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ABSTRACT

INVESTIGATION OF THEORIES FOR LAMINATED COMPOSITE PLATES

By

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High-order Shear Deformation Theories give good results for in-plane stresses but poor results for interlaminar stresses. However, Layerwise Theories give excellent results for both global and local distributions of displacement and stress (both in-plane and out-of-plane). A compromising theory, the so-called Generalized Zigzag Theory, is presented. It has two layer-dependent variables in the zeroth- and first-order terms. Due to its success in laminate analysis, the feasibility of assigning the two layer-dependent variables in the second- and third-order terms is examined, resulting in the Quasi-layerwise Theories. Unfortunately, a physical absurdity – coordinate dependency, takes place. It then requires a technique, the so-called Global-Local Superposition Technique, to formulate the laminate theory to be coordinate-independent for the numerical advantage. The recursive expressions presented in this study, though somewhat tedious, are necessary to achieve the numerical advantage. By examining the results based on the Superposition Theories, it is concluded that the completeness of the terms is meant two fold: not only can no low-order term be skipped, but more high-order terms are preferred.

The objective of completeness seems to conflict with the fundamental of two

continuity conditions in each coordinate direction. In order to satisfy both aspects, a special technique called the Hypothesis for Double Superposition is proposed. Several three-term theories, the so-called Double Superposition Theories, are examined. They give excellent values for in-plane displacement, in-plane stress, and transverse shear stress. However, because w is considered as constant in the examples, both transverse displacement and transverse normal stress are not as good as the remaining components.

Among all the theories examined in this thesis, it seems that the Generalized Zigzag Theory, with up to seventh-order terms, and the third-order Double Superposition Theories give the best agreement with Pagano's solution in all ranges of layer number for both symmetric and unsymmetrical laminates. Although they both are layer-number independent theories, the former has seven degrees-of-freedom while the latter has only three, provided w is considered to be constant through the laminate thickness. As a consequence, the Double Superposition Theories are concluded as the best selection for laminate analysis in this thesis.

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CHAPTER 1

INTRODUCTION

Because of their high stiffness and high strength with low density, fiber-reinforced polymer matrix composite laminates have been widely used for high-performance structures. When compared to conventional metals, the major characteristics of laminated composite materials include the orthotropy in the laminate plane, the low modulus in transverse shear, and the lamination through the thickness. In addition, it should be noted that most laminated composite materials are originally used in thin structures. As the technology of composite advances, laminated composites are also used for thick and moderately thick structures.

In the early days of study, the Classical Plate Theory (CPT) used for isotropic structures was directly applied to composite structures. In fact, in formulating the isotropic and orthotropic plates, the only difference lies in the constitutive equations. Modifying the CPT for orthotropic material applications does not create any further inaccuracy in the composite analysis. However, when applying the CPT to materials with low modulus in transverse shear and to thick plates, the results will be unsatisfactory because the transverse shear effect which is not considered in the CPT should be considered in both cases. As a consequence, the Shear Deformation Theories, being derived from the Classical Plate Theory, have been presented to improve the deficiency.

In composite structure analysis, both a stress approach and a displacement approach have been utilized by many investigators. An example of the stress approach is the hybrid-stress finite element method presented by Mau, Tong, and Pian [1]. By assuming a stress field which satisfies the equilibrium equations, and a simple displacement interpolation, the governing equations and associated boundary conditions can be obtained from a mixed variational principle [2]. Since the continuity conditions on the laminate interfaces are also satisfied, the results from the hybrid-stress finite element methods are in close agreement with the elasticity solution presented by Pagano [3]. However, the major disadvantage of the stress approach is the large number of degrees-of-freedom involved in the numerical solution.

Originating from Classical Plate Theory, the displacement approach, which includes the High-order Shear Deformation Theories (HSDT) [4], is much more efficient in numerical analysis than the stress approach. It is also recognized as being variationally consistent and is thus much more popular in composite structure analysis. In the displacement approach, the assumption of continuous functions is correct for out-of-plane displacement components. However, this assumption is not accurate for in-plane displacement components because the in-plane strains are discontinuous across the laminate interfaces. In fact, owing to the abrupt change of material properties across the laminate interfaces, the in-plane displacement components should be of zigzag distributions through the laminate thickness.

The inaccurate in-plane displacement assumptions can result in erroneous transverse stresses, especially on the laminate interfaces. With the continuous functions, their derivatives are also continuous. As a consequence, single-valued strains on laminate interfaces are created. Since different composite layers have different material properties, the transverse stresses obtained from the constitutive equations will be mistakenly double-valued. Most investigators tend to ignore and bypass this error. They utilize the equilibrium equations to 'recover' the transverse stresses [5]. Although the results from this post-processing technique seem to be promising, the technique itself is very controversial.

Another category of displacement based techniques is the so-called Layerwise Theories (LT) [6, 7]. They are developed from the High-order Shear Deformation Theories. Instead of modeling the whole laminate by the HSDT, each composite layer is described by individual displacement components. The displacement field of the whole composite laminate is nothing but the assembly of the displacement components of individual layers. The observation of the important roles of individual layers matches with the aforementioned recognition of the composite characteristic that the composite laminates are made of layers through their thicknesses. In addition, it should be noted that in assembling the composite layers, the continuity conditions of both displacements and transverse stresses are imposed on the laminate interfaces. As a consequence, the Layerwise Theories are recognized as the most accurate approaches toward laminated composite analysis.

Taking the properties of individual layers into consideration, the Layerwise Theories can satisfy all the major characteristics of laminated composite materials. Accordingly, it is obvious that they are superior to High-order Shear Deformation Theories. However, just because the LT are based on the assembly of individual layers, the total number of degrees-of-freedom is dependent on the number of layers. As the layer number increases, the computational requirement becomes very demanding. They then have the same disadvantages as those of hybrid-stress finite element methods. This major disadvantage may explain why the LT are rarely used for composite analysis, although they are the most accurate displacement approaches available in the literature.

At the same time that the Layerwise Theories were being developed, another major effort in the development of the composite laminate theories was being developed by attempting to remove the deficiencies in the High-order Shear Deformation Theories. These deficiencies were that the in-plane displacements and their derivatives are smoothly continuous through the laminate thickness instead of being zigzag distributions. The primary approach of this group of studies, the Zigzag Theories [8, 9], is based on assumed displacement functions with limited layer-dependent variables for individual layers. Many special functions aimed at presenting the zigzag distributions for in-plane displacements are presented and they all seem to be very promising for in-plane displacement predictions. However, since all the presented functions are specially selected and most of them do not satisfy the stress continuity conditions on the laminate interfaces, the predictions of the transverse stresses from the Zigzag Theories are no better than those from the HSDT. The recovering technique, as mentioned before, seems to be the only technique to bypass the errors.

By carefully examining the Zigzag Theories available in the literature, it can be found that they are just special cases of the Layerwise Theories. In addition, by comparing the advantages and disadvantages of the Layerwise Theories and the High-order Shear Deformation Theories, it can be concluded that a compromising theory may be an optimum choice for laminated composite analysis. In fact, a carefully selected Layerwise Theory which resembles the idea of the Zigzag Theories may be a candidate to satisfy the requirements of both computational efficiency and numerical accuracy. Thus, the objectives of this thesis are listed below:

1. Since the Classical Plate Theory is a special case of the High-order Shear Deformation Theories, and the Zigzag Theories and the High-order Shear Deformation Theories are special cases of the Layerwise Theories, the first objective is to propose a theory to unify all the laminate theories based on displacement assumption.

2. With the proposed theory, it is possible to judge the computational efficiency and numerical accuracy of various High-order Shear Deformation Theories and Zigzag Theories in a more systematic, consistent way.

3. Based on the proposed theory, it is possible to develop an optimum theory or various optimum theories for studying different laminated composite structures. Hence, a comprehensive study on all possible theories derived from the proposed theory is another goal.

With the above objectives, this thesis is divided into the following chapters. Chapter 2 starts with the Classical Plate Theory and the High-order Shear Deformation Theories. The evolution of HSDT is carefully documented. Eventually a generalized HSDT is presented. This generalized theory can be extended to theories of a much higher order, namely the Higher-order Shear Deformation Theories (HrSDT). Evaluations of HrSDT with various orders are of primary concern. In Chapter 3, a Generalized Zigzag Theory is presented. It is to demonstrate that the generalized theory envelops all the High-order Shear Deformation Theories and Zigzag Theories available in the literature and allows a systematic investigation for other potential theories. The Generalized Zigzag Theory is then extended to various quasi-layer-dependent theories. The Quasi-layerwise Theories are examined in Chapter 4. Comparisons among the various types of Quasi-layerwise

Theories are also made. However, unfortunately, the Quasi-layerwise Theories are found to be coordinate dependent. In order to remove this complexity, a global-local superposition technique is presented in Chapter 5. With this technique, many more theories, namely the Superposition Theories, are made available. They are later extended to cover more layer-dependent terms based on a hypothesis for double superposition. This hypothesis is proved to be very useful. As a result, many more composite theories are presented in Chapter 6. The selection of an efficient and accurate theory is also discussed. The last chapter of this thesis gives the conclusions of the investigation of composite laminate theories based on the criteria of computational efficiency and numerical accuracy. Recommendations for future studies are also presented in Chapter 7.

CHAPTER 2

SHEAR DEFORMATION THEORIES

2.1 Introduction

In the early days of study, the technique used for studying conventional plates was lent to analyze laminated composite plates. The Classical Plate Theory (CPT) was naturally the first tool for the investigations. As can be expressed by the following equations, the CPT is of a displacement approach:

$$
\begin{aligned}
u(x, y, z) &= u_0(x, y) - \frac{\partial w}{\partial x} z \\
v(x, y, z) &= v_0(x, y) - \frac{\partial w}{\partial y} z \\
w(x, y, z) &= w_0(x, y)
\end{aligned}
\tag{2.1}
$$

where u and v are in-plane displacements in the x and y directions, respectively, while w is the displacement in the plate thickness, the z, direction. The in-plane displacement components u_0 and v_0 can be viewed as translational components at the laminate midplane while $-\frac{\partial w}{\partial x}$ and $-\frac{\partial w}{\partial y}$ as rotational angles. In addition, considering the small deformation through the laminate thickness, w is assumed to be independent of the z-coordinate.

Because of its simplicity, the CPT is widely used in composite structure design and analysis. Reasonable results of displacements and in-plane stresses can be obtained for plates with aspect ratios (in-plane dimensions versus thickness) larger than twenty.

However, it should be recognized that the CPT is based on Kirchhoff's hypothesis which assumes that the lines perpendicular to the midplane before loading remain perpendicular after loading. The transverse shear effect is not considered in the analysis although it is significant to thick plate analysis and should not be neglected in materials with low shear moduli.

Reissner [10] and Mindlin [11] recognized the important role of transverse shear effect in plate bending. They replaced the rotational angles with more general variables to account for the shear effect, i.e.:

$$\begin{aligned} u(x, y, z) &= u_0(x, y) + \psi_x(x, y)\, z \\ v(x, y, z) &= v_0(x, y) + \psi_y(x, y)\, z \\ w(x, y, z) &= w_0(x, y) \end{aligned} \tag{2.2}$$

By closely examining Equations (2.2), it can be concluded that the in-plane displacements are expressed as first-order equations of z. Based on their pioneering works, high-order terms of z were gradually added to u, v, and even w. Many theories of this type were proposed in the past three decades and were called High-order Shear Deformation Theories (HSDT).

2.2 High-order Shear Deformation Theories (HSDT)

HSDT are also based on assumed displacement fields. Lo, Christensen, and Wu [5] unified the notations used in expressing various HSDT. They documented the development of HSDT according to the order of z of the polynomial equations for in-plane displacements. In addition to those mentioned in Lo, et. al. many more HSDT can be found in the review articles by Noor and Burton [12], and Kapania and Raciti [13]. As a summary, Lo, et. al. also presented a third-order shear deformation theory for laminated composite plate analysis, i.e.

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + \psi_x(x,y)\,z + \zeta_x(x,y)\,z^2 + \phi_x(x,y)\,z^3 \\
v(x,y,z) &= v_0(x,y) + \psi_y(x,y)\,z + \zeta_y(x,y)\,z^2 + \phi_y(x,y)\,z^3 \\
w(x,y,z) &= w_0(x,y) + \psi_z(x,y)\,z + \zeta_z(x,y)\,z^2
\end{aligned}
\tag{2.3}
$$

Apparently, all the HSDT available in the literature can be viewed as special cases of the above third-order theory. In addition, with the use of common notations, it is clear that Equations (2.3) are extended from Equations (2.1) and (2.2).

Although in their numerical studies, w was kept constant through the laminate thickness, Lo, et. al. verified that the third-order theory was superior to lower-order theories in deformation and stress analysis, especially of composite laminates with moderate thickness. With the assumed third-order equations for u and v as expressed in Equations (2.3), the displacements and their first derivatives with respect to the z-coordinate are continuous. However, it should be pointed out that although the in-plane displacements are continuous across the laminate interfaces when the interfaces are perfectly bonded, their first derivatives are discontinuous because of the abrupt change of material properties across the laminate interfaces. In other words, u and v should be of zigzag distributions through the laminate thickness.

In addition to the in-plane displacements and stresses, the transverse stresses are also very important in composite laminate analysis since interlaminar stresses take the primary responsibility for delamination, which is a unique and critical damage mode in laminated composite plates. Consequently, accurate interlaminar stresses are highly required for composite analysis and design. Since the third-order assumption for in-plane displacement gives continuous strain distributions through the laminate thickness, double-valued interlaminar stresses are obtained on the laminate interfaces when single-valued strains on the interfaces are multiplied by different material properties of different layers. To ignore the error, a post-process, the so-called recovering technique, which is based on equilibrium equations instead of constitutive equations was proposed (e.g. Lo, Christensen, and Wu [5]). Although relatively good transverse stresses can be obtained from this method, the approach itself is controversial. It then is desired to identify a technique which can give correct displacements and stresses (both in-plane and transverse) directly from the constitutive equations.

Despite the aforementioned fundamental defects, there is a primary advantage of using HSDT in composite plate analysis. By examining Equations (2.3) it can be concluded that eleven coefficients need to be determined regardless of the total number of layers of the composite laminate. The feature of layer-number independency is critically important to computational efficiency, especially when the layer number of the composite laminate becomes very large. Another way to express the layer-number independency of HSDT is to use the concept of "global sense." Instead of considering the composite layers individually, HSDT present each composite laminate as a whole piece. Since the interlaminar stresses are of "local" information, the reason that they cannot be calculated from global techniques such as HSDT is clear.

2.3 Higher-order Shear Deformation Theories (HrSDT)

As discussed in the previous section, almost all HSDT available in the literature are third-order theories or lower. There is no discussion on why the third-order seems to become the upper limit in laminate theories. Although Reissner [14] presented laminate theories of sixth-order, no numerical result was given to compare his higher-order theories with other lower-order ones. For completeness, it is necessary to further investigate HSDT of fourth-order and higher. Such higher-order theories will be called Higher-order Shear Deformation Theories (HrSDT) in this study.

HrSDT can be obtained by simply extending the third-order HSDT of Equations (2.3) to higher-orders. For convenience and uniformity, new notations are used in expressing the generalized displacement field:

$$
\begin{aligned}
u(x, y, z) &= \sum_{i=0}^{m} u_i(x, y)\, z^i \\
v(x, y, z) &= \sum_{i=0}^{m} v_i(x, y)\, z^i \\
w(x, y, z) &= \sum_{i=0}^{l} w_i(x, y)\, z^i
\end{aligned}
\tag{2.4}
$$

In the above equations, m is the order of in-plane displacement components and is usually used to name the order of the theory since w is frequently kept lower or constant. For m = 1, Equations (2.4) represent Reissner's and Mindlin's first-order theories while for m = 3, the equations are associated with the third-order HSDT. As an example, HrSDT of seventh-order can be written as follows:

$$
\begin{aligned}
u(x, y, z) &= u_0(x, y) + u_1(x, y)\, z + u_2(x, y)\, z^2 + u_3(x, y)\, z^3 + u_4(x, y)\, z^4 + \\
&\qquad u_5(x, y)\, z^5 + u_6(x, y)\, z^6 + u_7(x, y)\, z^7 \\
v(x, y, z) &= v_0(x, y) + v_1(x, y)\, z + v_2(x, y)\, z^2 + v_3(x, y)\, z^3 + v_4(x, y)\, z^4 + \\
&\qquad v_5(x, y)\, z^5 + v_6(x, y)\, z^6 + v_7(x, y)\, z^7 \\
w(x, y, z) &= w_0(x, y)
\end{aligned}
\tag{2.5}
$$

Apparently, as the order of the theory increases, the number of variables also increases, resulting in a better possibility to improve the accuracy of laminate analysis. However, it should also be recognized that the numerical effort will also become quite demanding.

2.4 Influence of High-order w

Because composite laminates are usually very thin, the displacement component in the z-direction is frequently assumed constant through the laminate thickness. In other words, the deformation of thickness is considered negligible when compared to the deformations in the plane. However, high-order terms can be added to w as those in u and v under the following conditions: (1) the thickness of a composite laminate is relatively high, (2) the composite laminate has low transverse shear modulus, and (3) in the case a more accurate prediction is highly desired.

As pointed out by Reissner [10], even-order terms of w play a more important role in thickness deformation than odd-order terms. If the polynomial equation of w is not to exceed the third-order of its in-plane counterparts, w can be assumed as follows:

$$w(x, y, z) = w_o(x, y) + w_1(x, y)z + w_2(x, y)z^2 \tag{2.6}$$

As a consequence, the following third-order HSDT (of u and v) is chosen for investigations of the influence of high-order w on composite laminate performance:

$$\begin{aligned} u(x, y, z) &= u_0(x, y) + u_1(x, y)z + u_2(x, y)z^2 + u_3(x, y)z^3 \\ v(x, y, z) &= v_0(x, y) + v_1(x, y)z + v_2(x, y)z^2 + v_3(x, y)z^3 \\ w(x, y, z) &= w_o(x, y) + w_1(x, y)z + w_2(x, y)z^2 \end{aligned} \tag{2.7}$$

2.5 Verification Technique

In view of the potential small discrepancies among the various theories to be presented, a high-sensitivity technique for verification is desired. Because the nature of the present study is of numerical analysis, an analytical technique for verification is more appropriate than an experimental one. Among the theoretical and numerical techniques available, Pagano's widely used elasticity solution [3] seems to be the most suitable, due not only to the consideration of convenience but also accuracy. In addition, it is recognized that Pagano's solution has been used for verification in almost all composite laminate theories presented in the literature. The comparisons made with Pagano's solution can be used to assess the theories presented in this thesis.

Pagano examined an infinitely long strip of laminated plate simply supported along both longer edges. As shown in Figure 2.1, a sinusoidal load, $q = q_o \sin px$, is exerted on the top surface of the laminate, resulting in cylindrical bending across the width of the laminate. As a consequence, this problem can be reduced to be a plane-strain type. By assuming linear strain-displacement relations and orthotropic constitutive equations, and imposing continuity conditions of displacements and transverse stresses on the laminate interfaces, the elasticity solution of the problem was identified by Pagano in the following forms:

$$
\begin{aligned}
\sigma_x^{(k)} &= \sin px \sum_{j=1}^{4} A_{jk} m_{jk}^2 e^{m_{jk} z_k} \\
\sigma_z^{(k)} &= -p^2 \sin px \sum_{j=1}^{4} A_{jk} e^{m_{jk} z_k} \\
\tau_{xz}^{(k)} &= -p \cos px \sum_{j=1}^{4} A_{jk} m_{jk} e^{m_{jk} z_k} \\
u^{(k)} &= -\frac{\cos px}{p} \sum_{j=1}^{4} A_{jk} \left(R_{13}^{(k)} p^2 - R_{11}^{(k)} m_{jk}^2 \right) e^{m_{jk} z_k} \\
w^{(k)} &= \sin px \sum_{j=1}^{4} A_{jk} \left(R_{13}^{(k)} m_{jk} - \frac{R_{33}^{(k)}}{m_{jk}} p^2 \right) e^{m_{jk} z_k}
\end{aligned}
\tag{2.8}
$$

where A_{jk}, m_{jk}, R_{13}, R_{11}, and R_{33} are constants associated with material and geometric properties of the k^{th} layer. Details of A_{jk} and m_{jk} can be found in Pagano's solution [3].

Pagano's study is valuable to all displacement-based theories since close-form solutions to all laminate theories can be obtained by assuming $u_i = U_i \cos px$ and $w_i = W_i \sin px$ $(i = 0, 1, 2, \dots)$. Thus, the simply-supported boundary conditions can be automatically satisfied while the coefficients U_i and W_i need to be determined from variational analysis.

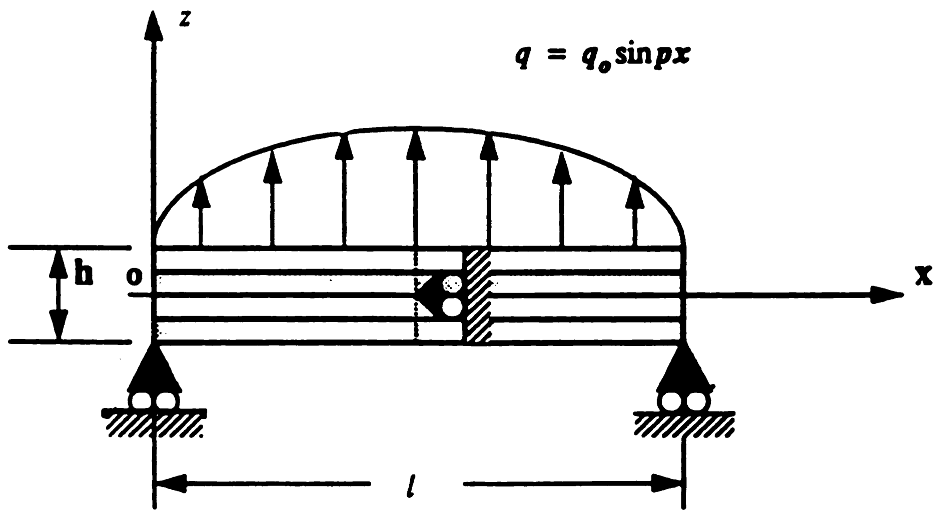

Figure 2.1 - Plane-strain problem: a laminated plate under cylindrical bending.

2.6 Numerical Results and Discussion

In numerical analysis, the same material properties examined by Pagano as listed below are also investigated in this thesis for verification of theories. Also presented in the following equations are the definitions of normalized in-plane stress, $\bar{\sigma}_x$, at midspan, normalized transverse shear stress, $\bar{\tau}_{xz}$, on the edge, normalized transverse normal stress, $\bar{\sigma}_z$, at midspan, normalized in-plane displacement, $\bar{u}$, on the edge, and normalized transverse normal displacement, $\bar{w}$, at midspan:

$$
\begin{gathered}
E_{TT} = 1 \times 10^6 \ psi, \quad E_{LL} = 25 \times 10^6 psi, \quad G_{LT} = 0.5 \times 10^6 \ psi \\
G_{TT} = 0.2 \times 10^6 \ psi, \quad \nu_{LT} = \nu_{TT} = 0.25 \\
\bar{\sigma}_x = \frac{\sigma_x\left(\frac{l}{2}, z\right)}{q_o}, \quad \bar{\sigma}_z = \frac{\sigma_z\left(\frac{l}{2}, z\right)}{q_o}, \quad \bar{\tau}_{xz} = \frac{\tau_{xz}(0, z)}{q_o} \\
\bar{u} = \frac{E_{TT} \times u(0, z)}{hq_o}, \quad \bar{w} = \frac{100 \times E_{TT} \times h^3 \times w\left(\frac{l}{2}, z\right)}{q_o \times l^4}, \quad \bar{z} = \frac{z}{h}
\end{gathered}
\tag{2.9}
$$

Both a two-layer laminate with a stacking sequence of [0/90] and a three-layer laminate of [0/90/0] are examined due to the availability of close-form solutions and their popularity in assessing various laminate theories. The aspect ratio $l/h = 4$ (length to thickness ratio) is chosen because of its ability to show the feasibility of a laminate theory in the analysis of both thin and thick composite laminates.

A. [0/90/0] Laminate

Results of the normalized stresses and displacement components of the [0/90/0] laminate are shown from Figures 2.2 to 2.6. It is found in Figure 2.2 that the in-plane stress $\bar{\sigma}_x$ is improved as the order of the theory increases. As can be seen from the diagram, the results from the third-order and the fourth-order are identical. Similarly, those from the fifth-order are the same as those from the sixth-order. These results may indicate the importance of the odd-order terms in this symmetric laminate.

As can be seen from Figure 2.3, the correct values of $\bar{\tau}_{xz}$ are not obtained. It is believed that this discrepancy is due to the fact that the continuity conditions of transverse shear stresses are not satisfied on the laminate interfaces.

Figure 2.4 shows the transverse normal stress, $\bar{\sigma}_z$. Although no continuity condition is imposed on the laminate interfaces, the discontinuity of $\bar{\sigma}_z$ at the interfaces seems to be very small. In addition, because no boundary condition of transverse normal stress is enforced, the result on the top surface is not equal to negative one and that on the bottom surface is not equal to zero. Moreover, it should be pointed out that although the results in Figure 2.4 seem to be different from the exact solutions, they are small quantities when compared with the dominant component, $\bar{\sigma}_x$.

Another important observation is focused on the in-plane displacement $\bar{u}$ (Figure 2.5). Although the fifth-order and above have significant effect on the transverse shear, along with other lower-order theories, they fail to give distinct kinks on the laminate interfaces. It is believed that the discrepancy cannot be improved simply by increasing the order of the theory.

The prediction of transverse normal displacement, $\bar{w}$, is shown in Figure 2.6. The results reveal that the prediction is improved as the order of the theory increases.

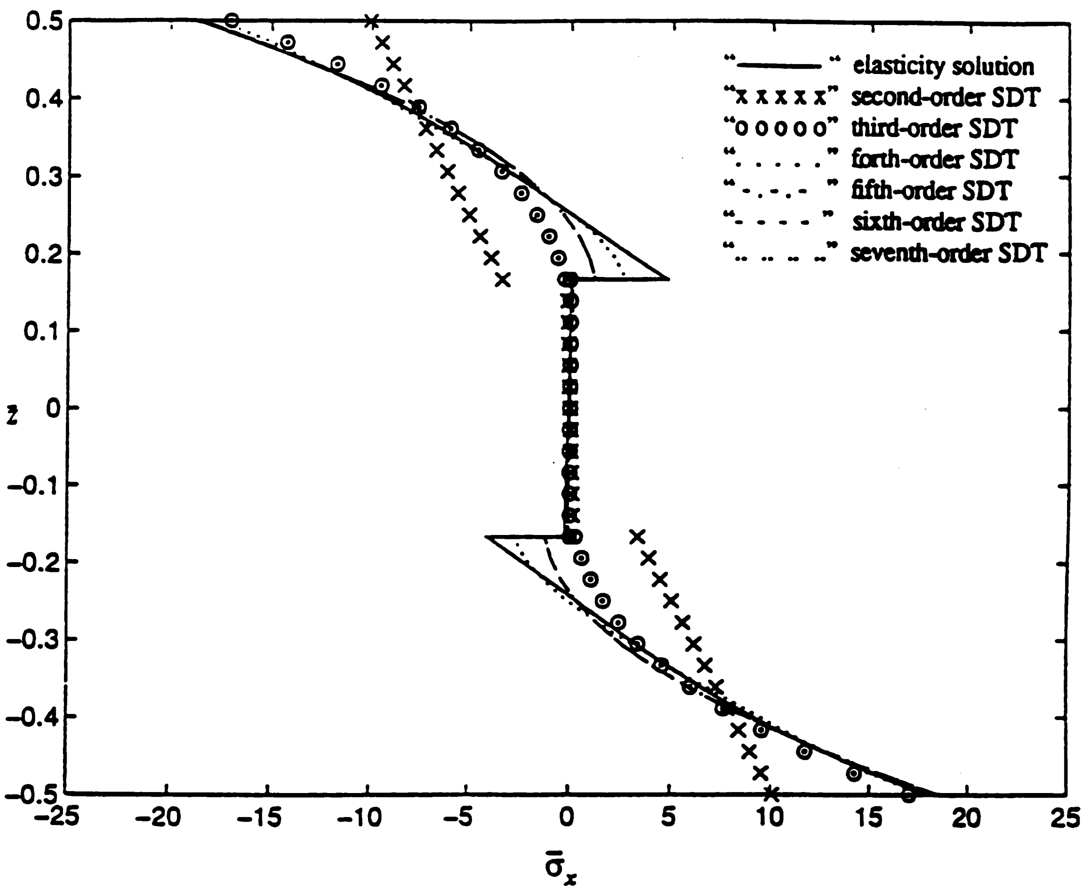

Figure 2.2 - Comparison of σ_x from Shear Deformation Theories of various orders for a [0/90/0] laminate.

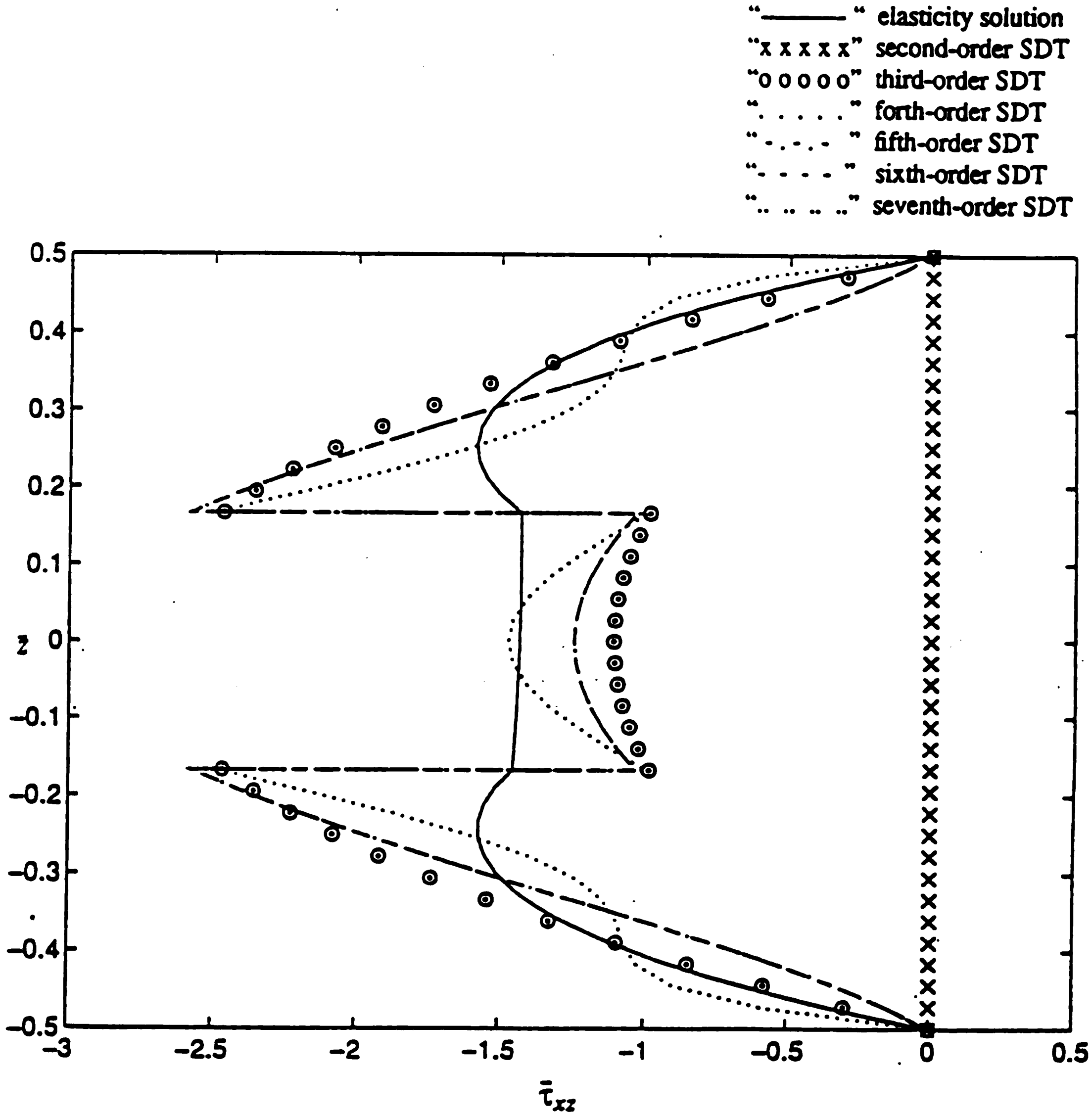

Figure 2.3 - Comparison of τ_{xz} from Shear Deformation Theories of various orders for a [0/90/0] laminate.

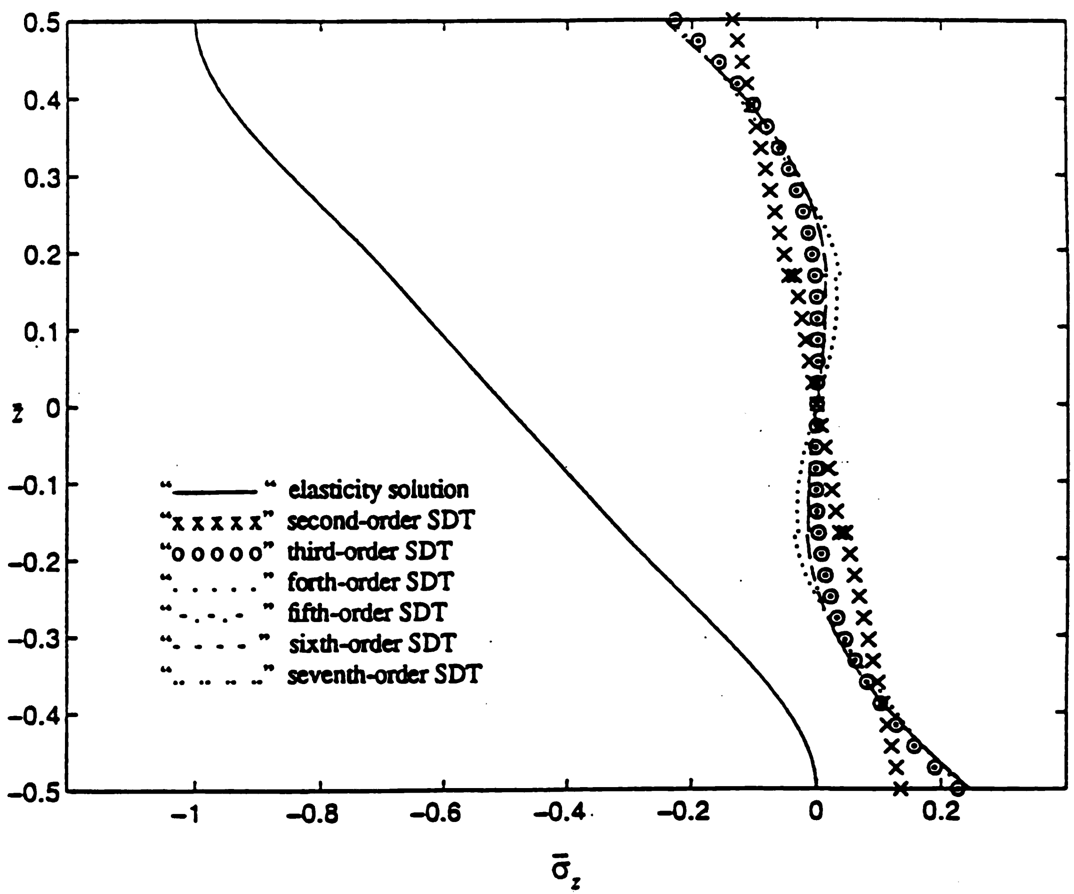

Figure 2.4 - Comparison of $\bar{\sigma}_z$ from Shear Deformation Theories of various orders for a [0/90/0] laminate.

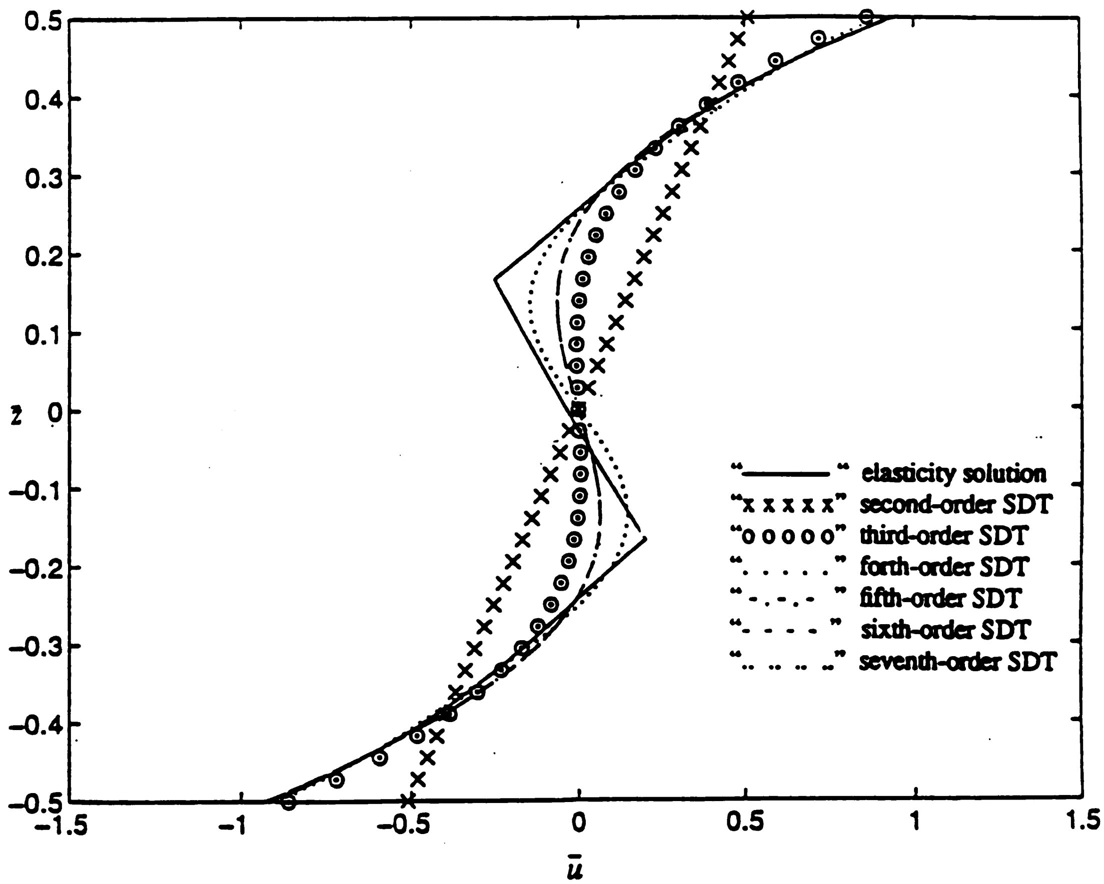

Figure 2.5 - Comparison of $\bar{u}$ from Shear Deformation Theories of various orders for a [0/90/0] laminate.

