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VARIATIONAL PROBLEMS ON COMPLEX CONTACT MANIFOLDS WITH APPLICATIONS TO TWISTOR SPACE THEORY

By

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ABSTRACT

VARIATIONAL PROBLEMS ON COMPLEX CONTACT MANIFOLDS WITH APPLICATIONS TO TWISTOR SPACE THEORY

By

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A complex contact manifold is a complex manifold M with complex dimension $2n+1$ and an open atlas $\mathcal{U} = \{\mathcal{O}_j\}$ such that:

1. On each $\mathcal{O}_j$, there exists a local holomorphic 1–form ω_j, called a *local contact form*, such that $\omega_j \wedge (d\omega_j)^n \neq 0$.
2. On $\mathcal{O}_j \cap \mathcal{O}_k$, there exists a holomorphic function $f_{jk} : \mathcal{O}_j \cap \mathcal{O}_k \to C$ such that $\omega_j = f_{jk}\omega_k$.

A famous example of this type of manifold is the twistor space of a quaternionic-Kähler manifold with nonzero scalar curvature. The kernel of the contact forms forms a $2n$–complex-dimensional, non-integrable subbundle $\mathcal{H}$ of TM. There is also a subbundle $\mathcal{V}$ with certain special properties such that $TM = \mathcal{V} \oplus \mathcal{H}$.

In this thesis, we investigate special metrics on compact complex contact manifolds by studying the critical conditions of various Riemannian functionals on a particular class of Riemannian metrics called the *associated metrics.* An associated metric is a Hermitian metric with respect to the complex structure on M, which makes $TM = \mathcal{V} \oplus \mathcal{H}$ an orthogonal splitting and gives $\mathcal{H}$ a quaternionic structure. These associated metrics generalize the Salamon-Bérard-Bergery metrics on the twistor spaces of quaternionic-Kähler manifolds. After defining and showing the existence of these metrics, we develop their structure equations.

We then define two Riemannian functionals, called the Ricci curvature of $\mathcal{V}$ and the $*$–Ricci curvature of $\mathcal{V}$. Using certain properties of the space of all associated metrics, we are able to find the critical conditions of these functionals. Finally, we

investigate the complex contact structure of twistor spaces by applying the previous results. This work allows us to characterize the Salamon-Bérard-Bergery metrics among all associated metrics.

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INTRODUCTION

A *complex contact manifold* is a complex manifold M with $dim_{\mathbb{C}}M = 2n+1$ and an atlas $\mathcal{U} = \{\mathcal{O}_j\}$ such that:

1. On each $\mathcal{O}_j$, there exists a maximally-ranked holomorphic 1-form ω_j, that is, $\omega_j \wedge (d\omega_j)^n \neq 0$ on $\mathcal{O}_j$;

2. On $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, $\omega_j = f_{jk}\omega_k$ where $f_{jk} : \mathcal{O}_j \cap \mathcal{O}_k \to \mathbb{C}^*$ is a holomorphic function.

The study of these geometric objects began with Kobayashi [Ko] in the late 1950's. The subject continued in the early 60's during which both Boothby in [Bo] and Wolf in [Wo] studied homogeneous complex contact manifolds. Further inquiry began again in the early 1980's with the work of Ishihara and Konishi in [IsS1],[IsS2], and [IsS3]. Since this time, research in complex contact geometry has experienced a resurgence with recent important results by Salamon [Sa], Bérard-Bergery [Bé], LeBrun [Le1], [Le2], and Morpianu and Semmelmann [MoS].

However, with the exclusion of the work of Ishihara and Konishi, complex contact manifolds have not yet been approached from a solely Riemannian perspective. This approach has been very successful in the study of real contact manifolds, e.g. Blair [Bl1]. In particular, by specifying a special class of Riemannian metrics called associated metrics on a real contact manifold, many properties of real contact manifolds can be exploited. Furthermore, much success has occurred through the study of associated metrics which are critical for various Riemannian functionals [Bl2],[Bl3], [Bl4], [Bl5], and [BL].

In the first chapter of this dissertation, we establish the definition and existence of an associated metric for a complex contact manifold. We continue by deriving structure equations for these metrics. In the second chapter, we describe the space of all associated metrics and derive the critical conditions for two Riemannian functionals on this space. Finally, in chapter three, we apply these results in order to study the complex contact structure of twistor spaces over quaternionic-Kähler manifolds with positive scalar curvature.

Chapter One

COMPLEX CONTACT MANIFOLDS AND ASSOCIATED METRICS

We establish the necessary notation in Section 1. In Section 2 and Section 3, we define and prove the existence of an associated metric of a complex contact structure. In section 4, we establish basic facts about a complex contact metric structure. Finally, we derive important structure equations for associated metrics in Section 5 and Section 6.

1.1 Notation and Basic Identities

In order to expedite many of the calculations involved ahead of us, we need to set up some fairly basic notation.

Definition Let (V, J) be a vector space with almost complex structure J. For any linear transformation $A : V \to V$, we define linear transformations:

$$A^s : V \to V,\ A^d : V \to V$$

by

$$A^s = \frac{1}{2}(A - JAJ),\ A^d = \frac{1}{2}(A + JAJ).$$

Then, we have the following facts:

1) $A^s J = JA^s$.

2) $A^d J = -JA^d$.

3) $A = A^s + A^d$.

4) Suppose g is a hermitian metric on (V, J). Then, A is (skew-)symmetric with respect to g, if and only if both A^s and A^d are (skew-)symmetric with respect to g.

5) For any two linear transformations $A, B : V \to V$, we have:

$$(AB)^s = A^s B^s + A^d B^d,$$

$$(AB)^d = A^s B^d + A^d B^s.$$

6) Let $V^{\mathbb{C}}$ be the complexification of V with

$$V^{1,0} = \{X \in V^{\mathbb{C}} : JX = iX\},\ V^{0,1} = \{X \in V^{\mathbb{C}} : JX = -iX\}.$$

Let $A : V \to V$ be any linear transformation. Extend A to be a linear transformation on $V^{\mathbb{C}}$. Then:

$$A^s(V^{0,1}) \subset V^{0,1};\ A^s(V^{1,0}) \subset V^{1,0}$$

$$A^d(V^{0,1}) \subset V^{1,0};\ A^d(V^{1,0}) \subset V^{0,1}.$$

In our case, $(V, J) = (TM, J)$ where J is an integrable complex structure on M. Although we could easily define analogous notation for other almost complex structures that will appear on M , we will not; ^{s} and ^{d} will always be defined with respect to the integrable structure J.

Let us now suppose that V has an inner product g. Let $A : V \otimes V \to \mathbb{R}$ be any $(0,2)$-tensor on V. Then we define $A^\sharp : V \to V$ to be the linear transformation given by:

$$g(A^\sharp X, Y) = A(X, Y)\ \forall\ X, Y \in V.$$

Also, if we fix a vector X, we define the $(0,1)-$tensor $\iota(X)A$ by:

$$(\iota(X)A)(Y) = A(X, Y),$$

for every vector $Y \in V$.

Let $T : V \to V$ be any linear transformation. Then we define two new linear transformations

$$sym(T) : V \to V,\ skew(T) : V \to V,$$

by:

$$g(sym(T)X,Y) = \frac{1}{2}(g(TX,Y) + g(TY,X));$$
$$g(skew(T)X,Y) = \frac{1}{2}(g(TX,Y) - g(TY,X))$$

Then $sym(T)$ is symmetric with respect to g, and $skew(T)$ is skew-symmetric with respect to g. Also, $T = sym(T) + skew(T)$.

Finally, a note about the component notation of endomorphisms. For $T : V \to V$ a linear transformation on V and $\{e_1, \ldots e_N\}$ an orthonormal basis of V, then we set $Te_j = T_j{}^k e_k$.

1.2 Basic Definitions and Constructions

Recall that we call a complex manifold M with $dim_{\mathbb{C}} M = 2n+1$ and complex structure J a *complex contact manifold*, if there exists an atlas $\mathcal{U} = \{\mathcal{O}_j\}$ such that:

1. On each $\mathcal{O}_j$, there exists a maximally-ranked holomorphic 1-form ω_j, that is, $\omega_j \wedge (d\omega_j)^n \neq 0$ on $\mathcal{O}_j$,

2. On $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset, \omega_j = f_{jk}\omega_k$ where $f_{jk} : \mathcal{O}_j \cap \mathcal{O}_k \to \mathbb{C}^*$ is a holomorphic function.

For any complex contact manifold M, let J denote its complex structure.

Set $\mathcal{H}_j = ker(\omega_j)$. Then, on $\mathcal{O}_j \cap \mathcal{O}_k, \mathcal{H}_j = \mathcal{H}_k$. So, $\mathcal{H} = \cup \mathcal{H}_j$ is a well-defined holomorphic, non-integrable subbundle on M, called the *contact subbundle* or the *horizontal subbundle*. Note: $dim_{\mathbb{R}} \mathcal{H} = 4n$.

Set $L = TM/\mathcal{H}$. Then L is a complex line bundle and

$$0 \to \mathcal{H} \to TM \stackrel{\omega}{\to} L \to 0$$

is a short, exact sequence. Let $\tilde{p} : L \to M$ be the natural projection. We may consider each $\omega_j = \zeta_j \circ \omega$, where each $\zeta_j : \tilde{p}^{-1}(\mathcal{O}_j) \to \mathcal{O}_j \times \mathbb{C}$ is a local trivialization of L and $\zeta_j \circ \zeta_k^{-1} = f_{jk}$.

Let $\{\eta_j : \tilde{p}^{-1}(\mathcal{O}_j) \to \mathcal{O}_j \times \mathbb{C}\}$ be a trivialization of L such that $\eta_j \circ \eta_k^{-1} = h_{jk}$ where $h_{jk} : \mathcal{O}_j \times \mathcal{O}_k \to S^1$. Set $\pi_j = \eta_j \circ \omega$. Then:

1. Since $ker(\pi_j) = ker(\omega_j)$, π_j is a complex-function multiple of ω_j.

2. $\pi_j \wedge (d\pi_j)^{2n} \wedge \bar{\pi}_j \wedge (d\bar{\pi}_j)^{2n} \neq 0$ on $\mathcal{O}_j$.

3. On $\mathcal{O}_j \cap \mathcal{O}_k, \pi_j = h_{jk}\pi_k$.

Set $\pi_j = u_j - iv_j$, where u_j, v_j are real 1-forms on $\mathcal{O}_j$. Then $v_j = u_j \circ J$. We call $\underline{\pi} = \{\pi_j\}$ a *normalized contact structure* with respect to $\underline{\omega} = \{\omega_j\}$.

It is easily seen that M has a global complex contact structure if and only if L is a trivial complex line bundle over M. Since Kobayashi [Ko] has shown that $c_1(M) = (n+1)c_1(L)$, we see that, if M is compact, then M has a global complex contact structure if and only if $c_1(M) = 0$, i.e. L is trivial, cf. Boothby [Bo].

From now on, if $\mathcal{O}_j$ is understood, we will suppress the subscripts of each local tensor.

Now, we wish to show the existence of a unique complex line subbundle $\mathcal{V}$ of TM which satisfies certain properties along with $TM = \mathcal{V} \oplus \mathcal{H}$.

On $\mathcal{O} \in \mathcal{U}$, set

$$N_p = \{X \in T_p\mathcal{O} : du(X,Y) = 0 \ \forall Y \in \mathcal{H}_p\}$$

$$\tilde{N}_p = \{X \in T_p\mathcal{O} : dv(X,Y) = 0 \ \forall Y \in \mathcal{H}_p\} \ \forall \ p \in \mathcal{O}.$$

Then both N and $\tilde{N}$ on $\mathcal{O}$ have real-dimension 2. Also, since du and dv have maximal rank on $\mathcal{H}$, we know that $N \cap \mathcal{H} = \tilde{N} \cap \mathcal{H} = (0)$, i.e. $u(X) \neq 0$ and $v(X) \neq 0 \ \forall X \in (N - (0)) \cup (\tilde{N} - (0))$.

Let $U, \tilde{V}$ be the unique vector fields on $\mathcal{O}$ such that:

$$U \in N, \tilde{V} \in \tilde{N}$$

$$u(U) = 1, v(U) = 0$$

$$u(\tilde{V}) = 0, v(\tilde{V}) = 1.$$

Set $V = -JU$. So, $JV = U$.

Lemma 1.2.1 For $X \in T\mathcal{O}, Y \in \mathcal{H}, dv(X,Y) = du(JX,Y)$.

Proof: Suppose $Y \in \mathcal{H}$ is an infinitesimal automorphism of J, i.e. $[Y, JX] = J[Y, X] \ \forall X \in TM$.

Thus, for $X \in \mathcal{H}, X = U, or X = V$,

$$\begin{aligned}
dv(X,Y) &= \frac{1}{2}(Xv(Y) - Yv(X) - v([X,Y])) \\
&= -\frac{1}{2}v([X,Y]) \\
&= \frac{1}{2}v([J^2X,Y]) \\
&= \frac{1}{2}v(J[JX,Y]) \\
&= \frac{1}{2}(u \circ J)(J[JX,Y]) \\
&= -\frac{1}{2}u([JX,Y]) \\
&= du(JX,Y)
\end{aligned}$$

Since $\mathcal{H}$ is a holomorphic subbundle of TM, we can choose a local basis $\underline{E}'$, of $\mathcal{H}$ such that each element of $\underline{E}'$ is a infinitesimal automorphism of J. Set $\underline{E} = \underline{E}' \cup \{U, V\}$. Then we know from the above calculation that $dv(X,Y) = du(JX,Y)$ for all $X \in \underline{E},\ Y \in \underline{E}'$. Since $\underline{E}$ is a local basis of TM and $\underline{E}'$ is a local basis of $\mathcal{H}$, this proves the lemma.

Thus, for $Y \in \mathcal{H},\ dv(V,Y) = du(JV,Y) = du(U,Y) = 0$. Also, $u(V) = u(-JU) = -v(U) = 0$ and $v(V) = -(u \circ J)(JU) = u(U) = 1$. Thus, $V = \tilde{V}$, and we have the following proposition.

Proposition 1.2.2 On each $\mathcal{O} \in \mathcal{U}$, there exist unique vector fields $U, V = -JU$ such that:

$$u(U) = 1,\ v(U) = 0,\ (\iota(U)du)|_{\mathcal{H}} = 0$$

$$u(V) = 0,\ v(V) = 1,\ (\iota(V)dv)|_{\mathcal{H}} = 0.$$

Set $\mathcal{V}_{\mathcal{O}} = \text{span}\{U, V\}\ \forall \mathcal{O} \in \mathcal{U}$. We shall now show that on $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset, \mathcal{V}_{\mathcal{O}_j} = \mathcal{V}_{\mathcal{O}_k}$.

