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ABSTRACT

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By

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The purpose of this work is to study the asymptotic behavior of the solutions of *Stochastic Evolution Equations.* More precisely, we investigate the stability of and the invariant measures for the mild and strong solutions of the equations.

A sufficient condition for such asymptotic behavior is the ultimate boundedness of the solutions. In the first part this concept is studied for the strong solution under coercivity condition with an eye towards applications to stochastic PDE's. In fact, under ultimate boundedness, we get recurrence behavior for the solution in the second part. Finally, we study asymptotic behavior of the mild solution through approximation by a sequence of strong solutions.

The main technique used is the construction of a Lyapunov function for linear equation and use it for non-linear equation through first order approximation. This makes our results applicable to stochastic PDE's. We derive asymptotic behavior specifically for Navier-Stokes, Parabolic Ito and random heat equations.

To: Ming Wu — my wife

and

our parents

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Contents

Introduction

The purpose of this work is to study the asymptotic behavior of the solutions of stochastic evolution equations. More specifically, we study the stability of and the invariant measure for the mild solution and the invariant measure for the strong solution. In the case of the strong solution, one needs coercivity condition on the coefficients if the initial value is "non-smooth", thus making the results applicable to stochastic partial differential equations (SPDEs). On the other hand, in the case of the mild solution, we can dispense with the coercivity condition.

In the case of the finite-dimensional stochastic differential equation (SDE), Wonham, Zakai and Miyahara have considered ultimate boundedness of the solution which guarantees not only the existence of an invariant measure, but also weak recurrence of the solution to closed bounded, thus compact sets. We study the above problems by considering ultimate boundedness of the solution of the SPDE. We use the method of Lyapunov functions for both the mild and strong solutions. This allows us to derive the known results in a very simple manner.

In the first part of this dissertation, we study ultimate boundedness and the existence of the invariant measure for the solution of the SPDE. In the work of Khasminskii and Mandrekar [15] and the work of Mandrekar [17], the exponential stability of the zero solution of the stochastic evolution equation was studied and a Lyapunov function was constructed under this condition. It is clear that exponential stability of the zero solution implies that the system has an invariant measure degenerate at zero. Thus two questions arise:

1. What is a general condition (less restrictive than exponential stability) under which one can construct a Lyapunov function?

2. Can one consider conditions under which a non-trivial finite invariant measure exist?

Following the ideas of Wonham [27] and Zakai [29], Miyahara [18] introduced the concept of exponential ultimate boundedness in mean square sense (m. s. s.) and constructed a Lyapunov function for the finite dimensional case. We generalize his work to the linear case for stochastic evolution equations and study the nonlinear case through first order approximation. We also give sufficient conditions in terms of a Lyapunov function for a weaker concept, namely, ultimate boundedness in the m. s. s.. This latter concept implies under appropriate condition on the Gelfand Triplet, the existence of invariant measures of the solutions and can be used along with the generalization of another theorem of Miyahara ([19], Th. 2) to obtain the boundedness of the second moment of the invariant measures.

The invariant measures for the mild solutions of stochastic evolution equations in the infinite-dimensional case was studied by Ichikawa [12] and was systematically taken up by Da Prato and Zabczyk [6], where the reader can find additional references. However, we use techniques of Ethier and Kurtz ([8], Ch.IV, Sec.9) to show the existence of invariant measures for the strong solutions. As special cases, we derive recent results on the invariant measures for Navier-Stokes equation [1], Parabolic Ito equation [3] and an improved version for stochastic heat equation([20], [25]), we also get the existence of invariant measure in the case of multiplicative noise for the random motion string introduced by Funaki [9]. In fact, we prove the ultimate boundedness of the solutions in these cases.

The weak recurrence property to a bounded set was studied by Miyahara [18] for the solutions of the stochastic differential equations in the finite dimensional case. For the solutions of stochastic evolution equations in a Hilbert space, Ichikawa [12] indicated that the same theorem held under the same condition as in Miyahara [18].

In the second part, we study the weak recurrence property to a compact set for the strong solutions of stochastic evolution equations under the coercivity condition in a Hilbert space. Under appropriate condition on the Gelfand Triplet, we conclude that the solution is weakly recurrent to a compact set if it is ultimately bounded in m. s. s. and weakly positive recurrent to a compact set if it is exponentially ultimately bounded in m. s. s.. These results extend the work of Miyahara [18], Wonham [27] and Zakai [29]. Using the results in chapter 2, we can give conditions in terms of a Lyapunov function for the weak and weakly positive recurrence to a compact set.

The purpose of the third part is to study the stability and ultimate boundedness of the mild solutions of stochastic semilinear evolution equations. The pioneering work in the field was done by Haussmann [10] in the linear case and Ichikawa [12, 11] for the semilinear case. A good exposition can be seen in book of Prato and Zabczyk [6]. The methods used by them were a direct attack on the problems. In [2], Chow suggested the use of Lyapunov functions in the study of the stability for the strong solution. However, this is not appropriate for the mild solution, furthermore, the Lyapunov function suggested by him for the linear problem in Haussmann [10] is not bounded below. In [15], Khasminskii and Mandrekar produced the correct Lyapunov function for the strong solution under coercivity condition and showed that non-linear problem could be studied through the first order linear approximation. It was shown in Mandrekar [17] that the sufficient conditions of Ichikawa for mild solution could be derived through a strong solution approximation. This also led to the study of stability in probability. We remove in this dissertation the coercivity condition. Through the strong solution approximation, we study the stability, exponential ultimate boundedness and stability in probability for the the mild solution. The main technique is again to construct an appropriate Lyapunov function. Once this is done, we can exploit the methods developed in [15] and the first part of this dissertation to obtain results for the mild solutions. As a consequence, we get simplified proofs of the results of Haussmann [10], Ichikawa [11, 12], Da Prato, Gatarek and Zabczyk [5].

Chapter 1

Preliminaries and Notations

The purpose of this chapter is to provide background material for the subsequent chapters.

1.1 Nuclear and Hilbert-Schmidt Operators

Let E, G be Banach spaces and let $L(E; G)$ be the Banach space of all linear bounded operators from E into G endowed with the usual operator norm $\|\cdot\|$. We denote by E^* and G^* the continuous dual spaces of E and G respectively. An element $T \in L(E, G)$ is said to be a *nuclear* operator if there exist two sequences $\{a_j\} \subset G, \{\varphi_j\} \subset E^*$ such that T has a representation

$$Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x), \quad x \in E.$$

with

$$\sum_{j=1}^{\infty} \|a_j\| \cdot \|\varphi_j\| < \infty$$

The space of all nuclear operators from E into G , endowed with the norm

$$\|T\|_1 = \inf\{\sum_{j=1}^{\infty} \|a_j\| \cdot \|\varphi_j\| : Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x)\},$$

is a Banach space, and will be denoted by $L_1(E, G)$.

Let H be a separable Hilbert space and let $\{e_j\}$ be a complete orthonormal system in H. We denote by $(\cdot,\cdot)$ the inner product in H. If $T \in L_1(H,H)$ then we define trace of T:

$$tr(T) = \sum_{j=1}^{\infty}(Te_j, e_j).$$

Proposition 1.1.1 *If $T \in L_1(H,H)$, then $tr(T)$ is a well-defined number independent of the choice of the orthonormal basis $\{e_j\}$.*

Note also that

$$|tr(T)| \leq \|T\|_1, T \in L_1(H).$$

Corollary 1.1.1 *If $T \in L_1(H,H)$ and $S \in L(H,H)$, then $TS, ST \in L_1(H,H)$ and*

$$tr(TS) = tr(ST) \leq \|T\|_1\|S\|.$$

Proposition 1.1.2 *A nonnegative operator $T \in L(H,H)$ is nuclear if and only if for an orthonormal basis $\{e_j\}$ on H*

$$\sum_{j=1}^{\infty}(Te_j, e_j) < \infty.$$

Moreover in this case $tr(T) = \|T\|_1$.

Because of this fact, we call a nuclear operator a trace class operator in this case.

Now, we introduce the *Hilbert – Schmidt* operator.

Let E and F be two separable Hilbert spaces with complete orthonormal bases $\{e_i\} \subset E, \{f_j\} \subset F$. A linear bounded operator $T : E \to F$ is said to be *Hilbert – Schmidt* if

$$\sum_{i=1}^{\infty} \|Te_i\|^2 < \infty$$

Since

$$\sum_{i=1}^{\infty} \|Te_i\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |(Te_i, f_j)|^2 = \sum_{j=1}^{\infty} \|T^* f_j\|^2$$

the definition of Hilbert - Schmidt operator, and the number

$$\|T\|_2 = (\sum_{i=1}^{\infty} \|Te_i\|^2)^{1/2}$$

is independent of the choice of the basis $\{e_i\}$. Moreover $\|T\|_2 = \|T^*\|_2$.

1.2 Hilbert Space Valued Wiener Processes

In this subsection, we will give the definition of a Wiener process on a separable Hilbert space K.

Throughout this dissertation, we assume that all the random variables, stochastic processes, probability measures are defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let K be another real separable Hilbert space.

We start with the definition of a Gaussian probability measure on the Hilbert space K.

Definition 1.2.1 *A probability measure μ on a Hilbert space $(K, \mathcal{B}(K))$ is a Gaussian measure with mean m and covariance Q, if for arbitrary $k \in K$ and $A \in \mathcal{B}(R^1)$,*

$$\mu\{x \in K : (k, x) \in A\} = \mathcal{N}((m, k), (Qk, k))(A),$$

where $\mathcal{N}((m, k), (Qk, k))(A)$ is a non degenerate Gaussian distribution with mean (m, k) and variance (Qk, k).

Proposition 1.2.1 *If μ is a Gaussian measure on a Hilbert space $(K, \mathcal{B}(K))$ with mean m and covariance Q, then*

(i) $\int_K (k, x)\mu(dx) = (m, k), \forall k \in K,$

(ii) $\int_K (k_1, x)(k_2, x)\mu(dx) - (m, k_1)(m, k_2) = (Qk_1, k_2), \quad \forall k_1, k_2 \in K.$

Proposition 1.2.2 *Let μ be a Gaussian probability measure with mean 0 and covariance Q, Then Q is a nonnegative symmetric trace class operator on K.*

Now we introduce the Wiener process on K.

Definition 1.2.2 *Suppose Q is a nonnegative symmetric trace class operator. A K-valued stochastic process $W(t), t \geq 0$, is called a Q-Wiener process or a Q-Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, if*

(i) $W(0) = 0$,

(ii) *W has a continuous trajectories,*

(iii) $W(t)$ *is adapted to* $\mathcal{F}_t \quad \forall t \geq 0$,

(iv) *W has independent increments,*

(v) $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q), \forall \quad t \geq s \geq 0.$

Since K is separable, there exists a complete orthonormal system $\{e_i\}$ in K, and a bounded sequence of nonnegative real numbers λ_i such that

$$Qe_i = \lambda_i e_i, i = 1, 2, \cdots.$$

We also have a similar decomposition for $W(t)$.

Proposition 1.2.3 *Assume Q is a nonnegative symmetric trace class operator. The following statements hold.*

(i) $E(W(t)) = 0, Cov(W(t)) = tQ \quad \forall \quad t \geq 0$,

(ii) $E(W(t), k_1)(W(s), k_2) = (t \wedge s)(Qk_1, k_2), \forall k_1, k_2 \in K$,

(iii) *For arbitrary t, W has the expansion*

$$W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i(t)e_i$$

where

$$\beta_i(t) = \frac{1}{\sqrt{\lambda_i}}(W(t), e_i), \quad j = 1, 2, \cdots,$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$ *and the series in (1.2.3) is convergent in* $L^2(\Omega, \mathcal{F}, P)$.

On the other hand, we have the following proposition:

Proposition 1.2.4 *For an arbitrary nonnegative symmetric trace class operator on a separable Hilbert space H, there exists a Q-Wiener process* $W(t), t \geq 0$.

1.3 Definition of Stochastic Integral

Suppose K and H are two separable Hilbert spaces. In this subsection, we will construct the following stochastic integral:

$$\int_0^t \Phi(s)dW(s), t \in [0,T]$$

where $W(t)$ is a K-valued Q-Brownian motion with respect to $\mathcal{F}_t$ as defined in the last subsection, and Φ is a process with values that are linear but not necessarily bounded operators from K to H. Let us fix $T \leq \infty$, and let $\mathcal{I} = [0,T]$

We define the stochastic integral in several steps.

A process $\Phi(t), t \in \mathcal{I}$ in $L(K,H)$ is called simple if it takes only a finite number of values, i. e., there exists a sequence $0 = t_0 < t_1 < \cdots < t_k = T$ and a sequence $\Phi_0, \Phi_1, \cdots, \Phi_{k-1}$ of $L(K,H)$-valued random variables such that Φ_m is $\mathcal{F}_{t_m}$-measurable and

$$\Phi(t) = \Phi_m, \quad \text{for} \quad t \in (t_m, t_{m+1}], m = 0, 1, \cdots, k-1.$$

For a simple process Φ we define the stochastic integral by the formula:

$$\int_0^t \Phi(s)dW(s) = \sum_{m=0}^{k-1} \Phi_m(W_{t_{m+1}\wedge t} - W_{t_m \wedge t})$$

and denote it by $\Phi \cdot W(t), t \in \mathcal{I}$

Now we introduce a Hilbert space $K_0 = Q^{1/2}(K)$ of a subspace of K which endowed with the inner product

$$(k_1, k_2)_0 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}(k_1, e_i)(k_2, e_i) = (Q^{-1/2}k_1, (Q^{-1/2}k_1).$$

Let $L_2^0 = L_2(K_0, H)$ be the space of all Hilbert - Schmidt operators from K_0 to H. It is also a separable Hilbert space, equipped with the norm

$$\begin{aligned} \|\Psi\|_{L_2^0}^2 &= \sum_{i,j=1}^{\infty} |(\Psi g_i, f_j)|^2 &= \sum_{i,j=1}^{\infty} \lambda_i |(\Psi e_i, f_j)|^2 \\ &= \|\Psi Q^{1/2}\|^2 &= tr(\Psi Q \Psi^*) \end{aligned}$$

where $\{g_i\}$ with $g_i = \sqrt{\lambda_i} e_i, \{e_i\}$ and $\{f_i\}$ are complete orthonormal bases in K_0, K and H respectively. Clearly, $L(K,H) \subset L_2^0$, but not all operators in L_2^0 can be regarded as restrictions of operators in $L(K,H)$. The space L_0^2 contains genuinely unbounded operators on K.

Let $\Phi(t), t \in \mathcal{I}$ be a measurable L_2^0-valued process, we define the norms

$$|||\Phi|||_t = \{E\int_0^t \|\Phi(s)\|_{L_0^2}^2 ds\}^{1/2} = \{E\int_0^t tr((\Phi(s)Q\Phi * (s))ds\}^{1/2}$$

for $t \in T$.

Proposition 1.3.1 *If a process Φ is simple and $|||\Phi|||_T < \infty$, then the process $\Phi \cdot W$ is a continuous, square integrable H-valued martingale on $[0,T]$ and*

$$E|\Phi \cdot W|^2 = |||\Phi|||_t^2, \quad 0 \le t \le T$$

Remark 1.3.1 *Note that the stochastic integral is an isometric transformation from the space of all simple processes equipped with the norm $|||\cdot|||_T$ into the space of all H-valued martingales.*

To extend the definition of the stochastic integral to more general processes it is convenient to regard integrands as random variables defined on the product space $\Omega_\infty = [0,\infty) \times \Omega$ (resp. $\Omega_T = [0,T) \times \Omega$), equipped with the product σ-field: $\mathcal{B}([0,\infty)) \times \mathcal{F}$ (resp. $\mathcal{B}([0,T)) \times \mathcal{F}$). The product of Lebesgue measure on $[0,T)$ (resp. $[0,T]$)) and the probability measure P is denoted by P_∞ (resp. P_T).

For the σ-field introduced just above, we consider the sub σ-field generated by the adapted simple processes, this sub σ-field is called the predictable σ-field, we denote it by $\mathcal{P}_\infty$ (resp. $\mathcal{P}_T$). It turns out that the proper class of integrands are predictable processes with values in L_2^0, more precisely, measurable mappings from $(\Omega_\infty, \mathcal{P}_\infty)$ (resp. $(\Omega_T, \mathcal{P}_T)$) into $(L_2^0, \mathcal{B}(L_2^0))$.

Proposition 1.3.2 *The following statements hold:*

(i) If a mapping Φ from Ω_T into $L(K,H)$ is $L(K,H)$-predictable, then it is also L_2^0-predictable. In particular, simple processes are L_2^0-predictable.

(ii) If Φ is a L_2^0-predictable process such that $|||\Phi|||_T < \infty$ then there exists a sequence $\{\Phi_n\}$ of simple processes such that $|||\Phi - \Phi_n|||_T \to 0$ as $n \to \infty$.

Now we are able to extend the definition of the stochastic integral to all L_2^0 predictable processes Φ such that $|||\Phi|||_T < \infty$. Note that they form a Hilbert space, we denoted it by $\mathcal{N}_W^2(0,T;L_2^0)$, and by the above proposition, simple processes are dense in $\mathcal{N}_W^2(0,T;L_2^0)$, by proposition (1.3.1), the stochastic integral $\Phi \cdot W$ is an isometric transformation from that dense set into the space of all $H-$valued martingales. Therefore, the definition of the stochastic integral can be immediately extended to all elements of $\mathcal{N}_W^2(0,T;L_2^0)$.

1.4 General Stochastic PDE

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, K a real separable Hilbert space and $\{W(t), t \geq 0\}$ a K-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, $Q-$Brownian motion defined on $(\Omega, \mathcal{F}, P)$.

Let $V \subseteq H$ be two real separable Hilbert spaces such that $V \subseteq H$ is dense and $V \hookrightarrow H$ is continuous, We identify H with its dual space, and denote by V^* the dual space of V, therefore, we have

$$V \subseteq H \subseteq V^*.$$

Denote by $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$ the norms in V, H and V^* respectively, by $<\cdot,\cdot>$ the duality product between V and V^*. In addition, we assume that for $v \in V$ and $v^* \in H$, $<v, v^*> = (v, v^*)$. The above triplet $V \subseteq H \subseteq V^*$ is called a Gelfand triplet.

Let $M^2(0,T;V)$ denote the space of all V-valued measurable processes satisfying:

(i) $u(t,\cdot)$ is $\mathcal{F}_t$-measurable; and,

(ii) $E\int_0^T \|u(t,\omega)\|_V^2 dt$ is finite.

We first study the following equation:

$$\begin{cases} u \in M^2(0,T;V) \\ du(t) = A(u(t))dt + B(u(t))dW(t) \\ u(0) = \varphi. \end{cases} \tag{1.1}$$

where $\varphi \in H$, $A : V \to V^*$ is an operator with $\|A(u)\|_{V^*} \leq a_1\|u\|_V, B(u) \in L(K,H)$ and $\|B(u)\|_{L(K,H)} \leq b_1\|u\|_V$ for $u \in V$, where $L(K,H)$ is the space of all bounded linear operators from K to H. Here A, B are in general nonlinear, a_1, b_1 are constants.

For the existence of solutions of the above equation, we need the following crucial condition:

coercivity condition: $\exists \alpha > 0, \lambda$ and γ, such that for $\forall v \in V$,

$$2<v, A(v)> + tr(B(v)QB^*(v)) \leq \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + \gamma, \tag{1.2}$$

and monotonicity condition: for $\forall u, v \in V$,

$$2<u-v, A(u) - A(v)> + tr((B(u) - B(v))Q(B(u) - B(v))^*) \leq \lambda\|u - v\|_H^2,$$

Theorem 1.4.1 *Under the above coercivity condition and monotonicity condition, equation (1.1) has a unique solution* $\{u^\varphi(t), t \geq 0\}$ *satisfying*

$$u^\varphi \in L^2(\Omega, C(0,T;H)) \bigcap M^2(0,T;V).$$

Furthermore, the solution is Markovian ([23], Ch. 3) and the corresponding semigroup is Feller.

The above solution is called a strong solution.

The major tool to study stochastic differential equation is Ito's formula, we quote it here for the ease of reference [21].

Let $\Psi : H \to R$ be a function satisfying:

(i) Ψ is twice (Frechet) differentiable with Ψ' and Ψ'' locally bounded.

(ii) Ψ, Ψ' are continuous on H

(iii) For all trace class operators T, $tr(T\Psi'(\cdot))$ is continuous on $H \to R$. (1.3)

(iv) If $v \in V$ then $\Psi'(v) \in V, u \to <\Psi'(u), v^*>$ is continuous for each $v^* \in V^*$.

