bundle map.

From this data we can form the family of operators

$$D_s = D + s\mathcal{A}$$

where $s \in \mathbb{R}$. Furthermore, assuming that the bundle E and F have metrics, we can form the adjoint A^*, and the formal L^2 adjoint $D_s^* = D^* + s\mathcal{A}^*$ of D_s. The symbol of D^* is $-\sigma^*$. The main point of this thesis is that such a family D_s is especially interesting when

$\mathcal{A}$ and the symbol σ are related in the following way.

Definition 1.0.1 (Concentrating pairs)**.** *In the above context, we say that $(\sigma, \mathcal{A})$ is a* concentrating pair *if it satisfies the algebraic condition*

$$\mathcal{A}^* \circ \sigma(u) = \sigma(u)^* \circ \mathcal{A} \qquad \textit{for every } u \in T^*X. \tag{1.0.1}$$

Lemma 1.0.2. *A pair $(\sigma, \mathcal{A})$ is concentrating if and only if the operator*

$$B_{\mathcal{A}} = D^* \circ \mathcal{A} + \mathcal{A}^* \circ D$$

has order 0, that is, is a bundle map. If so, then for each $\xi \in C^\infty(E)$,

$$\|D_s\xi\|_2^2 \;=\; \|D\xi\|_2^2 \;+\; s^2\|\mathcal{A}\xi\|_2^2 \;+\; s\langle \xi, B_{\mathcal{A}}\xi\rangle \tag{1.0.2}$$

where these are L^2 norms and inner products.

Proof. Given a tangent vector $u \in T_p^*X$, choose a smooth function f with $df|p = u$. Then for any smooth section ξ of E,

$$\begin{aligned} B_{\mathcal{A}}(f\xi) \;=\; D^*(f\mathcal{A}(\xi)) \;+\; \mathcal{A}^*(D(f\xi)) &\;=\; -\sigma^*(df)A\xi + fD^*\mathcal{A}\xi + A^*\sigma(df)\xi + f\mathcal{A}^*D\xi \\ &\;=\; \big(-\sigma^*(u)\mathcal{A} + \mathcal{A}^*\sigma(u)\big)\xi + fB_{\mathcal{A}}(\xi). \end{aligned}$$

Thus (1.0.1) holds if and only if $B_{\mathcal{A}}(f\xi) = fB_{\mathcal{A}}(\xi)$, which means that $B_{\mathcal{A}}$ is a zeroth order operator. To obtain (1.0.2), expand $|D + s\mathcal{A}|^2$ and integrate; this gives

$$\|D_s\xi\|_2^2 \;=\; \|D\xi\|_2^2 \,+\, s^2\,\|\mathcal{A}(\xi)\|_2^2 \,+\, s\,\langle D\xi, \mathcal{A}\xi\rangle \,+\, s\,\langle \mathcal{A}\xi, D\xi\rangle$$

where, after integrating by parts, the last two terms are equal to $s\,\langle \xi, B_{\mathcal{A}}\xi\rangle$. $\square$

Remark 1.0.3. Given D and A as above, one can always form the self-adjoint operators

$$\mathcal{D} \;=\; \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} \;=\; \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Then $(\sigma_{\mathcal{D}}, \mathcal{A})$ is a concentrating pair when $\sigma_{\mathcal{D}}(u)\circ\mathcal{A} = -\mathcal{A}\circ\sigma_{\mathcal{D}}(u),\ u \in T^*X$. This implies that both (σ, A) and $(-\sigma^*, A^*)$ are concentrating pairs.

The assumption that $D : \Gamma(E) \to \Gamma(F)$ is elliptic mans that the bundles E and F have the same rank. Thus a generic bundle map $\mathcal{A} : E \to F$ is an isomorphism at almost every point. In the anyalsis of the family $D + s\mathcal{A}$, a key role will be played by the *singular set of* $\mathcal{A}$, defined as

$$Z_{\mathcal{A}} \;:=\; \big\{x \in X \mid \ker \mathcal{A}(x) \neq 0\big\},$$

that is, the set where $\mathcal{A}$ fails to be injective.

The following theorem shows the importance of the concentrating condition 1.0.1. It shows that, under Condition 1.0.1, all solutions of $D_s\xi = 0$ concentrative along the singular set $Z_{\mathcal{A}}$. More generally, it shows that all solutions of the eigenvalue problem $D_s\xi = \lambda(s)\xi$ with $\lambda(s) = O(s)$ also concentrate along $Z_{\mathcal{A}}$.

Fix $\delta > 0$, set $Z(\delta)$ be the δ-neighborhood of $Z_{\mathcal{A}}$ and let $\Omega(\delta) = X \setminus Z(\delta)$ be its complement.

Theorem 1.0.4 (Concentration Principle)**.** *There exist $C' = C'(\delta, \mathcal{A}, C) > 0$, such that whenever $\xi \in C^\infty(E)$ is a section with L^2 norm 1 satisfying $\|D_s\xi\|_2^2 \leq C|s|$, one has the concentration estimate*

$$\int_{\Omega(\delta)} |\xi|^2 \, dv_g \; < \frac{C'}{|s|}. \tag{1.0.3}$$

Proof. By equation (1.0.2) and ξ as in assumption we get

$$C|s| \;\geq\; \|D_s\xi\|_2^2 \;\geq\; s \, \langle \xi, B_{\mathcal{A}}\xi \rangle \;+\; s^2 \, \|\mathcal{A}(\xi)\|_2^2.$$

By Lemma 1.0.2, $B_{\mathcal{A}}$ is a tensor and compactness implies that $M_1 = \sup_X |B_{\mathcal{A}}|$ is finite hence by Cauchy-Schwartz

$$|\langle \xi, B_{\mathcal{A}}(\xi) \rangle| \;\leq\; M_1 \int_X |\xi|^2 \, dv_g \;=\; M_1.$$

Also for each $x \in X \setminus Z_{\mathcal{A}}$, $\mathcal{A}$ is injective on fibers, so there is a positive constant $\kappa(x)$ with $|\mathcal{A}(\xi)| \geq \kappa(x)|\xi|$. By compactness, those inequalities hold with a constant $\kappa > 0$ uniform on the closure of $\Omega(\delta)$ and therefore

$$s^2 \, \|A(\xi)\|_2^2 \;\geq\; \kappa^2 s^2 \int_{\Omega(\delta)} |\xi|^2 \, dv_g.$$

Consequently

$$\int_{\Omega(\delta)} |\xi|^2 \, dv_g \;\leq\; \frac{M_1 + C}{\kappa^2 |s|}.$$

$\square$

Remark 1.0.5. The above proof can be refined if one has an estimate of the form $|\mathcal{A}(\xi)|^2 \geq r^a |\xi|^2$ on a tubular neighborhood of $Z_{\mathcal{A}}$, where r is the distance from $Z_{\mathcal{A}}$ and $a > 0$; this gives a bound on how the constant C' in (1.0.3) depends on δ. Assumption (1.0.9) below gives such an estimate with $a = 2$. Later, in Chapters 4 and 5, we will develop better estimates that show that the eigensections of D_s with low eigenvalues are well-approximated by Gaussians as $s \to \infty$.

It is convenient to work in the context of Dirac operators. Recall that a vector space V is a representation of the Clifford algebra $C(\mathbb{R}^n)$ if there is a linear map $c : \mathbb{R}^n \to \mathrm{End}(V)$ that satisfies the Clifford relations

$$c(u)c(v) + c(v)c(u) \;=\; -2\langle u, v\rangle \; Id. \tag{1.0.4}$$

for all $u, v \in V$. We will often denote $u. = c(u)$.

Lemma 1.0.6. *The concentration condition* (1.0.1) *with* $\sigma = c$ *is equivalent to*

$$c(u) \circ \mathcal{A}^* = \mathcal{A} \circ c(u)^* \qquad \forall u \in T^*X. \tag{1.0.5}$$

Hence $D + s\mathcal{A}$ *concentrates if and only if the adjoint operator* $D^* + s\mathcal{A}^*$ *does.*

Proof. Multiplying (1.0.1) on the left by $c(u)$ and on the right by $c(u)^*$ we get

$$|u|^2 \mathcal{A} \circ c(u)^* = c(u) \circ \mathcal{A}^* |u|^2$$

which gives (1.0.5). The proof in the opposite direction is similar. $\square$

Dirac Operator Assumptions:

1. We assume that E, F are of equal rank, $E \oplus F$ admits a $\mathbb{Z}_2$ graded Spin^c structure induced from the symbol c and a Spin^c connection preserving the grading so that $\nabla c = 0$.

2. $(c, \mathcal{A})$ is a concentrating pair: $\mathcal{A}^* c(u) = c(u)^* \mathcal{A}$ for every $u \in T^*X$.

Then the composition $D = c \circ \nabla$ is a Dirac operator

$$D : \Gamma(E) \to \Gamma(F). \tag{1.0.6}$$

We also impose two further conditions on $\mathcal{A}$ that will guarantee that the components Z_ℓ of the singular set $Z_{\mathcal{A}}$ are submanifolds and that the rank of $\mathcal{A}$ is constant on each Z_ℓ. For this, we regard $\mathcal{A}$ as a section of a subbundle $\mathcal{L}$ of $\mathrm{Hom}(E, F)$ as in the following diagram:

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook] \arrow[d] & \mathrm{Hom}(E,F) \supseteq \mathcal{F}^l \\ (X, g) \arrow[u, bend left, "\mathcal{A}"] & \end{tikzcd} \tag{1.0.7}$$

Here $\mathcal{L}$ is a bundle that parameterizes some family of linear maps $A : E \to F$ that satisfy the concentration condition (1.0.1) for the operator (1.0.6), that is, each $A \in \mathcal{L}$ satisfies $A^* \circ c(u) = c(u)^* \circ A$ for every $u \in T^*X$. Inside the total space of the bundle $\mathrm{Hom}(E, F)$, the set of linear maps with l-dimensional kernel is a submanifold $\mathcal{F}^l$; because E and F have the same rank, this submanifold has codimension l^2. Assume that $\mathcal{L} \cap \mathcal{F}^l$ is a manifold for every l (see the Appendix).

Transversality Assumptions:

3. As a section of $\mathcal{L}$, $\mathcal{A}$ is transverse to $\mathcal{L} \cap \mathcal{F}^l$ for every l, and these intersections occur at points where $\mathcal{L} \cap \mathcal{F}^l$ is a manifold.

4. Z_ℓ is closed for all ℓ.

As a consequence of the Implicit Function Theorem $\mathcal{A}^{-1}(\mathcal{L} \cap \mathcal{F}^l) \subseteq X$ will be a submanifold of X for every l. The singular set decomposes as a union of these submanifolds, and even further as a union of connected components Z_ℓ:

$$Z_A = \bigcup_l \mathcal{A}^{-1}(\mathcal{L} \cap \mathcal{F}^l) = \bigcup_\ell Z_\ell. \tag{1.0.8}$$

By Assumption 3, $\mathcal{A}$ has constant rank along each Z_ℓ, so $\ker \mathcal{A}$ and $\ker \mathcal{A}^*$ are bundles over Z_ℓ.

Our final assumption is a statement about the Taylor expansion of $\mathcal{A}$.

Non-degeneracy Assumption:

5. Let K be the bundle obtained by parallel translating $\ker \mathcal{A} \to Z_\ell$ along geodesics normal to Z_ℓ in a tubular neighborhood of the singular set $Z_{\mathcal{A}}$. We require

$$\mathcal{A}^*\mathcal{A}|_K \;=\; r^2 M + O(r^3) \tag{1.0.9}$$

where r is the distance function from $Z_{\mathcal{A}}$, and M is a positive-definite symmetric endomorphism of the bundle K.

Now fix a point $p \in Z_\ell$ and choose an orthonormal frame $\{e_\alpha\}$ of the normal bundle $N_\ell \to Z_\ell$ at p with dual frame $\{e^\alpha\}$. In Chapter 4 Lemma 4.1.1 we prove that the matrices

$$M_\alpha = -c(e^\alpha)\nabla_{e_\alpha}\mathcal{A}_p : \,\ker \mathcal{A}_p \to \ker \mathcal{A}_p \tag{1.0.10}$$

are a collection of commuting isomorphisms, and that each is self-adjoint (by Condition 1.0.1), and its spectrum is real, symmetric, and does not contain 0 (by Assumption 5). Hence there exist a common decomposition into eigenspaces $\ker \mathcal{A}_p = \bigoplus_i K_i$ that simultaneously diagonalizes the family $\{M_\alpha\}$. In this decomposition, it is the eigenspaces with *positive* eigenvalue that are important — this positivity ultimately translates into the fact that there are L^2 concentrating sections in these directions.

Definition 1.0.7. *For each component Z_ℓ of Z_A, let $\mathcal{K}_\ell \to Z_\ell$ be the bundle whose fiber at $p \in Z_\ell$ is*

$$\mathcal{K}_\ell|_p = \operatorname{span}\left\{ \varphi \in \ker \mathcal{A}_p \;\middle|\; \begin{array}{ll} \varphi & \textit{a common eigenvector of } \{M_\alpha\} \textit{ with every eigenvalue} \\ & \textit{positive} \end{array} \right\}.$$

There is a similar bundle $\hat{K}_\ell \to Z_\ell$ defined in the same way with the matrices M_α replaced by $\hat{M}_\alpha = -c(e^\alpha)\nabla_\alpha \mathcal{A}^*|_p : \ker \mathcal{A}_p^* \to \ker \mathcal{A}_p^*$.

In Chapter 4 Proposition 4.1.5 we use assumption 5 to prove that $\mathcal{K}$ and $\hat{\mathcal{K}}$ are bundles over each component of $Z_{\mathcal{A}}$. Also in Proposition 4.1.6 of the same chapter, we show that, for each component Z_ℓ, the bundle $\mathcal{K}_\ell \oplus \hat{\mathcal{K}}_\ell$ is associated with a canonical Spin^c structure c_ℓ on Z_ℓ i.e the restriction of the Spin^c structure on (X, g).

Finally we look at the pullback connection of the bundles

$$\begin{tikzcd} ((E \oplus F)|_{Z_\ell}, \bar{\nabla}) \arrow[r] \arrow[d] & (E \oplus F,\, \nabla) \arrow[d] \\ Z_\ell \arrow[r] & X \end{tikzcd}$$

In Chapter 3, we show that this connection preserves the sub-bundle $\mathcal{K}_\ell \oplus \hat{\mathcal{K}}_\ell$ (see Proposition 4.1.5 and Proposition 4.1.6). We can compose this connection with Clifford multiplication to construct Dirac operators for the bundles $\mathcal{K}_\ell$ and $\hat{\mathcal{K}}_\ell$ over Z_ℓ.

Definition 1.0.8. *On each component Z_ℓ, we define*

$$D^\ell \;=\; c_\ell \circ \bar{\nabla} : \Gamma(Z_\ell,\, \mathcal{K}_\ell) \to \Gamma(Z_\ell,\, \hat{\mathcal{K}}_\ell).$$

and we denote its adjoint by $\hat{D}^\ell$.

In this definition, the Clifford multiplication c_ℓ is compatible with the Levi-Civita connection on X, not with the Levi-Civita connection of the induced metric on Z_ℓ (the two differ by a term involving the second fundamental form of Z_ℓ).

The main result of this thesis is a converse of Theorem 1.0.4. Recall that Theorem 1.0.4 shows that, for each C, the eigensections ξ satisfying $D_s^* D_s \xi = \lambda(s)\xi$ with $|\lambda(s)| \leq C$, concentrate around $\bigcup Z_\ell$ for large s. The following Spectral Separation Theorem shows that these localized solutions can be reconstructed using local data obtained from $Z_{\mathcal{A}}$.

Spectral Separation Theorem. *Suppose that $D_s = D + s\mathcal{A}$ satisfies Assumptions 1-5 above. Let E_λ be the λ-eigenspace of the operator $D_s^* D_s$ and F_λ the corresponding space for $D_s D_s^*$. Then the low eigenspaces split according the decomposition* (1.0.8)*: there exist $\lambda_0 > 0$ and $s_0 > 0$ so that for every $s > s_0$, there exist vector space isomorphisms*

$$\bigoplus_{\lambda \leq \lambda_0} E_\lambda \xrightarrow{\cong} \bigoplus_\ell \ker D^\ell$$

and

$$\bigoplus_{\lambda \leq \lambda_0} F_\lambda \xrightarrow{\cong} \bigoplus_\ell \ker \hat{D}^\ell.$$

In particular, if a component Z_ℓ of Z_A is a point p, then $D^\ell = 0$.

As a corollary we get the following localization for the index :

Index Localization Theorem. *Suppose that $D_s = D + s\mathcal{A}$ satisfies Assumptions 1-5 above. Then the index of D can be written as a sum of local indices as*

$$\operatorname{index} D = \sum_\ell \operatorname{index} D^\ell.$$

Proof. By the Spectral Separation Theorem there exist $\lambda_0 > 0$ and $s_0 > 0$ so that for every $s > s_0$

$$\begin{aligned}
\operatorname{index} D_s &= \dim\ker D_s - \dim\ker D_s^* = \dim\ker D_s^* D_s - \dim\ker D_s D_s^* \\
&= \dim \bigoplus_{\lambda \le \lambda_0} E_\lambda - \dim \bigoplus_{\lambda \le \lambda_0} F_\lambda \\
&= \sum_\ell \operatorname{index} D^\ell
\end{aligned}$$

where the third equality holds because $D_s^* D_s$ and $D_s D_s^*$ have the same spectrum and their eigenspaces corresponding to a common non zero eigenvalue are isomorphic. Since D_s and D differ by a compact perturbation they have the same index. This finishes the proof. $\square$

Chapter 2

Examples

The concentration condition (1.0.1) is clearly an algebraic condition on the symbol c of the Dirac operator D. The existence of the perturbation term $\mathcal{A}$ and the construction of interesting examples of concentrating pairs $(c, \mathcal{A})$ is an algebraic problem about representations of Clifford algebras and their connection with geometry. In the next several sections, we start with basic examples and progressively built more elaborate ones.

2.1 First Examples

Our first two examples are in dimension two. These have the form $D_s = D + s\mathcal{A}$ where D is a $\overline{\partial}$ operator and $\mathcal{A}$ is a *conjugate-linear* zeroth-order operator. Thus D_s is a *real* operator, although in the examples it is convenient to write it using complex notation.

Example 1: For functions $f, g : \mathbb{C} \to \mathbb{C}$, consider the operators

$$D_s f = \overline{\partial} f + sz\bar{f} \quad \text{and} \quad D_s' g = -\partial g + sz\bar{g}.$$

These have the form $D + s\mathcal{A}$ where $\mathcal{A}$ is the self-adjoint real linear map $\mathcal{A}f = z\bar{f}$. Using

Lemma 1.0.2, the calculations

$$\mathcal{B}_{\mathcal{A}} \;=\; (\overline{\partial}^* \mathcal{A} + \mathcal{A}^* \overline{\partial}) f \;=\; -\partial(z\bar{f}) - z\overline{\overline{\partial} f} \;=\; -\bar{f}$$

and

$$\mathcal{B}'_{\mathcal{A}} \;=\; (\overline{\partial} \mathcal{A}^* - \mathcal{A}\partial) f \;=\; \overline{\partial}(z\bar{f}) - z\overline{\partial f} \;=\; 0 \tag{2.1.1}$$

show that both $(\sigma_D, \mathcal{A})$ and $(\sigma_{D'}, \mathcal{A}^*)$ are concentrating pairs. Theorem 1.0.4 then shows that as $s \to \infty$, all solutions of $Df = 0$ and $D'g = 0$ concentrate around the zero set of $\mathcal{A}$, which is the origin. For these equations, we can find the solutions explicitly:

- Equations (1.0.2) and (2.1.1) show that any solution of $D'_s g = 0$ satisfies

  $$0 = \|\overline{\partial}^* g + s\mathcal{A}^* g\|_2^2 = \|\partial g\|_2^2 + s^2 \|\mathcal{A}^* g\|_2^2.$$

  This means that g is an anti-holomorphic function that vanishes for $z \neq 0$, so $g = 0$. Thus $\ker D'_s = 0$ for all $s \neq 0$.

- If f satisfies $D_s f = 0$, we can apply the operator $-\partial$ to get $-\partial\overline{\partial} f - s\bar{f} + s^2|z|^2 f = 0$. Writing $f = f_1 + i f_2$, the imaginary part f_2 satisfies

  $$-\partial\overline{\partial} f_2 + s f_2 + s^2 |z|^2 f_2 = 0.$$

  Taking L^2 inner product with f_2 and integrating by parts, we see that $f_2 \equiv 0$, so

$f = f_1$ is a real-valued function. Finally, by completing the differential, we obtain

$$\overline{\partial}(e^{s|z|^2} f) = 0.$$

But the a real-valued holomorphic function is constant, so $f(z) = Ce^{-s|z|^2}$ for some $C \in \mathbb{R}$. Thus $\ker D_s$ is real and one-dimensional, and the non-zero solutions of $D_s f = 0$ clearly concentrate at the origin as $s \to \infty$.

Similarly the problem $\overline{\partial} f + s\bar{z}\bar{f} = 0$ has trivial solutions and its adjoint has a real one-dimensional kernel.

It is more interesting to consider *real* Dirac operators on Riemann surfaces. In Section 7 of [T1], C. H. Taubes showed a concentration property for perturbed $\overline{\partial}$-operators on complex line bundles over Riemann surfaces. The following example generalizes Taubes' observation to higher rank bundles.

Example 2: Let (Σ, g) be a closed Riemann surface with anticanonical bundle $\bar{K}$, and let E be a holomorphic bundle of rank r with a Hermitian metric $\langle \cdot, \cdot \rangle$ conjugate linear in the second argument. The direct sum of the $\overline{\partial}$-operator $\overline{\partial} : \Gamma(E) \to \Gamma(\bar{K}E)$ and its adjoint is a self-adjoint Dirac operator

$$D = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix} : \Gamma(E \oplus \bar{K}E) \to \Gamma(E \oplus \bar{K}E).$$

The symbol of D, applied to a $(0,1)$-form u is

$$c(u)(\xi) \;=\; u\wedge\xi \;-\; \iota_u\xi, \quad \xi\in E\oplus\bar{K}E. \tag{2.1.2}$$

One checks that this satisfies the Clifford relations (1.0.4), so defines a Clifford bundle structure on $E\oplus\bar{K}E$. Now choose

$$\overline{\mu}\in\Gamma(\Sigma,\ \bar{K}\otimes_{\mathbb{C}}\mathrm{Sym}^2_{\mathbb{C}}E).$$

Combined with the conjugate linear isomorphism $E\cong E^*$ defined by the hermitian metric, $overline\mu$ becomes a conjugate linear map $\overline{\mu}: E\to\bar{K}E$. Set

$$\mathcal{A}=\begin{pmatrix}0 & \mu^* \\ \mu & 0\end{pmatrix}\in\mathrm{End}_{\mathbb{R}}(E\oplus\bar{K}E).$$

Lemma 2.1.1. *$(c,\mathcal{A})$ is a concentrating pair.*

Proof. It suffices to fix a point $p\in\Sigma$ and verify that $c(u)\circ\mathcal{A}=-\mathcal{A}\circ c(u)$ for all $u\in T^*_p\Sigma$. This is equivalent to proving that μ and it's adjoint μ^* satisfy the two identities

$$\iota_u(\mu(\xi)) \;=\; \mu^*(u\wedge\xi) \qquad\text{and}\qquad u\wedge\mu^*(\eta) \;=\; \mu(\iota_u(\eta))$$

for all ξ in the fiber E_p and η in $(\bar{K}\otimes E)_p$. Choose orthonormal bases $\{e_i\}$ of E_p and $\bar{k}$ of $\bar{K}$. Then $\overline{\mu}=\bar{k}\mu^{ij}e_i\otimes e_j\in\bar{K}\otimes_{\mathbb{C}}\mathrm{Sym}^2_{\mathbb{C}}(E)$ corresponds to the map $\mu: E\to\bar{K}E$

defined by

$$\mu(\xi) \;=\; \bar{k}\langle e_i,\,\xi\rangle\mu^{ij}e_j.$$

Thus for $u = \lambda\bar{k}$, we have

$$\iota_u\mu(\xi) \;=\; \lambda\,\iota_{\bar{k}}(\bar{k}\mu^{ij}\langle e_i,\xi\rangle)e_i \;=\; \lambda\mu^{ij}\langle e_i,\,\xi\rangle e_j$$

and

$$\mu^*(u\wedge\xi) \;=\; \langle\mu^*(u\wedge\xi),\,e_i\rangle e_i = \overline{\langle u\wedge\xi,\,\mu(e_j)\rangle}e_i = \lambda\mu^{ij}\langle\bar{k}e_j,\,\bar{k}\xi\rangle e_i.$$

These are equal since $\mu^{ij} = \mu^{ji}$. The second identity is proved from the first one using Lemma 1.0.6. $\square$

Lemma 2.1.1 shows that Theorem 1.0.4 applies. Thus as $s\to\infty$ the low eigensections of the operator

$$D_s \;=\; D + s\mathcal{A} : \Gamma(E\oplus\bar{K}E)\to\Gamma(E\oplus\bar{K}E)$$

concentrate on the singular set $Z_{\mathcal{A}}$. The following lemma describes the structure of $Z_{\mathcal{A}}$.