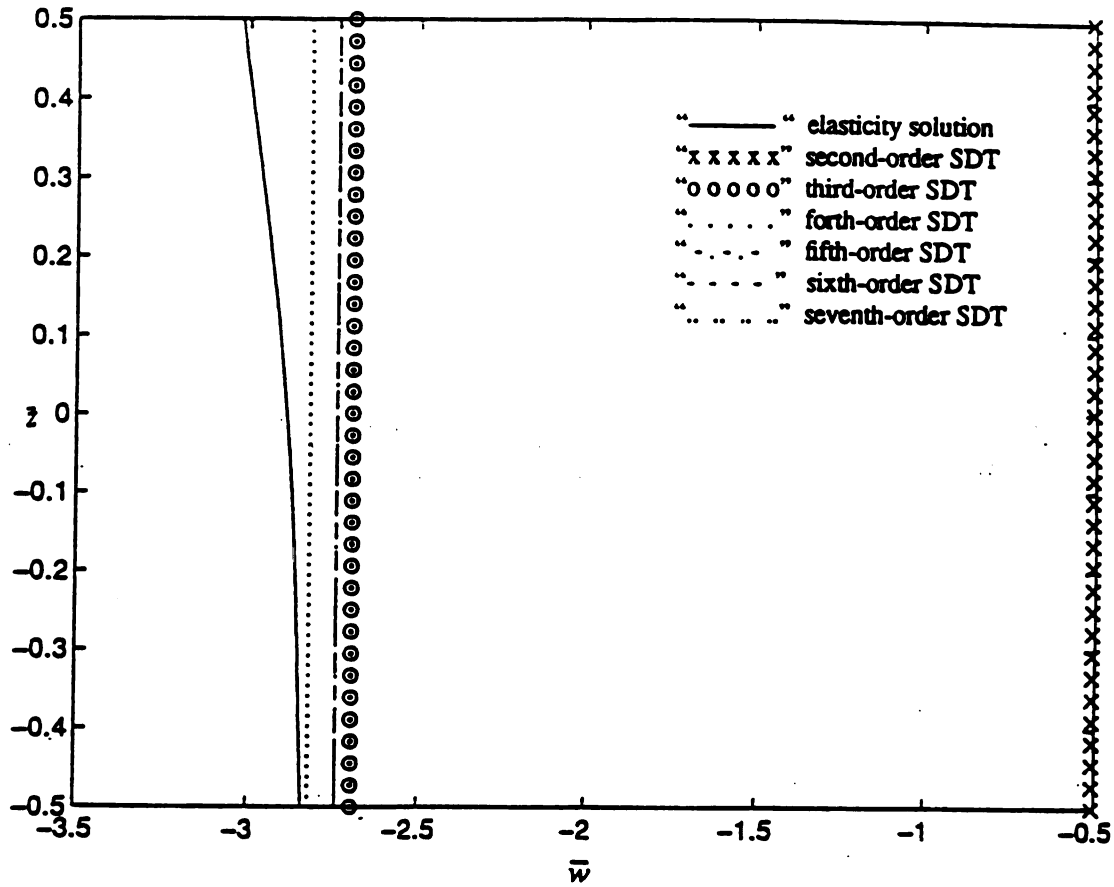

Figure 2.6 - Comparison of $\bar{w}$ from Shear Deformation Theories of various orders for a [0/90/0] laminate.

B. [0/90] Laminate

The significance of the results from the investigation of a [0/90] laminate is essentially the same as that mentioned in the study of the [0/90/0] laminate. That is, the theories of the fourth-order and above have an important effect on $\bar{\sigma}_x$, $\bar{u}$, and $\bar{w}$. The results for $\bar{\sigma}_z$ are fair since they are small when compared to $\bar{\sigma}_x$. Only $\bar{\tau}_{xz}$ is presented in Figure 2.7. It is found that even-order and odd-order terms both play important roles in the [0/90] laminate while only odd-order terms are critical to the [0/90/0] laminate.

C. Influence of High-order w

In order to identify the influence of high-order w on composite analysis, comparisons are made between a third-order HSDT with constant w and with second-order w. The results for $\bar{\sigma}_x$, $\bar{\tau}_{xz}$, $\bar{\sigma}_z$, $\bar{u}$ and $\bar{w}$ are shown in Figures 2.8, 2.9, 2.10, 2.11, and 2.12, respectively.

As shown in Figures 2.8, 2.9, 2.11, and 2.12, the high-order terms of w seem to have no significant effect on $\bar{\sigma}_x$, $\bar{\tau}_{xz}$, and $\bar{u}$. On the contrary, the values of $\bar{\sigma}_z$ are clearly improved with the addition of the high-order terms, as shown in Figure 2.10. However, it should be pointed out that the values of $\bar{\sigma}_z$ are small when compared with the dominant components such as $\bar{\sigma}_x$ and $\bar{\tau}_{xz}$. This may imply that the effort involved in adding the high-order terms for w is not well rewarded. Hence, it may suggest that the use of constant w in laminate theory is justified. Also, shown in Figures 2.8 to 2.12 are the results from the Zigzag Theory. They will be discussed in a later section.

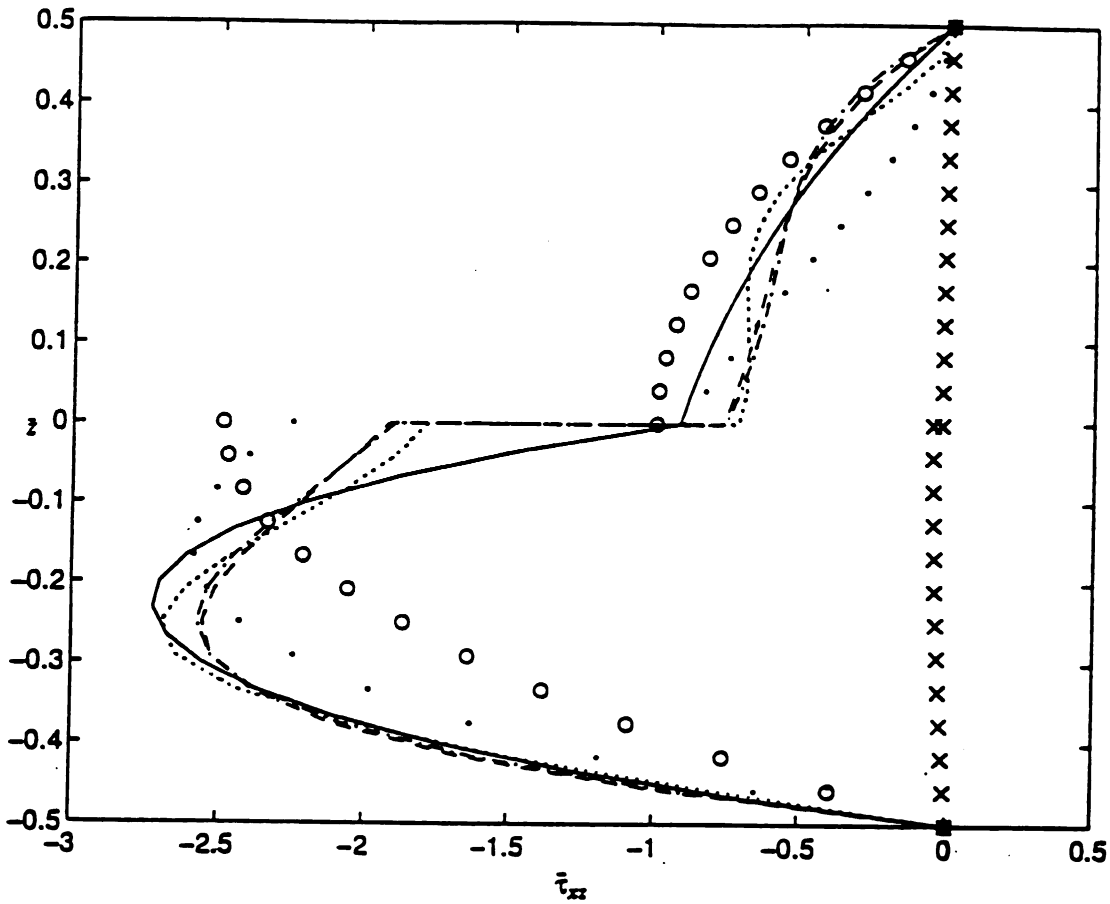

Figure 2.7 - Comparison of τ_{xz} from Shear Deformation Theories of various orders for a [0/90] laminates.

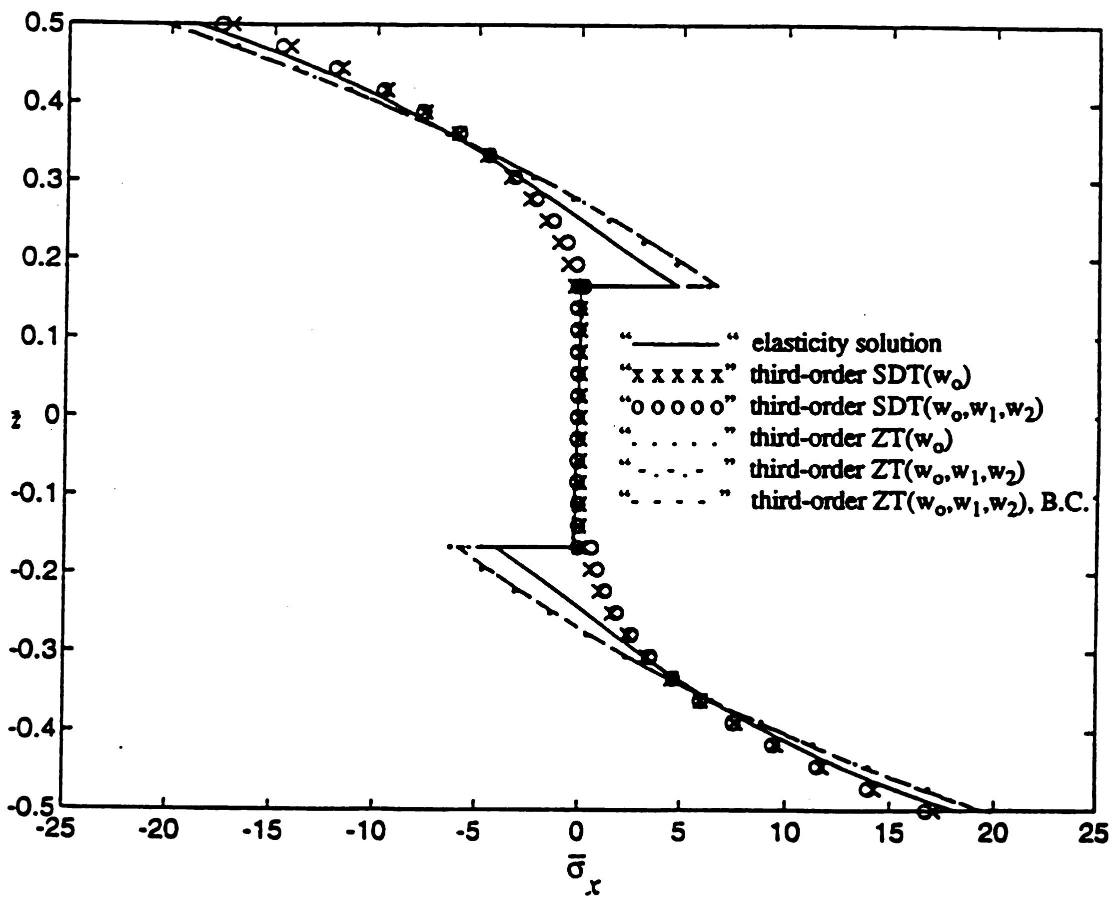

Figure 2.8 - Comparison of $\bar{\sigma}_x$ from various theories for a [0/90/0] laminate

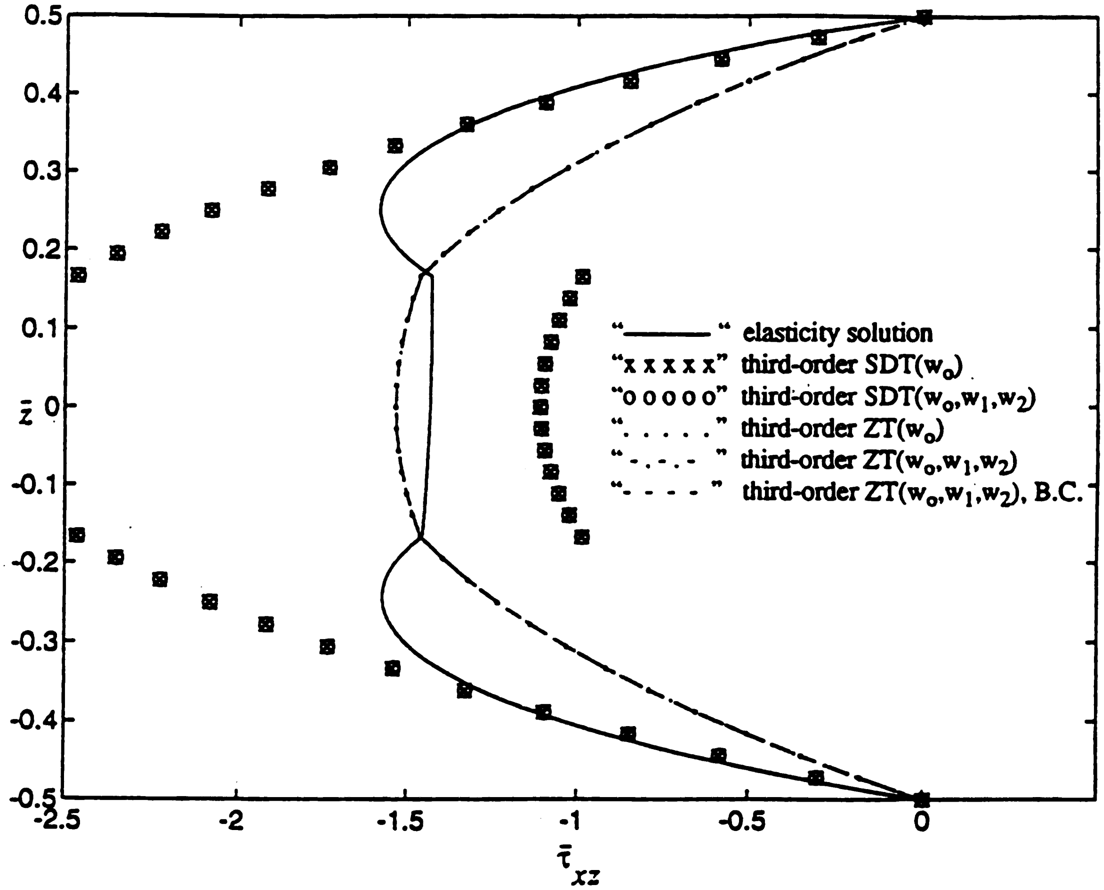

Figure 2.9 - Comparison of $\bar{\tau}_{xz}$ from various theories for a [0/90/0] laminate.

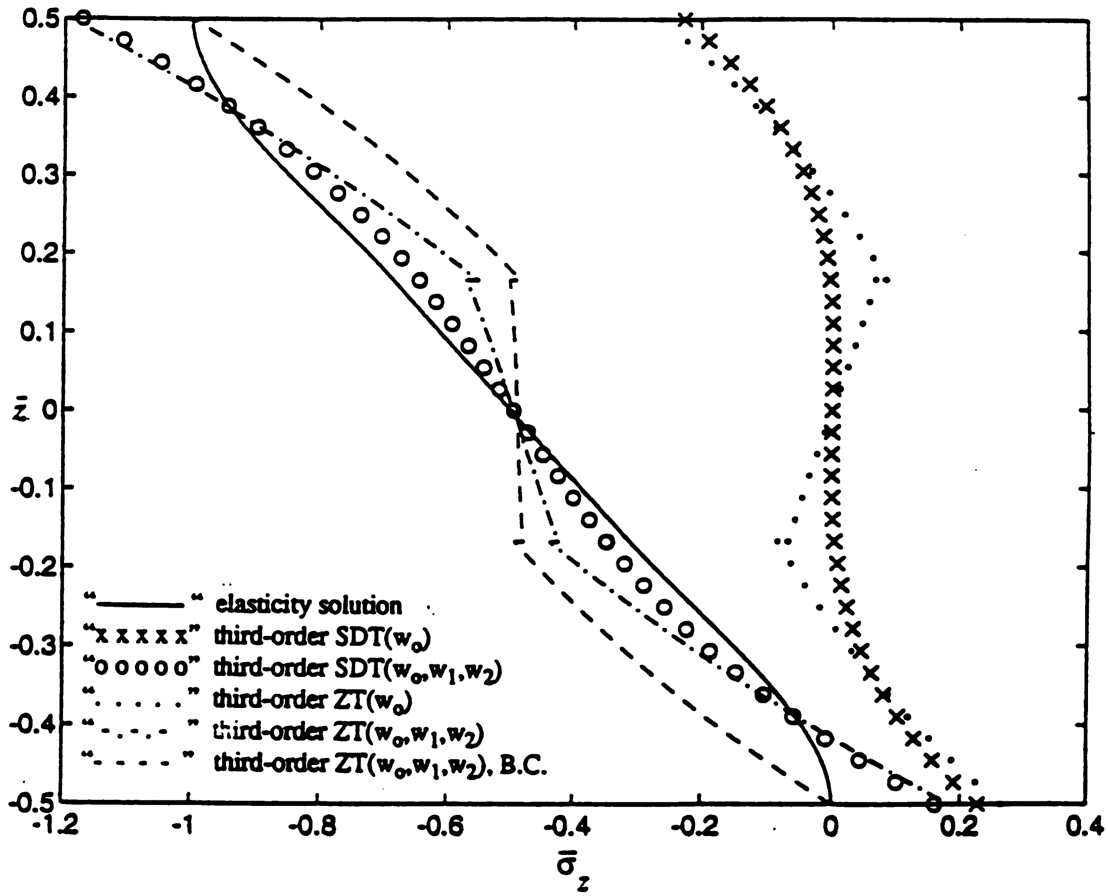

Figure 2.10 - Comparison of $\bar{\sigma}_z$ from various theories for a [0/90/0] laminate.

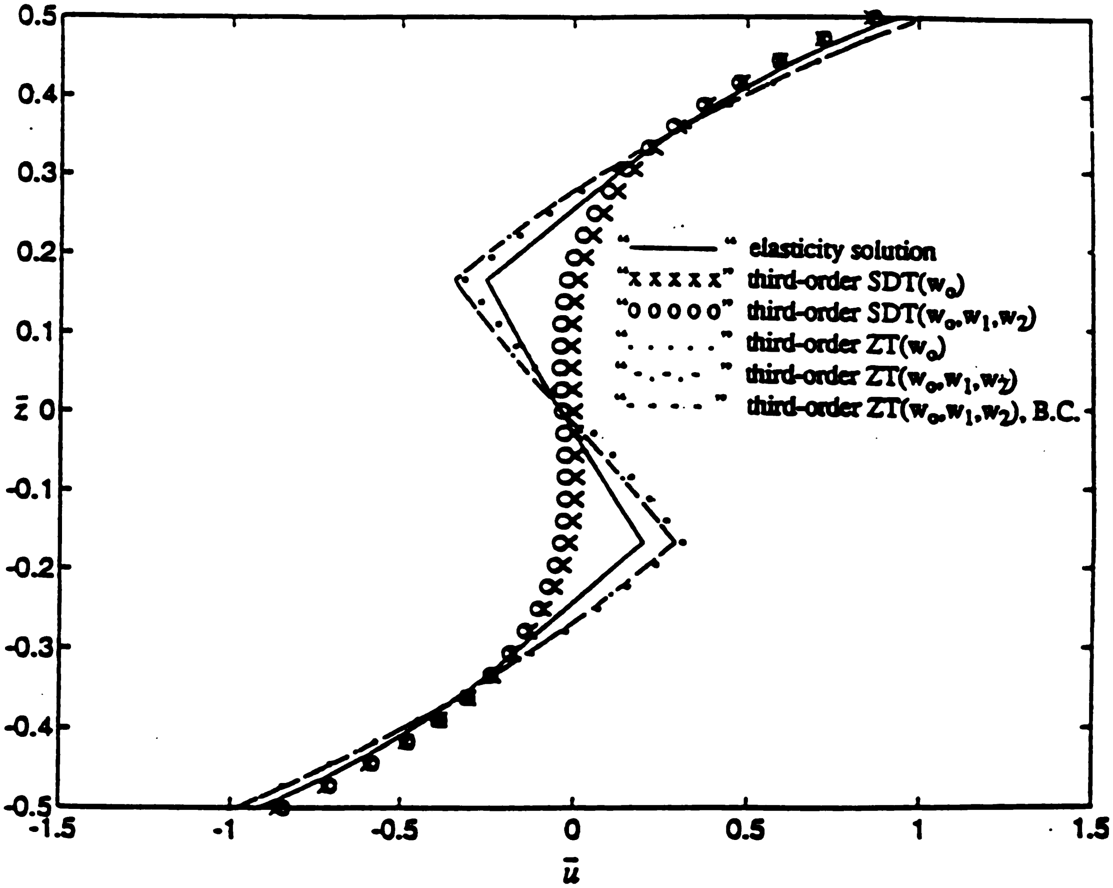

Figure 2.11 - Comparison of $\bar{u}$ from various theories for a [0/90/0] laminate.

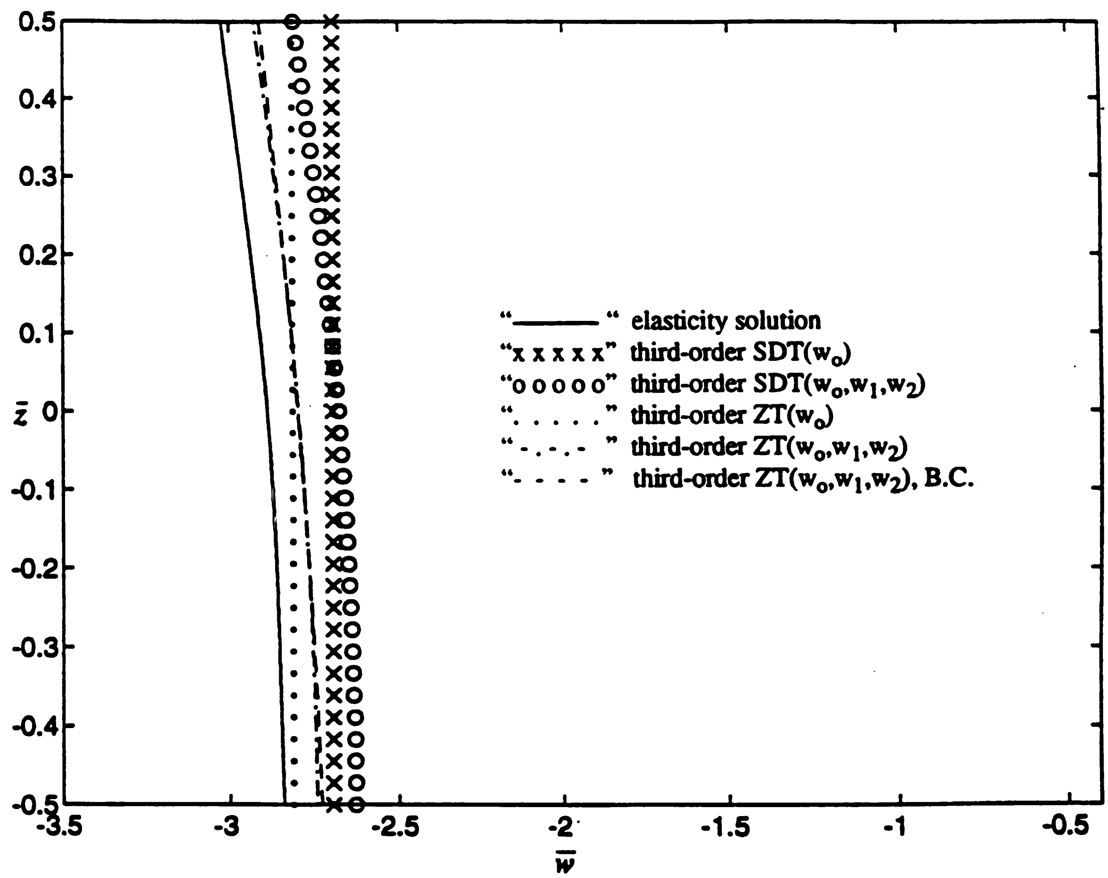

Figure 2.12 - Comparison of $\bar{w}$ from various theories for a [0/90/0] laminate.

2.7 Summary

Based on the above numerical results, the following conclusions can be summarized:

A. HSDT are layer-number independent theories. They model a composite laminate as a whole piece, disregarding the details of the laminate interfaces.

B. Equation (2.4) gives a generalized theory for HSDT. It covers all the available HSDT and provides the potential for development in this area.

C. HSDT give reasonable in-plane stress and displacement components. However, they fail to give zigzag distribution of in-plane displacement through the laminate thickness.

D. With HrSDT, it is not possible to obtain satisfactory transverse shear stresses directly from the constitutive equations because the continuity conditions of transverse shear stresses on the laminate interfaces are not satisfied.

E. Although the distribution of the transverse normal stresses is not in close agreement with the elasticity solution, the discrepancies seem to be acceptable, given the fact that they are very small numbers when compared with the in-plane and transverse shear stresses.

F. The assumption of a constant w in a displacement field seems to be acceptable due to the small values of $\bar{\sigma}_z$ and the insignificant difference in $\bar{w}$ between constant w and second-order w.

CHAPTER 3

LAYERWISE THEORIES

3.1 Introduction

As mentioned in Chapter 2, the major drawback of Shear Deformation Theories arises from the assumption of continuous functions for in-plane displacement components. This poor assumption subsequently results in (1) the incapability of presenting zigzag distribution of in-plane displacement through the laminate thickness and (2) the erroneous double-valued interlaminar stresses on the laminate interfaces. Apparently, the overall assumption of a displacement field for the entire composite laminate can always cause this type of error, regardless of the order of Shear Deformation Theory. In order to remove this fundamental defect, it is necessary to describe each composite laminate as an assembly of individual layers. Theories based on this layer-assembly technique are called Layerwise Theories.

A. Generalized Layerwise Theory (GLT)

In view of the important roles of the individual layers in overall performance of composite laminates, a layerwise approach was presented by Barbero and Reddy [7]. By assuming both translational and rotational displacement components for each composite layer, the displacement of the whole composite laminate is nothing but the assembly of the individual components. As a consequence, the total number of displacement variables is dependent on the layer number of the composite laminate. In their study, Barbero and

Reddy only imposed displacement continuity conditions on the laminate interfaces. Lu and Liu [15], and Lee and Liu [16] further employed the continuity conditions of interlaminar shear stresses and interlaminar normal stress for composite layer assembly.

In their study, Lee and Liu [16] assign two variables, a translational component and its derivative, for each of the displacement components at each surface. In order to model the displacement field of a composite layer, a two-node element is required with each node representing a surface of the layer. Consequently, as can be seen below, there are four variables in each displacement component. Hermitian cubic shape functions ϕ_i^k are imposed to assemble these four variables in each composite layer, i.e.

$$
\begin{aligned}
u^k &= U_{2k-2}\phi_1^k + T_{2k-2}\phi_2^k + U_{2k-1}\phi_3^k + T_{2k-1}\phi_4^k \\
v^k &= V_{2k-2}\phi_1^k + S_{2k-2}\phi_2^k + V_{2k-1}\phi_3^k + S_{2k-1}\phi_4^k \\
w^k &= W_{2k-2}\phi_1^k + R_{2k-2}\phi_2^k + W_{2k-1}\phi_3^k + R_{2k-1}\phi_4^k
\end{aligned}
\tag{3.1}
$$

The above equations result in a total of 12n variables for an n-layer composite laminate. The superscript k in the above equations represents the order of layer, while the subscript represents the order of interface in the thickness direction. In addition, U, V, and W denote the translational displacement components in x, y, and z directions, respectively, while T, S, and R denote the derivatives of U, V, and W with respect to x, y, and z, respectively.

In assembling the individual layers together, the continuity conditions of both displacement and transverse stress are employed at every laminate interface. Accordingly, the total number of variables becomes 6n + 6. The imposition of the continuity conditions through the laminate thickness not only largely reduces the total number of degrees-of-freedom, but also greatly improves the accuracy of displacement and stress predictions. The results from their Layerwise Theory are believed to be the most accurate among all the laminate theories available in the literature. However, it should be noted that computational efficiency is sacrificed in exchange for numerical accuracy. When compared to the Shear Deformation Theories, which have total numbers of degrees-of-freedom regardless of the number of layers, the computational efficiency in Layerwise Theories seems to be totally neglected. A laminate theory which accounts for both numerical accuracy and computational efficiency is highly desired.

The displacement field of Equations (3.1) is based on translational displacement components and their derivatives, namely angles, and has a very distinct physical meaning. However, this type of expression is not consistent with those used in the Shear Deformation Theories. For comparison and unification purposes, the notations used in the Shear Deformation Theories are utilized to express the Layerwise Theory. Since the Hermitian shape functions are of third-order, with respect to the thickness direction, and there are four variables in each displacement component, the following polynomial equations are equivalent to Equations (3.1):

$$
\begin{aligned}
u^k(x,y,z) &= u_0^k(x,y) + u_1^k(x,y)\,z + u_2^k(x,y)\,z^2 + u_3^k(x,y)\,z^3 \\
v^k(x,y,z) &= v_0^k(x,y) + v_1^k(x,y)\,z + v_2^k(x,y)\,z^2 + v_3^k(x,y)\,z^3 \\
w^k(x,y,z) &= w_0^k(x,y) + w_1^k(x,y)\,z + w_2^k(x,y)\,z^2 + w_3^k(x,y)\,z^3
\end{aligned}
\tag{3.2}
$$

Equations (3.2) are of a general form, they are not limited to Equations (3.1) which impose Hermitian cubic shape functions in layer assembly. Besides, they are based on a general coordinate system. As a result, they are called the Generalized Layerwise Theory. It can be seen that Equations (3.2) unifies the Shear Deformation Theories and all the Layerwise Theories mentioned before. In fact, the Shear Deformation Theories are just simplified cases of the Generalized Layerwise Theory.

B. Zigzag Theories (ZT)

In view of the incapability of presenting the zigzag distributions for in-plane displacement components through the laminate thickness in the Shear Deformation Theories, there were some efforts in improving the deficiency. Quite a few theories were presented and named Zigzag Theories (ZT), owing to the zigzag distributions of in-plane displacements through the laminate thickness. Di Sciuva [8] and Murakami [9] were among the most recent to present their own Zigzag Theories. Their approaches were based on assumed displacement components for individual layers. Similar to Barbero and Reddy [7], only the displacement continuity conditions on the laminate interfaces were considered. In their studies, the most important achievements of their theories were the ability to show the zigzag shape of in-plane displacements in the thickness direction and the associated improvement of the in-plane stress prediction. However, due to the low order of their assumed displacement fields, the transverse shear stresses were constant through the thickness in both cases. The aforementioned post-processing technique was then required for finding the true values.

Although high-order terms were later added to their theories (Di Sciuva [17, 18], Toledano and Murakami [19, 20]), their theories still suffered from the deficiency in predicting the correct transverse shear stresses. Similar studies could also be found in the articles by Lee, Senthilnathan, Lim, and Chow [21] and Soldatos [22]. It was not until Cho and Parmerter [23] presented a Zigzag Theory accounting for both the overall performance and the interlaminar continuity conditions of transverse shear stresses, that the direct, and correct, calculation for transverse shear stresses from constitutive equations became possible. However, it should be pointed out that similar to all other Zigzag Theories, their theory was also of a special case instead of a general form. Hence, it then is the first objective of this study to present a Generalized Zigzag Theory (GZT) which can not only give accurate displacement and stress efficiently, but can also be linked to the Generalized Layerwise Theory.

3.2 Formulation of Generalized Zigzag Theory

In addition to the ability of describing the zigzag distribution of in-plane displacement through the laminate thickness, the most important characteristic of the Zigzag Theories is their layer-independency. This is very critical to computational efficiency and should be considered as a requirement in developing a laminate theory.

A. Fundamental Equations

As mentioned above, all the Zigzag Theories available in the literature are of special types since they all are based on specially assumed displacement functions. By examining the Equations (3.2) of the Generalized Layerwise Theory, it can be found that if both displacement and transverse shear stress are to be continuous across the laminate interfaces, there should be two, and only two, layer-dependent variables in each in-plane displacement component. After analyzing all the Zigzag Theories available in the literature, it is concluded that they all designate the zeroth- and the first-order terms as layer-dependent variables. It is then only natural to define the following displacement field as a Generalized Zigzag Theory since it is of a general form instead of a special form:

$$
\begin{aligned}
u^k(x,y,z) &= u_0^k(x,y) + u_1^k(x,y)\,z + u_2(x,y)\,z^2 + u_3(x,y)\,z^3 \\
v^k(x,y,z) &= v_0^k(x,y) + v_1^k(x,y)\,z + v_2(x,y)\,z^2 + v_3(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{3.3}
$$

The displacement field shown above represents the displacement components of the k^{th} layer located between the interfaces $z = z_k$ and $z = z_{k+1}$. Apparently, the displacement components in both the x and the y directions are layer dependent, while in the z direction it is layer independent. Details of the coordinate system, layer order, and interface location are shown in Figure 3.1. In addition, the following linear strain-displacement relations are employed in this study.

$$
\begin{aligned}
&\varepsilon_x^k = \frac{\partial}{\partial x}u^k; \qquad \varepsilon_y^k = \frac{\partial}{\partial y}v^k; \qquad \varepsilon_z^k = \frac{\partial}{\partial z}w^k; \\
&\gamma_{xy}^k = \frac{\partial}{\partial y}u^k + \frac{\partial}{\partial x}v^k; \qquad \gamma_{yz}^k = \frac{\partial}{\partial z}v^k + \frac{\partial}{\partial y}w^k; \qquad \gamma_{xz}^k = \frac{\partial}{\partial z}u^k + \frac{\partial}{\partial x}w^k;
\end{aligned}
\tag{3.4}
$$

In view of the subsequent close-form verification, only orthotropic laminates made of cross-ply stacking sequences are considered. The constitutive equations for the k^{th} layer are written below,

$$
\begin{aligned}
&\begin{Bmatrix} \sigma_x^k \\ \sigma_y^k \\ \sigma_z^k \\ \tau_{xy}^k \end{Bmatrix} = \begin{bmatrix} Q_{11}^k & Q_{12}^k & Q_{13}^k & 0 \\ Q_{12}^k & Q_{22}^k & Q_{23}^k & 0 \\ Q_{13}^k & Q_{23}^k & Q_{33}^k & 0 \\ 0 & 0 & 0 & Q_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_x^k \\ \varepsilon_y^k \\ \varepsilon_z^k \\ \gamma_{xy}^k \end{Bmatrix} = \left[Q^k\right]_I \begin{Bmatrix} \varepsilon_x^k \\ \varepsilon_y^k \\ \varepsilon_z^k \\ \gamma_{xy}^k \end{Bmatrix} \\
&\begin{Bmatrix} \tau_{yz}^k \\ \tau_{xz}^k \end{Bmatrix} = \begin{bmatrix} Q_{44}^k & 0 \\ 0 & Q_{55}^k \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^k \\ \gamma_{xz}^k \end{Bmatrix} = \left[Q^k\right]_{II} \begin{Bmatrix} \gamma_{yz}^k \\ \gamma_{xz}^k \end{Bmatrix}
\end{aligned}
\tag{3.5}
$$

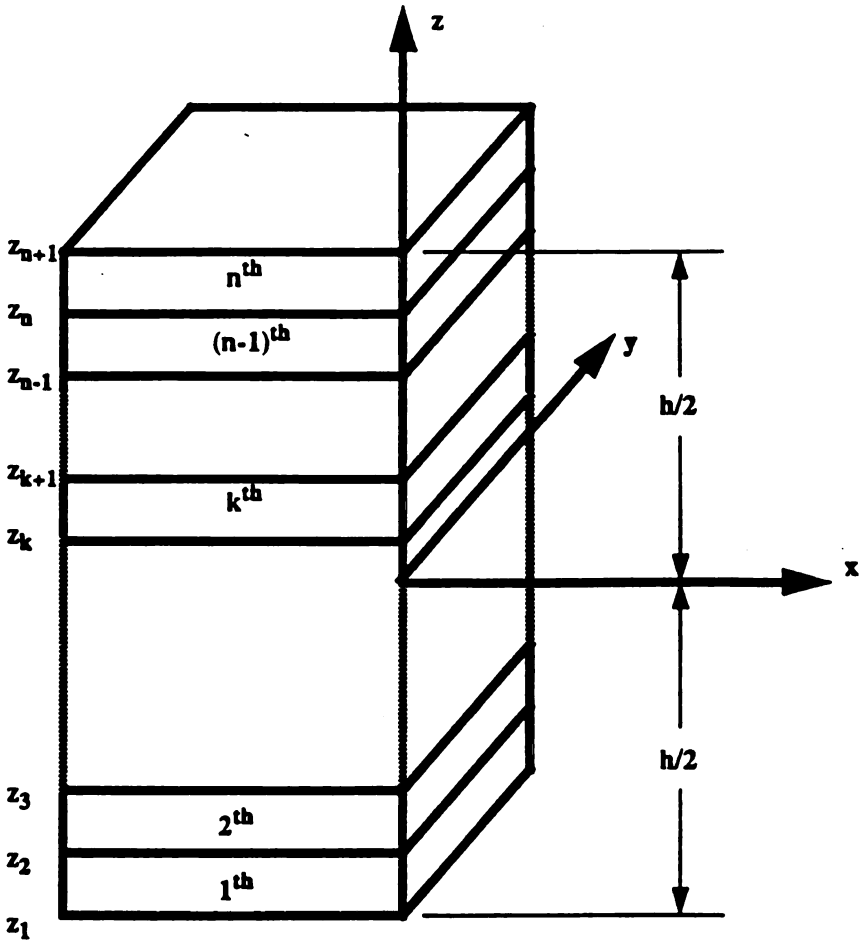

Figure 3.1 -Coordinate system, layer order, and interface locations.

where Q_{ij}^{k} are stiffness components of the k^{th} layer. Details of the definitions can be found in Vinson and Sierakowski [24].

B. Continuity Conditions

In Equations (3.3), the number of layer-dependent variables in an n-layer composite laminate is $4n$. These variables can be replaced with layer-independent variables through the enforcement of continuity conditions of displacement and transverse shear stress on the laminate interfaces.

If the n-layer composite laminate has continuous displacement across the interfaces, the following continuity conditions should be satisfied:

$$\begin{aligned} u^{k-1}\big|_{z=z_k} &= u^{k}\big|_{z=z_k} \\ v^{k-1}\big|_{z=z_k} &= v^{k}\big|_{z=z_k} \qquad k = 2, 3, \ldots, n \end{aligned} \tag{3.6}$$

By substituting Equations (3.3) into Equations (3.6), the following equations can be obtained:

$$\begin{aligned} u_o^k - u_o^{k-1} &= (u_1^{k-1} - u_1^k)\, z_k \\ v_o^k - v_o^{k-1} &= (v_1^{k-1} - v_1^k)\, z_k \qquad k = 2, 3, \ldots, n \end{aligned} \tag{3.7}$$

In addition, if $u_o^1 = u_o$ and $v_o^1 = v_o$ are defined, u_o^k and v_o^k can be expressed as the sums of u_1^k and v_1^k, respectively, i.e.:

$$\begin{aligned} u_o^k &= u_o + \sum_{j=2}^{k} (u_1^{j-1} - u_1^j)\, z_j \\ v_o^k &= v_o + \sum_{i=2}^{k} (v_1^{j-1} - v_1^j)\, z_j \qquad k = 2, 3, \ldots, n \end{aligned} \tag{3.8}$$

The number of layer-dependent variables is then reduced to $2n$.