(v) $\|\Psi'(v)\|_V \leq C_0(1 + \|v\|_V)$ for some $C_0 > 0, \forall v \in V$.

Theorem 1.4.2 *(Ito's formula):Suppose $\Psi : H \to R$ satisfies the above conditions and $\{u^\varphi(t), t \geq 0\}$ is a solution of (1.1) with $u^\varphi \in L^2(\Omega, C(0,T;H)) \bigcap M^2(0,T;V)$. Then*

$$\Psi(u^\varphi(t)) = \Psi(\varphi) + \int_0^t \mathcal{L}\ \Psi(u^\varphi(s))ds + \int_0^t (\Psi'(u^\varphi(s)), B(u^\varphi(s))dW(s)). \tag{1.4}$$

where $\mathcal{L}\ \Psi(u) = <\Psi'(u), A(u)> + \frac{1}{2}tr(\Psi''(u)B(u)QB^*(u))$.

1.5 Semilinear Stochastic PDE

When A is a semilinear operator, equation (1.1) is reduced to the following semilinear stochastic evolution equation on H:

$$\begin{cases} du = (Au + F(u))dt + B(u)dW(t) \\ \qquad u(0) = \varphi. \end{cases} \tag{1.5}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$ on H satisfying $\|S(t)\| \leq e^{\omega t}$ for some real number ω, F and B are in general nonlinear mappings from H to H and H to $L(K, H)$ satisfying the Lipschitz condition:

$$\begin{aligned} \|F(y) - F(z)\| + \|B(y) - B(z)\| \leq d\|y - z\|, \\ \|F(y)\| + \|B(y)\| \leq d(1 + \|y\|). \end{aligned} \tag{1.6}$$

for some constant c and all $y, z \in H$.

Besides the concept of a strong solution, for the semilinear case, we have the concept of mild solutions following [11]:

Definition 1.5.1 *A stochastic process $u(t), t \in \mathcal{I}$, is a mild solution of (1.5) if*

(i) $u(t)$ *is adapted to* $\mathcal{F}_t$,

(ii) $u(t)$ *is measurable and* $\int_0^T \|u(t)\|^2 dt < \infty$ *w.p. 1 and*

(iii) $u(t) = S(t)\varphi + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)B(u(s))dW(s)$

for all $t \geq 0$ *w.p. 1.*

In general the strong solution is rather stronger than the mild solution, for the relationship of these two solutions, we have the following propositions: [11]:

Proposition 1.5.1 *If $u(t), 0 \leq t \leq \infty$, is a strong solution of equation (1.5), then it is a mild solution.*

On the other hand, under some sufficient conditions, a mild solution can be a strong solution [11]:

Proposition 1.5.2 *Suppose that*

(a) $u(0) \in \mathcal{D}(A) w.p.1, S(t-r)F(u) \in \mathcal{D}(A), S(t-r)B(u)k \in \mathcal{D}(A) \quad \forall u \in H, k \in K, \quad and \quad t > r,$

(b) $\|AS(t-r)F(u)\| \leq g_1(t-r)\|u\|, g_1 \in \mathcal{L}_1(0,T)$,

(c) $\|AS(t-r)B(u)\| \leq g_2(t-r)\|u\|, g_2 \in \mathcal{L}_2(0,T)$.

Then a mild solution $u(t)$ is also a strong solution.

For the existence of the mild solution of equation (1.5), we have [11]:

Theorem 1.5.1 *Let φ be $\mathcal{F}_0$ measurable with $E\|\varphi\|^p < \infty$ for some integer $p \geq 2$. Under the hypothesis (1.6), (1.5) has a unique mild solution $u^{\varphi}(t)$ in $C(0,T; L_p(\Omega, \mathcal{F}, \mu; H))$.*

Corollary 1.5.1 *If φ is nonrandom, then there exists a unique mild solution of (1.5) in $C(0,T; L_p(\Omega, \mathcal{F}, \mu; H))$ for all $p \geq 2$.*

Without loss of generality, we assume the initial value φ is nonrandom throughout the dissertation.

Since we reduced the solution of equation (1.5) to H, the Ito's formula has a simpler form. Let's see the Ito's formula in this case.

Let $C^2(H)$ denote the space of all real-valued functions Ψ on H with properties:

(i) $\Psi(x)$ is twice (Frechet) differentiable,

(ii) $\Psi'(x)$ and $\Psi''(x)x_1$ for each $x_1 \in H$ are continuous.

By $C_b^2(H)$ denote the space of all functions in $C^2(H)$ with the first two derivatives bounded. We have the following Ito's formula [11]:

Theorem 1.5.2 *(Ito's formula):Suppose $\Psi \in C^2(H)$ and $\{u^\varphi(t), t \geq 0\}$ is a strong solution of (1.5). Then*

$$\Psi(u(t)) = \Psi(\varphi) + \int_0^t \mathcal{L}\ \Psi(u(s))ds + \int_0^t (\Psi'(u(s)), B(u(s))dW(s)). \tag{1.7}$$

where $\mathcal{L}\ \Psi(x) =< \Psi'(x), Ax + F(x) > +\frac{1}{2}tr(\Psi''(x)B(x)QB^*(x))$ is called the infinitesimal generator of equation (1.5).

Since Ito's formula is only applicable to the strong solution of (1.5), we introduce the approximating systems:

$$\begin{cases} du = Au + R(n)F(u(t))dt + R(n)B(u)dW(t) \\ \qquad\qquad u(0) = R(n)\varphi. \end{cases} \tag{1.8}$$

where $n \in \rho(A)$, the resolvent set of A and $R(n) = R(n, A) = (n - A)^{-1}$. The infinitesimal generator $\mathcal{L}_n$ corresponding to this equation is $\mathcal{L}_n\ \Psi(x) =< \Psi'(x), Ax + R(n)F(x) > +\frac{1}{2}tr(\Psi''(x)R(n)B(x)Q(R(n)B(x))^*)$

Theorem 1.5.3 *Under the hypotheses of Theorem 1.1, equation (1.8) has a unique strong solution $u_n^\varphi(t)$ in $C(0, T; L_p(\Omega, \mathcal{F}, \mu; H)$ for all T and $p \geq 2$. Moreover, $u_n^\varphi(t)$ converges to the mild solution $u^\varphi(t)$ of (1.5) in $C(0, T; L_p(\Omega, \mathcal{F}, \mu; H)$ as $n \to \infty$, i.e.:*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} E(\|u^\varphi(t) - u_n^\varphi(t)\|^p) = 0 \tag{1.9}$$

Chapter 2

Ultimate Boundedness and Invariant Measures of the Strong Solution

In this chapter we study necessary and sufficient conditions for exponentially ultimate boundedness of the strong solution of the stochastic evolution equation in terms of a Lyapunov function. We will explicitly construct the Lyapunov function in the linear case and derive sufficient conditions for the non-linear case through the first order approximation. We also will give conditions for ultimate boundedness of the solution of SPDE's and study the problem of the existence of invariant measures and their second moment. As application of our general result, we obtain recent results mentioned in the introduction.

2.1 Exponentially Ultimate Boundedness and Lyapunov Function

In [15], exponential stability in m. s. s. of the zero solution of (1.1) was considered and in the linear case a Lyapunov function was constructed. This function was then used to consider the stability through the first order approximation, in the nonlinear case. Following [18], we define

Definition 2.1.1 *The solution $\{u^\varphi(t), t \geq 0\}$ of (1.1) is exponentially ultimately bounded (in $\|\cdot\|_H$) in m. s. s. if there exist positive constants c, β, M such that*

$$E\|u^\varphi(t)\|_H^2 \leq ce^{-\beta t}\|\varphi\|_H^2 + M. \quad \textit{for } \forall \varphi \in H. \tag{2.1}$$

Remark 2.1.1 *If $M = 0$ we say that the zero solution is exponentially stable in m. s. s..*

Theorem 2.1.1 *Consider equation (1.1) satisfying the coercivity condition (1.2), and let $\{u^\varphi(t), t \geq 0\}$ be its solution. If there exists a function $\Lambda : H \to R$ which satisfies the following conditions:*

(i) condition (1.3),

(ii) $c_1\|\varphi\|_H^2 - k_1 \leq \Lambda(\varphi) \leq c_3\|\varphi\|_H^2 + k_3, \ \forall \varphi \in H$,

(iii) $\mathcal{L}\Lambda(\varphi) \leq -c_2\Lambda(\varphi) + k_2, \ \forall \varphi \in V$,

where $c_1(>0), c_2(>0), c_3(>0), k_1, k_2$ and k_3 are constants, then $\{u^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s.

Proof: Since $\Lambda(\varphi)$ satisfies (1.3), apply Ito's formula (1.4) to it and take expectation, we get

$$\begin{aligned} E\Lambda(u^\varphi(t)) - E\Lambda(u^\varphi(t') &= E\int_{t'}^t \mathcal{L}\Lambda(u^\varphi(s))ds \\ &\leq \int_{t'}^t (-c_2 E\Lambda(u^\varphi(s)) + k_2)ds \end{aligned}$$

Let $\Phi(t) = E\Lambda(u^\varphi(t))$ and use the fact that $\Phi(t)$ is continuous in t we have

$$\Phi'(t) \leq -c_2\Phi(t) + k_2.$$

Hence

$$\Phi(t) \leq \frac{k_2}{c_2} + (\Phi(0) - \frac{k_2}{c_2})e^{-c_2 t},$$

i.e.,

$$E\Lambda(u^\varphi(t)) \leq \frac{k_2}{c_2} + (\Lambda(\varphi) - \frac{k_2}{c_2})e^{-c_2 t}.$$

Using (ii), we have

$$\begin{aligned} c_1 E\|u^\varphi(t)\|_H^2 - k_1 \leq E\Lambda(u^\varphi(t)) &\leq \frac{k_2}{c_2} + (\Lambda(\varphi) - \frac{k_2}{c_2})e^{-c_2 t} \\ &\leq \frac{k_2}{c_2} + (c_3\|\varphi\|_H^2 + k_3 - \frac{k_2}{c_2})e^{-c_2 t}. \end{aligned} \tag{2.2}$$

From the above inequality we get

$$E\|u^\varphi(t)\|_H^2 \leq ce^{-\beta t}\|\varphi\|_H^2 + M \text{ for } \forall\varphi \in H,$$

for some constants c, β and M. So $u^\varphi(t)$ is exponentially ultimately bounded in m. s. s.

We note that (2.2) gives

Corollary 2.1.1 *If* $\Lambda : H \to R$ *satisfy* $(i), (iii)$ *in Theorem 2.1 and*

(ii)' $c_1\|\varphi\|_H^2 - k_1 \leq \Lambda(\varphi) \quad \forall\varphi \in H$

for some constants $c_1(> 0)$ *and* k_1*, then*

$$\limsup_{t\to+\infty} E\|u^\varphi(t)\|_H^2 \leq \frac{1}{c_1}(k_1 + \frac{k_2}{c_2}).$$

If $\{u^\varphi(t), t \geq 0\}$ satisfies the above condition, we say it is ***ultimately bounded in m. s. s.***. The function $\Lambda(\varphi)$ defined above is called a Lyapunov function. We now will construct a Lyapunov function if the solution of (1.1) under coercivity condition (1.2) is exponentially ultimately bounded in m. s. s..

Suppose the solution $\{u^\varphi(t), t \geq 0\}$ of (1.1) is exponentially ultimately bounded in m. s. s., i.e., we suppose (2.1) holds. Let

$$\Lambda(\varphi) = \int_0^T (\int_0^t E\|u^\varphi(s)\|_V^2 ds)dt \tag{2.3}$$

where T is a positive constant to be determined later.

Applying Ito's formula (1.4) to $\|\varphi\|_H^2$, taking expectation and applying coercivity condition (1.2), we get

$$\begin{aligned} E\|u^\varphi(t)\|_H^2 - \|\varphi\|_H^2 &= \int_0^t E\mathcal{L}\|u^\varphi(s)\|_H^2 ds \\ &\leq \lambda \int_0^t E\|u^\varphi(s)\|_H^2 ds - \alpha \int_0^t E\|u^\varphi(s)\|_V^2 ds + \gamma t \end{aligned}$$

hence

$$\int_0^t E\|u^\varphi(s)\|_V^2 ds \leq \frac{1}{\alpha}(\lambda \int_0^t E\|u^\varphi(s)\|_H^2 ds + \|\varphi\|_H^2 + \gamma t).$$

Since $\{u^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded and it satisfies (2.1)

$$\begin{aligned} \int_0^t E\|u^\varphi(s)\|_V^2 ds &\leq \frac{1}{\alpha}[\frac{c|\lambda|}{\beta}(1 - e^{-\beta t})\|\varphi\|_H^2 + \|\varphi\|_H^2 + |\lambda|Mt + \gamma t] \\ &\leq (\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha})\|\varphi\|_H^2 + \frac{\gamma + |\lambda|M}{\alpha} t. \end{aligned} \tag{2.4}$$

Therefore,

$$\begin{aligned} \Lambda(\varphi) &= \int_0^T (\int_0^t E\|u^\varphi(s)\|_V^2 ds)dt \\ &\leq (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T\|\varphi\|_H^2 + \frac{\gamma + |\lambda|M}{2\alpha} T^2. \end{aligned} \tag{2.5}$$

Now for $v \in V$,

$$\mathcal{L}\|v\|_H^2 = 2<v, A(v)> + tr(B(v)QB^*(v))$$

so

$$\begin{aligned} |\mathcal{L}\|v\|_H^2| &\leq 2a_1\|v\|_V^2 + \|B(v)\|_{L(K,H)}^2 tr(Q) \\ &\leq 2a_1\|v\|_V^2 + b_1^2 tr(Q)\|v\|_V^2 \\ &\leq c'\|v\|_V^2 \end{aligned}$$

for some positive constant c'.

hence

$$\mathcal{L}\|v\|_H^2 \geq -c'\|v\|_V^2.$$

Therefore, we have

$$\begin{aligned} E\|u^\varphi(t)\|_H^2 - \|\varphi\|_H^2 &= \int_0^t E\mathcal{L}\|u^\varphi(s)\|_H^2 ds \\ &\geq -c'\int_0^t E\|u^\varphi(s)\|_V^2 ds \end{aligned}$$

hence,

$$\begin{aligned} c'\int_0^t E\|u^\varphi(s)\|_V^2 ds &\geq \|\varphi\|_H^2 - E\|u^\varphi(t)\|_H^2 \\ &\geq \|\varphi\|_H^2 - M - ce^{-\beta t}\|\varphi\|_H^2 \\ &= (1-ce^{-\beta t})\|\varphi\|_H^2 - M \end{aligned}$$

therefore,

$$\begin{aligned} \Lambda(\varphi) &= \int_0^T(\int_0^t E\|u^\varphi(s)\|_V^2 ds)dt \\ &\geq \frac{1}{c'}[\int_0^T \|\varphi\|_H^2(1-ce^{-\beta t})dt - MT] \\ &= \frac{1}{c'}[T - \frac{c}{\beta}(1-e^{-\beta t})]\|\varphi\|_H^2 - \frac{MT}{c'} \\ &\geq \frac{1}{c'}(T-\frac{c}{\beta})\|\varphi\|_H^2 - \frac{MT}{c'} \end{aligned} \tag{2.6}$$

this proves (ii) if $T > \frac{c}{\beta}$.

Now we need the following lemma to continue:

Lemma 2.1.1 *If $f \geq 0$, and $f \in L^1[0,T]$ for any $T > 0$, then*

$$\lim_{\Delta t\to 0}\int_0^T \frac{\int_t^{t+\Delta t} f(s)ds}{\Delta t}dt = \int_0^T \lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f(s)ds}{\Delta t}dt = \int_0^T f(t)dt.$$

Proof: We are going to use Fubini theorem to change the order of integrals:

$$\int_0^T \frac{\int_t^{t+\Delta t} f(s)ds}{\Delta t}dt$$

$$\begin{aligned}
&= \frac{1}{\Delta t}\int_0^T\Big(\int_t^{t+\Delta t} f(s)ds\Big)dt \\
&= \frac{1}{\Delta t}\Big[\int_0^{\Delta t}\Big(\int_0^s f(s)dt\Big)ds + \int_{\Delta t}^T\Big(\int_{s-\Delta t}^s f(s)dt\Big)ds + \int_T^{T+\Delta t}\Big(\int_{s-\Delta t}^T f(s)dt\Big)ds\Big] \\
&= \frac{1}{\Delta t}\Big[\int_0^{\Delta t} sf(s)ds + \int_{\Delta t}^T f(s)\Delta t ds + \int_T^{T+\Delta t} f(s)(T+\Delta t - s)ds\Big] \\
&\le \frac{1}{\Delta t}\Big[\Delta t\int_0^{\Delta t} f(s)ds + \Delta t\int_{\Delta t}^T f(s)ds + \Delta t\int_T^{T+\Delta t} f(s)ds\Big] \\
&= \int_0^{\Delta t} f(s)ds + \int_{\Delta t}^T f(s)ds + \int_T^{T+\Delta t} f(s)ds
\end{aligned}$$

the first and the third term go to zero as $\Delta t \to 0$, so

$$\lim_{\Delta t\to 0}\int_0^T \frac{\int_t^{t+\Delta t} f(s)ds}{\Delta t}dt \le \int_0^T f(t)dt.$$

the other direction of the inequality follows from Fatou's lemma easily. This proves the lemma.