Lemma 2.1.2. *For generic μ, $Z_{\mathcal{A}}$ is a finite set of oriented points $\{p_\ell\}$. Furthermore,*

- *At each positive p_ℓ, $\mathcal{K}_\ell\cong\mathbb{R}$ and $\hat{K}_\ell = 0$, and*
- *At each negative p_ℓ, $\mathcal{K}_\ell = 0$ and $\hat{K}_\ell\cong\mathbb{R}$.*

Proof. The singular set of $\mathcal{A}$ is the set of points in Σ where $\mu : E\to\bar{K}E$ fails to be an isomorphism. Thus $Z_{\mathcal{A}}$ is the zero set of $\det\mu : \Lambda^r E\to\Lambda^r(\bar{K}E)$. Using the isomorphism

$\Lambda^r E \cong \Lambda^r E^*$ of the induced hermitian metric on $\Lambda^r E$, this becomes a complex map $\Lambda^r E^* \to \Lambda^r(\bar{K}E)$, or equivalently a section

$$\det \mu \in \Gamma(L)$$

of the complex line bundle

$$L = \bar{K}^r \otimes_{\mathbb{C}} \Lambda^r E \otimes_{\mathbb{C}} \Lambda^r E. \tag{2.1.3}$$

Note that while L is a holomorphic bundle, this section is only assumed to be smooth. For a generic choice of μ, the section $\det \mu$ will have only transverse zeros, which are therefore isolated points. By compactness the set $\{p_\ell\}$ of zeros is finite. At each p_ℓ, the derivative $(\nabla \det \mu)$ is an isomorphism from $T_{p_\ell}\Sigma$ to the fiber of L at p. Both of these spaces are oriented; p_ℓ is called positive if this isomorphism is orientation-preserving and is called negative if orientations are reversed.

Let z be a local holomorphic coordinate on Σ centered at $p \in \{p_\ell\}$. Because $\det \mu$ has a zero at p, there is a non-vanishing section e_1 of E so that $\mu(e_1)$ vanishes at $z = 0$. Since μ is conjugate-linear, the section $e_2 = ie_1$ also satisfies $\mu(e_2) = 0$ at $z = 0$. Hence we can choose real local framings of E and $\bar{K}E$ in which μ has the local expansion

$$\mu \;=\; \begin{pmatrix} H & 0 \\ 0 & * \end{pmatrix} \;+\; O(|z|^2)$$

where $*$ denotes an invertible $(n-2) \times (n-2)$ real matrix and

$$H : \ker \mu_0 \to \ker \mu_0^*.$$

is the real 2×2 matrix that corresponds to multiplication by $f \mapsto (\alpha z + \beta \bar{z})\bar{f}$ under the identification $\mathbb{C} = \mathbb{R}^2$.

For a generic section we have $|\alpha| \neq |\beta|$. It follows that $\det \mu$ has a positive zero at p if $|\alpha| > |\beta|$, and a negative zero if $|\alpha| < |\beta|$.

Suppose $|\alpha| < |\beta|$. By changing coordinates if necessary, we may assume that $\alpha = 0$ and $\beta = 1$. One then sees that $\mathcal{A}^*\mathcal{A}$ has the expansion (1.0.9), so all of the assumptions of the Spectral Separation Theorem hold. Write $z = x + iy$, and use the basis $\{e_1, e_2 = ie_1\}$ of $\ker \mu_0$ and $\{d\bar{z}e_1, d\bar{z}e_2\}$ of $\ker \mu_0^*$ to write $f = (f_1, f_2) \in \ker \mu_0$. Then

$$H(x, y) \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} = \big(xA_1 + yA_2\big) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In this basis, one can calculate that the Clifford multiplication (2.1.2) is given by

$$c(dx) = -\tfrac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad c(dy) = \tfrac{\sqrt{2}}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where these are maps $\ker \mu_0^* \to \ker \mu_0$. The corresponding matrices (1.0.10) are therefore

$$M_1 = -c(dx)A_1 = \tfrac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = -c(dy)A_2 = \tfrac{\sqrt{2}}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying Definition 1.0.7, one then sees that $\mathcal{K}_p = 0$ in this case. Analoguous calculations show that

$$\hat{M}_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{M}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and hence $\hat{\mathcal{K}}_p$ is one dimensional.

The case $|\alpha| > |\beta|$ is similar.

□

Corollary 2.1.3. *(Riemann-Roch) If E is a rank r holomorphic bundle over a complex curve C, then*

$$\operatorname{index} \bar{\partial}_E \;=\; 2c_1(E)[\Sigma] - r\chi(\Sigma). \tag{2.1.4}$$

Proof. Lemma 2.1.1 and the proof of Lemma 2.1.2 show that so all of the assumptions of the Index Localization Theorem 1 hold. In this case, $Z = \{p_\ell\}$ is the set of zeros of a generic section $\det \mu$ of the complex line bundle L defined by (2.1.3). By Lemma 2.1.2, each positive zero has local index $D^\ell = \dim \mathcal{K}_\ell - \dim \hat{\mathcal{K}}_\ell = 1$, and similarly each negative zero has index $D^\ell = -1$. The Index Localization Theorem therefore says that index D is given by the Euler number

$$\operatorname{index} \bar{\partial}_E \;=\; \chi(L)[\Sigma] \;=\; c_1(L)[\Sigma].$$

This Riemann-Roch formula (2.1.4) follows because

$$c_1(\Lambda^r E \otimes_{\mathbb{C}} \Lambda^r(\bar{K} \otimes_{\mathbb{C}} E) \;=\; 2c_1(\Lambda^r E) - c_1(K^r) \;=\; 2c_1(E) - rc_1(K)$$

and $c_1(K)[\Sigma] = \chi(\Sigma)$. $\square$

Example 3: On a closed Riemmanian manifold (X, g) the bundle $E \oplus F = \Lambda^{\mathrm{ev}} T^*X \oplus \Lambda^{odd} T^*X$ is a Clifford algebra bundle in two ways:

$$c(v) = v \wedge - \iota_{v^\#} \qquad \text{and} \qquad \hat{c}(w) = w \wedge + \iota_{w^\#} \tag{2.1.5}$$

for $v, w \in T^*X$. One checks that these anti-commute:

$$c(v)\hat{c}(w) \;=\; -\hat{c}(w)c(v). \tag{2.1.6}$$

Note that $D = d + d^*$ is a first-order operator whose symbol is c. Fix a 1-form γ with transverse zeros and set $\mathcal{A}_\gamma = \hat{c}(\gamma)$. Then Theorem 1.0.4 shows that the low eigenvectors of

$$D_s = D + s\mathcal{A}_\gamma = (d + d^*) + s\hat{c}(\gamma) : \Omega^{ev}(X) \to \Omega^{odd}(X)$$

concentrate around the zeros of γ.

This is the localization in E. Witten well-known paper on Morse Theory [W1]. After fixing a Morse function $f \in C^\infty(X)$ setting $D_s = d + d^* + s\hat{c}(df)$, Witten considered the corresponding Laplacian $\Delta_s = D_s^* D_s + D_s D_s^*$ on the space $\Omega^*(X)$ of differential forms on X. He showed that the q-forms are low eigenvectors of Δ_s concentrate at the index q critical points of f as $s \to \infty$. In fact, using the natural $\mathbb{Z}$-grading on $\Omega^*(X)$, Witten is able to prove a refined localization of the low eigenvectors: the low eigenvectors of Δ_s on $\Omega^q(X)$ localize around the critical points of f with index q.

Let $\{p_\ell\}$ be the set of critical points of f and choose one of them $p_{\ell_0} = p$ with index q. In Morse coordinates around p, df has the form $\sum \eta_\alpha\, x^\alpha\, dx^\alpha$ where $\eta_\alpha = 1$ for $\alpha = 1, \dots q$ and $\eta_\alpha = 1$ for $\alpha > q$. Then

$$M_\alpha = -\eta_\alpha c(dx^\alpha)\hat{c}(dx^\alpha) : \Lambda^{\mathrm{ev}} T_p^* X \to \Lambda^{\mathrm{ev}} T_p^* X \forall \alpha$$

are invertible self-adjoint matrices with symmetric spectrum of eigenvalues ± 1 that commute with each other. In particular if φ belongs in the $+1$- eigenspace of M_α then

$$K_\alpha^+ = \{c(dx^I)\varphi \;:\; |I| = \text{ even } \alpha \notin I\} \quad \text{and} \quad K_\alpha^- = \{c(dx^I)\varphi \;:\; |I| = \text{ even } \alpha \in I\}.$$

Lemma 2.1.4.
- *At each positive p_ℓ, $\mathcal{K}_\ell \cong \mathbb{R}$ and $\hat{K}_\ell = 0$, and*
- *At each negative p_ℓ, $\mathcal{K}_\ell = 0$ and $\hat{K}_\ell \cong \mathbb{R}$.*
- *If p_ℓ has index q_ℓ then a basis element for $\mathcal{K}_\ell \oplus \hat{\mathcal{K}}_\ell$ is a q_ℓ-form.*

Proof. The proof of the first two bullets is a verbatim of the proof of Lemma 2.3.7.

For the last bullet, suppose p is a positive zero of index q and $\varphi \in \mathcal{K}$ is a basis vector. φ is an even form and we use the notation $\alpha \in \varphi$ to denote that when $\varphi = \varphi_I dx^I$ then $\alpha \in I$. Using (2.1.5)

$$\begin{aligned}
\varphi \;&=\; M_\alpha \varphi \;=\; -\eta_\alpha c(dx^\alpha)\hat{c}(dx^\alpha)\varphi \;=\; -\eta_\alpha (dx^\alpha \wedge (\iota_{\partial_\alpha}\varphi) \;-\; \iota_{\partial_\alpha}(dx^\alpha \wedge \varphi) \\
&=\; \eta_\alpha(\varphi \;-\; 2dx^\alpha \wedge (\iota_{\partial_\alpha}\varphi)) \;=\; \begin{cases} -\eta_\alpha\varphi, & if\, \alpha \in \varphi \\ \eta_\alpha\varphi, & if \alpha \notin \varphi \end{cases}
\end{aligned}$$

where in the fourth equality we used the Cartan's identity. Hence we must have that $\alpha \in \{1, \ldots, q\}$ if and only if $\alpha \in \varphi$ i.e. φ has to be a q- form. A similar calculation shows φ to be a q-form when p is a negative zero. $\square$

The Index Localization Theorem then shows the well-known fact that the index of $d + d^* : \Omega^{ev}(X) \to \Omega^{odd}(X)$ is the Euler characteristic $\chi(X)$. Witten further showed how the Morse flow gives rise to "tunneling" maps between the spaces of low eigenvectors, and how this data enables one to compute the total homology of the manifold.

2.2 Clifford Pairs

Examples 1-3 can be extended and placed in a general context by working with Clifford algebra bundles. A bundle $W \to X$ is called a *Clifford algebra bundle* if it is equipped with a a bundle map $c : Cl(T^*X) \to \mathrm{End}(W)$ that is an algebra homomorphism, meaning that it satisfies the Clifford relation (1.0.4). For each connection ∇ on W, there is an associated Dirac operator $D = c \circ \nabla$ on $\Gamma(W)$ whose symbol is c. This section shows how interesting examples arise by taking W to be the direct sum of two Clifford bundles associated with different representations of the groups $\mathrm{Spin}(n)$ or $\mathrm{Spin}^c(n)$.

To describe the general context, let (E, c) and (E', c') be two Clifford algebra bundles on (X, g) with connection and with corresponding Dirac operators D and D'. Suppose there is a bundle map $\mathcal{P} : E' \to E$; one can then consider the diagram

$$\begin{tikzcd} E \arrow[r, "c(v)"] & E \\ E' \arrow[u, "\mathcal{P}"] \arrow[r, "c'(v)"] & E' \arrow[u, "\mathcal{P}"'] \end{tikzcd} \tag{2.2.1}$$

for each $v \in T^*X$. Then the perturbed operator

$$\mathcal{D}_s = \mathcal{D} + s\,\mathcal{A} : \Gamma(E \oplus E') \to \Gamma(E \oplus E')$$

with

$$\mathcal{D} \;=\; \begin{pmatrix} D & 0 \\ 0 & D' \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} \;=\; \begin{pmatrix} 0 & \mathcal{P} \\ -\mathcal{P}^* & 0 \end{pmatrix}$$

satisfies the concentration principle if and only if Diagram (2.2.1) is commutes for every $v \in T^*X$. Furthermore, if E and E' are reducible Clifford bundles, then one can restrict $\mathcal{D}_s$ to sub-bundles to produce additional examples of concentrating pairs.

The examples in this section are special cases in which we take E and E' to be of the form $W \otimes \Lambda^*(T^*X)$ where W is a bundle of spinors. We next describe this setup, beginning with some linear algebra.

Let Δ be the fundamental Spin^c representation of the group Spin$^c(n)$; Δ is irreducible for n odd and the sum $\Delta^+ \oplus \Delta^-$ of two irreducible representations for n even. Clifford multiplication is a linear map $c : \mathbb{R}^n \to \mathrm{End}_{\mathbb{C}}(\Delta)$; we will often use Hitchin's "lower dot" notation

$$v.\varphi := c(v)(\varphi).$$

There is also a Clifford algebra map $\hat{c} : \mathbb{R}^n \to \mathrm{End}\,(\Lambda^*\mathbb{R}^n)$ given by

$$\hat{c}(v) := \sigma_{d+d^*}(v) = (v \wedge \cdot) - \iota_v(\cdot).$$

Lemma 2.2.1. *Clifford multiplication extends to a* $\mathrm{Spin}^c(n)$*-equivariant linear map* $c : \Lambda^*\mathbb{R}^n \to \mathrm{End}_{\mathbb{C}}(\Delta)$ *that satisfies*

$$v.b.\psi = (\hat{c}(v)b).\psi \qquad \textit{for all } v \in \mathbb{R}^n,\ b \in \Lambda^*\mathbb{R}^n \textit{ and } \psi \in \Delta. \tag{2.2.2}$$

Proof. Define the extension $c : \Lambda^*\mathbb{R}^n \to \mathrm{End}_{\mathbb{C}}(\Delta)$ using the standard basis $\{e^j\}$ of $\mathbb{R}^n$ by

$$c(e^1 \wedge \cdots \wedge e^p) \;=\; e^1_. \ldots e^p_. \tag{2.2.3}$$

for each p-tuple $(i_1, \ldots, i_p)$ with $i_1 < \cdots < i_p$. This map is $\mathrm{Spin}(n)$-equivariant because for every $g \in$ Spin(n), $\eta \in \Lambda^*X$ we have that

$$c(\mathrm{Ad}(g)^*\eta) = g.c(\eta)g^{-1}_.$$

Indeed if $\eta = e^1 \wedge \cdots \wedge e^p$ then according to (2.2.3) and since $\{\mathrm{ad}(g)^*e^i\}$ is also an orthonormal coframe with the same orientation

$$\begin{aligned}
c(\mathrm{Ad}(g)^*\eta)(g.\psi) &= c(\mathrm{Ad}(g)^*e^1)c(\mathrm{Ad}(g)^*e^2)\dots c(\mathrm{Ad}(g)^*e^p)(g.\psi) \\
&= c(\mathrm{Ad}(g)^*e^1)c(\mathrm{Ad}(g)^*e^2)\dots(g.(c(e^p)\psi) \\
&= \dots = g.(c(e^1)c(e^2)\dots c(e^p)\psi \\
&= g.(c(\eta)\psi).
\end{aligned}$$

To verify (2.2.2) note that for all k, l

$$\begin{aligned}
e^l_{\cdot}(e^1\wedge\dots\wedge e^k).\psi &= e^l_{\cdot}e^1_{\cdot}\dots e^k_{\cdot}\psi \\
&= \begin{cases} (e^l\wedge e^1\wedge\dots\wedge e^k).\psi, & \text{if } l>k \\ (-1)^l(e^1\wedge\dots\wedge\hat{e^l}\wedge\dots\wedge e^k).\psi & \text{if } 1\le l\le k \end{cases} \\
&= \begin{cases} (e^l\wedge e^1\wedge\dots\wedge e^k).\psi, & \text{if } l>k \\ -(\iota_{e^l}(e^1\wedge\dots\wedge e^k)).\psi, & \text{if } 1\le l\le k \end{cases} \\
&= \hat{c}(e^l)(e^1\wedge\dots\wedge e^k).\psi
\end{aligned}$$

□

Because of $\mathrm{Spin}^c(n)$-equivariance, the map of Lemma 2.2.1 globalizes. Let (X,g) be an oriented Riemannian n-manifold with a Spin^c bundle W and Hermitian metric $\langle\cdot,\cdot\rangle$ conjugate linear in the second factor and determinant bundle $L=\det_{\mathbb{C}}(W)$. Clifford multiplication defines bundle maps

$$c : \Lambda^* T^* X \to \mathrm{End}_{\mathbb{C}}(W) \qquad \text{and} \qquad \hat{c} : T^* X \to \mathrm{End}\left(\Lambda^* T^* X\right) \tag{2.2.4}$$

that satisfy (2.2.2). Given a Hermitian connection A on L with curvature F_A we get an induced spin covariant derivative ∇^A on W compatible with the Levi-Civita connection ∇ on T^*X and a Dirac operator D_A on W.

Example 4: *Spinor-form pairs*

In the above context, consider the map

$$P : W \to \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^* T^*_{\mathbb{C}} X,\, W\Big) \;:\; \psi \mapsto c(\cdot)\psi. \tag{2.2.5}$$

For ψ a spinor on W we consider the operator

$$\mathcal{D}_s = \mathcal{D} + s\,\mathcal{A}_\psi : \Gamma(W \oplus \Lambda^*_{\mathbb{C}} T^* X) \to \Gamma(W \oplus \Lambda^*_{\mathbb{C}} T^* X)$$

with

$$\mathcal{D} \;=\; \begin{pmatrix} D_A & 0 \\ 0 & d + d^* \end{pmatrix} \qquad \text{and} \qquad \mathcal{A}_\psi \;=\; \begin{pmatrix} 0 & P_\psi \\ -{P_\psi}^* & 0 \end{pmatrix}$$

where ${P_\psi}^*$ denotes the complex adjoint of P_ψ.

Lemma 2.2.2. $(\sigma_{\mathcal{D}}, \mathcal{A}_\psi)$ *is s concentrating pair.*

Proof. The symbol of $\mathcal{D}$, applied to a covector v, and his adjoint are given by

$$\sigma_{\mathcal{D}}(v) \;=\; \begin{pmatrix} c(v) & 0 \\ 0 & \hat{c}(v) \end{pmatrix} \qquad \text{and} \qquad \sigma_{\mathcal{D}}(v)^* \;=\; \begin{pmatrix} -c(v)(\cdot) & 0 \\ 0 & -\hat{c}(v) \end{pmatrix}$$

Formula (2.2.2) expresses the fact that the diagram

$$\begin{tikzcd} W \arrow[r, "c(v)"] & W \\ \Lambda^* X \arrow[u, "P_\psi"] \arrow[r, "\hat{c}(v)"] & \Lambda^* X \arrow[u, "P_\psi"'] \end{tikzcd}$$

commutes for every $v \in T^*X$, which means that $(\sigma_{\mathcal{D}}, \mathcal{A}_\psi)$ is s concentrating pair. $\square$

Unfortunately, Lemma 2.2.2 does not automatically mean that the theorems in the introduction apply to general spinor-form pairs. The difficulty is seen when one examines the singular set

$$Z_{\mathcal{A}} \;=\; \{x \in X \mid \ker P_\psi \neq 0\}.$$

The dimension of the exterior algebra $\Lambda^*(\mathbb{R}^n)$ is 2^n, and the fundamental representation of $Spin(n)$ has complex dimension $2^{[\frac{n}{2}]}$ (see the chart). Thus if whenever $\dim X > 2$, every map $P_\psi : \Lambda^*(T^*X) \to W$ has a non-trivial kernel at each point, so $Z_{\mathcal{A}}$ is all of X.

n	2	3	4	5	6	7
$\dim_{\mathbb{R}} \Lambda^*(\mathbb{R}^n)$	4	8	16	32	64	128
$\dim_{\mathbb{R}} W$	4	4	8	8	16	16

Table 2.1: Dimension count

To avoid this difficulty, we look for sub-bundles $\mathcal{L}$ of $\mathrm{Hom}(\Lambda^*(T^*X), W)$ as in diagram (1.0.7). One way to obtain such sub-bundles is via bundle involutions.

Suppose that $T = \begin{pmatrix} \tau & 0 \\ 0 & \hat{\tau} \end{pmatrix}$ is a metric invariant bundle involution on $E \oplus F$ so that

$$\sigma_D(v)\tau \;=\; \pm\,\hat{\tau}\sigma_D(v) \quad \text{and} \quad \nabla T = 0 \tag{2.2.6}$$

for every covector v. Let $E = E^+ \oplus E^-$ and $F = F^+ \oplus F^-$ be the decompositions into ± 1 eigenspaces of τ and $\hat{\tau}$ with $p^\pm = \frac{1}{2}(1_E \pm \tau) : E \to E^\pm$ and $\hat{p}^\pm = \frac{1}{2}(1_F \pm \hat{\tau}) : F \to F^\pm$ the corresponding projections. Set

$$D^+ = \sigma_D \circ p^+ \nabla|_{E^+} \qquad \text{and} \qquad A^+ = \hat{p}^\pm A|_{E^+}$$

the restrictions of D and A to E^+ with values in F^+ or F^- depending on the sign of (2.2.6).

Lemma 2.2.3. *If D_s satisfies the concentration condition (1.0.1), then so does*

$$D_s^+ = D^+ + sA^+ : \Gamma(E^+) \to \Gamma(F^\pm). \tag{2.2.7}$$

Proof. The operator $p^+\nabla|_{E^+}$ defines a metric compatible connection on sections of $E^+ \to X$. Also

$$(A^+)^*\sigma_D(v)|_{E^+} + (\sigma_D(v)|_{E^+})^*A^+ = p^+(A^*\sigma_D(v) + \sigma_D^*(v)A)|_{E^+} = 0$$

for every $v \in T^*X$. $\square$

In the examples below, we will build involutions by combining three bundle maps. All three are defined when X is an oriented Riemannian n-manifold.

- The parity operator $(-1)^p$ that is $(-1)^p Id$ on p-forms.
- The Hodge star operator, which satisfies $*^2 = (-1)^{p(n-p)}$.
- Clifford multiplication by the volume form $d\mathrm{vol}$, which satisfies $(d\mathrm{vol}.)^2 = (-1)^{[n/2]}$.

The parity involution. When $\dim X = 2n$ is even, the endomorphism

$$\tau = \hat{\tau} = i^n d\mathrm{vol}. \oplus (-1)^{p+1} \in \mathrm{End}(W \oplus \Lambda^*_{\mathbb{C}} X)$$

is an involution; its ± 1 eigenbundles are

$$E^+ = W^+ \oplus \Lambda^{odd}_{\mathbb{C}} X \qquad \text{and} \qquad F^+ = W^- \oplus \Lambda^{ev}_{\mathbb{C}} X,$$

and $\sigma_{\mathcal{D}^*}(v)\tau = -\tau\sigma_{\mathcal{D}}(v)$. Furthermore, the restriction of (2.2.5) decomposes as

$$W^+ \to \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^{ev} T^*_{\mathbb{C}} X,\, W^+\Big) \oplus \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^{odd} T^*_{\mathbb{C}} X,\, W^-\Big).$$

Thus for any $\psi \in \Gamma(W^+)$, we can write $P_\psi = P^{ev}_\psi + P^{odd}_\psi$ under this decomposition, and set

$$\mathcal{A}^+_\psi = \begin{pmatrix} 0 & P^{odd}_\psi \\ -P^{ev*}_\psi & 0 \end{pmatrix}. \qquad (2.2.8)$$

Then by Lemma 2.2.3 the operator

$$\mathcal{D}_s^+ = \mathcal{D}^+ + s\,\mathcal{A}_\psi^+ : \Gamma(W^+ \oplus \Lambda_{\mathbb{C}}^{odd} X) \to \Gamma(W^- \oplus \Lambda_{\mathbb{C}}^{ev} X)$$

satisfies the concentration condition (1.0.1).