Enforcing the following continuity conditions of transverse shear stresses on laminate interfaces, the remaining 2n layer-dependent variables can be eliminated:

$$\begin{aligned} \tau_{xz}^{k-1}\Big|_{z=z_k} &= \tau_{xz}^{k}\Big|_{z=z_k} \\ \tau_{yz}^{k-1}\Big|_{z=z_k} &= \tau_{yz}^{k}\Big|_{z=z_k} \qquad k = 2, 3, 4, \ldots, n \end{aligned} \tag{3.9}$$

The displacement field shown by Equations (3.3) will then become a layer-number independent theory.

By combining Equations (3.8) and (3.3), utilizing Equations (3.4) and (3.5), and then substituting the transverse shear stresses into Equations (3.9), the following relations can be achieved:

$$\begin{aligned} Q_{55}^{k} u^{k}{}_{1} - Q_{55}^{k-1} u_{1}^{k-1} &= 2\Omega_k z_k u_2 + 3\Omega_k z_k^2 u_3 + \Omega_k w_{o,x} \\ Q_{44}^{k} v^{k}{}_{1} - Q_{44}^{k-1} v_{1}^{k-1} &= 2\Theta_k z_k v_2 + 3\Theta_k z_k^2 v_3 + \Theta_k w_{o,y} \end{aligned} \tag{3.10}$$

$$k = 2, 3, 4, \ldots, n$$

where,

$$\Omega_k = Q_{55}^{k-1} - Q_{55}^{k}, \quad \Theta_k = Q_{44}^{k-1} - Q_{44}^{k}.$$

Given the new definitions $u_1^1 = u_1$ and $v_1^1 = v_1$, Equations (3.10) can be rewritten as:

$$\begin{aligned} u_1^k &= F_1^k u_1 + F_2^k u_2 + F_3^k u_3 + F_4^k w_{o,x} \\ v_1^k &= L_1^k v_1 + L_2^k v_2 + L_3^k v_3 + L_4^k w_{o,y} \end{aligned} \tag{3.11}$$

where,

$$F_1^k = \frac{Q_{55}^1}{Q_{55}^k} \qquad L_1^k = \frac{Q_{44}^1}{Q_{44}^k}$$

$$F_2^k = \frac{2}{Q_{55}^k}\sum_{j=2}^{k}(\Omega_{j-1} - \Omega_j)\, z_j \qquad L_2^k = \frac{2}{Q_{44}^k}\sum_{j=2}^{k}(\Theta_{j-1} - \Theta_j)\, z_j$$

$$F_3^k = \frac{3}{Q_{55}^k}\sum_{j=2}^{k}(\Omega_{j-1} - \Omega_j)\, z_j^2 \qquad L_3^k = \frac{3}{Q_{44}^k}\sum_{j=2}^{k}(\Theta_{j-1} - \Theta_j)\, z_j^2$$

$$F_4^k = F_1^k - 1 \qquad L_4^k = L_1^k - 1$$

$$k = 2, 3, 4, \ldots, n$$

It should be noted that $F_1^1 = 1$, $F_2^1 = F_3^1 = F_4^1 = 0$, $L_1^1 = 1$, and $L_2^1 = L_3^1 = L_4^1 = 0$.

C. Boundary Conditions

The independent variables in Equations (3.11) can be further reduced by applying shear-free conditions on both surfaces of the laminate because of their popularity in laminate plate analysis, i.e.:

$$\tau_{xz}^1(z_1) = 0, \quad \tau_{xz}^n(z_{n+1}) = 0, \quad \tau_{yz}^1(z_1) = 0, \quad \tau_{yz}^n(z_{n+1}) = 0 \tag{3.12}$$

Utilizing Equations (3.4), (3.5), (3.11), and (3.12), the following equations can be achieved:

$$\begin{bmatrix} F_2^n + 2z_{n+1} & F_3^n + 3z_{n+1}^2 \\ 2z_1 & 3z_1^2 \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} -F_1^n & -\left(F_4^n + 1\right) \\ -1 & -1 \end{bmatrix} \begin{pmatrix} u_1 \\ w_{o,x} \end{pmatrix}$$

$$\begin{bmatrix} L_2^n + 2z_{n+1} & L_3^n + 3z_{n+1}^2 \\ 2z_1 & 3z_1^2 \end{bmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} -L_1^n & -\left(L_4^n + 1\right) \\ -1 & -1 \end{bmatrix} \begin{pmatrix} v_1 \\ w_{o,y} \end{pmatrix} \tag{3.13}$$

Solutions of u_2, u_3, v_2, and v_3 can be expressed in the following manner:

$$\begin{aligned} u_2 &= A_1 u_1 + A_2 w_{o,x} & u_3 &= B_1 u_1 + B_2 w_{o,x} \\ v_2 &= C_1 v_1 + C_2 w_{o,y} & v_3 &= D_1 v_1 + D_2 w_{o,y} \end{aligned} \tag{3.14}$$

where the coefficients A_i, B_i, C_i, and D_i can be easily identified from Equations (3.13). However, it should be recognized that in order to have unique solutions to Equations (3.13), it requires that,

(a) $z_1 \neq 0$, and

(b) $z_1 \neq \dfrac{2F_3^n + 6h^2}{3F_2^n - 6h}$ or $z_1 \neq \dfrac{2L_3^n + 6h^2}{3L_2^n - 6h}$. (3.15)

Substituting Equations (3.14) into Equations (3.11) and (3.8), it yields:

$$\begin{aligned} u_1^k &= R_1^k u_1 + R_2^k w_{o,x} & u_o^k &= u_o + S_1^k u_1 + S_2^k w_{o,x} \\ v_1^k &= O_1^k v_1 + O_2^k w_{o,y} & v_o^k &= v_o + P_1^k v_1 + P_2^k w_{o,y} \end{aligned} \tag{3.16}$$

where,

$$\begin{aligned} R_1^k &= F_1^k + A_1 F_2^k + B_1 F_3^k & R_2^k &= F_4^k + A_2 F_2^k + B_2 F_3^k \\ O_1^k &= L_1^k + C_1 L_2^k + D_1 L_3^k & O_2^k &= L_4^k + C_2 L_2^k + D_2 L_3^k \\ S_1^k &= \sum_{l=2}^{k} (R_1^{l-1} - R_1^l)\, z_l & S_2^k &= \sum_{l=2}^{k} (R_2^{l-1} - R_2^l)\, z_l \\ P_1^k &= \sum_{l=2}^{k} (O_1^{l-1} - O_1^l)\, z_l & P_2^k &= \sum_{l=2}^{k} (O_2^{l-1} - O_2^l)\, z_l \end{aligned}$$

By substituting Equations (3.16) into Equations (3.3), a displacement field which is free of layer-dependent variables but dependent on layer properties can be concluded:

$$
\begin{aligned}
u^k &= u_o + \Phi_1^k(z)\, u_1 + \Phi_2^k(z)\, w_{o,x} \\
v^k &= v_o + \Psi_1^k(z)\, v_1 + \Psi_2^k(z)\, w_{o,y} \\
w^k &= w_0
\end{aligned}
\tag{3.17}
$$

where,

$$
\Phi_1^k(z) = S_1^k + R_1^k z + A_1 z^2 + B_1 z^3 \qquad \Phi_2^k(z) = S_2^k + R_2^k z + A_2 z^2 + B_2 z^3
$$

$$
\Psi_1^k(z) = P_1^k + O_1^k z + C_1 z^2 + D_1 z^3 \qquad \Psi_2^k(z) = P_2^k + O_2^k z + C_2 z^2 + D_2 z^3
$$

It should be pointed out that R_i^k, S_i^k, O_i^k, P_i^k, A_i, B_i, C_i and D_i are only related to material properties and layer thicknesses. The total number of independent variables then becomes five.

D. Variational Formulation

In order to verify the Generalized Zigzag Theory, the cylindrical bending problem mentioned in Chapter 2 is to be investigated. If normal forces are exerted on the surfaces of the composite laminate, the governing equations can be obtained from the principle of virtual displacement:

$$
\int_\Omega \left(\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\begin{Bmatrix} \sigma_x^k \\ \sigma_y^k \\ \sigma_z^k \\ \tau_{xy}^k \end{Bmatrix}^T \begin{Bmatrix} \delta\varepsilon_x^k \\ \delta\varepsilon_y^k \\ \delta\varepsilon_z^k \\ \delta\gamma_{xy}^k \end{Bmatrix} + \begin{Bmatrix} \tau_{yz}^k \\ \tau_{xz}^k \end{Bmatrix}^T \begin{Bmatrix} \delta\gamma_{yz}^k \\ \delta\gamma_{xz}^k \end{Bmatrix} \right) dz - q^{\pm}\ (\delta w^{\pm}\ ) \right) dxdy = 0
\tag{3.18}
$$

For simplicity, the strains and displacements can be expressed in matrix forms, i.e.:

$$\varepsilon_x^k = \left[N_x^k\right]\{X\}, \qquad \varepsilon_y^k = \left[N_y^k\right]\{X\}, \qquad \varepsilon_z^k = \left[N_z^k\right]\{X\}$$

$$\gamma_{xy}^k = \left[N_{xy}^k\right]\{X\}, \quad \gamma_{xz}^k = \left[N_{xz}^k\right]\{X\}, \quad \gamma_{yz}^k = \left[N_{yz}^k\right]\{X\} \tag{3.19}$$

$$w^+ = \left[\ N_w^+\ \right]\{X\}, \qquad w^- = \left[N_w^-\right]\{X\}$$

where, $$\{X\} = \{u_o \ v_o \ u_1 \ v_1 \ w_o\}^T$$

$$\left[N_x^k\right] = \left[\frac{\partial}{\partial x} \quad 0 \quad \Phi_1^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_2^k\frac{\partial^2}{\partial x^2}\right]$$

$$\left[N_y^k\right] = \left[0 \quad \frac{\partial}{\partial y} \quad 0 \quad \Psi_1^k\frac{\partial}{\partial y} \quad \Psi_2^k\frac{\partial^2}{\partial y^2}\right]$$

$$\left[N_z^k\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$\left[N_{xy}^k\right] = \left[\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x} \quad \Phi_1^k\frac{\partial}{\partial y} \quad \Psi_1^k\frac{\partial}{\partial x} \quad (\Phi_2^k + \Psi_2^k)\frac{\partial^2}{\partial x \partial y}\right]$$

$$\left[N_{xz}^k\right] = \left[0 \quad 0 \quad \frac{d}{dz}\Phi_1^k \quad 0 \quad \left(\frac{d}{dz}\Phi_2^k + 1\right)\frac{\partial}{\partial x}\right]$$

$$\left[N_{yz}^k\right] = \left[0 \quad 0 \quad 0 \quad \frac{d}{dz}\Psi_1^k \quad \left(\frac{d}{dz}\Psi_2^k + 1\right)\frac{\partial}{\partial y}\right]$$

$$\left[N_w^+\ \right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

$$\left[N_w^-\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

Let

$$\left[N^k\right]_I = \begin{bmatrix} \left[N_x^k\right] \\ \left[N_y^k\right] \\ \left[N_z^k\right] \\ \left[N_{xy}^k\right] \end{bmatrix}, \qquad \left[N^k\right]_{II} = \begin{bmatrix} \left[N_{xz}^k\right] \\ \left[N_{yz}^k\right] \end{bmatrix} \tag{3.20}$$

the variational equation, Equation (3.19), can be rewritten as follows:

$$\int_{\Omega} \{\delta X\}^T \left(\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\left[N^k\right]_I^T \left[Q^k\right]_I \left[N^k\right]_I + \left[N^k\right]_{II}^T \left[Q^k\right]_{II} \left[N^k\right]_{II} \right) dz \right) \{X\}\, dxdy =$$

$$\int_{\Omega} \{\delta X\}^T \left(\quad q^+ \left[N_w^+ \right] \right) dxdy + \int_{\Omega} \left(q^- \left[N_w^- \right] \right) dxdy$$

(3.21)

E. Close-form Solution

As stated in Section 2.2, it is possible to have a close-form solution to the aforementioned cylindrical bending problem. If the composite laminate is of an infinitely long strip and simply supported along the two long edges, the problem can be reduced to a plane-strain type. In order to satisfy both the sinusoidal loading, $q^+\left(x, \frac{h}{2}\right) = q_o^+ \sin\frac{m\pi}{L}x$, on the top surface and the simply-supported boundary conditions, the following close-form solution can be assumed:

$$u_o = U_o \cos px \qquad u_1 = U_1 \cos px \qquad w_o = W_o \sin px \tag{3.22}$$

As a consequence, $\{X\}$ can be simplified and $\{\bar{X}\}$ can be defined as follows:

$$\{X\} = \{u_o \quad u_1 \quad w_o\}^T \text{ and } \{\bar{X}\} = \{U_o \quad U_1 \quad W_o\}^T \tag{3.23}$$

Substituting Equations (3.22) into (3.19), it yields:

$$\begin{aligned}
\left[N_x^k\right] &= \left[\bar{N}_x^k\right]\sin px = \left[-p \quad -p\Phi_1^k \quad -p^2\Phi_2^k\right]\sin px \\
\left[N_z^k\right] &= \left[\bar{N}_z^k\right]\sin px = \left[0 \qquad 0 \qquad 0\right]\sin px \\
\left[N_{xz}^k\right] &= \left[\bar{N}_{xz}^k\right]\cos px = \left[0 \quad \frac{d}{dz}\Phi_1^k \quad p\left(\frac{d}{dz}\Phi_2^k + 1\right)\right]\cos px \\
\left[N_w^+\right] &= \left[\bar{N}_w^+\right]\sin px = \left[0 \qquad 0 \qquad 1\right]\sin px
\end{aligned} \tag{3.24}$$

And the variational equation, Equation (3.21), then becomes:

$$\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\Big(Q_{11}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_x^k\right]+Q_{13}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_z^k\right]+Q_{13}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_x^k\right]+Q_{33}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_z^k\right]$$
$$+Q_{55}^k\left[\bar{N}_{xz}^k\right]^T\left[\bar{N}_{xz}^k\right]\Big)dz\,\{\bar{X}\} = q_o^+\left[N_w^+\right]^T \tag{3.25}$$

Equation (3.25) is a set of linear equations for three variables U_o, U_1, and W_o which can be determined by solving the equations simultaneously.

3.3 Generalized Zigzag Theory with Higher-order Terms

HrSDT can be combined with the Generalized Zigzag Theory to improve the accuracy of global response. This is especially efficient in composite laminates with lower numbers of layers such as two-layer and three-layer laminates. In this investigation, a Generalized Zigzag Theory of fifth-order is of interest, i.e.:

$$\begin{aligned}
u^k(x,y,z) &= u_0^k(x,y) + u_1^k(x,y)\,z + u_2(x,y)\,z^2 + u_3(x,y)\,z^3 + u_4(x,y)\,z^4 + u_5(x,y)\,z^5\\
v^k(x,y,z) &= v_0^k(x,y) + v_1^k(x,y)\,z + v_2(x,y)\,z^2 + v_3(x,y)\,z^3 + v_4(x,y)\,z^4 + v_5(x,y)\,z^5\\
w^k(x,y,z) &= w_0(x,y)
\end{aligned} \tag{3.26}$$

With a process similar to that shown in Section 3.2, the layer-dependent theory can be converted to the following layer-independent theory:

$$\begin{aligned}
u^k &= u_o + \Phi_1^k(z)\,u_1 + \Phi_2^k(z)\,u_2 + \Phi_3^k(z)\,u_3 + \Phi_4^k(z)\,w_{o,x}\\
v^k &= v_o + \Psi_1^k(z)\,v_1 + \Psi_2^k(z)\,v_2 + \Psi_3^k(z)\,v_3 + \Psi_4^k(z)\,w_{o,y}\\
w^k &= w_0
\end{aligned} \tag{3.27}$$

Furthermore, the matrices regarding the displacement components and their derivatives can be found:

$$\{X\} = \{u_o \ v_o \ u_1 \ v_1 \ u_2 \ v_2 \ \ u_3 \ v_3 \ w_o\}^T$$

$$\left[N_x^k\right] = \left[\frac{\partial}{\partial x} \quad 0 \quad \Phi_1^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_2^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_3^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_4^k\frac{\partial^2}{\partial x^2}\right]$$

$$\left[N_y^k\right] = \left[0 \quad \frac{\partial}{\partial y} \quad 0 \quad \Psi_1^k\frac{\partial}{\partial y} \quad 0 \quad \Psi_2^k\frac{\partial}{\partial y} \quad 0 \quad \Psi_3^k\frac{\partial}{\partial y} \quad \Psi_4^k\frac{\partial^2}{\partial y^2}\right]$$

$$\left[N_z^k\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$\left[N_{xy}^k\right] = \left[\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x} \quad \Phi_1^k\frac{\partial}{\partial y} \quad \Psi_1^k\frac{\partial}{\partial x} \quad \Phi_2^k\frac{\partial}{\partial y} \quad \Psi_2^k\frac{\partial}{\partial x} \quad \Phi_3^k\frac{\partial}{\partial y} \quad \Psi_3^k\frac{\partial}{\partial x} \quad (\Phi_4^k + \Psi_4^k)\frac{\partial^2}{\partial x \partial y}\right]$$

$$\left[N_{xz}^k\right] = \left[0 \quad 0 \quad \frac{d}{dz}\Phi_1^k \quad 0 \quad \frac{d}{dz}\Phi_2^k \quad 0 \quad \frac{d}{dz}\Phi_3^k \quad 0 \quad \left(\frac{d}{dz}\Phi_4^k + 1\right)\frac{\partial}{\partial x}\right]$$

$$\left[N_{yz}^k\right] = \left[0 \quad 0 \quad 0 \quad \frac{d}{dz}\Psi_1^k \quad 0 \quad \frac{d}{dz}\Psi_2^k \quad 0 \quad \frac{d}{dz}\Psi_3^k \quad \left(\frac{d}{dz}\Psi_4^k + 1\right)\frac{\partial}{\partial y}\right]$$

$$\left[N_w^+\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

$$\left[N_w^-\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

(3.28)

The close-form solution can be obtained by utilizing Equations (3.28) and Equation (3.25).

3.4 Numerical Results and Discussion

A. [0/90/0] Laminate

Results of the normalized stress and displacement components of the [0/90/0] laminate are shown from Figures 3.2 to 3.6. It is found that the results are improved as the order of the theory increases. As can be seen from the figures, the results from the third-order and the fourth-order are identical. Similarly, those from the fifth-order are the same as those from the sixth-order (see τ_{xz} in Figure 3.3). These results may indicate the importance of the odd-order terms in this symmetric laminate.

As can be seen from Figure 3.2, the fifth-order and above seem to give exact in-plane stress σ_x for the case under examination. However, the exact values of τ_{xz} are more difficult to obtain. Although the fifth-order and above have significant effect on the transverse shear stress distribution, they fail to show the distinct kinks on the laminate interfaces as given in the exact solution. It is believed that the deficiency cannot be improved simply by increasing the order of the theory.

Figure 3.4 shows the transverse normal stress, σ_z. Although no continuity condition is imposed on the laminate interfaces, the discrepancy at the interfaces seems to be very small, especially in the higher-order theories. In addition, because no boundary condition of transverse normal stress is enforced, the result on the top surface is not equal to negative one and on the bottom surface is not equal to zero. Although the results in Figure 3.4 seem to be different from the exact solutions, it should be recognized that they are small values when compared to the dominant component, σ_x.

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Another excellent result takes place in the in-plane displacement $\bar{u}$. The zigzag distribution can be perfectly represented by the theories of the third-order and above. This excellent agreement in conjunction with the excellent results in $\bar{\sigma}_x$ seems to imply that the higher-order (fifth and above) theories can give excellent in-plane displacement and stress components. The prediction of transverse normal displacement, $\bar{w}$, is very good for both the third- and the fourth-orders and excellent for the fifth-order and above.

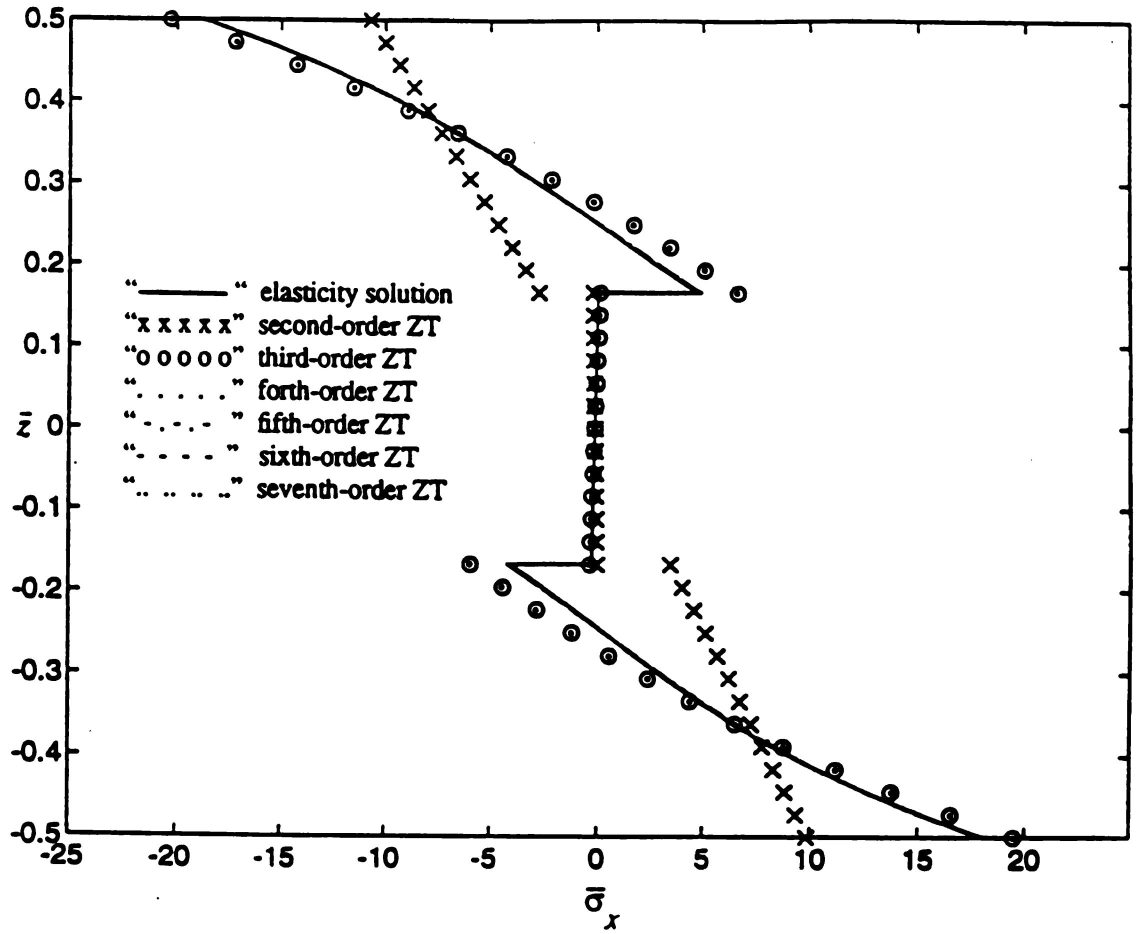

Figure 3.2 - Comparison of $\bar{\sigma}_x$ from Zigzag Theories of various orders for a [0/90/0] laminate.

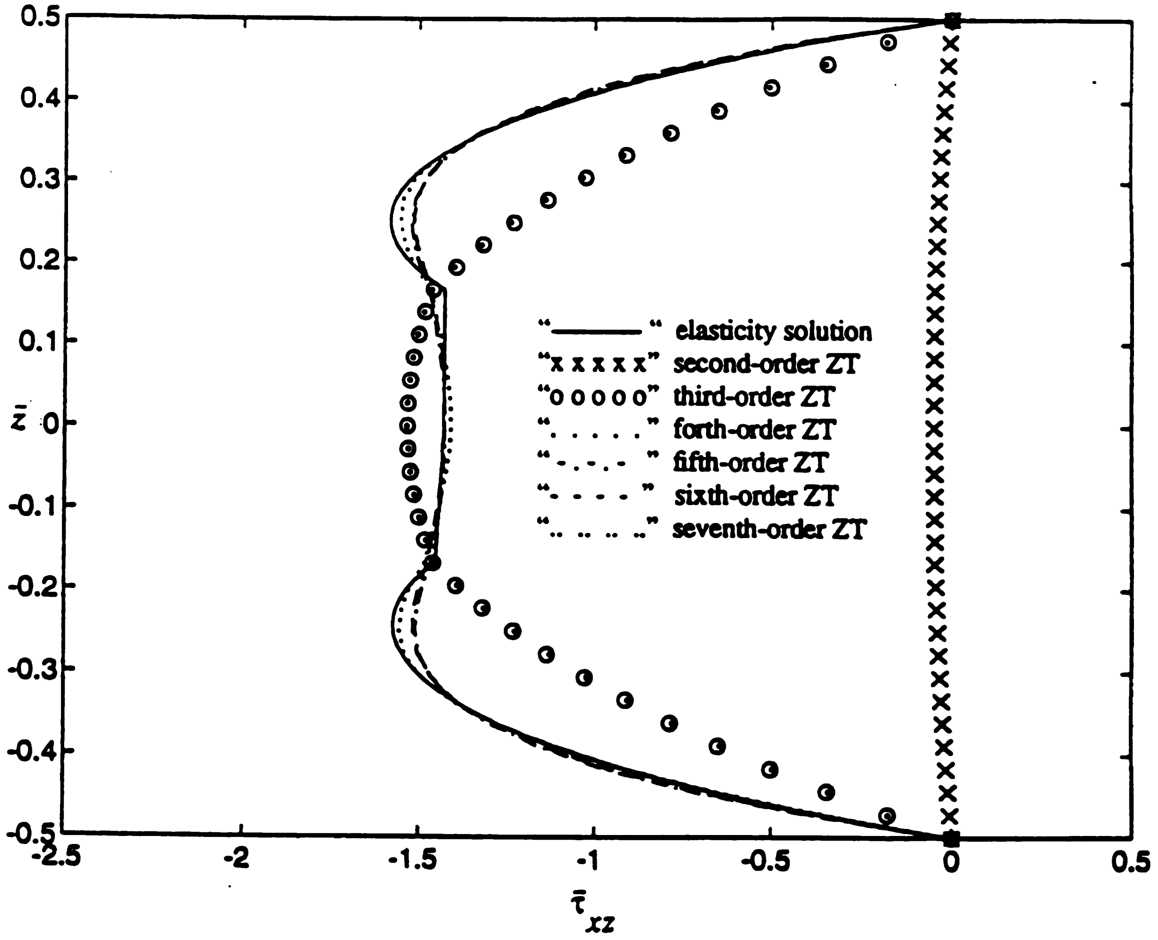

Figure 3.3 - Comparison of τ_{xz} from Zigzag Theories of various orders for a [0/90/0] laminate.

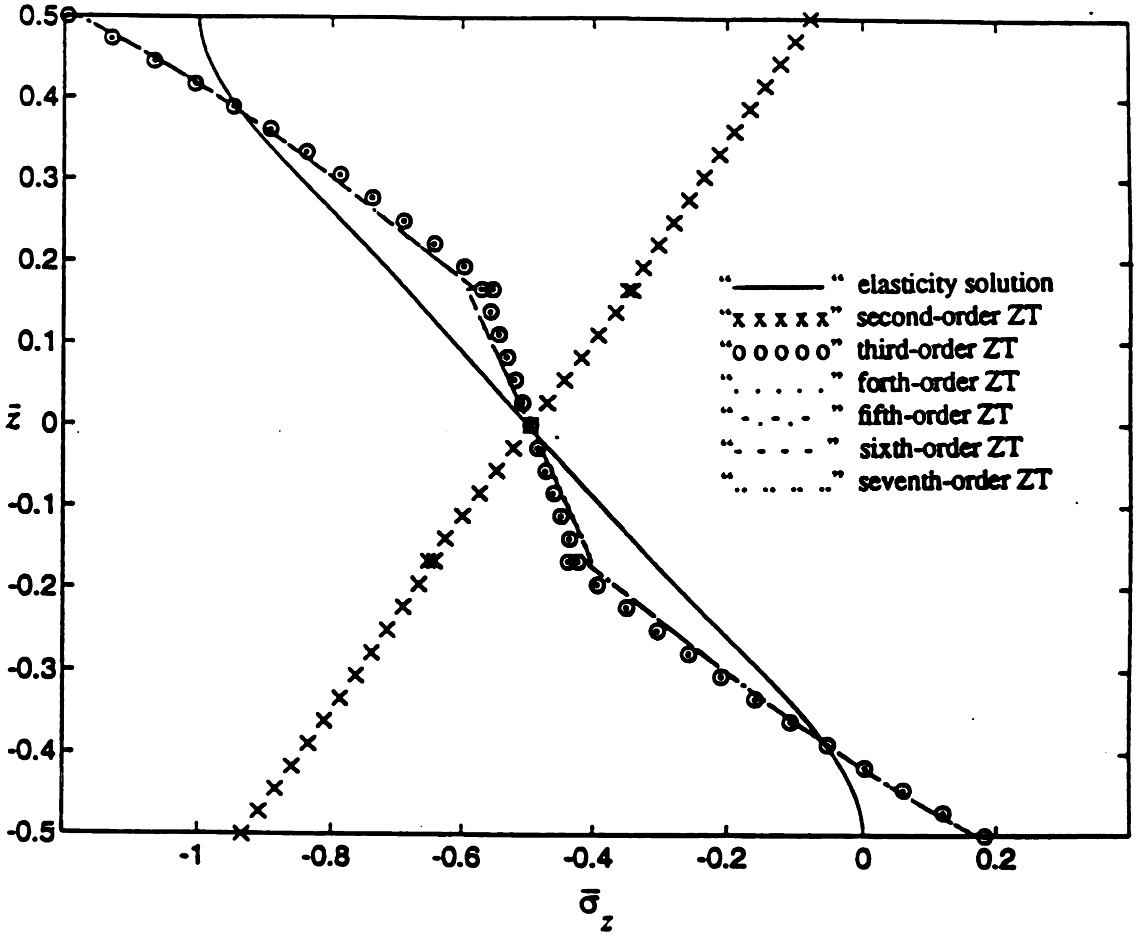

Figure 3.4 - Comparison of σ_z from Zigzag Theories of various orders for a [0/90/0] laminate.

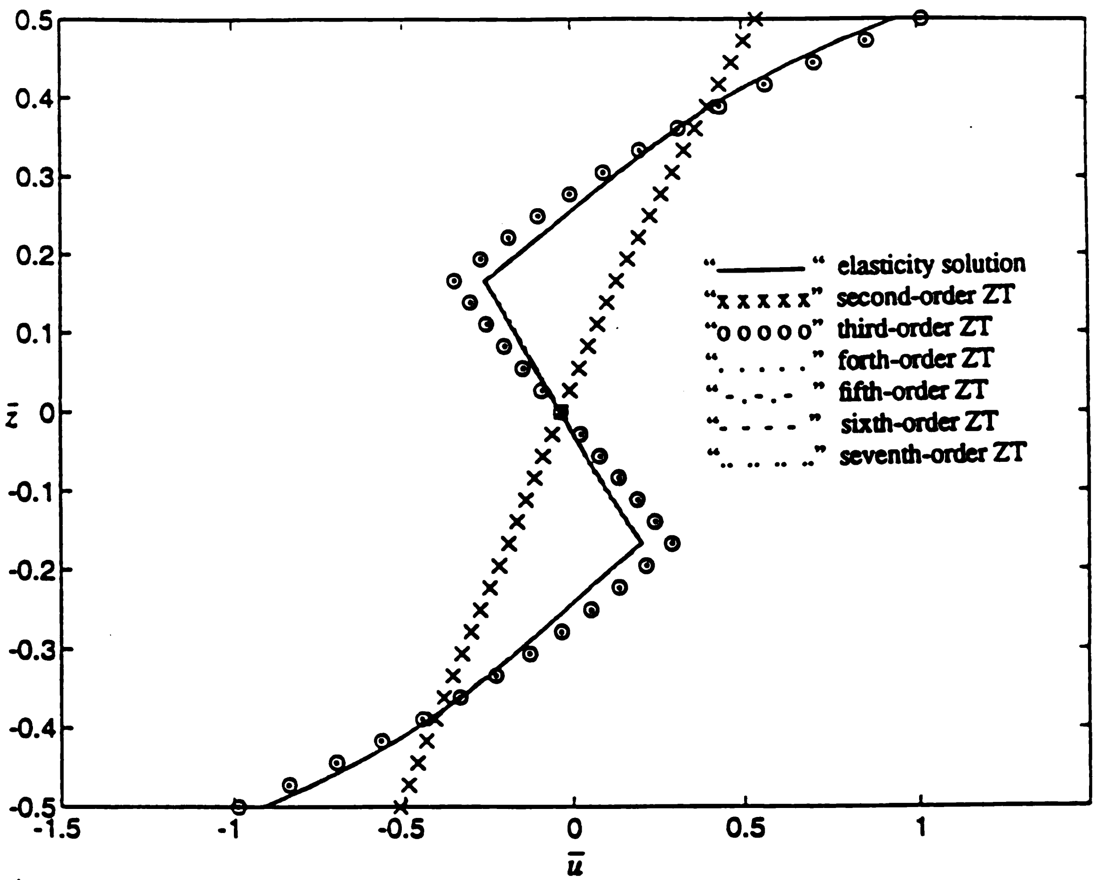

Figure 3.5 - Comparison of $\bar{u}$ from Zigzag Theories of various orders for a [0/90/0] laminate.

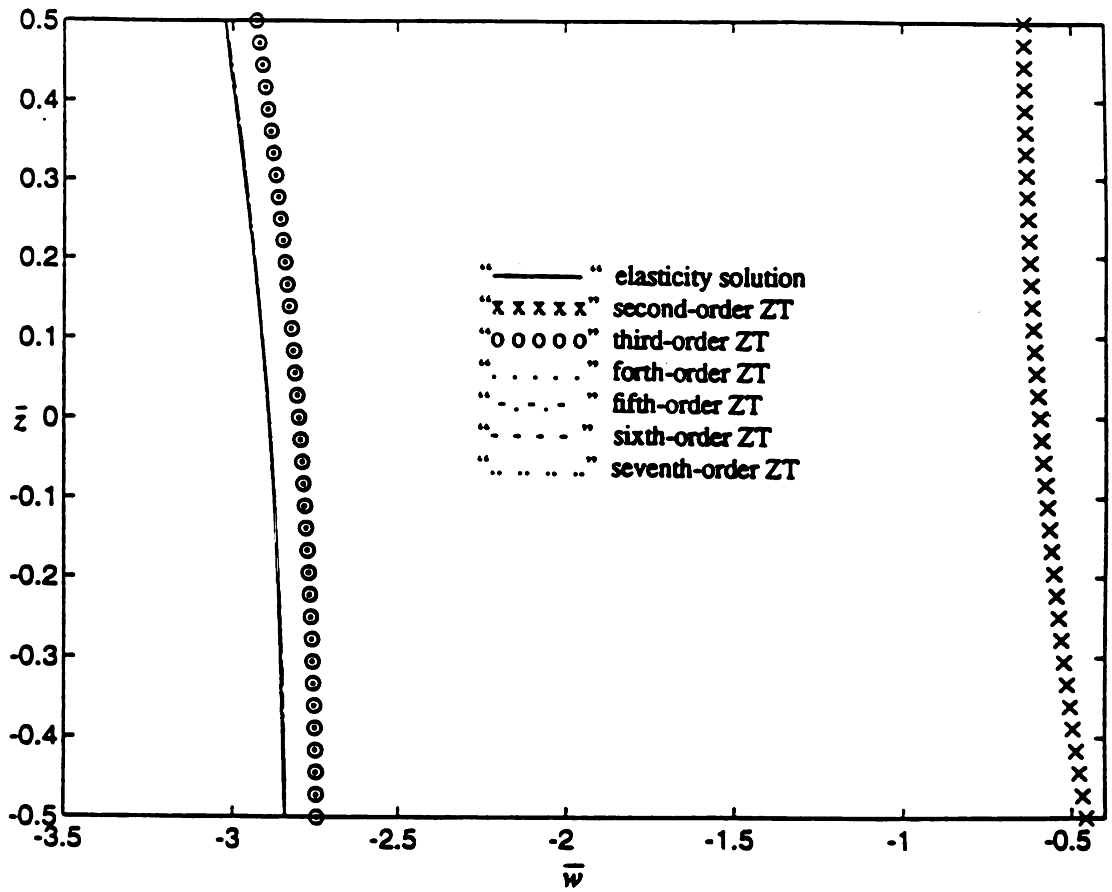

Figure 3.6 - Comparison of $\bar{w}$ from Zigzag Theories of various orders for a [0/90/0] laminate.

B. [0/90] Laminate

The significance of the results from the investigation of a [0/90] laminate is essentially the same as that mentioned in the study of the [0/90/0] laminate. That is, the theories of the fourth-order and above give excellent results for $\bar{\sigma}_x$, $\bar{u}$, and $\bar{w}$. The results for $\bar{\sigma}_z$ are fair since they are small when compared to $\bar{\sigma}_x$. Only $\bar{\tau}_{xz}$, presented in Figure 3.7, shows the distinct difference of the unsymmetrical [0/90] laminate from the symmetric [0/90/0] laminate. It is found that even-order and odd-order terms play equally important roles in the [0/90] laminate while only odd-order terms are critical to the [0/90/0] laminate.

C. Comparisons between GZT and HSDT

In order to further discuss the accuracy of the Generalized Zigzag Theory (GZT), comparisons of the five different theories shown below are made:

(a) a third-order HSDT with constant w,

(b) a third-order HSDT with second-order w,

(c) a third-order GZT with constant w,

(d) a third-order GZT with second-order w, and

(e) a third-order GZT with second-order w plus imposed transverse normal stress boundary conditions.

The results for normalized in-plane stress, transverse shear stress, transverse normal stress, in-plane displacement, and transverse normal displacement are shown in Figures 2.8, 2.9, 2.10, 2.11 and 2.12, respectively.

As shown in Figure 2.8, the cases of HSDT give continuous in-plane stress distribution in the thickness direction owing to the assumed continuous displacement field through the laminate thickness. On the contrary, the cases of GZT give correct discontinuous distribution. The superiority of GZT to HSDT is obvious.

None of the third-order theories give good results for the transverse shear stress depicted in Figure 2.9. However, the results from GZT are continuous while those from HSDT are not. As shown in Figure 3.3, it takes an order higher than three to show the two bulging distributions in the 0^{0} layers.

In Figure 2.10, the theories with second-order w give much better transverse normal stresses than those with a constant w assumption. However, only the case with imposed transverse normal stress boundary conditions gives the correct results at the surfaces although the efforts involved in enforcing the boundary conditions should also be recognized. In addition, it should be pointed out that in the cases of constant w, although the transverse normal strains are zero, the transverse normal stresses are not zero due to Poisson's effect.