Let's now suppose that $\Lambda(\varphi)$ satisfies (1.3). To prove the converse of Theorem 2.1, it remains to prove (iii). Observe

$$E\Lambda(u^\varphi(r)) = E\int_0^T\int_0^t E(\|u^{u^\varphi(r)}(s)\|_V^2|u^\varphi(r))dsdt$$

But by the Markov property of the solution of (1.1), this equals

$$\int_0^T\int_0^t E(E(\|u^{u^\varphi(r)}(s)\|_V^2|\mathcal{F}_r^u))dsdt$$

where $\mathcal{F}_r^u = \sigma\{u^\varphi(\tau), \tau \le r\}$. The uniqueness of the solution implies

$$E(\|u^{u^\varphi(r)}(s)\|_V^2|\mathcal{F}_r^u) = E(\|u^\varphi(s+r)\|_V^2|\mathcal{F}_r^u).$$

Hence

$$E\Lambda(u^\varphi(r)) = \int_0^T\Big(\int_0^t E\|u^\varphi(r+s)\|_V^2 ds\Big)dt = \int_0^T\Big(\int_r^{r+t} E\|u^\varphi(s)\|_V^2 ds\Big)dt.$$

Therefore,

$$\mathcal{L}\Lambda(\varphi) = \frac{d}{dr}(E\Lambda(u^\varphi(r)))|_{r=0}$$

$$
\begin{aligned}
&= \lim_{r\to 0} \frac{E\Lambda(u^{\varphi}(r)) - E\Lambda(\varphi)}{r} \\
&= \lim_{r\to 0} \int_0^T \frac{\int_r^{r+t} E\|u^{\varphi}(s)\|_V^2 ds - \int_0^t E\|u^{\varphi}(s)\|_V^2 ds}{r} dt \\
&= \lim_{r\to 0} \int_0^T \frac{\int_t^{r+t} E\|u^{\varphi}(s)\|_V^2 ds - \int_0^r E\|u^{\varphi}(s)\|_V^2 ds}{r} dt \\
&= \lim_{r\to 0} \int_0^T \frac{\int_t^{r+t} E\|u^{\varphi}(s)\|_V^2 ds}{r} dt - \lim_{r\to 0} \frac{T}{r} \int_0^r E\|u^{\varphi}(s)\|_V^2 ds \\
&= I_1 + I_2
\end{aligned} \tag{2.7}
$$

From the above lemma and (2.4)

$$
\begin{aligned}
I_1 &= \int_0^T \lim_{r\to 0} \frac{\int_t^{r+t} E\|u^{\varphi}(s)\|_V^2 ds}{r} dt \\
&= \int_0^T E\|u^{\varphi}(t)\|_V^2 dt \\
&\le \left(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha}\right)\|\varphi\|_H^2 + \frac{\gamma + |\lambda| M}{\alpha} T
\end{aligned} \tag{2.8}
$$

and since $V \hookrightarrow H$ is continuous, there exists a positive constant α_0, such that $\|v\|_H^2 \le \alpha_0 \|v\|_V^2$ for all $v \in V$, so

$$
I_2 \le -\lim_{r\to 0} \frac{T}{\alpha_0} \frac{1}{r} \int_0^r E\|u^{\varphi}(s)\|_H^2 ds
$$

and by the continuity of the map $s \to E\|u^{\varphi}(s)\|_H^2$ we have $I_2 \le -\frac{T}{\alpha_0}\|\varphi\|_H^2$

therefore,

$$
\mathcal{L}\Lambda(\varphi) \le \left(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha} - \frac{T}{\alpha_0}\right)\|\varphi\|_H^2 + \frac{\gamma + |\lambda| M}{\alpha} T \tag{2.9}
$$

Let $T > \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha})$, then $\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha} - \frac{T}{\alpha_0} < 0$, then we get *(iii)*

Up to now, we have proved the following theorem:

Theorem 2.1.2 *Consider the equation (1.1) satisfying (1.2),let the solution* $\{u^{\varphi}(t), t \ge 0\}$ *of it be exponentially ultimately bounded in m. s. s.. Suppose*

$$
\Lambda(\varphi) = \int_0^T \left(\int_0^t E\|u^{\varphi}(s)\|_V^2 ds\right) dt
$$

satisfies condition (1.3). Then $\Lambda(\varphi)$ *satisfies the conditions in theorem 2.1, i.e., there exist constants* $c_1(>0), c_2(>0), c_3(>0), k_1, k_2$ *and* k_3*, such that*

$$c_1\|\varphi\|_H^2 - k_1 \le \Lambda(\varphi) \le c_3\|\varphi\|_H^2 + k_3. \ for\ \forall\varphi \in H$$

and

$$\mathcal{L}\Lambda(\varphi) \le -c_2\Lambda(\varphi) + k_2. \ for\ \forall\varphi \in V$$

Remark 2.1.2 *In addition, if $s \to E\|u^\varphi(s)\|_V^2$ is continuous for $\forall\varphi \in V$, then $I_2 = -T\|\varphi\|_V^2$, using (2.8) and by the fact that $\|v\|_H^2 \le \alpha_0\|v\|_V^2$ for all $v \in V$, we have*

$$I_1 \le \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha})\|\varphi\|_V^2 + \frac{\gamma + |\lambda|M}{\alpha}T$$

therefore,

$$\mathcal{L}\Lambda(\varphi) \le \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha} - \frac{T}{\alpha_0})\|\varphi\|_V^2 + \frac{\gamma + |\lambda|M}{\alpha}T \quad for\ \varphi \in V \tag{2.10}$$

If $T > \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha})$, then $\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha} - \frac{T}{\alpha_0} < 0$ and use the fact $\|v\|_H^2 \le \alpha_0\|v\|_V^2$ for all $v \in V$ again, we also have

$$\mathcal{L}\Lambda(\varphi) \le (\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha} - \frac{T}{\alpha_0})\|\varphi\|_H^2 + \frac{\gamma + |\lambda|M}{\alpha}T \quad for\ \varphi \in V$$

Unfortunately, we do not know at this moment if $\Lambda(\varphi) = \int_0^T(\int_0^t E\|u^\varphi(s)\|_V^2 ds)dt$ satisfies condition(1.3) or not. Now we will restrict our consideration to the following linear SPDE:

$$\begin{cases} u \in M^2(0,T;V) \\ du(t) = A_0u(t)dt + B_0u(t)dW(t) \\ u(0) = \varphi. \end{cases} \tag{2.11}$$

We suppose the linear operators A_0, B_0 satisfy the same conditions as A, B in equation (1.1) with the same constants and the same coercivity condition, that is:

$$2\langle v, A_0v\rangle + tr(B_0vQB_0^*v) \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + \gamma. \tag{2.12}$$

Denote the solution of (2.11) by $\{u_0^\varphi(t), t \ge 0\}$ and let

$$\Lambda_0(\varphi) = \int_0^T(\int_0^t \|u_0^\varphi(s)\|_V^2 ds)dt \quad \text{for some } T.$$

Theorem 2.1.3 *If $\{u_0^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s., then $\Lambda_0(\varphi)$ defined above satisfies condition (1.3).*

Proof: Suppose $\{u_0^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s. and it satisfies (2.1). From the computation of (2.5), we have

$$\Lambda_0(\varphi) \leq (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T\|\varphi\|_H^2 + \frac{\gamma + |\lambda|M}{2\alpha}T^2$$

If $\|\varphi\|_H^2 = 1$, then

$$\Lambda_0(\varphi) \leq (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T + \frac{\gamma + |\lambda|M}{2\alpha}T^2$$

Since $u_0^\varphi(t)$ is linear in φ, for any positive constant K, we have

$$u_0^{k\varphi}(t) = ku_0^\varphi(t)$$

Hence,

$$\Lambda_0(k\varphi) = k^2\Lambda_0(\varphi)$$

Therefore, for any $\varphi \in H$

$$\Lambda_0(\varphi) = \|\varphi\|_H^2\Lambda_0(\frac{\varphi}{\|\varphi\|_H}) \leq [(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T + \frac{\gamma + |\lambda|M}{2\alpha}T^2]\|\varphi\|_H^2.$$

Let $c'' = (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T + \frac{\gamma+|\lambda|M}{2\alpha}T^2$ then $\Lambda_0(\varphi) \leq c''\|\varphi\|_H^2$ for $\forall\varphi \in H$. And let

$$\mathcal{T}(\varphi, \psi) = \int_0^T(\int_0^t E<u_0^\varphi(s), u_0^\psi(s)>_V ds)dt \text{ for } \varphi, \psi \in H$$

Then $\mathcal{T}$ is a bilinear form on H, and by using Schwartz inequality, we get

$$\begin{aligned}
|\mathcal{T}(\varphi,\psi)| &= |\int_0^T(\int_0^t E<u_0^\varphi(s), u_0^\psi(s)>_V ds)dt| \\
&\leq \int_0^T(\int_0^t (E\|u_0^\varphi(s)\|_V^2)^{\frac{1}{2}}(E\|u_0^\psi(s)\|_V^2)^{\frac{1}{2}} ds)dt \\
&\leq \int_0^T(\int_0^t E\|u_0^\varphi(s)\|_V^2 ds)^{\frac{1}{2}}(\int_0^t E\|u_0^\psi(s)\|_V^2 ds)^{\frac{1}{2}} dt \\
&\leq (\int_0^T(\int_0^t E\|u_0^\varphi(s)\|_V^2 ds)dt)^{\frac{1}{2}}(\int_0^T(\int_0^t E\|u_0^\psi(s)\|_V^2 ds)dt)^{\frac{1}{2}} \\
&= \Lambda_0(\varphi)^{\frac{1}{2}}\Lambda_0(\psi)^{\frac{1}{2}} \\
&\leq c''\|\varphi\|_H \cdot \|\psi\|_H.
\end{aligned}$$

Hence there exists a continuous linear operator $C : H \to H$, such that

$$\mathcal{T}(\varphi, \psi) = (C\varphi, \psi), \tag{2.13}$$

and

$$\|C\|_{L(H,H)} = \sup_{\|\varphi\|_H=1, \|\psi\|_H=1} |(C\varphi, \psi)| \leq c''$$

Since $\Lambda_0(\varphi) = \mathcal{T}(\varphi, \varphi) = (C\varphi, \varphi)$, so

$$\Lambda_0'(\varphi) = 2C\varphi \text{ and } \Lambda_0''(\varphi) = 2C$$

Hence, Λ_0, Λ_0' and Λ_0'' are locally bounded on H , Λ_0 and Λ_0' are continuous on H and

$$|\Lambda_0(\varphi)| \leq \|C\|_{L(H,H)} \|\varphi\|_H^2, \text{ and}$$

$$\|\Lambda_0'(\varphi)\|_H = \|2C\varphi\|_H \leq 2\|C\|_{L(H,H)} \|\varphi\|_H$$

This proves $(i), (ii)$ of (1.3). And for trace class $Q, Q\Lambda'' = 2QC$ and trace(QC) being constant is continuous, hence (iii) of (1.3) holds.

To prove (iv) of (1.3), we observe $|\mathcal{T}(\varphi, \psi)| \leq c''\|\varphi\|_H \|\psi\|_H$ and $\|\varphi\|_H^2 \leq \alpha_0 \|\varphi\|_V^2$ for any $\varphi, \psi \in V$. This implies $|\mathcal{T}(\varphi, \psi)| \leq c''\alpha_0 \|\varphi\|_V \|\psi\|_V$. Because $\mathcal{T}(\varphi, \psi)$ is bilinear on $V \times V$, there exists a continuous operator $\tilde{C} : V \to V$, such that

$$\mathcal{T}(\varphi, \psi) = (\tilde{C}\varphi, \psi) \text{ for all } \varphi, \psi \in V. \tag{2.14}$$

Hence $\Lambda'(\varphi) = 2\tilde{C}\varphi \in V$ for $\varphi \in V$ and $\varphi \to \tilde{C}\varphi$ is continuous on $V \to V$. Since

$$\|\Lambda'(\varphi)\|_V = 2\|\tilde{C}\varphi\|_V \leq 2\|\tilde{C}\|_{L(V,V)} \|\varphi\|_V \leq 2\|\tilde{C}\|_{L(V,V)} (\|\varphi\|_V + 1)$$

for any $\varphi \in V$, therefore, we have proved Λ satisfies (v) of (1.3) and proved the theorem.

Therefore, we have the following theorem:

Theorem 2.1.4 *Consider the linear equation(2.11) satisfying (2.12). Its solution* $\{u_0^\varphi(t), t \geq 0\}$ *is exponentially ultimately bounded in m. s. s. if and only if there exists a function* $\Lambda : H \to R$ *satisfying*

$$c_1\|\varphi\|_H^2 - k_1 \leq \Lambda(\varphi) \leq c_3\|\varphi\|_H^2 + k_3, \ \forall \varphi \in H$$

and

$$\mathcal{L}_0\Lambda(\varphi) \leq -c_2\Lambda(\varphi) + k_2, \ \forall \varphi \in V$$

for some constants $c_1(>0), c_2(>0), c_3(>0), k_1, k_2$ *and* k_3. *where*

$$\mathcal{L}_0\Lambda(\varphi) = <\Lambda'(\varphi), A_0(\varphi)> + \frac{1}{2}tr(\Lambda''(\varphi)B_0(\varphi)QB_0^*(\varphi)).$$

Furthermore, if we set $T_0 = \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha}) + \frac{c}{\beta}$, *then* $\Lambda_0(\varphi) = \int_0^{T_0}(\int_0^t E\|u_0^\varphi(s)\|_V^2 ds)dt$ *will be such a function.*

Now, we will consider the nonlinear case by linear approximation in case (2.3) does not satisfies (1.3). where $\{u^\varphi(t), t \geq 0\}$ is the solution of the nonlinear equation (1.1). We need the following lemma ([24], PP. 39):

Lemma 2.1.2 *Suppose* $\mathcal{X}$ *is a Hilbert space,* $T \in L(\mathcal{X}, \mathcal{X})$ *is a trace class operator. Define* $\tau(T) = tr(TT^*)^{\frac{1}{2}}$, *then it has the following properties:*

a) $|tr(T)| \leq \tau(T)$,

b) $\tau(TS) \leq \|S\|\tau(T)$ *and* $\tau(ST) \leq \|S\|\tau(T)$ *for all* $S \in L(\mathcal{X}, \mathcal{X})$.

Theorem 2.1.5 *Suppose the linear equation (2.11) satisfies coercivity condition (2.12) and its solution* $\{u_0^\varphi(t), t \geq 0\}$ *is exponentially ultimately bounded in m. s. s.. Let* $\{u^\varphi(t), t \geq 0\}$ *be the solution of the nonlinear equation (1.1). Furthermore, we suppose* $A(v) - A_0 v \in H$ *for all* $v \in V$. *If for* $v \in V$,

$$2\|v\|_H\|A(v) - A_0 v\|_H + \tau(B(v)QB^*(v) - B_0 vQB_0^* v) \leq \omega\|v\|_H^2 + k \qquad (2.15)$$

with ω, k *constants and*

$$\omega < \frac{c}{\alpha_0\beta[(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}) + \frac{\gamma+|\lambda|M}{2\alpha}(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta})^2]}. \qquad (2.16)$$

Then $\{u^\varphi(t), t \geq 0\}$ *is exponentially ultimately bounded in m. s. s.*

Proof: Let $\Lambda_0(\varphi) = \int_0^{T_0}(\int_0^t E\|u_0^\varphi(s)\|_V^2 ds)dt$, where $T_0 = \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha}) + \frac{c}{\beta}$, then $\Lambda_0(\varphi)$ satisfies (1.3), and

$$c_1\|\varphi\|_H^2 - k_1 \leq \Lambda_0(\varphi) \leq c_3\|\varphi\|_H^2 + k_3. \text{ for } \forall\varphi \in H$$

for some constants $c_1(>0), c_3(>0), k_1,$ and k_3 . It remains to show

$$\mathcal{L}\Lambda_0(\varphi) \leq -c_2\Lambda(\varphi) + k_2.\text{for } \forall\ \varphi \in V$$

for constants $c_2(>0), k_2$. Since $A(\varphi) - A_0\varphi \in H$, we have

$$\begin{aligned}
&\mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) \\
&= <\Lambda_0'(\varphi), A(\varphi) - A_0\varphi> + \frac{1}{2}tr(\Lambda_0''(\varphi)(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi)) \\
&= (\Lambda_0'(\varphi), A(\varphi) - A_0\varphi) + \frac{1}{2}tr(\Lambda_0''(\varphi)(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi))
\end{aligned}$$

with $\Lambda_0'(\varphi) = 2C\varphi,$ and $\Lambda_0''(\varphi) = 2C$ for $\varphi \in V$, where C as defined in (2.13) is a bounded positive operator from H to H, and

$$\|C\|_{L(H,H)} \leq (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T_0 + \frac{\gamma + |\lambda|M}{2\alpha}T_0^2$$

with T_0 defined as above. Hence,

$$\mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) = 2(C\varphi, A(\varphi) - A_0\varphi) + tr(C(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi))$$

Using the above lemma, we obtain:

$$\begin{aligned}
\mathcal{L}\Lambda_0(\varphi) \ \leq\ & \mathcal{L}_0\Lambda_0(\varphi) + 2\|C\|_{L(H,H)}\|\varphi\|_H\|A(\varphi) - A_0\varphi\|_H \\
& +\tau(C(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi)) \\
\leq\ & \mathcal{L}_0\Lambda_0(\varphi) + \|C\|_{L(H,H)}(2\|\varphi\|_H\|A(\varphi) - A_0\varphi\|_H \\
& +\tau(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi))
\end{aligned}$$

From the computation of (2.9), when $T = T_0$,

$$\mathcal{L}_0\Lambda_0(\varphi) \leq -\frac{c}{\alpha_0\beta}\|\varphi\|_H^2 + \frac{\gamma + |\lambda|M}{\alpha}T_0$$

therefore

$$\begin{aligned}\mathcal{L}\Lambda_0(\varphi) &\leq -\frac{c}{\alpha_0\beta}\|\varphi\|_H^2 + \frac{\gamma+|\lambda|M}{\alpha}T_0 + \|C\|_{L(H,H)}(\omega\|\varphi\|_H^2 + k) \\ &\leq (-\frac{c}{\alpha_0\beta} + \omega\|C\|_{L(H,H)})\|\varphi\|_H^2 + k\|C\|_{L(H,H)} + \frac{\gamma+|\lambda|M}{\alpha}T_0\end{aligned}$$

Since $-\frac{c}{\alpha_0\beta} + \omega\|C\|_{L(H,H)} < 0$ when ω satisfies (2.16), we get the required inequality. This proves the theorem.

Corollary 2.1.2 *Suppose the linear equation (2.11) satisfies coercivity condition (2.12) and its solution $\{u_0^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s.. Let $\{u^\varphi(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). Furthermore, we suppose for $v \in V$, $A(v) - A_0v \in H$, and*

$$\|A(v) - A_0v\|_H^2 + \tau(B(v)QB^*(v) - B_0vQB_0^*v) \leq K(1 + \|v\|_H^2)$$

for some constant $K > 0$. If for $v \in V$, as $\|v\|_H \to \infty$

$$\|A(v) - A_0v\|_H = o(\|v\|_H) \text{ and } \tau(B(v)QB^*(v) - B_0vQB_0^*v) = o(\|v\|_H^2)$$

then $\{u^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s.

Proof: By theorem 2.5, we just need to show that (2.15) holds for some constants ω and k with ω satisfying (2.16). Since for $v \in V$, as $\|v\|_H \to \infty$.

$$\|A(v) - A_0v\|_H = o(\|v\|_H) \text{ and } \tau(B(v)QB^*(v) - B_0vQB_0^*v) = o(\|v\|_H^2)$$

For any fixed ω satisfying (2.16), there exists an $R > 0$, such that

$$2\|v\|_H\|A(v) - A_0v\|_H + \tau(B(v)QB^*(v) - B_0vQB_0^*v) \leq \omega\|v\|_H^2$$

for $\forall v \in V$ and $\|v\|_H \geq R$. For $v \in V$ but $\|v\|_H \leq R$, by assumption, we have

$$\begin{aligned}&2\|v\|_H\|A(v) - A_0v\|_H + \tau(B(v)QB^*(v) - B_0vQB_0^*v) \\ \leq\ &\|v\|_H^2 + \|A(v) - A_0v\|_H^2 + \tau(B(v)QB^*(v) - B_0vQB_0^*v)) \\ \leq\ &\|v\|_H^2 + K(1 + \|v\|_H^2) \\ \leq\ &K + (K+1)R^2\end{aligned}$$

Therefore, for $\forall v \in V$

$$2\|v\|_H\|A(v) - A_0v\|_H + \tau(B(v)QB^*(v) - B_0vQB_0^*v) \leq \omega\|v\|_H^2 + (K+1)R^2 + K$$

This proves (2.15) with ω satisfying (2.16), thus the assertion holds.

Theorem 2.1.6 *Suppose the linear equation (2.11) satisfies coercivity condition (2.12) and its solution $\{u_0^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s., furthermore, we suppose $t \to E\|u_0^\varphi(t)\|_V^2$ is continuous for all $\varphi \in V$. Let $\{u^\varphi(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). If for $v \in V$,*

$$2\|v\|_V\|A(v) - A_0v\|_{V^*} + \tau(B(v)QB^*(v) - B_0vQB_0^*v) \leq \omega\|v\|_V^2 + k \qquad (2.17)$$

with ω, k constants and

$$\omega < \frac{c}{(\alpha_0+1)\beta[(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}) + \frac{\gamma+|\lambda|M}{2\alpha}(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta})^2]} \qquad (2.18)$$

then $\{u^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s.