The self-duality involution. In even dimensions there is a second self-duality involution on the bundles $W^\pm \oplus \Lambda_{\mathbb{C}}^{odd/ev} X$ namely $\tau = Id \oplus *$. The self-duality involution preserves the eigenspaces of the parity involution and the two involutions commute. Then $\sigma_{\mathcal{D}^*}(v)\tau = \tau\sigma_{\mathcal{D}}(v)$ and $\mathcal{A}_\psi^{+*}\tau = \tau^*\mathcal{A}_\psi^+$ for $\psi \in W^+$ in this case. Hence

$$\mathcal{D}^{++} + s\mathcal{A}_\psi^{++} : \Gamma(E) \to \Gamma(F) \tag{2.2.9}$$

satisfies the concentration condition (1.0.1), where

$$E = W^+ \oplus \left(\sum_{p=1}^{k} \Lambda_{\mathbb{C}}^{2p-1} X \right) \qquad \text{and} \qquad F = W^- \oplus \left(\Lambda_{\mathbb{C}}^{2k,+} X \oplus \sum_{p=0}^{k-1} \Lambda_{\mathbb{C}}^{2p} X \right).$$

when $\dim X = 4k$, and

$$E = W^+ \oplus \left(\Lambda_{\mathbb{C}}^{2k+1,+} X \oplus \sum_{p=1}^{k} \Lambda_{\mathbb{C}}^{2p-1} X \right) \qquad \text{and} \qquad F = W^- \oplus \left(\sum_{p=0}^{k} \Lambda_{\mathbb{C}}^{2p} X \right).$$

when $\dim X = 4k+2$. In the next section, we will use these involutions to construct spinor-form pairs that display the concentrating property with a non-trivial singular set $Z_{\mathcal{A}}$.

2.3 Self-dual spinor-form pairs in dimension 4

When X is an oriented Riemannian 4-manifold, the self-duality involution produces a Dirac operator (2.2.9) with the concentration property and with a singular set $Z_{\mathcal{A}}$ that, we will show next, is not all of X.

In dimension four, the self-dual spinor-form spaces from section 2.2 are

$$E = W^+ \oplus \Lambda^1 X \quad \text{and} \quad F = W^- \oplus (\Lambda^0 \oplus \Lambda^{2,+} X)$$

and the concentrating pair is given by

$$\sigma_{\mathcal{D}}(v) = \begin{pmatrix} c(v) & 0 \\ 0 & \hat{c}(v) \end{pmatrix} \quad \text{and} \quad \mathcal{A}_\psi = \begin{pmatrix} 0 & P_\psi^{odd} \\ -P_\psi^{ev*} & 0 \end{pmatrix}$$

where $\hat{c}(v)$ is the symbol map of the Dirac operator $\sqrt{2}d^+ + d^*$. In order for the diagram

$$\begin{tikzcd} W^+ \arrow[r, "c(v)"] & W^- \\ \Lambda^0 \oplus \Lambda^{2,+} \arrow[u, "P_\psi^{ev}"] \arrow[r, "\hat{c}(v)"] & \Lambda^1 \arrow[u, "P_\psi^{odd}"'] \end{tikzcd}$$

to commute and the concentration condition to hold, we have to slightly modify $\mathcal{A}_\psi$ from (2.2.8) by defining

$$P_\psi^{odd} : \Lambda^1 \to W^-, \quad b \mapsto b.\psi \qquad \text{and} \qquad P_\psi^{ev} : \Lambda^0 \oplus \Lambda^{2,+}_{\mathbb{C}} \to W^+ \quad (\rho, \theta) \mapsto (\rho + \tfrac{1}{\sqrt{2}}\theta).\psi.$$

Lemma 2.3.1. *W^+ is a Clifford bundle for the bundle of Clifford algebras $Cl(\Lambda^{2,+}(X))$.*

Proof. It suffices to show this at a point $p \in X$. Let now $\{e^i\}$ be an orthonormal coframe and define $\Lambda^{2,+}(X) = \mathrm{span}\{\eta_0,\, \eta_1,\, \eta_2\}$ where

$$\eta_0 \;=\; \tfrac{1}{\sqrt{2}}(e^1\wedge e^2 + e^3\wedge e^4), \qquad \eta_1 \;=\; \tfrac{1}{\sqrt{2}}(e^1\wedge e^3 + e^4\wedge e^2), \qquad \eta_2 \;=\; \tfrac{1}{\sqrt{2}}(e^1\wedge e^4 + e^2\wedge e^3)$$

are orthonormal. Note that

$$\eta_i.\eta_j. \;+\; \eta_j.\eta_i. \;=\; 2\langle \eta_i,\, \eta_j\rangle(d\mathrm{vol}. \;-\; Id_W) \tag{2.3.1}$$

for every i, j and so the same identity holds for every other two forms in $\Lambda^{2,+}(X)$. Restricted to W^+ we get

$$\eta.\theta. \;+\; \theta.\eta. \;=\; -4\langle \eta,\, \theta\rangle Id_{W^+} \tag{2.3.2}$$

for every $\eta,\, \theta \,\in\, \Lambda^{2,+}(X)$ which is an analog of the Clifford relation for the self dual 2-forms acting on W^+. This finishes the proof. $\square$

Remark 2.3.2. Choose $\psi \in W^+\backslash\{0\}$. Then from the above proof it follows that the set $\{\psi, \eta_k.\psi\} \subset W^+$ is orthogonal and $\eta_k.\eta_k.\psi = -2\psi$ which implies $|\eta_k.\psi|^2 = 2|\psi|^2$. Both E and F are 8-dimensional real vector bundles. The volume form acts on $\xi = \begin{pmatrix} \varphi \\ b \end{pmatrix} \in E_p$ by $d\mathrm{vol}.\xi = \begin{pmatrix} -\varphi \\ b \end{pmatrix}$. Hence choosing $|\varphi| = |b| = \frac{1}{\sqrt{2}}$, the Clifford action produces an orthonormal basis for E_p and F_p so that

$$E_p = \operatorname{span}\{e^I_\cdot\xi : I \,\text{even string}\} \quad \text{and} \quad F_p = \operatorname{span}\{e^J_\cdot\xi : J \,\text{odd string}\}.$$

Regard now W^+ as a real vector bundle of rank 4 with the induced metric. By considering the negative definite quadratic form produced by that metric we can form the algebra bundle $Cl^{0,4}(W^+)$. The perturbation $\mathcal{A}_\psi$ enjoys the following property:

Lemma 2.3.3. *The map* $W^+ \to \operatorname{End}(E \oplus F) : \psi \mapsto \begin{pmatrix} 0 & \mathcal{A}_\psi^* \\ \mathcal{A}_\psi & 0 \end{pmatrix}$ *defines a representation of the real Clifford algebra bundle* $Cl^{0,4}(W^+)$ *on* E.

Proof. Fix $\psi \in W_p^+$. By the Clifford relations, the sets $\{e^k_\cdot\psi\} \subset W^-$ and $\{\psi, \eta_k.\psi\} \subset W^+$ are orthogonal. Therefore for $b \in \Lambda^1_{\mathbb{C}}$,

$$P_\psi^{odd*} \circ P_\psi^{odd}(b) = b_l P_\psi^{odd*} \circ P_\psi^{odd}(e^l) = \Re\langle e^l_\cdot\psi,\, e^k_\cdot\psi\rangle b_l e^k = |\psi|^2 b$$

and similarly for $(\rho, \theta) \in \Lambda^0_{\mathbb{C}} \oplus \Lambda^{2,+}_{\mathbb{C}}$

$$\begin{aligned} P_\psi^{ev*} \circ P_\psi^{ev}(\rho,\theta) &= \rho|\psi|^2 + \frac{1}{\sqrt{2}}\theta_l\langle\eta_l.\psi,\psi\rangle + \frac{1}{\sqrt{2}}\rho\langle\psi,\eta_k.\psi\rangle\eta_k + \frac{1}{2}\theta_l\langle\eta_l.\psi,\eta_k.\psi\rangle\eta_k \\ &= \rho|\psi|^2 + \frac{1}{2}\langle\eta_k.\psi,\eta_k.\psi\rangle\theta_k\eta_k \\ &= |\psi|^2(\rho,\theta). \end{aligned}$$

This proves that

$$\mathcal{A}_\psi^* \circ \mathcal{A}_\psi \;=\; |\psi|^2 Id_E \qquad \text{and} \qquad \mathcal{A}_\psi \circ \mathcal{A}_\psi^* \;=\; |\psi|^2 Id_F.$$

Finally, polarization gives the relations

$$\mathcal{A}_{\psi_1}^* \circ \mathcal{A}_{\psi_2} + \mathcal{A}_{\psi_2}^* \circ \mathcal{A}_{\psi_1} \;=\; 2\Re\langle \psi_1,\, \psi_2\rangle Id_E$$

and

$$\mathcal{A}_{\psi_1} \circ \mathcal{A}_{\psi_2}^* + \mathcal{A}_{\psi_2} \circ \mathcal{A}_{\psi_1}^* \;=\; 2\Re\langle \psi_1,\, \psi_2\rangle Id_F$$

for every $\psi_1, \psi_2 \in W^+$. Hence get a well defined *algebra* map

$$Cl^{0,4}(W^+) \to \mathrm{End}(E \oplus F) : \, \psi \mapsto \begin{pmatrix} 0 & \mathcal{A}_\psi^* \\ \mathcal{A}_\psi & 0 \end{pmatrix}.$$

The result follows. $\square$

Corollary 2.3.4. *The mapping $\psi \mapsto \mathcal{A}_\psi$ defines an injection $W^+ \to \mathrm{Isom}(E, F)$.*

Proof. Let $\xi \in E$. Then by Lemma 2.3.3

$$|\mathcal{A}_\psi \xi|^2 \;=\; |\xi|^2 |\psi|^2.$$

implying that if $\xi \in \ker \mathcal{A}_\psi$ is nontrivial then $\psi = 0$. Therefore $\psi \neq 0$ if and only if $\mathcal{A}_\psi$ is non singular which implies the corollary. $\square$

Example 5: We would like to the study the operator $\mathcal{D} + s\mathcal{A}_\psi$. Choosing a transverse section $\psi : X \to W^+$ the singular set of $\mathcal{A}_\psi$ will be a finite set of oriented points.

Let $p \in X$ be such a point and let a coordinate chart $(U, \{x_\alpha\})$ with $x_\alpha(p) = 0$ and tangent frame and coframe $\{e_\alpha\}$ and $\{e^\alpha\}$ respectively. Expanding we get sections

$$\psi(x) = x_\alpha \psi_\alpha + O(|x|^2)$$

for some elements $\psi_\alpha \in W_p^+$. Extend these smoothly to sections, still called ψ_α, of W^+ near p. By transversality at p we have $\sum x_\alpha \psi_\alpha \neq 0$ for all $x \neq 0$. Setting

$$A_\alpha := \nabla_{e_\alpha} \mathcal{A}_\psi = \mathcal{A}_{\psi_\alpha},$$

we see that

$$x_\alpha A_\alpha : K_p = \ker(\mathcal{A}_{\psi(p)}) \to \operatorname{coker}(\mathcal{A}_{\psi(p)}) = \hat{K}_p \tag{2.3.3}$$

is an isomorphism for every $x \in T_pX - \{0\}$.

The following technical lemma assures that $\mathcal{A}_\psi$ can be perturbed to satisfy the non-degeneracy assumption (1.0.9).

Lemma 2.3.5. *We can modify ψ without changing its zero set $Z(\psi)$ to insure that $\{\psi_\alpha\}$ are orthonormal.*

Proof. Let $H : W_p^+ \to W_p^+$ be a real orientation preserving linear isomorphism with eigenvalues $\{\mu_\alpha\}$ such that $\{H\psi_\alpha\}$ is an orthonormal basis for W^+. We may assume that

H has no real eigenvalues otherwise replace H by λH where $\lambda \in S^1$ such that none of the numbers $\{\lambda\mu_\alpha\}$ is real. As a consequence there exist constant $C > 0$ such that

$$\inf_{(t,x)\in\mathbb{R}\times S^3} \Big| tH(x_\alpha\psi_\alpha) + (1-t)x_\alpha\psi_\alpha \Big| > C.$$

Let $B(0,2R) \subset U$ and ρ be a smooth cutoff function with $\mathrm{supp}(\rho) \subset B(0,2R)$ and $\rho|_{B(0,R)} \equiv 1$. We redefine ψ in $(U,\, x)$ as

$$\Psi(x) := \psi(x) + \rho(x)(H - Id)(x_\alpha\psi_\alpha) = \rho(x)(H - Id)(x_\alpha\psi_\alpha) + x_\alpha\psi_\alpha + O(|x|^2).$$

where $|O(|x|^2)| \le C_1|x|^2$ for $x \in U$. Clearly Ψ has a transverse zero at p and satisfies the conclusion of the lemma at p. Chossing $R < \frac{C}{2C_1}$ we get that for $x \in B(0,2R)\backslash\{0\}$

$$\frac{|O(|x|^2)|}{|x|} \le 2C_1R < C < \Big|\rho(x)H(\frac{x_\alpha}{|x|}\psi_\alpha) + (1-\rho(x))\frac{x_\alpha}{|x|}\psi_\alpha\Big|$$

therefore there are no other zeros of Ψ in U except at p. Repeating this proccess for each of the finitely many zeros of the original ψ we are done. $\square$

Recall the matrices $M_\alpha = -e^\alpha_\cdot A_\alpha \in \mathrm{End}(K_p)$ and $\hat{M}_\alpha = -e^\alpha_\cdot A^*_\alpha \in \mathrm{End}(\hat{K}_p)$. Let $K^\pm_\alpha$ and $\hat{K}^\pm_\alpha$ be the positive/negative eigenspaces of M_α and $\hat{M}_\alpha$ respectively. We are interested in describing their common positive eigenspaces

$$\mathcal{K}_p = \bigcap_\alpha K^+_\alpha \quad \text{and} \quad \hat{\mathcal{K}}_p = \bigcap_\alpha \hat{K}^+_\alpha.$$

Lemma 2.3.6. *The eigenvalues of M_α are $\lambda_\alpha = \pm 1$ and the corresponding eigenspaces* *can be described as*

$$K_\alpha^+ = \mathrm{span}\left\{ \begin{pmatrix} e^\alpha_\cdot b.\psi_\alpha \\ \\ b \end{pmatrix} : b \in \Lambda^1 X \right\} \quad \textit{and} \quad K_\alpha^- = \mathrm{span}\left\{ \begin{pmatrix} -e^\alpha_\cdot b.\psi_\alpha \\ \\ b \end{pmatrix} : b \in \Lambda^1 X \right\}$$

for every α.

Proof. By relation (1.0.1) and Corollary 2.3.4

$$M_\alpha^2 = e^\alpha_\cdot A_\alpha e^\alpha_\cdot A_\alpha = A_\alpha^* A_\alpha = Id$$

i.e. M_α has eigenvalues ± 1. Let now $\varphi = \begin{pmatrix} \xi \\ \\ b \end{pmatrix} \in K_p$ is a λ_α- eigenvector of M_α. Then

$$M_\alpha \varphi = -e^\alpha_\cdot A_\alpha \varphi = -\begin{pmatrix} -c(e^\alpha) & 0 \\ \\ 0 & -\hat{c}(e^\alpha) \end{pmatrix} \begin{pmatrix} b.\psi_\alpha \\ \\ -P^{ev*}_{\psi_\alpha} \xi \end{pmatrix} = \lambda_k \begin{pmatrix} \xi \\ \\ b \end{pmatrix}. \tag{2.3.4}$$

By comparing the first rows of (2.3.4) we see that

$$\xi = \frac{1}{\lambda_\alpha} e^\alpha_\cdot b.\psi_\alpha = \lambda_\alpha e^\alpha \cdot b.\psi_\alpha \tag{2.3.5}$$

since $\lambda_\alpha^2 = 1$. It remains to show that given $b \in \bigwedge^1 X$, the above choice of ξ gives equality of second rows of (2.3.4). Using (2.2.4)

$$\begin{aligned}
-\hat{c}(e^{\alpha})P^{ev*}_{\psi_\alpha}\xi &= (P^{ev}_{\psi_\alpha}\hat{c}(e^{\alpha}))^*\lambda_\alpha e^{\alpha}_{\cdot} b.\psi_\alpha \;=\; \lambda_\alpha(b.e^{\alpha}_{\cdot} P^{ev}_{\psi_\alpha}\hat{c}(e^{\alpha}))^*\psi_\alpha \\
&= -\lambda_\alpha(P^{ev}_{\psi_\alpha}\hat{c}(b))^*\psi_\alpha \\
&= \lambda_\alpha\hat{c}(b)P^{ev*}_{\psi_\alpha}\psi_\alpha.
\end{aligned}$$

Also for every $\eta \in \Lambda^0 X \oplus \Lambda^{2,+}X$

$$\Re\langle \eta,\, P^{ev*}_{\psi_\alpha}\psi_\alpha\rangle \;=\; \Re\langle \eta.\psi_\alpha,\, \psi_\alpha\rangle \;=\; \begin{cases} 0 & \text{if}\, \eta \in \Lambda^{2,+}X \\ \eta & \text{if}\, \eta \in \Lambda^0 X \end{cases}$$

showing that $P^{ev*}_{\psi_\alpha}\psi_\alpha = 1$. Hence

$$\lambda_\alpha\hat{c}(b)P^{ev*}_{\psi_\alpha}\psi_\alpha \;=\; \lambda_\alpha\hat{c}(b)1 \;=\; \lambda_\alpha b$$

proving equality of the second rows of (2.3.4). $\square$

In Chapter 4 we prove that the families $\{\hat{M}_\alpha\}$ and $\{M_\alpha\}$ are related as

$$e^I_{\cdot} M_\alpha = -\hat{M}_\alpha e^I_{\cdot} \quad \text{if} \quad \alpha \in I \qquad \text{and} \qquad e^I_{\cdot} M_\alpha = \hat{M}_\alpha e^I_{\cdot} \quad \text{if} \quad \alpha \notin I \tag{2.3.6}$$

for every string I of odd length. It follows

$$e^I_{\cdot} \in \mathrm{Hom}(K^{\pm}_\alpha,\, \hat{K}^{\mp}_\alpha) \quad \text{if} \quad \alpha \in I \quad \text{and} \qquad e^I_{\cdot} \in \mathrm{Hom}(K^{\pm}_\alpha,\, \hat{K}^{\pm}_\alpha) \quad \text{if} \quad \alpha \notin I.$$

Lemma 2.3.7. *The spaces $\bigcap_\alpha K_\alpha^+$ and $\bigcap_\alpha \hat{K}_\alpha^+$ are at most one dimensional. If $\nabla\psi : T_pX \to W_p^+$ preserves orientation then $\mathcal{K}_p = \bigcap_\alpha K_\alpha^+$ is non trivial and $\hat{\mathcal{K}}_p = \bigcap_\alpha \hat{K}_\alpha^+ = \{0\}$. If $\nabla\psi$ reverses orientation then $\bigcap_\alpha K_\alpha^+ = \{0\}$ and $\bigcap_\alpha \hat{K}_\alpha^+$ is nontrivial.*

Proof. Let $\varphi \in \bigcap_\alpha K_\alpha^+$. Then for every even string I and $\alpha \in I$

$$M_\alpha e^I_\cdot \varphi = -e^I_\cdot \varphi$$

which implies that $e^I_\cdot \varphi \in K_\alpha^-$. By Remark 2.3.2 $E_p = \operatorname{span}\{e^I\varphi : I = \text{even}\}$ therefore $\bigcap_\alpha K_\alpha^+ = \langle\varphi\rangle$ is at most one dimensional. The case with $\bigcap_\alpha \hat{K}_\alpha^+$ is analogous.

Suppose now that J is a string and $\varphi \in \left(\bigcap_{\alpha\in J} K_\alpha^-\right) \cap \left(\bigcap_{\alpha\in J^c} K_\alpha^+\right)$ is a nontrivial vector.

- J is an even string if and only if

$$M_\alpha(e^J_\cdot\varphi) = \begin{cases} e^J_\cdot M_\alpha\varphi = e^J_\cdot\varphi & \text{if } \alpha \notin J \\ -e^J_\cdot M_\alpha\varphi = e^J_\cdot\varphi & \text{if } \alpha \in J \end{cases}$$

  for every α so that $\bigcap_\alpha K_\alpha^+ = \langle e^J_\cdot\varphi\rangle$.

- J is an odd if and only if

$$\hat{M}_\alpha(e^J_\cdot\varphi) = \begin{cases} e^J_\cdot M_\alpha\varphi = e^J_\cdot\varphi & \text{if } \alpha \notin J \\ -e^J_\cdot M_\alpha\varphi = e^J_\cdot\varphi & \text{if } \alpha \in J \end{cases}$$

  for every α so that $\bigcap_\alpha \hat{K}_\alpha^+ = \langle e^J_\cdot\varphi\rangle$.

This dichotomy shows also that either $\bigcap_\alpha K_\alpha^+$ or $\bigcap_\alpha \hat{K}_\alpha^+$ should be nontrivial at each zero of ψ. Say that $\alpha \sim \beta$ iff $\alpha,\, \beta \in J$ or $\alpha,\, \beta \in J^c$. By Lemma (2.3.6) if $\alpha \sim \beta$ we can write $\varphi = \begin{pmatrix} \pm e^\alpha_{\cdot} b.\psi_\alpha \\ \\ b \end{pmatrix} = \begin{pmatrix} \pm e^\beta_{\cdot} b.\psi_\beta \\ \\ b \end{pmatrix}$ for some common $b \in \Lambda^1 X$ and if $\alpha \not\sim \beta$ then $\varphi = \begin{pmatrix} e^\alpha_{\cdot} b.\psi_\alpha \\ \\ b \end{pmatrix} = \begin{pmatrix} -e^\beta_{\cdot} b.\psi_\beta \\ \\ b \end{pmatrix}$ for the same $b \in \Lambda^1 X$. In particular we have a description of the orthonormal basis $\{\psi_\alpha\}$ in terms of ψ_1 and b as

$$\psi_\alpha = \begin{cases} b.e^\alpha_{\cdot} e^1_{\cdot} b.\psi_1 & \text{if}\, \alpha \sim 1 \\ \\ -b.e^\alpha_{\cdot} e^1_{\cdot} b.\psi_1 & \text{if}\, \alpha \not\sim 1 \end{cases}.$$

But $|J| + |J^c| = 4$ hence J, J^c are both even or both odd. Therefore $\{\psi_\alpha\}$ is positively oriented in W_p^+ for J even and negatively oriented for J odd. $\square$

Corollary 2.3.8. *As a consequence of Spectral Separation Theorem the index of* $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ *is the signed count of the zeros of* ψ *i.e.*

$$\operatorname{index} \mathcal{D} \;=\; c_2(W^+)[X]$$

the second Chern class of the bundle W^+ *evaluated on the fundamental class of* X*.*

Example 6: J-holomorphic curves in symplectic four-manifolds.

Recall the philosophy of Diagram 1.0.7: if we can find a sub-bundle $\mathcal{L}$ of $\mathrm{Hom}(E, F)$ whose sections satisfy the concentration condition, then we obtain concentrating operators D_s with singular sets $Z_{\mathcal{A}}$ of possibly different dimensions. This example illustrates this phenomenon in dimension four, by showing how a sub-bundle $\mathcal{L}$ can be constructed from a symplectic structure.

Let (X^4, ω) a closed symplectic manifold with a complex hermitian line bundle L and a section $\psi \in \Gamma(L)$ whose zero set is a transverse disjoint union $Z_\psi = \cup_\ell Z_\ell$ of symplectic submanifolds of X. Let N_ℓ be the symplectic normal bundle of Z_ℓ. Choose an almost complex structure J and a Riemannian metric g on X so that (ω, J, g) is a compatible triple and each Z_ℓ is J-holomorphic; the symplectic and metric normal bundles of Z_ℓ are the same.

As usual, write $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ and define the canonical bundle to be the complex line bundle $K = \Lambda^{2,0}X$. In this context, the direct sum $W = W^+ \oplus W^-$ of the complex rank 2 bundles

$$W^+ = L \oplus L\bar{K}, \qquad W^- = L \otimes \Lambda^{0,1}X$$

has a Spin^c structure and a Clifford multiplication $T^*M \otimes W \to W$ from wedging and contracting $(0,1)$ forms as in formula (2.1.2).

Let ∇^L be a hermitan connection on L and ∇^X the Levi-Civita connection on X. These can be used to build a Spin^c connection $\nabla = \nabla^L \oplus \nabla^{L\bar{K}}$ on W^+. There is also the projection to the $(0,1)$ part of T^*X of ∇^L namely $\overline{\partial}_L\psi := \frac{1}{2}(\nabla^L\psi + i\nabla^L\psi \circ J)$. Then a Dirac operator is defined by

$$D = \sqrt{2}(\overline{\partial}_L + \overline{\partial}_L^*) : L \oplus \bar{K}L \to \Lambda^{0,1}X \otimes L.$$

We would like to study the perturbed operator $\mathcal{D} + s\mathcal{A}_\psi$. Fix one component $Z = Z_\ell$ with normal bundle $N = N_\ell$. By the transversality of ψ, the map $\nabla^L\psi : N \to L$ is an $\mathbb{R}$ linear isomorphism.