Similar to the in-plane stresses, the in-plane displacements from GZT given in Figure 2.11 show excellent agreement with Pagano's solutions. The "zigzag" distribution is very distinct. This is believed to be the most distinguished contribution of the Zigzag Theories. Generally speaking, the results from the GZT are better than those from HSDT in the transverse normal displacement, as can be seen in Figure 2.12.

D. Multi-layered Laminates

As mentioned before, only cross-ply laminates are examined in this study due to the availability of close-form solutions. In addition to the widely investigated [0/90/0] and

[0/90] laminates, it is also interesting to look into the applications of the fifth-order Generalized Zigzag Theory for composite laminates with multiple 0^0-90^0 alternation. In this study, layer numbers of 2, 6, 14, and 30 for unsymmetrical laminates and 3, 7, 15, and 31 for symmetrical laminates are examined. Details of the stacking sequence for the composite laminates are shown below.

symmetric 3-layer: [0/90/0]

symmetric 7-layer: [0/90/0/90/0/90/0]

symmetric 15-layer: $[[0/90/]_7/0]$

symmetric 31-layer: $[[0/90]_{15}/0]$

unsymmetrical 2-layer: [0/90]

unsymmetrical 6-layer: [0/90/0/90/0/90]

unsymmetrical 14-layer: $[[0/90]_7]$

unsymmetrical 30-layer: $[[0/90]_{15}]$

Figures 3.8 and 3.9 are the results of u and τ_{xz} for even layer numbers. The elasticity solutions are also shown along with the numerical predictions for comparison. Apparently, the fifth-order Generalized Zigzag Theory can be used for composite laminates with all ply ranges of 0^0-90^0 alternation as shown in Figure 3.8. However, the predictions of transverse shear stresses are not as good as those of in-plane displacement. It can be concluded from Figure 3.9 that the higher the layer number, the closer the agreement between the elasticity solution and the Generalized Zigzag Theory prediction. It is believed that this is due to the assumption that the Generalized Zigzag Theory initially has layer-dependent terms for the zeroth-order and the first-order. In other words, the second-order term, which is responsible for curvature of the composite laminate, is of global sense. As the layer number increases, the composite laminate of 0^0-90^0 alternation is becoming more homogeneous through the laminate.

3.5 Summary

1. Both HSDT and GZT are layer-number independent theories. However, the former is based on an averaging sense in assembling the composite layers through the laminate thickness, while the latter considers the layer properties individually during the assembly stage.

2. This study gives a generalized theory for composite laminate analysis. It unifies all the available HSDT and ZT, and provides an efficient technique to examine various theories.

3. Zigzag Theories are superior to High-order Shear Deformation Theories in giving excellent in-plane stress and displacement. The zigzag shape of u is accurately predicted. This is where the name "Zigzag" comes from and is a major contribution of the Zigzag Theories to the composite laminate analysis.

4. With the Generalized Zigzag Theory of the fifth-order and above, it is possible to obtain satisfactory transverse shear stresses directly from constitutive equations. Although the distribution of the transverse normal stresses is not close, the discrepancy seems to be acceptable, especially when considering that they are calculated directly from the constitutive equations.

5. The fifth-order Generalized Zigzag Theory can be applied to laminates with a wide range of 0^0-90^0 alternation. It seems that the higher the layer number, the better the prediction as compared to the elasticity solution.

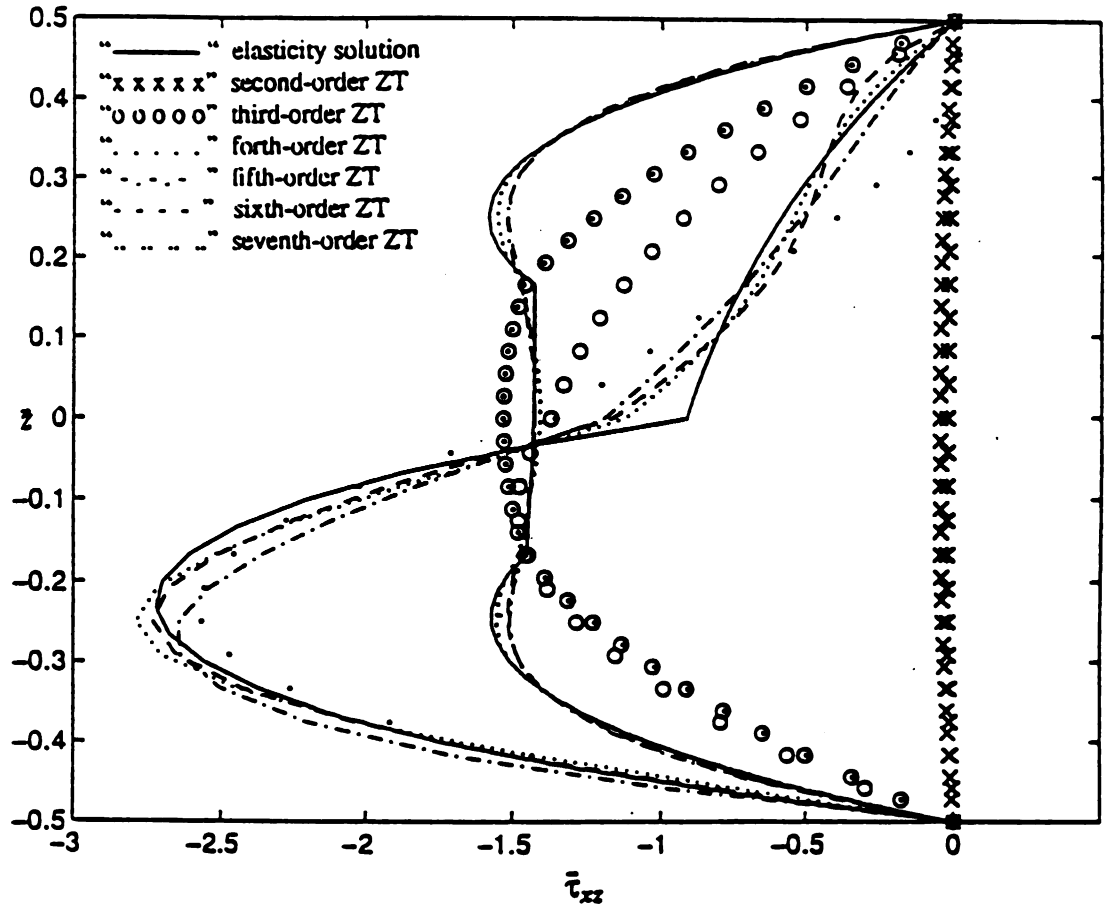

Figure 3.7 - Comparison of τ_{xz} from Generalized Zigzag Theory of various orders for both [0/90/0] and [0/90] laminates.

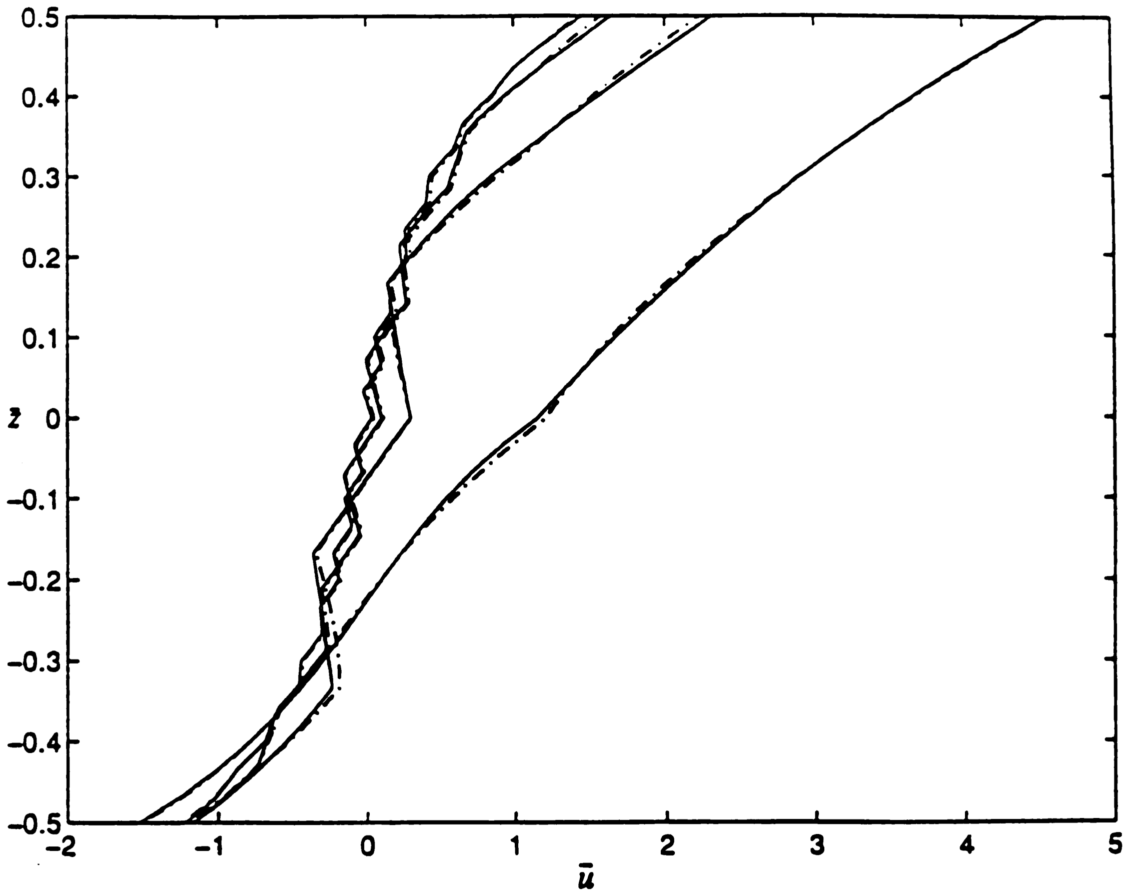

Figure 3.8 - Comparison of $\bar{u}$ from the fifth-order Generalized Zigzag Theory and elasticity solutions for 2-, 6-, 14-, and 30-layer unsymmetrical laminates.

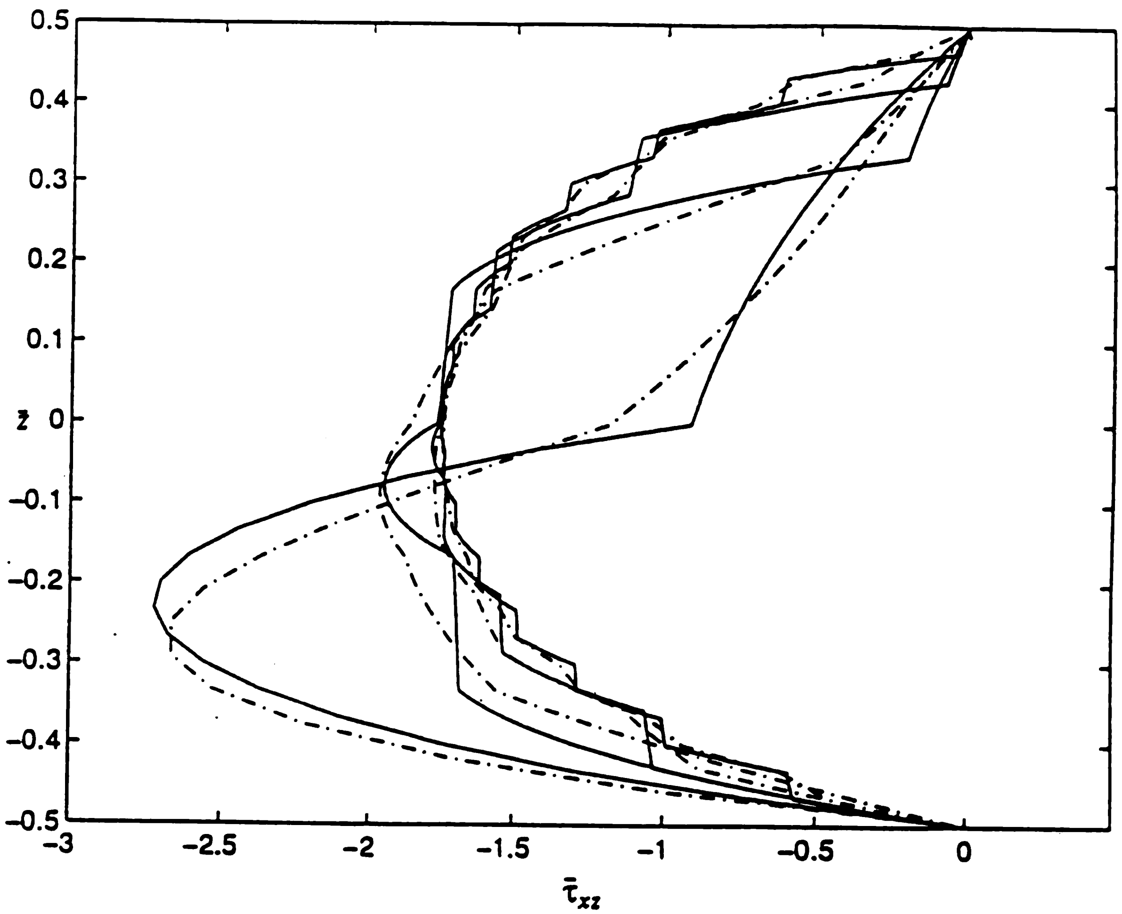

Figure 3.9 - Comparison of τ_{xz} from the fifth-order Generalized Zigzag Theory and elasticity solutions for 2-, 6-, 14-, and 30-layer unsymmetrical laminates.

CHAPTER 4

QUASI-LAYERWISE THEORIES

4.1 Introduction

Compared to the Generalized Layerwise Theory (GLT), Equations (3.2), the Generalized Zigzag Theory (GZT) of Equations (3.3) is just a simplified case of GLT, since the layer-dependent variables are designated to the zeroth-order and the first-order terms only. Other possibilities of simplified cases with two layer-dependent variables are those designated to the zeroth- and the second-order terms, the zeroth- and the third-order terms, the first- and the second-order terms, the first- and the third-order terms, and the second- and the third-order terms. Since initially each of these six theories has two layer-dependent variables, they are of the form of the Layerwise Theories. With the use of continuity conditions, they eventually all become layer-number independent theories. They are thus called Quasi-layerwise Theories. For convenience, these six theories will be named the 0-1, 0-2, 0-3, 1-2, 1-3, and 2-3 Quasi-layerwise Theories, respectively. Since the 0-1 Quasi-layerwise Theory was previously introduced as the Generalized Zigzag Theory in Equations (3.3), the remaining five theories are expressed below:

A. The 0-2 Quasi-layerwise Theory

$$\begin{aligned} u^k(x,y,z) &= u_0^k(x,y) + u_1(x,y)\,z + u_2^k(x,y)\,z^2 + u_3(x,y)\,z^3 \\ v^k(x,y,z) &= v_0^k(x,y) + v_1(x,y)\,z + v_2^k(x,y)\,z^2 + v_3(x,y)\,z^3 \\ w^k(x,y,z) &= w_0(x,y) \end{aligned} \tag{4.1}$$

B. The 0-3 Quasi-layerwise Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0^k(x,y) + u_1(x,y)\,z + u_2(x,y)\,z^2 + u_3^k(x,y)\,z^3 \\
v^k(x,y,z) &= v_0^k(x,y) + v_1(x,y)\,z + v_2(x,y)\,z^2 + v_3^k(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.2}
$$

C. The 1-2 Quasi-layerwise Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1^k(x,y)\,z + u_2^k(x,y)\,z^2 + u_3(x,y)\,z^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1^k(x,y)\,z + v_2^k(x,y)\,z^2 + v_3(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.3}
$$

D. The 1-3 Quasi-layerwise Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1^k(x,y)\,z + u_2(x,y)\,z^2 + u_3^k(x,y)\,z^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1^k(x,y)\,z + v_2(x,y)\,z^2 + v_3^k(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.4}
$$

E. The 2-3 Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1(x,y)\,z + u_2^k(x,y)\,z^2 + u_3^k(x,y)\,z^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1(x,y)\,z + v_2^k(x,y)\,z^2 + v_3^k(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.5}
$$

In a third-order displacement based theory, each of the four components has a distinct physical meaning. The zeroth-order term represents the translational component, while the first-order term represents the rotational component. The second-order term can be viewed as curvature, while the third-order term can be viewed as the third-order derivative of the displacement, or high-order rotation. Accordingly, different order terms play different roles in laminate performance. The selection of different order terms can strongly affect the foundation of a laminate theory. The objective of this chapter is to investigate the feasibility of the individual theories for laminated plate analysis. Similar to Section 3.3, in the present study, a constant w and a fifth-order u^k will be employed for the consideration of both computational efficiency and numerical accuracy. However, the details of formulation will be presented below by taking the 1-3 Quasi-layerwise Theory as an example, since there is a major difference between the 0-1 Quasi-layerwise Theory and the remaining five theories. In fact, a special numerical technique is required in formulating the remaining five Quasi-layerwise Theories.

By applying the continuity conditions, the layer-dependent variables can be reduced to layer-independent variables. However, it will also lead to a set of constraint equations which are generally coupled together. Consequently, implicit, instead of explicit, expressions are obtained, and a huge matrix will be generated in the solution scheme. The summation expression used in Chapter 3 is no longer feasible. In order to overcome this drawback, a recursive technique will be necessary. By virtue of this technique, the coupled equations can be resolved to be the desired explicit formulations.

4.2 Formulation of the 1-3 Quasi-layerwise Theory

Among the five theories given above from Equations (4.1) to (4.5), at least one layer-dependent term is higher than first-order. This implies that the curvature (the second derivative and above) and the high-order rotation may be associated with the layer properties. The analysis of these five theories may give an insight into the roles of individual terms in laminate analysis. However, the high-order layer-dependent terms can also create trouble in computation. For example, the numerical singularity may take place if the global coordinate system coincides with any of the layer interfaces. In laminate analysis, it is a general practice to choose the midplane of the laminate as the x-y plane. In order to avoid the potential numerical singularity, it is proposed to move the coordinate system out of the laminate before the numerical process, and move it back to the midplane position after the process. As mentioned before, the 1-3 Quasi-layerwise Theory will be taken as an example for numerical formulation:

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1^k(x,y)\,z + u_2(x,y)\,z^2 + u_3^k(x,y)\,z^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1^k(x,y)\,z + v_2(x,y)\,z^2 + v_3^k(x,y)\,z^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.6}
$$

A. Continuity Conditions

If all the interfaces of an n-layer composite laminate are perfectly bonded, the following interlaminar displacement continuity conditions should be satisfied:

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\begin{aligned}
u^{k-1}\Big|_{z = z_k} &= u^k\Big|_{z = z_k} \\
v^{k-1}\Big|_{z = z_k} &= v^k\Big|_{z = z_k} \qquad k = 2, 3, \ldots, n
\end{aligned}
\tag{4.7}
$$

By substituting Equations (4.6) into (4.7), the two layer-dependent variables can be correlated as follows:

$$
\begin{aligned}
\left(u_1^k - u_1^{k-1}\right) z_k + \left(u_3^k - u_3^{k-1}\right) z_k^3 &= 0 \\
\left(v_1^k - v_1^{k-1}\right) z_k + \left(v_3^k - v_3^{k-1}\right) z_k^3 &= 0
\end{aligned}
\tag{4.8}
$$

If the coordinate z_k is properly selected to avoid being equal to zero, Equations (4.8) can be rewritten as:

Compare with 3.7

$$
\begin{aligned}
u_3^k &= u_3^{k-1} + \frac{\left(u_1^{k-1} - u_1^k\right)}{z_k^2} \\
v_3^k &= v_3^{k-1} + \frac{\left(v_1^{k-1} - v_1^k\right)}{z_k^2}
\end{aligned}
\tag{4.9}
$$

The interlaminar shear stress continuity conditions have to always be true for laminated composite plates with any bonding conditions, thus:

$$
\begin{aligned}
\tau_{xz}^{k-1}\Big|_{z = z_k} &= \tau_{xz}^k\Big|_{z = z_k} \\
\tau_{yz}^{k-1}\Big|_{z = z_k} &= \tau_{yz}^k\Big|_{z = z_k} \qquad k = 2, 3, 4, \ldots, n
\end{aligned}
\tag{4.10}
$$

By substituting Equations (4.6) into (4.9) and then (4.10), and utilizing Equations (3.4) and (3.5), the following relations can be obtained:

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$$u_1^k = \frac{1}{2}(3-\alpha_k)\,u_1^{k-1} + \frac{3}{2}(1-\alpha_k)\,z_k^2 u_3^{k-1} - \beta_k z_k u_2 - \frac{1}{2}\beta_k w_{o,x}$$
$$v_1^k = \frac{1}{2}(3-\zeta_k)\,v_1^{k-1} + \frac{3}{2}(1-\zeta_k)\,z_k^2 v_3^{k-1} - \eta_k z_k v_2 - \frac{1}{2}\eta_k w_{o,y} \tag{4.11}$$

where,

$$\alpha_k = \frac{Q_{55}^{k-1}}{Q_{55}^k} \qquad \beta_k = \frac{Q_{55}^{k-1} - Q_{55}^k}{Q_{55}^k}$$
$$\zeta_k = \frac{Q_{44}^{k-1}}{Q_{44}^k} \qquad \eta_k = \frac{Q_{44}^{k-1} - Q_{44}^k}{Q_{44}^k}$$

For convenience, rewrite Equations (4.11) in another form:

$$u_1^k = F_1^k u_1 + F_2^k u_3^1 + F_3^k w_{o,x}$$
$$u_3^k = G_1^k u_1 + G_2^k u_3^1 + G_3^k w_{o,x}$$
$$v_1^k = L_1^k v_1 + L_2^k v_3^1 + L_3^k w_{o,y} \tag{4.12}$$
$$v_3^k = M_1^k v_1 + M_2^k v_3^1 + M_3^k w_{o,y}$$

Obviously, $G_1^1 = G_3^1 = 0$, $G_2^1 = 1$, $M_1^1 = M_3^1 = 0$, and $M_2^1 = 1$. The remaining coefficients are to be determined later.

B. Boundary Conditions

In this study, it is assumed that the bottom surface of the n-layer composite laminate also has free shear tractions, i.e.:

$$\tau_{xz}^1(z_1) = 0$$
$$\tau_{yz}^1(z_1) = 0 \tag{4.13}$$

It then results in,

$$u_1^1 + 2z_1 u_2 + 3z_1^2 u_3^1 + w_{o,x} = 0$$
$$v_1^1 + 2z_1 v_2 + 3z_1^2 v_3^1 + w_{o,y} = 0 \tag{4.14}$$

By utilizing Equations (4.14), the following coefficients of Equations (4.12) can be obtained:

$$
\begin{aligned}
F_1^1 &= -2z_1 \qquad & F_2^1 &= -3z_1^2 \qquad & F_3^1 &= -1 \\
L_1^1 &= -2z_1 \qquad & L_2^1 &= -3z_1^2 \qquad & L_3^1 &= -1
\end{aligned}
\tag{4.15}
$$

It is of great advantage to express the higher-order coefficients by the following recursive equations:

$$
\begin{aligned}
F_1^k &= \frac{1}{2}(3-\alpha_k)F_1^{k-1} + \frac{3}{2}(1-\alpha_k)\,z_k^2 G_1^{k-1} - \beta_k z_k \\
F_2^k &= \frac{1}{2}(3-\alpha_k)F_2^{k-1} + \frac{3}{2}(1-\alpha_k)\,z_k^2 G_2^{k-1} \\
F_3^k &= \frac{1}{2}(3-\alpha_k)F_3^{k-1} + \frac{3}{2}(1-\alpha_k)\,z_k^2 G_3^{k-1} - \frac{1}{2}\beta_k \\
G_1^k &= G_1^{k-1} + \left(F_1^{k-1} - F_1^k\right)\left(\frac{1}{z_k^2}\right) \\
G_2^k &= G_2^{k-1} + \left(F_2^{k-1} - F_2^k\right)\left(\frac{1}{z_k^2}\right) \\
G_3^k &= G_3^{k-1} + \left(F_3^{k-1} - F_3^k\right)\left(\frac{1}{z_k^2}\right) \\
L_1^k &= \frac{1}{2}(3-\zeta_k)L_1^{k-1} + \frac{3}{2}(1-\zeta_k)\,z_k^2 M_1^{k-1} - \eta_k z_k \\
L_2^k &= \frac{1}{2}(3-\zeta_k)L_2^{k-1} + \frac{3}{2}(1-\zeta_k)\,z_k^2 M_2^{k-1} \\
L_3^k &= \frac{1}{2}(3-\zeta_k)L_3^{k-1} + \frac{3}{2}(1-\zeta_k)\,z_k^2 M_3^{k-1} - \frac{1}{2}\eta_k \\
M_1^k &= M_1^{k-1} + \left(L_1^{k-1} - L_1^k\right)\left(\frac{1}{z_k^2}\right) \\
M_2^k &= M_2^{k-1} + \left(L_2^{k-1} - L_2^k\right)\left(\frac{1}{z_k^2}\right) \\
M_3^k &= M_3^{k-1} + \left(L_3^{k-1} - L_3^k\right)\left(\frac{1}{z_k^2}\right)
\end{aligned}
$$

where, $k = 2, 3, 4, \ldots, n$

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The top surface of the composite laminate is also assumed to be free of shear traction, i.e. $\tau_{xz}^{n}(z_{n+1}) = 0$ and $\tau_{yz}^{n}(z_{n+1}) = 0$. It is necessary to satisfy the following equations:

$$u_1^n + 2z_{n+1}u_2 + 3z_{n+1}^2 u_3^n + w_{o,x} = 0 \qquad v_1^n + 2z_{n+1}v_2 + 3z_{n+1}^2 v_3^n + w_{o,y} = 0 \tag{4.17}$$

By solving Equations (4.17), u_3^1 and v_3^1 can be expressed as:

$$u_3^1 = A_1 u_2 + B_1 w_{o,x} \qquad v_3^1 = A_2 v_2 + B_2 w_{o,y} \tag{4.18}$$

where,

$$A_1 = -\frac{F_1^n + 3z_{n+1}^2 G_1^n + 2z_{n+1}}{\Delta_1} \qquad B_1 = -\frac{F_3^n + 3z_{n+1}^2 G_3^n + 1}{\Delta_1}$$

$$A_2 = -\frac{L_1^n + 3z_{n+1}^2 M_1^n + 2z_{n+1}}{\Delta_2} \qquad B_2 = -\frac{L_3^n + 3z_{n+1}^2 M_3^n + 1}{\Delta_2}$$

$$with \quad \Delta_1 = F_2^n + 3z_{n+1}^2 G_2^n \qquad \Delta_2 = L_2^n + 3z_{n+1}^2 M_2^n$$

Substituting Equations (4.18) into Equations (4.12) to further simplify the expressions, it yields:

$$\begin{aligned} u_1^k &= R_1^k u_2 + R_2^k w_{o,x} \qquad & u_3^k &= S_1^k u_2 + S_2^k w_{o,x} \\ v_1^k &= O_1^k v_2 + O_2^k w_{o,y} \qquad & v_3^k &= P_1^k v_2 + P_2^k w_{o,y} \end{aligned} \tag{4.19}$$

where,

$$\begin{aligned} R_1^k &= F_1^k + A_1 F_2^k \qquad & S_1^k &= G_1^k + A_1 G_2^k \\ R_2^k &= F_3^k + B_1 F_2^k \qquad & S_2^k &= G_3^k + B_1 G_2^k \\ O_1^k &= L_1^k + A_2 L_2^k \qquad & P_1^k &= M_1^k + A_2 M_2^k \\ O_2^k &= L_3^k + B_2 L_2^k \qquad & P_2^k &= M_3^k + B_2 M_2^k \end{aligned}$$

Utilizing Equations (4.19) for Equations (4.6), a displacement field which is free of layerdepe

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dependent variables, but dependent on layer properties can be concluded:

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + \Phi_1^k(z)\,u_2 + \Phi_2^k(z)\,w_{o,x} \\
v^k(x,y,z) &= v_0(x,y) + \Psi_1^k(z)\,v_2 + \Psi_2^k(z)\,w_{o,y} \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{4.20}
$$

where,

$$
\Phi_1^k(z) = R_1^k z + z^2 + S_1^k z^3 \qquad \Phi_2^k(z) = R_2^k z + S_2^k z^3
$$

$$
\Psi_1^k(z) = O_1^k z + z^2 + P_1^k z^3 \qquad \Psi_2^k(z) = O_2^k z + P_2^k z^3
$$

For numerical analysis, Equations (4.20) can be substituted into the variational equation, Equation (3.21), with the matrices defined as follows:

$$
\begin{aligned}
\{X\} &= \{u_o \;\; v_o \;\; u_2 \;\; v_2 \;\; w_o\}^T \\
\left[N_x^k\right] &= \left[\frac{\partial}{\partial x} \quad 0 \quad \Phi_1^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_2^k\frac{\partial^2}{\partial x^2}\right] \\
\left[N_y^k\right] &= \left[0 \quad \frac{\partial}{\partial y} \quad 0 \quad \Psi_1^k\frac{\partial}{\partial y} \quad \Psi_2^k\frac{\partial^2}{\partial y^2}\right] \\
\left[N_z^k\right] &= \left[0 \quad 0 \quad 0 \quad 0 \quad 0\right] \\
\left[N_{xy}^k\right] &= \left[\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x} \quad \Phi_1^k\frac{\partial}{\partial y} \quad \Psi_1^k\frac{\partial}{\partial x} \quad (\Phi_2^k + \Psi_2^k)\frac{\partial^2}{\partial x \partial y}\right] \\
\left[N_{xz}^k\right] &= \left[0 \quad 0 \quad \frac{d}{dz}\Phi_1^k \quad 0 \quad \left(\frac{d}{dz}\Phi_2^k + 1\right)\frac{\partial}{\partial x}\right] \\
\left[N_{yz}^k\right] &= \left[0 \quad 0 \quad 0 \quad \frac{d}{dz}\Psi_1^k \quad \left(\frac{d}{dz}\Psi_2^k + 1\right)\frac{\partial}{\partial y}\right] \\
\left[N_w^+\right] &= \left[0 \quad 0 \quad 0 \quad 0 \quad 1\right] \\
\left[N_w^-\right] &= \left[0 \quad 0 \quad 0 \quad 0 \quad 1\right]
\end{aligned}
\tag{4.21}
$$

Close-form solutions based on the 1-3 Theory can be obtained by a similar manner as expressed in Section 3.2.

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4.3 Numerical Results and Discussion

As mentioned before, the similar plane-strain problem is examined. In applying the Quasi-layerwise Theories to the composite analysis, it is noted that the formulation is strongly dependent on the thickness coordinate, except for the 0-1 Quasi-layerwise Theory, i.e. the Generalized Zigzag Theory. As expressed in Equations (4.16), G_i^k are functions of $\frac{1}{z_k^2}$. Apparently, there will be numerical singularity if any z_k equals zero. In order to avoid the singularity problem, the thickness coordinate is shifted from the mid-plane of the composite laminate. However, it is found that the numerical results are dependent on the location of the thickness coordinate. In other words, with different z_k, different displacement and stress values will be obtained. Figures 4.1 to 4.5 show the results of the 1-3 Quasi-layerwise Theory based on various coordinate shifts (along the z direction). The shift ratio is defined as the shift of the z-coordinate to the thickness of the composite laminate. As can be seen, the results are unreasonably dependent on the thickness coordinate. Accordingly, only the results from the 0-1 Quasi-layerwise Theory are true values since they are not dependent on the coordinate system. These results are shown in Figures 4.6 to 4.10 for $\bar{\sigma}_x$, $\bar{\tau}_{xz}$, $\bar{\sigma}_z$, $\bar{u}$, and $\bar{w}$, with the results from the remaining five Quasi-layerwise Theories, which are adjusted to match with those of the 0-1 Theory as closely as possible through the manipulation of shifting the thickness coordinates. The shifting ratios are 1.2, 3.0, 1.6, 2.0, and 2.5, for the 0-2, 0-3, 1-2, 1-3, and 2-3 Quasi-layerwise Theories, respectively. Although all six theories can be very close to one another, the fatal deficiency of coordinate dependence of the 0-2, 0-3, 1-2, 1-3, and 2-3 Quasi-layerwise Theories is recognized.

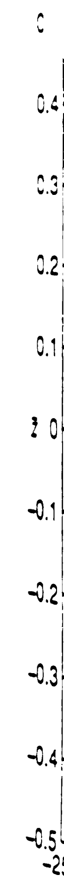

Figure 4

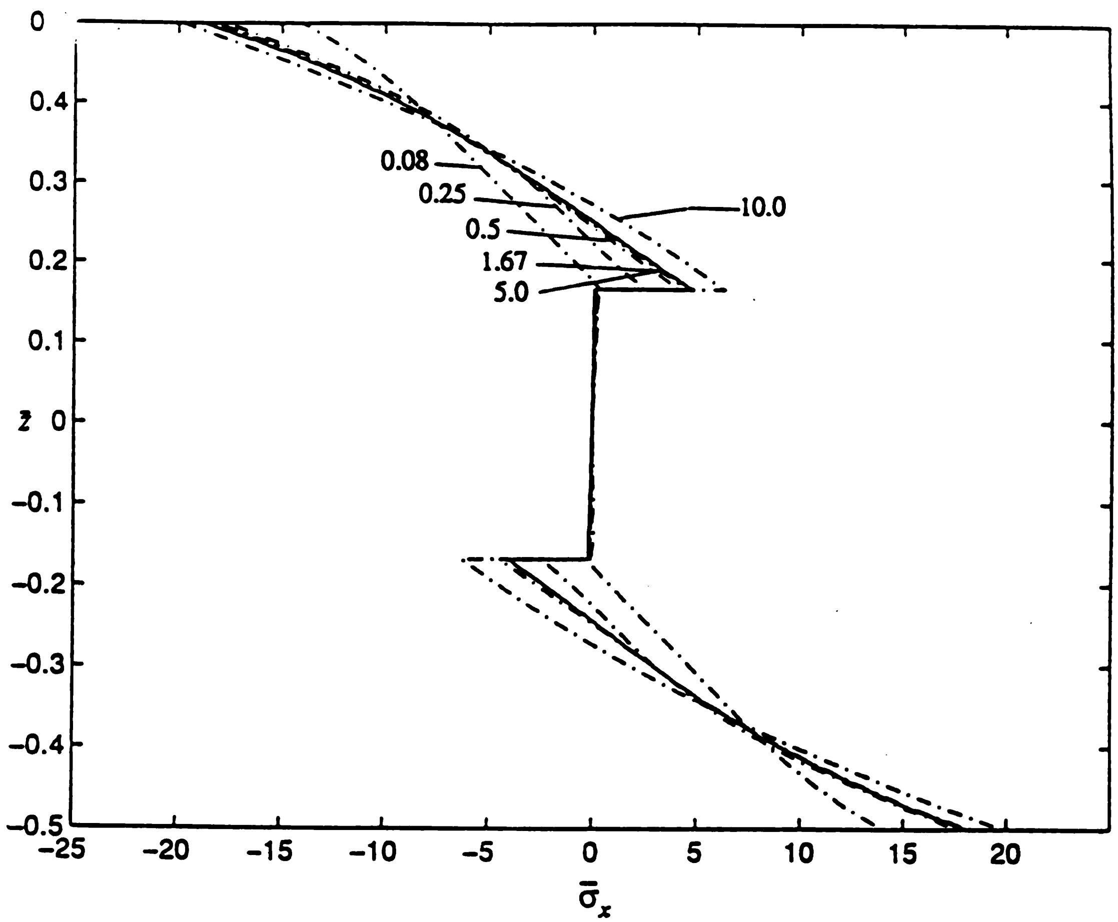

Figure 4.1 - Variation of $\bar{\sigma}_x$ from the 1-3 Quasi-layerwise Theory due to various shift ratios in a [0/90/0] laminate.

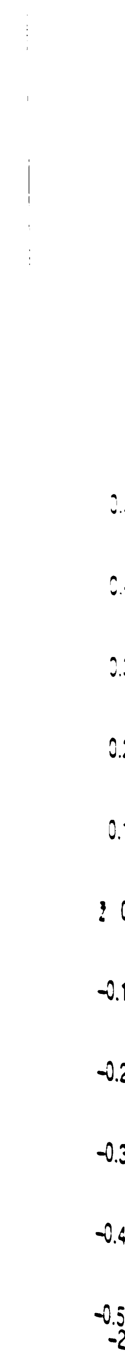

Figure 4

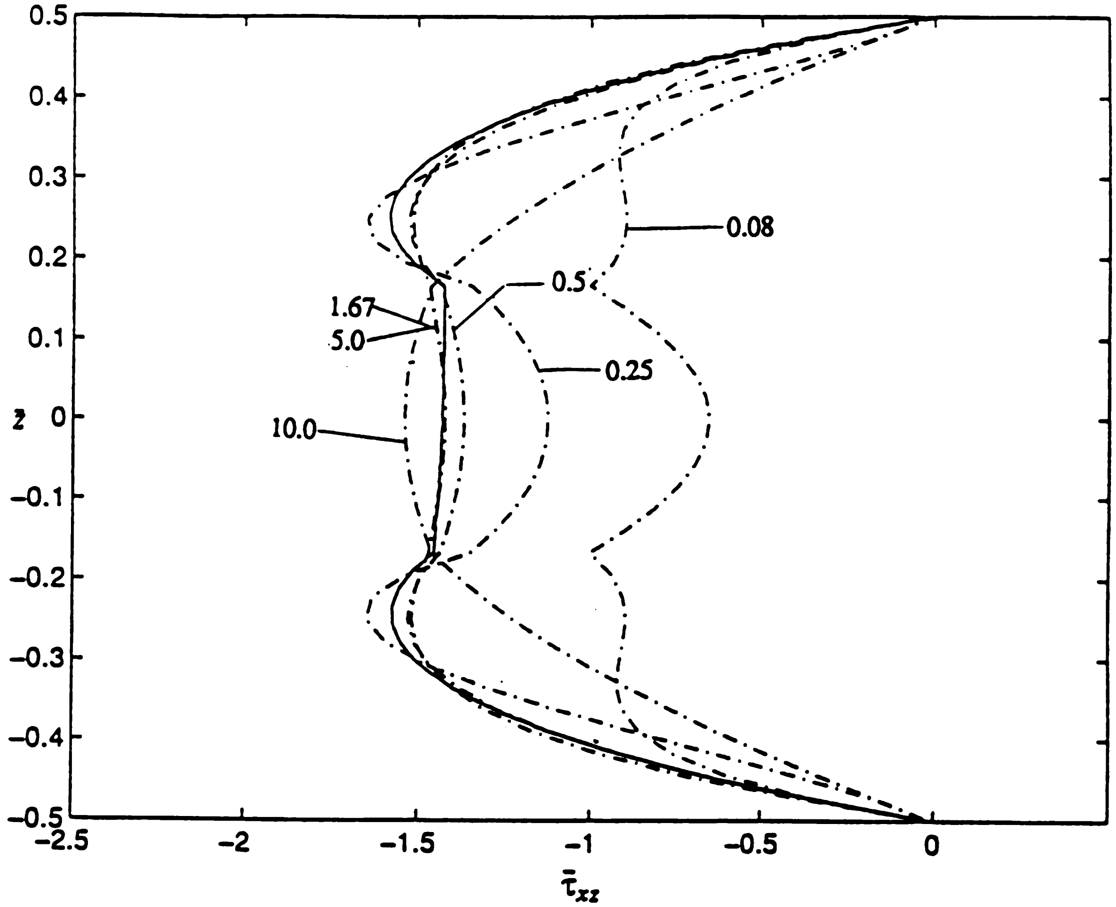

Figure 4.2 - Variation of τ_{xz} from the 1-3 Quasi-layerwise Theory due to various shifts ratios in a [0/90/0] laminate.