Proof: The proof is similar to that of the above theorem. Let $\Lambda_0(\varphi) = \int_0^{T_0}(\int_0^t E\|u_0^\varphi(s)\|_V^2 ds)dt$, where $T_0 = \alpha_0(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha}) + \frac{c}{\beta}$. We just need to show

$$\mathcal{L}\Lambda_0(\varphi) \leq -c_2\Lambda(\varphi) + k_2 \qquad \text{for } \forall\ \varphi \in V$$

for constants $c_2(>0), k_2$. Since

$$\mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) = <\Lambda_0'(\varphi), A(\varphi) - A_0\varphi> + \frac{1}{2}tr(\Lambda_0''(\varphi)(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi))$$

with $\Lambda_0'(\varphi) = 2\tilde{C}\varphi$, and $\Lambda_0''(\varphi) = 2C$ for $\varphi \in V$, where $\tilde{C}$ and C as defined in (2.14) and (2.13) are bounded positive operators from V to V and from H to H respectively, and

$$\|\tilde{C}\|_{L(V,V)} \leq \alpha_o((\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T_0 + \frac{\gamma+|\lambda|M}{2\alpha}T_0^2),$$

$$\|C\|_{L(H,H)} \leq (\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})T_0 + \frac{\gamma+|\lambda|M}{2\alpha}T_0^2$$

with T_0 defined as above. Hence,

$$\mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) = 2<\tilde{C}\varphi, A(\varphi) - A_0\varphi> + tr(C(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi)).$$

Using lemma 2.2, we get:

$$\begin{aligned}
\mathcal{L}\Lambda_0(\varphi) &\leq \mathcal{L}_0\Lambda_0(\varphi) + 2\|\tilde{C}\|_{L(V,V)}\|\varphi\|_V\|A(\varphi) - A_0\varphi\|_{V^*} \\
&\quad + \tau(C(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi)) \\
&\leq \mathcal{L}_0\Lambda_0(\varphi) + 2\|\tilde{C}\|_{L(V,V)}\|\varphi\|_V\|A(\varphi) - A_0\varphi\|_{V^*} \\
&\quad + \|C\|_{L(H,H)}\tau(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi) \\
&\leq \mathcal{L}_0\Lambda_0(\varphi) + (\|\tilde{C}\|_{L(V,V)} + \|C\|_{L(H,H)})(2\|\varphi\|_V\|A(\varphi) - A_0\varphi\|_{V^*} \\
&\quad + \tau(B(\varphi)QB^*(\varphi) - B_0\varphi QB_0^*\varphi)).
\end{aligned}$$

Since $t \to E\|u_0^\varphi(t)\|_V^2$ is continuous for all $\varphi \in V$, from the computation of (2.10), when $T = T_0$,

$$\mathcal{L}_0\Lambda_0(\varphi) \leq -\frac{c}{\beta}\|\varphi\|_V^2 + \frac{\gamma + |\lambda|M}{\alpha}T_0,$$

therefore

$$\begin{aligned}
\mathcal{L}\Lambda_0(\varphi) &\leq -\frac{c}{\beta}\|\varphi\|_V^2 + \frac{\gamma + |\lambda|M}{\alpha}T_0 + (\|\tilde{C}\|_{L(V,V)} + \|C\|_{L(H,H)})(\omega\|\varphi\|_V^2 + k) \\
&\leq (-\frac{c}{\beta} + \omega(\|\tilde{C}\|_{L(V,V)} + \|C\|_{L(H,H)}))\|\varphi\|_V^2 \\
&\quad + k(\|\tilde{C}\|_{L(V,V)} + \|C\|_{L(H,H)}) + \frac{\gamma + |\lambda|M}{\alpha}T_0
\end{aligned}$$

Since $-\frac{c}{\beta} + \omega(\|\tilde{C}\|_{L(V,V)} + \|C\|_{L(H,H)}) < 0$ when ω satisfies (2.18), we get the required inequality. This proves the theorem.

Corollary 2.1.3 *Suppose the linear equation (2.11) satisfies coercivity condition (2.12) and its solution $\{u_0^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s., furthermore, we suppose $t \to E\|u_0^\varphi(t)\|_V^2$ is continuous for all $\varphi \in V$. Let $\{u^\varphi(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). If for $v \in V$, as $\|v\|_V \to \infty$*

$$\|A(v) - A_0v\|_{V^*} = o(\|v\|_V) \text{ and } \tau(B(v)QB^*(v) - B_0vQB_0^*v) = o(\|v\|_V^2)$$

then $\{u^\varphi(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s.

Proof: By theorem 2.6, we just need to show (2.17) holds for some constants ω, k with ω satisfying (2.18). Since for $v \in V$, as $\|v\|_V \to \infty$,

$$\|A(v) - A_0 v\|_{V^*} = o(\|v\|_V) \text{ and } \tau(B(v)QB^*(v) - B_0 v Q B_0^* v) = o(\|v\|_V^2)$$

For any fixed ω satisfying (2.18), there exists an $R > 0$, such that

$$2\|v\|_V \|A(v) - A_0 v\|_{V^*} + \tau(B(v)QB^*(v) - B_0 v Q B_0^* v) \leq \omega \|v\|_V^2$$

for all $\|v\|_V \geq R$. By the assumption,

$$\|A(v)\|_{V^*}, \quad \|A_0 v\|_{V^*} \leq a_1 \|v\|_V \quad \text{and}$$

$$\|B(v)\|_{L(K,H)}, \quad \|B_0 v\|_{L(K,H)} \leq b_1 \|v\|_V,$$

thus for $v \in V$ and $\|v\|_V \leq R$,

$$\begin{aligned}
&2\|v\|_V \|A(v) - A_0 v\|_{V^*} + \tau(B(v)QB^*(v) - B_0 v Q B_0^* v) \\
&\leq 2\|v\|_V (\|A(v)\|_{V^*} + \|A_0 v\|_{V^*}) + \tau(B(v)QB^*(v)) + \tau(B_0 v Q B_0^* v)) \\
&\leq 4a_1 \|v\|_V^2 + \|B(v)\|_{L(K,H)}^2 \tau(Q) + \|B_0 v\|_{L(K,H)}^2 \tau(Q) \\
&\leq 4a_1 \|v\|_V^2 + 2b_1^2 \tau(Q) \|v\|_V^2 \\
&\leq (4a_1 + 2b_1^2 \tau(Q)) \|v\|_V^2 \\
&\leq (4a_1 + 2b_1^2 \tau(Q)) R^2
\end{aligned}$$

Therefore, for $\forall v \in V$

$$2\|v\|_V \|A(v) - A_0 v\|_{V^*} + \tau(B(v)QB^*(v) - B_0 v Q B_0^* v) \leq \omega \|v\|_V^2 + (4a_1 + 2b_1^2 \tau(Q)) R^2$$

This proves (2.17) with ω satisfying (2.18), thus the assertion holds.

Example 2.1.1 *Consider the following stochastic evolution equation:*

$$du(t) = A_0 u(t)dt + F(u(t))dt + B(u(t))dW_t \tag{2.19}$$

with initial condition

$$u(0) = \varphi \in H$$

Suppose A_0, F and B satisfy the following conditions:

(i) $A_0 : V \to V^*$ *is coercive so that there exist constants* $\alpha > 0$ *and* λ, *for* $\forall v \in V$,

$$2\langle v, A_0 v\rangle \leq \lambda\|v\|_H^2 - \alpha\|v\|_V^2$$

(ii) $F : H \to H$ *and* $B : H \to L(K, H)$ *satisfy: for* $v \in H$.

$$\|F(v)\|_H^2 + \|B(v)\|_{L(K,H)}^2 \leq K(1 + \|v\|_H^2)$$

(iii) For $u, v \in H$,

$$\|F(u) - F(v)\|_H^2 + tr((B(u) - B(v))Q(B^*(u) - B^*(v))) \leq \lambda\|u - v\|_H^2.$$

If the solution $\{u_0(t), t \geq 0\}$ *of* $du(t) = A_0 u(t)dt$ *is exponentially stable(or even exponentially ultimately bounded), and as* $\|v\|_H \to \infty$

$$\|F(v)\|_H = o(\|v\|_H), \quad \|B(v)\|_{L(k,H)} = o(\|v\|_H).$$

then the solution $\{u(t), t \geq 0\}$ *of (2.19) is exponentially ultimately bounded in m. s. s.*

Proof: Let $A(v) = A_0 v + F(v)$ for $v \in V$. Since $F(v) \in H$,

$$\begin{aligned}
&2\langle v, A(v)\rangle + tr(B(v)QB^*(v)) \\
&= 2\langle v, A_0 v\rangle + 2\langle v, F(v)\rangle + tr(B(v)QB^*(v)) \\
&= 2\langle v, A_0 v\rangle + 2(v, F(v)) + tr(B(v)QB^*(v)) \\
&\leq \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + 2\|v\|_H\|F(v)\|_H + \|B(v)\|_{L(K,H)}^2 tr(Q) \\
&\leq \lambda'\|v\|_H^2 - \alpha\|v\|_V^2 + \gamma
\end{aligned}$$

for some constants λ' and γ, hence equation (2.19) is coercive. Under additional assumptions $(ii), (iii)$, the strong solution $\{u(t), t \geq 0\}$ of (2.19) exists ([21], Th 3.1). By assumption (ii)

$$\begin{aligned}
\|F(v)\|_H^2 + \tau(B(v)QB^*(v)) &\leq \|F(v)\|_H^2 + \|B(v)\|_{L(K,H)}^2 \tau(Q) \\
&\leq (1 + \tau(Q))K(1 + \|v\|_H^2)
\end{aligned}$$

and since

$$\|F(v)\|_H = o(\|v\|_H), \tau(B(v)QB^*(v)) \leq \|B(v)\|^2_{L(K,H)}\tau(Q) = o(\|v\|^2_H)$$

as $\|v\|_H \to \infty$, the assertion follows from corollary 2.2.

Remark 2.1.3 *the above example extends to infinite dimensions the corresponding results in Zakai [29] and Miyahara [18]. As an application, we derive the following.*

Example 2.1.2 Stochastic heat equation. *Let S^1 be the unit circle and $W(\cdot,\cdot)$ a Brownian sheet on $[0,\infty) \times S^1$. We consider the following stochastic heat equation:*

$$\frac{\partial X(t)}{\partial t}(\xi) = \frac{\partial^2 X(t)}{\partial \xi^2}(\xi) - \alpha X(t)(\xi) + f(X(t)(\xi)) + b(X(t)(\xi))\frac{\partial^2 W}{\partial t \partial \xi}, \tag{2.20}$$

with initial condition

$$X(0)(\cdot) = x(\cdot) \in L^2(S^1),$$

where α is a constant and f, b are real-valued functions.

Let

$$H = L^2(S^1), \qquad V = W^{1,2}(S^1), \qquad A_0(x) = (\frac{\partial^2}{\partial \xi^2} - \alpha)x,$$

and F and B given for $\xi \in S^1$ and $x, y \in L^2(S^1)$ are defined by

$$F(x)(\xi) = f(x(\xi)), \qquad B(x)[y](\xi) = b(x(\xi))y(\xi).$$

Let

$$\|x\|_H = (\int_{S^1} x^2 d\xi)^{\frac{1}{2}} \quad \text{for } x \in H$$
$$\|x\|_V = (\int_{S^1} (x^2 + (\frac{\partial x}{\partial \xi})^2) d\xi)^{\frac{1}{2}} \quad \text{for } x \in V.$$

Then

$$2 < x, A_0 x >= -2\|x\|^2_V + (-2\alpha + 2)\|x\|^2_H \leq -2\|x\|^2_H + (-2\alpha + 2)\|x\|^2_H = -2\alpha\|x\|^2_H.$$

Therefore, by Theorem 2.1.1 and Remark 2.1.1 the solution of $dx(t) = A_0x(t)dt$ is exponentially stable if $\alpha > 0$. furthermore, if, in addition we assume f and b are both Lipschitz continuous and bounded, then from Example 2.1.1, the solution of (2.20) is exponentially ultimately bounded in m. s. s..

Example 2.1.3 *Consider the following SPDE:*

$$d_tu(t,x) = (\alpha^2\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + \gamma u + g(x))dt + (\sigma_1\frac{\partial u}{\partial x} + \sigma_2 u)dW(t)$$

with initial condition

$$u(0,x) = \phi(x) \in L^2(-\infty,\infty)\bigcap L^1(-\infty,\infty),$$

where $W(t)$ is a one-dimensional standard Brownian Motion.

Let

$$\begin{aligned}
H &= L^2(-\infty,\infty), & V = H_0^1(-\infty,\infty)\\
A(u) &= \alpha^2\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + \gamma u + g & B(u) = \sigma_1\frac{\partial u}{\partial x} + \sigma_2 u\\
\|u\|_H &= (\int_{-\infty}^{\infty} u^2 dx)^{\frac{1}{2}} \qquad \text{for } u \in H\\
\|u\|_V &= (\int_{-\infty}^{\infty} (u^2 + (\frac{\partial u}{\partial x})^2)dx)^{\frac{1}{2}} \qquad \text{for } u \in V
\end{aligned}$$

Suppose $g(x) \in L^2(-\infty,\infty)\bigcap L^1(-\infty,\infty)$. For $v \in V$.

$$\begin{aligned}
&2<v, A(v)> + tr(BvQB^*v)\\
&= 2\int_{-\infty}^{\infty}(v, \alpha^2\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial x} + \gamma v + g)dx + \int_{-\infty}^{\infty}(\sigma_1\frac{\partial v}{\partial x} + \sigma_2 v)^2 dx\\
&= (-2\alpha^2 + \sigma_1^2)\|v\|_V^2 + (2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2)\|v\|_H^2 + 2\int_{-\infty}^{\infty}(v,g)dx\\
&\le (-2\alpha^2 + \sigma_1^2)\|v\|_V^2 + (2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2 + \epsilon)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2
\end{aligned}$$

for $\forall\epsilon > 0$. Similarly for $u, v \in V$,

$$\begin{aligned}
&2<u-v, A(u) - A(v)> + tr(B(u-v)QB^*(u-v))\\
&= (-2\alpha^2 + \sigma_1^2)\|u-v\|_V^2 + (2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2)\|u-v\|_H^2
\end{aligned}$$

By ([21], Th. 3.1), if $-2\alpha^2 + \sigma_1^2 < 0$, there exists a unique strong solution

$$u^\varphi(t) \in L^2(\Omega, C(0,T;H)) \bigcap M^2(0,T:V).$$

Now we want to find its Lyapunov function explicitly. Taking Fourier transform of the SPDE:

$$\begin{aligned} d_t\widehat{u}(t,\lambda) &= (-\alpha^2\lambda^2\widehat{u}(t,\lambda) + i\lambda\beta\widehat{u} + \gamma\widehat{u}(t,\lambda) + \widehat{g}(\lambda))dt \\ &\quad +(i\sigma_1\lambda\widehat{u}(t,\lambda) + \sigma_2\widehat{u}(t,\lambda))dW(t) \\ &= ((-\alpha^2\lambda^2 + i\lambda\beta + \gamma)\widehat{u}(t,\lambda) + \widehat{g}(\lambda))dt \\ &\quad +(i\sigma_1\lambda + \sigma_2)\widehat{u}(t,\lambda)dW(t) \end{aligned}$$

Now for fixed λ, let

$$\begin{aligned} a &= -\alpha^2\lambda^2 + i\lambda\beta + \gamma \\ b &= \widehat{g}(\lambda) \\ c &= i\sigma_1\lambda + \sigma_2 \end{aligned}$$

By the result in the appendix,

$$\begin{aligned} E|\widehat{u}(t,\lambda)|^2 \quad &= \{E|\widehat{\varphi}(\lambda)|^2 + 2Re(\frac{b\bar{b} + \bar{b}\widehat{\varphi}(\lambda)(a+\bar{a}+c\bar{c})}{(a+\bar{a}+c\bar{c})(\bar{a}+c\bar{c})})\}e^{(a+\bar{a}+c\bar{c})t} \\ &\quad -2Re(\frac{\bar{b}(a\widehat{\varphi}(\lambda)+b)}{a(\bar{a}+c\bar{c})}e^{at}) + 2Re(\frac{b\bar{b}}{a(a+\bar{a}+c\bar{c})}) \end{aligned}$$

By the Plancheral theorem, with $H = L^2(-\infty,\infty)$

$$\|u^\varphi(t,\cdot)\|_H^2 = \|\widehat{u}^\varphi(t,\cdot)\|_H^2$$

Hence

$$\begin{aligned} E\|u^\varphi(t)\|_H^2 &= E\|\widehat{u}^\varphi(t)\|_H^2 = E\int_{-\infty}^{\infty} |\widehat{u}(t,\lambda)|^2 d\lambda \\ &= \int_{-\infty}^{\infty} E|\widehat{u}(t,\lambda)|^2 d\lambda \end{aligned}$$

$$E\|u^\varphi(t)\|_V^2 = E\|u^\varphi(t)\|_H^2 + E\|\frac{\partial u^\varphi(t)}{\partial x}\|_H^2 = \int_{-\infty}^{\infty}(1+\lambda^2)E|\widehat{u}(t,\lambda)|^2 d\lambda$$

For a suitable $T > 0$,

$$\begin{aligned}\Lambda(\varphi) &= \int_0^T \int_0^t E\|u^{\varphi}(s)\|_V^2 ds dt \\ &= \int_0^T \int_0^t \int_{-\infty}^{+\infty} (1+\lambda^2) E|\widehat{u}(t,\lambda)|^2 d\lambda ds dt \\ &= \int_{-\infty}^{+\infty} (1+\lambda^2) \int_0^T \int_0^t E|\widehat{u}(t,\lambda)|^2 ds dt d\lambda\end{aligned}$$

The above computation of the Lyapunov function is very complicated, but if we use Corollary 2.3, we just need to compute the Lyapunov function of the linear SPDE, which is much simpler. let $\{u_0(t), t \geq 0\}$ be the solution of

$$d_t u(t,x) = (\alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + \gamma u)dt + (\sigma_1 \frac{\partial u}{\partial x} + \sigma_2 u) dW(t)$$

then

$$d_t \widehat{u}_0(t,\lambda) = a\widehat{u}_0(t,\lambda)dt + c\widehat{u}_0(t,\lambda)dW(t)$$

a, c are defined as above. We can solve $\widehat{u}_0(t,\lambda)$ explicitly:

$$\widehat{u}_0(t,\lambda) = \widehat{u}_0(0,\lambda)e^{at-\frac{1}{2}c^2t+cW(t)} = \widehat{\varphi}(\lambda)e^{at-\frac{1}{2}c^2t+cW(t)}$$

$$E|\widehat{u}_0(t,\lambda)|^2 = |\widehat{\varphi}(\lambda)|^2 e^{(a+\bar{a}+c\bar{c})t}$$

It is easy to see

$$t \to E\|u_0^{\varphi}(t)\|_V^2 = \int_{-\infty}^{+\infty} (1+\lambda^2)|\widehat{\varphi}(\lambda)|^2 e^{(a+\bar{a}+c\bar{c})t} d\lambda$$

is continuous for $\forall \varphi \in V$, and

$$\begin{aligned}\|A(v) - A_0(v)\|_{V^*} = \|g\|_{V^*} &= o(\|v\|_V) \quad \text{as } \|v\|_V \to +\infty \\ \tau(B(v)QB^*(v) - B_0vQB_0^*v) &= 0\end{aligned}$$

since B is linear. Therefore if $\{u_0(t), t \geq 0\}$ is exponentially ultimately bounded in m. s. s., the Lyapunov function $\Lambda_0(\varphi)$ of the linear system is also a Lyapunov function of the nonlinear system , and for a suitable $T > 0$,

$$\Lambda_0(\varphi) = \int_{-\infty}^{\infty} (1+\lambda^2) \int_0^T \int_0^t E|\widehat{u}_0(t,\lambda)|^2 ds dt d\lambda$$

$$= \int_{-\infty}^{\infty}(1+\lambda^2)|\widehat{\varphi}(\lambda)|^2\Big(\frac{e^{\{(-2\alpha^2+\sigma_1^2)\lambda^2+2\gamma+\sigma_2^2\}T}}{\{(-2\alpha^2+\sigma_1^2)\lambda^2+2\gamma+\sigma_2^2\}^2}$$
$$-\frac{T}{(-2\alpha^2+\sigma_1^2)\lambda^2+2\gamma+\sigma_2^2}-\frac{1}{\{(-2\alpha^2+\sigma_1^2)\lambda^2+2\gamma+\sigma_2^2\}^2}\Big)d\lambda$$

Therefore the solution of the nonlinear system SPDE is also exponentially ultimately bounded in m. s. s..

Remark 2.1.4 *From the above computation we see, if we replace* $E\|u^{\varphi}(s)\|_V^2$ *by* $E\|u^{\varphi}(s)\|_H^2$ *in* $\Lambda_0(\varphi)$*, then the leading term of* $\Lambda_0(\varphi)$ *is*

$$T\int_{-\infty}^{\infty}\frac{|\widehat{\varphi}(\lambda)|^2}{(2\alpha^2-\sigma_1^2)\lambda^2-2\gamma-\sigma_2^2}d\lambda.$$

this does not satisfy the first inequality of (ii) of theorem 2.1.1.

2.2 Ultimate Boundedness and Invariant Measures

In the previous section, we considered ultimate boundedness:

$$\limsup_{t\to+\infty} E\|u^{\varphi}(t)\|_H^2 \le M \text{ for } \forall\varphi\in H \tag{2.21}$$

for the solution of (1.1) and gave a sufficient condition for (2.21) in terms of a Lyapunov function in Corollary 2.1.1. In this section will study the existence of invariant measures for $\{u(t)\}$ under ultimate boundedness. First we will see the result in $H = R^n$.