In order for the nondegeneracy condition (1.0.9) to hold we need the following:

Lemma 2.3.9. *We can change ψ without changing its zero set so that $\nabla^L\psi : N \to L$ becomes orthogonal.*

Proof. We consider the bundles N with the induced metric and $L|_Z$ as a real vector bundle with the induced metric h from the hermitian metric. Let $O(N, L|_Z) = \{H \in \mathrm{Hom}(N, L|_Z) | H^*h = g\}$, a deformation retraction of $\mathrm{Hom}(N, L|_Z)$. Therefore there is a smooth path of bundle maps $[0, 1] \ni t \to H_t \in \mathrm{Hom}(N,\, L|_Z)$ so that $H_0 = \nabla^L\psi$ and $H_1 \in O(N, L|_Z)$. This path can be chosen so that H_t is invertible for every $t \in [0, 1]$. As a consequence there exist constant $C > 0$ such that

$$\inf_{(t,v)\in[0,1]\times S} \Big| H_t(v) \Big| \,>\, C$$

where S is the unit sphere bundle of the normal bundle N.

Now use the exponential map on the normal bundle N of Z to define a tubular neigbroorhood $\mathcal{N}$, a parallel transport map $\tau : L|_Z \to L|_{\mathcal{N}}$ along normal geodesics and set $x = \exp(v)$. Let $B(Z, R) \subset \mathcal{N}$ and ρ be a smooth cutoff function with $\mathrm{supp}(\rho) \subset B(Z, 2R)$ and $\rho|_{B(Z,R)} \equiv 1$. We redefine ψ in $B(Z, 2R)$ as

$$\Psi(v) := \,\psi(v) \,+\, \tau(H_{\rho(v)}(v) \,-\, \nabla^L_v\psi|_Z).$$

Note that

$$|\psi(v) - \tau\nabla_v^L\psi|_Z| = O(|v|^2) \leq C_1|v|^2$$

for $v \in \mathcal{N}$. Clearly Ψ has also transverse intersection with the zero section at Z and satisfies the conclusion of the lemma at Z. Choosing $R < \frac{C}{2C_1}$ we get that for $v \in B(Z, 2R)\backslash\{0\}$

$$\frac{|O(|v|^2)|}{|v|} \leq 2C_1R < C < \left|\tau H_{\rho(v)}(\frac{v}{|v|})\right|$$

therefore there are no other zeros of Ψ in U except at p. Repeating this proccess for each component of the singular set of the original ψ we are done. $\square$

Fix now $p \in Z$ and local coordinates $\{x_i\}$ in X so that $Z = \{x_1 = x_2 = 0\}$ and orthonormal frames $\{e_1, e_2 = J(e_1)\}$ and $\{e_3, e_4 = J(e_3)\}$ trivializing N and TZ respectively around p. By the Lemma $\{\psi_i = \nabla_{e_i}^L\psi\}$ is an orthonormal frame trivializing $L|_Z$ around p and we extend it to the normal directions to a frame trivializing L. Then ψ expands in the normal directions of Z as

$$\psi = x_1\psi_1 + x_2\psi_2 + O(|x|^2).$$

Denote $\nabla_{e_i}\mathcal{A}\psi = \mathcal{A}_{\nabla_i^L\psi} = A_i$.

We now have to consider the matrices $M_\alpha = -e_\cdot^\alpha A_\alpha$, $\alpha = 1, 2$ and their common positive spectrum. By Lemma 2.3.6 the positive eigenspaces are given by

$$K_\alpha^+ = \mathrm{span}\left\{\begin{pmatrix} e_\cdot^\alpha b.\psi_\alpha \\ b \end{pmatrix} : b \in \Lambda^1 X\right\},\ \alpha = 1, 2.$$

There are two cases:

- The map $\nabla^L\psi : N \to L|_Z$ preserves the natural orientations as an $\mathbb{R}$ linear map. Then

$$e_2 = J(e_1) \quad \text{and} \quad \nabla^L_{J(e_1)}\psi = \psi_2 = i\psi_1 = i\nabla^L_{e_1}\psi$$

so that $e^1_{\cdot}\psi_1 + e^2_{\cdot}\psi_2 = D\psi|_Z = \overline{\partial}_L\psi|_Z = 0$. Then

$$\mathcal{K} = K_1^+ \cap K_2^+ = \mathrm{span}\left\{ \begin{pmatrix} \psi_1 \\ e^1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ e^2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \nabla^L_{v^\sharp}\psi \\ v \end{pmatrix} : v \in N^* \right\} \simeq N^*. \quad (2.3.7)$$

Also

$$\hat{\mathcal{K}} = \mathrm{span}\{e^3_{\cdot}\mathcal{K}, e^4_{\cdot}\mathcal{K}\} \simeq \text{ orthogonal complement of } \omega \text{ in } \Lambda^{2,+}X \simeq K^X|_Z \simeq T^*Z \otimes N^*.$$

Hence the local operator is

$$D^Z = \bar{\partial}_{N^*} : \Gamma(N^*) \to \Gamma(T^*Z_i \otimes N^*)$$

and by Riemann-Roch

$$\mathrm{index}\, D^Z = 2N^2 - 2(g-1) = (L|_Z)^2 - 2(g-1) \quad \text{where } g = \text{ genus of } Z.$$

Also if since $\nabla^L\psi|_Z : N \to L|_Z$ preserves orientation then the adjunction formula applies to give

$$2(g-1) = (L|_Z)^2 + L|_Z K.$$

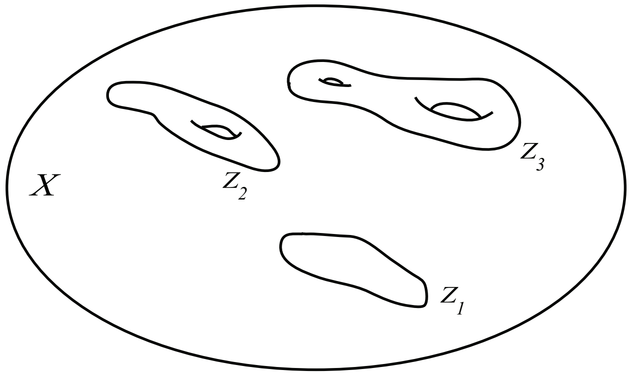

Figure 2.1: The zero set of ψ

Hence $\operatorname{index} D^Z = (L|_Z)^2 - K|_Z \cdot L|_Z$ in this case.

- If $\nabla^L \psi|_Z$ reverses orientation then adjunction formula gives

$$2(g-1) = (\bar{L}|_Z)^2 + K\bar{L}_Z = (L|_Z)^2 - K \cdot L|_Z$$

and a similar calculation computes the operator in this case $D^Z = \bar{\partial}^* : T^*Z \to \mathbb{C}$. By Riemann-Roch we then have

$$\operatorname{index} D^Z = 2(g-1) = (L|Z)^2 - K \cdot L|_Z.$$

Applying the Index Localization Theorem and using the contributions of the local indices from all the components Z_ℓ with $L_\ell = L|_{Z_\ell}$ we get

$$c_2(W^+)[X] = \operatorname{index} D = \sum_\ell (L_\ell^2 - K \cdot L_\ell) = L^2 - KL.$$

This is a familiar formula is SW theory. It describes the dimension of the SW moduli space in terms of the bundled K and L.

Example 7: *Spinor/form pairs twisted by $SU(2)$-bundles.*

Let $(X, g, W^{\pm}, c(\cdot))$ be a Riemmannian manifold with a Spin^c- structure and $(E, h) \to X$ be a Hermitian $SU(2)$ - bundle. Set also $\mathfrak{su}(E) := \{A \in \mathrm{End}_{\mathbb{C}}(E) : A + A^* = 0,\ \mathrm{tr}_{\mathbb{C}} A = 0\}$ where A^* is the Hermitian adjoint of A. Differences of Hermitian connections on E are sections of $\Lambda^1 X \otimes \mathfrak{su}(E)$. Equip E with a Hermitian connection ∇_E and W with a Spin^c connection. We get induced connections $\nabla^{W \otimes E}$ on $W \otimes E$ and ∇ on $\Lambda^* X \otimes \mathfrak{su}(E)$. The symbol maps c and $\hat{c}$ extend as

$$c(v) \otimes id_E : W^+ \otimes E \to W^- \otimes E \quad \text{and} \quad \hat{c}(v) \otimes id_{\mathfrak{su}(E)} : \Lambda^{odd} X \otimes \mathfrak{su}(E) \to \Lambda^{ev} X \otimes \mathfrak{su}(E).$$

Finally we get operators

$$D_E = (c \otimes id_E) \circ \nabla^{W \otimes E} \quad \text{and} \quad d_E + d_E^* = (\hat{c} \otimes id_{\mathfrak{su}(E)}) \circ \nabla.$$

We define a Clifford multiplication c_E to include End(E)- valued forms by

$$\begin{aligned} c_E : \Lambda^*(X) \otimes \mathrm{End}(E) &\to \mathrm{End}(W \otimes E) \\ \eta \otimes A &\mapsto \eta . \otimes A \end{aligned} \tag{2.3.8}$$

The restriction of c_E to the subspace $\Lambda^*(X) \otimes \mathfrak{su}(E)$ defines maps

$$P^{ev} :\ W^+ \otimes E \to \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^{ev}(X) \otimes \mathfrak{su}(E),\, W^+ \otimes E\Big)$$

and

$$P^{odd}: \ W^+ \otimes E \to \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^{odd}(X) \otimes \mathfrak{su}(E),\, W^- \otimes E\Big)$$

both given by $\psi \otimes e \mapsto c_E(\cdot)\psi \otimes e$.

Proposition 2.3.10. *For fixed $\Psi \in W^+ \otimes E$ the perturbed operator*

$$\mathcal{D}_s = \mathcal{D} + s\,\mathcal{A}_\Psi : \Gamma(W^+ \otimes E) \oplus \Omega^{odd}(X, \mathfrak{su}(E)) \to \Gamma(W^- \otimes E) \oplus \Omega^{ev}(X, \mathfrak{su}(E))$$

with

$$\mathcal{D} \ = \begin{pmatrix} D_E & 0 \\ 0 & d_E + d_E^* \end{pmatrix} \qquad \text{and} \qquad \mathcal{A}_{\psi\otimes e} \ = \begin{pmatrix} 0 & P_\Psi^{odd} \\ -P_\Psi^{ev*} & 0 \end{pmatrix}$$

satisfies the concentration relation 1.0.1. Here P_Ψ^{ev*} denotes the adjoint of P_Ψ^{ev}.

Proof. It suffices to show the proposition for $\Psi = \psi \otimes e$. The symbol of $\mathcal{D}$, applied to a covector v, and his adjoint are given by

$$\sigma_{\mathcal{D}}(v) \ = \begin{pmatrix} c(v) \otimes id_E & 0 \\ 0 & \hat{c}(v) \otimes id_{\mathfrak{su}(E)} \end{pmatrix}$$

and

$$\sigma_{\mathcal{D}}(v)^* \ = \begin{pmatrix} -c(v) \otimes id_E & 0 \\ 0 & -\hat{c}(v) \otimes id_{\mathfrak{su}(E)} \end{pmatrix}.$$

Checking the concentration relation is just proving the identity

$$P^{ev}_{\psi\otimes e} \circ (\hat{c}(v) \otimes id_{\mathfrak{su}(E)}) \,=\, (c(v) \otimes id_E) \circ P^{odd}_{\psi\otimes e}.$$

By linearity it is enough to check the identity for $b \otimes B \in \Lambda^{odd} X \otimes \mathfrak{su}(E)$. Then

$$(c(v) \otimes id_E) \circ P^{odd}_{\psi\otimes e}(b \otimes B) = (c(v) \otimes id_E)(c(b)\psi \otimes B(e)) = (c(v)c(b)\psi) \otimes B(e)$$

and

$$\begin{aligned} P^{ev}_{\psi\otimes e} \circ (\hat{c}(v) \otimes id_{\mathfrak{su}(E)})(b \otimes B) &= P^{ev}_{\psi\otimes e}(\hat{c}(v)b \otimes B) = (c(\hat{c}(v)b)\psi) \otimes B(e) \\ &= (c(v)c(b)\psi) \otimes B(e) \end{aligned}$$

where in the third equality we used relation (2.2.2). □

Finally the map c_E has an interesting property:

Lemma 2.3.11. *On $\Lambda^*(X) \otimes \mathrm{End}(E)$ the bracket*

$$[\eta_1 \otimes A_1,\, \eta_2 \otimes A_2] \,= \hat{c}(\eta_1)\eta_2 \otimes A_1 A_2 \,-\, \hat{c}(\eta_2)\eta_1 \otimes A_2 A_1 \tag{2.3.9}$$

defines a Lie algebra structure. The map c_E becomes then a Lie algebra homomorphism.

Proof. On $\Lambda^* X$ the bracket

$$(\eta_1,\, \eta_2) \,\mapsto\, \hat{c}(\eta_1)\eta_2 \,-\, \hat{c}(\eta_2)\eta_1$$

defines a Lie algebra structure. Then $[\cdot,\cdot]$ is an extended Lie algebra from $\Lambda^*(X)$ and $\mathrm{End}(E)$. Using (2.3.9) we see that

$$c_E([\eta_1 \otimes A_1,\, \eta_2 \otimes A_2]) \,=\, c_E(\eta_1 \otimes A_1) \circ c_E(\eta_2 \otimes A_2) \,-\, c_E(\eta_2 \otimes A_2) \circ c_E(\eta_1 \otimes A_1).$$

$\square$

Example 8: *Witten's deformations twisted by $SU(2)$-bundles.*

Let (X, g) closed Riemannian and, as in the previous example, $(E, h) \to X$ be a Hermitian $SU(2)$ - bundle with Hermitian connection ∇_E and set

$$\mathrm{Sym}(E) := \{A \in \mathrm{End}_{\mathbb{C}}(E) : A = A^*\}.$$

a subspace of $\mathrm{End}(E)$ with the trace product. Recall the two Clifford representations c and $\hat{c}$ of $\Lambda^*(X)$ described in (2.1.5) and extend them to maps

$$\sigma(v) = c(v) \otimes id_{\mathrm{End}(E)} : \Lambda^{odd}(X) \otimes \mathrm{End}(E) \;\to\; \Lambda^{ev}(X) \otimes \mathrm{End}(E)$$

and, fixing $\alpha \times A \in \Lambda^1(X) \otimes \mathrm{Sym}(E)$

$$\begin{aligned} \mathcal{A}_{\alpha\otimes A} : \;\; \Lambda^{odd}(X) \otimes \mathrm{End}(E) \;&\to\; \Lambda^{ev}(X) \otimes \mathrm{End}(E) \\ \beta \otimes B \;&\mapsto\; \hat{c}(v)\beta \otimes A \circ B \end{aligned}$$

for every $v \in T^*X$ and $A \in \mathrm{Sym}(E)$. Finally the induced connection ∇ on $\Lambda^*(X) \otimes$

$\mathrm{End}(E)$ gives operator

$$D_E = \sigma \circ \nabla : \Gamma(\Lambda^{odd}(X) \otimes \mathrm{End}(E)) \to \Gamma(\Lambda^{ev}(X) \otimes \mathrm{End}(E)).$$

Proposition 2.3.12. *The perturbed operator*

$$D_s = D_E + s\,\mathcal{A}_{\alpha\otimes A}$$

satisfies the concentration condition (1.0.1).

Proof. By linearity it is enough to check the identity for $b \otimes B \in \Lambda^{odd} X \otimes \mathrm{End}(E)$. Then

$$\begin{aligned}
(c(v)^* \otimes id_{\mathrm{End}(E)}) \circ \mathcal{A}_{\alpha\otimes A}(b \otimes B) &= (c(v)^* \otimes id_{\mathrm{End}(E)}(\hat{c}(\alpha)\beta \otimes AB) \\
&= c(v)^* \hat{c}(\alpha)\beta \otimes AB
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}^*_{\alpha\otimes A} \circ (c(v) \otimes id_{\mathrm{End}(E)})(b \otimes B) &= \mathcal{A}_{\alpha\otimes A}(c(v)b \otimes B) \\
&= \hat{c}(\alpha)^* c(v) b \otimes A^* B.
\end{aligned}$$

Since $A^* = A$ and by relation (2.1.6) the two lines are equal. $\square$

Chapter 3

Transverse Concentration

3.1 Structure of $\mathcal{A}$ near the singular set

In proving the Spectral Separation Theorem we will have to analyze the geometry of the operator

$$D_s = c \circ \nabla + s\mathcal{A} : \Gamma(E) \to \Gamma(F)$$

near the singular set $Z_{\mathcal{A}}$. The idea is to expand into Taylor series along the normal directions of each component Z_ℓ of the singular set $Z_{\mathcal{A}}$.

Fix a $Z_\ell =: Z$ an m-dimensional submanifold of the n dimensional manifold X. Let $\pi : N \to Z$ be the normal bundle of Z in X with $p_Z : TX|_Z \to TZ$ and $p_N : TX|_Z \to N$ the orthogonal projections along Z. The Levi-Civita connection ∇^X, when restricted on sections of $TX|_Z$ decomposes to $p^Z\nabla^X$ and $p^N\nabla^X$ i.e. the Levi Civita connection ∇^Z of Z and a connection ∇^N of the normal bundle respectively. Our first task is to understand the perturbation term $\mathcal{A}$ on a tubular neighborhood $\mathcal{N}$ of Z. For that purpose we introduce the following coordinates:

Fix normal coordinates $(U, \{z_i\})$ centered at $p \in Z$ and choose orthonormal moving frames $\{e_\alpha\}$ parallel at p with respect to ∇^N on $N|_U$. The frame $\{e_\alpha\}$ at z identifies an open subset $\mathcal{N}_z \subset N_z$ with an open subset of $\mathcal{N}_p \subset \mathbb{R}^{n-m}$ with coordinates $\{y_\alpha\}$. We get the chart

$$U \times \mathcal{N}_p \to \mathcal{N}_U \subset X,\ (z,y) \to \exp_z(y_\alpha e_\alpha) \tag{3.1.1}$$

with tangent frame and coframe $\{\partial_i\}$ and $\{dx_i\}$ and on the normal fibers $\{\partial_\alpha\}$ and $\{dy_\alpha\}$ so that $\partial_a|_{U\times\{0\}} = e_\alpha$ and $\partial_i|_{U\times\{0\}} =: e_i$. These are normal coordinates, the distance function r from Z writes in this chart $r(z,y) = (\sum_\alpha z_a^2)^{1/2}$ and $g = \exp^* g_X$ has the form $g_{\alpha\beta} = \delta_{\alpha\beta} + O(z, r^2)$ in normal directions.

The Levi Civita connection from X pullback to $U \times \mathcal{N}_p$ and writes

$$\nabla^X_{\partial_i}\partial_\alpha = \Gamma^j_{i\alpha}\partial_j + \Gamma^\beta_{i\alpha}\partial_\beta.$$

where $\Gamma^\beta_{i\alpha} = 0$ at p.

Next introduce the rank l - subbundles

$$K_z := \ker \mathcal{A}_z \le E_z \qquad \text{and} \qquad \hat{K}_z := \ker \mathcal{A}^*_z \le F_z$$

of $E|_Z$ and $F|_Z$ as z runs in Z. By ∇^E- parallel transporting along the normal fibers of $\mathcal{N}$ we create subbundles $K \subseteq E|_{\mathcal{N}}$ and $\hat{K} \subseteq F|_{\mathcal{N}}$. Recall now the non-degeneracy assumption

$$\mathcal{A}^*\mathcal{A}|_K \;=\; r^2 M + O(r^3)$$

where M is positive definite and symmetric. We choose orthonormal frame $\{\sigma_k\}$ that diagonalize M at E_p and extend locally in $U \subset \Sigma$ to a parallel frame trivializing $E|_U =$

$(K \oplus K^{\perp})|_U$. Extend over $\mathcal{N}_U$ by parallel transporting along the normal radial geodesics.

Since index $\mathcal{A} = 0$ a consequence of the concentration condition (1.0.1) is

$$u.K|_Z = \hat{K}|_Z \qquad \text{and} \qquad u.K^{\perp}_Z = \hat{K}|^{\perp}_Z \tag{3.1.2}$$

for every $u \in T^*X|_Z$. In particular the bundle $(K \oplus \hat{K})|_Z$ over Z has a natural $\mathbb{Z}_2$- graded Spin^c structure. Derivating relations (1.0.1) and (1.0.5) along Z we also get

$$u.\nabla_\alpha \mathcal{A} = -\nabla_\alpha \mathcal{A}^* u. \quad \text{and} \quad \nabla_\alpha \mathcal{A}\, u. = -u.\nabla_\alpha \mathcal{A}^* \tag{3.1.3}$$

for every $u \in T^*X|_Z$. Transversality condition (see Appendix) of $\mathcal{A}$ along the normal directions of $U \subset Z$ gives

$$\nabla_\alpha \mathcal{A}(K|_U) \subseteq \hat{K}|_U \qquad \text{and} \qquad \nabla_\alpha \mathcal{A}^*(\hat{K}|_U) \subseteq K|_U.$$

Here the second relation is obtained by the first one using (3.1.3). Hence

$$\nabla_\alpha \mathcal{A} = \begin{pmatrix} A_\alpha & 0 \\ 0 & *(z) \end{pmatrix} : (K \oplus K^{\perp})|_U \to (\hat{K} \oplus \hat{K}^{\perp})_U.$$

Hence the Taylor expansion with respect to the decompositions $E|_{\mathcal{N}} = K \oplus K^{\perp}$ and $F|_{\mathcal{N}} = \hat{K} \oplus \hat{K}^{\perp}$ of the perturbation term $\mathcal{A}$ along the normal directions of $\mathcal{N}_U$ write

$$\mathcal{A} = A_0 + y_\alpha \begin{pmatrix} A_\alpha & 0 \\ 0 & *(z) \end{pmatrix} + \frac{1}{2} y_\alpha y_\beta \nabla_\alpha \nabla_\beta \mathcal{A} + O(r^3)(z) \tag{3.1.4}$$

where $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & *(z) \end{pmatrix}$ is the evaluation of A at $z \in Z$. We have the very useful technical lemma:

Lemma 3.1.1. *The restriction of ∇^E and ∇^F to $Z \subset X$ preserve the splittings $E = K \oplus K^\perp$ and $F = \hat{K} \oplus \hat{K}^\perp$. Since $\nabla \mathcal{A}$ preserves those splittings all the covariant derivatives $\nabla^k \mathcal{A}$ preserve these splittings.*

Proof. Let $\xi \in \Gamma(\mathcal{N}, K)$ with $\xi|_Z \in K|_Z$. By (3.1.4) at p

$$\mathcal{A}(\nabla^E_\alpha \xi) \;=\; \nabla^F_\alpha (\mathcal{A}\xi) \;-\; (\nabla_\alpha \mathcal{A})\xi \;=\; A_\alpha \xi \;-\; A_\alpha \xi \;=\; 0$$

and since $\mathcal{A}\xi|_Z \equiv 0$

$$\mathcal{A}(\nabla^E_i \xi) \;=\; -(\nabla_i \mathcal{A})\xi \;=\; 0$$

i.e. $\nabla^E_A \xi|_p \in K_p$ for $A = i, \alpha$. But the last conclusion is an independent statement of the frame hence it holds in Z. Also the Riemannian metric on E is parallel with respect to ∇^E therefore ∇^E satisfies the same property with the bundle $K^\perp$. The case where E is replaced by F is the same. $\square$

Proposition 3.1.2. *The 2 -jets of $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}$ along Z satisfy*

$$\nabla^2_{v,v}(\mathcal{A}^*\mathcal{A})|_K > 0, \qquad \nabla^2_{v,w}(\mathcal{A}^*\mathcal{A})|_K = 0 \qquad \textit{and} \qquad \nabla^2_{u,v}\mathcal{A}|_K = 0 \tag{3.1.5}$$

for every $u \in TZ$ and $v, w \in N \setminus \{0\}$ with $v \perp w$. The 1- jet of the perturbation $\mathcal{A}$ along

$Z \cap \mathcal{N}_U$ satisfies

$$A_\alpha \in \mathrm{Isom}(K, \hat{K}), \qquad A_\alpha^* A_\alpha = A_\beta^* A_\beta, \qquad \textit{and} \qquad A_\alpha^* A_\beta + A_\beta^* A_\alpha = 0 \qquad (3.1.6)$$

for every i, α, β with $\alpha \neq \beta$.

Proof. The first couple of relations are a direct consequence of the assumption of (1.0.9). Now if R denotes the curvature on $\mathrm{End}(E, F)$

$$R^F(\partial_i, \partial_\alpha)(\mathcal{A}\xi) \;=\; (R(\partial_i, \partial_\alpha)\mathcal{A})\xi \;+\; \mathcal{A}(R^E(\partial_i, \partial_\alpha)\xi).$$

By Lemma 3.1.1 $R^E(e_i, e_\alpha)\xi|_p \in K|_p$ and $\mathcal{A}\xi_p = 0$ hence $(R(e_i, e_\alpha)\mathcal{A})\xi|_p = 0$. Also $(\nabla_i \mathcal{A})\xi|_p = 0$ and $[e_i, e_\alpha]|_p = [\partial_i, \partial_\alpha]|_p = 0$ therefore

$$0 \;=\; (R(e_i, e_\alpha)\mathcal{A})\xi|_p \;=\; (\nabla_i \nabla_\alpha \mathcal{A})\xi|_p = (\nabla^2_{i,\alpha}\mathcal{A})\xi_p.$$

This proves the last relation of the Hessian of $\mathcal{A}$.