Now, $\pi_j = u_j - iv_j = h_{jk}\pi_k$, and $\pi_k = u_k - iv_k$. Set $h_{jk} = a + ib$, where a and b are real functions on $\mathcal{O}_j \cap \mathcal{O}_k$ such that $a^2 + b^2 = 1$. Then,

$$u_j = au_k + bv_k,$$

$$v_j = -bu_k + av_k.$$

Let $X \in \mathcal{H}|_{\mathcal{O}_j \cap \mathcal{O}_k}$. Then,

$$\begin{aligned}
du_j(aU_k + bV_k, X) &= d(au_k + bv_k)(aU_k + bV_k, X) \\
&= ad(au_k + bv_k)(U_k, X) + bd(au_k + bv_k)(V_k, X) \\
&= a^2 du_k(U_k, X) + ab dv_k(U_k, X) + ab du_k(V_k, X) + b^2 dv_k(V_k, X) \\
&\quad + a(da \wedge u_k)(U_k, X) + a(db \wedge v_k)(U_k, X) \\
&\quad + b(da \wedge u_k)(V_k, X) + b(db \wedge v_k)(V_k, X) \\
&= ab(dv_k(U_k, X) + du_k(V_k, X)) - \frac{1}{2} ada(X) - \frac{1}{2} bdb(X) \\
&= ab(du_k(JU_k, X) - du_k(JU_k, X)) - \frac{1}{2} X(a^2 + b^2) \\
&= 0.
\end{aligned}$$

$$u_j(aU_k + bV_k) = (au_k + bv_k)(aU_k + bV_k)$$

$$= a^2 + b^2$$

$$= 1.$$

$$v_j(aU_k + bV_k) = (-bu_k + av_k)(aU_k + bV_k)$$

$$= -ab + ab$$

$$= 0.$$

Therefore, $U_j = aU_k + bV_k$. Consequently, $V_j = -bU_k + aV_k$. So,

$$U_j - iV_j = (aU_k + bV_k) - i(-bU_k + aV_k)$$

$$= (a + ib)(U_k - iV_k)$$

$$= h_{jk}(U_k - iV_k),$$

$$\text{i.e. } U_j + iV_j = h_{jk}^{-1}(U_k + iV_k).$$

Thus, $\mathcal{V}$, given locally by span$\{U, V\}$, is a well-defined, J-invariant subbundle of TM, which can also be seen as a complex line bundle over M with transition functions given by $\{h_{jk}^{-1}\}$.

We have shown:

Theorem 1.2.3 There is a unique two-dimensional, J-invariant, subbundle $\mathcal{V}$ of TM such that:

1. $TM \cong \mathcal{H} \oplus \mathcal{V}$
2. $\mathcal{V} \cong L$ as complex line bundles.
3. There is a local basis of $\mathcal{V}, \{U, V = -JU\}$ with :
   a. $u(U) = 1, v(U) = 0, u(V) = 0, v(V) = 1,$
   b. $du(U, X) = dv(V, X) = 0 \ \forall X \in \mathcal{H}.$

We call $\mathcal{V}$ the *vertical subbundle* of the contact structure. Let $p : TM \to \mathcal{H}$, $q : TM \to \mathcal{V}$ be the projections with respect to the splitting $TM \cong \mathcal{H} \oplus \mathcal{V}$. Note that, on $\mathcal{O} \in \mathcal{U}, q = u \otimes U + v \otimes V$. In other words, $\pi = \phi \circ q$, where $\phi : \mathcal{V}|_{\mathcal{O}} \to \mathcal{O} \times \mathbb{C}$, is a particular local trivialization of $\mathcal{V}$. Also, note that, since both $\mathcal{H}$ and $\mathcal{V}$ are preserved by J, $p \circ J = J \circ p$

and $q \circ J = J \circ q$. Since we will need such notation in the future, we will define $J' = pJ$, i.e. J restricted to $\mathcal{H}$ and $J'' = qJ$, i.e. J restricted to $\mathcal{V}$.

From this point hence, we will assume that $\mathcal{V}$ is, in fact, a foliation, i.e. it is an integrable sub-bundle of TM. Although it is still unproven whether the vertical subbundle of any complex contact manifold is integrable, every known example of a complex contact manifold has an integrable vertical subbundle. Furthermore, the twistor spaces over quaternionic-Kaehler manifolds with positive Ricci curvature have this sort of vertical subbundle. Thus, for our work, we will lose no generality by making this assumption.

Now, each $\mathcal{O} \in \mathcal{U}$, define a local, $\mathbb{C}$-valued 2-form Ω by:

$$\Omega = d\pi(pX, pY) \ \forall X, Y \in T\mathcal{O}.$$

Let $\tilde{G} = \mathrm{Re}\ \Omega$ and $\tilde{H} = -\mathrm{Im}\ \Omega$. So, by Lemma 1.2.1, we know: $\tilde{H}(X,Y) = \tilde{G}(JX,Y)$ $\forall\ X, Y \in T\mathcal{O}$.

Suppose $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$ with $\pi_j = h_{jk}\pi_k$. Then, $d\pi_j = dh_{jk} \wedge \pi_k + h_{jk} d\pi_{jk}$. So, $\Omega_j = h_{jk}\Omega_k$, since $\pi_k \circ p = 0$. Thus, $\tilde{G}_j = a\tilde{G}_k + b\tilde{H}_k$, $\tilde{H}_j = -b\tilde{G}_k + a\tilde{H}_k$. Since $du_j(X,Y) = \Omega_j(X,Y)$ and $du_j(U_j, X) = 0$ for all $X, Y \in \mathcal{H}$, we know that $\tilde{G}_j = du_j + \alpha_j \wedge v_j$ for some real 1-form α_j. Similarly, we have that $\tilde{H}_j = dv_j + \beta_j \wedge u_j$ for some real 1-form β_j.

Suppose X is perpendicular to V_j. Then

$$\begin{aligned} 0 &= \tilde{G}_j(X, V_j) \\ &= du_j(X, V_j) + \alpha_j \wedge v_j(X, V_j) \\ &= du_j(X, V_j) + \frac{1}{2}\alpha_j(X). \end{aligned}$$

Thus, $\alpha_j(X) = -2du_j(X, V_j)$. Similarly, $\beta(Y_j) = -2dv_j(X, U_j)\ \forall Y \perp U_j$.

Now, suppose $X \in \mathcal{H}$. Then, by Lemma 1.2.1, $\frac{1}{2}\beta_j(X) = -dv_j(X, U_j) = -du_j(X, JU_j) = du_j(X, V_j) = -\frac{1}{2}\alpha_j(X)$,

$$\text{i.e. } (\beta_j)|_{\mathcal{H}} = -(\alpha_j)|_{\mathcal{H}}.$$

Set

$$\sigma_j(X) = \beta_j(X) \ \forall X \in \mathcal{H},$$

$$\sigma_j(U_j) = \alpha_j(U_j),$$

$$\sigma_j(V_j) = \beta_j(V_j),$$

and linearize σ_j. Then we have:

$$\tilde{G}_j = du_j - \sigma_j \wedge v_j$$

$$\tilde{H}_j = dv_j + \sigma_j \wedge u_j \ ;$$

$$\text{Or } \Omega_j = d\pi_j - i \ \sigma_j \wedge \pi_j.$$

Now, on $\mathcal{O}_j \cap \mathcal{O}_k, \Omega_j = h_{jk}\Omega_k$. So, we have:

$$d\pi_j - i\sigma_j \wedge \pi_j = h_{jk}(d\pi_k - i\sigma_k \wedge \pi_k)$$

$$d(h_{jk}\pi_k) - i\sigma_j \wedge \pi_j = h_{jk}d\pi_k - 1\sigma \wedge (h_{jk}\pi_k)$$

$$dh_{jk} \wedge \pi_k + h_{jk}d\pi_k - i\sigma_j \wedge \pi_j = h_{jk}d\pi_k - i\sigma_k \wedge \pi_j$$

$$h_{jk}^{-1}dh_{jk} \wedge \pi_j - i\sigma_j \wedge \pi_j + i\sigma_k \wedge \pi_j = 0$$

$$(h_{jk}^{-1}dh_{jk} - i\sigma_j + i\sigma_k) \wedge \pi_j = 0$$

So, on $\mathcal{H}$, we have:

$$h_{jk}^{-1}dh_{jk}^{-1} - i\sigma_j + i\sigma_k = 0.$$

Since h_{jk} has values in S^1, we know that, for any $Y \in T(\mathcal{O}_j \cap \mathcal{O}_k), h_{jk}^{-1}dh_{jk}(Y) \in \ iR$. Also, recall that $\pi_j(U_j) = 1$ and $\pi_j(V_j) = -i$. So, we know:

$$0 = 2(h_{jk}^{-1}dh_{jk} - i\sigma_j + i\sigma_k) \wedge \pi_j(U_j, V_j)$$

$$= (h_{jk}^{-1}dh_{jk}(V_j) - i\sigma_j(V_j) + i\sigma_k(V_j)) + i(h_{jk}^{-1}dh_{jk}(U_j) - i\sigma_j(U_j) + i\sigma_k(U_j))$$

Taking the real and imaginary parts of the above equation, we see:

$$h_{jk}^{-1}dh_{jk}(U_j)^{-1} - i\sigma_j(U_j) + i\sigma_k(U_j) = 0,$$

$$h_{jk}^{-1}dh_{jk}(V_j) - i\sigma_j(V_j) + i\sigma_k(V_j) = 0.$$

Since $\{U_j, V_j\}$ spans $\mathcal{V}$ on $\mathcal{O}_j \cap \mathcal{O}_k$, we have:

$$h_{jk}^{-1} dh_{jk} - i\sigma_j + i\sigma_k = 0,$$

that is, the $\{\sigma_j\}$ is the set of local 1-forms for a connection on $\mathcal{V}$. We call this connection the *Ishihara-Konishi connection.*

Now, we define a complex almost contact structure. This is an analogous definition in the complex category of an almost contact structure in the real category.

Definition A *complex almost contact structure* on a complex manifold (M, J) with complex dimension $2n+1$ is a maximal atlas $\mathcal{U} = \{\mathcal{O}\}$ and a Hermitian metric g such that, on each $\mathcal{O}$, there are local tensors: $G : T\mathcal{O} \to T\mathcal{O}, u : T\mathcal{O} \to R$ and a local vector field U, which satisfy these properties:

1) $G^2 = -id$ on $\mathrm{span}(U, JU)$

$G \circ J = -J \circ G$

$GU = 0,\ u(U) = 0,\ u \circ G = 0$

2) $u(X) = g(U, X)$

$g(X, GY) = -g(GX, Y)$

3) On $\mathcal{O} \cap \mathcal{O}'$, there exists $h : \mathcal{O} \cap \mathcal{O}' \to S^1$ with :

$u - i\ u \circ J = h(u' - i\ u' \circ J),$

$G - i\ GJ = h(G' - i\ G'J)$

For a fixed open set $\mathcal{O} \subset M$, we call $\{G, H, U, V, u, v, g, \mathcal{O}\}$ a *local complex almost contact structure.*

Finally, we define the associated metric of a normalized complex contact structure.

Definition For a normalized complex contact structure $\{\pi = u - i\ u \circ J\}$, on M and local vertical vector fields $\{U, V = -JU\}$, we call a metric g *associated* to the contact structure, if there exist local endomorphisms $\{G\}$ of TM such that:

1) $\{G, H = GJ, U, V, u, v, g\}$ is a complex almost contact structure on M.

2) $g(X, GY) - ig(X, GJY) = d\pi(X, Y)\ \forall\ X, Y \in \mathcal{H}$.

1.3 Construction of an Associated Metric

The following work is due to Ishihara and Konishi in [IsS3]. We will show that every complex contact structure admits an associated metric. First, though, we will need the following theorem, cf. Chevalley [Ch], Hatakayama [Ha].

Theorem 1.3.1 Let $Gl(n, \mathbb{R})$ be the general linear group of $\mathbb{R}^n$, $O(n)$ be the orthogonal group of $\mathbb{R}^n$, and $H(n)$ be the groups of positive definite symmetric $n \times n$ matrices. Then there is an analytic isomorphism

$$\Phi : Gl(n; \mathbb{R}) \to O(n) \times H(n),$$

whose inverse is given by matrix multiplication.

We will now construct an associated metric on an arbitrary complex contact manifold M. Let $\pi = u - iv$ be the local normalized contact form and U and V be the local vertical vector fields associated to π, as explained before. Let $p : TM \to \mathcal{H}, q : TM \to \mathcal{V}$ be the usual projections.

Let $\tilde{\tilde{g}}$ be any Hermitian metric on M. We define a new Hermitian metric $\tilde{g}$ locally by

$$\tilde{g}(X, Y) = \tilde{\tilde{g}}(pX, pY) + u \otimes U + v \otimes V.$$

Then, since $u \otimes U + v \otimes V$ is a global (1,1)-tensor as is p, we know that $\tilde{g}$ is, in fact, a well-defined global Hermitian metric on M. Furthermore, we have locally:

$$\tilde{g}(U, X) = u(X);$$

$$\tilde{g}(V, X) = v(X).$$

Out of $\tilde{g}$, we will now construct an associated metric g. Let $\mathcal{O}$ be an open subset of M with local normalized contact form $\pi = u - iv$. Let U, V be the vertical vector fields on $\mathcal{O}$ as in Theorem 1.2.3. Let $\underline{E} = \{E_1, JE_1, \ldots, E_{2n}, JE_{2n}\}$ be an orthonormal basis of $\mathcal{H}$ with respect to $\tilde{g}$. So, $\underline{E} \cup \{U, V\}$ is an orthonormal basis of $T\mathcal{O}$.

With respect to $\underline{E}$, we can represent $\tilde{G}$ by a $(4n+2) \times (4n+2)$ matrix:

$$\tilde{\Phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix},$$

where $\phi \in Gl(4n; R)$. By the above theorem, we know that there are unique matrices $\alpha \in O(4n), \beta \in H(4n)$ such that $\phi = \alpha\beta$.

Set

$$\tilde{\beta} = \left(\begin{array}{c|cc} \beta & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right); \quad \tilde{\alpha} = \left(\begin{array}{c|cc} \alpha & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 0 \\ & 0 & 0 \end{array}\right).$$

Then $\tilde{\beta}$ defines a local metric, and $\tilde{\alpha}$ defines a local endomorphism $\mathcal{H} \to \mathcal{H}$. Now, $\tilde{\Phi}$ is a skew-symmetric matrix. So, we have:

$$^t\phi = -\phi$$

$$^t(\alpha\ \beta) = -\alpha\ \beta$$

$$^t\beta\ ^t\alpha = -\alpha\ \beta$$

$$\beta\ ^t\alpha = -\alpha\ \beta$$

$$\beta = -\alpha\ \beta\ \alpha$$

$$\beta = -\alpha^2 \quad ^t\alpha\ \beta\ \alpha,$$

since $\alpha \in O(4n)$ and $\beta \in H(4n)$. So,

$$\beta = \underbrace{(-\alpha^2)}_{\in O(4n)} \cdot \underbrace{(^t\alpha\beta\alpha)}_{\in H(4n)}.$$

By uniqueness of the $O(4n) \times H(4n)$ decomposition of $Gl(4n; R)$, we have:

$$\beta =^t \alpha\beta\alpha$$

$$I = -\alpha^2,$$

i.e.

$$\alpha\ \beta = \beta\alpha$$

$$^t\alpha = -\alpha.$$

So, $\tilde{\alpha}$ represents a local endomorphism $G : T\mathcal{O} \to \mathcal{O}$ such that $G|\text{span}_{\{U,V\}} = 0$ and $G^2 = -p$. And $\tilde{\beta}$ represents a local metric, g, such that $g(X, GY) = -g(GX, Y)\ \forall\ X, Y \in T\mathcal{O}$

since, for $X, Y \in TO$,

$$\begin{aligned} g(X, GY) &= {}^t[X]\ \tilde{\beta}\ [GY] \\ &= {}^t[X]\ \tilde{\beta}\tilde{\alpha}\ [Y] \\ &= {}^t[X]\ \tilde{\Phi}\ [Y] \\ &= \tilde{G}(X, Y) \\ &= -\tilde{G}(Y, X) \\ &= -{}^t[Y]\ \tilde{\Phi}\ [X] \\ &= -g(Y, GX) \\ &= -g(GX, Y), \end{aligned}$$

where $[X], [Y]$ are the column representations of $X, Y \in TO$ with respect to $\underline{E}$. Recall: $\tilde{G}(X, Y) = -\tilde{H}(JX, Y)$; $\tilde{H} = \tilde{G}(JX, Y)$. With respect to $\underline{E}$, J has the matrix form:

$$\tilde{\Gamma} = \left(\begin{array}{c|c} \Gamma & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right),$$

where $\Gamma = \left(\begin{array}{c|c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right)$. Furthermore, $\tilde{H}$ has the matrix form

$$\tilde{\Psi} = \left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & 0 \end{array}\right),$$

such that $\psi = {}^t\Gamma\phi = -\Gamma\phi$. Thus, $\psi = -\Gamma\phi = -\Gamma\alpha\beta = \delta\beta$, where we define $\delta = -\Gamma\alpha$.