Let $I_R(x)$ denote the indicator function of the set$\{x \in H, \|x\|_H > R\}$, with $R > 0$, we have the following result, see ([13], PP 72):

Theorem 2.2.1 *If* $H = R^n$*, and a Markovian semigroup* (P_t) *is Feller, then an invariant measure* μ *for* (P_t) *exists if and only if for some element* $x \in H$*,*

$$\lim_{R\to+\infty}\liminf_{T\to+\infty}\frac{1}{T}\int_0^T P_tI_R(x)dt = 0.$$

the key in the proof of the sufficient condition of this theorem is that $\{x \in H, \|x\|_H \leq R\}$ is compact when $H = R^n$, but this fails to hold when H is a Hilbert space. But if

$$V \hookrightarrow H$$

is compact, then $\{x : \|x\|_V \leq R\}$ is compact in H, therefore, we can have the following counterpart result in Hilbert spaces as in R^n.

Theorem 2.2.2 *Let $\tilde{I}_R(x)$ denote the indicator function of the set $\{x \in H, \|x\|_V > R\}$, with $R > 0$. Suppose $V \hookrightarrow H$ is compact and a Markovian semigroup (P_t) is Feller. Then a sufficient condition for an invariant measure μ for (P_t) exists is there exists some element $\varphi \in H$, such that*

$$\lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_t\tilde{I}_R(\varphi)dt = 0. \tag{2.22}$$

On the other hand, if there exists an invariant measure μ for (P_t) with support in V, then (2.22) is also necessary.

For equation (1.1), the semigroup

$$P_t f(x) = \int_H f(y)p(t,x,y)dy$$

is Markovian and Feller, therefore, we can apply the above theorem to the solutions of (1.1) and get:

Theorem 2.2.3 *Suppose $V \hookrightarrow H$ is compact. Then a sufficient condition for an invariant measure μ for the solutions of (1.1) exists is there exists some element $\varphi \in H$, such that*

$$\lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{1}{T}\int_0^T P\{\|u^\varphi(t)\|_V > R\}dt = 0. \tag{2.23}$$

On the other hand, if there exists an invariant measure μ for the solutions of (1.1) with support in V, then (2.23) is also necessary.

Now if we use the coercivity condition, we can get the following sufficient condition for the existence of invariant measures of the solutions of (1.1):

Theorem 2.2.4 *Suppose $V \hookrightarrow H$ is compact, and the solution $\{u(t), t \geq 0\}$ of (1.1) under coercivity condition (1.2) is ultimately bounded (in $\|\cdot\|_H$ norm). Then there exists an invariant measure μ for $\{u(t), t \geq 0\}$*

Proof: Applying Ito's formula (1.4) to $\|\varphi\|_H^2$, taking expectation and applying coercivity condition (1.2), we get

$$\begin{aligned} E\|u^\varphi(t)\|_H^2 - \|\varphi\|_H^2 &= \int_0^t E\mathcal{L}\|u^\varphi(s)\|_H^2 ds \\ &\leq \lambda \int_0^t E\|u^\varphi(s)\|_H^2 ds - \alpha \int_0^t E\|u^\varphi(s)\|_V^2 ds + \gamma t \end{aligned}$$

hence

$$\int_0^t E\|u^\varphi(s)\|_V^2 ds \leq \frac{1}{\alpha}(\lambda \int_0^t E\|u^\varphi(s)\|_H^2 ds + \|\varphi\|_H^2 + \gamma t),$$

therefore,

$$\begin{aligned} &\frac{1}{T}\int_0^T P\{\|u^\varphi(t)\|_V > R\}dt \\ &\leq \frac{1}{T}\int_0^T \frac{E\|u^\varphi(t)\|_V^2}{R^2} dt \\ &\leq \frac{1}{\alpha R^2}(\frac{|\lambda|}{T}\int_0^T E\|u^\varphi(t)\|_H^2 dt + \frac{\|\varphi\|_H^2}{T} + \gamma). \end{aligned}$$

Since we assume $\{u^\varphi(t), t \geq 0\}$ is ultimately bounded, for fixed φ_0, there exist two constants T_0 and M, such that

$$E\|u^{\varphi_0}(t)\|_H^2 \leq M \text{ for } t \geq T_0$$

Therefore,

$$\begin{aligned} &\lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{1}{T}\int_0^T P\{\|u^{\varphi_0}(t)\|_V > R\}dt \\ &\leq \lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{|\lambda|}{\alpha R^2}\frac{1}{T}\int_0^T E\|u^{\varphi_0}(t)\|_H^2 dt \\ &\leq \lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{|\lambda|}{\alpha R^2}\frac{1}{T}(\int_0^{T_0} E\|u^{\varphi_0}(t)\|_H^2 dt + \int_{T_0}^T E\|u^{\varphi_0}(t)\|_H^2 dt) \\ &\leq \lim_{R\to+\infty} \liminf_{T\to+\infty} \frac{|\lambda|}{\alpha R^2}\frac{1}{T}(\int_0^{T_0} E\|u^{\varphi_0}(t)\|_H^2 dt + M(T - T_0)) \\ &= 0 \end{aligned}$$

Therefore the assertion of the theorem follows.

For the application of the above theorem, let us see the following example.

Example 2.2.1 *(Stochastic Navier-Stokes Equation [26]) Let $D \subseteq R^2$ be a bounded domain with smooth boundary ∂D. Consider the equation:*

$$\begin{cases} \frac{\partial v_i(t,x)}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P(t,x)}{\partial x_i} + \nu \sum_{j=1}^{2} \frac{\partial^2 v_i}{\partial x_j^2} + \sigma_i \dot{W}_t^i(x) \\ \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j} = 0, \qquad x \in D, i = 1,2, \qquad \nu > 0 \end{cases}$$

Let $\mathcal{C}_0^\infty = \{v \in [C_0^\infty(D)]^2 : \nabla \cdot v = 0\}$ ($\nabla\cdot$ is gradient) and H the closure of $\mathcal{C}_0^\infty$ in $[L^2(D)]^2$, $V = \{v \in [H_0^1(D)]^2 : \nabla \cdot v = 0\}$. It is known [26] that

$$[L^2(D)]^2 = H \oplus H^\perp$$

Where $H^\perp$ is the orthogonal complement of H characterized by

$$H^\perp = \{v = \nabla(p), \quad \text{for some} \quad p \in H^1(D)\}$$

Denote by Π orthogonal projection from $[L^2(D)]^2$ to $H^\perp$ and define for $v \in \mathcal{C}_0^\infty$,

$$B(v) = \nu \Pi \Delta v - \Pi[(v \cdot \nabla)v]$$

Then B can be extended as a continuous operator on V to V^*, and $V \subseteq H \subseteq V^*$ is a Gelfand Triplet with $V \hookrightarrow H$ compact. The equation can be recast as a stochastic evolution equation in the form:

$$\begin{cases} du(t) = B(u(t))dt + \sigma dW(t) \\ u(0) = \xi, \qquad \xi \in V \qquad a.e. \end{cases}$$

where W(t) is a H-valued Q-Brownian Motion. We observe ([26],PP. 347) that the above equation has an unique strong solution $\{u^\xi(t), t \geq 0\}$ satisfying:

$$E\|u^\xi(T)\|_H^2 + \nu E \int_0^T \sum_{i=1}^{2} \|\frac{\partial u^\xi(t)}{\partial x_i}\|_H^2 dt \leq E\|\xi\|_H^2 + \frac{T}{2} tr(Q)$$

Hence using the fact that $\|u^\xi(t)\|_V$ is equivalent to $(\sum_{i=1}^{2} \|\frac{\partial u^\xi(t)}{\partial x_i}\|_H^2)^{\frac{1}{2}}$, we get

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T E(\|u^\xi(t)\|_V^2 dt \leq \frac{c}{2\nu} tr(Q)$$

where c is a constant. Using Chebychev's inequality we get

$$\lim_{R\to+\infty}\liminf_{T\to\infty}\frac{1}{T}\int_0^T P(\|u^\xi(t)\|_V > R)dt = 0$$

By the above remarks we get that invariant measure exists and the support of it is in V.

The idea used in the above example is the relationship between ultimate boundedness of $\{u^\varphi(t)\}$ in $H-$norm and boundedness of $\frac{1}{T}\int_0^T E(\|u^\varphi(t)\|_V^2 dt$ in addition to the compactness of embedding of $V \hookrightarrow H$.

Remark 2.2.1 *As a consequence we easily get a result on the existence of the invariant measure of the stochastic heat equation ([20], [25]). As we see in Example 2.2, the solution of the stochastic heat equation is ultimately bounded in m. s. s., and since $V \hookrightarrow H$ is compact by Sobolev embedding theorem, the existence of a invariant measure follows.*

Example 2.2.2 *We consider the equation of the form:*

$$du(t) = -Au(t)dt + F(u(t))dt + B(u(t))dW(t), \qquad u(0) = \varphi \in H,$$

where F, B satisfy the conditions in Example 2.1.1.

The above model with $A = -\Delta$ occurs in the work of Funaki [9] on the random motion of string problem. Funaki gave an explicit form of the invariant measure in the case $B \equiv 1$. However since A is coercive [14], we get that the solution is ultimately bounded in m. s. s.. In view of the fact that Δ has pure point spectrum with eigenvalues $\lambda_k \sim -k^2$, we get by [12] that it has an invariant measure.

Furthermore, we get conditions on the finiteness of the second moment of invariant measures as in the following theorem.

Theorem 2.2.5 *Suppose $V \hookrightarrow H$ is compact, and the solution $\{u(t), t \geq 0\}$ of (1.1) under coercivity condition (1.2) is ultimately bounded (in $\|\cdot\|_H$ norm). Then any* *invariant measure* μ *of* $\{u(t), t \geq 0\}$ *satisfies*

$$\int_V \|x\|_V^2 \mu(dx) < \infty$$

Proof: Let $f(x) = \|x\|_V^2$, and $f_n(x) = \chi_{[0,n]}(f(x))$, where χ is a characteristic function. We note that $f_n(x) \in L^1(V, \mu)$, By the use of Ergodic theorem for Markov process with invariant measure([28],PP. 388), there exists the limit

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t f_n(x) dt = f_n^*(x) \qquad (\mu - a.e.)$$

and

$$E_\mu f_n^* = E_\mu f_n,$$

where $E_\mu f_n = \int_V f_n(x) \mu(dx)$.

From the assumption of ultimate boundedness of $\{u^x(t), t \geq 0\}$, there exists a positive constant $M > 0$, such that

$$\limsup_{t \to +\infty} E\|u^x(t)\|_H^2 \leq M \text{ for } \forall x \in H.$$

By the same argument as in the above theorem, we have

$$\limsup_{T \to +\infty} \frac{1}{T} \int_0^T E\|u^x(t)\|_V^2 dt \leq \limsup_{T \to +\infty} \frac{1}{\alpha} \left(\frac{|\lambda|}{T} \int_0^T E\|u^x(t)\|_H^2 dt + \frac{\|x\|_H^2}{T} + \gamma \right) \leq \frac{M|\lambda|}{\alpha}$$

hence,

$$\begin{aligned} f_n^*(x) &= \lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t f_n(x) dt \leq \limsup_{T \to +\infty} \frac{1}{T} \int_0^T P_t f(x) dt \\ &= \limsup_{T \to +\infty} \frac{1}{T} \int_0^T E\|u^x(t)\|_V^2 dt \leq \frac{M|\lambda|}{\alpha} \end{aligned}$$

therefore, the fact that $f_n(x) \uparrow f(x)$ implies

$$E_\mu f = \lim_{n \to +\infty} E_\mu f_n = \lim_{n \to +\infty} E_\mu f_n^* \leq \frac{M|\lambda|}{\alpha}$$

This proves the assertion.

This improves the result in [3] on the parabolic Ito equation and gives more information about the invariant measure studied in ([25], [20]). The existence of invariant measure for parabolic Ito equation can be proved using Corollary 2.1.1 with $\Lambda(\varphi) = \|\varphi\|_H^2$ and Theorem 2.2.4.

Chapter 3

Weak (Weakly Positive) Recurrence of the Strong Solution

In this chapter, we will study the weak recurrence and weakly positive recurrence properties to compact sets for the strong solution of the stochastic evolution equation (1.1) under the the condition of ultimate boundedness. Using the results in Chapter 2, we study the problem in terms of Lyapunov functions.

3.1 Ultimate Boundedness and Weak Recurrence

In this section, we study weak (positive) recurrence of the solution of (1.1) to a compact set under the condition that it is (exponentially) ultimately bounded in m. s. s..

Definition 3.1.1 *A stochastic process $X(t)$ defined on H is weakly recurrent to a compact set if there exists a compact set C, such that*

$$P_x\{\omega : X(t) \in C \text{ for some } t \geq 0\} = 1 \quad \text{for } \forall x \in H.$$

where P_x stands for the conditional probability under the initial condition $X(0) = x$, the set C is said to be a recurrent region.

Remark 3.1.1 *Throughout this paper, weakly (positive) recurrent means weakly (positive) recurrent to a compact set, instead of to a bounded set as in [18, 12].*

Theorem 3.1.1 *Suppose $V \hookrightarrow H$ is compact, and the solution $\{u(t), t \geq 0\}$ of (1.1) under coercivity condition (1.2) is ultimately bounded in m. s. s.. Then $\{u(t), t \geq 0\}$ is weakly recurrent.*

This theorem is proved through a series of lemmas

Lemma 3.1.1 *Let $X(t)$ be a strong Markov process on H, if there exist a positive Borel measurable function $\rho(x)$ defined on H, a compact set C and a positive constant δ, such that*

$$P_x\{\omega : X(\rho(x)) \in C\} \geq \delta > 0 \quad \text{for} \quad \forall x \in H. \tag{3.1}$$

then the process $X(t)$ is weakly recurrent and C is a recurrent region.

Proof: For fixed $x \in H$, let

$$\begin{aligned}
\tau_1 &= \rho(x), \\
\Omega_1 &= \{\omega : X(\tau_1) \notin C\}, \\
\tau_2 &= \tau_1 + \rho(X(\tau_1)), \\
\Omega_2 &= \{\omega : X(\tau_2) \notin C\}, \\
\tau_3 &= \tau_2 + \rho(X(\tau_2)), \\
\Omega_3 &= \{\omega : X(\tau_3) \notin C\}, \\
&\cdots \\
\Omega_\infty &= \bigcap_{i=1}^{\infty} \Omega_i.
\end{aligned}$$

Since $\{\omega : X(t, \omega) \notin C \text{ for any } t \geq 0\} \subseteq \Omega_\infty$, it is sufficient to show $P_x(\Omega_\infty) = 0$. By the assumption,

$$P_x(\Omega_1) \leq 1 - \delta < 1,$$

and since $\rho : H \to R$ is Borel measurable, τ_i is a stopping time for each i, by the use of strong Markov property, we have

$$\begin{aligned} P_x(\Omega_1 \bigcap \Omega_2) &= E^x(E^x(\chi_{\Omega_1}\chi_{\Omega_2}|\mathcal{F}_{\tau_1})) \\ &= E^x(\chi_{\Omega_1}E^x(\chi_{\Omega_2}|\mathcal{F}_{\tau_1})) \\ &= E^x(\chi_{\Omega_1}E^x(\chi_{\Omega_2}|X(\tau_1))) \\ &= E^x(\chi_{\Omega_1}P_{X(\tau_1)}\{\omega : X(\rho(X(\tau_1))) \notin C\}). \end{aligned}$$

By assumption (3.1),

$$P_{X(\tau_1)}\{\omega : X(\rho(X(\tau_1))) \notin C\} \leq 1 - \delta,$$

hence

$$P_x(\Omega_1 \bigcap \Omega_2) \leq (1 - \delta)^2.$$

In the same manner,

$$P_x(\bigcap_{i=1}^{n} \Omega_i) \leq (1 - \delta)^n \to 0 \qquad \text{as} \quad n \to \infty.$$

This proves $P_x(\Omega_\infty) = 0$ and $X(t)$ is weakly recurrent.

Lemma 3.1.2 *Let $X(t)$ be a continuous strong Markov process on H, if there exist a positive Borel measurable function $\gamma(x)$ defined on H, a closed set C and a positive constant δ, such that*

$$\int_{\gamma(x)}^{\gamma(x)+1} P_x\{\omega : X(t) \in C\} \geq \delta > 0 \; for \, \forall x \in H. \tag{3.2}$$

then there exists a positive Borel measurable function $\rho(x)$ defined on H, such that

$$\gamma(x) \leq \rho(x) \leq \gamma(x) + 1, \quad and$$

$$P_x\{\omega : X(\rho(x)) \in C\} \geq \delta > 0. \quad for \quad \forall x \in H.$$

Proof: By (3.2), $\int_{\gamma(x)}^{\gamma(x)+1} P_x\{\omega : X(t) \in C\} \geq \delta > 0$ for $\forall x \in H$, hence there exists $t_x \in [\gamma(x), \gamma(x)+1)$, such that

$$P_x\{\omega : X(t_x) \in C\} \geq \delta.$$

Define

$$\rho(x) = \inf\{t \in [\gamma(x), \gamma(x)+1), P_x\{\omega : X(t) \in C\} \geq \delta\}.$$

Since the characteristic function of a closed set is upper semicontinuous and $t \to X(t)$ is continuous, $t \to P_x\{\omega : X(t) \in C\}$ is upper semicontinuous for each fixed $x \in H$, therefore,

$$P_x\{\omega : X(\rho(x)) \in C\} \geq \delta.$$

Now what we need to show is $x \to \rho_n(x)$ is Borel measurable.

For each $t \geq 0$, define $\mathcal{B}_t(H) = \mathcal{B}(H)$. For any fixed $T > 0$, since $X(t)$ is a Markov process, the map $(t, x) \to P_x\{\omega : X(t) \in C\}$ of $[0, T] \times H$ into $(R^1, \mathcal{R}^1)$ is $\mathcal{B}([0, T]) \times \mathcal{B}(H)$ measurable, hence it is $\mathcal{B}([0, T]) \times \mathcal{B}_T(H)$ measurable, therefore $(t, x) \to P_x\{\omega : X(t) \in C\}$ is a progressive process w. r. t. $\{\mathcal{B}_t(H)\}_{t\geq 0}$, by ([7], Cor. 1.6.12), $x \to \rho_n(x)$ is Borel measurable. This proves the lemma

Let $D_r = \{x : \|x\|_V \leq r\}$ for any real number r and let $\overline{D}_r$ be the closure of D_r in $(H, \|\cdot\|_H)$, D_r^o the interior of D_r in $(H, \|\cdot\|_H)$, and $D_r^c = H - D_r$, then $(\overline{D}_r)^c = (D_r^c)^o$.

Lemma 3.1.3 *Suppose the solution $\{u(t), t \geq 0\}$ of (1.1) under coercivity condition (1.2) is ultimately bounded in m. s. s., i.e., (2.21) holds. Let $M_1 = M + 1$, then there exists a positive Borel measurable function $\rho(\varphi)$ defined on H, such that*

$$P_\varphi\{\omega : u(\rho(\varphi)) \in (D_r^c)^o\} \leq \frac{1}{\alpha r^2}(|\lambda| M_1 + M_1 + \gamma) \tag{3.3}$$

for any positive number r and any $\varphi \in H$.

Proof: Since $\limsup_{t\to\infty} E^\varphi \|u(t)\|_H^2 \leq M < M_1$ for $\forall \varphi \in H$, hence for each $\varphi \in H$, there exists a positive number T_φ, such that

$$E^\varphi \|u(t)\|_H^2 \leq M_1 \qquad \text{for} \quad t \geq T_\varphi.$$

Let

$$\gamma(\varphi) = \inf\{t : E^{\varphi}\|u(s)\|_H^2 \leq M_1 \quad \text{for all} \quad s \geq t\}.$$

Since $t \to E^{\varphi}\|u(s)\|_H^2$ is continuous, $E^{\varphi}\|u(\gamma(\varphi))\|_H^2 \leq M_1$, and

$$\begin{aligned}\{\varphi : \gamma(\varphi) \leq t\} &= \{\varphi : E^{\varphi}\|u(s)\|_H^2 \leq M_1 \quad \text{for all} s \geq t\} \\ &= \bigcap_{s\geq t, s\in Q} \{\varphi : E^{\varphi}\|u(s)\|_H^2 \leq M_1\}.\end{aligned}$$

and since $\varphi \to E^{\varphi}\|u(s)\|_H^2$ is Borel measurable, $\{\varphi : \gamma(\varphi) \leq t\} \in \mathcal{B}(H)$, therefore, $\varphi \to \gamma(\varphi)$ is Borel measurable.