For relations on the 1 -jet we just notice by using (3.1.4) to expand $\mathcal{A}^*\mathcal{A}$

$$A_\alpha^* A_\alpha = \nabla^2_{\alpha,\alpha}(\mathcal{A}^*\mathcal{A})|_K > 0$$

so A_α is invertible for every α. Also $e_\alpha \pm e_\beta$ are orthogonal and the relations (3.1.5) give

$$0 \;=\; \nabla_{e_\alpha + e_\beta} \nabla_{e_\alpha - e_\beta}(\mathcal{A}^*\mathcal{A})|_K \;=\; \nabla^2_{\alpha,\alpha}(\mathcal{A}^*\mathcal{A})|_K \;-\; \nabla^2_{\beta,\beta}(\mathcal{A}^*\mathcal{A})|_K \;=\; A_\alpha^* \mathcal{A}_\alpha - \mathcal{A}_\beta^* \mathcal{A}_\beta.$$

Similarly by (3.1.4)

$$0 = \nabla^2_{\alpha,\beta}\mathcal{A}|_K = A^*_\alpha A_\beta + A^*_\beta A_\alpha$$

i.e the last relation. $\square$

Finally we can change $\mathcal{A}$ so that the following happens :

Lemma 3.1.3. *We can choose our perturbation $\mathcal{A}$ so that $\nabla^2_{u,v}\mathcal{A}|_Z \equiv 0$ for every $u, v \in N$.*

Proof. According to properties (3.1.6) of the family $\{A_\alpha\}$ we see that

$$(y_\alpha A_\alpha)^*(y_\beta A_\beta) = \sum_{\alpha \neq \beta}(A^*_\alpha A_\beta + A^*_\beta A_\alpha) + \sum_\alpha y^2_\alpha A^*_\alpha A_\alpha = \sum_\alpha y^2_\alpha A^*_\alpha A_\alpha$$

and since $A^*_\alpha A_\alpha$ is positive definite for every α there exist $C > 0$ so that

$$|y_\alpha A_\alpha \xi| \geq C|y||\xi|$$

for every $\xi \in K|_Z$. In particular $y_\alpha A_\alpha : K|_Z \to \hat{K}|_Z$ is an isomorphism for every $y \neq 0$ and

$$|(y_\alpha A_\alpha)^{-1}| \leq \frac{C}{|y|}.$$

Hence there exist $\epsilon_1 > 0$ so that for every $0 < |y| < \epsilon_1$

$$A_0 + y_\alpha \nabla_\alpha \mathcal{A} : E|_Z \to F|_Z$$

is invertible with

$$|(A_0 + y_\alpha \nabla_\alpha \mathcal{A})^{-1}| \le \frac{C}{|y|}.$$

Introduce now a cut off function supported on $\mathcal{N}$, a tubular neighborhood around Z of radius ϵ to be chosen later. Let $\hat{\rho} : [0,\infty) \to [0,1]$ smooth cut off with $\hat{\rho}^{-1}(\{0\}) = [1,\infty)$, $\hat{\rho}^{-1}(\{1\}) = [0,1/2]$ and strictly decreasing in $[1/2,1]$, define $\rho(q) = \hat{\rho}(\frac{r(q)}{\epsilon})$ on $\mathcal{N}$ and extend as 0 on $X - \mathcal{N}$. We can form the bundle map

$$\mathcal{B} : E \to F, \quad \mathcal{B}(q) = \frac{\rho(q)}{2} \nabla^2_{v,v} \mathcal{A}|_Z, \quad q = \exp_z(v).$$

Derivating relation (3.1.3) we get that

$$u.\nabla_\alpha \nabla_\beta \mathcal{A} = -\nabla_\alpha \nabla_\beta \mathcal{A}^* u.$$

hence

$$u.\mathcal{B} = -\mathcal{B}^* u.$$

for every $u \in T^*X|_Z$. Hence $\mathcal{A} - \mathcal{B}$ satisfies (1.0.1) and using (3.1.7) on $\mathcal{N}_U$

$$\mathcal{A} - \mathcal{B} \;=\; A_0 \;+\; y_\alpha \nabla_\alpha \mathcal{A} \;+\; \frac{1-\rho(y)}{2} y_\alpha y_\beta \nabla_\alpha \nabla_\beta \mathcal{A} \;+\; O(|y|^3)(z).$$

Choose $0 < \epsilon < \epsilon_1$ so that for every $0 < |y| < \epsilon$

$$\Big|\frac{1-\rho(y)}{2} y_\alpha y_\beta \nabla_\alpha \nabla_\beta \mathcal{A} \;+\; O(|y|^3)\Big| < \frac{|y|}{2C} < \frac{1}{2}\big|(A_0 + y_\alpha \nabla_\alpha \mathcal{A})^{-1}\big|^{-1}.$$

Then $\mathcal{A}-\mathcal{B}$ is invertible on $\mathcal{N}-Z$ and agrees with $\mathcal{A}$ outside $\mathcal{N}$ therefore $Z_{\mathcal{A}-\mathcal{B}} = Z_{\mathcal{A}}$. Also by construction

$$\nabla^2_{u,v}(\mathcal{A}-\mathcal{B}) \equiv 0 : E|_Z \to F|_Z$$

for every $u, v \in N$. Finally we have only changed the 2-jet of $\mathcal{A}$ around Z to produce $\mathcal{A}-\mathcal{B}$. Hence condition (1.0.9) still holds for $\mathcal{A}-\mathcal{B}$ since it relates only to the 1-jet of $\mathcal{A}$ on Z. Replacing $\mathcal{A}$ with $\mathcal{A}-\mathcal{B}$ on every component Z of $Z_{\mathcal{A}}$ we are done. $\square$

According to Lemma 3.1.3 on a sufficiently small tubular neighborhood $\mathcal{N}$ around Z

$$\mathcal{A} = A_0 + y_\alpha \begin{pmatrix} A_\alpha & 0 \\ 0 & *(z) \end{pmatrix} + O(r^3)(z) \tag{3.1.7}$$

3.2 Structure of $D + s\mathcal{A}$ along the normal fibers

Our next task is understanding D_s near the set Z. Denote by g^Z and g^N the metric g^X restricted on TZ and on N respectively. Recall the chart (3.1.1) centered at $p \in Z$ with $(\{x_i\}, \{y_\alpha\})$ being the horizontal and vertical coordinates respectively and the frame $\{\sigma_k\}$ on $E|_{\mathcal{N}_U}$. Denote by $\partial^{V/H}$ the local directional derivatives of those vertical and horizontal tangent frames in this chart.

Except for the tangent frame on $T\mathcal{N}$ we can parallel transport the local frames $\{e_\alpha\}$ and $\{e_i\}$ from U to $\mathcal{N}|_U$ along the normal radial geodesics using ∇^X to construct new frames $\{\tau_a\}$ and $\{\tau_i\}$ with dual frames τ^α and τ^i respectively.

Comparing the parallel frame with the tangent frame at p we see

$$\tau_\alpha - \partial_\alpha = O(r^2)^V + O(r^2)^H \qquad \text{and} \qquad \tau_i - \partial_i = O(r^2)^V + O(r)^H$$

hence, action on the local differentials will write

$$\partial_{\tau_\alpha} = \partial_\alpha + O(r^2)\partial^V + O(r^2)\partial^H \qquad \text{and} \qquad \partial_{\tau_i} = \partial_i + O(r^2)\partial^V + O(r)\partial^H \quad (3.2.1)$$

where the symbols $O(r)^V$ and $O(r)^H$ are used to denote decomposition in the vertical and horizontal frames respectively of order r. Also

$$\nabla^E_{\tau_A}(f\sigma_k) = (\partial_{\tau_A} f)\sigma_k + f\omega^l_{Ak}\sigma_l$$

for $A = i, \alpha$ and note that the Taylor expansions of the connection components is

$$\omega_{\alpha k} = O(r^2) \qquad \text{and} \qquad \omega_{ik} = \omega^0_{ik} + O(r).$$

Definition 3.2.1. *The dilation operators from $\Gamma(\mathcal{N}, E|_{\mathcal{N}}) \to \Gamma(\mathcal{N}, F|_{\mathcal{N}})$ are defined as*

$$D^N = \tau^\alpha_. \nabla^E_{\tau_\alpha} \qquad \textit{and} \qquad D^H = \tau^i_. \nabla^E_{\tau_i}.$$

Those are globally defined operators on the tubular neighborhood $\mathcal{N}$ of Z independent of the choice of the above frames and $D = D^N + D^H$. Also for fixed $z \in Z$, the Euclidean

Clifford action $e^\alpha_\cdot : E_z \to F_z$ induce a Euclidean Dirac operator

$$\not{D}_0 = e^\alpha_\cdot \partial_\alpha : \Gamma(N_z, E_z) \to \Gamma(N_z, F_z).$$

Finally observe that the coframe $\{e^i\}$ acts on $(E \oplus F)|_Z$ making it a $\mathbb{Z}_2$ - graded Spin^c bundle over Z. The restriction of the connection components $\{\omega_{ik}\}$ on U give rise to connections denoted both as $\bar{\nabla}_i \sigma_k = \omega^0_{ik}$ regarded otherwise as the pullback connections of the bundles

$$\begin{tikzcd}
(\pi^*(E|_Z), \bar{\nabla}) \arrow[r] \arrow[d] & (E|_Z, \bar{\nabla}) \arrow[r] \arrow[d] & (E|_{\mathcal{N}}, \nabla^E) \arrow[d] \\
N \arrow[r, "\pi"] & Z \arrow[r] & \mathcal{N}
\end{tikzcd}$$

Definition 3.2.2.

$$D^Z : \Gamma(N, \pi^*(E|_Z)) \to \Gamma(N, \pi^*(F|_Z)) : \xi \mapsto e^i_\cdot \bar{\nabla}_i \xi.$$

The definitions for $\hat{D}^Z$ is analogous.

Notice that D^N and D^H are only defined on sections $\xi : \mathcal{N} \to E|_{\mathcal{N}}$ while $\not{D}_0$ and D^Z are defined on sections $\xi : N \to \pi^*(E|_Z)$. In order to relate D^N with $\not{D}_0$ and D^H to D^Z we introduce the parallel transport map along the radial geodesics of $\mathcal{N}_z$

$$\begin{aligned} \tau : \quad \Gamma(\mathcal{N}_z, (E \oplus F)_z) \quad &\to \quad \Gamma(\mathcal{N}_z, (E \oplus F)|_{\mathcal{N}_z}) \\ f(z,w)\sigma_k(z,0) \quad &\mapsto \quad f(z,w)\sigma_k(z,w) \end{aligned} \tag{3.2.2}$$

for every $z \in Z$. Hence τ operates from sections of $\pi^*(E \oplus F)|_Z$ to give sections of

$(E \oplus F)|_{\mathcal{N}}$. Also since the symbol map is ∇^X- parallel

$$(\tau.^A \sigma_k)(z,v) = \tau(e.^A \sigma_k(z,0)),\ A = \alpha,\, i$$

for every $(z,v) \in \mathcal{N}$ where τ is the parallel transport map defined by (3.2.2).

Proposition 3.2.3. *Let $\xi : \mathcal{N} \to \pi^*(E|_K)$. We have the relations*

$$D^N(\tau\xi) \;=\; \tau \not{D}_0 \xi \;+\; O(r^2\partial^V + r^2\partial^H + r^2)\tau\xi$$

and

$$D^H(\tau\xi) \;=\; \tau D^Z \xi \;+\; O(r^2\partial^V + r\partial^H + r)\tau\xi$$

Proof. By linearity and compactness it suffices to work in the chart $\mathcal{N}_U$ centered at $p \in Z$ with $\tau\xi = f\sigma_k$. Since the rates of the Taylor expansion do not depend on the choice of the frame we can restrict our calculations at p and use (3.2.1).

For the vertical operator $D^N = \tau.^\alpha \nabla^E_{\tau_\alpha}$ we estimate

$$\begin{aligned} D^N(\tau\xi) &= \tau.^\alpha((\partial_\alpha f)\sigma_k \;+\; O(r^2(\partial^V f) + r^2(\partial^H f) + r^2 f)\sigma_k) \\ &= \tau \not{D}_0 \xi \;+\; O(r^2\partial^V + r^2\partial^H + r^2)\tau\xi. \end{aligned}$$

Also

$$\begin{aligned} D^H(\tau\xi) &= \tau.^i \nabla^E_{\tau_i}(\tau\xi) \;=\; \tau.^i((\partial_i f)\sigma_k + f\tau\omega^0_{ik} \;+\; O(r^2(\partial^V f) + r(\partial^H f) + rf)\sigma_k) \\ &= \tau D^Z \xi \;+\; O(r^2\partial^V + r\partial^H + r)\tau\xi. \end{aligned}$$

$\square$

Combining Proposition 3.2.3 with expansion 3.1.7 we get:

Corollary 3.2.4. *D_s expands along the normal directions of the singular set Z with respect to the decompositions $E|_{\mathcal{N}} = K \oplus K^{\perp}$ and $F|_{\mathcal{N}} = \hat{K} \oplus \hat{K}^{\perp}$ as*

$$(D + s\mathcal{A})\tau\xi \;=\; \tau(\not{D}_s + D^Z)\xi \;+\; sA_0\tau\xi \;+\; O(r^2\partial^V + r\partial^H + r + sr^3)\tau\xi.$$

Here

$$\not{D}_s := e^{\alpha}_{\cdot}\partial_\alpha + sy_\alpha \begin{pmatrix} A_\alpha & 0 \\ 0 & *(z) \end{pmatrix}.$$

The same construction holds for the dual operator $D_s^* : \Gamma(F|_{\mathcal{N}}) \to \Gamma(E|_{\mathcal{N}})$ with the normal and horizontal operators denoted as $\not{D}_s^*$ and $\hat{D}^Z$ respectively.

Proposition 3.2.4 shows the rates of each of the horizontal and vertical derivatives along Z as $s >> 0$. In particular, for large s, sections $(\xi_1, \xi_2) : \mathcal{N} \to K \oplus K^{\perp}$ satisfying

$$D_s \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \;=\; 0$$

are well-approximated by sections $\xi = (\xi_1, 0)$ of the bundle K in $\ker \not{D} \cap \ker D^Z$. We will construct approximate solutions of $D_s\xi = 0$ by finding solutions to the first order approximation

$$(\not{D}_s \,+\, D^Z)\xi_1 = 0$$

where $\xi_1 : \mathcal{N} \to \pi^*(K|_Z)$. Solving the first order approximation is the next main topic.

Chapter 4

Constructing approximate solutions

In this chapter we explicitly describe solutions to

$$(\not{D}_s + D^Z)\xi = 0$$

where $\xi : \mathcal{N} \to \pi^*(K|_Z)$. By Proposition 4.2.1 this amounts to finding solutions of the system

$$\not{D}_s\xi = 0 \qquad \text{and} \qquad D^Z\xi = 0.$$

Freezing $z \in Z$ the first equation can be solved in sections $\xi_z : N_z \to K_z$, since we have only derivatives in the normal directions of Z. The family of solutions spaces over $z \in Z$ form the so called bundle of vertical solutions and can be constructed using the bundle $\mathcal{K} \to Z$ introduced in Definition 1.0.7. An analogue bundle is constructed in the dual case of $\not{D}_s^*$ using $\hat{\mathcal{K}} \to Z$ introduced in the same definition. Then D^Z restricts to the sections of those solution bundles to give Dirac type operator there. The second equation can be interpreted as the kernel of that operator.

4.1 The bundle of vertical solutions

Recall the normal coframe $\{e^\alpha\}$ on $N|_U$ introduced in Section 3.1and the terms $A_\alpha = \nabla_{e_\alpha}\mathcal{A}|_K$ in the Taylor expansion of $\mathcal{A}$ around Z. For this section we will be using w coordinates moving in the whole fiber of the normal bundle N. One can think of those coordinates as the blown up coordinates $w = \sqrt{s}y$ when $s = \infty$ (compare with Remark 1.0.5).

Fix $z \in U$. The Euclidean Clifford action $e^\alpha_\cdot : K_z \to \hat{K}_z$ induce Euclidean Dirac operators

$$\not{D}_0 : \Gamma(N_z, K_z) \to \Gamma(N_z, \hat{K}_z), \quad \text{by} \quad \not{D}_0\xi = (e^\alpha_\cdot \partial_\alpha)\xi \tag{4.1.1}$$

and

$$\not{D} : \Gamma(N_z, K_z) \to \Gamma(N_z, \hat{K}_z), \quad \text{by} \quad \not{D}\xi = (e^\alpha_\cdot \partial_\alpha)\xi + w_\alpha A_\alpha \xi. \tag{4.1.2}$$

The purpose of this section is to count the dimension of decaying solutions $\xi : N_z \to K_z$ of the equation

$$\not{D}\xi = 0 \tag{4.1.3}$$

and show that they form a bundle as z varies in Z.

We start by noticing that $\{A_\alpha\}$ is a family of invertible matrices satisfying the relations

$$e^{\beta}_{\cdot} A_\alpha + A^*_\alpha e^{\beta}_{\cdot} = 0, \quad A^*_\alpha A_\alpha = A^*_\beta A_\beta, \ \forall a, b \quad \text{and} \quad A^*_\alpha A_\beta + A^*_\beta A_\alpha = 0, \ a \neq b. \tag{4.1.4}$$

The first of these is obtained by differentiating the concentration condition (1.0.1), while the other two are due to Proposition 3.1.2.

Lemma 4.1.1. *Under the relations (4.1.4) the set of matrices*

$$\{M_\alpha = -e^{\alpha}_{\cdot} \nabla_{e_\alpha} \mathcal{A}|_K\} \subseteq \mathrm{End}(K_z)$$

constitutes of commuting invertible self-adjoint endomorphisms satisfying

$$e^{\alpha}_{\cdot} e^{\beta}_{\cdot} M_\beta \ = \ M_\beta e^{\beta}_{\cdot} e^{\alpha}_{\cdot} \qquad \textit{and} \qquad M^2_\alpha = M^2_\beta \tag{4.1.5}$$

for every α, β. Each M_α has symmetric spectrum and opposite eigenvalues have the same multiplicity. Furthermore for every string I of even length $e^{I}_{\cdot}$ satisfies

$$e^{I}_{\cdot} \in \mathrm{Hom}(K^\pm_\alpha, K^\pm_\alpha) \quad \textit{if} \quad \alpha \notin I \qquad \textit{and} \qquad e^{I}_{\cdot} \in \mathrm{Hom}(K^\pm_\alpha, K^\mp_\alpha) \quad \textit{if} \quad \alpha \in I. \tag{4.1.6}$$

where $K^\pm_\alpha$ denote the eigenspaces of M_α corresponding to the $\pm\mu_\alpha$ - eigenvalues respectively.

Proof. The first and second of relations (4.1.4) directly imply the first and second relations of (4.1.5) respectively and show that M_α is self-adjoint. Finally the third of relations

(4.1.4) implies that $\{M_a\}$ is a commuting family. Also

$$e^\alpha_\cdot e^\beta_\cdot M_\gamma \;=\; e^\alpha_\cdot e^\gamma_\cdot e^\beta_\cdot e^\gamma_\cdot M_\gamma \;=\; M_\gamma e^\gamma_\cdot e^\alpha_\cdot e^\gamma_\cdot e^\beta_\cdot \;=\; M_\gamma e^\alpha_\cdot e^\beta_\cdot$$

for $\alpha, \beta \neq \gamma$. This imply the relations

$$M_\alpha e^I_\cdot = e^I_\cdot M_\alpha \quad \text{if} \quad \alpha \notin I \qquad \text{and} \qquad e^I_\cdot M_\alpha = -M_\alpha e^I_\cdot \quad \text{if} \quad \alpha \in I$$

showing (4.1.6). $\square$

Remark 4.1.2. 1) Relations (4.1.4) show that $\{e^\alpha_\cdot \varphi,\, A_b \varphi\}_{\alpha,\beta}$ is a family of orthogonal vectors when $\varphi \in K_z$ is non trivial.

2) Using (4.1.5), and the commutativity of $\{M_\alpha\}$

$$\begin{aligned}
\sum_{\alpha,\beta} \langle w_\beta A_\beta \xi,\, w_\alpha A_\alpha \psi\rangle \;&=\; \sum_{\alpha,\beta} w_\alpha w_\beta \langle e^\beta_\cdot M_\beta \xi,\, e^\alpha_\cdot M_\alpha \psi\rangle \;=\; -\sum_{\alpha,\beta} w_\alpha w_\beta \langle M_\alpha e^\alpha_\cdot e^\beta_\cdot M_\beta \xi,\, \psi\rangle \\
&=\; -\sum_{\alpha\neq\beta} w_\alpha w_\beta \langle e^\beta_\cdot e^\alpha_\cdot M_\alpha M_\beta \xi,\, \psi\rangle \;+\; \sum_\alpha w_\alpha^2 \langle M_\alpha^2 \xi,\, \psi\rangle \\
&=\; \sum_{\alpha<\beta} w_\alpha w_\beta \langle e^\alpha_\cdot e^\beta_\cdot [M_\alpha,\, M_\beta] \xi,\, \psi\rangle \;+\; \sum_\alpha w_\alpha^2 \langle M_\alpha^2 \xi,\, \psi\rangle
\end{aligned}$$

for every $w \in N_z$ and every $\xi, \psi \in K|_Z$. In particular, if the M_α commute then

$$(w_\alpha A_\alpha)^*(w_\beta A_\beta)\xi \;=\; w_\alpha^2 M_\alpha^2 \xi. \tag{4.1.7}$$

Let $\bigoplus K_i$ be the decomposition of K_z into the common eigenspaces of the family $\{M_\alpha\}$. Hence $K_i = K_\alpha^+$ when viewed as eigenspace where M_α has a positive eigenvalue or $K_i = K_\alpha^-$ otherwise.

Since $M_\alpha^2 = M_\beta^2$ the eigenvalues $\mu_{\alpha i}$ of M_α on K_i are equal in absolute value to a common number μ_i. Fixing now a summand K_i let tr_i be the normalized trace over $\mathrm{End}(K_i)$. We have the following:

Lemma 4.1.3 (Exponential decay estimates)**.** *Set*

$$M_w = -\sum_{\alpha,\beta} w_\alpha w_\beta e^\alpha_\cdot(\nabla_{e_\beta}\mathcal{A})|_K \in \mathrm{End}(K_z),$$

and let $\xi : N_z \to K_i$ be a C^2 section, decaying at infinity and satisfying (4.1.3). Then there exist a constant $M_0 > 0$ such that

$$|\xi(w)| \leq M_0 e^{\frac{1}{2}tr_i(M_w)}.$$

Proof. Let Δ denote the analyst's Laplacian on $N_z \simeq \mathbb{R}^{n-m}$ and ∇ the Euclidean gradient. Applying $\not{D}^*$ to equation 4.1.3 and using Remark 4.1.2 (2), we obtain

$$\begin{aligned} 0 &= \langle \not{D}^*\not{D}\xi,\, \xi\rangle = \langle \not{D}_0^*\not{D}_0\xi, \xi\rangle + \langle e_\alpha . A_\alpha\xi,\, \xi\rangle + |w_\alpha A_\alpha \xi|^2 \\ &= \langle \nabla^*\nabla\xi,\, \xi\rangle - \sum_\alpha \langle M_\alpha\xi, \xi\rangle + \sum_\alpha w_\alpha^2 |M_\alpha\xi|^2 \\ &= \langle \nabla^*\nabla\xi,\, \xi\rangle + \sum_\alpha (w_\alpha^2\mu_{\alpha i}^2 - \mu_{\alpha i})|\xi|^2. \end{aligned}$$

Therefore

$$\Delta|\xi|^2 = 2|\nabla\xi|^2 - 2\Re\langle \nabla^*\nabla\xi,\, \xi\rangle = 2|\nabla\xi|^2 + 2\sum_\alpha (w_\alpha^2\mu_{\alpha i}^2 - \mu_{\alpha i})|\xi|^2. \qquad (4.1.8)$$

Combining with the identity

$$\Delta|\xi|^2 = \sum_j \partial_j(2|\xi|\partial_j|\xi|) = 2|\xi|\Delta|\xi| + 2|\nabla|\xi||^2$$

and Kato's inequality $|\nabla|\xi|| \leq |\nabla\xi|$ gives

$$|\xi|\Delta|\xi| \geq \sum_\alpha (w_\alpha^2\mu_{\alpha i}^2 - \mu_{\alpha i})|\xi|^2.$$

Fix now $\epsilon > 0$ and define

$$F : N_z \to \mathbb{R},\ F(w) = M_\epsilon e^{\frac{1}{2}\mathrm{tr}_i M_w},$$

where M_ϵ is to be defined. Assuming $\{\sigma_l\}$ is an orthonormal base for K_i and using Remark 4.1.2 (1) we calculate

$$\begin{aligned} \mathrm{tr}_i M_w &= -w_\alpha w_\beta \Re\langle e^\alpha_\cdot A_\beta \sigma_l, \sigma_l\rangle = -w_\alpha^2 \Re\langle e^\alpha_\cdot A_\alpha \sigma_l, \sigma_l\rangle \\ &= -w_\alpha^2 \mu_{\alpha i}. \end{aligned} \tag{4.1.9}$$

Therefore

$$\Delta F = \sum_\alpha (w_\alpha^2\mu_{\alpha i}^2 - \mu_{\alpha i})F$$

and the difference satisfies:

$$|\xi|\Delta(F - |\xi|) \leq \sum\nolimits_\alpha (w_\alpha^2\mu_{\alpha i}^2 - \mu_{\alpha i})|\xi|(F - |\xi|). \tag{4.1.10}$$

Now set

$$R = \frac{1}{\mu_i}(\sum_\alpha |\mu_{\alpha i}|)^{1/2} \quad \text{and} \quad M_\epsilon = (\epsilon + \sup_{|w|=R} |\xi|) e^{\frac{1}{2}\mu_i R^2}$$

When $|w| = R$

$$\mathrm{tr}_i(M_w) + \mu_i R^2 \geq w_\alpha^2(|\mu_{\alpha i}| - \mu_{\alpha i}) \geq 0$$

so $(F - |\xi|)\big|_{|w|=R} \geq \epsilon > 0$ and when $|w| > R$ the term $\sum_\alpha (w_\alpha^2 \mu_{\alpha i}^2 - \mu_{\alpha i})$ is strictly positive. Set $V := \{w : |w| > R, |\xi(w)| > F(w)\}$, an open set that satisfies the properties:

(i) $\bar{V} \cap \{|w| = R\} = \emptyset$

(ii) $|\xi| > 0$ on $\bar{V}$. Enlarging slightly V we may assume that ∂V is smooth, properties (i) and (ii) are satisfied, and furthermore

(iii) $(F - |\xi|)\big|_{\partial V} > 0$.