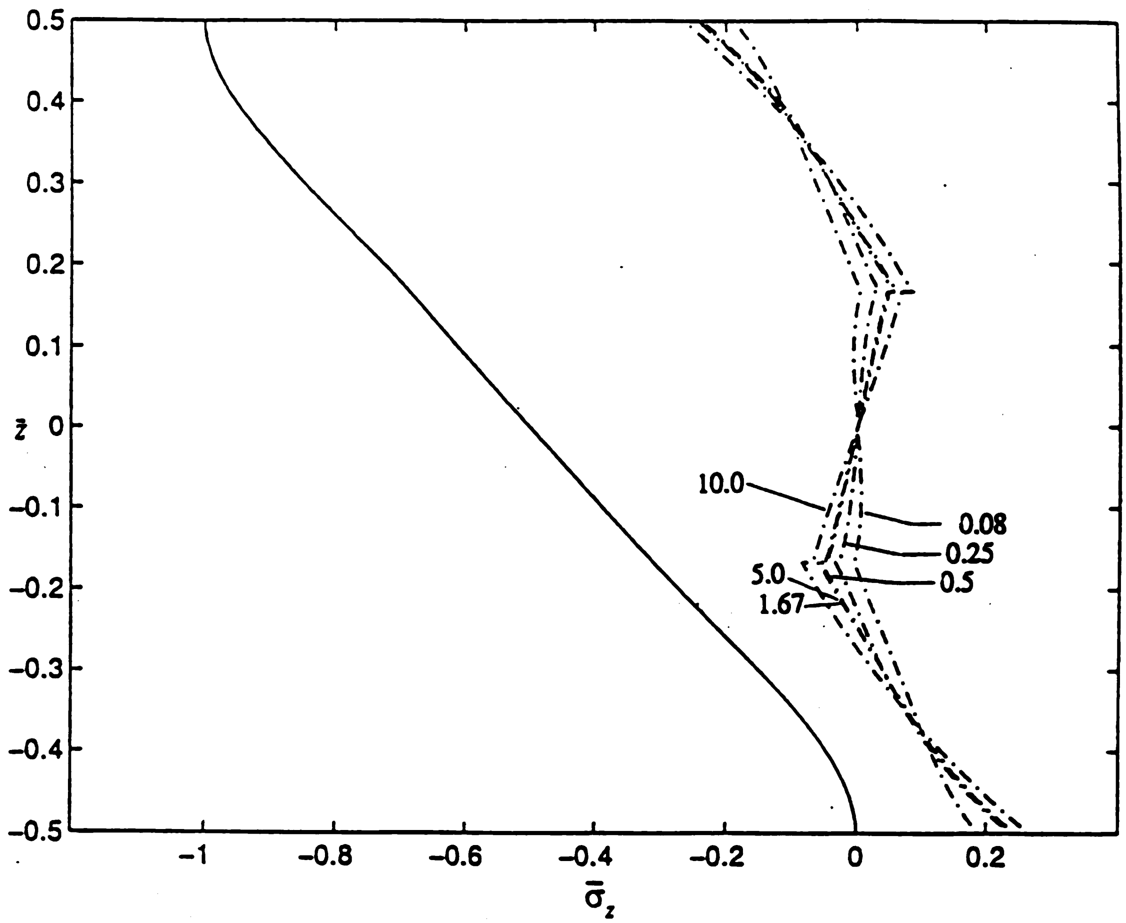

Figure 4.3 - Variation of σ_z from the 1-3 Quasi-layerwise Theory due to various shifts ratios in a [0/90/0] laminate.

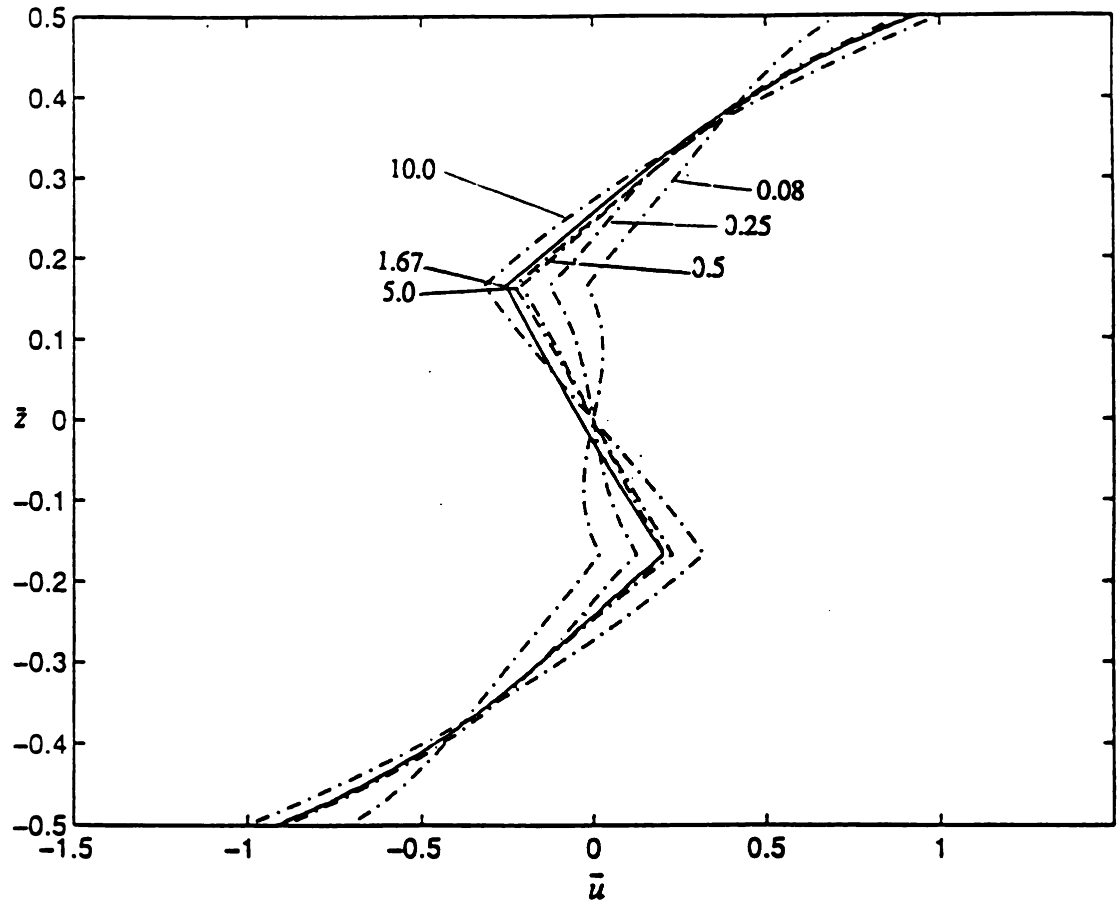

Figure 4.4 - Variation of $\bar{u}$ from the 1-3 Quasi-layerwise Theory due to various shifts ratios in a [0/90/0] laminate.

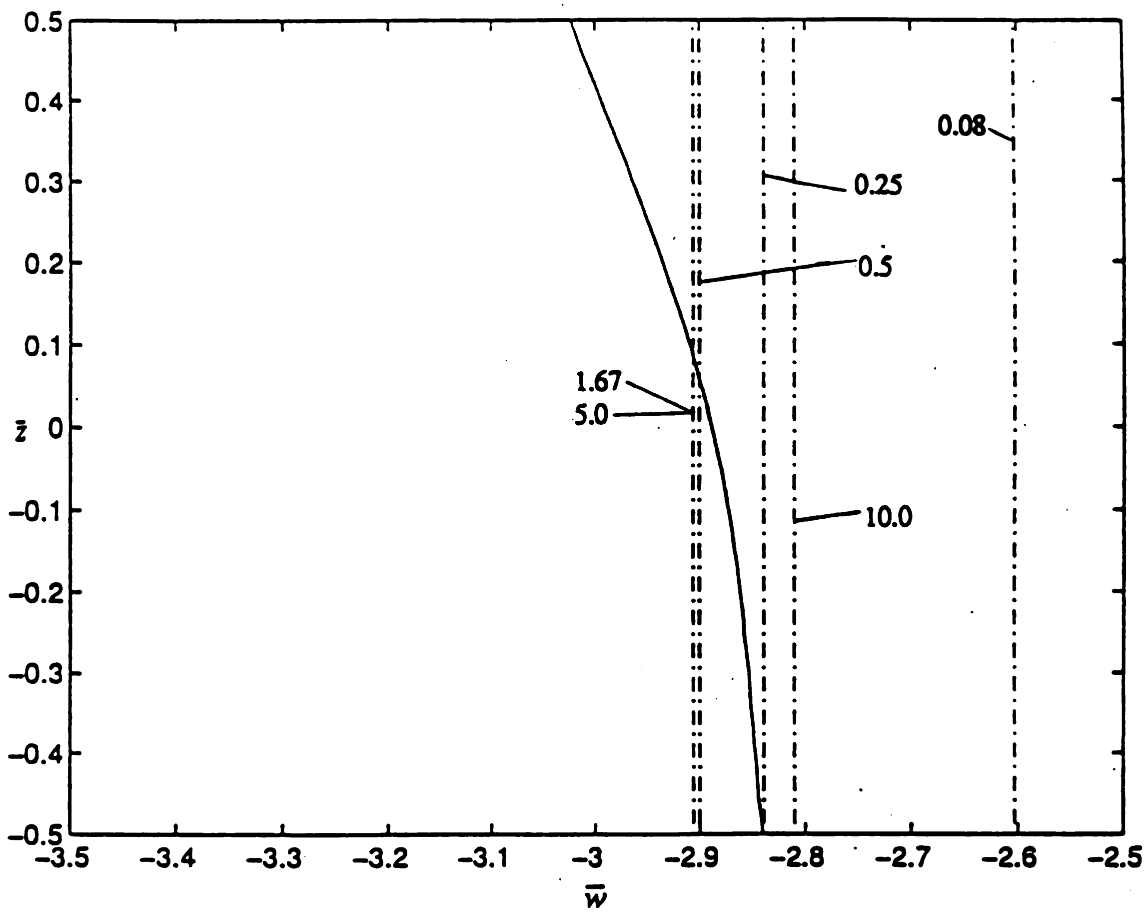

Figure 4.5 - Variation of $\bar{w}$ from the 1-3 Quasi-layerwise Theory due to various shifts ratios in a [0/90/0] laminate.

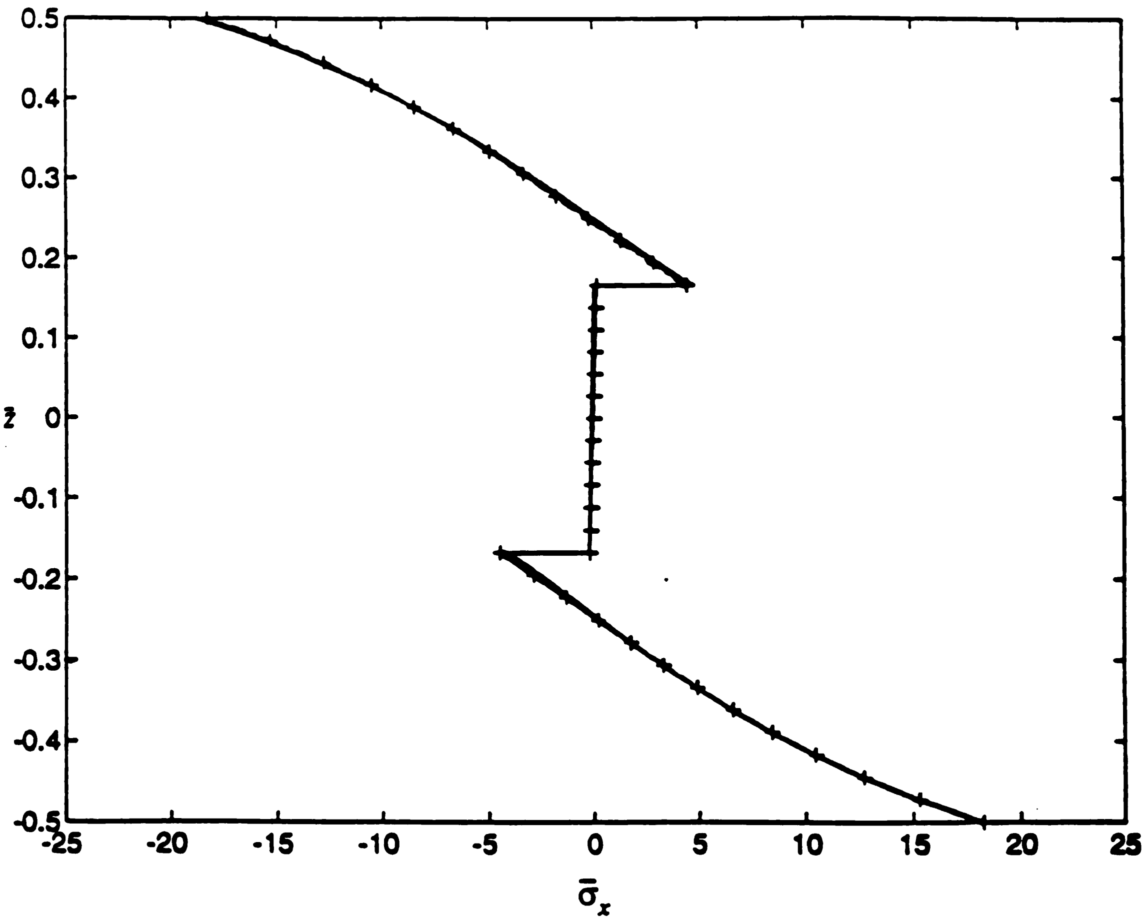

Figure 4.6 - Results of $\bar{\sigma}_x$ for a [0/90/0] laminate from various Quasi-layerwise Theories obtained by adjusting the thickness coordinate.

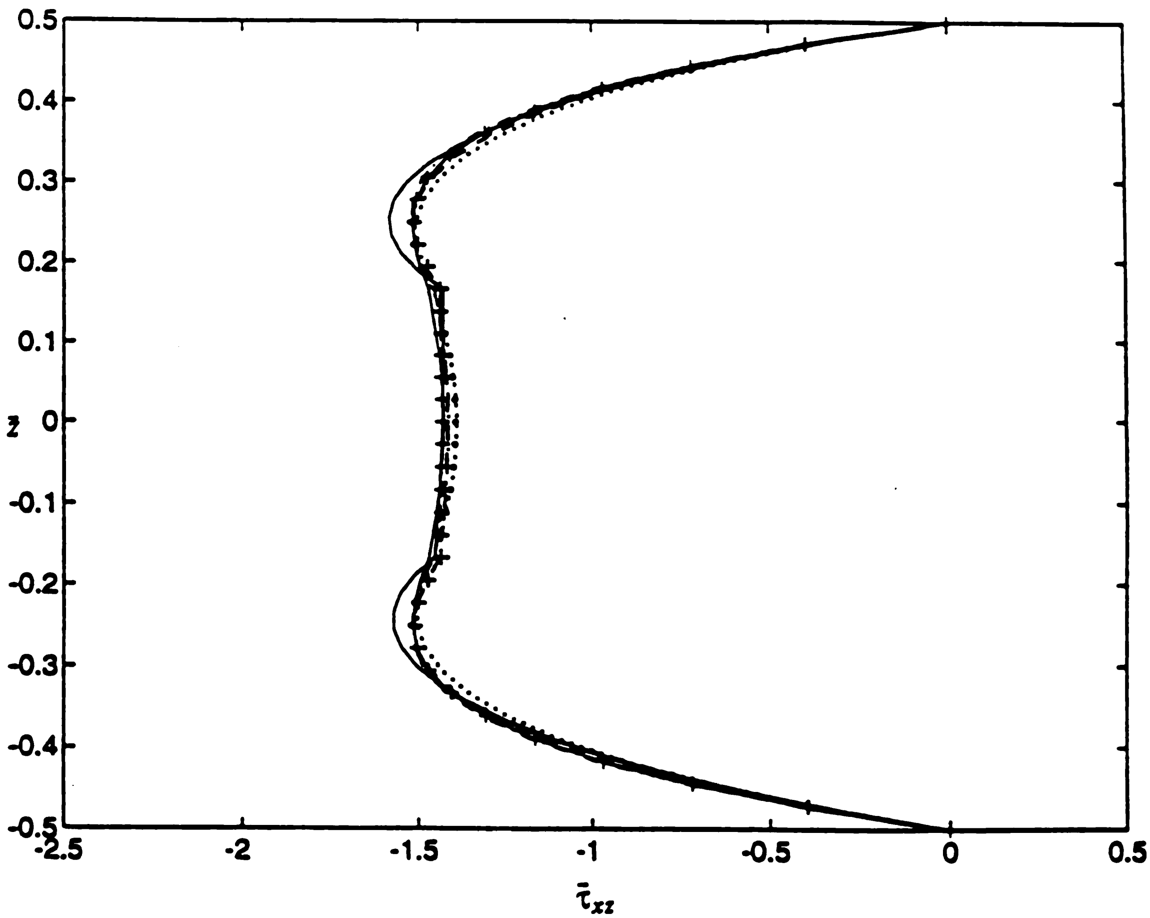

Figure 4.7 - Results of τ_{xz} for a [0/90/0] laminate from various Quasi-layerwise Theories obtained by adjusting the thickness coordinate.

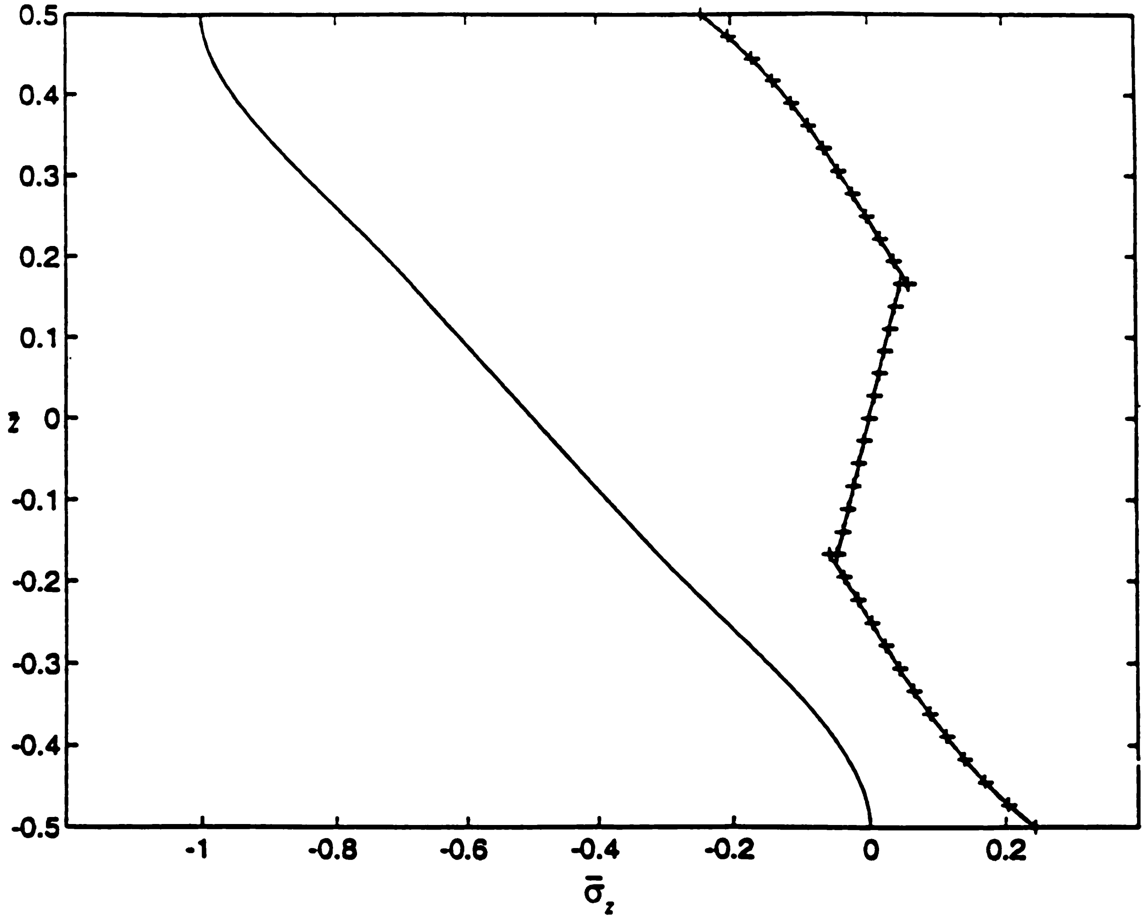

Figure 4.8 - Results of $\bar{\sigma}_z$ for a [0/90/0] laminate from various Quasi-layerwise Theories obtained by adjusting the thickness coordinate.

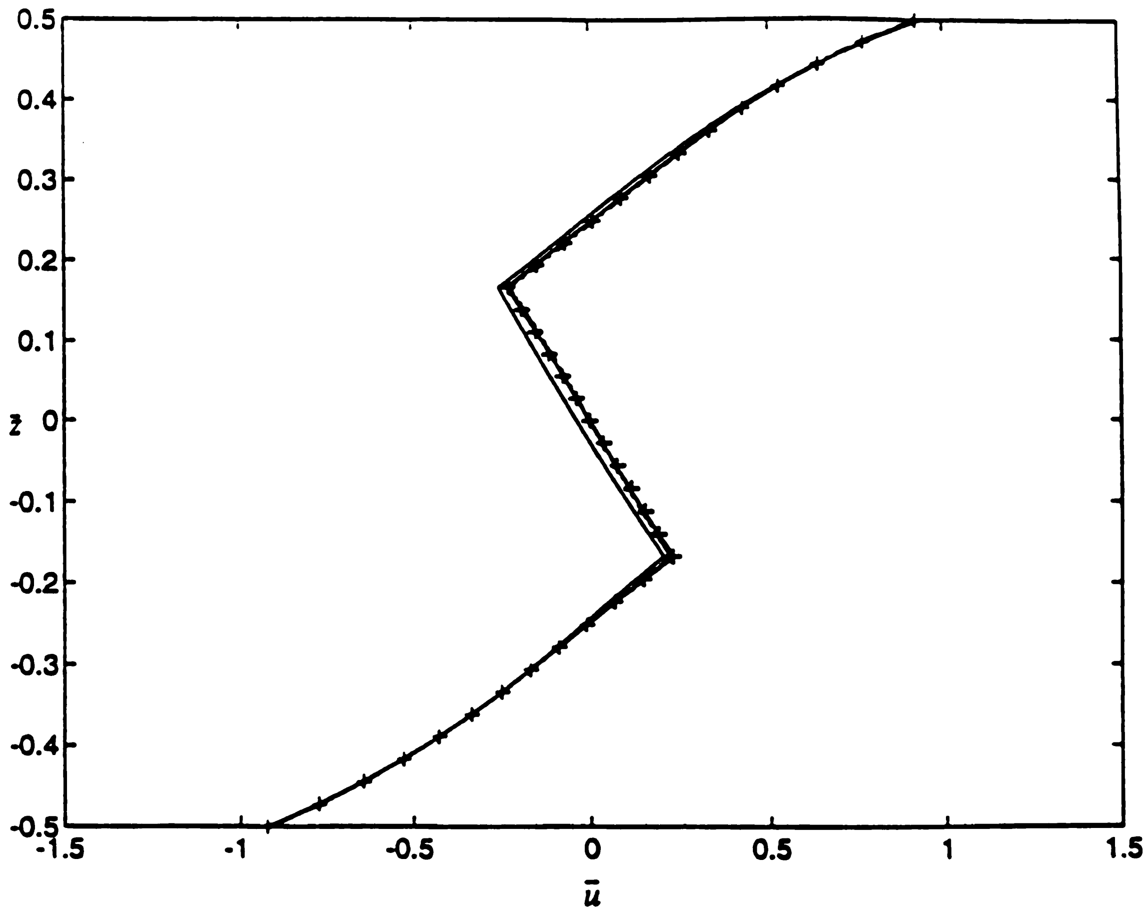

Figure 4.9 - Results of $\bar{u}$ for a [0/90/0] laminate from various Quasi-layerwise Theories obtained by adjusting the thickness coordinate.

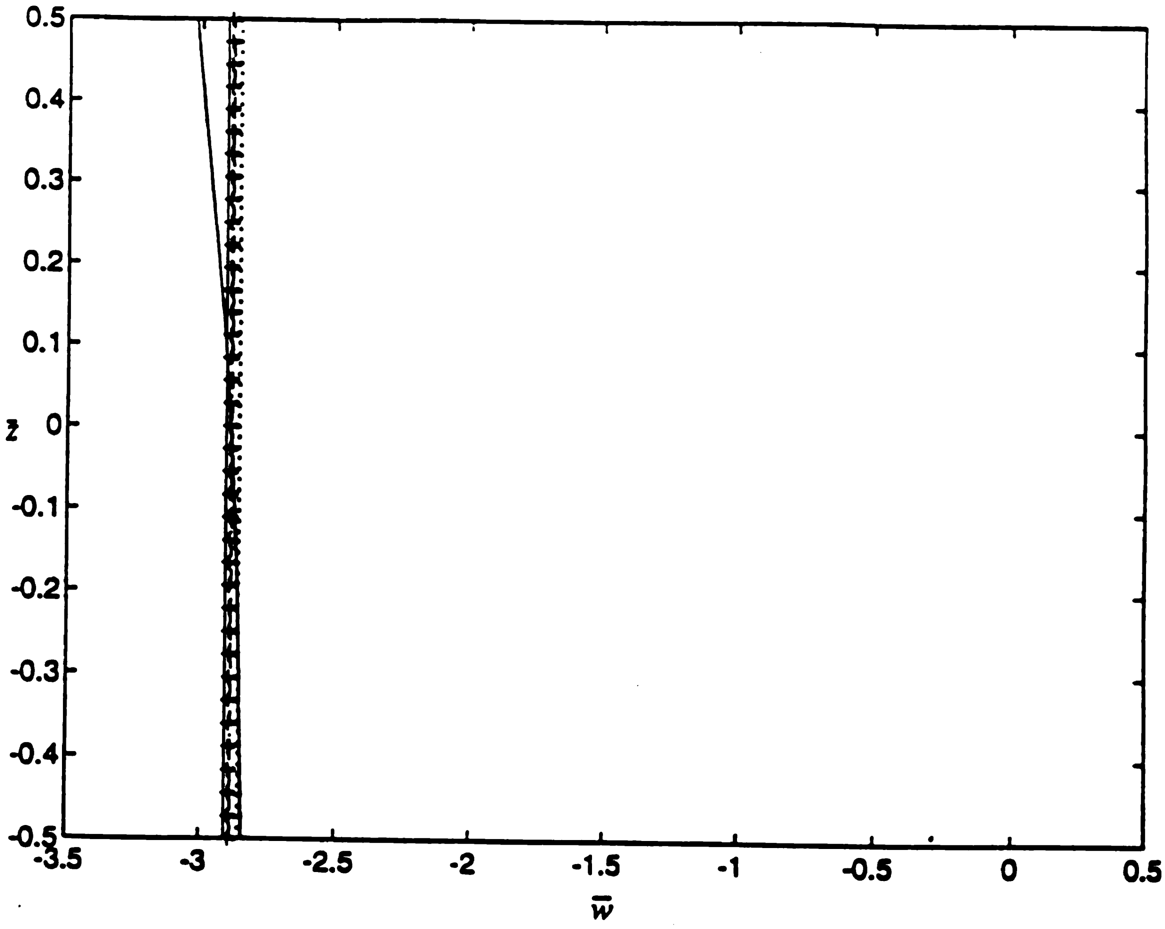

Figure 4.10 - Results of $\bar{w}$ for a [0/90/0] laminate from various Quasi-layerwise Theories obtained by adjusting the thickness coordinate.

4.4 Summary

Of the six Quasi-layerwise Theories analyzed, only the 0-1 Quasi-layerwise Theory, i.e. the Generalized Zigzag Theory, is correct. The other five theories are overly sensitive to variations in the coordinate system. In other words, their results are dependent on the thickness coordinate. This coordinate dependency is not reasonable and not acceptable.

CHAPTER 5

GLOBAL-LOCAL SUPERPOSITION THEORIES

5.1 Introduction

The Quasi-layerwise Theories are superior to the Shear Deformation Theories in numerical accuracy if the thickness coordinate is properly selected. They are also superior to the Layerwise Theories in computational efficiency. However, they suffer from a serious defect — coordinate dependency. Only the 0-1 Quasi-layerwise Theory, i.e. the Generalized Zigzag Theory, is not coordinate dependent. The remaining five theories are sensitive to the thickness coordinate. If the coordinate system is not properly selected, singularity can take place. Although it is possible to extract reasonable results from manipulating the thickness coordinate for the aforementioned Pagano's problem, the procedure does not seem to be practical for general applications. As a consequence, it is necessary to develop a new technique to avoid this numerical dilemma.

Based on a global displacement field, the Shear Deformation Theories give reasonable results for the overall performance of composite laminates, such as in-plane deformation and stress. On the contrary, due to their assumptions for individual layers, the Layerwise Theories are excellent for local behaviors such as interlaminar stresses. The Quasi-layerwise Theories of third-order, e.g. the 0-1 Quasi-layerwise Theory, have two terms for layer independency and two terms for layer dependency. In fact, they are mixtures of the third-order Shear Deformation Theory and the first-order Generalized Layerwise Theory. As a consequence, these theories can be clearly divided into a global component of shear deformation type and a local component of layerwise type, and assembled by a superposition principle.

Taking the 0-1 Quasi-layerwise theory as an example, it can be rewritten as:

$$
\begin{aligned}
u^k(x, y, z) &= \underset{\sim}{u}(x, y, z) + \underline{u}^k(x, y, z) \\
v^k(x, y, z) &= \underset{\sim}{v}(x, y, z) + \underline{v}^k(x, y, z) \\
w^k(x, y, z) &= \underset{\sim}{w}(x, y, z)
\end{aligned}
\tag{5.1}
$$

where,

$$
\begin{aligned}
\underset{\sim}{u}(x, y, z) &= u_2(x, y)\, z^2 + u_3(x, y)\, z^3 \\
\underset{\sim}{v}(x, y, z) &= v_2(x, y)\, z^2 + v_3(x, y)\, z^3 \\
\underset{\sim}{w}(x, y, z) &= w_0(x, y)
\end{aligned}
\tag{5.2}
$$

are global components of shear deformation type, and

$$
\begin{aligned}
\underline{u}^k(x, y, z) &= u_0^k(x, y) + u_1^k(x, y)\, z \\
\underline{v}^k(x, y, z) &= v_0^k(x, y) + v_1^k(x, y)\, z
\end{aligned}
\tag{5.3}
$$

are local components of layerwise type.

5.2 Global-Local Superposition Technique

By utilizing the technique of superposition for the global and local displacement fields, a wide range of new laminate theories can be proposed. However, since only four continuity conditions (two displacement components and two interlaminar shear stresses on each laminate interface) need to be satisfied, only four layer-dependent terms (two for u^k and two for v^k) are allowed in a laminate theory if layer-number independency is a primary concern. In addition, by superimposing the global and local components through the composite laminate, two coordinate systems are required. For the global description, z is used, while for the local description, a linear coordinate ξ_k for the k^{th} layer is assumed. As a consequence, the 0-1 Quasi-layerwise Theory can be rewritten as the 0-1 Superposition Theory:

$$
\begin{aligned}
u^k(x, y, z) &= u_0(x, y) + u_1(x, y)\, z + u_2(x, y)\, z^2 + u_3(x, y)\, z^3 + u_0^k + u_1^k \xi_k \\
v^k(x, y, z) &= v_0(x, y) + v_1(x, y)\, z + v_2(x, y)\, z^2 + v_3(x, y)\, z^3 + v_0^k + v_1^k \xi_k \\
w^k(x, y, z) &= w_0(x, y)
\end{aligned} \tag{5.4}
$$

Similarly, the remaining five Superposition Theories can be rewritten as follows:

A. The 0-2 Superposition Theory

$$
\begin{aligned}
u^k(x, y, z) &= u_0(x, y) + u_1(x, y)\, z + u_2(x, y)\, z^2 + u_3(x, y)\, z^3 + u_0^k + u_2^k \xi_k^2 \\
v^k(x, y, z) &= v_0(x, y) + v_1(x, y)\, z + v_2(x, y)\, z^2 + v_3(x, y)\, z^3 + v_0^k + v_2^k \xi_k^2 \\
w^k(x, y, z) &= w_0(x, y)
\end{aligned} \tag{5.5}
$$

B. The 0-3 Superposition Theory

$$
\begin{aligned}
u^k(x, y, z) &= u_0(x, y) + u_1(x, y)\, z + u_2(x, y)\, z^2 + u_3(x, y)\, z^3 + u_0^k + u_3^k \xi_k^3 \\
v^k(x, y, z) &= v_0(x, y) + v_1(x, y)\, z + v_2(x, y)\, z^2 + v_3(x, y)\, z^3 + v_0^k + v_3^k \xi_k^3 \\
w^k(x, y, z) &= w_0(x, y)
\end{aligned} \tag{5.6}
$$

C. the 1-2 Superposition Theory

$$
\begin{aligned}
u^k(x, y, z) &= u_0(x, y) + u_1(x, y)\, z + u_2(x, y)\, z^2 + u_3(x, y)\, z^3 + u_1^k \xi_k + u_2^k \xi_k^2 \\
v^k(x, y, z) &= v_0(x, y) + v_1(x, y)\, z + v_2(x, y)\, z^2 + v_3(x, y)\, z^3 + v_1^k \xi_k + v_2^k \xi_k^2 \\
w^k(x, y, z) &= w_0(x, y)
\end{aligned} \tag{5.7}
$$

D. The 1-3 Superposition Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1(x,y)\,z + u_2(x,y)\,z^2 + u_3(x,y)\,z^3 + u_1^k\xi_k + u_3^k\xi_k^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1(x,y)\,z + v_2(x,y)\,z^2 + v_3(x,y)\,z^3 + v_1^k\xi_k + v_3^k\xi_k^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{5.8}
$$

E. The 2-3 Superposition Theory

$$
\begin{aligned}
u^k(x,y,z) &= u_0(x,y) + u_1(x,y)\,z + u_2(x,y)\,z^2 + u_3(x,y)\,z^3 + u_2^k\xi_k^2 + u_3^k\xi_k^3 \\
v^k(x,y,z) &= v_0(x,y) + v_1(x,y)\,z + v_2(x,y)\,z^2 + v_3(x,y)\,z^3 + v_2^k\xi_k^2 + v_3^k\xi_k^3 \\
w^k(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{5.9}
$$

The 0-1 Superposition Theory is actually identical to the 0-1 Quasi-layerwise Theory since they are not coordinate dependent. However, the remaining five Superposition Theories are different from their Quasi-layerwise counterparts because of the numerical advantage. In fact, all the Superposition Theories are not layer dependent. As an example, the 1-3 Superposition Theory is selected in the following section for numerical formulation. Similar procedures can be applied to the remaining five cases.

5.3 Formulation of the 1-3 Superposition Theory

A. Continuity Conditions

For simplicity, the local coordinate ξ_k of the k^{th} layer is defined as a linear function of z, i.e.:

$$\xi_k = a_k z + b_k \tag{5.10}$$

where, $a_k = \dfrac{2}{z_{k+1} - z_k}$ and $b_k = -\dfrac{z_{k+1} + z_k}{z_{k+1} - z_k}$.

It should be noted that $\xi_k(z_k) = -1$ and $\xi_{k-1}(z_k) = 1$ for the interface located between the k^{th} layer and the $(k-1)^{th}$ layer. To meet the continuity conditions of displacement, the following conditions should hold:

$$\begin{aligned} u_3^k &= -u_1^k - u_1^{k-1} - u_3^{k-1} \\ v_3^k &= -v_1^k - v_1^{k-1} - v_3^{k-1} \\ k &= 2, 3, 4, \ldots, n \end{aligned} \tag{5.11}$$

In this study, linear strain-displacement relations and orthotropic constitutive equations are considered. Thus, the transverse shear stresses can be expressed in terms of the displacement variables:

$$\begin{aligned} \tau_{yz}^k &= Q_{44}^k (v_1 + 2v_2 z + 3v_3 z^2 + a_k v_1^k + 3a_k v_3^k \xi_k^2 + w_{o,y}) \\ \tau_{xz}^k &= Q_{55}^k (u_1 + 2u_2 z + 3u_3 z^2 + a_k u_1^k + 3a_k u_3^k \xi_k^2 + w_{o,x}) \end{aligned} \tag{5.12}$$

To meet the continuity conditions of τ_{yz}^k and τ_{xz}^k at interface $z = z_k$, it yields:

$$\begin{aligned} & a_k Q_{44}^k v_1^k - a_{k-1} Q_{44}^{k-1} v_1^{k-1} + 3a_k Q_{44}^k v_3^k - 3a_{k-1} Q_{44}^{k-1} v_3^{k-1} = \\ & \qquad \Theta_k v_1 + 2\Theta_k z_k v_2 + 3\Theta_k z_k^2 v_3 + \Theta_k w_{o,y} \\ & a_k Q_{55}^k u_1^k - a_{k-1} Q_{55}^{k-1} u_1^{k-1} + 3a_k Q_{55}^k u_3^k - 3a_{k-1} Q_{55}^{k-1} u_3^{k-1} = \\ & \qquad \Omega_k u_1 + 2\Omega_k z_k u_2 + 3\Omega_k z_k^2 u_3 + \Omega_k w_{o,x} \\ & k = 2, 3, 4, \ldots, n \end{aligned} \tag{5.13}$$

where, $\Theta_k = Q_{44}^{k-1} - Q_{44}^k$ and $\Omega_k = Q_{55}^{k-1} - Q_{55}^k$.

Substituting Equations (5.11) into Equations (5.13), the following relations can be concluded:

$$
\begin{aligned}
u_1^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)u_1^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)u_3^{k-1}-\frac{1}{2}\beta_k u_1-\beta_k z_k u_2-\frac{3}{2}\beta_k z_k^2 u_3-\frac{1}{2}\beta_k w_{o,x} \\
v_1^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)v_1^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)v_3^{k-1}-\frac{1}{2}\eta_k v_1-\eta_k z_k v_2-\frac{3}{2}\eta_k z_k^2 v_3-\frac{1}{2}\eta_k w_{o,y} \\
k &= 2, 3, 4, \ldots, n
\end{aligned}
\tag{5.14}
$$

where, $\alpha_k = \dfrac{a_{k-1}Q_{55}^{k-1}}{a_k Q_{55}^k}$, $\beta_k = \dfrac{\Omega_k}{a_k Q_{55}^k}$, $\zeta_k = \dfrac{a_{k-1}Q_{44}^{k-1}}{a_k Q_{44}^k}$, and $\eta_k = \dfrac{\Theta_k}{a_k Q_{44}^k}$.