Now, ${}^t(-\Gamma\alpha)(-\Gamma\alpha) = {}^t\alpha{}^t\Gamma\ \Gamma\alpha = {}^t\alpha\alpha = id_{4n}$. This means that $\delta = -\Gamma\alpha \in O(4n)$, and so $\psi = \delta\beta$ is the unique $O(4n) \times H(4n)$ decomposition of $\psi \in Gl(4n; R)$. Also, ψ is the matrix representation of the endomorphism H given by:

$$g(X, HY) = \tilde{H}(X, Y),$$

with respect to the basis $\underline{E}$. Note: $g(X, HY) = \tilde{H}(X, Y) = \tilde{G}(X, JY) = g(X, GJY)\ \forall X, Y \in TO$. Thus, $H = GJ$. Furthermore, since $\tilde{H}$ is a 2-form, we know that ψ is a skew-symmetric matrix. So,

$$
\begin{aligned}
{}^t\psi &= -\psi \\
{}^t({}^t\Gamma\phi) &= -{}^t\Gamma\phi \\
{}^t\phi\Gamma &= -{}^t\Gamma\phi \\
-\phi\Gamma &= -{}^t\Gamma\phi \\
\phi\Gamma &= {}^t\Gamma\phi.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tilde{G}(X,JY) &= \tilde{G}(JX,Y) \\
g(X,GJY) &= g(JX,GY) \\
g(X,GJY) &= -g(GJX,Y) \\
g(X,HY) &= -g(HX,Y)
\end{aligned}
$$

When restricted to $\mathcal{H}$, g is a Hermitian metric with respect to G and H. Thus, for $X,Y \in \mathcal{H}, g(X,Y) = g(HX,HY) = g(GJX,GJY) = g(JX,JY)$. So, when restricted to $\mathcal{H}, g$ is a Hermitian metric with respect to J. Furthermore, from the definition of g as the metric corresponding to the matrix $\tilde{\beta}$, we know that when restricted to $\mathcal{V}$ g is a Hermitian metric with respect to J and that the splitting $\mathcal{H} \oplus \mathcal{V}$ is orthogonal with respect to g. Thus, g is a Hermitian metric on $\mathcal{O}$ with respect to J.

Now, all we need to do is to show:

1)g as defined locally above, actually defines a global metric.

2)The local endomorphisms $\{G\}$ have the correct transition functions.

Let γ denote the transformation of adapted frames $\underline{E} = \{E_1, JE_1, \ldots, E_{2n}, JE_{2n}, U, V\}$ and $\underline{\check{E}} = \{\check{E}_1, J\check{E}_1, \ldots, \check{E}_{2n}, J\check{E}_{2n}, \check{U}, \check{V}\}$ on the open set $\mathcal{O} \cap \check{\mathcal{O}}$. Let h denote the transition function of π and $\check{\pi}$, i.e. $\pi = h\check{\pi}$. Let a and b be the real functions given by $h = a + ib$. Since both $\underline{E}$ and $\underline{\check{E}}$ are orthonormal bases with respect to $\tilde{g}$, we have $\gamma^{-1} = {}^t\gamma$.

Now, we know that $\tilde{G} - i\tilde{H} = h((\tilde{\check{G}}) - i(\tilde{\check{H}}))$. In paricular, we have:

$$\tilde{\phi} = \gamma(a(\tilde{\check{\phi}}) + b(\tilde{\check{\psi}}))^t\gamma;$$

$$\phi = \gamma(a\check{\phi} + b\check{\psi})^t\gamma$$

$$\alpha\beta = \gamma(a\check{\alpha}\check{\beta} + b\check{\delta}\check{\beta})^t\gamma$$

$$\alpha\beta = \gamma(a\check{\alpha} + b\check{\delta})\check{\beta}^t\gamma$$

$$\alpha\beta = \underbrace{(\gamma(a\check{\alpha} + b\check{\delta})^t\gamma)}_{\in O(4n)}\underbrace{(\gamma\check{\beta}^t\gamma)}_{\in H(4n)}$$

Thus,

$$\alpha = \gamma(a\check{\alpha} + b\check{\delta})^t\gamma$$

$$\beta = \gamma\check{\beta}^t\gamma.$$

The first equation tells us that, on $\mathcal{O}\cap\check{\mathcal{O}}, G = a\check{G} + b\check{H}$. So, $H = GJ = -b\check{G} + a\check{H}$. The second equation tells us that β defines a global metric on $\mathcal{H}$. So, $[g]_{\underline{E}} = \beta + [u\otimes u + v\otimes v]_{\underline{E}}$ defines a global metric on $TM = \mathcal{H}\oplus\mathcal{V}$. Therefore, $\{G, H, U, V, u, v, g\}$ is a complex almost contact metric structure on M. And, thus, we have shown that every complex contact structure has at least one complex contact metric structure. We shall see in the future that there are infinitely many of these metric structures.

1.4 Basic Facts and Structure Equations of a Complex Contact Metric Structure

For this section, we will assume $\{G_j, H_j, U_j, V_j, u_j, v_j, g\}$ is a complex contact metric structure on M for atlas $\{O_j\}$ such that g is an associated metric. We will omit the indices when it is possible to do so without confusion. One of the first things we learn about such a structure is that the particular unit "trivializations" of $\mathcal{V}$ are not important.

Proposition 1.4.1 Let $\tilde{U}$ be a unit section of $\mathcal{V}$. Let $\tilde{V} = -J\tilde{U}$. Define $\tilde{u}$ be the 1-form on the domain of $\tilde{U}$ given by:

$$\tilde{u}(X) = g(\tilde{U}, X).$$

Let $\tilde{v} = \tilde{u} \circ J$. Set $\tilde{\pi} = \tilde{u} - i\tilde{v}$. Define the local endomorphism $\tilde{G}$ by

$$g(X, \tilde{G}Y) = d\tilde{u}(pX, pY).$$

Then:

1) $\{\pi_j\} \cup \{\tilde{\pi}\}$ is a normalized contact structure on M.

2) $\{U_j, u_j, G_j\} \cup \{\tilde{U}, \tilde{u}, \tilde{G}\}$ is a complex almost contact structure on M.

3) $\{\tilde{\pi}, \tilde{U}, \tilde{G}, g\}$ is a local complex contact metric structure, which is compatible with the original complex contact metric structure.

This proposition follows easily from the fact that $\tilde{U} = aU + bJU$, for any unit vertical vector field on the overlap of the domains of $\tilde{U}$ and U and, thus, $\tilde{u} = au + b(u \circ J)$. This proposition means that we need only to choose a local unit vertical vector field U; and, on its domain, we have a complex almost contact structure $\{G, H, U, V, u, v, g\}$.

Proposition 1.4.2 $\mathcal{V}$ is totally geodesic.

Proof: Let $\{U, V = -JU\}$ be a local orthonormal basis of $\mathcal{V}$. Let $X \in \mathcal{H}$. Since $\mathcal{V}$ is integrable, $[U, V] \in \mathcal{V}$. So,

$$0 = g([U, V], X) = g(\nabla_U V - \nabla_V U, X).$$

Also,

$$\begin{aligned}
0 &= dv(U,X) + du(V,X) \\
&= -\frac{1}{2}v([U,X]) - \frac{1}{2}u([V,X]) \\
&= \frac{1}{2}(-g(V,\nabla_U X) + g(V,\nabla_X U) - g(U,\nabla_V X) + g(U,\nabla_X V)) \\
&= \frac{1}{2}(g(\nabla_U V,X) + g(\nabla_V U,X)) \\
&= \frac{1}{2}g(\nabla_U V + \nabla_V U,X)
\end{aligned}$$

Therefore, $g(\nabla_U V,X) = 0$. So, $p\nabla_U V = 0$. Similarly, $p\nabla_V U = 0$.

Also, for $X \in \mathcal{H}$,

$$\begin{aligned}
0 &= 2du(U,X) \\
&= -u([U,X]) \\
&= -g(U,\nabla_U X) + g(U,\nabla_X U) \\
&= -g(U,\nabla_U X) \\
&= g(\nabla_U U,X)
\end{aligned}$$

Thus, $p\nabla_U U = 0$. Similarly, $p\nabla_V V = 0$. Therefore, $\mathcal{V}$ is a totally geodesic foliation of TM with respect to g.

Corollary 1.4.3 On $\mathcal{O}$, $\sigma(X) = g(\nabla_X U,V)$.

Proof: We know $\tilde{G} = du - \sigma \wedge v$. So, for $X \in \mathcal{H}$ or $X = U$,

$$\begin{aligned}
du(X,V) &= \sigma \wedge v(X,V) \\
&= \frac{1}{2}\sigma(X)
\end{aligned}$$

Also,

$$\begin{aligned}
du(X,V) &= -\frac{1}{2}u([X,V]) \\
&= \frac{1}{2}(-g(U,\nabla_X V) + g(U,\nabla_V X)) \\
&= \frac{1}{2}g(\nabla_X U,V).
\end{aligned}$$

Thus, $\sigma(X) = g(\nabla_X U,V)$ for $X \in \mathcal{H}$ or $X = U$. Similarly, using $\tilde{H} = dv + \sigma \wedge u$, we get that $\sigma(X) = g(\nabla_X U,V)$ for $X \in \mathcal{H}$ or $X = V$. Therefore, $\sigma(X) = g(\nabla_X U,V)$.

It is important to note that σ depends solely on the choice of U. By choosing unit vertical vector field U, we get a local almost contact structure $\{G,u\}$ along with

σ. At times, we will need to emphasize this dependence. For this purpose, we will set $\sigma_U = \sigma$. So, for any unit vertical vector field U,

$$\sigma_U(X) = -g(\nabla_X U, JU).$$

Note: For a unit vertical vector field U,

$$\begin{aligned}\sigma_{JU}(X) &= -g(\nabla_X JU, JJU)\\ &= g(\nabla_X JU, U)\\ &= -g(JU, \nabla_X U)\\ &= \sigma_U(X).\end{aligned}$$

So, the dependence on U is not as rigid as one would initially suppose. We will be using this particular fact quite a few times.

We now would like to describe the basic structure equations of a complex contact metric manifold. We will now assume that g is an associated metric of the complex contact structure of M.

Let U be a unit vertical vector field with $\mathcal{O}$ = domain of U. Let $\{G, H, U, V, u, v, g, \mathcal{O}\}$ be the almost contact structure corresponding to U as given by Proposition 1.4.1.

We define two local endomorphisms $h_U, k_U : T\mathcal{O} \to T\mathcal{O}$ by:

$$g(h_U X, Y) = \frac{1}{2}(g(\nabla_{pX} U, pY) + g(\nabla_{pY} U, pX);$$
$$g(k_U X, Y) = \frac{1}{2}(g(\nabla_{pX} U, pY) - g(\nabla_{pY} U, pX),$$

for any $X, Y \in T\mathcal{O}$.

Then we have:

$$\nabla_X U = \underbrace{\sigma(X)V}_{\in\mathcal{V}} + \underbrace{h_U X + k_U X}_{\in\mathcal{H}}, \ \forall X \in T\mathcal{O}.$$

So, h_U and k_U represent the symmetric and skew-symmetric parts of the linear transformation $X \mapsto p\nabla_X U$.

Now, suppose X, Y are horizontal vector fields. Then:

$$\begin{aligned} g(k_U X, Y) &= \frac{1}{2} g(\nabla_X U, Y) - \frac{1}{2} g(\nabla_Y U, X) \\ &= -\frac{1}{2} g(U, \nabla_X Y - \nabla_Y X) \\ &= -\frac{1}{2} u([X, Y]) \\ &= du(X, Y) \\ &= g(X, GY) \\ &= -g(GX, Y). \end{aligned}$$

Thus, $k_U = -G$, i.e.

$$p\nabla_X U = -GX + h_U X.$$

Similarly, we find that $k_{JU} = H$, so that

$$p\nabla_X V = -HX + h_V X.$$

Note: $k_{JU} = H = -JG = Jk_U$. This is very much reminiscent of the real contact case where we have the relation: $\nabla_X \xi = -\phi X - \phi h X$.

Also, note that we can define h_U and k_U for any vertical vector field, U, regardless of whether it is unit or not. Thus, we have, in fact, two vector bundle maps $\mathcal{V} \to Hom(TM, TM)$ given by:

$$U \mapsto h_U$$

$$U \mapsto k_U.$$

Now, in fact, like the real case, h_U has a very geometric interpretation:

Proposition 1.4.4 For any unit vertical vector field $U, h_U \equiv 0$ if and only if $(\mathcal{L}_U g)|_{\mathcal{H}} \equiv 0$.

Proof: Let $X, Y \in \mathcal{H}$.

$$\begin{aligned} (\mathcal{L}_U g)(X, Y) &= Ug(X, Y) - g([U, X], Y) - g(X, [U, Y]) \\ &= g(\nabla_U X, Y) + g(X, \nabla_U Y) - g(\nabla_U X, Y) + g(\nabla_X U, Y) - g(X, \nabla_U Y) + g(X, \nabla_Y U) \\ &= g(\nabla_X U, Y) + g(X, \nabla_Y U) \\ &= 2g(h_U X, Y) \end{aligned}$$

Locally, since $\mathcal{V}$ is a foliation, we can take open sets $\mathcal{O}$ of M and fibre out their vertical parts:

$$\begin{array}{c}\mathcal{O}\\ \downarrow\rho\\ \tilde{\mathcal{O}},\end{array}$$

with $\mathcal{V}|_{\mathcal{O}} = \ker(\rho_*)$. Then, from Ishihara [Is], we know that there exists a metric $\tilde{g}$ on $\tilde{\mathcal{O}}$ such that $g = \rho^*(\tilde{g}) + u \otimes u + v \otimes v$, i.e. g is "projectable," if and only if $(\mathcal{L}_U g)|_{\mathcal{H}} \equiv 0$ for all unit vertical vector fields U. Thus, we see that h_U is the obstruction of the "projectability" of the associated metric g.

We now would like to give some rather basic lemmas dealing with h_U, h_{JU}. The first also deals with k_U and k_{JU}.

Lemma 1.4.5 The vector bundle map $\mathcal{V} \to Hom(TM, TM)$ given by: $U \mapsto (h_U + k_U)$ is a vector bundle homomorphism, i.e. it is linear in the variable U. And, thus, the maps $U \mapsto h_U, U \mapsto k_U$ are both linear in the variable U.