Suppose that $V \neq \emptyset$. If $F - |\xi|$ had a negative minimum $w_0 \in V$ then, by (4.1.10) and the maximum principle, $F(w_0) > |\xi(w_0)|$ which is a contradiction. By Property (iii) above and the assumption that ξ decays at infinity, the function $F - |\xi|$ cannot attend a negative minimum in the boundary of V or at infinity, if V is unbounded, therefore $V = \emptyset$ and $|\xi(w)| \leq M_\epsilon e^{\frac{1}{2}\mathrm{tr}_i M_w}$ for every $\epsilon > 0$ and $|w| > R$ and the result follows. $\square$

Denote by $\mathcal{K}_z$ is the sum of those K_i's where $M_\alpha|_{K_i} = \mu_i Id_{K_i}$ for every α. This will essentially be the space of $L^{1,2}$ solutions of the equation (4.1.3):

Theorem 4.1.4. *Suppose that $X = \mathbb{R}^n$ with coordinates $\{w_\alpha\}$ and A_α is a collection of matrices satisfying (4.1.4). All $L^{1,2}$ solutions of the equation*

$$\sum_\alpha (e^\alpha_\cdot \partial_\alpha + w_\alpha A_\alpha)\xi = 0 \tag{4.1.11}$$

are linear combinations of sections of the form

$$\xi_i(w) = e^{-\frac{1}{2}\mu_i |w|^2}\varphi$$

where $\varphi : N_z \to K_i \subset \mathcal{K}_z$ is a constant section.

Proof. Given $\xi \in \Gamma(N_z, K_z)$ using (4.1.7) we compute

$$\begin{aligned} \not{D}^*\not{D}\xi &= \not{D}_0^*\not{D}_0\xi + e^\alpha_\cdot A_\alpha \xi + w_\alpha w_\beta A_\alpha^* A_\beta \xi \\ &= \not{D}_0^*\not{D}_0\xi - \sum_\alpha M_\alpha \xi + \sum_\alpha w_\alpha^2 M_\alpha^2 \xi. \end{aligned} \tag{4.1.12}$$

Now when ξ is a $L^{1,2}$ solution of (4.1.11) then by regularity ξ is a C^2 decaying solution and equivalently ξ evaluates the above presentation to 0. Since M_α are all simultaneously diagonalizable we can decompose $\xi = \sum_i \xi_i$ according to the decomposition $K_z = \bigoplus K_i$. Because of linear interdependency $\xi_i : N_z \to K_i$ will be an $L^{1,2}$ solution of (4.1.11) for each i and we can assume that $\xi = \xi_i$ for some i.

Now recall $F(w) = e^{\frac{1}{2}\mathrm{tr}_i(M_w)} : N_z \to \mathbb{R}$ and note that by calculation (4.1.9) F satisfies the system $\partial_\alpha F + \mu_{\alpha i} w_\alpha F = 0$ for every α. Replacing ξ by $F\xi$ we calculate

$$\begin{aligned} \not{D}(F\xi) &= \sum_\alpha \Big(e^\alpha_\cdot \partial_\alpha (F\xi) + w_\alpha e^\alpha_\cdot M_\alpha (F\xi)\Big) = \sum_\alpha \Big(\partial_\alpha F + w_\alpha \mu_{\alpha i} F\Big) e^\alpha_\cdot \xi + F\not{D}_0 \xi \\ &= F\not{D}_0\xi. \end{aligned}$$

Hence if $\xi : N_z \to K_i$ obey (4.1.11)

$$0 = \not{D}\xi = \not{D}(FF^{-1}\xi) = F\not{D}_0(F^{-1}\xi)$$

i.e. $\not{D}_0(F^{-1}\xi) = 0$. But by Lemma 4.1.3 the section $F^{-1}\xi$ is bounded and harmonic on the entire N_z in the Euclidean sense hence $F^{-1}\xi = \varphi \in K_i$ is a constant vector and so $\xi = F\varphi : N_z \to K_i$ where φ now is viewed as a constant section. This section belongs in L^2 iff $\varphi = 0$ or if $K_i \subset \mathcal{K}_z$. In the later case $\mathrm{tr}_i(M_w) = -\mu_i |w|^2$. $\square$

Proposition 4.1.5. *The spaces $\{\mathcal{K}_z : z \in Z\}$ are independent of the choice of $\{e^\alpha\}$, and they form a subbundle of K over Z.*

Proof. Let $\{e'_\alpha\}$ be a second orthonormal frame of N_z with $\{M'_\alpha\}$ the corresponding family of commuting self-adjoint matrices:

Claim: When all $\{M_a\}$ have same eigenvalue $\pm\mu_i$ on K_i then all $\{M'_\alpha\}$ have the same eigenvalue on K_i.

Suppose $e'_\alpha = d_{\alpha\beta} e_\beta$ for some orthogonal matrix d and let $\sigma \in K_i$ so that $M_a \sigma = \mu_i \sigma$

for every α. Then $d_{\alpha\beta}d_{\gamma\beta} = \delta_{\alpha\gamma}$ and therefore

$$M'_\alpha\sigma \,=\, -d_{\alpha\beta}d_{\alpha\gamma}e^\beta_\cdot e^\gamma_\cdot M_\gamma\sigma \,=\, \mu_i\sum_\beta d^2_{\alpha\beta}\sigma \,-\, \mu_i\sum_{\beta\neq\gamma} d_{\alpha\beta}d_{\alpha\gamma}e^\beta_\cdot e^\gamma_\cdot\sigma \,=\, \mu_i\sigma$$

for every M'_α. This proves the claim.

Suppose now $p \in Z$ and that $K_i \subset \mathcal{K}_p$, so that the $\{M_\alpha\}$ have common eigenvalue μ_i on K_i. To construct local bundles recall chart 3.1.1 with $U \subset Z$ a geodesic ball of center p with orthonormal frame $\{e_\alpha\}$ of N_p. This induce the family $\{M_\alpha\} \subset \mathrm{End}(K_p)$ and $\{\sigma_k\}$ of K_p so that the set partition to orthonormal frames of each K_i. We extend both frames to moving frames over U centered at p and we extend the family $\{M_\alpha\}$ accordingly by means of $\{e_\alpha\}$. Let $\gamma(t)$ be a geodesic of U with $\gamma(0) = p$ and $\dot\gamma(0) \in T_pZ$. Fix a parallel local section σ so that $\sigma(p) \in K_i$. By the last relation of (3.1.5)

$$\nabla^E_{\dot\gamma(t)}(M_\alpha\sigma) \,=\, -e^\alpha_\cdot(\nabla_{\dot\gamma(t)}A_\alpha)\sigma \,=\, -e^\alpha_\cdot(\nabla^2_{\dot\gamma(t),\alpha}\mathcal{A})\sigma \,=\, 0$$

i.e. $M_\alpha\sigma$ is also parallel hence a linear combination of $\{\sigma_k\}$. Since at the origin $M_\alpha\sigma = \mu_i\sigma$ this equality holds in U proving that M_α has constant spectrum over U with $\{\sigma_k\}$ being the eigenvectors. Hence each $K_i \subset \mathcal{K}_p$ extend over U providing a local bundle of solutions. But by the previous claim in the intersections of the various U's each local version K_i is independent of the frame $\{e_\alpha\}$ used and the various local versions of the K_i patch together to give a global bundle K_i over Z. Hence also $\mathcal{K}$ is a well defined bundle of solutions along Z, and a subbundle of $K|_Z$. $\square$

In the dual case $\hat K = \ker\mathcal{A}^*$ and we define similarly $\hat M_\alpha = -e^\alpha_\cdot A^*_\alpha = e^\alpha_\cdot M_\alpha e^\alpha_\cdot \in$

$\mathrm{End}(\hat{K})$. Then the family $\{\hat{M}_\alpha\}$ satisfy the same relations as the family $\{M_\alpha\}$ and give rise to a well defined bundle $\hat{\mathcal{K}}$ of solutions of $\not{D}^*$.

Proposition 4.1.6. *$\mathcal{K} \oplus \hat{\mathcal{K}}$ is a $\mathbb{Z}_2$-graded* Spin^c *bundle over Z.*

Proof. To start we define similarly $\hat{K}_\alpha^\pm$ to be the $\pm\mu_\alpha$-eigenspaces of $\hat{M}_\alpha$ for every α and notice that the frame $\{e^A\}, A = i, \alpha$ acts analogously to relations to (4.1.6)

$$e^I_\cdot M_\alpha = -\hat{M}_\alpha e^I_\cdot \quad \text{if} \quad \alpha \in I \qquad \text{and} \qquad e^I_\cdot M_\alpha = \hat{M}_\alpha e^I_\cdot \quad \text{if} \quad \alpha \notin I \tag{4.1.13}$$

for every string I of odd length implying that M_α has the same spectrum and multiplicities as $\hat{M}_\alpha$ and furthermore

$$e^I_\cdot \in \mathrm{Hom}(K^\pm_\alpha, \hat{K}^\pm_\alpha) \quad \text{if} \quad \alpha \notin I \quad \text{and} \qquad e^I_\cdot \in \mathrm{Hom}(K^\pm_\alpha, \hat{K}^\mp_\alpha) \quad \text{if} \quad \alpha \in I.$$

In particular $\mathcal{K} \oplus \hat{\mathcal{K}} \subset (K \oplus \hat{K})|_Z$ is preserved by the Clifford action of the tangent coframe $\{e^i\}$ to Z thus being a $\mathbb{Z}_2$- graded Spin^c-bundle over Z. $\square$

Remark 4.1.7. We end up this paragraph with a remark. The non - degeneracy assumption (1.0.9) can be weaken for the proof of the Spectral Separation Theorem. It is included for making the construction of the bundle of solutions simpler. In general one has to examine the various normal vanishing rates of the eigenvalues of $\mathcal{A}^*\mathcal{A}$. In particular the bundles of solutions examined here will have a layer structure corresponding to those rates and possibly will have a jumping locus in their dimension.

4.2 The operators D^Z and $\hat{D}^Z$

Recall the Spin^c bundle

$$\pi^*(E \oplus F)|_Z \to N$$

with connection $\bar{\nabla}$ introduced in Section 2.2. In view of Lemma 3.1.1, this connection restricts to a connection of the bundle $\pi^*(K \oplus \hat{K})|_Z \to N$. Accordingly, the restrictions

$$\not{D} : \Gamma(N_z, K_z) \to \Gamma(N_z, \hat{K}_z), \quad \forall z \in Z$$

and

$$D^Z : \Gamma(N, \pi^*(K|_Z)) \to \Gamma(N, \pi^*(\hat{K}|_Z))$$

are well-defined with adjoints $\not{D}^*$ and $\hat{D}^Z$ and satisfy the following:

Proposition 4.2.1. *For every section $\xi : N \to \pi^*(K|_Z)$, we have*

$$(\not{D}^* D^Z + \hat{D}^Z \not{D})\xi = 0,$$

and therefore

$$\|(\not{D} + D^Z)\xi\|_2^2 = \|\not{D}\xi\|_2^2 + \|D^Z\xi\|_2^2.$$

Proof. Recall the chart (3.1.1) centered at p with tangent frame $\{\partial_A\}, A = i, \alpha$ so that $\partial_A|_U = e_A$ and the parallel frame $\{\sigma_k\}$ on $E|_{\mathcal{N}_U}$ so that $\bar{\nabla}_i \sigma_k = \partial_i + \omega_{ik}^0$. Then by (2.2.2) $e^i_\cdot \nabla_i^X e^\alpha_\cdot = (d + d^*)e^\alpha_\cdot = ((d + d^*)dy_\alpha|_U)_\cdot = 0$. Moreover the last relation of (3.1.5) evaluated at p gives $\nabla_i A_\alpha = 0$. Hence if $\xi = f\sigma_k$ at p

$$\begin{aligned}
\not{D}^* D^Z \xi &= (e^\alpha_\cdot \partial_\alpha + w_\alpha A^*_\alpha) e^i_\cdot \bar{\nabla}_i \xi \\
&= -e^i_\cdot \bar{\nabla}_i (e^\alpha_\cdot (\partial_\alpha f)\sigma_k \,+\, e^i_\cdot (\nabla^X_i e^\alpha).(\partial_\alpha f)\sigma_k \\
&- \; w_\alpha e^i_\cdot \bar{\nabla}_i (A_\alpha f \sigma_k) \,+\, w_\alpha e^i_\cdot (\nabla_i A_\alpha) f \sigma_k \\
&= -\hat{D}^Z \not{D} \xi
\end{aligned}$$

Since p was arbitrary this holds everywhere. The last identity follows since the cross terms are zero by the first. $\square$

Note that $\bar{\nabla}$ is really a restriction of ∇ to sections of the above bundles and that $\mathcal{K}$ and $\hat{\mathcal{K}}$ can be viewed as subbundles. Recall the construction of the bundles K_i, $\hat{K}_i$ from Proposition 4.1.5. It follows that $\bar{\nabla}$ preserves them inducing in that way a connection on sections of $K_i \oplus \hat{K}_i$ and hence a well defined connection on their direct sum bundle $\mathcal{K} \oplus \hat{\mathcal{K}} \to Z$. This connection will not in general be compatible with the Spin^c structure on $\mathcal{K} \oplus \hat{\mathcal{K}}$ unless the second fundamental form of the embedding $Z \hookrightarrow X$ is trivial.

Definition 4.2.2. *The restriction operator*

$$D^Z = e^i_\cdot \bar{\nabla}_i \; : \Gamma(Z, \mathcal{K}) \to \Gamma(Z, \hat{\mathcal{K}})$$

is well-defined Dirac operator.

Hence solutions $\xi : \mathcal{N} \to \pi^*(K|_Z)$ of the equation

$$(\not{D}_s \,+\, D^Z)\xi = 0$$

are explicitly given in terms of the distance r from Z by

$$\xi = \sum_i e^{-\frac{1}{2}s\mu_i r^2}\varphi_i \tag{4.2.1}$$

where $\mu_i > 0$ and where $\varphi_i \in \Gamma(Z, K_i)$ satisfies $D^Z\varphi_i = 0$. In Chapter 5 we use these Gaussian sections (see Definition 5.1.2) to construct a space of approximate solutions of the equation

$$D_s\xi = 0.$$

Chapter 5

Approximate eigenvectors

5.1 Low-High separation of the spectrum

Our main goal for this chapter is to prove the Spectrum Separation Theorem stated in the introduction. For that purpose we will use the bundles $\mathcal{K}_\ell$ introduced in Chapter 1 and define a space of approximate solutions to the equation $D_s\xi = 0$. The space of approximate solutions is linearly isomorphic to a certain "thickening" of $\ker D_s$ by "low" eigenspaces of $D_s^* D_s$ for large s. The same result will apply to $\ker D_s^*$. The "thickening" will occur by a phenomenon of separation of the spectrum of $D_s^* D_s$ into low and high eigenvalues for large s. The following lemma makes this idea precise:

Lemma 5.1.1. *Let $L : H \to H'$ be a densely defined closed operator with between the Hilbert spaces H, H' so that L^*L has descrete spectrum. Denote E_μ the μ- eigenspace of L^*L. Suppose V is an k- dimensional subspace of H so that*

$$|Lv|^2 \leq C_1|v|^2, \forall v \in V \qquad \textit{and} \qquad |Lw|^2 \geq C_2|w|^2, \forall w \in V^\perp.$$

Then there exist consecutive eigenvalues μ_1, μ_2 of L^*L so that $\mu_1 \leq C_1$, $\mu_2 \geq C_2$ and if *in addition* $4C_1 < C_2$*, the* **orthogonal projection**

$$P : \bigoplus_{\mu \le \mu_1} E_\mu \to V$$

is an isomorphism.

Proof. Let μ_1 be the k-th eigenvalue of the self-adjoint operator L^*L with counted multiplicity and μ_2 be the next eigenvalue. Denote by $G_k(H)$ the set of k- dimensional subspaces of H and set $W = \oplus_{\mu \le \mu_1} E_\mu$, also k-dimensional. By the Rayleigh quotients we have

$$\mu_2 = \max_{S \in G_k(H)} \left\{ \inf_{v \in S^\perp, |v|=1} |Lv|^2 \right\} \ge \inf_{v \in V^\perp, |v|=1} |Lv|^2 \ge C_2.$$

and also

$$\mu_1 = \max_{S \in G_{k-1}(H)} \left\{ \inf_{v \in S^\perp, |v|=1} |Lv|^2 \right\}.$$

But for any $k-1$-dimensional subspace $S \subset H$ there exist a unit $v_S \in S^\perp \cap V$ so

$$\mu_1 \le \max_{S \in G_{k-1}(H)} \left\{ |Lv_S|^2 \right\} \le C_1.$$

Finally, given $w \in W$ write $w = v_0 + v_1$ with $v_0 = P(w)$ and $v_1 \in V^\perp$. Then

$$C_2 |w - P(w)|^2 = C_2 |v_1|^2 \le |Lv_1|^2 \le 2(|Lw|^2 + |Lv_0|^2) \le 2(\mu_1 + C_1)|w|^2 \le 4C_1 |w|^2$$

and so $|id_W - P|^2 \le 4\frac{C_1}{C_2}$. If additionally $4C_1 < C_2$ and $P(w) = 0$ for some $w \ne 0$ then

$$|w|^2 = |w - P(w)|^2 \le |id_W - P|^2 |w|^2 < |w|^2$$

a contradiction. Hence P is injective and by dimension count an isomorphism. $\square$

We have to construct an appropriate space V that will be viewed as the space of approximate solutions to the problem $L\xi = D_s\xi = 0$. It is enough to describe the construction for a fixed m - dimensional component $Z = Z_\ell$. For this purpose we introduce a cutoff function ρ near Z.

Let $\mathcal{N} = \mathcal{N}_{2\epsilon}$ be the tubular neighborhood of radius 2ϵ around Z. The exponential map $\mathcal{N} \to \exp(\mathcal{N})$, is a radial isometry along the vertical directions and the distance function r from Z. Let $\hat{\rho} : [0,\infty) \to [0,1]$ be a smooth cut off with $\hat{\rho}^{-1}(\{0\}) = [1,\infty)$, $\hat{\rho}^{-1}(\{1\}) = [0, \frac{1}{2}]$ and strictly decreasing in $[\frac{1}{2}, 1]$ with $|\hat{\rho}'| \leq 3$ and define

$$\rho(p) = \begin{cases} \hat{\rho}(\frac{r(p)}{\epsilon}) & p = \exp(z,v) \in \exp(\mathcal{N}) \\ 0 & p \in X \setminus \exp(\mathcal{N}). \end{cases}$$

Let $\mathcal{K}_\ell$ be the bundle over Z_ℓ as in Definition 1.0.7; $\mathcal{K}_\ell$ is the direct sum of common eigenspaces K_i of the family $\{M_\alpha = -e^\alpha_\cdot A_\alpha\}$ of positive common eigenvalues μ_i. As in Definition 4.2.2, the Dirac operator D^ℓ acts on sections of $\mathcal{K}_\ell$ along each component Z_ℓ of $Z_{\mathcal{A}}$.

Definition 5.1.2. *For each component Z_ℓ of $Z_{\mathcal{A}}$, set*

$$V^s_\ell = \mathrm{span}\big\{\rho \cdot e^{-\frac{1}{2}s\mu_i r^2}\tau\varphi_i \;\big|\; \varphi_i \in \Gamma(Z_\ell, K_i),\, \mu_i > 0,\; D^{Z_\ell}\varphi_i = 0\big\},$$

where τ is the parallel transport map defined by (3.2.2). Taking the direct sum over all *components of $Z_{\mathcal{A}}$ we construct the space of approximate solutions along $Z_{\mathcal{A}}$*

$$V_s = \bigoplus_{\ell} V_\ell^s \subset L^{1,2}(X,E).$$

and denote by $V_s^\perp$ the closure of its L^2-perpendicular in the $L^{1,2}$-norm:

$$V_s^\perp := V_s^{\perp L^2} \cap L^{1,2}(X,E).$$

Notice that $V_{Z_1}^s$ is L^2-perpendicular to $V_{Z_2}^s$ for $Z_1 \neq Z_2$ since their corresponding sections have disjoint supports. There are completely analogous constructions of the spaces $\hat{V}_{Z_\ell}^s$, $\hat{V}_s$ for D_s^* and $L^{1,2}(X,F) = \hat{V}_s \oplus \hat{V}_s^\perp$.

Theorem 5.1.3. *There exist an $s_0 > 0$ and constants $C_i = C_i(s_0) > 0$, $i = 1,2$ so that when $s > s_0$*

(a) For every $\eta \in V_s$,

$$\|D_s\eta\|_2^2 \leq \frac{C_1}{s}\|\eta\|_2^2. \tag{5.1.1}$$

(b) For every $\eta \in V_s^\perp$,

$$\|D_s\eta\|_2^2 \geq C_2\|\eta\|_2^2. \tag{5.1.2}$$

The same estimates hold for the L^2-adjoint operator D_s^*.

In proving estimate (5.1.1) we will use the following growth rates of Lemma 5.1.4. The proof of estimate (5.1.2) will be given in the next section.

Lemma 5.1.4. *For every $k > 0$ there exist a constant $C > 0$ depending on the eigenvalues $\{\mu_i\}$ of the family $\{M_\alpha\}$ so that for every $\rho \cdot \xi$ approximate solution with $\|\xi\|_2 = 1$*

$$\int_N |y|^{2k}(|\xi|^2 + |\partial^H \xi|^2) dydz \leq Cs^{-k} \quad \textit{and} \quad \int_N |y|^{2k}|\partial^V \xi|^2 dydz \leq Cs^{1-k}$$

Furthermore there exist an $s_0 = s_0(\epsilon)$ so that

$$\int_{\mathcal{N}} |\rho(z,y) \cdot \xi(z,y)|^2 dydz \geq \frac{1}{2}$$

for every $s > s_0$ uniformly on $\|\xi\|_2 = 1$. Here $r = |y|$ is the distance from the singular set $Z_{\mathcal{A}}$.