B. Boundary Conditions

Imposing the free shear traction condition on the bottom surface of the composite laminate, i.e. $\tau_{xz}^1(z_1) = 0$ and $\tau_{yz}^1(z_1) = 0$, and noting that $\xi_1(z_1) = -1$, the following two equations are satisfied:

$$
\begin{aligned}
u_1+2z_1u_2+3z_1^2u_3+w_{o,x}+a_1u_1^1+3a_1u_3^1 &= 0 \\
v_1+2z_1v_2+3z_1^2v_3+w_{o,y}+a_1v_1^1+3a_1v_3^1 &= 0
\end{aligned}
\tag{5.15}
$$

By examining Equations (5.14) and (5.15), it can be concluded that u_1^k, v_1^k, u_3^k, and v_3 can be expressed as functions of u_1, u_2, u_3, u_3^1, and $w_{0,x}$, i.e.:

$$
\begin{aligned}
u_1^k &= F_1^k u_1 + F_2^k u_2 + F_3^k u_3 + F_4^k u_3^1 + F_5^k w_{o,x} \\
u_3^k &= G_1^k u_1 + G_2^k u_2 + G_3^k u_3 + G_4^k u_3^1 + G_5^k w_{o,x} \\
v_1^k &= L_1^k v_1 + L_2^k v_2 + L_3^k v_3 + L_4^k v_3^1 + L_5^k w_{o,y} \\
v_3^k &= M_1^k v_1 + M_2^k v_2 + M_3^k v_3 + M_4^k v_3^1 + M_5^k w_{o,y}
\end{aligned}
\tag{5.16}
$$

After comparing Equations (5.15) and (5.16), the coefficients for k = 1 can be identified:

$$
\begin{aligned}
&F_1^1 = -\frac{1}{a_1},\ F_2^1 = -\frac{2z_1}{a_1},\ F_3^1 = -\frac{3z_1^2}{a_1},\ F_4^1 = -3,\ F_5^1 = -\frac{1}{a_1} \\
&G_1^1 = 0,\ G_2^1 = 0,\ G_3^1 = 1,\ G_4^1 = 0,\ G_5^1 = 0 \\
&L_1^1 = -\frac{1}{a_1},\ L_2^1 = -\frac{2z_1}{a_1},\ L_3^1 = -\frac{3z_1^2}{a_1},\ L_4^1 = -3,\ L_5^1 = -\frac{1}{a_1} \\
&M_1^1 = 0,\ M_2^1 = 0,\ M_3^1 = 1,\ M_4^1 = 0,\ M_5^1 = 0
\end{aligned}
\tag{5.17}
$$

Assuming k = 1 for the first two equations of Equations (5.16) and subsequently substituting them into Equations (5.14) for k = 2, it can be revealed that F_1^2 is a function of F_i^1 and G_i^1 (where $i = 1,2,..5$). By further examining the terms for k = 3 and above, the following recursive equations can be achieved:

$$
\begin{aligned}
F_1^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)F_1^{k-1} - \left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)G_1^{k-1} - \frac{1}{2}\beta_k \\
F_2^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)F_2^{k-1} - \left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)G_2^{k-1} - \beta_k z_k \\
F_3^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)F_3^{k-1} - \left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)G_3^{k-1} - \frac{3}{2}\beta_k z_k^2 \\
F_4^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)F_4^{k-1} - \left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)G_4^{k-1} \\
F_5^k &= -\left(\frac{3}{2}+\frac{1}{2}\alpha_k\right)F_5^{k-1} - \left(\frac{3}{2}+\frac{3}{2}\alpha_k\right)G_5^{k-1} - \frac{1}{2}\beta_k
\end{aligned}
$$

and,

$$
\begin{aligned}
L_1^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)L_1^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)M_1^{k-1}-\frac{1}{2}\eta_k \\
L_2^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)L_2^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)M_2^{k-1}-\eta_k z_k \\
L_3^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)L_3^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)M_3^{k-1}-\frac{3}{2}\eta_k z_k^2 \\
L_4^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)L_4^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)M_4^{k-1} \\
L_5^k &= -\left(\frac{3}{2}+\frac{1}{2}\zeta_k\right)L_5^{k-1}-\left(\frac{3}{2}+\frac{3}{2}\zeta_k\right)M_5^{k-1}-\frac{1}{2}\eta_k
\end{aligned}
\tag{5.18}
$$

By virtue of Equations (5.11), it is possible to further establish the following relations:

$$
\begin{aligned}
G_1^k &= -F_1^{k-1}-F_1^k-G_1^{k-1}, & M_1^k &= -L_1^{k-1}-L_1^k-M_1^{k-1} \\
G_2^k &= -F_2^{k-1}-F_2^k-G_2^{k-1}, & M_2^k &= -L_2^{k-1}-L_2^k-M_2^{k-1} \\
G_3^k &= -F_3^{k-1}-F_3^k-G_3^{k-1}, & M_3^k &= -L_3^{k-1}-L_3^k-M_3^{k-1} \\
G_4^k &= -F_4^{k-1}-F_4^k-G_4^{k-1}, & M_4^k &= -L_4^{k-1}-L_4^k-M_4^{k-1} \\
G_5^k &= -F_5^{k-1}-F_5^k-G_5^{k-1}, & M_5^k &= -L_5^{k-1}-L_5^k-M_5^{k-1}
\end{aligned}
\tag{5.19}
$$

By imposing the free shear traction condition on the top surface of the n-layer composite laminate, i.e. $\tau_{xz}^n(z_{n+1}) = 0$ and $\tau_{yz}^n(z_{n+1}) = 0$, another two equations can be obtained with the recognition of $\xi_n(z_{n+1}) = 1$:

$$
\begin{aligned}
u_1+2z_{n+1}u_2+3z_{n+1}^2u_3+w_{o,x}+a_nu_1^n+3a_nu_3^n &= 0 \\
v_1+2z_{n+1}v_2+3z_{n+1}^2v_3+w_{o,y}+a_nv_1^n+3a_nv_3^n &= 0
\end{aligned}
\tag{5.20}
$$

These two equations can be rewritten by virtue of the notation defined in Equations (5.16), i.e.:

$$
\begin{gathered}
(1 + a_n F_1^n + 3a_n G_1^n)\, u_1 + (2z_{n+1} + a_n F_2^n + 3a_n G_2^n)\, u_2 + \left(3z_{n+1}^2 + a_n F_3^n + 3a_n G_3^n\right) u_3 \\
+ (a_n F_4^n + 3a_n G_4^n)\, u_3^1 + (1 + a_n F_5^n + 3a_n G_5^n)\, w_{o,x} = 0 \\
(1 + a_n L_1^n + 3a_n M_1^n)\, v_1 + (2z_{n+1} + a_n L_2^n + 3a_n M_2^n)\, v_2 + \left(3z_{n+1}^2 + a_n L_3^n + 3a_n M_3^n\right) v_3 \\
(a_n L_4^n + 3a_n M_4^n)\, v_3^1 + (1 + a_n L_5^n + 3a_n M_5^n)\, w_{o,y} = 0
\end{gathered}
\tag{5.21}
$$

The above equations imply that two more dependent variables can also be eliminated from the formulation. Assume these two variables are u_3^1 and v_3^1, with following definitions:

$$
\begin{aligned}
u_3^1 &= A_1 u_1 + B_1 u_2 + C_1 u_3 + D_1 w_{o,x} \\
v_3^1 &= A_2 v_1 + B_2 v_2 + C_2 v_3 + D_2 w_{o,y}
\end{aligned}
\tag{5.22}
$$

where,

$$
A_1 = \frac{1 + a_n F_1^n + 3a_n G_1^n}{\Delta_1} \qquad B_1 = \frac{2z_{n+1} + a_n F_2^n + 3a_n G_2^n}{\Delta_1}
$$

$$
C_1 = \frac{3z_{n+1}^2 + a_n F_3^n + 3a_n G_3^n}{\Delta_1} \qquad D_1 = \frac{1 + a_n F_5^n + 3a_n G_5^n}{\Delta_1}
$$

$$
A_2 = \frac{1 + a_n L_1^n + 3a_n M_1^n}{\Delta_2} \qquad B_2 = \frac{2z_{n+1} + a_n L_2^n + 3a_n M_2^n}{\Delta_2}
$$

$$
C_2 = \frac{3z_{n+1}^2 + a_n L_3^n + 3a_n M_3^n}{\Delta_2} \qquad D_2 = \frac{1 + a_n L_5^n + 3a_n M_5^n}{\Delta_2}
$$

$$
\Delta_1 = a_n F_4^n + 3a_n G_4^n \qquad \Delta_2 = a_n L_4^n + 3a_n M_4^n
$$

Utilizing Equations (5.22), Equation (5.12) can be simplified as follows:

$$
\begin{aligned}
u_1^k &= R_1^k u_1 + R_2^k u_2 + R_3^k u_3 + R_4^k w_{o,x} \qquad & u_3^k &= S_1^k u_1 + S_2^k u_2 + S_3^k u_3 + S_4^k w_{o,x} \\
v_1^k &= O_1^k v_1 + O_2^k v_2 + O_3^k v_3 + O_4^k w_{o,y} \qquad & v_3^k &= P_1^k v_1 + P_2^k v_2 + P_3^k v_3 + P_4^k w_{o,y}
\end{aligned}
\tag{5.23}
$$

where,

$$
\begin{aligned}
R_1^k &= F_1^k + A_1 F_4^k & S_1^k &= G_1^k + A_1 G_4^k \\
R_2^k &= F_2^k + B_1 F_4^k & S_2^k &= G_2^k + B_1 G_4^k \\
R_3^k &= F_3^k + C_1 F_4^k & S_3^k &= G_3^k + C_1 G_4^k \\
R_4^k &= F_5^k + D_1 F_4^k & S_4^k &= G_5^k + D_1 G_4^k \\
O_1^k &= L_1^k + A_2 L_4^k & P_1^k &= M_1^k + A_2 M_4^k \\
O_2^k &= L_2^k + B_2 L_4^k & P_2^k &= M_2^k + B_2 M_4^k \\
O_3^k &= L_3^k + C_2 L_4^k & P_3^k &= M_3^k + C_2 M_4^k \\
O_4^k &= L_5^k + D_2 L_4^k & P_4^k &= M_5^k + D_2 M_4^k
\end{aligned}
$$

Substituting Equations (5.23) into Equations (5.8), the assumed displacement field can be expressed in terms of the layer-independent variables:

$$
\begin{aligned}
u^k &= u_0 + \Phi_1^k(z)\,u_1 + \Phi_2^k(z)\,u_2 + \Phi_3^k(z)\,u_3 + \Phi_4^k(z)\,w_{o,x} \\
v^k &= v_0 + \Psi_1^k(z)\,v_1 + \Psi_2^k(z)\,v_2 + \Psi_3^k(z)\,v_3 + \Psi_4^k(z)\,w_{o,y} \\
w^k &= w_0
\end{aligned}
\tag{5.24}
$$

where,

$$
\begin{aligned}
\Phi_1^k(z) &= z + R_1^k(a_k z + b_k) + S_1^k(a_k z + b_k)^3 \\
\Phi_2^k(z) &= z^2 + R_2^k(a_k z + b_k) + S_2^k(a_k z + b_k)^3 \\
\Phi_3^k(z) &= z^3 + R_3^k(a_k z + b_k) + S_3^k(a_k z + b_k)^3 \\
\Phi_4^k(z) &= R_4^k(a_k z + b_k) + S_4^k(a_k z + b_k)^3 \\
\Psi_1^k(z) &= z + O_1^k(a_k z + b_k) + P_1^k(a_k z + b_k)^3 \\
\Psi_2^k(z) &= z^2 + O_2^k(a_k z + b_k) + P_2^k(a_k z + b_k)^3 \\
\Psi_3^k(z) &= z^3 + O_3^k(a_k z + b_k) + P_3^k(a_k z + b_k)^3 \\
\Psi_4^k(z) &= O_4^k(a_k z + b_k) + P_4^k(a_k z + b_k)^3
\end{aligned}
$$

It should be noted that all Φ_i^k and Ψ_i^k in the above equations are associated only with material properties and thickness coordinates. As a consequence, Equations (5.24) have eleven independent variables. They represent a layer-independent theory and have the same computational efficiency as those of HSDT. In fact, the Superposition 1-3 Theory is like a third-order Shear Deformation Theory, although the total number of independent variables is higher by four than its Shear Deformation Theory counterpart. However, being different from the HSDT, the coefficients of Equations (5.24) are dependent on the individual layers, instead of just coefficients to be determined from a variational process.

The characteristic of layer-dependence of the Superposition Theories is different from that of the Layerwise Theories, whose coefficients are also layer-dependent but need to be determined from variational process. By examining Equations (5.8), it can be seen that not only can the distributions of in-plane displacement and transverse shear stresses be zigzag through the thickness, but also that the curvature of transverse shear stresses are dependent on the individual layers. These characteristics are similar to those of the Layerwise Theories. In summary, the Superposition Theories are like the Quasi-layerwise Theories and have the advantage of coordinate independency and numerical accuracy.

5.4 Numerical Solution

The procedures to obtain the solution to a composite laminate under bending are similar to those stated in Section 4.4. First of all, the strain and displacement components are expressed in matrix forms:

$$\varepsilon_x^k = \left[N_x^k\right]\{X\}, \quad \varepsilon_y^k = \left[N_y^k\right]\{X\}, \quad \varepsilon_z^k = \left[N_z^k\right]\{X\}$$
$$\gamma_{xy}^k = \left[N_{xy}^k\right]\{X\}, \quad \gamma_{xz}^k = \left[N_{xz}^k\right]\{X\}, \quad \gamma_{yz}^k = \left[N_{yz}^k\right]\{X\} \tag{5.25}$$
$$w^+ = \left[N_w^+\right]\{X\}, \qquad w^- = \left[N_w^-\right]\{X\}$$

where,

$$\{X\} = \{u_o \;\; v_o \;\; u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; w_o\}^T$$

$$\left[N_x^k\right] = \left[\frac{\partial}{\partial x} \quad 0 \quad \Phi_1^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_2^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_3^k\frac{\partial}{\partial x} \quad 0 \quad \Phi_4^k\frac{\partial^2}{\partial x^2}\right]$$

$$\left[N_y^k\right] = \left[0 \quad \frac{\partial}{\partial y} \quad 0 \quad \Psi_1^k\frac{\partial}{\partial y} \quad 0 \quad \Psi_2^k\frac{\partial}{\partial y} \quad 0 \quad \Psi_3^k\frac{\partial}{\partial y} \quad \Psi_4^k\frac{\partial^2}{\partial y^2}\right]$$

$$\left[N_z^k\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$\left[N_{xy}^k\right] = \left[\frac{\partial}{\partial y} \quad \frac{\partial}{\partial x} \quad \Phi_1^k\frac{\partial}{\partial y} \quad \Psi_1^k\frac{\partial}{\partial x} \quad \Phi_2^k\frac{\partial}{\partial y} \quad \Psi_2^k\frac{\partial}{\partial x}\right.$$
$$\left.\Phi_3^k\frac{\partial}{\partial y} \quad \Psi_3^k\frac{\partial}{\partial x} \quad (\Phi_4^k + \Psi_4^k)\frac{\partial^2}{\partial x \partial y}\right]$$

$$\left[N_{xz}^k\right] = \left[0 \quad 0 \quad \frac{d}{dz}\Phi_1^k \quad 0 \quad \frac{d}{dz}\Phi_2^k \quad 0\right.$$
$$\left.\frac{d}{dz}(\Phi_3^k) \quad 0 \quad \left(\frac{d}{dz}\Phi_4^k + 1\right)\frac{\partial}{\partial x}\right]$$

$$\left[N_{yz}^k\right] = \left[0 \quad 0 \quad 0 \quad \frac{d}{dz}\Psi_1^k \quad 0 \quad \frac{d}{dz}\Psi_2^k\right.$$
$$\left.0 \quad \frac{d}{dz}\Psi_3^k \quad \left(\frac{d}{dz}\Psi_4^k + 1\right)\frac{\partial}{\partial y}\right]$$

$$\left[N_w^+\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

$$\left[N_w^-\right] = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\right]$$

By defining

$$
\left[N^k\right]_I = \begin{bmatrix} N_x^k \\ N_y^k \\ N_z^k \\ N_{xy}^k \end{bmatrix} \qquad \left[N^k\right]_{II} = \begin{bmatrix} N_{xz}^k \\ N_{yz}^k \end{bmatrix} \tag{5.26}
$$

and substituting them into Equations (3.21), the variational equation becomes:

$$
\begin{aligned}
&\int_\Omega \left(\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\left[N^k\right]_I^T \left[Q^k\right]_I \left[N^k\right]_I + \left[N^k\right]_{II}^T \left[Q^k\right]_{II} \left[N^k\right]_{II} \right) dz \right) \{X\}\, dxdy = \\
&\qquad + \int_\Omega \left(q^+ \left[N_w^+\right] \right) dxdy + \int_\Omega \left(q^- \left[N_w^-\right] \right) dxdy
\end{aligned} \tag{5.27}
$$

If Pagano's cylindrical bending problem of plane-strain type is of interest, both the loading on the top surface of the composite laminate, $q^+\left(x, \frac{h}{2}\right) = q_o^+ \sin\frac{m\pi}{L}x$, and the displacement variables can be assumed as follows:

$$
u_o = U_o \cos px; \quad u_1 = U_1 \cos px; \quad u_2 = U_2 \cos px; \quad u_3 = U_3 \cos px
$$
$$
w_o = W_o \sin px
$$

As a consequence, the simply supported boundary conditions are automatically satisfied and the matrices expressed in Equations (5.25) can be further simplified as follows:

$$
\{X\} = \{u_o \quad u_1 \quad u_2 \quad u_3 \quad w_o\}^T
$$

$$
\{\bar{X}\} = \{U_o \quad U_1 \quad U_2 \quad U_3 \quad W_o\}^T
$$

$$\left[N_x^k\right] = \left[\bar{N}_x^k\right]\sin px = \left[-p \quad -p\Phi_1^k \quad -p\Phi_2^k \quad -p\Phi_3^k \quad -p^2\Phi_4^k\right]\sin px$$
$$\left[N_z^k\right] = \left[\bar{N}_z^k\right]\sin px = \left[0 \quad 0 \quad 0 \quad 0 \quad 0\right]\sin px$$
$$\left[N_{xz}^k\right] = \left[\bar{N}_{xz}^k\right]\cos px = \left[0 \quad \frac{d}{dz}\Phi_1^k \quad \frac{d}{dz}\Phi_2^k \quad \frac{d}{dz}\Phi_3^k \quad p\left(\frac{d}{dz}\Phi_4^k + 1\right)\right]\cos px \tag{5.28}$$
$$\left[N_w^+\right] = \left[\bar{N}_w^+\right]\sin px = \left[0 \quad 0 \quad 0 \quad 0 \quad 1\right]\sin px$$

Substituting Equations (5.28) into (5.27), it yields:

$$\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\Big(Q_{11}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_x^k\right] + Q_{13}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_z^k\right] + Q_{13}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_x^k\right] + Q_{33}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_z^k\right] + Q_{55}^k\left[\bar{N}_{xz}^k\right]^T\left[\bar{N}_{xz}^k\right]\Big)dz\,\{\bar{X}\} = q_o^+\left[\bar{N}_w^+\right]^T \tag{5.29}$$

Once the independent variables U_o, U_1, U_2, U_3, and W_o are identified by simultaneously solving the equations, the displacement and stress components can be obtained through the associated equations.

5.5 Results and Discussion

The results for the six Superposition Theories are shown from Figure 5.1 to Figure 5.5. As depicted in Figure 5.1, it seems that the Superposition Theories of 1-2, 1-3, and 2-3 types are superior to the Superposition Theories of 0-1, 0-2, and 0-3 types in giving $\bar{\sigma}_x$. For the displacement, $\bar{u}$, illustrated in Figure 5.4, both the 1-2 and 1-3 Superposition Theories give good results. However, only the 1-3 Superposition Theory gives excellent results of τ_{xz}, as shown in Figure 5.2. Since the boundary condition of the transverse normal stress σ_z is not satisfied and constant w is assumed through the composite laminate, the differences of $\bar{\sigma}_z$ and $\bar{w}$ as shown in Figure 5.3 and 5.5, respectively, are not significant.

Based on the above results, it seems that the 1-3 Superposition Theory is the best of the six theories. Among the four coefficients, it seems that the zeroth-order term, which is the midplane displacement of each layer, is probably the least important one because the Superposition Theories of 0-1, 0-2, and 0-3 types give worse results when compared with the remaining three theories. It is believed that a layer-independent zeroth-order term combined with displacement continuity conditions and variational process should be enough to give accurate midplane displacement for each layer.

Among the first-, second-, and third-order terms, it seems that the first-order term is the most important one because the 1-2 Superposition Theory and the 1-3 Superposition Theory are better than the 2-3 Superposition Theory in giving displacement, and the 1-3 Superposition Theory is the best one for transverse shear stress. A possible interpretation is that the first-order term is the most fundamental one for both displacement and stress components.

Between the second- and the third-order terms, the latter seems to be more important since the 1-3 Superposition Theory gives excellent τ_{xz}. It is recognized that the second-order term becomes the first-order after being differentiated with respect to z, while the third-order becomes the second-order. Since the transverse shear stress is known to be of parabolic distribution, a layer-dependent third-order term certainly is very important.

Unfortunately, the above interpretation is not validated by further numerical results.

Shown in Figures 5.6 and 5.7 are the results for composite laminates made of 7, 15, and 31 layers. Though $\bar{u}$ seems to be reasonable, non-negligible errors still exist in the transverse shear stress. The oscillations of τ_{xz} is believed to be attributed to the absence of the second-order layer-dependent term. The numerical error seems to become more distinct as the number of composite layers increases. It is thought that the boundary conditions, the free shear tractions on both surfaces, strongly constrain the final distribution of τ_{xz} in the [0/90/0] laminate, resulting in excellent agreement with the elasticity solution. However, as the number of layer increases, the influence from the boundary conditions becomes weaker, and the important role of the second-order term becomes very distinct. As can be seen in Figure 5.2, comparing the 0-2 and 1-2 Superposition Theories with the 0-3 and 1-3 Superposition Theories, it is obvious that one is convex to the left and the other to the right. If both terms are considered in a theory, there will be an opportunity of balancing the extremities.

5.6 Summary

The zeroth-order term can be omitted from the local component of a displacement field. The first-order is the most fundamental one and can not be neglected. For higher-order terms, it is concluded that the completeness of a displacement field is very important in general composite analysis.

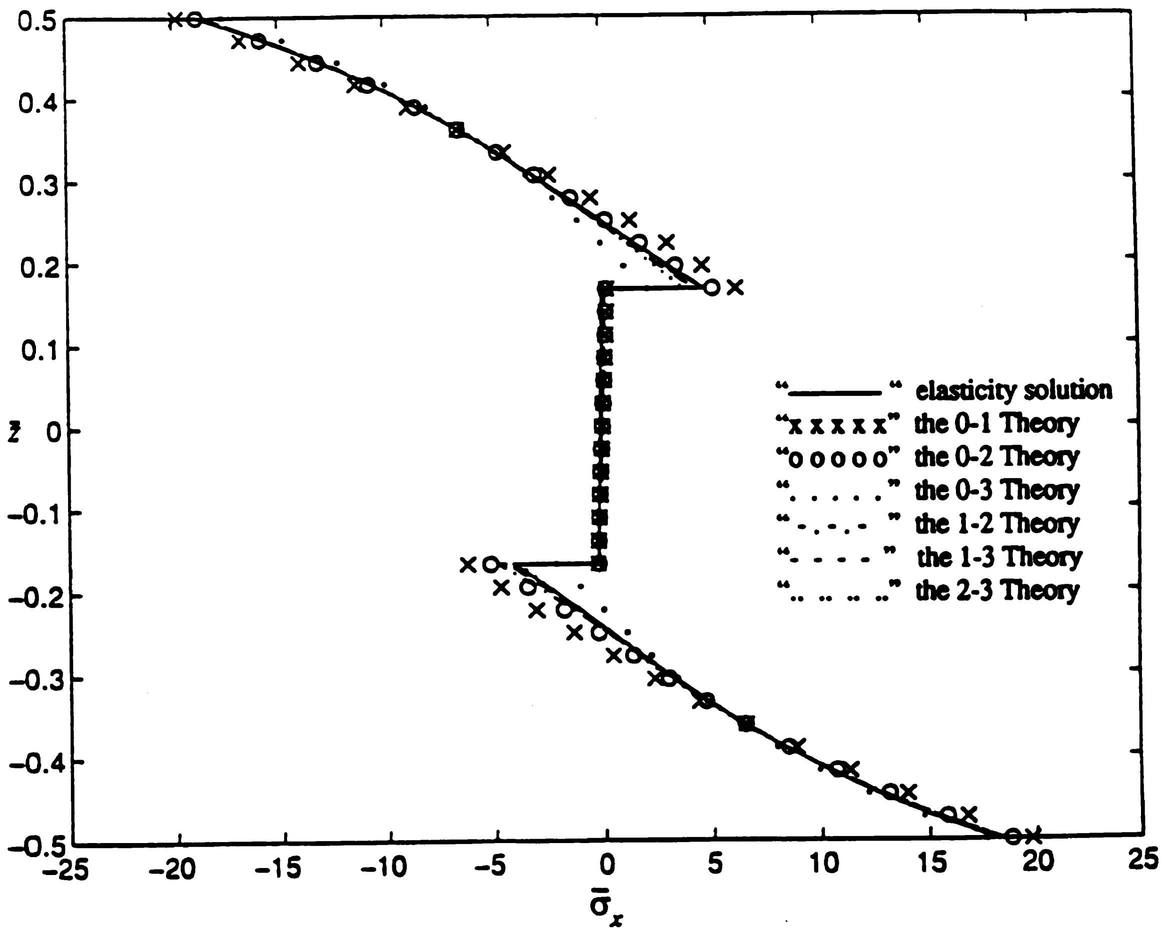

Figure 5.1 - Comparison of $\bar{\sigma}_x$ from various Superposition Theories for a [0/90/0] laminate.

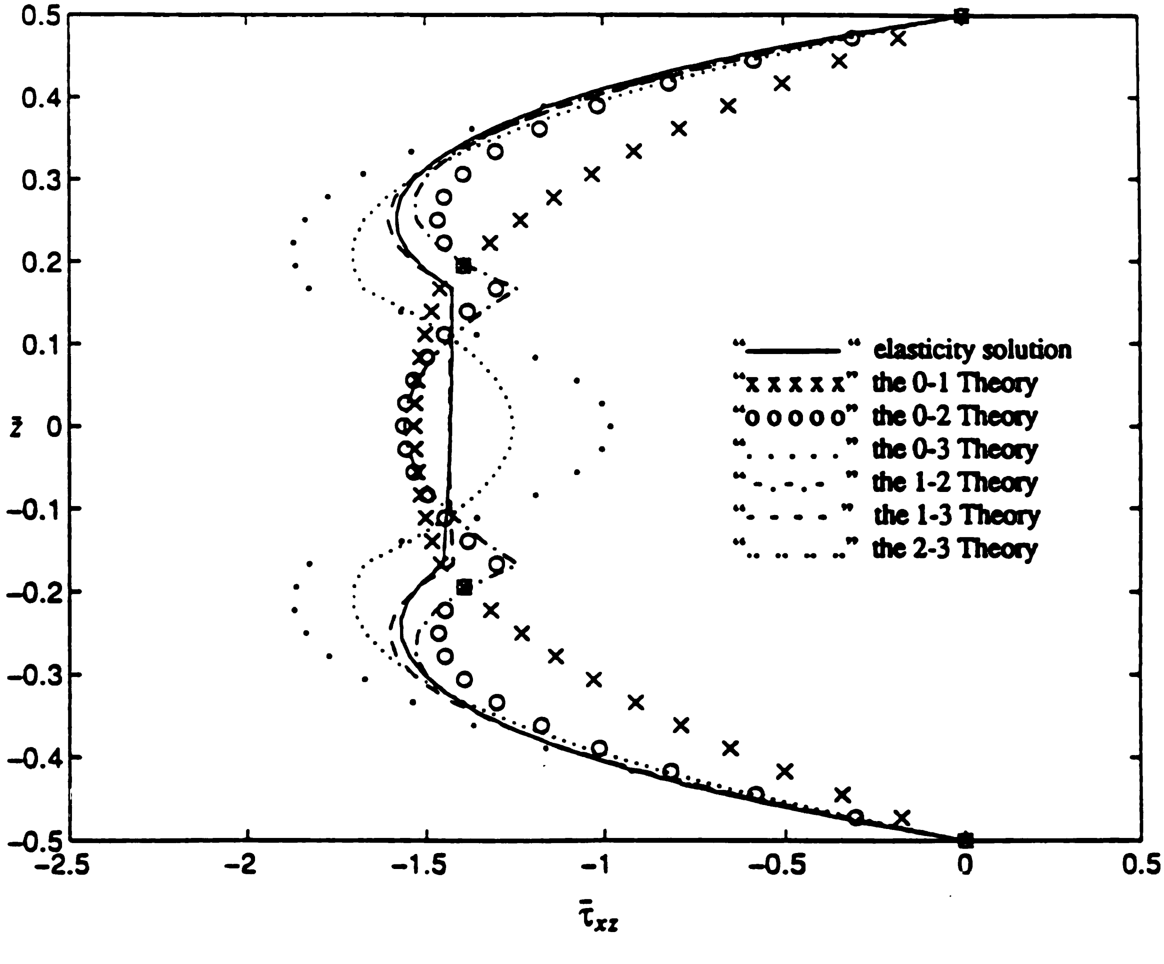

Figure 5.2 - Comparison of τ_{xz} from various Superposition Theories for a [0/90/0] laminate.

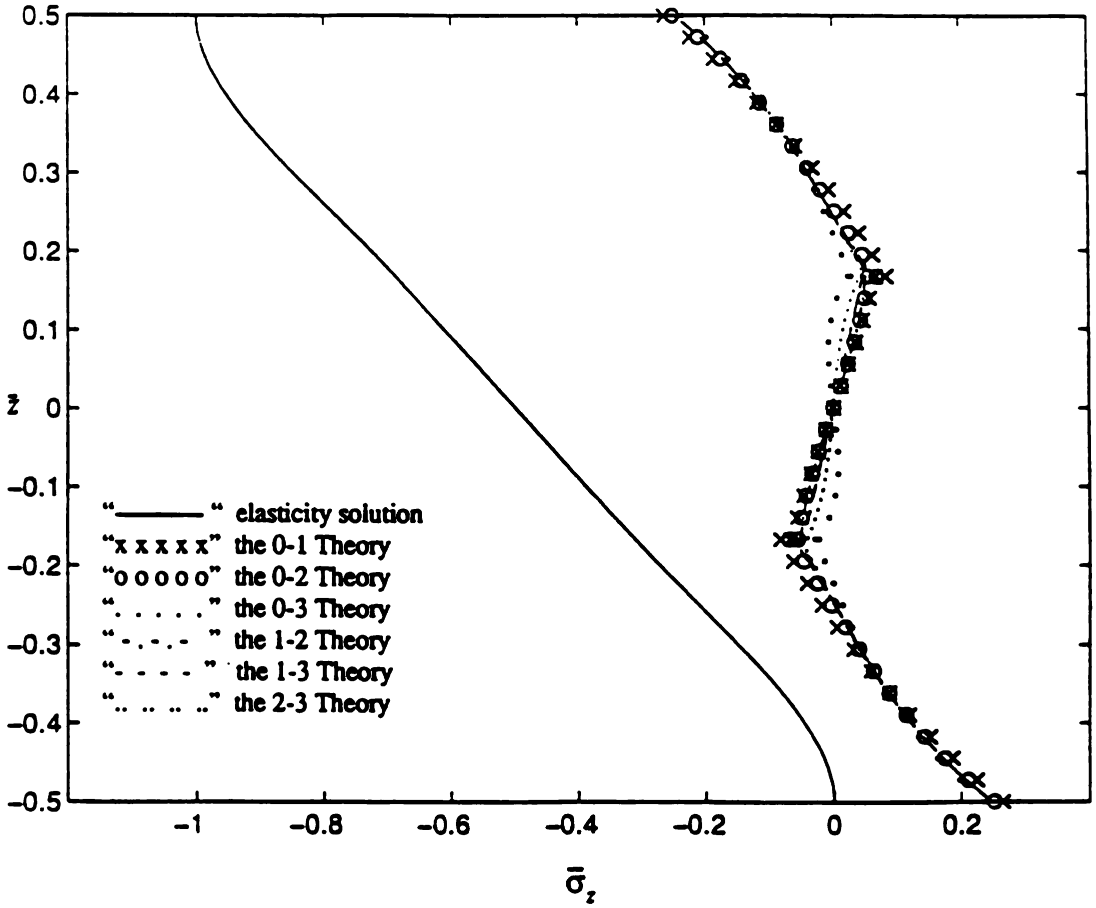

Figure 5.3 - Comparison of $\bar{\sigma}_z$ from various Superposition Theories for a [0/90/0] laminate.

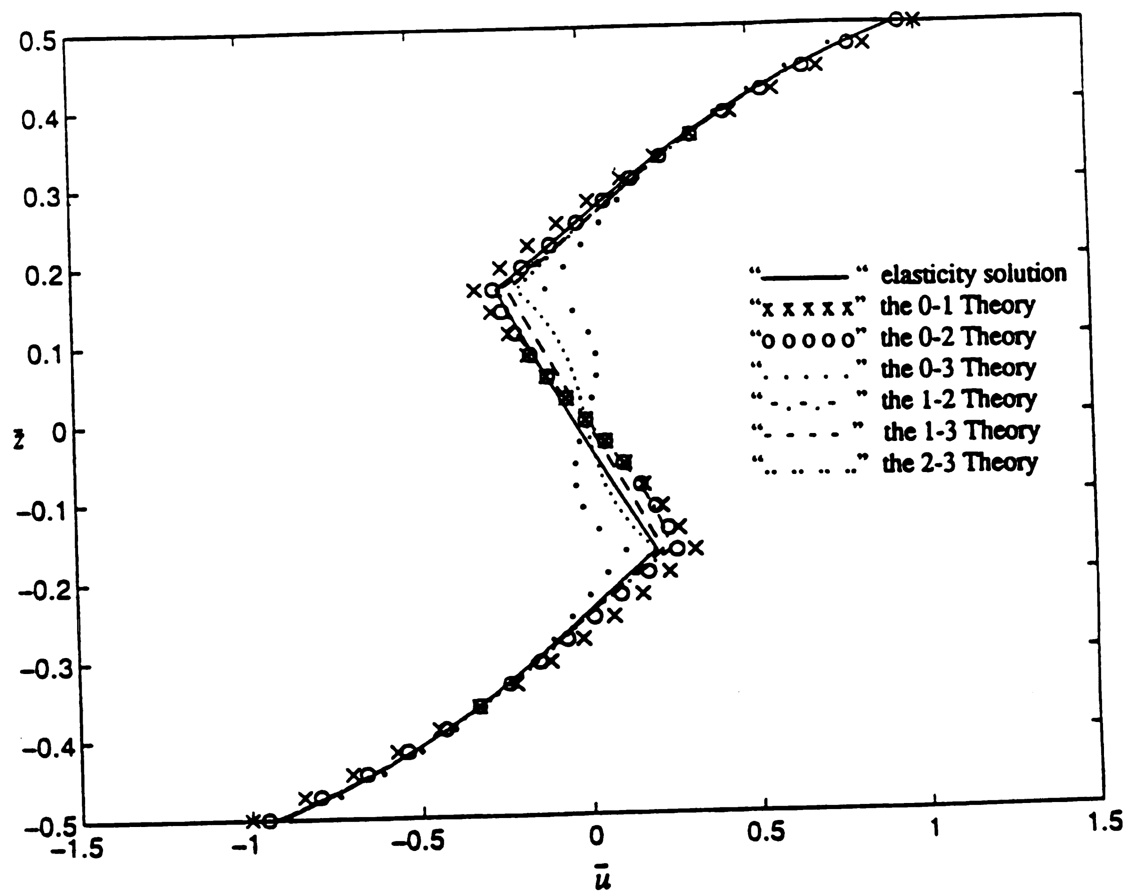

Figure 5.4 - Comparison of $\bar{u}$ from various Superposition Theories for a [0/90/0] laminate.

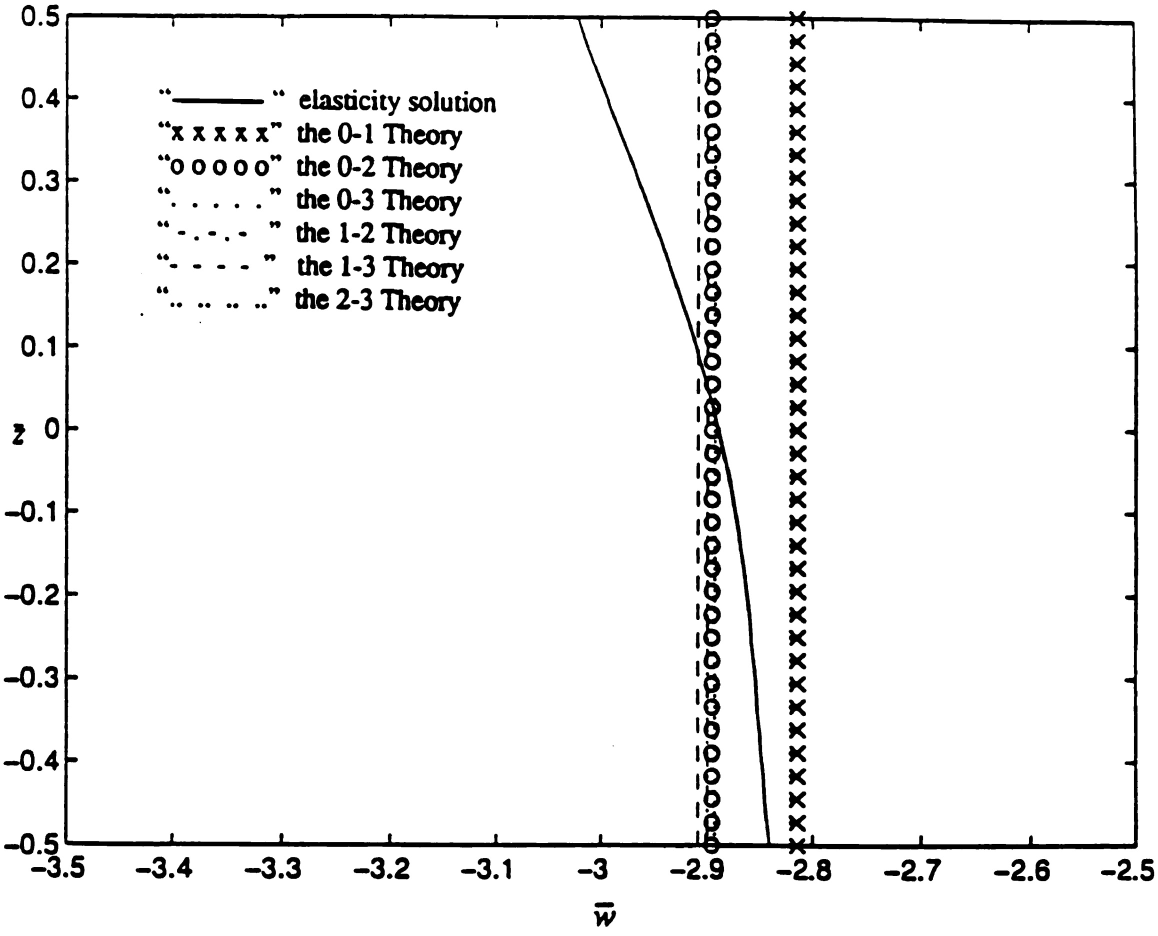

Figure 5.5 - Comparison of $\bar{w}$ from various Superposition Theories for a [0/90/0] laminate.