Proof: Let U, W be any vertical vector fields with the same domain. Let X be a vector field defined on the same domain as U and W. Then:

$$\begin{aligned}(h_{fU+gW} + k_{fU+gW})X &= p\nabla_X(fU + gW)\\ &= p(Xf)U + fp\nabla_X U + p(Xg)W + gp\nabla_X W\\ &= fp\nabla_X U + gp\nabla_X W\\ &= fh_U X + fk_U X + gh_W X + gk_W X\\ &= f(h_U + k_U)X + g(h_W + k_W)X.\end{aligned}$$

Lemma 1.4.6 $p(\nabla_X J)U = h_{JU}X - Jh_U X.$

Proof:

$$\begin{aligned}p(\nabla_X J)U &= p\nabla_X(JU) - pJ\nabla_X U\\ &= h_{JU}X + k_{JU}X - Jh_U X - Jk_U X\\ &= h_{JU}X - Jh_U X.\end{aligned}$$

For any vertical vector field U, set $A_U X = p(\nabla_X J)U$. Now, since J is an integrable complex structure and g is Hermitian, we know $\nabla_{JX} J = J\nabla_X J$ for any vector X. So, $A_U \circ J = J \circ A_U$ $\forall U \in \mathcal{V}$. Or $A_U^a = 0$. This gives us two relationships:

$$p(\nabla_X J)U = h^s_{JU} X - J h^s_U X;$$

$$h^a_{JU} = J h^a_U.$$

It is important to note that h^s_{JU} is symmetric with respect to g and that $-Jh^s_U$ is skew-symmetric with respect to g. Thus, $A_U \equiv 0$ if and only if both $h^s_{JU} \equiv 0$ and $h^s_U \equiv 0$. In particular, if g is Kähler, then $h^s_U \equiv 0$ for all $U \in \mathcal{V}$.

We will finish this section with a couple of very elementary results concerning the Nijenhuis torsion of G. Recall for any $(1,1)$–tensor, Φ, on a manifold, we define the Nijenhuis torsion to be a $(1,2)$–tensor, $[\Phi,\Phi]$, given by:

$$[\Phi,\Phi](X,Y) = \Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y].$$

Lemma 1.4.7 For $X, Y \in \mathcal{H}, W \in \mathcal{V}$, we have:

1) $p[G,G](X,GY) = -G[G,G](X,Y) = p[G,G](GX,Y)$.

2) $[G,G](W,X) = -G(\mathcal{L}_W G)X$.

Proof:

1)
$$\begin{aligned} p[G,G](X,GY) &= G^2[X,GY] + p[GX,GGY] - G[GX,GY] - G[X,GGY] \\ &= G(G[X,GY] + G[GX,Y] - [GX,GY] + [X,Y]) \\ &= G(-G^2[X,Y] - [GX,GY] + G[X,GY] + G[GX,Y]) \\ &= -G[G,G](X,Y) \\ &= G[G,G](Y,X) \\ &= -p[G,G](Y,GX) \\ &= p[G,G](GX,Y) \end{aligned}$$

2)

$$
\begin{aligned}
[G,G](W,X) &= G^2[X,W] + [GX,GW] - G[GX,W] - G[X,GW] \\
&= -G^2[W,X] + G[W,GX] \\
&= G(-G\mathcal{L}_W X + \mathcal{L}_W(GX)) \\
&= G(\mathcal{L}_W G)X.
\end{aligned}
$$

1.5 A description of ∇G

Before we can continue, we will need a description of ∇G with respect to the various structure tensors of the complex contact metric structure. We will actually end up with two descriptions, the second being a refinement of the first.

In order to do this, we need the following two equations that we get from basic Riemannian geometry. The first is the invariant description of the Levi-Civita connection of a Riemannian metric, g :

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)$$

$$+ g([X,Y],Z) + g([Z,X]Y) - g([Y,Z],X).$$

The second is the invariant description of the exterior derivative of a given 2-form Φ :

$$3d\Phi(X,Y,Z) = X\Phi(Y,Z) + Y\Phi(Z,X) + Z\Phi(X,Y)$$

$$- \Phi([X,Y],Z) - \Phi([Z,X],Y) - \Phi([Y,Z],X).$$

Proposition 1.5.1

$$2g((\nabla_X G)Y,Z) = g([G,G](Y,Z),GX) - 3v \wedge d\sigma(X,GY,GZ) + 3v \wedge d\sigma(X,Y,Z)$$

$$- 2\sigma(X)\tilde{H}(Y,Z) + 4v(X)g(Y,J'Z)$$

$$- \sigma(Y)\tilde{H}(Z,X) + \sigma(GY)g(Z,J'X) - 2u(Y)g(X,pZ) - 2v(Y)g(Z,J'X)$$

$$+ \sigma(Z)\tilde{H}(Y,X) - \sigma(GZ)g(Y,J'X) + 2u(Z)g(X,pY) + 2v(Z)g(Y,J'X),$$

where $J' = pJ$.

Proof:

$$2g((\nabla_X G)Y,Z) = 2g(\nabla_X(GY),Z) + 2g(\nabla_X Y,GZ)$$

$$= Xg(GY,Z) + (GY)g(X,Z) - Zg(X,GY)$$

$$+ g([X,GY],Z) + g([Z,X],GY) - g([GY,Z],X)$$

$$+ Xg(Y,GZ) + Yg(X,GZ) - (GZ)g(X,Y)$$

$$+ g([X,Y],GZ) + g([GZ,X],Y) - g([Y,GZ],X)$$

$$= -X\tilde{G}(Y,Z) + (GY)(\tilde{G}(GZ,X) + u(Z)u(X) + v(Z)v(X)) - Z\tilde{G}(X,Y)$$

$$- \tilde{G}([X,GY],GZ) + u([X,GY])u(Z) + v([X,GY])v(Z)$$

$$+ \tilde{G}([Z,X],Y) - g(G[GY,Z],GX) + u(X)u([Z,GY]) + v(X)v([Z,GY])$$

$$+ X\tilde{G}(GY,GZ) - Y\tilde{G}(Z,X) - (GZ)(\tilde{G}(GY,X) + u(Y)u(X) + v(Y)v(X))$$

$$+ \tilde{G}([X,Y],Z) - \tilde{G}([GZ,X],GY) + u([GZ,X])u(Y) + v([GZ,X])v(Y)$$

$$- g(G[Y,GZ],GX) + u(X)u([GZ,Y]) + v(X)v([GZ,Y])$$

$$+ \tilde{G}([Y,Z],X) - g([Y,Z],GX)$$

$$- \tilde{G}([GY,GZ],X) + g([GY,GZ],GX)$$

$$= X\tilde{G}(GY,GZ) + (GY)\tilde{G}(GZ,X) + (GZ)\tilde{G}(X,GY)$$

$$- \tilde{G}([X,GY],GZ) - \tilde{G}([GZ,X],GY) - \tilde{G}([GY,GZ],X)$$

$$- X\tilde{G}(Y,Z) - Y\tilde{G}(Z,X) - Z\tilde{G}(X,Y)$$

$$+ \tilde{G}([X,Y],Z) + \tilde{G}([Z,X],Y) + \tilde{G}([Y,Z],X)$$

$$- g([Y,Z],GX) + g([GY,GZ],GX) - g(G[Y,GZ],GX) - g(G[GY,Z],GX)$$

$$+ (GY)(u(Z)u(X) + v(Z)v(X)) + u([X,GY])u(Z) + v([X,GY])v(Z)$$

$$+ u(X)u([X,GY]) + v(X)v([Z,GY]) - (GZ)(u(Y)u(X) + v(Y)v(X))$$

$$+ u([GZ,X])u(Y) + v([GZ,X])v(Y) + u(X)u([GZ,Y]) + v(X)v([GZ,Y])$$

$$= 3d\tilde{G}(X,GY,GZ) - 3d\tilde{G}(X,Y,Z) + g([G,G](Y,Z),GX)$$

$$+ (GY)(u(Z)u(X) + v(Z)v(X)) + u([X,GY])u(Z) + v([X,GY])v(Z)$$

$$+ u(X)u([X,GY]) + v(X)v([Z,GY]) - (GZ)(u(Y)u(X) + v(Y)v(X))$$

$$+ u([GZ,X])u(Y) + v([GZ,X])v(Y) + u(X)u([GZ,Y]) + v(X)v([GZ,Y])$$

$$= 3d\tilde{G}(X,GY,GZ) - 3d\tilde{G}(X,Y,Z) + g([G,G](Y,Z),GX)$$

$$+ \big(\mathcal{L}_{GY}(u(Z))\big)u(X) + u(X)\big(\mathcal{L}_{GY}(u(X))\big) + \big(\mathcal{L}_{GY}(v(Z))\big)v(X) + v(X)\big(\mathcal{L}_{GY}(v(X))\big)$$

$$+ u(\mathcal{L}_X(GY))u(Z) + v(\mathcal{L}_X(GY))v(Z)$$

$$+ u(X)u(\mathcal{L}_Z(GY)) + v(X)v(\mathcal{L}_Z(GY))$$

$$+ \mathcal{L}_{GY}(u(Z))u(X) + u(X)\mathcal{L}_{GY}(u(X)) + \mathcal{L}_GY(v(Z))v(X) + v(X)\mathcal{L}_{GY}(v(X))$$

$$+ u(\mathcal{L}_{GZ}X)u(Y) + v(\mathcal{L}_{GZ}X)v(Y)$$

$$+ u(X)u(\mathcal{L}_{GZ}Y) + v(X)v(\mathcal{L}_{GZ}Y)$$

$$= 3d\tilde{G}(X,GY,GZ) - 3d\tilde{G}(X,Y,Z) + g([G,G](Y,Z),GX)$$

$$+ u(X)\big(\mathcal{L}_{GY}(u(Z)) - u(\mathcal{L}_{GY}Z) - \mathcal{L}_{GZ}(u(Z)) + u(\mathcal{L}_{GZ}Y)\big)$$

$$+ v(X)\big(\mathcal{L}_{GY}(v(Z)) - v(\mathcal{L}_{GY}Z) - \mathcal{L}_{GZ}(v(Z)) + v(\mathcal{L}_{GZ}Y)\big)$$

$$+ u(Z)\big(\mathcal{L}_{GY}(u(X)) - u(\mathcal{L}_{GY}X)\big) + v(Z)\big(\mathcal{L}_{GY}(v(X)) - v(\mathcal{L}_{GY}X)\big)$$

$$- u(Y)\big(\mathcal{L}_{GZ}(u(X)) - u(\mathcal{L}_{GZ}X)\big) - v(Y)\big(\mathcal{L}_{GZ}(v(X)) - v(\mathcal{L}_{GZ}X)\big)$$

Thus,

$$\begin{aligned} 2g((\nabla_X G)Y,Z) &= 3d\tilde{G}(X,GY,GZ) - 3d\tilde{G}(X,Y,Z) + g([G,G](Y,Z),GX) \\ &\quad + u(X)((\mathcal{L}_{GY}u)(Z) - (\mathcal{L}_{GZ}u)(Y)) + 2u(Z)du(GY,X) - 2u(Y)du(GZ,X) \\ &\quad + v(X)((\mathcal{L}_{GY}v)(Z) - (\mathcal{L}_{GZ}v)(Y)) + 2v(Z)dv(GY,X) - 2v(Y)dv(GZ,X) \end{aligned}$$

We will now refine this result by analyzing the various terms in the above equation.

1)

$$\begin{aligned} d\tilde{G} &= d(du - \sigma \wedge v) \\ &= -d\sigma \wedge v + \sigma \wedge dv \\ &= -d\sigma \wedge v + \sigma \wedge \tilde{H}. \end{aligned}$$

Therefore,

$$3d\tilde{G}(X,GY,GZ) = -3d\sigma\wedge v(X,GY,GZ) + 3\sigma\wedge\tilde{H}(X,GY,GZ)$$

$$= -v(X)d\sigma(GY,GZ) + \sigma(X)\tilde{H}(GY,GZ)$$

$$+ \sigma(GY)\tilde{H}(GZ,X) + \sigma(GZ)\tilde{H}(X,GY)$$

$$= -v(X)d\sigma(GY,GZ) + \sigma(X)g(GY,HGZ)$$

$$+ \sigma(GY)g(GZ,HX) + \sigma(GZ)g(X,HGY)$$

$$= -v(X)d\sigma(GY,GZ) - \sigma(X)g(Y,HZ)$$

$$+ \sigma(GY)g(Z,J'X) + \sigma(GZ)(X,J'Y)$$

$$= -v\wedge d\sigma(X,GY,GZ)$$

$$- \sigma(X)g(Y,HZ) + \sigma(GY)g(J'X,Z) - \sigma(GZ)(J'X,Y),$$

Also,

$$-3d\tilde{G}(X,Y,Z) = 3d\sigma\wedge v(X,Y,Z) - 3\sigma\wedge\tilde{H}(X,Y,Z)$$

$$= 3d\sigma\wedge v(X,Y,Z)$$

$$- \sigma(X)\tilde{H}(Y,Z) \quad - \sigma(Y)\tilde{H}(Z,X) - \sigma(Z)\tilde{H}(X,Y)$$

$$= 3d\sigma\wedge v(X,Y,Z)$$

$$- \sigma(X)g(Y,HZ) \quad - \sigma(Y)g(Z,HX) - \sigma(Z)g(X,HY).$$

2) Set $M_1(X,Y) = (\mathcal{L}_{GY}u)(Z) - (\mathcal{L}_{GZ}u)(Y)$ and $M_2(Y,Z) = (\mathcal{L}_{GY}v)(Z) - (\mathcal{L}_{GZ}v)(Y)$. So, M_1 and M_2 are 2-forms on $\mathcal{O}$. Let $Y,Z\in\mathcal{H}$. Then,

$$M_1(Y,Z) = (\mathcal{L}_{GY}u)(Z) - (\mathcal{L}_{GZ}u)(Y)$$

$$= -u(\mathcal{L}_{GY}Z) + u(\mathcal{L}_{GZ}Y)$$

$$= 2du(GY,Z) - 2du(GZ,Y)$$

$$= 2g(GY,GZ) - 2g(GZ,GY)$$

$$= 0;$$

$$M_1(U,V) = 0;$$

$$
\begin{aligned}
M_1(U,Z) &= -(\mathcal{L}_{GZ}u)(U) \\
&= u[GZ,U] \\
&= -2du(GZ,U) \\
&= 0; \\
M_1(V,Z) &= -(\mathcal{L}_{GZ}u)(V) \\
&= -2du(GZ,V) \\
&= -2\sigma \wedge v(GZ,V) \\
&= -\sigma(GZ).
\end{aligned}
$$

$$
\begin{aligned}
\text{Also, } M_2(Y,Z) &= 2dv(GY,Z) - 2dv(GZ,Y) \\
&= 4g(Y,J'Z); \\
M_2(U,V) &= 0; \\
M_2(V,Z) &= 0; \\
M_2(U,Z) &= -(\mathcal{L}_{GZ}v)(U) \\
&= v([GZ,U]) \\
&= -2dv(GZ,U) \\
&= 2(\sigma \wedge u)(GZ,U) \\
&= \sigma(GZ).
\end{aligned}
$$