Now we apply Ito's formula (1.4) to $\|x\|_H^2$, take expectation and make the use of coercivity condition (1.2), we get

$$\begin{aligned}&E^{\varphi}\|u(\gamma(\varphi)+1)\|_H^2 - E^{\varphi}\|u(\gamma(\varphi))\|_H^2 \\ = &\int_{\gamma(\varphi)}^{\gamma(\varphi)+1} E\mathcal{L}\|u(s)\|_H^2 ds \\ \leq &\lambda \int_{\gamma(\varphi)}^{\gamma(\varphi)+1} E^{\varphi}\|u(s)\|_H^2 ds - \alpha \int_{\gamma(\varphi)}^{\gamma(\varphi)+1} E^{\varphi}\|u(s)\|_V^2 ds + \gamma\end{aligned}$$

hence

$$\begin{aligned}\int_{\gamma(\varphi)}^{\gamma(\varphi)+1} E^{\varphi}\|u(s)\|_V^2 ds &\leq \frac{1}{\alpha}(\lambda \int_{\gamma(\varphi)}^{\gamma(\varphi)+1} E^{\varphi}\|u(s)\|_H^2 ds + E^{\varphi}\|u(\gamma(\varphi))\|_H^2 + |\gamma|) \\ &\leq \frac{1}{\alpha}(|\lambda| M_1 + M_1 + |\gamma|)\end{aligned}$$

Using Chebychev's inequality we get

$$\begin{aligned}\int_{\gamma(\varphi)}^{\gamma(\varphi)+1} P_{\varphi}\{\omega : \|u(t)\|_V > r\} dt &\leq \int_{\gamma(\varphi)}^{\gamma(\varphi)+1} \frac{E^{\varphi}\|u(t)\|_V^2}{r^2} dt \\ &\leq \frac{1}{\alpha r^2}(|\lambda| M_1 + M_1 + |\gamma|),\end{aligned}$$

hence

$$\int_{\gamma(\varphi)}^{\gamma(\varphi)+1} P_{\varphi}\{\omega : u(t) \in (D_r^c)^o\} dt \leq \frac{1}{\alpha r^2}(|\lambda| M_1 + M_1 + |\gamma|),$$

therefore

$$\int_{\gamma(\varphi)}^{\gamma(\varphi)+1} P_{\varphi}\{\omega : u(t) \in \overline{D}_r\} dt \geq 1 - \frac{1}{\alpha r^2}(|\lambda| M_1 + M_1 + |\gamma|).$$

By Lemma 2.2, there exists a positive Borel measurable function $\rho(\varphi)$ defined on H, such that

$$\gamma(\varphi) \leq \rho(\varphi) \leq \gamma(\varphi) + 1, \quad and$$

$$P_\varphi\{\omega : u(\rho(\varphi)) \in \overline{D}_r\} \geq 1 - \frac{1}{\alpha r^2}(|\lambda|M_1 + M_1 + |\gamma|) \quad \text{for} \quad \forall \varphi \in H. \tag{3.4}$$

Therefore

$$P_\varphi\{\omega : u(\rho(\varphi)) \in (D_r^c)^o\} \leq \frac{1}{\alpha r^2}(|\lambda|M_1 + M_1 + |\gamma|)$$

for any positive number r and any $\varphi \in H$.

Proof of Theorem 3.1.1: From (3.4), we can choose r large enough such that:

$$P_\varphi\{\omega : u(\rho(\varphi)) \in \overline{D}_r\} \geq \frac{1}{2} \quad \text{for} \quad \forall \varphi \in H.$$

Since $V \hookrightarrow H$ is compact, $\overline{D}_r$ is a compact set in H, by Lemma 3.1.1, $u(t)$ is weakly recurrent to $\overline{D}_r$.

now we consider weakly positive recurrence of the solution of (1.1) to a compact set under the condition that it is exponentially ultimate boundedness in m. s. s..

Definition 3.1.2 *A stochastic process $X(t)$ defined on H is weakly positive recurrent to a compact set if there exists a compact set C, such that $X(t)$ is weakly recurrent to C and the first hitting time to C has finite expectation for any $x = X(0) \in H$.*

Theorem 3.1.2 *Suppose $V \hookrightarrow H$ is compact, and the solution $\{u(t), t \geq 0\}$ of (1.1) under coercivity condition (1.2) is exponentially ultimately bounded in m. s. s.. Then $\{u(t), t \geq 0\}$ is weakly positive recurrent.*

Proof: Since the solution $\{u(t), t \geq 0\}$ of (1.1) is exponentially ultimately bounded in m. s. s., we suppose (2.1) is satisfied. Let $M_1 = M + 1$, then it is easy to see if $t \geq \frac{1}{\beta}\ln(1 + c\|\varphi\|_H^2)$, then $E^\varphi\|u(t)\|_H^2 \leq M_1$. Let

$$W(l) = \frac{1}{\beta}\ln(1 + cl^2),$$

then $W(l)$ satisfies:

$$E^{\varphi}\|u(t)\|_H^2 \le M_1 \qquad \text{for } \forall\varphi \in H \text{ and } t \ge W(\|\varphi\|_H),$$

and

$$\sum_{i=1}^{\infty} \frac{W((i+1)N)}{i^2} < \infty \qquad \text{for any } N \ge 0. \tag{3.5}$$

Let

$$\begin{aligned}
K &= \frac{1}{\sqrt{\alpha}}\sqrt{|\lambda|M_1 + M_1 + |\gamma|}(1+\epsilon), \\
E_0 &= \overline{D}_K \\
E_l &= \overline{D}_{(l+1)K} - \overline{D}_{lK} = \overline{D}_{(l+1)K} \bigcap (D_{lK}^c)^o \quad \text{for} \quad l \ge 1, \\
W'(l) &= W(lK\alpha_0) + 1
\end{aligned} \tag{3.6}$$

where α_0 is the constant such that $\|x\|_H \le \alpha_0\|x\|_V$ for $\forall x \in V$. As in the proof of Lemma 2.3, there exists a Borel measurable function $\rho(\varphi)$ defined on H, such that

$$\begin{aligned}
W(\|\varphi\|_H) \quad \le \quad \rho(\varphi) &\le W(\|\varphi\|_H) + 1 \qquad \text{and} \\
P_{\varphi}\{\omega : u(\rho(\varphi)) \in (D_{lK}^c)^o\} &\le \frac{1}{\alpha(lK)^2}(|\lambda|M_1 + M_1 + |\gamma|) \\
&= \frac{1}{l^2(1+\epsilon)^2} \qquad \text{for} \quad \forall\varphi \in H.
\end{aligned} \tag{3.7}$$

Let

$\tau_1 = \rho(x)$,

$x_1(\omega) = u(\tau_1, \omega)$,

$\Omega_1 = \{\omega : x_1(\omega) \notin E_0\}$,

$\tau_2 = \tau_1 + \rho(x_1(\omega))$,

$x_2(\omega) = u(\tau_2, \omega)$,

$\Omega_2 = \{\omega : x_2(\omega) \notin E_0\}$,

$\tau_3 = \tau_2 + \rho(x_2(\omega))$,

$x_3(\omega) = u(\tau_3, \omega)$,

$$\Omega_3 = \{\omega : x_3(\omega) \notin E_0\},$$

$$\cdots$$

$$\Omega_\infty = \bigcap_{i=1}^{\infty} \Omega_i.$$

By the proof of Lemma 3.1.1, when $\epsilon > 0$, we know

$$P_\varphi(\bigcap_{i=1}^{\infty} \Omega_i) = 0$$

therefore,

$$\Omega = \bigcup_{i=0}^{\infty} \Omega_i^c = \bigcup_{i=0}^{\infty} \{\omega : x_i(\omega) \in E_0\} \qquad a.e. \qquad (P_\varphi).$$

Let

$$\begin{aligned} A_i &= \Omega_i^c - \bigcup_{i=0}^{i-1} \Omega_i^c = \Omega_i^c \bigcap (\bigcap_{i=0}^{i-1} \Omega_i) \\ &= \{\omega : x_1(\omega) \notin E_0, \cdots, x_{i-1}(\omega) \notin E_0, x_i(\omega) \in E_0\}. \end{aligned}$$

then

$$\Omega = \sum_{i=0}^{\infty} A_i \qquad a.e. \qquad (P_\varphi).$$

For $i \geq 2$, let's further divide A_i as

$$A_i = \sum_{l_1, \cdots, l_{i-1} \geq 1} A_{i, l_1, \cdots, l_{i-1}}$$

where $A_{i,l_1,\cdots,l_{i-1}} = \{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-1} \in E_{l_{i-1}}, x_i(\omega) \in E_0\}$.

Let $\tau(\omega)$ be the first hitting time to E_0, then for $\omega \in A_1 = \Omega_1^c$,

$$\tau(\omega) \leq \rho(\varphi) \leq W(\|\varphi\|_H) + 1,$$

for $\omega \in A_{i,l_1,\cdots,l_{i-1}}$,

$$\tau(\omega) \leq \tau_i(\omega) = \tau_{i-1}(\omega) + \rho(x_{i-1}(\omega)).$$

Since when $\omega \in A_{i,l_1,\cdots,l_{i-1}}$,

$$x_{i-1}(\omega) \in E_{l_{i-1}} \subseteq \overline{D}_{(l_{i-1}+1)K},$$

hence

$$\|x_{i-1}(\omega)\|_H \le \alpha_0 \|x_{i-1}(\omega)\|_V \le \alpha_0(l_{i-1}+1)K$$

then

$$\rho(x_{i-1}(\omega)) \le W(\|(x_{i-1}(\omega)\|_H) + 1 \le W(\alpha_0(l_{i-1}+1)K) + 1 = W'(l_{i-1}+1)$$

then

$$\tau(\omega) \le \tau_{i-1} + W'(l_{i-1}+1).$$

Therefore by induction, for $\omega \in A_{i,l_1,\cdots,l_{i-1}}$,

$$\tau(\omega) \le W(\|\varphi\|_H) + 1 + W'(l_1+1) + \cdots + W'(l_{i-1}+1).$$

On the other hand, by the strong Markov property

$$\begin{aligned}
& P_\varphi(A_{i,l_1,\cdots,l_{i-1}}) \\
= \ & P_\varphi\{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-1}(\omega) \in E_{l_{i-1}}, x_i(\omega) \in E_0\} \\
\le \ & P_\varphi\{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-1}(\omega) \in E_{l_{i-1}}\} \\
\le \ & P_\varphi(\{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-2}(\omega) \in E_{l_{i-2}}\} \bigcap \{\omega : x_{i-1}(\omega) \in E_{l_{i-1}}\}) \\
= \ & E^\varphi\{\chi_{\{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-2}(\omega) \in E_{l_{i-2}}\}} \cdot P_{x_{i-2}(\omega)}\{\omega' : u(\rho(x_{i-2}(\omega), \omega') \in E_{l_{i-1}}\}\}.
\end{aligned}$$

Since $E_{l_{i-1}} = \overline{D}_{(l_{i-1}+1)K} \bigcap (D^c_{l_{i-1}K})^o$, by (3.7) we have

$$\begin{aligned}
& P_{x_{i-2}(\omega)}\{\omega' : u(\rho(x_{i-2}(\omega), \omega') \in E_{l_{i-1}}\} \\
\le \ & P_{x_{i-2}(\omega)}\{\omega' : u(\rho(x_{i-2}(\omega), \omega') \in (D^c_{l_{i-1}K})^o\} \\
\le \ & \frac{1}{l^2_{i-1}(1+\epsilon)^2},
\end{aligned}$$

hence

$$P_\varphi(A_{i,l_1,\cdots,l_{i-1}}) \le \frac{1}{l^2_{i-1}(1+\epsilon)^2} \cdot P_\varphi\{\omega : x_1(\omega) \in E_{l_1}, \cdots, x_{i-2}(\omega) \in E_{l_{i-2}}\},$$

by induction,

$$P_\varphi(A_{i,l_1,\cdots,l_{i-1}}) \le \frac{1}{(1+\epsilon)^{2(i-1)}} \frac{1}{l_1^2 \cdots l_{i-1}^2},$$

and

$$P_\varphi(A_1) < 1.$$

Therefore,

$$\begin{aligned}
& E^\varphi[\tau] \\
\le & \sum_{i,l_1,\cdots,l_{i-1}\ge 1} P_\varphi(A_{i,l_1,\cdots,l_{i-1}})[W(\|\varphi\|_H)+1+W'(l_1+1)+\cdots+W'(l_{i-1}+1)] \\
\le & W(\|\varphi\|_H)+1+\sum_{i=2}^\infty \frac{1}{(1+\epsilon)^{2(i-1)}} \\
& \cdot \sum_{l_1,\cdots,l_{i-1}\ge 1} \frac{W(\|\varphi\|_H)+1+W'(l_1+1)+\cdots+W'(l_{i-1}+1)}{l_1^2\cdots l_{i-1}^2} \\
= & W(\|\varphi\|_H)+1+\sum_{i=2}^\infty \frac{1}{(1+\epsilon)^{2(i-1)}}\{(W(\|\varphi\|_H)+1)\sum_{l_1,\cdots,l_{i-1}\ge 1}\frac{1}{l_1^2\cdots l_{i-1}^2} \\
& +(i-1)\sum_{l_1,\cdots,l_{i-1}\ge 1}\frac{W'(l_1+1)}{l_1^2\cdots l_{i-1}^2} \\
= & W(\|\varphi\|_H)+1+\sum_{i=2}^\infty \frac{1}{(1+\epsilon)^{2(i-1)}}\{(W(\|\varphi\|_H)+1)A^{i-1}+(i-1)A^{i-2}B\} \\
= & (W(\|\varphi\|_H)+1)(1+\sum_{i=2}^\infty(\frac{A}{(1+\epsilon)^2})^{i-1})+\frac{B}{(1+\epsilon)^2}\sum_{i=2}^\infty(\frac{A}{(1+\epsilon)^2})^{i-2}(i-1)
\end{aligned}$$

where $A=\sum_{l=1}^\infty \frac{1}{l^2}$ and $B=\sum_{l=1}^\infty \frac{1}{l^2}W'(l+1)$ which is convergent by (3.5) and (3.6). Hence we see if we choose ϵ large enough, $E^\varphi[\tau]$ is finite. Since $V \hookrightarrow H$ is compact, E_0 is compact, the assertion of the theorem holds.

3.2 Weak Recurrence and Lyapunov Functions

In chapter 2, we studied the relationship of ultimate boundedness and Lyapunov functions. Combining Theorem 2.1.5, 2.1.6 and the corollaries there and Theorem 3.1.1, 3.1.2 here, we immediately get the following results, these results give conditions in terms of Lyapunov function for weak and weakly positive recurrence.

Theorem 3.2.1 *Suppose $V \hookrightarrow H$ is compact. Let $\{u(t), t \ge 0\}$ be the solution of equation (1.1) satisfying coercivity condition (1.2). If there exists a function $\Lambda : H \to$*

R satisfying the following conditions:

(i) Λ *satisfies (1.3),*

(ii) $c_1\|\varphi\|_H^2 - k_1 \leq \Lambda(\varphi) \leq c_3\|\varphi\|_H^2 + k_3, \textit{for } \forall\varphi \in H$,

(iii) $\mathcal{L}\Lambda(\varphi) \leq -c_2\Lambda(\varphi) + k_2, \textit{for } \forall\varphi \in V$,

where $c_1(>0), c_2(>0), c_3(>0), k_1, k_2$ *and* k_3 *are constants. Then* $\{u(t), t \geq 0\}$ *is weakly positive recurrent.*

Theorem 3.2.2 *Let* Λ *satisfy* $(i), (iii)$ *in Theorem 3.1 and*

(ii)' $c_1\|\varphi\|_H^2 - k_1 \leq \Lambda(\varphi)$ *for* $\forall\varphi \in H$

for some constants $c_1(>0)$ *and* k_1*, then* $\{u(t), t \geq 0\}$ *is weakly recurrent.*

The weakly positive recurrence of the solution of the nonlinear equation (1.1) can also be studied through its first order approximation. Let $\{u_0(t), t \geq 0\}$ be the solution of the linear SPDE (2.11). We suppose that the linear operators A_0, B_0 satisfy the coercivity condition (2.12) and the other conditions posted there.

Theorem 3.2.3 *Suppose* $V \hookrightarrow H$ *is compact and the solution* $\{u_0(t), t \geq 0\}$ *of the linear equation (2.11) satisfying coercivity condition (2.12) is exponentially ultimately bounded in m. s. s.. Let* $\{u(t), t \geq 0\}$ *be the solution of the nonlinear equation (1.1). Furthermore, we suppose* $A(v) - A_0 v \in H$ *for all* $v \in V$*. If for* $v \in V$,

$$2\|v\|_H\|A(v) - A_0 v\|_H + \tau(B(v)QB^*(v) - B_0 v Q B_0^* v) \leq \omega\|v\|_H^2 + k \tag{3.8}$$

with ω, k *constants and*

$$\omega < \frac{c}{\alpha_0\beta[(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}) + \frac{\gamma+|\lambda|M}{2\alpha}(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta})^2]}. \tag{3.9}$$

Then $\{u(t), t \geq 0\}$ *is weakly positive recurrent.*

Corollary 3.2.1 *Suppose $V \hookrightarrow H$ is compact and the solution $\{u_0(t), t \geq 0\}$ of the linear equation (2.11) satisfying coercivity condition (2.12) is exponentially ultimately bounded in m. s. s.. Let $\{u(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). Furthermore, we suppose for $v \in V$, $A(v) - A_0 v \in H$, and*

$$\|A(v) - A_0 v\|_H^2 + \tau(B(v)QB^*(v) - B_0 vQB_0^* v) \leq K(1 + \|v\|_H^2)$$

for some constant $K > 0$. If for $v \in V$, as $\|v\|_H \to \infty$

$$\|A(v) - A_0 v\|_H = o(\|v\|_H) \text{ and } \tau(B(v)QB^*(v) - B_0 vQB_0^* v) = o(\|v\|_H^2).$$

Then $\{u(t), t \geq 0\}$ is weakly positive recurrent.

Theorem 3.2.4 *Suppose $V \hookrightarrow H$ is compact and the solution $\{u_0(t), t \geq 0\}$ of the linear equation (2.11) satisfying coercivity condition (2.12) is exponentially ultimately bounded in m. s. s.. Furthermore, we suppose $t \to E\|u_0(t)\|_V^2$ is continuous for all $\varphi \in V$. Let $\{u(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). If for $v \in V$,*

$$2\|v\|_V \|A(v) - A_0 v\|_{V^*} + \tau(B(v)QB^*(v) - B_0 vQB_0^* v) \leq \omega\|v\|_V^2 + k \tag{3.10}$$

with ω, k constants and

$$\omega < \frac{c}{(\alpha_0 + 1)\beta[(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta})(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}) + \frac{\gamma + |\lambda| M}{2\alpha}(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta})^2]}. \tag{3.11}$$

Then $\{u(t), t \geq 0\}$ is weakly positive recurrent.

Corollary 3.2.2 *Suppose $V \hookrightarrow H$ is compact and solution $\{u_0(t), t \geq 0\}$ of the linear equation (2.11) satisfying coercivity condition (2.12) is exponentially ultimately bounded in m. s. s.. Furthermore, we suppose $t \to E\|u_0(t)\|_V^2$ is continuous for all $\varphi \in V$. Let $\{u(t), t \geq 0\}$ be the solution of the nonlinear equation (1.1). If for $v \in V$, as $\|v\|_V \to \infty$*

$$\|A(v) - A_0 v\|_{V^*} = o(\|v\|_V) \text{ and } \tau(B(v)QB^*(v) - B_0 vQB_0^* v) = o(\|v\|_V^2).$$

Then $\{u(t), t \geq 0\}$ is weakly positive recurrent.

3.3 Parabolic Ito Equations and Examples

Let $D \subset R^n$ be a bounded domain with smooth boundary ∂D, r be a positive integer. Let

$$V = W^{r,2}(D), \qquad H = W^{0,2}(D).$$

By Sobolev imbedding theorem, $V \hookrightarrow H$ is compact.

Let

$$A_0(x) = \sum_{|\alpha| \leq 2r} a_\alpha(x) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \tag{3.12}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Garding's inequality ([22], Th. 7.2.2) says that if A_0 is a strongly elliptic operator, then it is coercive.