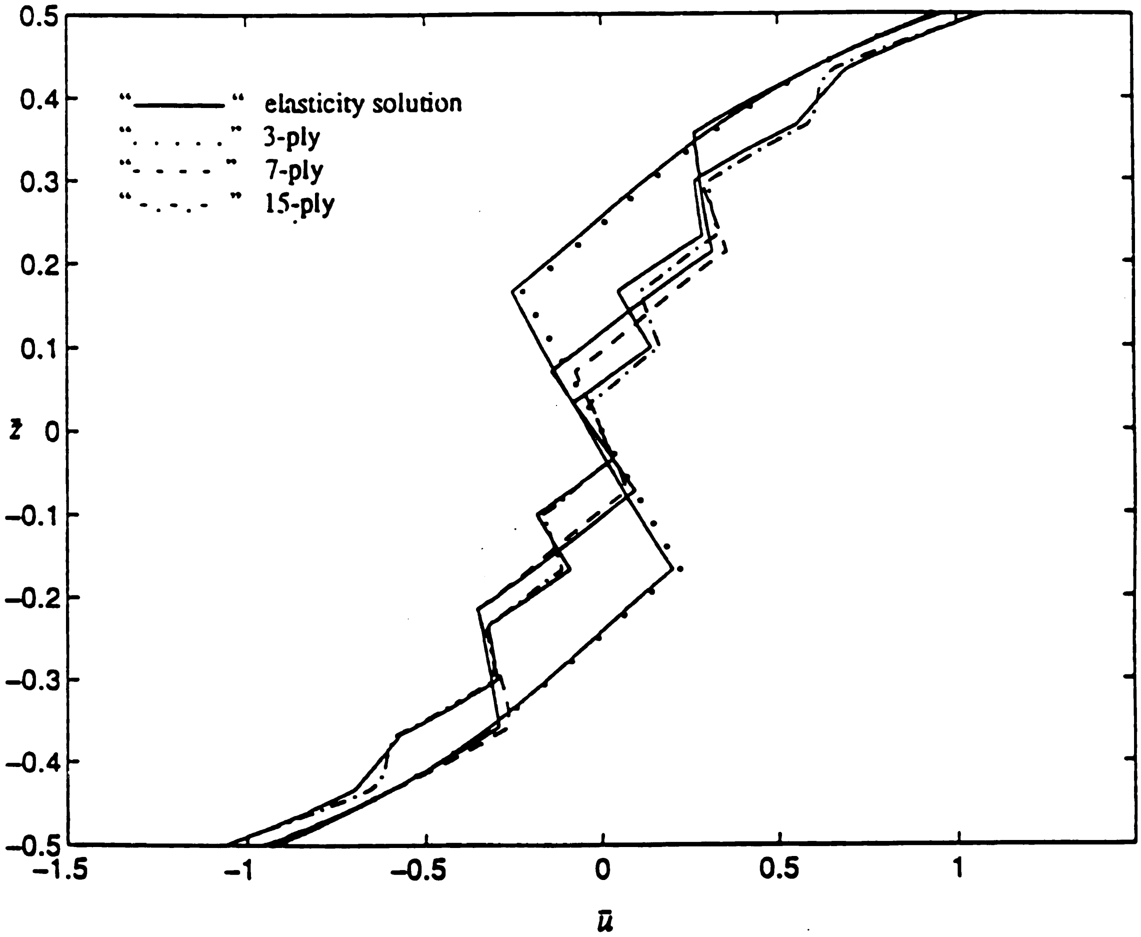

Figure 5.6 - Comparison of $\bar{u}$ from the 1-3 Superposition Theory and elasticity solutions for 7-, 15-, and 31-layer symmetric laminates.

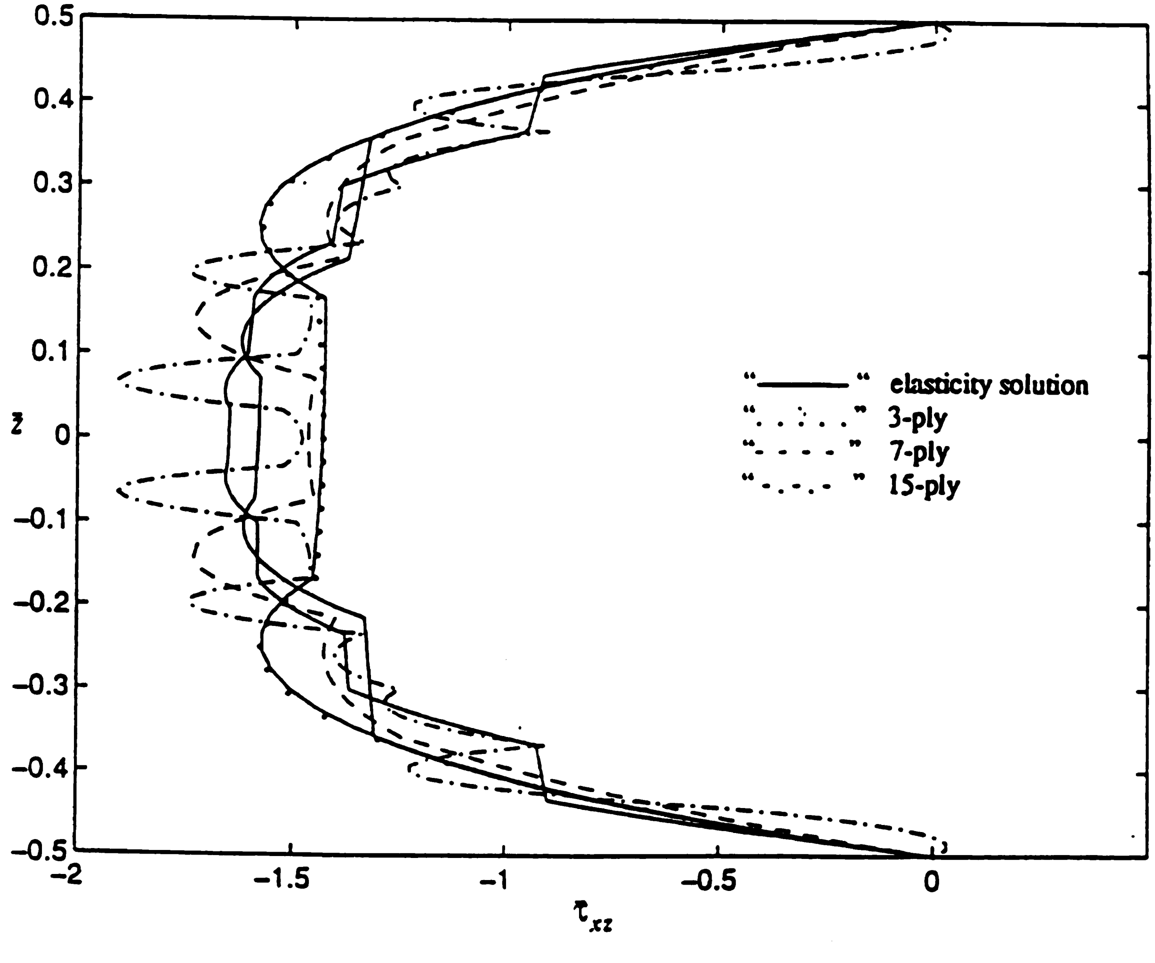

Figure 5.7 - Comparison of τ_{xz} from the 1-3 Superposition Theory and elasticity solutions for 7-, 15-, and 31-layer symmetric laminates.

CHAPTER 6

DOUBLE SUPERPOSITION THEORIES

6.1 Introduction

In Chapter 5, the global-local superposition technique is used to combine a global component and a local component together. The global component is actually the third-order Shear Deformation Theory and the local component resembles the Generalized Layerwise Theory. However, due to the fact that only two continuity conditions need to be satisfied in each coordinate-direction, the local component is allowed to have only two layer-dependent terms. As shown in Chapter 5, the six possible combinations give different results. It is believed that each local term has a distinct contribution to laminate performance. The zeroth-order term is the midplane displacement of each composite layer. It can be omitted from layer-dependent displacement assumption since the continuity conditions have been satisfied on the laminate interfaces, resulting in the uselessness of assuming zeroth-order terms for individual layers. The first-order term is of rotational angle. It is too fundamentally important to be ignored. The second-order term represents the curvature of the displacement distribution of the composite laminate, while the third-order term can be associated with the curvature of the transverse stress distribution. They are equally important to composite performance, especially when a laminate has a large number of layers. As a consequence, the numerical results are very sensitive to the various combinations of the terms, and the selection of the terms is very critical to the success of a theory.

In view of the fundamental roles of the individual terms, it is believed that not only the completeness of the terms, but also the inclusion of as many terms as possible, is important to a laminate theory. It then is the goal of this study to look into a theory which can satisfy the requirement of completeness and include all the first-, second-, and third-order layer-dependent terms in an assumed displacement field.

Since only two continuity conditions can be satisfied in composite layer assembly, the number of variables associated with the local behavior are limited to two. If the completeness of the local component is of concern, the only allowable global-local combination is the 0-1 Superposition Theory. However, if more terms are to be included, a special technique such as the following Hypothesis for Double Superposition should be proposed:

The Global-Local Superposition Technique is applied to the three local terms twice, once for grouping two local terms and the other for one local term.

The application of the Hypothesis for Double Superposition in proposing new laminate theories is demonstrated in the following sections.

A laminate theory whose global component is of a third-order Shear Deformation Theory is considered:

$$\begin{aligned}
u_G(x,y,z) &= u_0(x,y) + u_1(x,y)\,z + u_2(x,y)\,z^2 + u_3(x,y)\,z^3 \\
v_G(x,y,z) &= v_0(x,y) + v_1(x,y)\,z + v_2(x,y)\,z^2 + v_3(x,y)\,z^3 \\
w_G(x,y,z) &= w_o(x,y)
\end{aligned} \tag{6.1}$$

and the local component has two groups, one with two local terms and the other with only one local term. The total displacement field can be summarized as follows:

$$
\begin{aligned}
u^k &= u_G + \underline{u}_L^k + \underset{\sim}{u}_L^k \\
v^k &= v_G + \underline{v}_L^k + \underset{\sim}{v}_L^k \qquad (6.2) \\
w &= w_G
\end{aligned}
$$

where u_G, v_G, and w_G are global terms; $\underline{u}_L^k$ and $\underline{v}_L^k$ are of two-term local groups; and $\underset{\sim}{u}_L^k$ and $\underset{\sim}{v}_L^k$ are of one-term local groups. Assuming the local component is also limited to the third-order, the three possible combinations for grouping the three local terms twice, resulting in three Double Superposition Theories, can be listed below:

A. The 1,2-3 Double Superposition Theory

In this theory, the local component can be divided into two groups: the first group contains the first- and second-order terms while the second group contains the third-order term, i.e.:

$$
\begin{aligned}
\underline{u}_L^k (x, y, \xi_k) &= u_1^k (x, y)\, \xi_k + u_2^k (x, y)\, \xi_k^2 \\
\underline{v}_L^k (x, y, \xi_k) &= v_1^k (x, y)\, \xi_k + v_2^k (x, y)\, \xi_k^2 \\
\underset{\sim}{u}_L^k (x, y, \xi_k) &= u_3^k (x, y)\, \xi_k^3 \qquad (6.3) \\
\underset{\sim}{v}_L^k (x, y, \xi_k) &= v_3^k (x, y)\, \xi_k^3
\end{aligned}
$$

B. The 1,3-2 Double Superposition Theory

$$
\begin{aligned}
\underline{u}_L^k (x, y, \xi_k) &= u_1^k (x, y)\, \xi_k + u_3^k (x, y)\, \xi_k^3 \\
\underline{v}_L^k (x, y, \xi_k) &= v_1^k (x, y)\, \xi_k + v_3^k (x, y)\, \xi_k^3
\end{aligned}
$$

$$\underset{\sim}{u}_L^k (x, y, \xi_k) = u_2^k (x, y)\, \xi_k^2 \qquad (6.4)$$
$$\underset{\sim}{v}_L^k (x, y, \xi_k) = v_2^k (x, y)\, \xi_k^2$$

C. The 2,3-1 Double Superposition Theory

$$\underline{u}_L^k (x, y, \xi_k) = u_2^k (x, y)\, \xi_k^2 + u_3^k (x, y)\, \xi_k^3$$
$$\underline{v}_L^k (x, y, \xi_k) = v_2^k (x, y)\, \xi_k^2 + v_3^k (x, y)\, \xi_k^3$$
$$\underset{\sim}{u}_L^k (x, y, \xi_k) = u_1^k (x, y)\, \xi_k \qquad (6.5)$$
$$\underset{\sim}{v}_L^k (x, y, \xi_k) = v_1^k (x, y)\, \xi_k$$

6.2 Formulation of the 1,2-3 Double Superposition Theory

Taking the 1,2-3 Double-Superposition Theory as an example to demonstrate the process of Hypothesis for Double Superposition, the global component is of a third-order Shear Deformation Theory as given in Equations (6.1). The local component is divided into two groups. The first group includes the first-order and the second-order terms while the second group includes the third-order term. As a consequence, the displacement field can be written as follows:

$$\begin{aligned} u^k (x, y, z) &= u_0 (x, y) + u_1 (x, y)\, z + u_2 (x, y)\, z^2 + u_3 (x, y)\, z^3 + \\ &\quad u_1^k (x, y)\, \xi_k + u_2^k (x, y)\, \xi_k^2 + u_3^k (x, y)\, \xi_k^3 \\ v^k (x, y, z) &= v_0 (x, y) + v_1 (x, y)\, z + v_2 (x, y)\, z^2 + v_3 (x, y)\, z^3 + \\ &\quad v_1^k (x, y)\, \xi_k + v_2^k (x, y)\, \xi_k^2 + v_3^k (x, y)\, \xi_k^3 \\ w^k (x, y, z) &= w_0 (x, y) \end{aligned} \qquad (6.6)$$

where a linear relation for transformation between the local and the global coordinates is assumed, i.e.:

$$\xi_k = a_k z + b_k \tag{6.7}$$

and,

$$a_k = \frac{2}{z_{k+1} - z_k}; \quad b_k = -\frac{z_{k+1} + z_k}{z_{k+1} - z_k}$$

As mentioned before, it should be noted that $\xi_k(z_k) = -1$ and $\xi_{k-1}(z_k) = 1$ on the laminate interfaces.

A. Continuity Conditions

The continuity conditions of displacement on the laminate interfaces should be satisfied. By imposing the continuity condition for the two-term group, the following two equations can be identified:

$$\begin{aligned} u_2^k &= u_1^k + u_1^{k-1} + u_2^{k-1} \\ v_2^k &= v_1^k + v_1^{k-1} + v_2^{k-1} \end{aligned} \tag{6.8}$$

In addition, by enforcing the continuity conditions for the one-term group, another two equations are obtained:

$$\begin{aligned} u_3^k &= -u_3^{k-1} \\ v_3^k &= -v_3^{k-1} \end{aligned} \tag{6.9}$$

It should be pointed out that the above equations are valid for $k = 2, 3, 4, \ldots, n$.

Assuming linear strain-displacement relations and utilizing three-dimensional constitutive equations for cross-ply laminates, the transverse shear stresses for the k^{th} layer are:

$$\begin{aligned}
\tau_{xz}^k &= Q_{55}^k \left(u_1 + 2u_2 z + 3u_3 z^2 + a_k u_1^k + 2a_k u_2^k \xi_k + 3a_k u_3^k \xi_k^2 + w_{o,x}\right) \\
\tau_{yz}^k &= Q_{44}^k \left(v_1 + 2v_2 z + 3v_3 z^2 + a_k v_1^k + 2a_k v_2^k \xi_k + 3a_k v_3^k \xi_k^2 + w_{o,y}\right)
\end{aligned} \tag{6.10}$$

In order to meet the continuity conditions of τ_{yz}^k and τ_{xz}^k on interface $z = z_k$, the following two equations need to be satisfied:

$$\begin{aligned}
-a_k Q_{55}^k u_1^k &= \left(2a_k Q_{55}^k + a_{k-1} Q_{55}^{k-1}\right) u_1^{k-1} + 2\left(a_k Q_{55}^k + a_{k-1} Q_{55}^{k-1}\right) u_2^{k-1} \\
&\quad + 3\left(a_k Q_{55}^k + a_{k-1} Q_{55}^{k-1}\right) u_3^{k-1} + \Omega_k u_1 + 2\Omega_k z_k u_2 + 3\Omega_k z_k^2 u_3 + \Omega_k w_{o,x} \\
-a_k Q_{44}^k v_1^k &= \left(2a_k Q_{44}^k + a_{k-1} Q_{44}^{k-1}\right) v_1^{k-1} + 2\left(a_k Q_{44}^k + a_{k-1} Q_{44}^{k-1}\right) v_2^{k-1} \\
&\quad + 3\left(a_k Q_{44}^k + a_{k-1} Q_{44}^{k-1}\right) v_3^{k-1} + \Theta_k v_1 + 2\Theta_k z_k v_2 + 3\Theta_k z_k^2 v_3 + \Theta_k w_{o,y}
\end{aligned} \tag{6.11}$$

where, $\Theta_k = Q_{44}^{k-1} - Q_{44}^k$, and $\Omega_k = Q_{55}^{k-1} - Q_{55}^k$.

Rearranging the terms, Equations (6.10) can be expressed in the explicit forms:

$$\begin{aligned}
u_1^k &= -(2 + \alpha_k)\, u_1^{k-1} - 2\,(1 + \alpha_k)\, u_2^{k-1} - 3\,(1 + \alpha_k)\, u_3^{k-1} \\
&\quad -\beta_k u_1 - 2\beta_k z_k u_2 - 3\beta_k z_k^2 u_3 - \beta_k w_{o,x} \\
v_1^k &= -(2 + \zeta_k)\, v_1^{k-1} - 2\,(1 + \zeta_k)\, v_2^{k-1} - 3\,(1 + \zeta_k)\, v_3^{k-1} \\
&\quad -\eta_k v_1 - 2\eta_k z_k v_2 - 3\eta_k z_k^2 v_3 - \eta_k w_{o,y}
\end{aligned} \tag{6.12}$$

where, $\alpha_k = \dfrac{a_{k-1} Q_{55}^{k-1}}{a_k Q_{55}^k}$, $\beta_k = \dfrac{\Omega_k}{a_k Q_{55}^k}$, $\zeta_k = \dfrac{a_{k-1} Q_{44}^{k-1}}{a_k Q_{44}^k}$, and $\eta_k = \dfrac{\Theta_k}{a_k Q_{44}^k}$.

B. Boundary Conditions

Free shear traction conditions are imposed in this analysis for both surfaces. Therefore, on the bottom surface, both $\tau_{xz}^1(z_1) = 0$ and $\tau_{yz}^1(z_1) = 0$ should be satisfied. Since $\xi_1(z_1) = -1$, these two boundary conditions become:

$$
\begin{aligned}
u_1 + 2z_1u_2 + 3z_1^2u_3 + w_{o,x} + a_1u_1^1 - 2a_1u_2^1 + 3a_1u_3^1 &= 0 \\
v_1 + 2z_1v_2 + 3z_1^2v_3 + w_{o,y} + a_1v_1^1 - 2a_1v_2^1 + 3a_1v_3^1 &= 0
\end{aligned}
\tag{6.13}
$$

By examining these equations with Equations (6.12), u_1^k, v_1^k, u_2^k, v_2^k, u_3^k, and v_3^k can be assumed as:

$$
\begin{aligned}
u_1^k &= F_1^ku_1^1 + F_2^ku_2^1 + F_3^ku_1 + F_4^ku_2 + F_5^ku_3 + F_6^kw_{o,x} \\
u_2^k &= G_1^ku_1^1 + G_2^ku_2^1 + G_3^ku_1 + G_4^ku_2 + G_5^ku_3 + G_6^kw_{o,x} \\
u_3^k &= H_1^ku_1^1 + H_2^ku_2^1 + H_3^ku_1 + H_4^ku_2 + H_5^ku_3 + H_6^kw_{o,x} \\
v_1^k &= L_1^kv_1^1 + L_2^kv_2^1 + L_3^kv_1 + L_4^kv_2 + L_5^kv_3 + L_6^kw_{o,y} \\
v_2^k &= M_1^kv_1^1 + M_2^kv_2^1 + M_3^kv_1 + M_4^kv_2 + M_5^kv_3 + M_6^kw_{o,y} \\
v_3^k &= N_1^kv_1^1 + N_2^kv_2^1 + N_3^kv_1 + N_4^kv_2 + N_5^kv_3 + N_6^kw_{o,y}
\end{aligned}
\tag{6.14}
$$

When compared to Equations (6.13), the coefficients for k = 1 can be easily identified:

$$
\begin{aligned}
&F_1^1 = 1,\ F_2^1 = F_3^1 = F_4^1 = F_5^1 = F_6^1 = 0 \\
&G_2^1 = 1,\ G_1^1 = G_3^1 = G_4^1 = G_5^1 = G_6^1 = 0 \\
&L_1^1 = 1,\ L_2^1 = L_3^1 = L_4^1 = L_5^1 = L_6^1 = 0 \\
&M_2^1 = 1,\ M_1^1 = M_3^1 = M_4^1 = M_5^1 = M_6^1 = 0 \\
&H_1^1 = -\frac{1}{3},\ H_2^1 = \frac{2}{3},\ H_3^1 = -\frac{1}{3a_1} \\
&H_4^1 = -\frac{2z_1}{3a_1},\ H_5^1 = -\frac{z_1^2}{a_1},\ H_6^1 = -\frac{1}{3a_1} \\
&N_1^1 = -\frac{1}{3},\ N_2^1 = \frac{2}{3},\ N_3^1 = -\frac{1}{3a_1} \\
&N_4^1 = -\frac{2z_1}{3a_1},\ N_5^1 = -\frac{z_1^2}{a_1},\ N_6^1 = -\frac{1}{3a_1}
\end{aligned}
$$

In a manner similar to that used in obtaining Equations (5.18) and (5.19), the following recursive equations can be obtained:

$$
\begin{aligned}
F_1^k &= -(2+\alpha_k)F_1^{k-1} - 2(1+\alpha_k)G_1^{k-1} - 3(1+\alpha_k)H_1^{k-1} \\
F_2^k &= -(2+\alpha_k)F_2^{k-1} - 2(1+\alpha_k)G_2^{k-1} - 3(1+\alpha_k)H_2^{k-1} \\
F_3^k &= -(2+\alpha_k)F_3^{k-1} - 2(1+\alpha_k)G_3^{k-1} - 3(1+\alpha_k)H_3^{k-1} - \beta_k \\
F_4^k &= -(2+\alpha_k)F_4^{k-1} - 2(1+\alpha_k)G_4^{k-1} - 3(1+\alpha_k)H_4^{k-1} - 2\beta_k z_k \\
F_5^k &= -(2+\alpha_k)F_5^{k-1} - 2(1+\alpha_k)G_5^{k-1} - 3(1+\alpha_k)H_5^{k-1} - 3\beta_k z_k^2 \\
F_6^k &= -(2+\alpha_k)F_6^{k-1} - 2(1+\alpha_k)G_6^{k-1} - 3(1+\alpha_k)H_6^{k-1} - \beta_k \\
L_1^k &= -(2+\zeta_k)L_1^{k-1} - 2(1+\zeta_k)M_1^{k-1} - 3(1+\zeta_k)N_1^{k-1} \\
L_2^k &= -(2+\zeta_k)L_2^{k-1} - 2(1+\zeta_k)M_2^{k-1} - 3(1+\zeta_k)N_2^{k-1} \\
L_3^k &= -(2+\zeta_k)L_3^{k-1} - 2(1+\zeta_k)M_3^{k-1} - 3(1+\zeta_k)N_3^{k-1} - \eta_k \\
L_4^k &= -(2+\zeta_k)L_4^{k-1} - 2(1+\zeta_k)M_4^{k-1} - 3(1+\zeta_k)N_4^{k-1} - 2\eta_k z_k \\
L_5^k &= -(2+\zeta_k)L_5^{k-1} - 2(1+\zeta_k)M_5^{k-1} - 3(1+\zeta_k)N_5^{k-1} - 3\eta_k z_k^2 \\
L_6^k &= -(2+\zeta_k)L_6^{k-1} - 2(1+\zeta_k)M_6^{k-1} - 3(1+\zeta_k)N_6^{k-1} - \eta_k
\end{aligned}
$$

$$
\begin{aligned}
G_1^k &= F_1^{k-1} + F_1^k + G_1^{k-1} & M_1^k &= L_1^{k-1} + L_1^k + M_1^{k-1} \\
G_2^k &= F_2^{k-1} + F_2^k + G_2^{k-1} & M_2^k &= L_2^{k-1} + L_2^k + M_2^{k-1} \\
G_3^k &= F_3^{k-1} + F_3^k + G_3^{k-1} & M_3^k &= L_3^{k-1} + L_3^k + M_3^{k-1} \\
G_4^k &= F_4^{k-1} + F_4^k + G_4^{k-1} & M_4^k &= L_4^{k-1} + L_4^k + M_4^{k-1} \\
G_5^k &= F_5^{k-1} + F_5^k + G_5^{k-1} & M_5^k &= L_5^{k-1} + L_5^k + M_5^{k-1} \\
G_6^k &= F_6^{k-1} + F_6^k + G_6^{k-1} & M_6^k &= L_6^{k-1} + L_6^k + M_6^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
H_1^k &= -H_1^{k-1} & N_1^k &= -N_1^{k-1} \\
H_2^k &= -H_2^{k-1} & N_2^k &= -N_2^{k-1} \\
H_3^k &= -H_3^{k-1} & N_3^k &= -N_3^{k-1} \\
H_4^k &= -H_4^{k-1} & N_4^k &= -N_4^{k-1} \\
H_5^k &= -H_5^{k-1} & N_5^k &= -N_5^{k-1} \\
H_6^k &= -H_6^{k-1} & N_6^k &= -N_6^{k-1}
\end{aligned}
\qquad k = 2, 3, 4, \ldots, n
$$

(6.15)

Although the recursive equations seem to be somewhat complex, their advantage of numerical efficiency should be recognized.

For free shear traction on the top surface of a composite laminate, another two conditions need to be satisfied. By imposing $\tau^n_{xz}(z_{n+1}) = 0$ and $\tau^n_{yz}(z_{n+1}) = 0$ and recognizing that $\xi_n(z_{n+1}) = 1$, the following two equations can be achieved:

$$\begin{aligned} u_1 + 2z_{n+1}u_2 + 3z^2_{n+1}u_3 + w_{o,x} + a_n u^n_1 + 2a_n u^n_2 + 3a_n u^n_3 &= 0 \\ v_1 + 2z_{n+1}v_2 + 3z^2_{n+1}v_3 + w_{o,y} + a_n v^n_1 + 2a_n v^n_2 + 3a_n v^n_3 &= 0 \end{aligned} \tag{6.16}$$

With the use of recursive equations, these two boundary conditions become:

$$\begin{aligned} &(a_nF^n_1 + 2a_nG^n_1 + 3a_nH^n_1)\,u^1_1 + (a_nF^n_2 + 2a_nG^n_2 + 3a_nH^n_2)\,u^1_2 \\ &+ (a_nF^n_3 + 2a_nG^n_3 + 3a_nH^n_3 + 1)\,u_1 + (a_nF^n_4 + 2a_nG^n_4 + 3a_nH^n_4 + 2z_{n+1})\,u_2 \\ &+ \left(a_nF^n_5 + 2a_nG^n_5 + 3a_nH^n_5 + 3z^2_{n+1}\right)u_3 + (a_nF^n_6 + 2a_nG^n_6 + 3a_nH^n_6 + 1)\,w_{o,x} = 0 \end{aligned}$$

$$\begin{aligned} &(a_nL^n_1 + 2a_nM^n_1 + 3a_nN^n_1)\,v^1_1 + (a_nL^n_2 + 2a_nM^n_2 + 3a_nN^n_2)\,v^1_2 \\ &+ (a_nL^n_3 + 2a_nM^n_3 + 3a_nN^n_3 + 1)\,v_1 + (a_nL^n_4 + 2a_nM^n_4 + 3a_nN^n_4 + 2z_{n+1})\,v_2 \\ &\left(a_nL^n_5 + 2a_nM^n_5 + 3a_nN^n_5 + 3z^2_{n+1}\right)v_3 + (a_nL^n_6 + 2a_nM^n_6 + 3a_nN^n_6 + 1)\,w_{o,y} = 0 \end{aligned}$$

$$\tag{6.17}$$

These equations imply that another two independent variables can be further eliminated in Equations (6.14), i.e.:

$$\begin{aligned} u^1_2 &= A_1u^1_1 + B_1u_1 + C_1u_2 + D_1u_3 + E_1w_{o,x} \\ v^1_2 &= A_2v^1_1 + B_2v_1 + C_2v_2 + D_2v_3 + E_2w_{o,y} \end{aligned} \tag{6.18}$$

where,

$$A_1 = -\frac{a_n F_1^n + 2a_n G_1^n + 3a_n H_1^n}{\Delta_1} \qquad A_2 = -\frac{a_n L_1^n + 2a_n M_1^n + 3a_n N_1^n}{\Delta_2}$$

$$B_1 = -\frac{1 + a_n F_3^n + 2a_n G_3^n + 3a_n H_3^n}{\Delta_1} \qquad B_2 = -\frac{1 + a_n L_3^n + 2a_n M_3^n + 3a_n N_3^n}{\Delta_2}$$

$$C_1 = -\frac{2z_{n+1} + a_n F_4^n + 2a_n G_4^n + 3a_n G_4^n}{\Delta_1} \qquad C_2 = -\frac{2z_{n+1} + a_n L_4^n + 2a_n M_4^n + 3a_n N_4^n}{\Delta_2}$$

$$D_1 = -\frac{3z_{n+1}^2 + a_n F_5^n + 2a_n G_5^n + 3a_n G_5^n}{\Delta_1} \qquad D_2 = -\frac{3z_{n+1}^2 + a_n L_5^n + 2a_n M_5^n + 3a_n N_5^n}{\Delta_2}$$

$$E_1 = -\frac{1 + a_n F_6^n + 2a_n G_6^n + 3a_n G_6^n}{\Delta_1} \qquad E_2 = -\frac{1 + a_n L_6^n + 2a_n M_6^n + 3a_n N_6^n}{\Delta_2}$$

with $\Delta_1 = a_n F_2^n + 2a_n G_2^n + 3a_n H_2^n$; $\Delta_2 = a_n L_2^n + 2a_n M_2^n + 3a_n N_2^n$.

By utilizing Equations (6.18), Equations (6.14) can be rewritten as follows:

$$\begin{aligned}
u_1^k &= R_1^k u_1^1 + R_2^k u_1 + R_3^k u_2 + R_4^k u_3 + R_5^k w_{o,x} \\
u_2^k &= S_1^k u_1^1 + S_2^k u_1 + S_3^k u_2 + S_4^k u_3 + S_5^k w_{o,x} \\
u_3^k &= T_1^k u_1^1 + T_2^k u_1 + T_3^k u_2 + T_4^k u_3 + T_5^k w_{o,x} \\
v_1^k &= O_1^k v_1^1 + O_2^k v_1 + O_3^k v_2 + O_4^k v_3 + O_5^k w_{o,y} \\
v_2^k &= P_1^k v_1^1 + P_2^k v_1 + P_3^k v_2 + P_4^k v_3 + P_5^k w_{o,y} \\
v_3^k &= Q_1^k v_1^1 + Q_2^k v_1 + Q_3^k v_2 + Q_4^k v_3 + Q_5^k w_{o,y}
\end{aligned} \tag{6.19}$$

where,

$$\begin{aligned}
R_1^k &= F_1^k + A_1 F_2^k & S_1^k &= G_1^k + A_1 G_2^k & T_1^k &= H_1^k + A_1 H_2^k \\
R_2^k &= F_3^k + B_1 F_2^k & S_2^k &= G_3^k + B_1 G_2^k & T_2^k &= H_3^k + B_1 H_2^k \\
R_3^k &= F_4^k + C_1 F_2^k & S_3^k &= G_4^k + C_1 G_2^k & T_3^k &= H_4^k + C_1 H_2^k \\
R_4^k &= F_5^k + D_1 F_2^k & S_4^k &= G_5^k + D_1 G_2^k & T_4^k &= H_5^k + D_1 H_2^k \\
R_5^k &= F_6^k + E_1 F_2^k & S_5^k &= G_6^k + E_1 G_2^k & T_5^k &= H_6^k + E_1 H_2^k
\end{aligned}$$

$$
\begin{array}{lll}
O_1^k = L_1^k + A_2 L_2^k & P_1^k = M_1^k + A_2 M_2^k & Q_1^k = N_1^k + A_2 N_2^k \\
O_2^k = L_3^k + B_2 L_2^k & P_2^k = M_3^k + B_2 M_2^k & Q_2^k = N_3^k + B_2 N_2^k \\
O_3^k = L_4^k + C_2 L_2^k & P_3^k = M_4^k + C_2 M_2^k & Q_3^k = N_4^k + C_2 N_2^k \\
O_4^k = L_5^k + D_2 L_2^k & P_4^k = M_5^k + D_2 M_2^k & Q_4^k = N_5^k + D_2 N_2^k \\
O_5^k = L_6^k + E_2 L_2^k & P_5^k = M_6^k + E_2 M_2^k & Q_5^k = N_6^k + E_2 N_2^k
\end{array}
$$

It should be noted that Q_i^k and E_i^k in Equations (6.18) and (6.19) are not the widely used material constants Q_{ij}^k and E_{ij}^k. By virtue of Equations (6.19), the displacement field then becomes:

$$
\begin{aligned}
u^k &= u_0 + \Phi_1^k(z)\, u_1^1 + \Phi_2^k(z)\, u_1 + \Phi_3^k(z)\, u_2 + \Phi_4^k(z)\, u_3 + \Phi_5^k(z)\, w_{o,x} \\
v^k &= v_0 + \Psi_1^k(z)\, v_1^1 + \Psi_2^k(z)\, v_1 + \Psi_3^k(z)\, v_2 + \Psi_4^k(z)\, v_3 + \Psi_5^k(z)\, w_{o,y} \\
w^k &= w_0
\end{aligned}
\qquad (6.20)
$$

where,

$$
\begin{aligned}
\Phi_1^k(z) &= R_1^k(a_k z + b_k) + S_1^k(a_k z + b_k)^2 + T_1^k(a_k z + b_k)^3 \\
\Phi_2^k(z) &= z + R_2^k(a_k z + b_k) + S_2^k(a_k z + b_k)^2 + T_2^k(a_k z + b_k)^3 \\
\Phi_3^k(z) &= z^2 + R_3^k(a_k z + b_k) + S_3^k(a_k z + b_k)^3 + T_3^k(a_k z + b_k)^3 \\
\Phi_4^k(z) &= z^3 + R_4^k(a_k z + b_k) + S_4^k(a_k z + b_k)^3 + T_4^k(a_k z + b_k)^3 \\
\Phi_5^k(z) &= R_5^k(a_k z + b_k) + S_5^k(a_k z + b_k)^3 + T_5^k(a_k z + b_k)^3 \\
\Psi_1^k(z) &= O_1^k(a_k z + b_k) + P_1^k(a_k z + b_k)^2 + Q_1^k(a_k z + b_k)^3 \\
\Psi_2^k(z) &= z + O_2^k(a_k z + b_k) + P_2^k(a_k z + b_k)^2 + Q_2^k(a_k z + b_k)^3 \\
\Psi_3^k(z) &= z^2 + O_3^k(a_k z + b_k) + P_3^k(a_k z + b_k)^2 + Q_3^k(a_k z + b_k)^3 \\
\Psi_4^k(z) &= z^3 + O_4^k(a_k z + b_k) + P_4^k(a_k z + b_k)^2 + Q_4^k(a_k z + b_k)^3 \\
\Psi_5^k(z) &= O_5^k(a_k z + b_k) + P_5^k(a_k z + b_k)^2 + Q_5^k(a_k z + b_k)^3
\end{aligned}
$$

In Equations (6.20), it can be seen that the number of total variables of the 1,2-3 Double Superposition Theory is independent of the number of layers. When compared with the Shear Deformation Theory of the same order, there are six additional layer independent variables. However, when compared with the Generalized Layerwise Theory, coefficients of higher-order terms (of z) are related to the properties and the thickness coordinates of the individual layers instead of unknown variables. In addition, based on the results from the Superposition Theories discussed in Chapter 5, it is expected that the through-the-thickness distributions of in-plane displacement will be kinky across laminate interfaces and the curvature of the transverse shear stress distribution will be layer dependent in the 1,2-3 Double Superposition Theory.