Thus,

$$
M_1(Y,Z) = 2(\sigma \circ G) \wedge v(Y,Z),
$$

$$
M_2(Y,Z) = 4g(Y,J'Z) - 2(\sigma \circ G) \wedge u(Y,Z).
$$

3) $$
\begin{aligned}
du(GY,X) &= (\tilde{G} + \sigma \wedge v)(GY,X) \\
&= \tilde{G}(GY,X) + \sigma \wedge v(GY,X) \\
&= g(Y,pX) + \frac{1}{2}\sigma \wedge v(GY,X)
\end{aligned}
$$

$$dv(GY,X) = (\tilde{H} - \sigma \wedge u)(GY,X)$$

$$= \tilde{H}(GY,X) - \sigma \wedge u(GY,X)$$

$$= g(GY,HX) - \frac{1}{2}\sigma(GY)u(X).$$

We now have:

$$\begin{aligned}
2g((\nabla_X G)Y,Z) = {} & g([G,G](Y,Z),GX) \\
& - 3d\sigma \wedge v(X,GY,GZ) + 3d\sigma \wedge v(X,Y,Z) \\
& - \sigma(X)g(Y,HZ) + \sigma(GY)g(J'X,Z) - \sigma(GZ)g(J'X,Y) \\
& - \sigma(X)g(Y,HZ) - \sigma(Y)g(HX,Z) + \sigma(Z)g(HX,Y) \\
& + 2u(X)((\sigma \circ G) \wedge v)(Y,Z) \\
& + 2u(Z)g(pX,Y) + u(Z)v(X)\sigma(GY) \\
& - 2u(Y)g(pX,Z) - u(Y)v(X)\sigma(GZ) \\
& + v(X)(4g(Y,J'Z) - 2((\sigma \circ G) \wedge u)(Y,Z)) \\
& + 2v(Z)g(Y,J'X) - v(Z)u(X)\sigma(GY) \\
& - 2v(Y)g(Z,J'X) + v(Y)u(X)\sigma(GZ)
\end{aligned}$$

$$\begin{aligned}
= {} & g([G,G](Y,Z),GX) \\
& - 3d\sigma \wedge v(X,GY,GZ) + 3d\sigma \wedge v(X,Y,Z) \\
& - \sigma(X)g(Y,HZ) + \sigma(GY)g(J'X,Z) - \sigma(GZ)g(J'X,Y) \\
& - \sigma(X)g(Y,HZ) - \sigma(Y)g(HX,Z) + \sigma(Z)g(HX,Y) \\
& + 2u(Z)g(pX,Y) - 2u(Y)g(pX,Z) \\
& + 4v(X)g(Y,J'Z) \\
& + 2v(Z)g(Y,J'X) - 2v(Y)g(Z,J'X)
\end{aligned}$$

$$= g([G,G](Y,Z),GX) - 3d\sigma\wedge v(X,GY,GZ) + 3d\sigma\wedge v(X,Y,Z)$$

$$-2\sigma(X)g(Y,HZ) + \sigma(GY)g(J'X,Z) - \sigma(GZ)g(J'X,Y)$$

$$-\sigma(Y)g(HX,Z) + \sigma(Z)g(HX,Y)$$

$$-2u(Y)g(pX,Z) + 2u(Z)g(pX,Y)$$

$$+4v(X)g(Y,J'Z)$$

$$-2v(Y)g(Z,J'X) + 2v(Z)g(Y,J'X)$$

Therefore,

$$2g((\nabla_X G)Y,Z) = g([G,G](Y,Z),GX) - 3d\sigma\wedge v(X,GY,GZ) + 3d\sigma\wedge v(X,Y,Z)$$

$$-2\sigma(X)\tilde{H}(Y,Z) + \sigma(GY)g(Z,J'X) - \sigma(GZ)g(Y,J'X)$$

$$-\sigma(Y)\tilde{H}(Z,X) + \sigma(Z)\tilde{H}(Y,X)$$

$$-2u(Y)g(pX,Z) + 2u(Z)g(pX,Y)$$

$$+4v(X)g(Y,J'Z)$$

$$-2v(Y)g(Z,J'X) + 2v(Z)g(Y,J'X)$$

$$= g([G,G](Y,Z),GX) - 3d\sigma\wedge v(X,GY,GZ) + 3d\sigma\wedge v(X,Y,Z)$$

$$-2\sigma(X)\tilde{H}(Y,Z) + 4v(X)g(Y,J'Z)$$

$$-\sigma(Y)\tilde{H}(Z,X) + \sigma(GY)g(Z,J'X) - 2u(Y)g(pX,Z) - 2v(Y)g(Z,J'X)$$

$$+\sigma(Z)\tilde{H}(Y,X) - \sigma(GZ)g(Y,J'X) + 2u(Z)g(pX,Y) + 2v(Z)g(Y,J'X)$$

This proposition gives a few very important structure equations dealing with complex contact metric structure.

Corollary 1.5.2 Let U be a unit vertical vector with corresponding complex almost contact structure $\{G,H,U,V,u,v,\mathcal{O},g\}$. Then:

a. $\nabla_U G = \sigma(U)H$, and $\nabla_V H = -\sigma(V)G$.

b. $G(\nabla_U J) = -(\nabla_U J)G$.

c. $Gh_U = -h_U G$.

d. $p(\mathcal{L}_U G)p = \underbrace{2Gh_U}_{\text{symmetric}} + \underbrace{\sigma(U)H}_{\text{skew-symmetric}}$.

e. $tr(h_U) = 0$.

f. $d\sigma(W, X) = 0$ for all $X \in \mathcal{H}, W \in \mathcal{V}$.

Proof:

a. Clearly, $q(\nabla_U G)q = 0$, since $qG = Gq = 0$. Also, for $X \in \mathcal{H}$ and $W \in \mathcal{V}$, we have:

$$\begin{aligned} g((\nabla_U G)X, W) &= g(\nabla_U(GX), W) - g(G\nabla_U X, W) \\ &= g(\nabla_U(GX), W) + g(\nabla_U X, GW) \\ &= g(\nabla_U(GX), W) \\ &= -g(GX, \nabla_U W) \\ &= 0, \end{aligned}$$

since $\mathcal{V}$ is totally geodesic, and so $\nabla_U W \in \mathcal{V}$. Therefore, $q(\nabla_U G)p = 0$; and, since $\nabla_U G$ is skew-symmetric with respect to g, $p(\nabla_U G)q = 0$.

Let $Y, Z \in \mathcal{H}$. Substituting $X = U$ into the above equation, we get:

$$\begin{aligned} 2g((\nabla_U G)Y, Z) &= -3v \wedge d\sigma(U, GY, GZ) + 3v \wedge d\sigma(U, Y, Z) \\ &\quad - 2\sigma(U)\tilde{H}(Y, Z) \\ &= -2\sigma(V)g(Y, HZ) \\ &= 2\sigma(V)g(HY, Z). \end{aligned}$$

And, so, $\nabla_U G = \sigma(U)H$.

Thus, for each unit vertical vector field U, we have: $-\nabla_U k_U = \sigma_U(U)\ k_J U$. Substituting JU for U in this equation, we get:

$$-\nabla_{JU} k_{JU} = -\sigma_{JU}(JU)\ k_U, \text{ or } \nabla_V k_{JU} = \sigma_U(V)\ k_U.$$

Thus, $\nabla_V H = -\sigma(V)G$.

b.
$$\begin{aligned} G(\nabla_U J) &= \nabla_U(GJ) - (\nabla_U G)J \\ &= -\nabla_U(JG) - \sigma(U)HJ \\ &= -\nabla_U(JG) + \sigma(U)JH \\ &= -\nabla_U(JG) + J(\nabla_U G) \\ &= -(\nabla_U J)G. \end{aligned}$$

c. & d. Let $X, Z \in \mathcal{H}$. Note: $G(\nabla_X G)U = -G^2\nabla_X U = p\nabla_X U = -GX + h_U X$.

Also,

$$2g(G(\nabla_X G)U, Z) = -2g((\nabla_X G)U, GZ)$$
$$= -g([G,G](U,GZ), GX) + \sigma(U)\tilde{H}(GZ,X) + 2g(X,GZ)$$
$$= g(G(\mathcal{L}_U G)GZ, GX) + \sigma(U)g(GZ,HX) + 2g(X,GZ)$$
$$= g(G(\mathcal{L}_U G)GZ, GX) + \sigma(U)g(GZ,HX) - 2g(GX,Z).$$

Combining these equations, we get:

$$2g(h_U X, Z) = g(G(\mathcal{L}_U G)GZ, GX) + \sigma(U)g(Z, JX)$$
$$g((\mathcal{L}_U G)GZ, X) = 2g(X, h_U Z) + \sigma(U)g(X, JZ)$$
$$p(\mathcal{L}_U G)G = 2h_U + \sigma(U)J'$$
$$p(\mathcal{L}_U G)p = -2h_U G - \sigma(U)J'G$$
$$p(\mathcal{L}_U G)p = -2h_U G + \sigma(U)H$$

Now, all we need to show is that h_U and G anti-commute. This will not only give us the equation, but it will also show that $2Gh_U$ is a symmetric operator with respect to g. It is easily seen that $p(\mathcal{L}_U p)p = 0$. So, we have $0 = -p(L_U(G^2))p = -p(L_U G)G - G(L_U G)p$ $= -p(L_U G)pG - Gp(L_U G)p$. Thus, we see that $p(L_U G)p$ and H both anti-commute with G; and, so, $-2h_U G$ anti-commutes with G. Therefore, h_U anti-commutes with G.

e. If e.is true for a unit vertical vector, then it is clearly true for any vertical vector. Thus, we need only show that it is true when $|U| = 1$.

$$tr(h_U) = tr(ph_U p) = -tr(G^2 h_U) = tr(G h_U G) = tr(G^2 h_U) = -tr(h_U),$$

since h_U and G anti-commute.

f. Let $X \in \mathcal{H}, W \in \mathcal{V}$. Since $p(\nabla_U G)p = (\nabla_U G)$,

$$0 = 2g((\nabla_U G)Y, V)$$
$$= 3v \wedge d\sigma(U, X, V)$$
$$= d\sigma(U, X).$$

So,$0 \equiv (\iota(U)d\sigma_U)|_{\mathcal{H}}$ for any unit vertical vector field U. Fixing a unit vertical field U and substituting JU for U, we get $0 \equiv (\iota(JU)d\sigma_{JU})|_{\mathcal{H}}$. Now, $\sigma_{JU} = \sigma_U$. So, we have $0 \equiv d\sigma(JU, X)$ for any horizontal vector X. Since the equation is true for both U and JU, it is true for any $W \in \mathcal{V}$.

1.6 Curvature Identities

We will now cover some curvature identities, which are true for any associated metric of M. We define the Riemannian curvature of the metric g by:

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

With respect to any basis $\underline{e} = \{e_1, \ldots, e_{4n+2}\}$, we let $R_{ijk}{}^l$ denote the components of R by:

$$R_{ijk}{}^l e_l = R_{e_i e_j} e_k.$$

For future reference, we define:

$$S(X,Y) = \text{trace}(Z \mapsto R_{ZX}Y),$$

$$g(X, QY) = S(X,Y),$$

$$Ric(X) = S(X,X),$$

$$\tau_g = \text{Tr}(Q),$$

for any two vectors X, Y on M.

Proposition 1.6.1 Let X be a horizontal vector field, U a unit vertical vector field. Then:

$$pR_{UX}U = -X + h_U^2 X + p(\nabla_U h_U)X + \sigma(U)h_{JU}X.$$

Proof: Let $\{G, H, U, V, u, v, g, \mathcal{O}\}$ be the local complex almost contact structure corresponding to U. Then,

$$\begin{aligned} pR_{UX}U &= p\nabla_U \nabla_X U - p\nabla_X \nabla_U U - p\nabla_{[U,X]}U \\ &= p\nabla_U(\sigma(X)V + k_U X + h_U X) - p\nabla_X(\sigma(U)V) - p\nabla_{\nabla_U X}U + p\nabla_{\nabla_X U}U \\ &= p\nabla_U(k_U X) + p\nabla_U(h_U X) + \sigma(U)p\nabla_X(JU) - k_U\nabla_U X - h_U \nabla_U X \\ &\quad + k_U(\nabla_X U) + h_U(\nabla_X U) \end{aligned}$$

$$= p(\nabla_U k_U)X + pk_U\nabla_U X + p(\nabla_U h_U)X + ph_U\nabla_U X + \sigma(U)h_{JU}X + \sigma(U)k_{JU}X$$

$$- k_U\nabla_U X - h_U\nabla_U X + k_U(k_U X + h_U X) + h_U(k_U X + h_U X)$$

$$= -(\nabla_U G)X + p(\nabla_U h_U)X + \sigma(U)HX + \sigma(U)h_{JU}X - X + (h_U)^2 X$$

$$= -\sigma(U)HX + p(\nabla_U h_U)X + \sigma(U)HX + \sigma(U)h_{JU}X - X + (h_U)^2 X$$

$$= p(\nabla_U h_U)X + \sigma(U)h_{JU}X - X + (h_U)^2 X.$$

Proposition 1.6.2 Let X be a horizontal vector field, U a unit vertical vector field. Then:

$$pR_{UX}JU = p(\nabla_U k_{JU})X + p(\nabla_U h_{JU})X - \sigma_U(U)(k_U X + h_U X)$$

$$- JX + k_{JU}h_U X + h_{JU}k_U X + h_{JU}h_U X.$$

Or

$$pR_{UX}JU = -(\nabla_U J)G + p(\nabla_U h_{JU})X - \sigma(U)h_U X - JX + Hh_U{}^{*}X - h_{JU}{}^{*}GX + h_{JU}h_U X,$$

where $\{G, H, U, V, u, v, g, O\}$ is the local complex almost complex structure corresponding to U.

Proof:

$$pR_{UX}JU = p\nabla_U\nabla_X(JU) - p\nabla_X\nabla_U(JU) - p\nabla_{\nabla_U X}(JU) + p\nabla_{\nabla_X U}(JU)$$

$$= p\nabla_U(k_{JU}X + h_{JU}X + \sigma(X)U) - p\nabla_X(\sigma(U)U)$$

$$- k_{JU}\nabla_U X - h_{JU}\nabla_U X + k_{JU}\nabla_X U + h_{JU}\nabla_X U$$

$$= p\nabla_U(k_{JU}X) + p\nabla_U(h_{JU}X) - \sigma(U)(k_U X + h_U X)$$

$$- k_{JU}\nabla_U X - h_{JU}\nabla_U X + k_{JU}(k_U X + h_U X) + h_{JU}(k_U X + h_U X)$$

$$= p(\nabla_U k_{JU})X + k_{JU}\nabla_U X + p(\nabla_U h_{JU})X + h_{JU}\nabla_U X$$

$$- \sigma(U)(k_U X + h_U X) - k_{JU}\nabla_U X - h_{JU}\nabla_U X$$

$$- JX + k_{JU}h_U X + h_{JU}k_U X + h_{JU}h_U X$$

$$= p(\nabla_U k_{JU})X + p(\nabla_U h_{JU})X - \sigma(U)(k_U X + h_U X)$$

$$- JX + k_{JU}h_U X + h_{JU}k_U X + h_{JU}h_U X$$

Now,

$$\begin{aligned}
\nabla_U(k_{JU}) &= \nabla_U(Jk_U) \\
&= (\nabla_U J)k_U + J(\nabla_U k_U) \\
&= (\nabla_U J)k_U - J(\nabla_U G) \\
&= (\nabla_U J)k_U - \sigma(U)JH \\
&= (\nabla_U J)k_U - \sigma(U)G \\
&= -(\nabla_U J)G - \sigma(U)G.
\end{aligned}$$

Also, $h_{JU}{}^d G = Jh_U{}^d G = -JGh_U{}^d = Hh_U{}^d$. So,

$$\begin{aligned}
pR_{UX}JU &= -(\nabla_U J)G - \sigma(U)G + p(\nabla_U h_{JU})X - \sigma(U)(k_U X + h_U X) \\
&\quad - JX + k_{JU}h_U X + h_{JU}k_U X + h_{JU}h_U X \\
&= -(\nabla_U J)G - \sigma(U)G + p(\nabla_U h_{JU})X + \sigma(U)GX - \sigma(U)h_U X \\
&\quad - JX + Hh_U X - h_{JU}GX + h_{JU}h_U X \\
&= -(\nabla_U J)G + p(\nabla_U h_{JU})X - \sigma(U)h_U X \\
&\quad - JX + Hh_U{}^s X - h_{JU}{}^s G + h_{JU}h_U X.
\end{aligned}$$

This last identity allows us to derive a fairly satisfactory description of $d\sigma$ when restricted to $\mathcal{H}$ in terms of other structure tensors.