Example 3.3.1 *Consider the parabolic Ito equation of the form:*

$$\begin{cases} du(t,x) = A_0 u(t,x)dt + f(u(t,x))dt + B(u(t,x))dW(t) \\ u(0,x) = \varphi \in H \\ u|_{\partial D} = 0 \end{cases} \tag{3.13}$$

where A_0, f and B satisfy the following conditions:

(i) $A_0 : V \to V^*$ is a strongly elliptic operator

(ii) $f : H \to H$ and $B : H \to L_2(K,H)$ satisfy: for $v \in H$.

$$\|f(v)\|_H^2 + \|B(v)\|_{L(K,H)}^2 \leq K(1 + \|v\|_H^2)$$

(iii) For $u, v \in H$,

$$\|f(u) - f(v)\|_H^2 + tr((B(u) - B(v))Q(B^*(u) - B^*(v))) \leq \lambda \|u - v\|_H^2.$$

If the solution of equation $du(t,x) = A_0 u(t,x)dt$ is exponentially ultimately bounded in m. s. s., and as $\|v\|_H \to \infty$

$$\|f(v)\|_H = o(\|v\|_H), \quad \|B(v)\|_{L(k,H)} = o(\|v\|_H),$$

then the solution $\{u(t), t \geq 0\}$ *of (3.13) is exponentially ultimately bounded in m. s. s.(example 2.1.1), hence it is weakly positive recurrent.*

Example 3.3.2 *Consider the following 1-dimensional parabolic Ito equation:*

$$\begin{cases} du(t,x) = (\alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + \gamma u + g(x))dt + (\sigma_1 \frac{\partial u}{\partial x} + \sigma_2 u)dW(t) \\ u(0,x) = \phi(x) \in L^2(D) \bigcap L^1(D) \\ u|_{\partial D} = 0 \end{cases} \tag{3.14}$$

where $D = [0,1] \subseteq R^1$, $W(t)$ *is a 1-dimensional standard Brownian Motion. Let*

$$V = W^{1,2}(D), \qquad H = W^{0,2}(D).$$

Suppose $g \in L^2(D) \bigcap L^1(D)$. Take $\Lambda(x) = \|x\|_H^2$ for $x \in H$, for $v \in V$,

$$\begin{aligned} \mathcal{L}\Lambda(v) &= 2{<}v, A(v){>} + tr(BvQB^*v) \\ &= 2\int_D (v, \alpha^2 \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial x} + \gamma + g) + \int_D (\sigma_1 \frac{\partial v}{\partial x} + \sigma_2 v)^2 dx \\ &= (-2\alpha^2 + \sigma_1^2)\|v\|_V^2 + (2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2)\|v\|_H^2 + 2\int_D (v, g)dx \\ &\leq (-2\alpha^2 + \sigma_1^2)\|v\|_V^2 + (2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2 + \epsilon)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2 \end{aligned}$$

for $\forall \epsilon > 0$. Hence if $-2\alpha^2 + \sigma_1^2 < 0$, then the coercivity condition (1.2) satisfied. Furthermore,

$$\mathcal{L}\Lambda(v) \leq (-2\alpha^2 + \sigma_1^2)\|\frac{\partial v}{\partial x}\|_H^2 + (2\gamma + \sigma_2^2 + \epsilon)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2,$$

by Poincare Lemma, $\|\frac{\partial v}{\partial x}\|_H^2 \geq 8\|v\|_H^2$, thus,

$$\mathcal{L}\Lambda(v) \leq (-16\alpha^2 + 8\sigma_1^2 + 2\gamma + \sigma_2^2 + \epsilon)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2,$$

Therefore if

$$-16\alpha^2 + 8\sigma_1^2 + 2\gamma + \sigma_2^2 < 0$$

then Theorem 3.2.1 says that the solution $\{u(t), t \geq 0\}$ of (3.14) is weakly positive recurrent.

Chapter 4

Stability and Ultimate Boundedness of the Mild Solution

In this chapter we will study the stability, exponentially ultimate boundedness and stability in probability for the the mild solution of the stochastic semilinear evolution equation. The main technique is to construct an appropriate Lyapunov function. Once this is done, we will exploit the methods developed in chapter 2 of this dissertation and those in [15] to obtain results for the mild solution.

4.1 Exponential Stability in the Mean Square Sense

The exponential stability in the mean square sense of the mild solution of (1.5) was undertaken in a systematic manner in [10, 2, 6], and was continued in [15] for the strong solution under coercivity condition. An example was given in [15] to show that the usual Lyapunov function was not bounded below. In this section, we construct a new Lyapunov function and show that the existence of such a Lyapunov function is a necessary and sufficient condition for the mild solution of (1.5) to be exponentially stable in the m. s. s.. Then we use this bounded below Lyapunov function to study the problem of the stability in probability, we conclude that exponential stability in the m. s. s. implies stability in probability for the mild solution of the semilinear evolution equation (1.5).

Let us assume that $u^{\varphi}(t)$ is the mild solution of (1.5), we say it is exponentially stable in the m. s. s. if there exist positive constants c, β, such that

$$E\|u^{\varphi}(t)\|^2 \leq ce^{-\beta t}\|\varphi\|^2. \quad \text{for all } \varphi \in H \text{ and } t > 0. \tag{4.1}$$

The next theorem gives a sufficient condition for $u^{\varphi}(t)$ to be exponentially stable in the m. s. s., it was proved in [11], we quote it here for the ease of reference.

Theorem 4.1.1 *The mild solution $u^{\varphi}(t)$ of (1.5) is exponentially stable in the m. s. s. if there exists a function $C_b^2(H) \ni \Lambda : H \to R$ satisfying the following conditions:*

$$(i) \quad c_1\|\varphi\|^2 \leq \Lambda(\varphi) \leq c_3\|\varphi\|^2, \tag{4.2}$$

$$(ii) \quad \mathcal{L}\Lambda(\varphi) \leq -c_2\Lambda(\varphi), \tag{4.3}$$

for $\forall\varphi \in H$, where c_1, c_2, c_3 are positive constants.

Proof: Apply Ito's formula (1.7) to $e^{c_2 t}\Lambda(\varphi)$ and $u_n(t)$ and take expectation, where $u_n(t)$ is the strong solution of (1.8), then

$$e^{c_2 t}E\Lambda(u_n^{\varphi}(t)) - \Lambda(u_n^{\varphi}(0)) = E\int_0^t e^{c_2 s}(c_2 + \mathcal{L}_n)\Lambda(u_n^{\varphi}(s))ds$$

By (ii),

$$\begin{aligned} & c_2\Lambda(\varphi) + \mathcal{L}_n\Lambda(\varphi) \leq -\mathcal{L}\Lambda(\varphi) + \mathcal{L}_n\Lambda(\varphi) \\ = \; & < \Lambda'(\varphi), (R(n) - I)F(\varphi) > + \frac{1}{2}tr(\Lambda''(\varphi)(R(n)B(\varphi)Q(R(n)B(\varphi))^* - B(\varphi)QB^*(\varphi))) \end{aligned}$$

therefore,

$$\begin{aligned} & e^{c_2 t}E\Lambda(u_n^{\varphi}(t)) - \Lambda(u_n^{\varphi}(0)) \\ \leq \; & E\int_0^t e^{c_2 s}(< \Lambda'(u_n^{\varphi}(s)), (R(n) - I)F(u_n^{\varphi}(s)) > \\ + \; & \frac{1}{2}tr(\Lambda''(u_n^{\varphi}(s))(R(n)B(u_n^{\varphi}(s))Q(R(n)B(u_n^{\varphi}(s)))^* - B(u_n^{\varphi}(s))QB^*(u_n^{\varphi}(s)))))ds. \end{aligned}$$

Let $n \to \infty$, by the dominated convergence theorem and Theorem 1.5.3, we get

$$e^{c_2 t} E\Lambda(u^{\varphi}(t)) \leq \Lambda(\varphi),$$

hence by(i), we have:

$$c_1 E\|u^{\varphi}(t)\|^2 \leq E\Lambda(u^{\varphi}(t)) \leq e^{-c_2 t}\Lambda(\varphi) \leq c_3 e^{-c_2 t}\|\varphi\|^2.$$

This proves the theorem.

Now we want to construct a Lyapunov function if the solution $u^{\varphi}(t)$ of (1.5) is exponentially stable in the m. s. s..

First, let us consider the following linear case. Suppose $F \equiv 0$ and $B = B_0$ is linear, Then equation (1.5) has the following form:

$$\begin{cases} du = Audt + B_0 udW(t) \\ \qquad u(0) = \varphi. \end{cases} \tag{4.4}$$

We assume $\|B_0 x\| \leq d\|x\|$ for $\forall x \in H$ and the solution of this equation is $u_0^{\varphi}(t)$. The infinitesimal generator $\mathcal{L}_0$ corresponding to this equation is $\mathcal{L}_0\ \Lambda(\varphi) =< \Lambda'(\varphi), A\varphi >$ $+\frac{1}{2}tr(\Lambda''(\varphi)B_0\varphi Q(B_0\varphi)^*)$.

Theorem 4.1.2 *If the solution $u_0^{\varphi}(t)$ of equation (4.4) is exponentially stable in the m. s. s., then there exits a function $\Lambda_0 \in C_b^2(H)$ satisfying (4.2) and (4.3) with $\mathcal{L}$ replaced by $\mathcal{L}_0$.*

Proof: Let

$$\Lambda_0(\varphi) = \int_0^{\infty} E\|u_0^{\varphi}(t)\|^2 dt + \alpha\|\varphi\|^2 \tag{4.5}$$

where α is a constant to be determined later. Since $u_0^{\varphi}(t)$ is exponentially stable in the m. s. s., $\int_0^{\infty} E\|u_0^{\varphi}(t)\|^2 dt$ is well defined and there exists a symmetric and nonnegative operator $R \in L(H)$ [6], such that

$$\int_0^{\infty} E\|u_0^{\varphi}(t)\|^2 dt =< R\varphi, \varphi >$$

and

$$\mathcal{L}_0 < R\varphi, \varphi >= -\|\varphi\|^2.$$

Hence,

$$\Lambda_0(\varphi) =< R\varphi, \varphi > +\alpha\|\varphi\|^2. \tag{4.6}$$

It is obvious that $\Lambda_0 \in C_b^2(H)$ and $\alpha\|\varphi\|^2 \leq \Lambda_0(\varphi) \leq (\|R\|+\alpha)\|\varphi\|^2$, this proves (4.2). To prove (4.3) with $\mathcal{L}$ replaced by$\mathcal{L}_0$, we note that A is the infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \leq e^{\omega t}$, there exists a constant λ, without loss of generality, we assume it is positive, such that $< \varphi, A\varphi >\leq \lambda\|\varphi\|^2$ ([12]), hence we have,

$$\mathcal{L}_0\|\varphi\|^2 = 2 < \varphi, \varphi > +tr(B_0\varphi Q(B_0\varphi)^*) \leq (2\lambda + d^2tr(Q))\|\varphi\|^2. \tag{4.7}$$

Hence,

$$\begin{aligned} \mathcal{L}_0\Lambda_0(\varphi) &= \mathcal{L}_0 < R\varphi, \varphi > +\alpha\mathcal{L}_0\|\varphi\|^2 \\ &\leq -\|\varphi\|^2 + \alpha(2\lambda + d^2tr(Q))\|\varphi\|^2 \\ &\leq (-1 + \alpha(2\lambda + d^2tr(Q)))\|\varphi\|^2. \end{aligned} \tag{4.8}$$

Therefore, if α is small enough, (4.3) holds with $\mathcal{L}$ replaced by$\mathcal{L}_0$. This proves the theorem.

Remark 4.1.1 *In [10], Haussmann proved a stability theorem under conditions* H_1 : $\exists c > 0, \gamma > 0$ *such that* $\|S(t)\| < ce^{-\gamma t}$ *for* $\forall t > 0$ *and* H_2 : $\| \int_0^\infty S_t^*\Delta(I)S_t dt\| < 1$, *where* $< \Delta(I)\varphi, \psi >= tr(B(\varphi)QB^*(\psi))$.

Define $\Lambda_0(\varphi) = \int_0^\infty E\|u_0^\varphi(t)\|^2dt + \alpha\|\varphi\|^2$, *This is well defined because of* H_1 *and* H_2. *From our theorem, use* $\Lambda_0(\varphi)$ *as a Lyapunov function, the result follows.*

For the nonlinear equation (1.5), to assure zero is a solution, we need to assume $F(0) = 0, B(0) = 0$. If the solution $u^\varphi(t)$ is exponentially stable in the m. s. s., we can still construct a Lyapunov as in (4.5):

$$\Lambda(\varphi) = \int_0^\infty E\|u^\varphi(t)\|^2dt + \alpha\|\varphi\|^2$$

But it may not be in $C_b^2(H)$. If we assume it is in $C_b^2(H)$, we claim that it satisfy (4.2) and (4.3). Now, let us prove this claim.

Since $u^\varphi(t)$ is exponentially stable in the m. s. s., we assume it satisfies(4.1), hence $\int_0^\infty E\|u^\varphi(t)\|^2 dt \le \frac{c}{\beta}\|\varphi\|^2$ for all $x \in H$, therefore $\alpha\|\varphi\|^2 \le \Lambda_0(\varphi) \le (\frac{c}{\beta}+\alpha)\|\varphi\|^2$, this proves (4.2).

To prove (4.3), let

$$\Psi(\varphi) = \int_0^\infty E\|u^\varphi(t)\|^2 dt.$$

Observe

$$E\Psi(u^\varphi(r)) = E\int_0^\infty E(\|u^{u^\varphi(r)}(s)\|^2 | u^\varphi(r))ds$$

But by the Markov property of the solution of (1.5), this equals

$$\int_0^\infty E(E(\|u^{u^\varphi(r)}(s)\|^2|\mathcal{F}_r^u))ds$$

where $\mathcal{F}_r^\varphi = \sigma\{u^\varphi(\tau), \tau \le r\}$. The uniqueness of the solution implies

$$E(\|u^{u^\varphi(r)}(s)\|^2|\mathcal{F}_r^\varphi) = E(\|u^\varphi(s+r)\|^2|\mathcal{F}_r^\varphi).$$

Hence

$$E\Psi(u^\varphi(r)) = \int_0^\infty E\|u^\varphi(r+s)\|^2 ds = \int_r^\infty E\|u^\varphi(s)\|^2 ds. \tag{4.9}$$

By the continuity of $t \to E\|u^\varphi(t)\|^2$, we get:

$$\begin{aligned}
\mathcal{L}\Psi(\varphi) &= \frac{d}{dr}(E\Psi(u^\varphi(r)))|_{r=0} \\
&= \lim_{r\to 0}\frac{E\Psi(u^\varphi(r)) - \Psi(\varphi)}{r} \\
&= \lim_{r\to 0} -\frac{1}{r}\int_0^r E\|u^\varphi(s)\|^2 ds \\
&= -\|\varphi\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}\Lambda(\varphi) &= \mathcal{L}\Psi(\varphi) + \alpha\mathcal{L}\|\varphi\|^2 \\
&= -\|\varphi\|^2 + \alpha(2 < x, Ax + F(\varphi) > +tr(B(\varphi)Q(B(\varphi))^*) \\
&\le -\|\varphi\|^2 + 2\alpha\lambda\|\varphi\|^2 + \alpha(2 < x, F(\varphi) > +tr(B(\varphi)Q(B(\varphi))^*).
\end{aligned}$$

Since we assume $F(0) = 0, B(0) = 0$, using the Lipschitz condition (1.6), we get

$$\mathcal{L}\Lambda(\varphi) \leq -\|\varphi\|^2 + \alpha(2\lambda + 2d + d^2 tr(Q))\|\varphi\|^2.$$

Hence if α is small enough, $\Lambda(\varphi)$ satisfies (4.3). Therefore, we have proved the following theorem:

Theorem 4.1.3 *If the solution $u^\varphi(t)$ of (1.5) is exponentially stable in the m. s. s., furthermore, $F(0) = 0, B(0) = 0$, and $\Psi(\varphi) = \int_0^\infty E\|u^\varphi(t)\|^2 dt$ is in $C_b^2(H)$. Then the function $\Lambda(\varphi)$ constructed above satisfies (4.2) and (4.3).*

As in the case of strong solution, we also have difficulty to show $\Psi(\varphi) \in C_b^2(H)$. We thus turn to use the first order approximation to study the exponential stability in the m. s. s. of the solution of the nonlinear equation (1.5).

Theorem 4.1.4 *Suppose the solution $u_0^\varphi(t)$ of the equation (4.4) is exponentially stable in the m. s. s., and it satisfies (4.1). Then the solution $u^\varphi(t)$ of (1.5) is exponentially stable in the m. s. s. if*

$$2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*) < \frac{\beta}{c}\|\varphi\|^2 \tag{4.10}$$

Proof: Let $\Lambda_0(\varphi) = < R\varphi, \varphi > +\alpha\|\varphi\|^2$ as defined in (4.6). Since $u_0^\varphi(t)$ satisfies (4.1), $\|R\| \leq \frac{c}{\beta}$. Since $\Lambda_0(\varphi) \in C_b^2(H)$ and satisfies (4.2), if we can show that $\Lambda_0(\varphi)$ satisfies (4.3), then by using Theorem 4.1.1, we are done. Since

$$\begin{aligned}
& \mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) \\
= \; & < \Lambda_0'(\varphi), F(\varphi) > +\frac{1}{2}tr(\Lambda_0''(\varphi)(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*)) \\
= \; & 2 < (R+\alpha)x, F(\varphi) > +tr((R+\alpha)(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*)) \\
\leq \; & 2(\|R\| + \alpha)\|\varphi\|\|F(\varphi)\| + (\|R\| + \alpha)\tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*) \\
= \; & (\|R\| + \alpha)(2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*)). \qquad (4.11)
\end{aligned}$$

here we used lemma 2.1.2. By (4.8) and the assumption (4.10), $\mathcal{L}\Lambda_0(\varphi)$ satisfies (4.3) if we choose α small enough. This proves the theorem.

The following example shows that the usual Lyapunov function is not bounded below.

Example 4.1.1 *Consider the following SPDE:*

$$d_t u(t,x) = (\alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + \gamma u)dt + \sigma u dW(t)$$

with initial condition

$$u(0,x) = \phi(x) \in L^2(-\infty,\infty) \bigcap L^1(-\infty,\infty).$$

where $W(t)$ is a one-dimensional standard Brownian Motion.

Let

$$\begin{aligned} H &= L^2(-\infty,\infty), & A(u) &= \alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + \gamma u, \\ B(u) &= \sigma u, & \|u\| &= (\int_{-\infty}^{\infty} u^2 dx)^{\frac{1}{2}}. \end{aligned}$$

Now we compute $E\|u^\phi(t)\|^2$ explicitly. Taking Fourier transform of the SPDE:

$$\begin{aligned} d_t \hat{u}(t,\lambda) &= (-\alpha^2\lambda^2\hat{u}(t,\lambda) + i\lambda\beta\hat{u}(t,\lambda) + \gamma\hat{u}(t,\lambda))dt + \sigma\hat{u}(t,\lambda)dW(t) \\ &= (-\alpha^2\lambda^2 + i\lambda\beta + \gamma)\hat{u}(t,\lambda)dt + \sigma\hat{u}(t,\lambda)dW(t). \end{aligned}$$

Solving the equation we get for each fixed λ:

$$E|\hat{u}(t,\lambda)|^2 = |\hat{\phi}(\lambda)|^2 e^{(-2\alpha^2\lambda^2+2\gamma+\sigma^2)t}.$$

By Plancheral theorem, with $H = L^2(-\infty,\infty)$

$$\|u^\phi(t,\cdot)\|^2 = \|\hat{u}^\phi(t,\cdot)\|^2$$

we have

$$\begin{aligned} E\|u^\phi(t)\|^2 &= E\|\hat{u}^\phi(t)\|^2 = E\int_{-\infty}^{\infty} |\hat{u}(t,\lambda)|^2 d\lambda \\ &= \int_{-\infty}^{\infty} |\hat{\phi}(\lambda)|^2 e^{(-2\alpha^2\lambda^2+2\gamma+\sigma^2)t} d\lambda \end{aligned}$$

If we assume $2\gamma + \sigma^2 < 0$, then we get

$$E\|u^{\phi}(t)\|^2 \leq \|\phi\|^2 e^{(2\gamma+\sigma^2)t},$$

hence the solution of the SPDE is stable. But,

$$\int_0^{\infty} E\|u^{\phi}(t)\|^2 dt = \int_{-\infty}^{\infty} |\widehat{\phi}(\lambda)|^2 \frac{1}{2\alpha^2\lambda^2 - (2\gamma + \sigma^2)} d\lambda$$

thus the usual Lyapunov function $\int_{-\infty}^{\infty} E\|u^{\phi}(t)\|^2 dt$ is not bounded below.