Proof. We write

$$\xi(z,y) = s^{\frac{n-m}{4}} \sum_i e^{-\frac{1}{2}s\mu_i r^2} \varphi_i$$

with $D^Z \varphi_i = 0$ and $\mu_i > 0$ for all i. Denote by V_{n-m} the measure of the $n-m-1$ - dimensional unit sphere. Then there exist $C > 0$ with

$$\begin{aligned}
\int_N |y|^{2k}|\xi|^2 dydz &= s^{\frac{n-m}{2}} \sum_i \int_N |y|^{2k} e^{-s\mu_i |y|^2} |\varphi_i|^2 dydz \\
&= s^{-k} V_{n-m} \sum_i \int_0^\infty r^{n-m+2k-1} e^{-\mu_i r^2} dr \int_Z |\varphi_i|^2 dz \\
&\leq Cs^{-k} V_{n-m} \sum_i \int_0^\infty r^{n-m-1} e^{-\mu_i r^2} dr \int_Z |\varphi_i|^2 dz \\
&= Cs^{-k}.
\end{aligned}$$

and

$$\begin{aligned}\int_N |y|^{2k}|\partial^V\xi|^2 dydz &= s^{1-k}V_{n-m}\sum_i \int_0^\infty \mu_i^2 r^{2k+n-m+1}e^{-\mu_i r^2}dr\int_Z |\varphi_i|^2 dz \\ &\le Cs^{1-k}V_{n-m}\sum_i \int_0^\infty r^{n-m-1}e^{-\mu_i r^2}dr\int_Z |\varphi_i|^2 dz \\ &= Cs^{1-k}.\end{aligned}$$

Elliptic regularity gives

$$\int_Z |\partial^H\varphi_i|^2 dz \le C_1\int_Z |\varphi_i|^2 dz$$

for every i. Using $2ab \le a^2+b^2$

$$\begin{aligned}\int_N |y|^{2k}|\partial^H\xi|^2 dydz &\le s^{-k}V_{n-m}\sum_{i,j}\int_0^\infty r^{2k+n-m-1}e^{-\frac{\mu_i+\mu_j}{2}r^2}dr\int_Z\langle\partial\varphi_i,\partial\varphi_j\rangle dz \\ &\le Cs^{-k}V_{n-m}\sum_i \int_0^\infty r^{n-m-1}e^{-\mu_i r^2}dr\int_Z |\varphi_i|^2 dz \\ &= Cs^{-k}.\end{aligned}$$

For the last part we change to w-variables and estimate

$$\begin{aligned}\int_{\mathcal{N}} |\rho(z,y)\xi(z,y)|^2 dy &= \int_{\mathcal{N}(\sqrt{s})}\rho^2(\frac{w}{\sqrt{s}})|\xi|^2 dw \ge \int_{\mathcal{N}(\frac{\sqrt{s}}{2})}|\xi|^2 dw \\ &= V_{n-m}\sum_i \int_0^{\epsilon\sqrt{s}} r^{n-m-1}e^{-\mu_i r^2}dr\int_Z |\varphi_i|^2 dz\end{aligned}$$

where $\mathcal{N}(\sqrt{s})$ denote the tubular neighborhood in the normal bundle of radius $\epsilon\sqrt{s}$. But there exist $s_0 = s_0(\epsilon)$ so that $\int_0^{\epsilon\sqrt{s}} r^{n-m-1}e^{-\mu_i r^2}dr > \frac{1}{2}\int_0^\infty r^{n-m-1}e^{-\mu_i r^2}dr$ for every i and every $s > s_0$. The result follows. $\square$

Proof of estimate (5.1.1) in Theorem 5.1.3. Recall the tubular neighborhood $\mathcal{N}$ of Z, the chart $(\mathcal{N}_U, \{z_i, y_\alpha\})$ described at (3.1.1), the frames $\{\partial_i|_U = e_i\}$, $\{\partial_\alpha|_U = e_\alpha\}$, the parallel transport map τ and the bundles $K = \ker \mathcal{A}$ and $\hat{K} = \ker \mathcal{A}^*$ from Chapter 2.

Choose $\eta = \rho \cdot \xi \in V_s$ with $\|\xi\|_2 = 1$. With respect the decompositions $E|_{\mathcal{N}} = K \oplus K^\perp$ and $F|_{\mathcal{N}} = \hat{K} \oplus \hat{K}^\perp$

$$\eta = \rho \cdot \xi = \begin{pmatrix} \rho \cdot s^{\frac{n-m}{4}} e^{-\frac{1}{2} s \mu_i r^2} \tau \varphi_i \\ \\ 0 \end{pmatrix}, \quad D^Z \varphi_i = 0.$$

The Taylor expansion from Corollary 3.2.4 gives

$$\begin{aligned}
(D + s\mathcal{A})\eta \;&=\; d\rho.\xi \,+\, \rho \cdot e^{-\frac{1}{2} s \mu_i r^2} \tau D^Z \varphi_i \\
&+\; \rho \cdot \tau \Big(e^\alpha_. \partial_\alpha \,+\, s y_\alpha \begin{pmatrix} A_\alpha & 0 \\ \\ 0 & *(z) \end{pmatrix} \Big) \begin{pmatrix} e^{-\frac{1}{2} s \mu_i r^2} \varphi_i \\ \\ 0 \end{pmatrix} \\
&+\; \rho \cdot O\big(r^2 \partial^V + r \partial^H + r + s r^3 \big) \xi \\
&=\; d\rho.\xi \,+\, \rho \cdot O\big(r^2 \partial^V + r \partial^H + r + s r^3 \big) \xi. \qquad (5.1.3)
\end{aligned}$$

Because $d\rho$ has support outside the $\epsilon\frac{\sqrt{s}}{2}$-neighborhood of $Z_{\mathcal{A}}$, the L^2 norm of the first term on the right hand side is bounded as

$$\int_{\mathcal{N}} |d\rho.\tau\xi|^2 dydz \;\leq\; C \sum_i \int_{\frac{\epsilon\sqrt{s}}{2}}^{\infty} r^{n-m-1} e^{-\mu_i r^2} dr \;\leq\; \frac{C}{s}.$$

Also, by Lemma 5.1.4 the squared L^2- norm of the error term on the right hand side of (5.1.3) is bounded by $\frac{C}{s}$. Furthermore, there is an $s_0 > 0$, independent of that choice of

η, so that $\|\eta\|_2^2 > \frac{1}{2}$ for every $s > s_0$. This proves the result. $\square$

Applying Lemma 5.1.1 we get a proof of Spectrum Separation Theorem stated in the Introduction:

Proof of Spectral Separation Theorem. Choose $s_0 > 0$ so that the constants of Theorem 5.1.3 satisfy $4\frac{C_1}{s} < C_2$ for every $s > s_0$. Then apply Lemma 5.1.1 for $L = D_s$ with $H = L^2(X, E)$, $H' = L^2(X, F)$ and V_s constructed above. But by construction

$$\begin{aligned} V^s_{Z_\ell} &\simeq \ker\{D^{Z_\ell} : \Gamma(Z_\ell, \mathcal{K}_\ell) \to \Gamma(Z_\ell, \hat{\mathcal{K}}_\ell)\} \\ \hat{V}^s_{Z_\ell} &\simeq \ker\{\hat{D}^{Z_\ell} : \Gamma(Z_\ell, \hat{\mathcal{K}}_\ell) \to \Gamma(Z_\ell, \mathcal{K}_\ell)\} \end{aligned}$$

for every ℓ. This completes the proof. $\square$

Remark 5.1.5. Combining Theorem 5.1.3 with the proof of Lemma 5.1.1 we actually get the stronger statement that if $\|\xi\|_2 = 1$ and $D_s\xi = 0$ then

$$\|\xi - P(\xi)\|_2 \leq \frac{4C_1}{sC_2} \to 0 \quad \text{as} \quad s \to \infty.$$

5.2 A Poincaré-type inequality

This section is entirely devoted to the proof of estimate (5.1.2) of Theorem 5.1.3. Recall the tubular neighborhood $\mathcal{N}$ and the chart (3.1.1). $\mathcal{N}$ is a Riemannian manifold with two equivalent metrics: g^X and $g^Z \times g^N$. These induce two different densities $|dvol|$ and $dzdy$ which, under the exponential map, are related by $|dvol| = kdzdy$ for some map $k : \mathcal{N} \to \mathbb{R}^+$. Henceforth we will be using the density $dzdy$ and the constants of equivalence will be suppressed in the calculations. Note that when $w = \sqrt{s}y$ then $dzdw = s^{\frac{n-m}{2}} dzdy$.

For the proof of estimate (5.1.2) Theorem 5.1.3 we need the following lemma:

Lemma 5.2.1. *If estimate (5.1.2) is true for $\eta \in V_s^\perp$ supported in $\mathcal{N} = \mathcal{N}(2\epsilon)$, a tubular neighborhood of Z_ℓ, then it is true for every $\eta \in V_s^\perp$.*

Proof. Let $\rho_4 : X \to [0, 1]$ a bump function supported in $B(Z_\ell, 2\epsilon)$ with $\rho_4 \equiv 1$ in $B(Z_\ell, \epsilon)$. Write $\eta = \rho_4\eta + (1 - \rho_4)\eta = \eta_1 + \eta_2$ with supp $\eta_1 \subset B(Z_\ell, 2\epsilon)$ and supp $\eta_2 \subset X \backslash B(Z_\ell, \epsilon)$. Then

$$\|D_s\eta\|^2 = \|D_s\eta_1\|^2 + \|D_s\eta_2\|^2 + 2\langle D_s\eta_1, D_s\eta_2\rangle. \tag{5.2.1}$$

Since $\rho_4 \cdot \rho = \rho$ we have $\eta_1 \in V_s^\perp$ and by assumption there exist $C_0 = C_0(\epsilon) > 0$ and $s_0 = s_0(\epsilon) > 0$ so that

$$\|D_s\eta_1\|^2_{L^2} \geq C_0\|\eta_1\|^2_{L^2}$$

for every $s > s_0$. Also since η_2 is supported away of Z_A, by Proposition 1.0.2

$$\|D_s\eta_2\|^2 \geq s^2\|\mathcal{A}\eta_2\|^2 - s|\langle\eta_2, B_A\eta_2\rangle| \geq (s^2k^2\epsilon^2 - sM)\|\eta_2\|^2.$$

To estimate the cross terms we calculate

$$D_s\eta_1 \;=\; \rho_4 D_s\eta + (d\rho_4).\eta, \qquad D_s\eta_2 \;=\; (1-\rho_4)D_s\eta - (d\rho_4).\eta$$

and hence

$$\begin{aligned}\langle D_s\eta_1, D_s\eta_2\rangle \;&=\; \int_X \rho_4(1-\rho_4)\big|D_s\eta\big|^2 dv_g + \int_X (1-2\rho_4)\langle (d\rho_4).\eta, D_s\eta\rangle dv_g \\ &-\; \int_X \big|(d\rho_4).\eta\big|^2 dv_g \;\geq\; -\frac{1}{2}\|D_s\eta\|^2 - \frac{3}{2}\int_X \big|(d\rho_4).\eta\big|^2 dv_g\end{aligned}$$

for every $s, \epsilon > 0$. We used that $|ab| \leq \frac{1}{2}(a^2+b^2)$ and that $(1-2\rho_4)^2 \leq 1$. But $(d\rho_4).\eta$ is supported in $X\ B(Z,\epsilon)$ hence by Proposition 1.0.2 applied again

$$\int_X \big|(d\rho_4).\eta\big|^2 dv_g \leq C_\epsilon \int_{B(Z_\ell,\epsilon)^c} |\eta|^2 dv_g \leq \frac{C_\epsilon}{s^2k^2\epsilon^2 - sM}\|D_s\eta\|_2^2 \leq \frac{1}{3}\|D_s\eta\|_2^2$$

for s large enough. Hence

$$\langle D_s\eta_1, D_s\eta_2\rangle \geq -\|D_s\eta\|_2^2.$$

Substituting to (5.2.1) and absorbing the first term in the left hand side there is an $s_1 = s_1(\epsilon)$ with

$$\begin{aligned}3\|D_s\eta\|^2 \;&\geq\; \|D_s\eta_1\|^2 + (s^2k^2\epsilon^2 - sM)\|\eta_2\|^2 \\ &\geq\; C_0(\|\eta_1\|_{L^2}^2 + \|\eta_2\|_{L^2}^2) \\ &\geq\; C_0\|\eta\|^2\end{aligned}$$

for every $s \geq s_1$. $\square$

Since L^2-norms are additive on sections with disjoint supports it is clear that we can work with $\eta \in V_s^\perp$ so that supp $\eta \subset \mathcal{N}$ for some individual tubular neighborhood of some individual singular component $Z_\ell = Z$.

Proof of estimate (5.1.2) in Theorem 5.1.3. This is a Poincaré type inequality and we prove it by contradiction. Suppose there exist s sequence $\{s_j\} \to \infty$ of positive numbers with no accumulation point and a sequence $\{\eta_j\} \subset L^{1,2}(\mathcal{N}, E|_{\mathcal{N}})$ so that $\eta_j \in V_Z^{s_j}$ has $\|\eta_j\|_2 = 1$ and $\|D_{s_j}\eta_j\|_2^2 \to 0$ as $s_j \to \infty$.

Recall the tubular neighborhood $\mathcal{N} = B(Z, 2\epsilon)$ of Z, the chart $(\mathcal{N}_U, \{z_i, y_\alpha\})$ in (3.1.1). In this chart we have the frames $\{\partial_i|_U = e_i\}$ and $\{\partial_\alpha|_U = e_\alpha\}$, the parallel transport map τ, and the decompositions $E|_Z = (K \oplus K^\perp)|_Z$ and $F|_Z = (\hat{K} \oplus \hat{K}^\perp)|_Z$ from Chapter 2. Introduce

$$\tau\xi_j(z,w) = s_j^{\frac{m-n}{4}} \eta_j\left(\frac{w}{\sqrt{s_j}}\right) \quad \text{and} \quad \xi_j^T(z,w) = \xi_j(z,w)\gamma(z,w) \quad \text{with} \quad \gamma(z,w) = \rho\left(\tfrac{\epsilon w}{T}\right)$$

for $T > 0$ in $w = \sqrt{s_j}y$ coordinates. We have the decomposition $\xi_j^T = \xi_{j1}^T + \xi_{j2}^T$ where

$$\xi_{j1}^T : B(Z,T) \to \pi^*(K|_Z) \quad \text{and} \quad \xi_{j2}^T : B(Z,T) \to \pi^*(K^\perp|_Z)$$

both supported on $B(Z,T)$.

Now in w-coordinates

$$
\begin{aligned}
(D\!\!\!/_{s_j}\xi_j^T)(z,y) &= (e_\cdot^\alpha \partial_{y_\alpha} + s_j y_\alpha \nabla_\alpha \mathcal{A})\xi_j^T(z,y) \\
&= \sqrt{s_j}(e_\cdot^\alpha \partial_{w_\alpha} + w_\alpha \nabla_\alpha \mathcal{A})\xi_j^T(z,w) \\
&= \sqrt{s_j}(D\!\!\!/\xi_j^T)(z,w).
\end{aligned}
$$

Hence in w-coordinates Corollary 3.2.4 shows that

$$
s_j A_0(\xi_{j2}^T) + \tau(\sqrt{s_j}D\!\!\!/ + D^Z)\xi_j^T = D_{s_j}(\xi_j^T) + \frac{1}{\sqrt{s_j}} O\big(|w|^2\partial^V + |w|\partial^H + |w|\big)\xi_j^T \quad (5.2.2)
$$

for every $|w| < T$. The L^2 norm of the left hand side is

$$
\begin{aligned}
\int |s_j A_0(\xi_{j2}^T) + (\sqrt{s_j}D\!\!\!/ + D^Z)\xi_j^T|^2 dwdz &= \int |(s_j A_0 + \sqrt{s_j}D\!\!\!/ + D^Z)\xi_{j2}^T|^2 dwdz \\
&+ \int |(\sqrt{s_j}D\!\!\!/ + D^Z)\xi_{j1}^T|^2 dwdz. \qquad (5.2.3)
\end{aligned}
$$

But by the concentration condition (1.0.1) for A_0 there is a C_1 so that

$$
|(\sqrt{s_j}D\!\!\!/ + D^Z)^* A_0 + A_0^*(\sqrt{s_j}D\!\!\!/ + D^Z)\xi_{j2}^T|^2 \le s_j C_1 |\xi_{j2}^T|^2.
$$

Also there exist $C_2 > 0$ so that $|A_0\xi_{j2}^T|^2 \ge C_2|\xi_{j2}^T|^2$. Hence there exist $C_3 > 0$ so that for all large s_j

$$
\int |(s_j A_0 + (\sqrt{s_j}D\!\!\!/ + D^Z))\xi_{j2}^T|^2 dwdz \ge s_j^2 C_3 \int |\xi_{j2}^T|^2 dwdx + \int |(\sqrt{s_j}D\!\!\!/ + D^Z)\xi_{j2}^T|^2 dwdz.
$$

Substituting back to (5.2.3)

$$\int |s_j A_0(\xi_{j2}^T) + (\sqrt{s_j}\not{D} + D^Z)\xi_j^T|^2 dwdz \geq s_j^2 C_3 \int |\xi_{j2}^T|^2 dwdz + \int |(\sqrt{s_j}\not{D} + D^Z)\xi_j^T|^2 dwdz. \tag{5.2.4}$$

By ellipticity of $\not{D} + D^Z$ the L^2 norm of the error term of (5.2.2) is bounded by

$$\|O\big(|w|^2\partial^V + |w|\partial^H + |w|\big)\xi_j^T\|_2^2 \leq C_T\big(\|(\not{D} + D^Z)\xi_j^T\|_2^2 + \|\xi_j^T\|_2^2\big).$$

Therefore, taking L^2 norms of (5.2.2), substituting (5.2.4), using Proposition 4.2.1 and absorbing terms in the left hand side, one obtains

$$s_j^2 C_3 \int_{B(Z,T)} |\xi_{j2}^T|^2 dwdz + (s_j - \frac{C_T}{s_j})\|\not{D}\xi_j^T\|_2^2 + (1 - \frac{C_T}{s_j})\|D^Z\xi_j^T\|_2^2 \leq \frac{\epsilon^2}{T^2}\int_X |d\rho.\xi_j|^2 dzdw + \|D_{s_j}\eta_j\|_2^2. \tag{5.2.5}$$

By assumption, the right hand side is bounded in j, hence $\|\xi_{j2}^T\|_2 \to 0$, $\|\not{D}\xi_j^T\|_2 \to 0$, and the sequence

$$\|(\not{D} + D^Z)\xi_j^T\|_2^2 \tag{5.2.6}$$

is uniformly bounded in j. By elliptic regularity for the operator $\not{D} + D^Z$, the sequence $\{\xi_j^T\} \subset L^{1,2}$ is bounded. By Rellich Theorem there is a subsequence, denoted again as $\{\xi_j^T\}$, that converges to $\xi^T : B(Z,T) \to \pi^*(K|_Z)$,where ξ^T is a compactly supported section with $\|\xi^T\|_2 \leq 5\|\xi_i\|_2 = 5$. By the weak compactness of the unit ball in $L^{1,2}$, we can also assume that $\xi_i^T \to \xi^T$ weakly in $L^{1,2}$. In this context, (5.2.5) shows that for every smooth section ψ

$$\begin{aligned}\int_{B(Z,T)} \langle D^Z \xi^T, \psi \rangle dwdz &= \int_{B(Z,T)} \langle \xi^T, \hat{D}^Z \psi \rangle dwdz = \lim_i \int_{B(Z,T)} \langle \xi_i^T, \hat{D}^Z \psi \rangle dwdz \\ &= \lim_i \int_{B(Z,T)} \langle D^Z \xi_i^T, \psi \rangle dwdz \leq \lim_i \|D^Z \xi_i^T\|_2 \|\psi\|_2 \\ &\leq \frac{C_\epsilon}{T} \|\psi\|_2.\end{aligned}$$

Therefore ξ^T satisfies

$$\|D^Z \xi^T\|_2 \leq \frac{C_\epsilon}{T}, \qquad \not{D} \xi^T = 0 \qquad \text{and} \qquad \|\xi^T\|_2 \leq 5$$

for every $T > 0$. Now notice that when $T' > T$ then $\xi^{T'}$ agrees with ξ^T in $B(Z, \frac{T}{2})$. Hence there is a well-defined section $\xi : N \to \pi^*(K|_Z)$ with $\xi = \xi^T$ on every neighborhood $B(Z, \frac{T}{2}) \subset N$. Furthermore, by Lemma 5.2.2 below, there is an estimate

$$\|\xi^T\|_{1,2} \leq C(\|(\not{D} + D^Z)\xi^T\|_2 + \|\xi^T\|_2) \leq C \left(\frac{C_\epsilon}{T} + 5 \right) \tag{5.2.7}$$

where the constant C is independent of T. Letting $T \to \infty$ we see that ξ is in fact an $L^{1,2}$-section satisfying

$$\not{D} \xi = 0 \quad \text{and} \quad D^Z \xi = 0. \tag{5.2.8}$$

Claim : $\xi \equiv 0$

By assumption $\eta_i \perp V_Z^{s_i}$. Using Definition 5.1.2 this condition translates in $w = \sqrt{s_i} y$

coordinates as

$$\eta_i \perp \rho \cdot s_i^{\frac{n-m}{4}} e^{-\frac{1}{2}s_i\mu|y|^2}\tau\psi \iff \xi_i \perp \rho_{s_i} \cdot e^{-\frac{1}{2}\mu|w|^2}\psi$$

for every section $\psi : Z \to \mathcal{K}$ with $D^Z\psi = 0$.

The sequence ξ_i^T is supported in $B(Z,T)$ and since $\|\xi_i\|_2 = 1$

$$\begin{aligned}
\int_{B(Z,T)} \rho_{s_i} \cdot e^{-\frac{1}{2}\mu|w|^2}\langle \xi_i^T, \psi\rangle dwdz &= \int_{B(Z,T)} \rho_{s_i} \cdot e^{-\frac{1}{2}\mu|w|^2}\langle \xi_i^T - \xi_i, \psi\rangle dwdz \\
&= \int_{B(Z,T)} \rho_{s_i} \cdot e^{-\frac{1}{2}\mu|w|^2}\langle (\gamma - 1)\xi_i, \psi\rangle dwdz \\
&\leq \int_{B(Z,T/2)^c} e^{-\frac{1}{2}\mu|w|^2}|\psi|^2 dzdw
\end{aligned}$$

for every i, T. Hence passing to the L^2 limit as $i \to \infty$

$$\int_{B(Z,T)} e^{-\frac{1}{2}\mu|w|^2}\langle \xi^T, \psi\rangle dwdz \leq \int_{B(Z,T/2)^c} e^{-\frac{1}{2}\mu|w|^2}|\psi|^2 dzdw.$$

Letting $T \to \infty$ we see that ξ is L^2 orthogonal to $e^{-\frac{1}{2}\mu|w|^2}\psi$. But ξ is in $L^{1,2}$ and satisfies (5.2.8) therefore, by (4.2.1) $\xi \equiv 0$. This proves the claim.

It now follows that for every $T > 0$, as $i \to \infty$

$$\lim_{i\to\infty} \int_{B(Z,\frac{T}{2\sqrt{s_i}})} |\eta_i|^2 dydz = \lim_{i\to\infty} \int_{B(Z,T/2)} |\xi_i|^2 dwdz = \int_{B(Z,T/2)} |\xi|^2 dwdz = 0.$$

Finally, we obtain a contradiction from the concentration estimate. By the nondegeneracy assumption (1.0.9) the bundle map $\mathcal{A}$ satisfies $\mathcal{A}^*\mathcal{A}|_K = |y|^2 M + O(|y|^3)$. Hence by the proof of Theorem 1.0.4 we estimate

$$\begin{aligned}
\|D_{s_i}\eta_i\|_2^2 &\geq s_i^2 \int_{B(Z,\frac{T}{2\sqrt{s_i}})^c} |\mathcal{A}(\eta_i)|^2 dydz - C_1 s_i \int_{B(Z,\frac{T}{2\sqrt{s_i}})^c} |\eta_i|^2 dydz \\
&\geq s_i(\frac{C_0}{4}T^2 - C_1) \int_{B(Z,\frac{T}{2\sqrt{s}})^c} |\eta_i|^2 dydz \\
&= s_i(\frac{C_0}{4}T^2 - C_1)\big(1 - \int_{B(Z,\frac{T}{2\sqrt{s_i}})} |\eta_i|^2 dydz\big).
\end{aligned}$$

But then for a fixed $T > 0$ large enough, as $i \to \infty$ we have $\liminf_i \|D_{s_i}\eta_i\|_2 = \infty$ contrary to our original assumption that $\|D_{s_i}\eta_i\|_2 \to 0$. $\square$

The following Lemma was used above to obtain estimate (5.2.7).

Lemma 5.2.2. *For any compactly supported section $\xi : B(Z,T) \to \pi^*(K|_Z)$, there is an elliptic estimate*

$$\|\xi\|_{1,2} \leq C(\|(\not{D} + D^Z)\xi\|_2 + \|\xi\|_2)$$

for some constant $C > 0$ independent of T.