Proposition 1.6.3 Suppose X, Y are vector fields in $\mathcal{H}$ and U is a unit vertical vector field with corresponding local complex almost contact structure $\{G, H, U, V, u, v, g, \mathcal{O}\}$. Then

$$d\sigma(X,Y) = 2g(JX,Y) + g((\nabla_U J)GX, Y) - 2g(Hh_U^d X, Y).$$

Proof: For endomorphisms S, T, define $[S,T] = ST - TS$. Then the Bianchi identity tells us that,

$$
\begin{aligned}
g(R_{UJU}X, Y) &= -g(R_{XU}JU, Y) - g(R_{JUX}U, Y) \\
&= g(R_{UX}JU, Y) - g(R_{JUX}U, Y) \\
&= g(R_{UX}JU, Y) - g(R_{UY}JU, X) \\
&= -g((\nabla_U J)GX, Y) - g(JX, Y) + g(h_{JU}h_U X, Y) \\
&\quad + g((\nabla_U J)GY, X) + g(JY, X) - g(h_{JU}h_U Y, X) \\
&= -2g(\nabla_U J)GX, Y) - 2g(JX, Y) + g([h_{JU}, h_U]X, Y),
\end{aligned}
$$

where $[S,T] = ST - TS$ for any endomorphisms S, T. Thus,

$$g(R_{UV}X, Y) = 2g((\nabla_U J)GX, Y) + 2g(JX, Y) + g([h_V, h_U]X, Y).$$

Hence,

$$
\begin{aligned}
2d\sigma(X, Y) &= X\sigma(Y) - Y\sigma(X) - \sigma([X, Y]) \\
&= Xg(\nabla_Y U, V) - Yg(\nabla_X U, V) - g(\nabla_{[X,Y]}U, V) \\
&= g(\nabla_X \nabla_Y U, V) + g(\nabla_Y U, \nabla_X V) \\
&\quad - g(\nabla_Y \nabla_X U, V) - g(\nabla_X U, \nabla_Y V) - g(\nabla_{[X,Y]}U, V) \\
&= g(R_{XY}U, V) + g(\nabla_Y U, \nabla_X V) - g(\nabla_X U, \nabla_Y V)
\end{aligned}
$$

Now,

$$
\begin{aligned}
g(\nabla_Y U, \nabla_X V) &= g(-GY + h_U Y, -HX + h_V X) \\
&= g(GY, HX) - g(GY, h_V X) - g(h_U Y, HX) + g(h_U Y, h_V X) \\
&= g(Y, JX) + g(Y, Gh_V X) - g(Y, h_U HX) + g(Y, h_U h_V X)
\end{aligned}
$$

$$
\begin{aligned}
-g(\nabla_X U, \nabla_Y V) &= -g(X, JY) - g(X, Gh_V Y) + g(X, h_U HY) - g(X, h_U h_V Y) \\
&= g(JX, Y) + g(h_V GX, Y) - g(Hh_U X, Y) - g(h_V h_U X, Y).
\end{aligned}
$$

So,

$$
\begin{aligned}
g(\nabla_Y U, \nabla_X V) - g(\nabla_X U, \nabla_Y V) &= 2g(JX, Y) + g([h_U, h_V]X, Y) \\
&\quad + g((Gh_V + h_V G)X, Y) - g((Hh_V + h_V H)X, Y).
\end{aligned}
$$

But

$$\begin{aligned} Gh_V + h_V G &= Gh^d_V + h^d_V G \\ &= 2Gh^d_V \\ &= -2GJh^d_U \\ &= -2Hh^d_U \\ &= -(Hh_U + h_U H). \end{aligned}$$

Thus,

$$g(\nabla_Y U, \nabla_X V) - g(\nabla_X U, \nabla_Y V) = 2g(JX,Y) - 4g(Hh^d_U X, Y) + g([h_U, h_V]X, Y).$$

Therefore,

$$\begin{aligned} d\sigma(X,Y) &= \frac{1}{2}g(R_{XY}U, V) + g(JX,Y) - 2g(Hh^d_U X, Y) + \frac{1}{2}g([h_U, h_V]X, Y) \\ &= g((\nabla_U J)GX, Y) + g(JX,Y) + \frac{1}{2}g([h_V, h_U]X, Y) + g(JX,Y) \\ &\quad - 2g(Hh^d_U X, Y) + \frac{1}{2}g([h_U, h_V]X, Y) \\ &= g((\nabla_U J)GX, Y) + g(JX,Y) + \frac{1}{2}g([h_V, h_U]X, Y) + g(JX,Y) \\ &\quad - 2g(Hh^d_U X, Y) - \frac{1}{2}g([h_V, h_U]X, Y) \\ &= g((\nabla_U J)GX, Y) + 2g(JX,Y) - 2g(Hh^d_U X, Y). \end{aligned}$$

We will finish this section with an application of this proposition:

Proposition 1.6.4 $p(\mathcal{L}_V G)p = \underbrace{(-2h^s_V G)}_{\text{symmetric}} + \underbrace{(2J' + (\nabla_U J)G + \sigma(V)H)}_{\text{skew-symmetric}}.$

Proof: Let $X, Y \in \mathcal{H}$. Then $G(\nabla_X G)V = G\nabla_X(GV) - G^2\nabla_X V = p\nabla_X V$.

This implies:

$$
\begin{aligned}
2g(\nabla_X V, Y) &= 2g(G(\nabla_X G)V, Y) \\
&= -2g((\nabla_X G)V, GY) \\
&= -g([G,G](V,GY), GX) - 3v \wedge d\sigma(X, V, GY) \\
&\quad + \sigma(V)g(GY, HX) + 2g(GY, JX) \\
&= g(G(\mathcal{L}_V G)GY, GX) - d\sigma(GY, X) \\
&\quad + \sigma(V)g(Y, JX) - 2g(Y, HX) \\
&= g((\mathcal{L}_V G)GY, X) - d\sigma(GY, X) \\
&\quad - \sigma(V)g(JY, X) + 2g(HY, X)
\end{aligned}
$$

Also, $p\nabla_X V = -HX + h_V X$. Thus, we have:

$$
\begin{aligned}
2g(\nabla_X V, Y) &= -2g(HX, Y) + 2g(h_V X, Y) \\
&= 2g(X, HY) + 2g(X, h_V Y).
\end{aligned}
$$

Recall: for any 2-form, ϕ, we define the $(1,1)$–tensor $\phi^\sharp$ by:

$$g(\phi^\sharp X, Y) = \phi(X, Y).$$

Hence,

$$2H + 2h_V = p(\mathcal{L}_V G)G - p(d\sigma)^\sharp G + \sigma(V)J' + 2H$$

$$p(\mathcal{L}_V G)G = p(d\sigma)^\sharp G + \sigma(V)J' + 2h_V$$

$$p(\mathcal{L}_V G)p = p(d\sigma)^\sharp + \sigma(V)H - 2h_V G$$

$$p(\mathcal{L}_V G)p = 2J' + p(\nabla_U J)G - 2Hh_U^d + \sigma(V)H - 2h_V G$$

$$p(\mathcal{L}_V G)p = 2J' + p(\nabla_U J)G + \sigma(V)H - 2h_V^s G,$$

since $-h_V^d G = Jh_U^d G = -JGh_U^d = Hh_U^d$.

1.7 Identities concerning h_U and its covariant derivative

We will finish this chapter with a few identities which describe the covariant derivative of h_U in the direction U and JU. These will be very important in the upcoming chapters.

Proposition 1.7.1 For any unit vertical vector field U, we have:

$$\nabla_{JU}h_U - \nabla_U h_{JU} = -\sigma_U(U)h_U - \sigma_U(JU)h_{JU} + 2k_{JU}h_U^s + 2h_{JU}^s k_U.$$

Proof: Let $X \in H, U \in \Gamma^\infty(\mathcal{V})$. Then, by Proposition 1.6.2,

$$\begin{aligned} pR_{UX}JU =& p(\nabla k_{JU})X + p(\nabla h_{JU})X - \sigma_U(U)(k_U X + h_U X) \\ & - JX + k_{JU}h_U X + h_{JU}k_U X + h_{JU}h_U X. \end{aligned}$$

Since $\nabla_U k_{JU}, k_U$, and J are all skew-symmetric with respect to g, we know that for any horizontal vector field X,

$$g((\nabla_U k_{JU})X, X) = 0,$$

$$\sigma_U(U)g(k_U X, X) = 0,$$

$$g(JX, X) = 0.$$

Therefore,

$$\begin{aligned} g(R_{UX}JU, X) =& g((\nabla_U h_{JU})X, X) - \sigma_U(U)g(h_U X, X) \\ & + g(k_{JU}h_U X, X) + g(h_{JU}k_U X, X) + g(h_{JU}h_U X, X). \end{aligned}$$

This statement is true for any unit vertical vector U. In particular, it is true for JU when we have specified U. Substituting 'JU' for 'U' in the above equation, we get:

$$\begin{aligned} -g(R_{JUX}U, X) =& -g((\nabla_{JU}h_U)X, X) - \sigma_{JU}(JU)g(h_{JU}X, X) \\ & - g(k_U h_{JU}X, X) - g(h_U k_{JU}X, X) - g(h_U h_{JU}X, X), \end{aligned}$$

that is,

$$\begin{aligned} g(R_{JUX}U, X) =& g((\nabla_{JU}h_U)X, X) + \sigma_U(JU)g(h_{JU}X, X) \\ & + g(k_U h_{JU}X, X) + g(h_U k_{JU}X, X) + g(h_U h_{JU}X, X) \\ =& g((\nabla_{JU}h_U)X, X) + \sigma_U(JU)g(h_{JU}X, X) \\ & - g(k_{JU}h_U X, X) - g(h_{JU}k_U X, X) + g(h_{JU}h_U X, X). \end{aligned}$$

Furthermore, we know that

$$g(R_{JUX}U, X) = g(R_{UX}JU, X).$$

So, we have:

$$g((\nabla_{JU}h_U)X, X) + \sigma_U(JU)g(h_{JU}X, X) - g(k_{JU}h_UX, X) - g(h_{JU}k_UX, X)$$

$$=g((\nabla_U h_{JU})X, X) - \sigma_U(U)g(h_UX, X) + g(k_{JU}h_UX, X) + g(h_{JU}k_UX, X).$$

Therefore,

$$g((\nabla_{JU}h_U)X, X) - g((\nabla_U h_{JU}X, X) = -\sigma_U(JU)g(h_{JU}X, X) - \sigma_U(U)g(h_UX, X)$$

$$+ 2g(k_{JU}h_UX, X) + 2g(h_{JU}k_UX, X).$$

Now $\nabla_U h_{JU}, \nabla_{JU}h_U, h_U,$ and h_{JU} are all g-symmetric. Thus,

$$g((\nabla_{JU}h_U)X, Y) - g((\nabla_U h_{JU}X, Y) = -\sigma_U(JU)g(h_{JU}X, Y) - \sigma_U(U)g(h_UX, Y)$$

$$+ g(k_{JU}h_UX, Y) + g(h_{JU}k_UX, Y)$$

$$+ g(k_{JU}h_UY, X) + g(h_{JU}k_UY, X)$$

$$= -\sigma_U(JU)g(h_{JU}X, Y) - \sigma_U(U)g(h_UX, Y)$$

$$+ g([k_{JU}, h_U]X, Y) + g([h_{JU}, k_UX], Y)$$

$$= -\sigma_U(JU)g(h_{JU}X, Y) - \sigma_U(U)g(h_UX, Y)$$

$$+ 2g(k_{JU}h_U^sX, Y) + g(h_{JU}^sk_UX, Y),$$

since

$$[k_{JU}, h_U] = k_{JU}h_U - h_Uk_{JU}$$

$$= k_{JU}h_U^s - h_U^sk_{JU} + k_{JU}h_U^d - h_U^dk_{JU}$$

$$= k_{JU}h_U^s - h_U^sk_{JU} + k_{JU}h_U^d - k_{JU}h_U^d$$

$$= k_{JU}h_U^s - h_U^sk_{JU}.$$

This proves the proposition.

Now, both operators $k_{JU}h_U^s = Hh_U{}^s$ and $h_{JU}^sk_U = -h_{JU}{}^sG$ anti-commute with J. So, combining these facts with Propostion 1.7.1, we have the following corollary:

Corollary 1.7.2 For any unit vertical vector field U,

$$(\nabla_{JU} h_U)^s - (\nabla_U h_{JU})^s = -\sigma_U(U) h_U^s - \sigma_U(JU) h_{JU}^s.$$

We close this section with some additional identities involving ${h_U}^d$ and its covariant derivative.

Proposition 1.7.3 For any vertical vector fields U and W, $(\nabla_W h_U^d)^s = \frac{1}{2}[\nabla_{JW} J, h_U^d]$.

Proof:

$$J(\nabla_W h_U^d) J = \nabla_W (J h_U^d J) - (\nabla_W J) h_U^d J - J h_U^d (\nabla_W J)$$
$$= \nabla_W h_U^d - (\nabla_{JW} h_U^d + h_U^d(\nabla_{JW} J).$$

This proves the proposition.

The previous two results give us the following proposition.