4.2 Stability in Probability

Stability in probability is studied in [15] for the strong solution under coercivity condition. The key in [15] is the construction of a *bounded below* Lyapunov function. In this chapter, we will use the bounded below Lyapunov function constructed in the previous section to study the problem of the stability in probability. We conclude that the exponentially stable in the m. s. s. implies the stability in probability for the mild solution of (1.5). We first present a general result following [15].

Theorem 4.2.1 *Let $u^{\varphi}(t)$ be the solution of equation (1.5). If there exists a function $\Lambda(\varphi) \in C_b^2(H)$ having the following properties:*

(i) $\Lambda(\varphi) \to 0$ *as* $\|\varphi\| \to 0$,

(ii) $\inf_{\|\varphi\|>\epsilon} \Lambda(\varphi) = \lambda_{\epsilon} > 0$,

(iii) $\mathcal{L}\Lambda(\varphi) \leq 0$ *when* $\|\varphi\| < \delta$ *for some small* δ.

Then

$$\lim_{\|\varphi\|\to 0} P\{\sup_t \|u^{\varphi}(t)\| > \epsilon\} = 0 \, for\ each\, \epsilon > 0$$

i.e., zero solution of equation (1.5) is stable in probability.

Proof: We first obtain the inequality

$$P\{\sup_t \|u^\varphi(t)\| > \epsilon\} \leq \frac{\Lambda(\varphi)}{\lambda_\epsilon} \text{ for } \varphi \in H.$$

To prove this, let $O_\epsilon = \{x \in H : \|\varphi\| < \epsilon\}, \tau_\epsilon = \inf\{t : \|u^\varphi(t)\| > \epsilon\}$. Using the same technical as in Theorem 4.1.1 and condition $(i), (ii)$, we get

$$\Lambda(\varphi) \geq E\Lambda(u^\varphi(t \wedge \tau_\epsilon)) \geq \lambda_\epsilon P(\tau_\epsilon < t).$$

this proves the inequality. Now let $x \to 0$, we get the assertion.

The function constructed in Theorem 2.2 for the linear equation (4.4) satisfies the conditions of Theorem 2.5, hence we get the following theorem.

Theorem 4.2.2 *The solution $u_0^\varphi(t)$ of the linear equation (4.4) is stable in probability if it is exponentially stable in the m. s. s..*

For the stability in probability of the zero solution of the nonlinear equation (1.5), we have the following theorem.

Theorem 4.2.3 *If the solution $u_0^\varphi(t)$ of the linear equation (4.4) is exponentially stable in the m. s. s., and*

$$2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*) < \omega\|\varphi\|^2 \tag{4.12}$$

for ω small enough in a sufficiently small neighborhood of $\varphi = 0$. Then the zero solution of the nonlinear equation (1.5) is stable in probability.

Proof: Since the solution $u_0^\varphi(t)$ of the linear equation (4.4) is exponentially stable in the m. s. s., we define $\Lambda_0(\varphi) =< R\varphi, \varphi > +\alpha\|\varphi\|^2$ as in (4.6). By (4.11) and assumption(4.12), we get

$$\mathcal{L}\Lambda_0(\varphi) \leq 0.$$

Obviously, $\Lambda_0(\varphi)$ satisfies the other conditions of Theorem 2.4, therefore our assertion holds.

4.3 Exponentially Ultimate Boundedness in the Mean Square Sense

Exponentially ultimate boundedness in the m. s. s was studied by Wonham [27], Zakai [29] and Miyahara [18] in terms of a Lyapunov function for the finite dimensional case, and Miyahara constructed a Lyapunov function if the solution of the stochastic differential equation is exponentially ultimately bounded. Ichikawa [12] gave a sufficient condition for the mild solution of a semilinear stochastic evolution equation to be exponentially ultimately bounded in terms of a Lyapunov function. In chapter 2 of this dissertation, we studied the same problem for the strong solution of SPDE under coercivity condition, and get a necessary and sufficient condition in terms of a Lyapunov function for the linear case and use the first order approximation to study the nonlinear case. In this section, we study this problem for the mild solution of (1.5) and also give a necessary and sufficient condition in terms of a Lyapunov function for the linear case and use the first order approximation to study the nonlinear case.

For exponential ultimately boundedness in the m. s. s. we have a similar result as Theorem 4.1.1 for exponential stability in the m. s. s..

Theorem 4.3.1 *The mild solution $u^\varphi(t)$ of (1.5) is exponentially ultimately bounded in the m. s. s. if there exists a function $C_b^2(H) \ni \Lambda : H \to R$ satisfying the following conditions:*

$$(i) \quad c_1\|\varphi\|^2 - k_1 \leq \Lambda(\varphi) \leq c_3\|\varphi\|^2 - k_3, \tag{4.13}$$

$$(ii) \quad \mathcal{L}\Lambda(\varphi) \leq -c_2\Lambda(\varphi) + k_2, \tag{4.14}$$

for $\forall \varphi \in H$, where $c_1(>0), c_2(>0), c_3(>0), k_1, k_2$ and k_3 are constants.

Proof: The proof of this theorem is similar to that of Theorem 4.1.1.

For the converse problem, we first see the linear equation (4.4). We have the following theorem.

Theorem 4.3.2 *If the solution $u_0^{\varphi}(t)$ of equation (4.4) is exponentially ultimately bounded in the m. s. s., then there exits a function $\Lambda_0 \in C_b^2(H)$ satisfying (4.13) and (4.14) with $\mathcal{L}$ replaced by $\mathcal{L}_0$.*

Proof: Suppose the solution $u_0^{\varphi}(t)$ of (4.4) is exponentially ultimately bounded in m. s. s., i.e., we suppose (2.1) holds. Let

$$\Lambda_0(\varphi) = \int_0^T E\|u_0^{\varphi}(s)\|^2 ds + \alpha\|\varphi\|^2 \tag{4.15}$$

where T is a positive constant to be determined later.

First Let us show $\Lambda_0 \in C_b^2(H)$. let

$$\Psi_0(\varphi) = \int_0^T E\|u_0^{\varphi}(t)\|^2 dt.$$

Using (2.1),

$$\Psi_0(\varphi) = \int_0^T (ce^{-\beta t}\|\varphi\|^2 + M)dt \leq \frac{c}{\beta}\|\varphi\|^2 + MT. \tag{4.16}$$

If $\|\varphi\|^2 = 1$, then

$$\Psi_0(\varphi) \leq \frac{c}{\beta} + MT.$$

Since $u_0^{\varphi}(t)$ is linear in x, for any positive constant k, we have

$$u_0^{k\varphi}(t) = ku_0^{\varphi}(t)$$

hence,

$$\Psi_0(k\varphi) = k^2\Psi_0(\varphi)$$

therefore, for any $x \in H$

$$\Psi_0(\varphi) = \|\varphi\|^2\Psi_0(\frac{x}{\|\varphi\|}) \leq (\frac{c}{\beta} + MT)\|\varphi\|^2.$$

Let $c' = \frac{c}{\beta} + MT$ then $\Psi_0(\varphi) \leq c'\|\varphi\|^2$ for $\forall\varphi \in H$.

Let

$$\mathcal{T}(\varphi,\psi) = \int_0^T E < u_0^{\varphi}(s), u_0^{\psi}(s) > ds \text{ for } \varphi,\psi \in H$$

then $\mathcal{T}$ is a bilinear form on H, and by using Schwartz inequality, we get

$$\begin{aligned}
|\mathcal{T}(\varphi,\psi)| &= |\int_0^T E < u_0^\varphi(s), u_0^\psi(s) > ds| \\
&\leq \int_0^T (E\|u_0^\varphi(s)\|^2)^{\frac{1}{2}}(E\|u_0^\psi(s)\|^2)^{\frac{1}{2}} ds \\
&\leq (\int_0^T E\|u_0^\varphi(s)\|^2 ds)^{\frac{1}{2}}(\int_0^T E\|u_0^\psi(s)\|^2 ds)^{\frac{1}{2}} \\
&= \Psi_0(\varphi)^{\frac{1}{2}}\Psi_0(\psi)^{\frac{1}{2}} \\
&\leq c'\|\varphi\| \cdot \|\psi\|.
\end{aligned}$$

Hence there exists a continuous linear operator $C \in L(H,H)$, such that

$$\mathcal{T}(\varphi,\psi) = (C\varphi,\psi). \tag{4.17}$$

and

$$\|C\| = \sup_{\|\varphi\|=1,\|\psi\|=1} |(C\varphi,\psi)| \leq c'.$$

Since $\Psi_0(\varphi) = \mathcal{T}(\varphi,\varphi) = (C\varphi,\varphi)$, so

$$\Psi_0'(\varphi) = 2C\varphi \text{ and } \Psi''(\varphi) = 2C$$

Hence, $\Psi_0 \in C_b^2(H)$ and $\Lambda(\varphi) \in C_b^2(H)$.

By (4.16) and the fact that $\Lambda_0(\varphi) \geq \alpha\|\varphi\|^2$, (4.13) satisfies.

By the same reason as we get (4.9), we have

$$E\Psi_0(u_0^\varphi(r)) = \int_0^T E\|u_0^\varphi(r+s)\|^2 ds = \int_r^{T+r} E\|u_0^\varphi(s)\|^2 ds.$$

By the continuity of $t \to E\|u_0^\varphi(t)\|^2$, we get:

$$\begin{aligned}
\mathcal{L}_0\Psi_0(\varphi) &= \frac{d}{dr}(E\Psi_0(u_0^\varphi(r)))|_{r=0} \\
&= \lim_{r\to 0}\frac{E\Psi_0(u_0^\varphi(r)) - E\Psi_0(\varphi)}{r} \\
&= \lim_{r\to 0}(-\frac{1}{r}\int_0^r E\|u_0^\varphi(s)\|^2 ds + \frac{1}{r}\int_T^{r+T} E\|u_0^\varphi(s)\|^2 ds) \\
&= -\|\varphi\|^2 + E\|u_0^\varphi(T)\|^2 \\
&\leq -\|\varphi\|^2 + ce^{-\beta T}\|\varphi\|^2 + M \\
&\leq (-1 + ce^{-\beta T})\|\varphi\|^2 + M
\end{aligned}$$

Thus, using (4.7),

$$\begin{aligned}\mathcal{L}_0\Lambda_0(\varphi) &= \mathcal{L}_0\Psi_0(\varphi) + \alpha\mathcal{L}_0\|\varphi\|^2 \\ &\leq (-1 + ce^{-\beta T})\|\varphi\|^2 + \alpha(2\lambda + d^2 tr(Q))\|\varphi\|^2 + M. \qquad (4.18)\end{aligned}$$

Therefore, if $T > \frac{\ln c}{\beta}$, then we can choose α small enough such that $\Lambda_0(\varphi)$ satisfies(4.14) with $\mathcal{L}$ replaced by $\mathcal{L}_0$.

Consider the solution of the nonlinear equation (1.5). If it is exponentially ultimately bounded in m. s. s., using ideas similar to the stability problem, we can still construct the Lyapunov function as $\Lambda(\varphi) = \int_0^T E\|u^\varphi(s)\|^2 ds + \alpha\|\varphi\|^2$, but it may not be in $C_b^2(H)$. But if it is in $C_b^2(H)$, follow the proof of this theorem and Theorem 2.3, we can show it satisfies (4.13) and (4.14). Therefore, we have the following theorem.

Theorem 4.3.3 *If the solution $u^\varphi(t)$ of (1.5) is exponentially ultimately bounded in the m. s. s., and $\Psi(\varphi) = \int_0^T E\|u^\varphi(t)\|^2 dt$ is in $C_b^2(H)$ for some big $T > 0$, then there exits a Lyapunov function for $u^\varphi(t)$ satisfies (4.13) and (4.14).*

Now we use the first order approximation to study the properties of exponentially ultimate boundedness in the m. s. s. of the solution of the nonlinear equation based on the same property of the solution of the linear equation. As in Theorem 2.5, we have the following theorem.

Theorem 4.3.4 *Suppose the solution $u_0^\varphi(t)$ of the equation (4.4) is exponentially ultimately bounded in the m. s. s., and it satisfies (2.1). Then the solution $u^\varphi(t)$ of (1.5) is exponentially ultimately bounded in the m. s. s. if*

$$2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*) < \omega\|\varphi\|^2 + M_1 \qquad (4.19)$$

for any constant M_1 and

$$\omega < \max_{s > \frac{\ln c}{\beta}} \frac{1 - ce^{-\beta s}}{\frac{c}{\beta} + Ms}. \qquad (4.20)$$

Proof: Let $\Lambda_0(\varphi)$ be the Lyapunov function as defined in (4.15) with $T > \frac{\ln c}{\beta}$ such that (4.20) gets its maximum at T. We just need to show that $\Lambda_0(\varphi)$ satisfies (4.14).

Since $\Lambda_0(\varphi) = (C\varphi, \varphi + \alpha\|\varphi\|^2$ for some $C \in L(H,H)$ with $\|C\| \leq \frac{c}{\beta} + MT$ and α very small. Following (4.11) we have

$$\begin{aligned} & \mathcal{L}\Lambda_0(\varphi) - \mathcal{L}_0\Lambda_0(\varphi) \\ \leq \;& (\|C\| + \alpha)(2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)Q(B(\varphi))^* - B_0\varphi Q(B_0\varphi)^*)) \\ \leq \;& (\frac{c}{\beta} + MT + \alpha)(\omega\|\varphi\|^2 + M_1). \end{aligned}$$

Using (4.18),

$$\begin{aligned} & \mathcal{L}\Lambda_0(\varphi) \\ \leq \;& (-1 + ce^{-\beta T})\|\varphi\|^2 + \alpha(2\lambda + d^2 tr(Q))\|\varphi\|^2 + M \\ & +(\frac{c}{\beta} + MT + \alpha)(\omega\|\varphi\|^2 + M_1) \\ = \;& (-1 + ce^{-\beta T} + \omega(\frac{c}{\beta} + MT))\|\varphi\|^2 \\ & +\alpha(2\lambda + d^2 tr(Q) + \omega)\|\varphi\|^2 + M + (\frac{c}{\beta} + MT + \alpha)M_1 \end{aligned}$$

Since ω satisfies (4.20), $-1 + ce^{-\beta T} + \omega(\frac{c}{\beta} + MT) < 0$, we can choose α small enough such that (4.14) is satisfied.

Corollary 4.3.1 *Suppose the solution $u_0^\varphi(t)$ of equation (4.4) is exponentially ultimately bounded in the m. s. s.. If as $\|\varphi\| \to \infty$*

$$\|F(\varphi)\| = o(\|\varphi\|) \text{ and } \tau(B(\varphi)QB^*(\varphi) - B_0\varphi Q(B_0\varphi)^*) = o(\|\varphi\|^2)$$

then the solution $u^\varphi(t)$ of (1.5) is exponentially ultimately bounded in the m. s. s

Proof: Since as $\|\varphi\| \to \infty$

$$\|F(\varphi)\| = o(\|\varphi\|) \text{ and } \tau(B(\varphi)QB^*(\varphi) - B_0\varphi Q(B_0\varphi)^*) = o(\|\varphi\|^2)$$

For any fixed ω satisfying (4.20), there exists an $K > 0$, such that

$$2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)QB^*(\varphi) - B_0\varphi Q(B_0\varphi)^*) \leq \omega\|\varphi\|^2$$

for all $\|\varphi\| \geq K$.

But for $\|\varphi\| \leq K$, by the Lipschitz condition,

$$\begin{aligned} & 2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)QB^*(\varphi) - B_0\varphi Q(B_0\varphi)^*) \\ \leq \; & \|\varphi\|^2 + \|F(\varphi)\|^2 + \tau(B(\varphi)QB^*(\varphi)) + \tau(B_0\varphi Q(B_0\varphi)^*) \\ \leq \; & \|\varphi\|^2 + \|F(\varphi)\|^2 + (\|B(\varphi)\|^2 + \|B_0(\varphi)\|^2)tr(Q) \\ \leq \; & \|\varphi\|^2 + 2d^2(1 + \|\varphi\|^2)(1 + 2tr(Q)) \\ \leq \; & K^2 + 2d^2(1 + K^2)(1 + 2tr(Q)). \end{aligned}$$

Therefore,

$$2\|\varphi\|\|F(\varphi)\| + \tau(B(\varphi)QB^*(\varphi) - B_0\varphi Q(B_0\varphi)*) \leq \omega\|\varphi\|^2 + R^2 + 2d^2(1 + K^2)(1 + 2tr(Q)).$$

The assertion follows from Theorem 3.4.

Chapter 5

Appendix

Computation of $E|X(t)|^2$ of the following stochastic differential equation:

$$dX(t) = (aX(t) + b)dt + cX(t)dW(t)$$

where a, b and c are complex constants, $W(t)$ is a standard one-dimensional real Brownian Motion.

If $b = 0$, then $X(t)$ can be computed explicitly:

$$X(t) = X(0)e^{at-\frac{1}{2}c^2t+cW(t)}.$$

So we suppose $b \neq 0$. Since

$$\begin{aligned} dX(t) &= (aX(t) + b)dt + cX(t)dW(t) \\ d\overline{X(t)} &= (\overline{a}\overline{X(t)} + \overline{b})dt + \overline{c}\overline{X(t)}dW(t) \end{aligned}$$

Where $\bar{\cdot}$ means the complex conjugate. By Ito's formula, we get

$$\begin{aligned} |X(t)|^2 &= X(t)\overline{X(t)} \\ &= X(0)\overline{X(0)} + \int_0^t X(s)d\overline{X(s)} + \int_0^t \overline{X(s)}dX(s) + \int_0^t d\langle X, \overline{X}\rangle_s \\ &= |X(0)|^2 + (a + \overline{a} + c\overline{c})\int_0^t |X(s)|^2 ds + \overline{b}\int_0^t X(s)ds + b\int_0^t \overline{X(s)}ds \\ &\quad (c + \overline{c})\int_0^t |X(s)|^2 dW(s) \end{aligned}$$

Let $\varphi(t) = E|X(t)|^2$, take expectation to the above equation:

$$\varphi(t) = \varphi(0) + (a + \overline{a} + c\overline{c}) \int_0^t \varphi(s)ds + \overline{b} \int_0^t EX(s)ds + b \int_0^t E\overline{X(s)}ds$$

Lemma: If $\frac{dy(t)}{dt} = ay(t) + g(t)$, then

$$y(t) = y(0)e^{at} + \int_0^t e^{a(t-s)}g(s)ds$$

Proof: Proof is elementary.

Using this lemma, we have

$$\begin{aligned} \varphi(t) &= \varphi(0)e^{(a+\overline{a}+c\overline{c})t} + \int_0^t e^{(a+\overline{a}+c\overline{c})(t-s)}(\overline{b}EX(s) + bE\overline{X(s)})ds \\ &= \varphi(0)e^{(a+\overline{a}+c\overline{c})t} + 2Re\int_0^t e^{(a+\overline{a}+c\overline{c})(t-s)}\overline{b}EX(s)ds \end{aligned}$$

Now we compute $EX(t)$, since

$$dX(t) = (aX(t) + b)dt + cX(t)dW(t)$$

$$X(t) = X(0) + \int_0^t (aX(s) + b)dt + c\int_0^t X(s)dW(s)$$

hence

$$EX(t) = EX(0) + a\int_0^t EX(s)dt + bt$$

Using the above lemma we get

$$EX(t) = -\frac{b}{a} + (EX(0) + \frac{b}{a})e^{at}$$

Thus

$$\begin{aligned} \varphi(t) &= \varphi(0)e^{(a+\overline{a}+c\overline{c})t} + 2Re\{-\frac{b\overline{b}}{a(a+\overline{a}+c\overline{c})}(e^{(a+\overline{a}+c\overline{c})t} - 1) \\ &\quad + \frac{\overline{b}(EX(0) + \frac{b}{a})}{\overline{a} + c\overline{c}}(e^{(a+\overline{a}+c\overline{c})t} - e^{at})\} \end{aligned}$$

therefore

$$\begin{aligned} E|X(t)|^2 &= (E|X(0)|^2 + 2Re\frac{b\overline{b} + \overline{b}EX(0)(a+\overline{a}+c\overline{c})}{(a+\overline{a}+c\overline{c})(\overline{a}+c\overline{c})})e^{(a+\overline{a}+c\overline{c})t} \\ &\quad -2Re(\frac{\overline{b}(aEX(0)+b)}{a(\overline{a}+c\overline{c})}e^{at}) + 2Re\frac{b\overline{b}}{a(a+\overline{a}+c\overline{c})} \end{aligned}$$

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