6.3 Variational Equation

The procedures to obtain the variational equation for the 1,2-3 Double Superposition Theory is similar to those used in previous chapters. The corresponding matrices are defined as follows:

$$\varepsilon_x^k = \left[N_x^k\right]\{X\}, \quad \varepsilon_y^k = \left[N_y^k\right]\{X\}, \quad \varepsilon_z^k = \left[N_z^k\right]\{X\}$$
$$\gamma_{xy}^k = \left[N_{xy}^k\right]\{X\}, \quad \gamma_{xz}^k = \left[N_{xz}^k\right]\{X\}, \quad \gamma_{yz}^k = \left[N_{yz}^k\right]\{X\}$$
$$w^+ = \left[N_w^+\right]\{X\}, \qquad w^- = \left[N_w^-\right]\{X\} \tag{6.21}$$

where,

$$\{X\} = \{u_o \;\; v_o \;\; u_1^1 \;\; v_1^1 \;\; u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; w_o\}^T$$

$$\left[N_x^k\right] = \left[\frac{\partial}{\partial x} \;\; 0 \;\; \Phi_1^k\frac{\partial}{\partial x} \;\; 0 \;\; \Phi_2^k\frac{\partial}{\partial x} \;\; 0 \;\; \Phi_3^k\frac{\partial}{\partial x} \;\; 0 \;\; \Phi_4^k\frac{\partial}{\partial x} \;\; 0 \;\; \Phi_5^k\frac{\partial^2}{\partial x^2}\right]$$

$$\left[N_y^k\right] = \left[0 \;\; \frac{\partial}{\partial y} \;\; 0 \;\; \Psi_1^k\frac{\partial}{\partial y} \;\; 0 \;\; \Psi_2^k\frac{\partial}{\partial y} \;\; 0 \;\; \Psi_3^k\frac{\partial}{\partial y} \;\; 0 \;\; \Psi_4^k\frac{\partial}{\partial y} \;\; \Psi_5^k\frac{\partial^2}{\partial y^2}\right]$$

$$\left[N_z^k\right] = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]$$

$$\left[N_{xy}^k\right] = \left[\frac{\partial}{\partial y} \;\; \frac{\partial}{\partial x} \;\; \Phi_1^k\frac{\partial}{\partial y} \;\; \Psi_1^k\frac{\partial}{\partial x} \;\; \Phi_2^k\frac{\partial}{\partial y} \;\; \Psi_2^k\frac{\partial}{\partial x} \;\; \Phi_3^k\frac{\partial}{\partial y} \;\; \Psi_3^k\frac{\partial}{\partial x} \right.$$
$$\left. \Phi_4^k\frac{\partial}{\partial y} \;\; \Psi_4^k\frac{\partial}{\partial x} \;\; (\Phi_5^k + \Psi_5^k)\frac{\partial^2}{\partial x \partial y}\right]$$

$$\left[N_{xz}^k\right] = \left[0 \;\; 0 \;\; \frac{d}{dz}\Phi_1^k \;\; 0 \;\; \frac{d}{dz}\Phi_2^k \;\; 0 \;\; \frac{d}{dz}\Phi_3^k \;\; 0 \right.$$
$$\left. \frac{d}{dz}\Phi_4^k \;\; 0 \;\; \left(\frac{d}{dz}\Phi_5^k + 1\right)\frac{\partial}{\partial x}\right]$$

$$\left[N_{yz}^k\right] = \left[0 \;\; 0 \;\; 0 \;\; \frac{d}{dz}\Psi_1^k \;\; 0 \;\; \frac{d}{dz}\Psi_2^k \;\; 0 \;\; \frac{d}{dz}\Psi_3^k \right.$$
$$\left. 0 \;\; \frac{d}{dz}\Psi_4^k \;\; \left(\frac{d}{dz}\Psi_5^k + 1\right)\frac{\partial}{\partial y}\right]$$

$$\left[N_w^+\right] = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1\right]$$

$$\left[N_w^-\right] = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1\right]$$

By defining

$$\left[N^k\right]_I = \begin{bmatrix} N_x^k \\ N_y^k \\ N_z^k \\ N_{xy}^k \end{bmatrix} \qquad\qquad \left[N^k\right]_{II} = \begin{bmatrix} N_{xz}^k \\ N_{yz}^k \end{bmatrix} \tag{6.22}$$

and substituting Equations (6.22) into (3.21), it yields:

$$\int_{\Omega}\left(\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\left[N^k\right]_I^T\left[Q^k\right]_I\left[N^k\right]_I+\left[N^k\right]_{II}^T\left[Q^k\right]_{II}\left[N^k\right]_{II}\right)dz\right)\{X\}\,dxdy = \tag{6.23}$$

$$+\int_{\Omega}\left(q^+\left[N_w^+\right]\right)dxdy+\int_{\Omega}\left(q^-\left[N_w^-\right]\right)dxdy$$

For a simply-supported, infinitely long, laminated strip under cylindrical bending, the loading can be expressed as $q^+\left(x,\frac{h}{2}\right) = q_o^+\sin px$, where $p = \frac{m\pi}{L}$. By assuming

$$u_o = U_o\cos px \qquad u_1^1 = U_1^1\cos px \qquad u_1 = U_1\cos px \qquad u_2 = U_2\cos px \tag{6.24}$$

$$u_3 = U_3\cos px \qquad w_o = W_o\sin px$$

the simply-supported boundary conditions are satisfied automatically. This is a plane-strain problem with $\varepsilon_y^k = \gamma_{xy}^k = \gamma_{yz}^k = 0$. The six variables to be determined are of the following matrix:

$$\{X\} = \{u_o \quad u_1^1 \quad u_1 \quad u_2 \quad u_3 \quad w_o\}^T \tag{6.25}$$

By substituting Equations (6.25) into Equations (6.24), it is obvious that

$$\{\bar{X}\} = \{U_o \quad U_1^1 \quad U_1 \quad U_2 \quad U_3 \quad W_o\}^T \tag{6.26}$$

and

$$\left[N_x^k\right] = \left[\bar{N}_x^k\right]\sin px = \left[-p \quad -p\Phi_1^k \quad -p\Phi_2^k \quad -p\Phi_3^k \quad -p\Phi_4^k \quad -p^2\Phi_5^k\right]\sin px$$

$$\left[N_z^k\right] = \left[\bar{N}_z^k\right]\sin px = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]\sin px$$

$$\left[N_{xz}^k\right] = \left[\bar{N}_{xz}^k\right]\cos px = \left[0 \quad \frac{d}{dz}\Phi_1^k \quad \frac{d}{dz}\Phi_2^k \quad \frac{d}{dz}\Phi_3^k \quad \frac{d}{dz}\Phi_4^k \right.$$
$$\left. p\left(\frac{d}{dz}\Phi_5^k + 1\right)\right]\cos px$$

$$\left[N_w^+\right] = \left[\bar{N}_w^+\right]\sin px = \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\right]\sin px$$

Substituting Equations (6.26) into Equations (3.25), it yields:

$$\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(Q_{11}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_x^k\right] + Q_{13}^k\left[\bar{N}_x^k\right]^T\left[\bar{N}_z^k\right] + Q_{13}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_x^k\right] + Q_{33}^k\left[\bar{N}_z^k\right]^T\left[\bar{N}_z^k\right]\right.$$
$$\left. + Q_{55}^k\left[\bar{N}_{xz}^k\right]^T\left[\bar{N}_{xz}^k\right]\right)dz\,\{\bar{X}\} = q_o^+\left[\bar{N}_w^+\right]^T \tag{6.27}$$

By solving Equations (6.27), U_o, U_1^1, U_1, U_2, U_3, and W_o can be obtained.

6.4 Numerical Results and Discussion

The results from all three theories turn out to be identical for a [0/90/0] laminate. They are shown from Figure 6.1 to Figure 6.5. Excellent agreement with the exact solution is obtained for $\bar{\sigma}_x$, $\bar{\tau}_{xz}$, and $\bar{u}$. Since constant w is assumed and no boundary condition for transverse normal stress is enforced in the analysis, $\bar{w}$ and $\bar{\sigma}_z$ are not good, although they both have small values.

The 1,2-3 Double Superposition Theory is also applied to both even-numbered and odd-numbered laminates. The results for the former are shown in Figures 6.6 and 6.7 for

2, 6, and 14 layers, while in Figures 6.8 and 6.9 results for 3, 7, and 15 layers are shown. Good results seem to be achieved. The reason that the 30-ply and the 31-ply laminates are not shown in corresponding figures is because of poor results attributed to numerical round-off error.

6.5 Summary

The importance of the completeness and the roles of high-order terms in a displacement field is demonstrated in this analysis. The application of Hypothesis for Double Superposition, though not verified mathematically, seems to be effective in giving a good laminate theory.

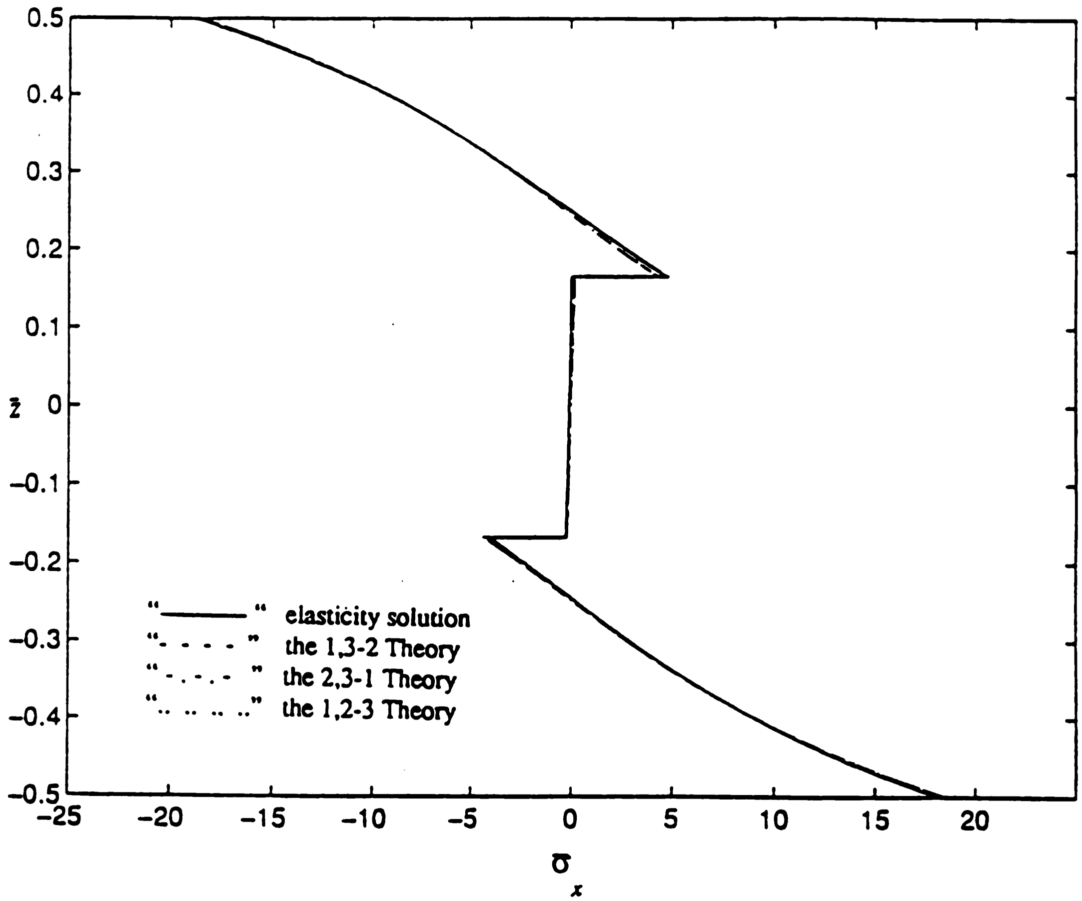

Figure 6.1 - Comparison of σ_x from the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories with the elasticity solutions for a [0/90/0] laminate.

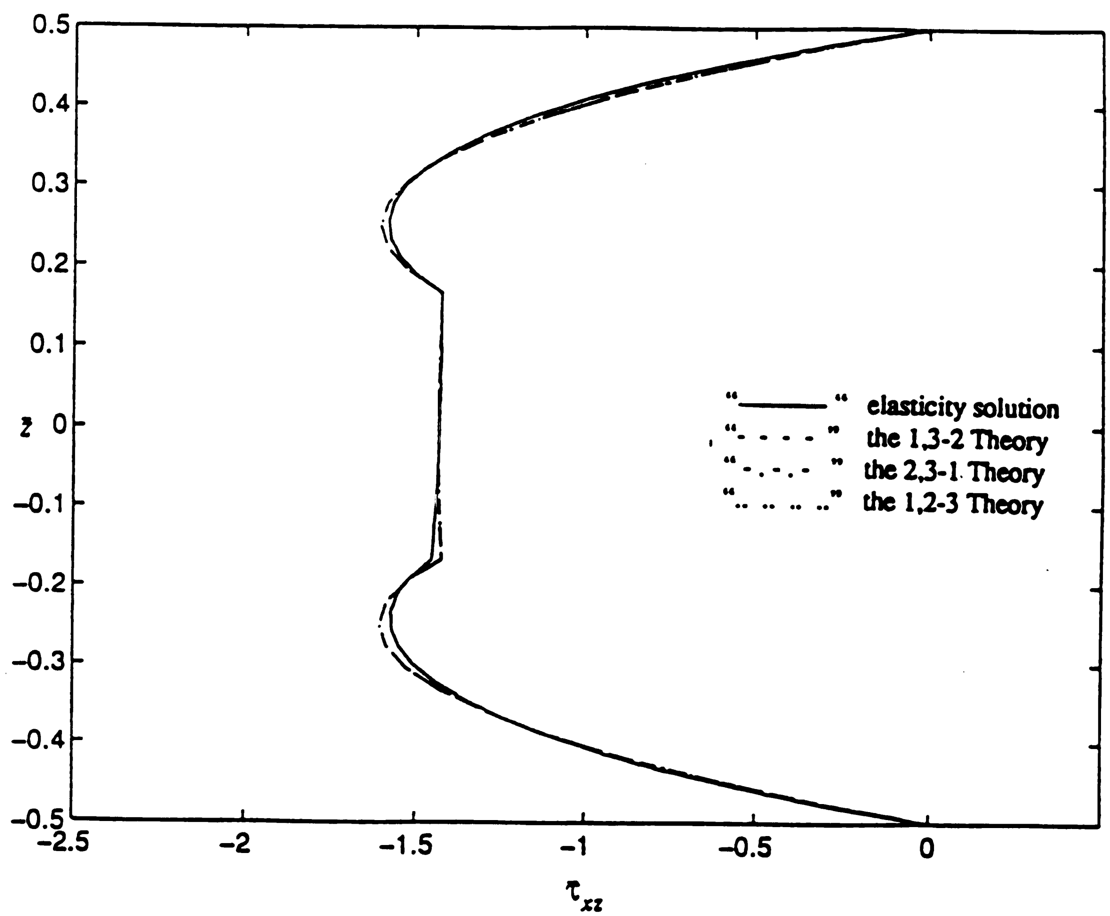

Figure 6.2 - Comparison of τ_{xz} from the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories with the elasticity solutions for a [0/90/0] laminate.

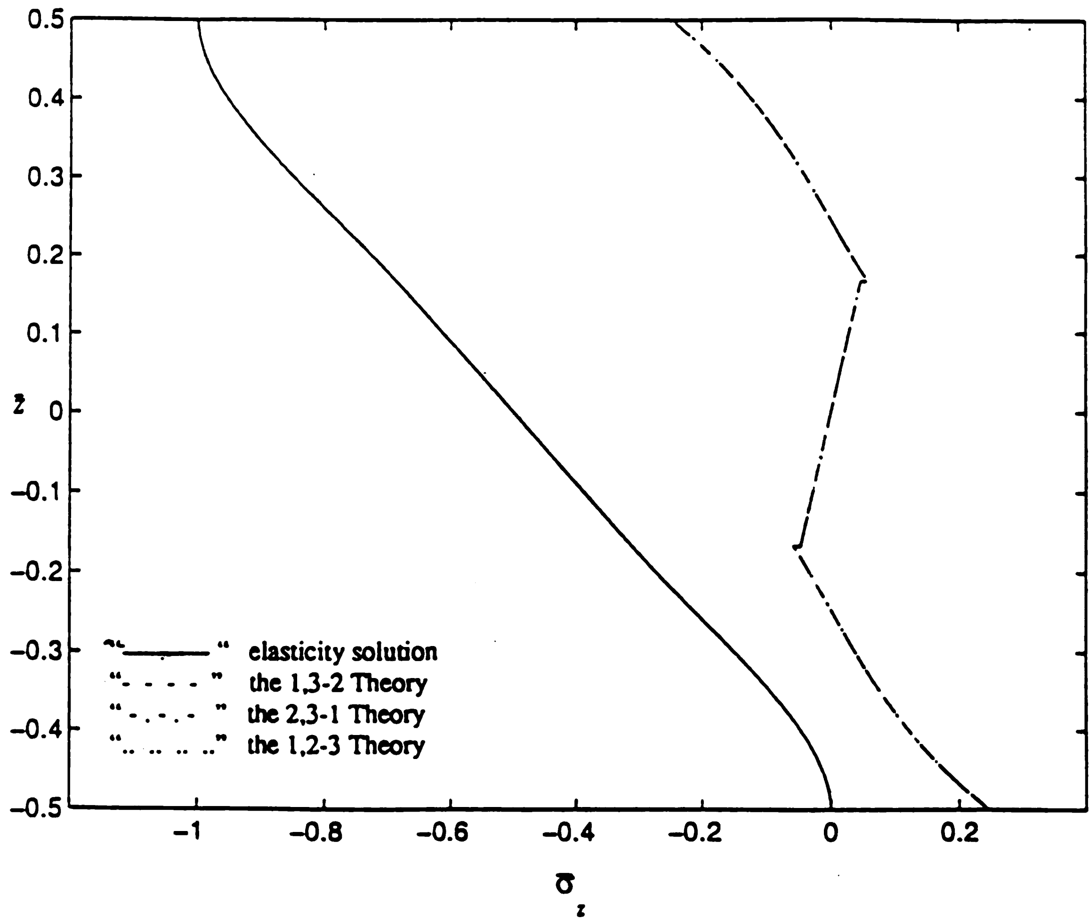

Figure 6.3 - Comparison of σ_z from the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories with the elasticity solutions for a [0/90/0] laminate.

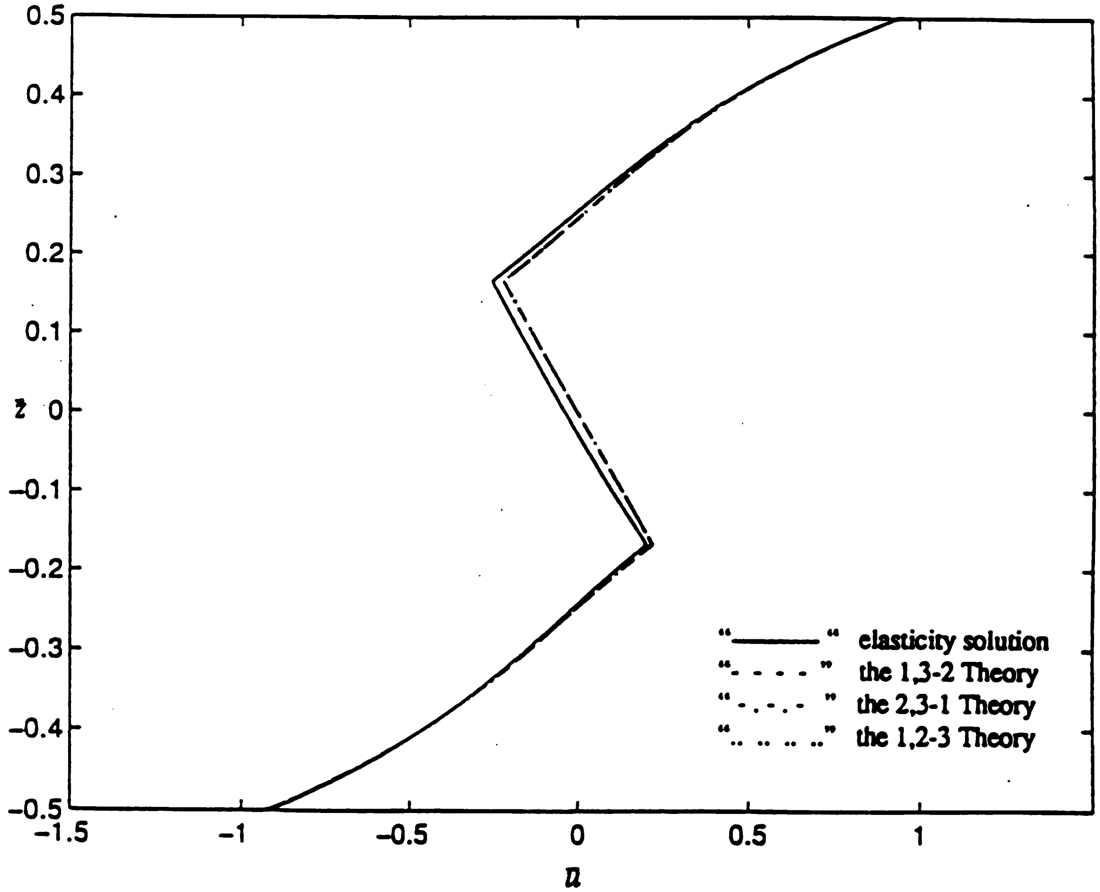

Figure 6.4 - Comparison of u from the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories with the elasticity solutions for a [0/90/0] laminate.

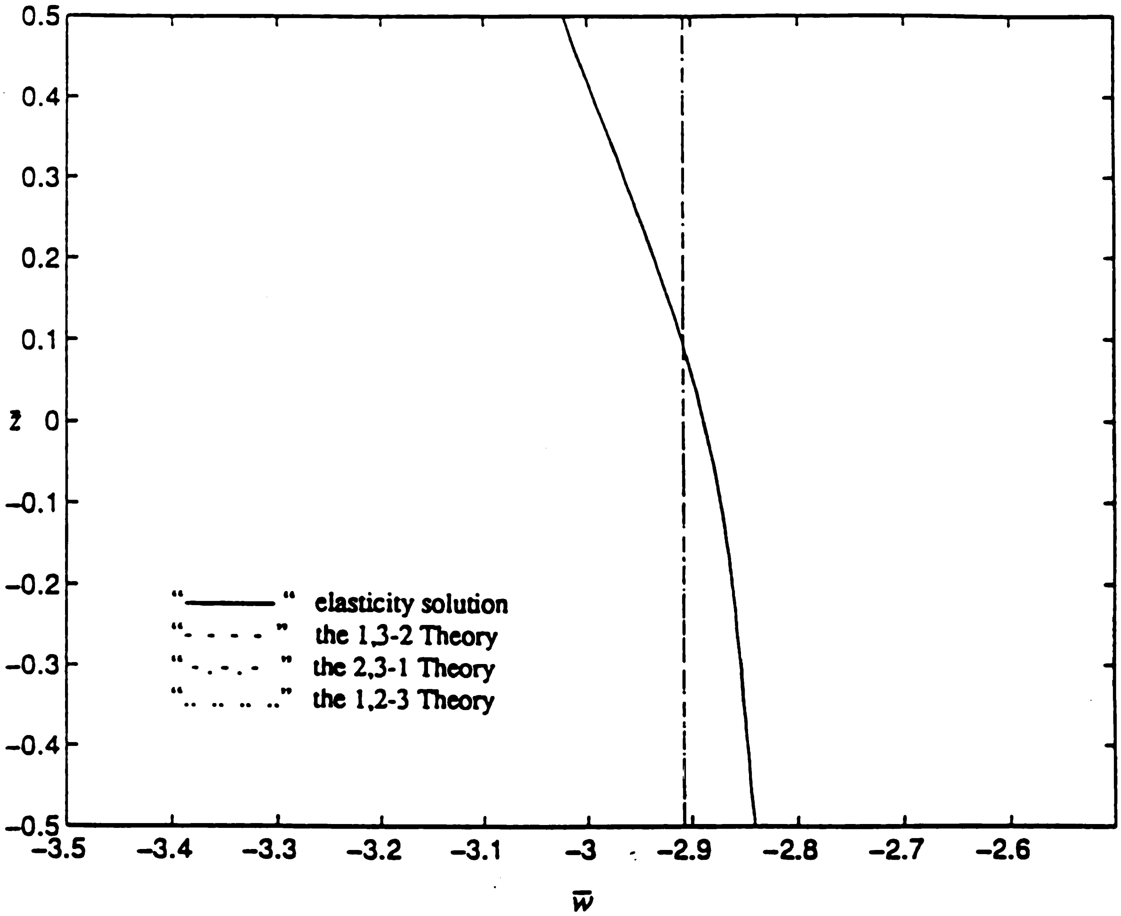

Figure 6.5 - Comparison of $\bar{w}$ from the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories with the elasticity solutions for a [0/90/0] laminate.

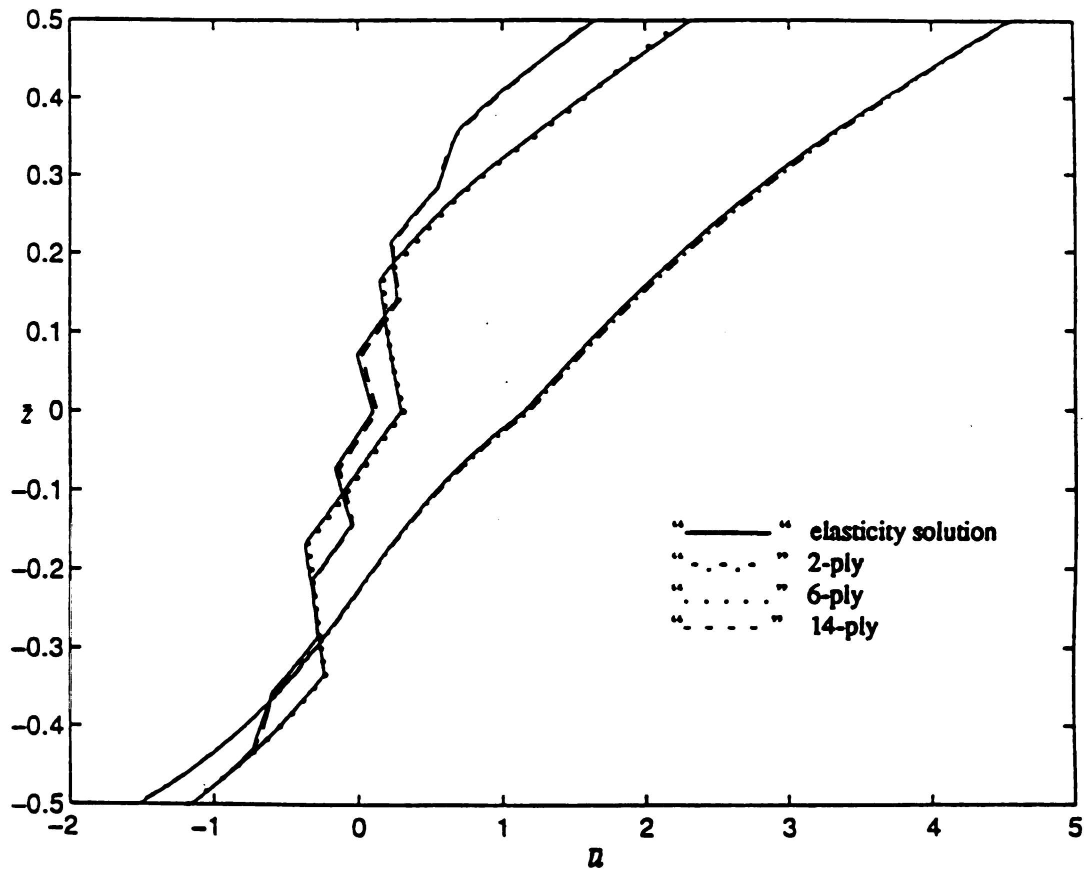

Figure 6.6 - Comparison of u from the 1,2-3 Double Superposition Theory and elasticity solutions for 2-, 6-, and 14-layer unsymmetrical laminates.

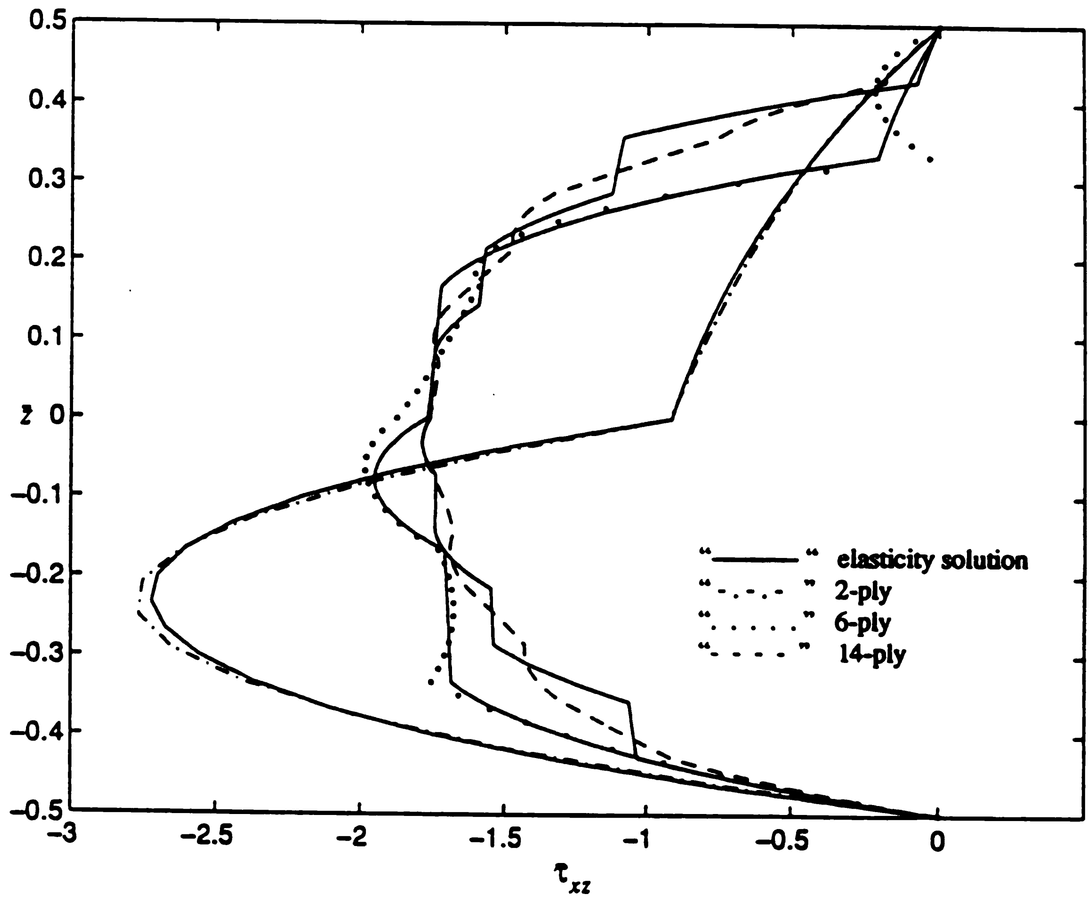

Figure 6.7 - Comparison of τ_{xz} from the 1,2-3 Double Superposition Theory and elasticity solutions for 2-, 6-, and 14-layer unsymmetrical laminates.

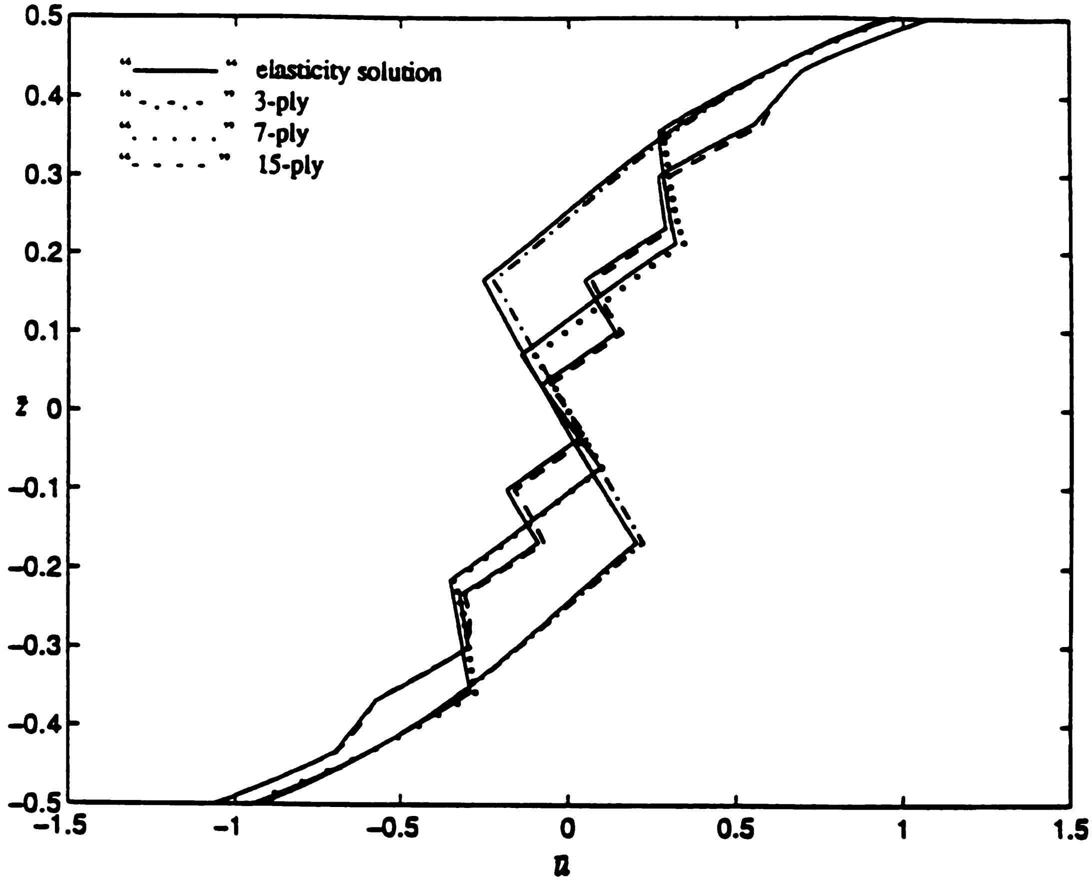

Figure 6.8 - Comparison of u from the 1,2-3 Double Superposition Theory and elasticity solutions for 3-, 7-, and 15-layer symmetric laminates.

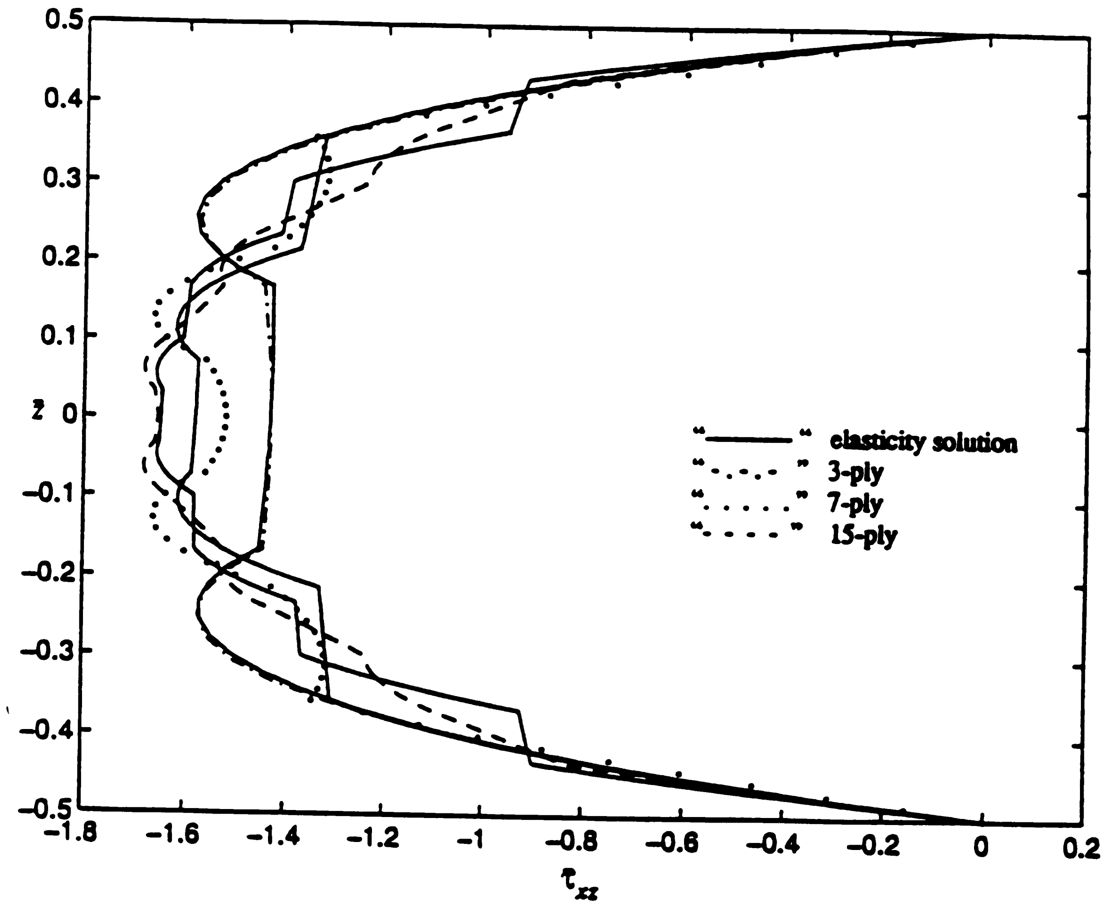

Figure 6.9 - Comparison of τ_{xz} from the 1,2-3 Double Superposition Theory and elasticity solutions for 3-, 7-, and 15-layer symmetric laminates.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

The Shear Deformation Theories give good results for global distribution of in-plane stresses but poor results for local distribution of interlaminar stresses. Layerwise Theories give excellent results for both global and local distributions of displacement and stress (both in-plane and out-of-plane). The former group of theories has the advantage of numerical efficiency because the total number of variables is independent of the layer number, while the latter group suffers from a numerical crisis if the layer number becomes too large. A compromising theory, the so-called the Generalized Zigzag Theory, is presented. With high-order terms, which are not necessary to be layer-dependent, both in-plane stress and transverse shear stress distributions are greatly improved. Moreover, the zigzag distribution of in-plane displacement can be accurately predicted.

The Generalized Zigzag Theory is a layer-independent theory, though having two layer-dependent terms for both the zeroth- and the first-order terms initially. Due to its success in laminate analysis, the feasibility of assigning the two layer-dependent variables in high-order terms (i.e. the second- and the third-order terms) is examined, resulting in the Quasi-layerwise Theories. Unfortunately, a physical impossibility — coordinate dependency, takes place. It then requires the use of the Global-Local Superposition Technique to formulate the laminate theory to be coordinate-independent. In addition, it is necessary to express the displacement components in an explicit manner to have the advantage of numerical efficiency. The recursive expressions presented in this study, though somewhat tedious, are aimed at this purpose. By examining the results based on the Superposition Theories, an important question regarding the complete selection of the high-order terms is raised. It is concluded that the completeness is two fold: not only can no low-order terms be skipped, but more high-order terms are preferred.

The objective of completeness seems to conflict with the fundamental of two continuity conditions in each coordinate direction. In order to satisfy both aspects, a special technique, namely the Hypothesis for Double Superposition, is proposed. Several three-term theories, called the Double Superposition Theories, are examined. Since these are three layer-dependent terms in a displacement field instead of two, an extra continuity condition of displacement is added twice to the formulation through the application of the superposition principle. All the Double-Superposition Theories are shown to have identical results for a [0/90/0] laminate. They give excellent values for in-plane displacement, in-plane stress, and transverse shear stress. However, because w is considered as constant in the example, both transverse displacement and normal stress are not as good as the remaining components.

Among all the theories examined in this thesis, it seems that the Generalized Zigzag Theory with up to the seventh-order term and the third-order Double Superposition Theories give the best agreement with the Pagano's solution in all ranges of layer number for both symmetric and unsymmetrical laminates. Although they both are layer-number independent theories, the former has seven degrees-of-freedom while the latter has only three, provided w is considered to be constant through the laminate thickness. As a consequence, the Double Superposition Theories are concluded to be the best selection for laminate analysis in this thesis.

7.2 Recommendations

This study presents three laminate theories for laminated composite analysis — the 1,2-3, 1,3-2, and 2,3-1 Double Superposition Theories. However, the success is primarily referred to the dominant displacement and stress components, namely in-plane displacement, in-plane stress, and transverse shear stress. Apparently, both the transverse normal displacement and stress are not carefully considered in the analysis. It is suggested that these two components should be included in the future development of laminate theories.

In addition, it should be pointed out that the assembly of the composite layer through the laminate thickness is based on the assumption that the interfacial bonding is perfectly rigid. Due to the complexity involved in the composite fabrication, it is possible to have non-rigid bonding such as shear slipping and normal separation on laminate interfaces. It is then required to include special stress-displacement relations in the laminate formulation. Besides, geometrical irregularities can also occur in composite fabrication, e.g. wavy fibers in composite laminates. In order to model the geometrical irregularity, special strain-displacement relation may be required. Certainly the application of the laminate theories to composite structures with complex geometry such as shells of various shapes, is another example.

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