Proposition 1.7.4 For any unit vertical vector field U,

$${h_U}^d(\nabla_U J) = (\nabla_U J){h_U}^d.$$

Proof: Let U be a unit vertical vector field with corresponding complex almost contact structure $\{G, H, U, V, u, v, g\}$. Then we know by Corollary 1.7.2 that

$$(\nabla_{JU} h_U)^s - (\nabla_U h_{JU})^s = -\sigma(U){h_U}^s - \sigma(JU){h_{JU}}^s.$$

The right-hand side of this equation anti-commutes with G, so the left-hand side must also anti-commute with G. In particular,

$$(\nabla_{JU} h_U)^s - (\nabla_U h_{JU})^s - G((\nabla_{JU} h_U)^s - (\nabla_U h_{JU})^s) G = 0,$$

i.e.

$$(\nabla_{JU} h_U)^s - G((\nabla_{JU} h_U)^s) G = (\nabla_U h_{JU})^s - G((\nabla_U h_{JU})^s) G.$$

Now,

$$\begin{aligned}
-G(\nabla_{JU}h_U)^s)G &= -(G(\nabla_{JU}h_U)G)^s \\
&= -(\nabla_{JU}(Gh_UG) - (\nabla_{JU}G)h_UG - Gh_U(\nabla_{JU}G))^s \\
&= -(\nabla_{JU}h_U - (\nabla_{JU}G)h_UG - Gh_U(\nabla_{JU}G))^s \\
&= -(\nabla_{JU}h_U)^s + (\nabla_{JU}G)^s {h_U}^d G + (\nabla_{JU}G)^d {h_U}^s G \\
&\quad + G{h_U}^s(\nabla_{JU}G)^d + G{h_U}^d(\nabla_{JU}G)^s.
\end{aligned}$$

Also,

$$\begin{aligned}
\nabla_{JU}G &= \nabla_{JU}(JH) \\
&= (\nabla_{JU}J)H + J(\nabla_{JU}H) \\
&= -(\nabla_U J)G - \sigma(JU)JG \\
&= -(\nabla_U J)G + \sigma(JU)H.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\nabla_{JU}G)^s &= -(\nabla_U J)G; \\
(\nabla_{JU}G)^d &= \sigma(JU)H.
\end{aligned}$$

Using these facts, we get

$$\begin{aligned}
-G(\nabla_{JU}h_U)^s G &= -(\nabla_{JU}h_U)^s - (\nabla_U J)G{h_U}^d G + \sigma(JU)H{h_U}^s G \\
&\quad + \sigma(JU)G{h_U}^s H - G{h_U}^d(\nabla_U J)G \\
&= -(\nabla_{JU}h_U)^s - (\nabla_U J){h_U}^d + {h_U}^d(\nabla_U J).
\end{aligned}$$

Therefore,

$$(\nabla_{JU}h_U)^s - G((\nabla_{JU}h_U)^s)G = -(\nabla_U J){h_U}^d + {h_U}^d(\nabla_U J).$$

Also,

$$\begin{aligned}
-G(\nabla_U h_{JU})^s G &= -(G(\nabla_U h_{JU})G)^s \\
&= -(\nabla_U(Gh_{JU}G) - (\nabla_U G)h_{JU}G - Gh_{JU}(\nabla_U G))^s \\
&= -(\nabla_U(Gh_{JU}G))^s + \sigma(U)H{h_{JU}}^s G + \sigma(U)G{h_{JU}}^s H \\
&= -(\nabla_U(Gh_{JU}G))^s.
\end{aligned}$$

So,

$$\begin{aligned}(\nabla_U h_{JU})^s - G(\nabla_U h_{JU})^s G &= (\nabla_U h_{JU})^s - (\nabla_U (G h_{JU} G))^s \\ &= (\nabla_U (h_{JU} - G h_{JU} G))^s \\ &= 2(\nabla_U h_{JU}{}^d)^s.\end{aligned}$$

Therefore, $2(\nabla_U h_{JU}{}^d)^s = -(\nabla_U J)h_U{}^d + h_U{}^d(\nabla_U J)$. But, by Proposition 1.7.3,

$$\begin{aligned}2(\nabla_U h_{JU}{}^d)^s &= [\nabla_{JU} J, h_{JU}{}^d] \\ &= (\nabla_{JU} J)h_{JU}{}^d - h_{JU}{}^d(\nabla_{JU} J) \\ &= J(\nabla_U J)J h_U{}^d - J h_U{}^d J(\nabla_U J) \\ &= (\nabla_U J)h_U{}^d - h_U{}^d(\nabla_U J).\end{aligned}$$

Thus, $0 = (\nabla_U h_{JU}{}^d)^s = (\nabla_U J)h_U{}^d - h_U{}^d(\nabla_U J)$.

Chapter Two

THE SPACE OF ASSOCIATED METRICS

In this chapter, we describe the set of all associated metrics. In Section 1, we derive some properties of this space, including a complete description of its tangent space. In Section 2, we use this tangent space to relate the structure tensors of any two associated metrics. In Section 3, we begin the groundwork to analyzing the critical conditions of Riemannian functionals of associated metrics. Finally, in Section 4 and Section 5, we define and derive the critical conditions for two Riemannian functionals.

2.1 The Space of all Associated Metrics

For this chapter we will assume that M is a compact complex contact manifold with normalized contact structure given by $\underline{\pi} = \{\pi\}$.

Let $\mathcal{A}$ = space of all metrics associated to the normalized contact structure, $\underline{\pi}$. Then $\mathcal{A}$ is, of course, contained in the space of all Hermitian metrics on M, which is, in turn, contained in the space of all Riemannian metrics on M. Now, since it is clear that given an associated metric g on M , there is a unique complex almost contact structure $\{G, U, u, g\}$ that comes with g, we see that $\mathcal{A}$ is, in fact, the space of all complex almost contact structures on M, which are derived from $\underline{\pi}$. We shall now study $\mathcal{A}$ in more detail.

Proposition 2.1.1 For any $g, g' \in \mathcal{A}, dV_g = dV_{g'}$.

Proof: Let $g \in \mathcal{A}$. Let $\mathcal{O} \subset M$ be an open set with local complex almost contact structure $\{G, H, U, V, u, v, g, \mathcal{O}\}$. Let $\{X_j, GX_j, JX_j, HX_j\}_{j=1}^{n} \cup \{U, V\}$ be a local orthonormal basis of TO with corresponding dual basis $\{\tau_j, \tau_j^*, \tau_j^{**}, \tau_j^{***}\} \cup \{u, v\}$ with respect to g.

Then $dV_g = u \wedge v \wedge (\Lambda_{j=1}^n(\tau_j \wedge \tau_j^* \wedge \tau_j^{**} \wedge \tau_j^{***})$. Now, $\tilde{G}(X,Y) = g(X, GY)\ \forall\ X, Y \in TO$. So,

$$\tilde{G} = \sum_{j=1}^n (\tau_j \wedge \tau_j^* + \tau_j^{**} \wedge \tau_j^{***}).$$

And so,

$$\begin{aligned} du &= \tilde{G} + \sigma \wedge v \\ &= \sum_{j=1}^n (\tau_j \wedge \tau_j^* + \tau_j^{**} \wedge \tau_j^{***}) + \sigma \wedge v. \end{aligned}$$

Therefore,

$$\begin{aligned} u \wedge v \wedge (du)^{2n} &= u \wedge v \wedge \Big(\sum_{j=1}^n (\tau_j \wedge \tau_j^* + \tau_j^{**} \wedge \tau_j^{***}) + \sigma \wedge v\Big)^{2n} \\ &= u \wedge v \wedge \Big(\sum_{j=1}^n (\tau_j \wedge \tau_j^* + \tau_j^{**} \wedge \tau_j^{***})\Big)^{2n} \\ &= u \wedge v \wedge A_n(\tau_1 \wedge \tau_1^* \wedge \tau_1^{**} \wedge \tau_1^{***} \wedge \ldots \wedge \tau_n \wedge \tau_n^* \wedge \tau_n^{**} \wedge \tau_n^{***}) \\ &= A_n u \wedge v \wedge \big(\Lambda_{j=1}^n(\tau_j \wedge \tau_j^* \wedge \tau_j^{**} \wedge \tau_j^{***})\big) \\ &= A_n dV_g, \end{aligned}$$

where A_n is a constant depending only on n. Hence,

$$dV_g = \frac{1}{A_n} u \wedge v \wedge (du)^{2n}.$$

Thus, we have shown that dV_g is a $(4n+2)$–form independent of g; and, so, the volume elements of any two associated metrics are the same.

We have established that all associated metrics give the same volume for M. For a fixed real number a, let $\mathcal{R}_a =$ space of all Riemannian metrics g on M such that $\int_M dV_g = a$. Then we know that, for $g \in \mathcal{R}_a, T_g\mathcal{R}_a = \{D \in \mathrm{Hom}(TM, TM) : g(DX, Y) = g(X, DY),\ \int_M \mathrm{Tr}(D) dV_g = 0\}$. In particular, $\mathcal{A} \subset \mathcal{R}_a$ for some fixed a, cf. Ebin [Eb]. Thus, we know that for $g \in \mathcal{A}$ and $D \in T_g\mathcal{A}, D$ is symmetric with respect to g and $\int_M \mathrm{Tr}(D) dV_g = 0$.

Note that, once a particular metric g is fixed we will be identifying $(1,1)$-tensors with $(0,2)$-tensors by the identification:

$$D(X,Y) = g(X, DY).$$

The work above gives us part of the following theorem:

Theorem 2.1.2 Let $g \in \mathcal{A}$ with local complex almost contact structure given by $\{G, U, u, \mathcal{O}\}$. Then $D \in T_g\mathcal{A}$ if and only if

1) D is symmetric with respect to g,

2) $DJ = JD$,

3) $DG = -GD$ on $\mathcal{O}$,

4) $DU = 0$ on $\mathcal{O}$.

So, with respect to a local basis $\underline{E} = \underline{E}' \cup \{U, V\}$, where $\underline{E}'$ is a local basis of $\mathcal{H}$,

$$D = \left(\begin{array}{c|c} \tilde{D} & 0 \\ \hline 0 & 0 \end{array}\right),$$

where $\tilde{D} = D|_{\mathcal{H}}, \tilde{D}J = J\tilde{D}$, and $\tilde{D}G = -G\tilde{D}$.

Proof: Let $t \mapsto g_t$ be a path in $\mathcal{A}$ with $g = g_0$. Then we define $D \in T_g\mathcal{A}$ by

$$\frac{d}{dt}(g_t(X,Y))|_{t=0} = D(X,Y) = g(X, DY).$$

So,

$$g_t(X,Y) = g(X,Y) + tg(X,DY) + O(t^2).$$

Note that, by definition of $T_g\mathcal{A}$, any element of $T_g\mathcal{A}$ can be realized by such a path. Now, each g_t is Hermitian with respect to J; so, for each $X, Y \in \mathcal{O}$,

$$g_t(X, JY) = g(X, JY) + tg(X, DJY) + O(t^2)$$

$$= -g_t(JX, Y) = -g(JX, Y) - tg(JX, DY) + O(t^2),$$

Hence, for $t \neq 0$, we have: $tg(X, DJY) + O(t^2) = -tg(JX, DY) + O(t^2)$. Thus, $g(X, DJY) + O(t) = -g(JX, DY) + O(t)$. And, by letting $t \to 0$, we get: $g(X, DJY) = -g(JX, DY)$ or $g(X, DJY) = g(X, JDY)$. Thus, $DJ = JD$.

Also, for $X \in T\mathcal{O}$,

$$\begin{aligned} g(X,U) &= u(X) \\ &= g_t(X,U) \\ &= g(X,U) + tg(X,DU) + O(t^2). \end{aligned}$$

So, $0 = tg(X, DU) + O(t^2)$; or, for $t \neq 0$, $0 = g(X, DU) + O(t)$. Let $t \to 0$, and we get:

$$0 = g(X, DU) \ \forall\ X \in T\mathcal{O}.$$

Thus, $DU = 0$.

Now, D is symmetric with respect to g, and $TM = \mathcal{H} \oplus \mathcal{V}$ is an orthogonal splitting. So, we know $D(\mathcal{H}) \subset \mathcal{H}$, i.e. with respect to a local basis $\underline{E} = \underline{E}' \cup \{U, V\}$, where $\underline{E}'$ is a local basis of $\mathcal{H}$,

$$D = \left(\begin{array}{c|c} \tilde{D} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Let $X, Y \in \mathcal{H}$. Then:

$$g(X, GY) = du(X, Y) = g_t(X, G_t Y) = g(X, G_t Y) + tg(X, DG_t Y) + O(t^2),$$

i.e.

$$G = G_t + tDG_t + O(t^2).$$

By applying G_t on the right and G on the left, we have:

$$G_t = G + tGD + O(t^2).$$

Squaring this, we get: $GDG - pDp = 0$. So, $GDp = pDG$. Since $D(\mathcal{H}) \subset \mathcal{H}$, we know that $pDp = D$; thus, $GD = -DG$.

Suppose D is a g-symmetric $(1,1)$-tensor on M such that $DJ = JD, DU = 0, DG = -GD$. Set $g_t(X, Y) = g(X, e^{tD}Y), G_t = G \circ e^{tD}$, where

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots.$$

Note: $e^{tD}J = Je^{tD}$; $e^{tD}G = Ge^{-tD}$; $e^{tD}U = U$. Also, $e^A e^{-A} = I$. If A is g-symmetric, then e^A is as well. If A is skew-symmetric with respect to g, then we have: $g(X, e^A Y) = g(e^{-A}X, Y)$. Therefore,

$$\begin{aligned} g_t(X, JY) &= g(X, e^{tD}JY) \\ &= g(X, Je^{tD}Y) \\ &= -g(JX, e^{tD}Y) \\ &= -g_t(JX, Y). \end{aligned}$$

Thus, each g_t is Hermitian with respect to J.

For $X, Y \in \mathcal{H}$,

$$\begin{aligned} g_t(X, G_tY) &= g(X, e^{tD}Ge^{tD}Y) \\ &= g(X, e^{tD}e^{-tD}GY) \\ &= g(X, GY) \\ &= du(X, Y). \end{aligned}$$

This also tells us that g_t is Hermitian with respect to G_t, as well. Additionally,

$$G_tJ = Ge^{tD}J = GJe^{tD} = -JGe^{tD} = -JG_t,$$

$$G_t^2 = (Ge^{tD})(Ge^{tD}) = Ge^{tD}e^{-tD}G = G^2 = -p.$$

Furthermore, it is clear that the local endomorphisms $G_t, H_t = G_t \circ J$ transform exactly as the original ones, $G, H = G \circ J$, do. Finally, for $X \in T\mathcal{O}, g_t(X, U) = g(X, e^{tD}U) = g(X, U) = u(X)$.

Thus, $\{G_t, H_t = G_t \circ J, U, V, u, v, g_t, \mathcal{O}\}$ forms a complex contact metric structure on M. Thus, $D \in T_g\mathcal{A}$. This proves the theorem.

The last result of this section concerns the connectivity of $\mathcal{A}$. This proposition not only tells us that $\mathcal{A}$ is path-connected; it also tells us that $\mathcal{A}$ is geodesically connected.

Proposition 2.1.3 Let $g, g' \in \mathcal{A}$. Then there exists $D \in T_g\mathcal{A}$ such that $g' = ge^D$, i.e.

$$g'(X, Y) = g(X, e^DY).$$

Proof: We will denote the local complex almost contact structure corresponding to g by $\{G, H, U, V, u, v, g, \mathcal{O}\}$ and that corresponding to g' by $\{G', H', U', V', u', v', g', \mathcal{O}'\}$. Note that we can assume that $\mathcal{O} = \mathcal{O}'$, so that $U = U', u = u'$.

We need to find $D \in T_g\mathcal{A}$ such that:

1) $g' = ge^D$,

2) $DJ = JD, DG = -GD, DU = 0$,

3) D is symmetric with respect to g.

Let

$$\mathcal{G} = \left(\begin{array}{c|c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right).$$

Then $\mathcal{G} \in Gl(4n; R)$. Let $\underline{X} = \{X_j\}_{j=1}^{4n} \subset \mathcal{H}$ be a local g-orthonormal basis of $\mathcal{H}$ such that $[G] = \mathcal{G}$. Here, for any $(1,1)$-tensor or $(0,2)$-tensor on $\mathcal{H}, A$, we denote its matrix representation with respect to $\underline{X}$ by $[A]$. Then

$$\begin{aligned} \mathcal{G} &= [G] \\ &= [du|_{\mathcal{H}}] \\ &= (du(X_i, X_j)) \\ &= (g'(X_j, G'X_j)) \\ &= [g'][G'] \end{aligned}$$

.