Proof. Recall the calculation (4.1.12)

$$\not{D}^* \not{D}\xi = -\sum_\alpha \partial_\alpha^2 \xi - \sum_\alpha M_\alpha \xi + \sum_\alpha w_\alpha^2 M_\alpha^2 \xi$$

where M_α are self-adjoint. Taking L^2 inner products with ξ, we obtain

$$\begin{aligned}
\|\not{D}\xi\|_2^2 &= \sum_\alpha \|\partial_\alpha \xi\|_2^2 - \sum_\alpha \int \langle M_\alpha \xi, \xi\rangle dwdz + \sum_\alpha \int w_\alpha^2 |M_\alpha \xi|^2 dwdz \qquad (5.2.9)\\
&\geq \sum_\alpha \|\partial_\alpha \xi\|_2^2 - \sum_\alpha \int \langle M_\alpha \xi, \xi\rangle dwdz\\
&\geq \sum_\alpha \|\partial_\alpha \xi\|_2^2 - C_1 \|\xi\|_2^2
\end{aligned}$$

with C_1 independent of T. Also, there is a Weitzenbock identity for $D^Z : \Gamma(Z, K|_Z) \to \Gamma(Z, \hat{K}|_Z)$ of the form

$$\hat{D}^Z D^Z \xi = \bar{\nabla}^* \bar{\nabla} \xi + R\xi$$

for some curvature term R. Again taking L^2 inner products with ξ, there is a constant $C_2 > 0$ with

$$\int_Z |\bar{\nabla}\xi|^2 dz \leq C \int_Z (|D^Z \xi|^2 + |\xi|^2) dz.$$

When ξ is instead, a section supported in $B(Z,T)$ we integrate this inequality with respect to the w-variable to get

$$\|\partial^H \xi\|_2 \leq C(\|D^Z \xi\|_2 + \|\xi\|_2) \qquad (5.2.10)$$

Combining (5.2.9) and (5.2.10) we get the result. $\square$

Chapter 6

Nonlinear concentration

6.1 The nonlinear model

There is a simple way to "nonlinearize" concentrating Dirac operators. Suppose

$$D_s = D + s\mathcal{A}_\psi : \Gamma(W) \to \Gamma(\widehat{W})$$

is a concentrating Dirac operator determined by $\psi \in \Gamma(\mathcal{L})$. Suppose that W decomposes as a sum $\oplus_j W_j$ of bundles, one of which, say $W_1 = \mathcal{L}$. Then we can convert the linear PDE

$$D\xi + s\mathcal{A}_\psi \xi = \xi_0 \qquad \xi = (\xi_1, \ldots, \xi_k)$$

to a non-linear equation simply by taking $\psi = \xi_1$. The resulting equation

$$D\xi + s\mathcal{A}_\xi \xi = \xi_0$$

is quadratic in ξ. It is interesting to study the concentrating properties of the solutions to this problem.

For example, let $(X, g, W^+ \oplus W^-, c)$ a $2n$-dimensional Spin^c Riemannian manifold with determinant bundle $L = \det_{\mathbb{C}}(W^+)$. A fixed Hermitian connection A_0 on L determines a Dirac operator D_0 acting on spinors. Fixing additionally $\psi \in \Gamma(W^+)$, special instance of the Clliford pairs example is

$$\mathcal{D}\xi + \frac{1}{2}\mathcal{A}_\psi \xi = \xi_0 \tag{6.1.1}$$

where $\xi = \begin{pmatrix} \varphi \\ \alpha \end{pmatrix} \in \Gamma(W^+ \oplus \Lambda^{odd}X)$ and $\xi_0 \in \Gamma(W^- \oplus \Lambda^{ev}X)$. Since $\psi, \varphi \in \Gamma(W^+)$ we can take $\psi = \varphi$ to get a nonlinear example.

6.2 Examples

In examples 8-10 below, we take W_1 to be a spin bundle and W_2 to be a bundle whose sections are connections, as in the linear examples 4 and 5 of Chapter 2. However, we restrict the operators $P_\psi^{ev/odd}$ to the real subbundle $\Lambda^{ev/odd}X \otimes i\mathbb{R}$ of $\Lambda^*_{\mathbb{C}}X$ and we treat the term α of ξ as a connection 1-form. Introduce a new connection A of L satisfying

$$A - A_0 = \alpha, \quad F_A - F_{A_0} = d\alpha \qquad \text{and} \qquad D_A - D_{A_0} = \frac{1}{2}\alpha.$$

for $\alpha \in \Omega^1(X, i\mathbb{R})$.

Finally it worth to be noted than when we restrict $P_\psi^{ev/odd}$ to $\Lambda^{ev/odd}X$ the map

$$W^+ \to \mathrm{End}(E^+ \oplus E^-) : \psi \mapsto \begin{pmatrix} 0 & \mathcal{A}_\psi^* \\ \mathcal{A}_\psi & 0 \end{pmatrix}$$

defines a real Clifford representation of W^+ to E.

Example 8: Let (Σ, ω, J) be a closed Riemman surface with canonical bundle K_Σ and a holomorphic line bundle L giving a Spin^c structure on Σ. Then $W^+ = L$, $W^- = K_\Sigma^{-1} \otimes L$ and $D_{A_0} = \bar{\partial}_{A_0}$. For $\xi_0 = \begin{pmatrix} 0 \\ \frac{ir}{2}\omega - F_{A_0} \end{pmatrix}$ equation (6.1.1) is rewritten as

$$\begin{aligned} \bar{\partial}_{A_0}\psi + \frac{1}{2}\alpha.\psi &= 0 \\ (d + d^*)\alpha - \frac{1}{2}P_\psi^{ev*}\psi - \frac{ir}{2}\omega + F_{A_0} &= 0. \end{aligned}$$

But for every $\rho \in \mathbb{R}$

$$\langle P_\psi^{ev*}\psi, (\rho, \omega)\rangle = \rho|\psi|^2 - i|\psi|^2$$

since $\omega.\psi = i\psi$. So $P_\psi^{ev*}\psi = |\psi|^2 - i|\psi|^2\omega$ and the projection to imaginary part is $-i|\psi|^2\omega$. Therefore the above equations are rewritten as

$$\begin{aligned} \bar{\partial}_A\psi &= 0 \\ F_A &= -\frac{i}{2}(|\psi|^2 - r)\omega \\ d^*\alpha &= 0. \end{aligned}$$

The first two equations are the r-vortex equations and the third one is a Coulomb condition defining a slice of the $U(1)$- gauge action on $\Gamma(\mathfrak{u}(L))$. Vortex equations have been studied thoroughly over $\mathbb{C}$ by C. Taubes (see [JT]) and over closed Riemann Surfaces and more general line bundles over closed Kähler manifolds by O. Garcia-Prada (see [O]) and S.

Bradlow (see [B]).

Example 9: On a symplectic 4-manifold (X, ω, J), adopting the definitions of Example 4 of the linear case for $\mathcal{D}$ and $\mathcal{A}_\psi$, using as $\xi_0 = \begin{pmatrix} 0 \\ -\frac{ir}{\sqrt{2}}\omega - \sqrt{2}F^+_{A_0} \end{pmatrix}$ for $r \in \mathbb{R}$, equation (6.1.1) is rewritten as

$$\begin{aligned} D_{A_0}\psi + \frac{1}{2}\alpha.\psi &= 0 \\ (\sqrt{2}d^+ + d^*)\alpha - \frac{1}{2}P^{ev*}_\psi\psi + \frac{ir}{\sqrt{2}}\omega + \sqrt{2}F^+_{A_0} &= 0. \end{aligned}$$

However a simple calculation shows that $P^{ev*}_\psi\psi = \sqrt{2}\tau(\psi \otimes \bar{\psi})$ therefore

$$\begin{aligned} D_A\psi &= 0 \\ F^+_A &= \frac{1}{2}(\tau(\psi \otimes \bar{\psi}) - ir\omega) \\ d^*\alpha &= 0. \end{aligned} \tag{6.2.1}$$

The first two are the perturbed symplectic Seiberg-Witten equations and the last one is defining a slice of the $U(1)$- gauge action on $\Gamma(\mathfrak{u}(L))$ as in the previous example. If (X, g) is a 4-manifold with out a symplectic structure then for $\xi_0 = \begin{pmatrix} 0 \\ -\sqrt{2}F^+_{A_0} \end{pmatrix}$ we get the unperturbed Seiberg-Witten equations.

Example 10: The following example is the next interesting perturbation on a non Kähler complex surface with positive geometric genus. This case has been studied by O. Biquard (see [OL]). If (X, J, g) is a complex surface with $\dim H^{2,0}_{\mathbb{C}}(X) > 0$ then we can choose a holomorphic section b of the anticanonical bundle K_X^{-1} of X and take $\xi_0 = \begin{pmatrix} 0 \\ \frac{r}{\sqrt{2}}(b+\bar{b}) - \sqrt{2}F^+_{A_0} \end{pmatrix}$. Equation (6.1.1) will then correspond to the equations

$$\begin{aligned} D_A\psi &= 0 \\ F^+_A &= \frac{1}{2}\tau(\psi\otimes\bar{\psi}) + \frac{r}{2}(b+\bar{b}) \\ d^*\alpha &= 0. \end{aligned}$$

Remark: *(L^2-concentration for the symplectic SW equations.)* The perturbation $ir\omega$ by the symplectic form was introduced by Witten for Kähler manifolds, then studied in detail by Taubes, who showed solutions localize as $r\to\infty$. We include this remark with the L^2-nonlinear concentration (see also [D]). For the subtle pointwise estimates see [T2] and for the subtle behavior of the concentrating set see [T1] and [T2] This perturbation is different from the perturbation $s\mathcal{A}_\psi$ in the concentration Theorem 1.0.4.

In the following calculation we basically use a parameter δ to interpolate between the Weitzenbock formulas of nonlinear Examples 7 and 8. This enables one to analyze the nonlinear concentration for the pair of equations (6.2.1).

With respect to an orthonormal basis $\{1, \frac{1}{2}d\bar{z}_1\wedge d\bar{z}_2\}$, we denote by $(F_A)_\omega$ the part of F_A parallel to ω, $F^{0,2}_A = (F_A)^{0,2}d\bar{z}_1\wedge d\bar{z}_2$, $\beta = \frac{1}{2}(\beta)d\bar{z}_1\wedge d\bar{z}_2$ and one can write $\psi = \sqrt{r}(a,(\beta))$ on $W^+ = L\oplus L\bar{K}$. Then

$$\psi\otimes\bar{\psi} - \frac{1}{2}|\psi|^2 Id = \begin{pmatrix} \frac{1}{2}(|a|^2-|\beta|^2) & a(\bar{\beta}) \\ \bar{a}(\beta) & \frac{1}{2}(|\beta|^2-|a|^2) \end{pmatrix}$$

and

$$c(F_A^+) = \begin{pmatrix} -2i(F_A)_\omega & -4(F_A)^{2,0} \\ 4(F_A)^{0,2} & 2i(F_A)_\omega \end{pmatrix}.$$

Equations (6.2.1) are then rewritten as

$$i(F_A)_\omega = \frac{r}{4}(2 - |a|^2 + |\beta|^2) \qquad (6.2.2)$$

$$F_A^{0,2} = \frac{r}{2}\bar{a}\beta. \qquad (6.2.3)$$

If ∇_A and $\bar{\partial}_A$ are the connection with it's (0,1) - part on $L \to X$, then $D_A = \sqrt{2}(\bar{\partial}_A, \bar{\partial}_A^*)$ is the Dirac operator on W^+ and we have the identities:

$$\bar{\partial}_A^2 a = \frac{1}{2}F_A^{0,2}a - \frac{1}{4}\partial_A a \circ N_J \qquad (6.2.4)$$

$$\bar{\partial}_A^*\bar{\partial}_A a = \frac{1}{2}(\nabla_A^*\nabla_A a - i(F_A)_\omega a) \qquad (6.2.5)$$

$$\bar{\partial}_A\bar{\partial}_A^*\beta = \frac{1}{2}\nabla_A^*\nabla_A\beta + i(R + \frac{1}{2}F_A)_\omega\beta \qquad (6.2.6)$$

$$c_1(L)[\omega] = \frac{i}{2\pi}\int_X \frac{1}{2}(F_A)_\omega\omega \wedge \omega = \frac{i}{2\pi}\int_X (F_A)_\omega dv_g \qquad (6.2.7)$$

where R denotes the curvature of $\bar{K}$. Observe that the difference in the i signs of formulas (6.2.5) and (6.2.6) comes from the fact that ω. has eigenspaces L and $L\bar{K}$ with eigenvalues $-2i$ and $2i$ respectively. Finally we compute:

$$0 = \frac{1}{2}D_A^*D_A\psi = \begin{pmatrix} \bar{\partial}_A^* \\ \bar{\partial}_A \end{pmatrix} (\bar{\partial}_A a + \bar{\partial}_A^*\beta) = \begin{pmatrix} \bar{\partial}_A^*\bar{\partial}_A a & + & \bar{\partial}_A^*\bar{\partial}_A^*\beta \\ \bar{\partial}_A^2 a & + & \bar{\partial}_A\bar{\partial}_A^*\beta \end{pmatrix}. \qquad (6.2.8)$$

Taking L^2 - product with a on the first row element of (6.2.8) and using (6.2.4) and

(6.2.5), for $0 < \delta < 1$:

$$
\begin{aligned}
0 &= \langle \bar{\partial}_A^* \bar{\partial}_A a,\, a\rangle + \langle \bar{\partial}_A^* \bar{\partial}_A^* \beta,\, a\rangle = (1-\delta)\|\bar{\partial}_A a\|^2 + \frac{\delta}{2}\|\nabla_A a\|^2 \\
&- \frac{i\delta}{2}\int_X (F_A)_\omega |a|^2 dv_g + \langle \beta,\, \bar{\partial}_A^2 a\rangle \\
&= (1-\delta)\|\bar{\partial}_A a\|^2 + \frac{\delta}{2}\|\nabla_A a\|^2 - i\delta \int_X (F_A)_\omega dv_g + \frac{i\delta}{2}\int_X (F_A)_\omega (2 - |a|^2) dv_g \\
&+ \frac{1}{2}\langle \beta,\, F_A^{0,2} a\rangle - \frac{1}{4}\langle \beta,\, \partial_A a \circ N_J\rangle.
\end{aligned}
$$

Using now equations (6.2.2), (6.2.3) and (6.2.7) and after a few calculations:

$$
\begin{aligned}
0 &= (1-\delta)\|\bar{\partial}_A a\|^2 + \frac{\delta}{2}\|\nabla_A a\|^2 + \frac{r\delta}{4}\|\beta\|^2 + \frac{r}{8}(2-\delta)\|a\beta\|^2 + \frac{r\delta}{8}\int_X (2-|a|^2)^2 dv_g \\
&- 2\pi\delta c_1(L)[\omega] - \frac{1}{4}\langle \beta,\, \partial_A a \circ N_J\rangle. \qquad (6.2.9)
\end{aligned}
$$

Similarly taking L^2 - product with β on the second row element of (6.2.8) and using (6.2.4) and (6.2.6):

$$
\begin{aligned}
0 &= \langle \bar{\partial}_A^2 a,\, \beta\rangle + \langle \bar{\partial}_A \bar{\partial}_A^* \beta,\, \beta\rangle = \frac{1}{2}\langle F_A^{0,2} a,\, \beta\rangle - \frac{1}{4}\langle \partial_A a \circ N_J,\, \beta\rangle + \frac{1}{2}\|\nabla_A \beta\|^2 \\
&+ i\int_X R_\omega |\beta|^2 dv_g + \frac{i}{2}\int_X (F_A)_\omega |\beta|^2 dv_g.
\end{aligned}
$$

Using again equations (6.2.2) and (6.2.3) we conclude:

$$
\begin{aligned}
0 &= \frac{1}{2}\|\nabla_A \beta\|^2 + \frac{r}{4}\|\beta\|^2 + \frac{r}{8}\|a\beta\|^2 + \frac{r}{8}\int_X |\beta|^4 dv_g + i\int_X R_\omega |\beta|^2 dv_g \\
&- \frac{1}{4}\langle \partial_A a \circ N_J,\, \beta\rangle. \qquad (6.2.10)
\end{aligned}
$$

Finally, adding up (6.2.9) and (6.2.10), the Weitzenbock formula from (6.2.8) will read:

$$\begin{aligned}(1-\delta)\|\bar{\partial}_A a\|^2 \;&+\; \frac{\delta}{2}\|\nabla_A a\|^2 + \frac{1}{2}\|\nabla_A \beta\|^2 + (1+\delta)\frac{r}{4}\|\beta\|^2 \\ &+\; (3-\delta)\frac{r}{8}\|a\beta\|^2 + \frac{r}{8}\int_X (|\beta|^4 + \delta(2-|a|^2)^2)dv_g \\ &=\; 2\pi\delta c_1(L)[\omega] + \frac{1}{2}\Re\langle \beta,\, \partial_A a \circ N_J\rangle - i\int_X R_\omega |\beta|^2 dv_g \quad (6.2.11)\end{aligned}$$

for every $0 < \delta < 1$. On the other hand, there exist $C = C(X, \omega, J) > 0$ such that

$$\begin{aligned}\frac{1}{2}\Re\langle \beta,\, \partial_A a \circ N_J\rangle \;&-\; i\int_X R_\omega |\beta|^2 dv_g \leq \frac{C}{\delta}\|\beta\|^2 + \frac{\delta}{2}\|\partial_A a\|^2 + C\|\beta\|^2 \\ &=\; (1+\delta)\frac{C}{\delta}\|\beta\|^2 + \frac{\delta}{2}\|\partial_A a\|^2.\end{aligned}$$

Substituting to (6.2.11) and absorbing the terms to left hand side we get

$$\begin{aligned}(1-\delta)\|\bar{\partial}_A a\|^2 \;&+\; \frac{1}{2}\|\nabla_A \beta\|^2 + (1+\delta)(\frac{r}{4} - \frac{C}{\delta})\|\beta\|^2 + (3-\delta)\frac{r}{8}\|a\beta\|^2 \\ &\leq\; \frac{r}{8}\int_X (|\beta|^4 + \delta(2-|a|^2)^2)dv_g + 2\pi\delta c_1(L)[\omega].\end{aligned}$$

For r large and $\delta = \frac{5C}{r}$, the terms in the left hand side are all positive and we get the desired concentration when $c_1(L)[\omega] > 0$ i.e. $L \to X$ is a nontrivial line bundle and the zero set of it's section a is nonempty. From this we see that as $r \to \infty$

$$\|\beta\|_{L^{1,2}} \to 0 \qquad \|\bar{\partial}_A \alpha\|_{L^2} \to 0 \qquad \|2-|\alpha|^2\|_{L^2} \to 0.$$

$\square$

APPENDIX

Manifold structures on the space of Fredholm operators

Let E, E' be separable Hilbert spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let also $\mathcal{B} := \mathcal{B}(E, E')$ be the set of bounded linear operators, $\mathcal{F} := \mathcal{F}(E, E')$ the space of Fredholm operators and $\mathcal{F}_i := \mathcal{F}_i(E, E')$ the space of Fredholm operators of index i. Then $\mathcal{F}_i$ are the open connected components of the set $\mathcal{F}$, an open subset of the set $\mathcal{B}$. Then $\mathcal{F}_i = \bigcup_{k \geq 0} \mathcal{F}_i^k$ where $\mathcal{F}_i^k$ are Fredholm operators with k - dimensional kernel and $(k - i)$ - dimensional cokernel $(k \geq i)$.

Proposition .0.1. *Each $\mathcal{F}_i^k$ is a $k(k-i)$ - codimensional submanifold of $\mathcal{F}_i$ and for $D \in \mathcal{F}_i^k$*

$$T_D \mathcal{F}_i^k = \{P \in \mathcal{B} : P(\ker D) \subseteq \mathrm{Im} D\}.$$

Proof. Let $E = \ker D \oplus \mathrm{Im} D^*$, $E' = \ker D^* \oplus \mathrm{Im} D$ where D^* is the corresponding adjoint of D. Then with respect to this decomposition D and an arbitrary D' will be written as

$$D = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad D' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $d \in \mathcal{F}_0^0(\mathrm{Im} D^*, \mathrm{Im} D)$. We would like to parametrize $\mathcal{F}_i^k$ near D. To do that suppose there exist $k_1 : \ker D \to (\ker D)^\perp = \mathrm{Im} D^*$ with adjoint k_1^* and $k_2 : \operatorname{coker} D = \ker D^* \to \mathrm{Im} D = (\ker D^*)^\perp$ with adjoint k_2^*. Then

$$\{(x, k_1(x)) : x \in \ker D\} = \ker D' \qquad \text{and} \qquad \{(y, k_2(y)) : y \in \ker D^*\} = \ker (D')^*$$

for some operator $D' \in \mathcal{F}_i^k$ since given a pair of different finite dimensional subspaces we can always construct operator having them as kernel and cokernel, a consequence of Hahn -Banach Theorem. We also get the extra set theoretic relations

$$\{(-k_1^*(x), x)) : x \in (\ker D)^{\perp}\} = (\ker D')^{\perp} \quad \text{and} \quad \{(-k_2^*(y), y)) : y \in \mathrm{Im} D\} = \mathrm{Im} D'.$$

Since $D' : (\ker D')^{\perp} \to \mathrm{Im} D'$ will be an isomorphism, for every $y \in \mathrm{Im} D$ there exist unique $x \in (\ker D)^{\perp}$ with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} -k_1^*(x) \\ x \end{pmatrix} = \begin{pmatrix} -k_2^*(y) \\ y \end{pmatrix}$$

so we get

$$-\alpha \circ k_1^*(x) + \beta(x) = -k_2^*(y) \tag{.0.1}$$

$$-\gamma \circ k_1^*(x) + \delta(x) = y \tag{.0.2}$$

and furthermore $-\gamma \circ k_1^* + \delta : (\ker D)^{\perp} \to \mathrm{Im} D$ has to be isomorphism. Eliminating y from (.0.1) and (.0.2) we get

$$-\alpha \circ k_1^* + \beta = k_2^* \circ \gamma \circ k_1^* - k_2^* \circ \delta \tag{.0.3}$$

on $(\ker D)^{\perp}$. On the other hand, for every $x \in \ker D$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ k_1(x) \end{pmatrix} = 0$$

so

$$\begin{aligned} \alpha + \beta \circ k_1 &= 0 && (.0.4) \\ \gamma + \delta \circ k_1 &= 0. && (.0.5) \end{aligned}$$

Substituting γ from (.0.5) into (.0.2) we get

$$-\gamma \circ k_1^* + \delta = \delta \circ (k_1 \circ k_1^* + id) \qquad (.0.6)$$

Furthermore substituting α and γ from (.0.4) and (.0.5) respectively to (.0.3) we get

$$\beta \circ (k_1 \circ k_1^* + id) = -k_2^* \circ \delta \circ (k_1 \circ k_1^* + id). \qquad (.0.7)$$

However since E and E' are separable and $k_1 \circ k_1^*$ is selfadjoint, by spectral theory, $k_1 \circ k_1^* + id : (\ker D)^{\perp} \to (\ker D)^{\perp}$ is an isomorphism. Therefore we get that

- using (.0.6) $\delta : (\ker D)^{\perp} \to \mathrm{Im} D$ is an isomorphism
- using (.0.7) $\beta = -k_2^* \circ \delta$

Finally

$$
\begin{aligned}
D' &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} k_2^* \circ \delta \circ k_1 & -k_2 \circ \delta \\ -\delta \circ k_1 & \delta \end{pmatrix} \\
&= \begin{pmatrix} -k_2^* & k_2^* \\ id & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -k_1 & id \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

If D' is close to D in the operator topology then we can see that $\ker D'$ and $\mathrm{Im}D'$ can be described uniquely as graphs via some k_1 and k_2 respectively giving the above decomposition for D' in terms of D. Conversely given

- $k_1 \in \mathcal{B}(\ker D, (\ker D)^{\perp}) \subseteq \mathcal{G}_k(E)$
- $k_2 \in \mathcal{B}((\mathrm{Im}D)^{\perp}, \mathrm{Im}D) \subseteq \mathcal{G}_{k-i}(E')$
- $\delta \in \mathcal{F}_0^0((\ker D)^{\perp}, \mathrm{Im}D)$

we can form the map $\Phi(k_1, k_2, \delta) = D'$ as described in the above decomposition with the resulting $D' \in \mathcal{F}_i^k$. This will be the required chart of the space $\mathcal{F}_i^k(E, E')$ around D. Using this chart it is easy to verify the last description of the tangent space $T_D\mathcal{F}_i^k$. $\square$

Remark .0.2. 1. For $k = 1$ a complement of the space $T_D\mathcal{F}_0^1$ is the family of spaces

$$\mathcal{C}_D^C = \{P \in \mathcal{B}(E, E') : P(\ker D) = C\}$$

for C any 1 - dimensional complement of $\mathrm{Im}D$ in E'. If $D \in \mathcal{F}_1^1$ then $T_D\mathcal{F}_1^1 = \mathbb{B}$ so we have trivial complement in this case.

2. In a sufficiently small neighborhood $U \subset \mathcal{F}(E, E')$ of D, using the decompositions $E = \ker D \oplus \mathrm{Im} D^*$ and $E' = \ker D^* \oplus \mathrm{Im} D$, the map

$$F_k : U \to \mathrm{Hom}(\ker D, \ker D^*) : D' = \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} \mapsto A - B \circ \Delta^{-1} \circ \Gamma$$

satisfies the relation $F_k^{-1}(\{0\}) = U \cap \mathcal{F}_i^k$ giving a local model structure of a subvariety to $\mathcal{F}_i^k \subset \mathcal{F}(E, E')$. Actually the various $\{F_k : k \geq 1\}$ give to $\bigcup_{k \geq 1} \mathcal{F}_i^k$ the structure of a stratified subvariety of $\mathcal{B}(E, E')$.

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