So, $[g'] = -\mathcal{G}[G']$, or $[G'] = \mathcal{G}[g']$. This shows that $-\mathcal{G}[G']$ is a positive definite, symmetric matrix. Thus, there exists a unique $\mathcal{D}$ such that $e^{\mathcal{D}} = -\mathcal{G}[G']$. Let D be the linear transformation on TM given by the matrix

$$D = \left(\begin{array}{c|c} \mathcal{D} & 0 \\ \hline 0 & 0 \end{array}\right),$$

with respect to the basis $\underline{X} \cup \{U, V\}$. Then $e^{\mathcal{D}} = [e^{D}|_{\mathcal{H}}]$, and, by definition, $D|_{\mathcal{V}} \equiv 0$. We have now shown that $g' = ge^{D}$. At this point, we only need to show that $D \in T_g\mathcal{A}$.

Set $\mathcal{J} = [J]$. Then $[g']\mathcal{J} = -{}^t\mathcal{J}[g'] = \mathcal{J}[g']$, since g' is Hermitian with respect to J and $\underline{X}$ is a J-basis. Then, $e^{\mathcal{D}}\mathcal{J} = \mathcal{J}e^{\mathcal{D}}$;and so $\mathcal{D}\mathcal{J} = \mathcal{J}\mathcal{D}$. Thus, $DJ = JD$, since $D|_{\mathcal{V}} \equiv 0$. Also,

$$-I = [G']^2 = \mathcal{G}[g']\mathcal{G}[g'] = \mathcal{G}e^{\mathcal{D}}\mathcal{G}e^{\mathcal{D}}.$$

Lastly, we have:

$$\mathcal{G} = e^{\mathcal{D}}\mathcal{G}e^{\mathcal{D}};$$

$$\mathcal{G}e^{-\mathcal{D}} = e^{-\mathcal{D}}\mathcal{G};$$

$$\mathcal{G}\mathcal{D} = -\mathcal{D}\mathcal{G};$$

$$GD = -DG,$$

which completes the proof.

2.2 Relations between Associated Metrics

By Proposition 2.1.3, we know that any two associated metrics on M can be connected by a geodesic in $\mathcal{A}$. We now would like to describe the relationship between the structure tensors of these two associated metrics, in terms of the tangent vector, D, of the path connecting them.

Propostion 2.2.1 Suppose $D \in T_gA$ and $g'(X,Y) = g(X, e^DY)$ for all $X, Y \in TM$. Let ∇' and ∇ be the Levi-Civita connections for g' and g, respectively. Let $k', k : V \to End(H)$ be the skew-symmetric operators of the corresponding metrics; and let $h', h : V \to End(H)$ be the symmetric operators. Then, for $X \in \mathcal{H}$:

1) $k'_UX = k_Ue^DX = e^{-D}k_UX$.

2) $h'_UX = \frac{1}{2}pe^{-D}(\nabla_Ue^D)X + \frac{1}{2}(k_U - e^{-D}k_Ue^D)X + \frac{1}{2}(h_U + e^{-D}h_Ue^D)X$,

3) $(\nabla'_UJ)X = (\nabla_UJ)X + 2e^{-D}k_UJX + pe^{-D}(\nabla_Ue^D)^dJX + e^{-D}[h^d_U, e^D]JX - e^{-D}\{k_U, e^D\}JX$.

Proof: Let $X, Z \in \mathcal{H}, U \in \mathcal{V}$. By definition,

$$\begin{aligned} g'(k'_UX, Z) &= \frac{1}{2}g'(\nabla'_XU, Z) - \frac{1}{2}g'(\nabla'_ZU, X) \\ &= -\frac{1}{2}u(\nabla'_XZ - \nabla'_ZX) \\ &= -\frac{1}{2}u([X, Z]) \\ &= g(k_UX, Z). \end{aligned}$$

Close scrutiny of this equation gives us: $k'_U = k_Ue^D = e^{-D}k_U$. This proves the first part of the proposition.

$$\begin{aligned} \text{Now, } 2g'(\nabla'_XU, Z) &= Ug'(X, Z) + g'([X, U], Z) + g'([Z, X], U) + g'(X, [Z, U]) \\ &= Ug(X, e^DZ) + g([X, U], e^DZ) + g([Z, X], e^DU) + g(X, e^D[Z, U]) \\ &= Ug(X, e^DZ) + g([X, U], e^DZ) - g([X, Z], U) + g(e^DX, [Z, U]) \\ &= g(\nabla_UX, e^DZ) + g(X, \nabla_U(e^DZ)) + g(\nabla_XU, e^DZ) - g(\nabla_UX, e^DZ) \\ &\quad + 2g(k_UX, Z) + g(e^DX, \nabla_ZU) - g(e^DX, \nabla_UZ). \end{aligned}$$

So,

$$
\begin{aligned}
2g'(\nabla'_X U, Z) =& 2g(k_U X, Z) + g(X, \nabla_U(e^D Z)) - g(X, e^D \nabla_U Z) \\
& + g(\nabla_X U, e^D Z) + g(\nabla_Z U, e^D X) \\
=& 2g(k_U X, Z) + g(X, (\nabla_U e^D) Z) \\
& + g(k_U X, e^D Z) + g(h_U X, e^D Z) + g(X, e^D k_U Z) + g(X, e^D h_U Z) \\
=& 2g(k_U X, Z) + g(X, (\nabla_U e^D) Z) \\
& + g(e^D k_U X, Z) + g(e^D h_U X, Z) - g(k_U e^D X, Z) + g(h_U e^D X, Z) \\
=& 2g(k_U X, Z) + g(X, (\nabla_U e^D) Z) \\
& + g([e^D, k_U] X, Z) + g(\{e^D, h_U\} X, Z),
\end{aligned}
$$

where $\{A, B\} = AB + BA$ and $[A, B] = AB - BA$, for any two linear transformations A and B. Also,

$$
\begin{aligned}
2g'(\nabla'_X U, Z) &= 2g'(k'_U X, Z) + 2g'(h'_U X, Z) \\
&= 2g(k_U X, Z) + 2g(h'_U X, e^D Z) \\
&= 2g(k_U X, Z) + 2g(e^D h'_U X, Z).
\end{aligned}
$$

Thus, we have two expressions for $2g'(\nabla'_X U, Z)$. Setting these equal to each other, we get:

$$
2e^D h'_U = \nabla_U e^D + [e^D, k_U] + \{e^D, h_U\}
$$

$$
\text{Or, } h'_U = \frac{1}{2} e^{-D} (\nabla_U e^D) + \frac{1}{2}(k_U - e^{-D} k_U e^D) + \frac{1}{2}(h_U + e^{-D} h_U e^D).
$$

Thus, the second part of the proposition is proven.

Let $Z, W \in \mathcal{H}$ be unit vectors. Then

$$
\begin{aligned}
2g'(\nabla'_U Z, W) =& U g'(Z, W) + g'([U, Z], W) + g'([W, U], Z) + g'(U, [W, Z]) \\
=& U g(Z, e^D W) + g([U, Z], e^D W) + g([W, U], e^D Z) + g(U, e^D [W, Z]) \\
=& g(\nabla_U Z, e^D W) + g(Z, \nabla_U(e^D W)) + g(\nabla_U Z, e^D W) - g(\nabla_Z U, e^D W) \\
& + g(\nabla_W U, e^D Z) - g(\nabla_U W, e^D Z) + g(e^D U, [W, Z]) \\
=& 2g(\nabla_U Z, e^D W) + g(Z, \nabla_U(e^D W)) - g(Z, e^D \nabla_U W) \\
& - g(\nabla_Z U, e^D W) + g(\nabla_W U, e^D Z) + g(U, [W, Z])
\end{aligned}
$$

$$=2g(\nabla_U Z, e^D W) + g(Z, (\nabla_U e^D)W)$$

$$- g(h_U Z, e^D W) - g(k_U Z, e^D W) + g(h_U W, e^D Z) + g(k_U W, e^D Z) + 2g(W, k_U Z)$$

$$=2g(\nabla_U Z, e^D W) + g(Z, (\nabla_U e^D)W)$$

$$- g(h_U Z, e^D W) + g(e^D Z, h_U W)$$

$$- g(k_U Z, e^D W) + g(e^D Z, k_U W) + 2g(k_U Z, W)$$

$$=2g(e^D \nabla_U Z, W) + g((\nabla_U e^D)Z, W) - g(e^D h_U Z, W) + g(h_U e^D Z, W)$$

$$- g(e^D k_U Z, W) - g(k_U e^D Z, W) + 2g(k_U Z, W)$$

$$=g(2e^D \nabla_U Z + (\nabla_U e^D)Z + [h_U, e^D]Z - \{k_U, e^D\}Z, W) + 2g(k_U Z, W).$$

Since $g'(\nabla'_U Z, W) = g(e^D \nabla'_U Z, W)$, we get:

$$2pe^D \nabla'_U Z = 2pe^D \nabla_U Z + p(\nabla_U e^D)Z + p[h_U, e^D]Z - p\{k_U, e^D\}Z + 2k_U Z$$

$$\text{Or,}\quad p\nabla'_U Z = p\nabla_U Z + e^{-D} k_U Z + \frac{1}{2} pe^{-D}(\nabla_U e^D + [h_U, e^D] - \{k_U, e^D\})Z.$$

However, $q\nabla'_U Z = q\nabla_U Z = 0$. Thus,

$$\nabla'_U Z = \nabla_U Z + e^{-D} k_U Z + \frac{1}{2} pe^{-D}(\nabla_U e^D + [h_U, e^D] - \{k_U, e^D\})Z.$$

Hence, for any horizontal vector X,

$$(\nabla'_U J)X = \nabla'_U(JX) - J\nabla'_U X$$

$$=\nabla_U(JX) + e^{-D} k_U JX + \frac{1}{2} pe^{-D}(\nabla_U e^D + [h_U, e^D] - \{k_U, e^D\})JX$$

$$- J\nabla_U X - Je^{-D} k_U X - \frac{1}{2} Jpe^{-D}(\nabla_U e^D + [h_U, e^D] - \{k_U, e^D\})X$$

$$=(\nabla_U J)X + e^{-D} k_U JX - e^{-D} J k_U X + \frac{1}{2} pe^{-D}((\nabla_U e^D)J - J(\nabla_U e^D))X$$

$$+ \frac{1}{2} pe^{-D}([h_U, e^D]J - J[h_U, e^D])X - \frac{1}{2} pe^{-D}(\{k_U, e^D\}J - J\{k_U, e^D\})X$$

$$=(\nabla_U J)X + 2e^{-D} k_U JX + \frac{1}{2} pe^{-D} 2(\nabla_U e^D)^d JX$$

$$+ \frac{1}{2} pe^{-D} 2[h_U, e^D]^d JX - \frac{1}{2} pe^{-D} 2\{k_U, e^D\}^d JX$$

$$=(\nabla_U J)X + 2e^{-D} k_U JX + pe^{-D}(\nabla_U e^D)^d JX$$

$$+ pe^{-D}[h^d_U, e^D]JX - pe^{-D}\{k_U, e^D\}JX.$$

This proves the last part of the proposition.

2.3 Riemannian Functionals on $\mathcal{A}$.

We will be interested in analyzing certain functions on $\mathcal{A}$. In particular, we are interested in characterizing the critical points of various functions. In order to do this, we will need the following lemma.

Lemma 2.3.1 Let $g \in \mathcal{A}$. Suppose that T is a $(1,1)$-tensor field, which is symmetric with respect to g. Then:

$$\int_M tr(TD)dV_g = 0 \;\forall\; D \in T_g\mathcal{A},$$

if and only if $p(TJ + JT)p = HTG - GTH$ on each $\mathcal{O}$.

The last condition can be written as:

$$pT^s p = -GT^s G.$$

Writing T as a $(0,2)$-tensor $(T(X,Y) = g(X,TY))$, this is equivalent to:

$$T(X,Y) + T(JX,JY) - T(GX,GY) - T(HX,HY) = 0 \;\forall\; X,Y \in \mathcal{H}.$$

Proof: Let T be a g-symmetric $(1,1)$-tensor field such that $\int_M tr(TD)dV_g = 0$ for any $D \in T_g\mathcal{A}$. Let $\{G, H, U, V, u, v, g, \mathcal{O}\}$ be a local almost contact structure with respect to g and f a C^∞ function with compact support in $\mathcal{O}$. Let $\underline{X}' = \{X_j, GX_j, JX_j, HX_j\}$ be a local orthonormal basis of $\mathcal{H}$ on $\mathcal{O}$, so that $\underline{X} = \underline{X}' \cup \{U,V\}$ is a local orthornormal basis of $T\mathcal{O}$.

We define a linear transformation $D : TM \to TM$ by:

$$D = f\left(\begin{array}{cccc|c} 0&0&0&1& \\ 0&0&1&0& \\ 0&1&0&0& 0 \\ 1&0&0&0& \\ \hline &&0&&0 \end{array}\right),$$

with respect to $\underline{X}$. Then D is a globally-defined, symmetric $(1,1)$-tensor on M, and that $DU = 0, DJ = JD, DG = -GD$. Thus, $D \in T_g\mathcal{A}$. And, so, $\int_M tr(TD) = dV_g$.

Set $T = (T_{ij})$ as a matrix with respect to $\underline{X}$. Then $tr(TD) = 2f(T_{14} + T_{32})$. In particular, $0 = \int_M f(T_{14} + T_{32})dV_g$ for any C^∞ function f with support in $\mathcal{O}$. Then, on $\mathcal{O}, T_{14} + T_{32} = 0$; or $T(X_1, HX_1) + T(GX_1, JX_1) = 0$.

Now, X_1 can be any unit horizontal vector field on $\mathcal{O}$. This means that, for any horizontal vector X, $T(X, HX) + T(GX, JX) = 0$. Writing T as a $(1,1)$-tensor, we get:

$$g(X, THX) = -g(GX, TJX) \ \forall \ X \in \mathcal{H}.$$

Substituting $X + Y$ for X in the above equation and using the g-symmetry of T, we get:

$$g(X, THY - HTY) = g(X, GTJY) + g(X, JTGY) \ \forall \ X, Y \in \mathcal{H}.$$

Hence,

$$pTH - HTp = GTJp + pJTG;$$

$$pTp + HTH = pJTJp - GTG;$$

$$pTp - pJTJp = -GTG - HTH;$$

$$p(T - JTJ)p = -GTG - GJTGJ;$$

$$p(T - JTJ)p = -G(T - JTJ)G;$$

$$pT^{*}p = -GT^{*}G.$$

Now, suppose $pTp - pJTJp = -GTG - HTH$ for every local complex almost contact structure $\{G, U, u, g, \mathcal{O}\}$. Let $D \in T_g\mathcal{A}$. Since $DU = 0$ and $DV = -DJU = -JDU = 0$, we know that $Dq = 0$. In particular, $pDq = 0$. Since D is symmetric with respect to g, we know also that $qDp = 0$. Thus, $D = pDp$.

Therefore,

$$\begin{aligned} TD &= (p+q)T(p+q)D \\ &= (p+q)TpDp \\ &= pTpDp + qTpDp. \end{aligned}$$

So, $tr(TD) = tr(pTpDp)$.

Now, $pTp = pJTJp - GTG - HTH$. So,

$$pTpDp = pJTJpDp - GTGDp - HTHDp$$

$$= JpTpDpJ + GTpDG + HTpDH.$$

But,

$$tr(JpTpDpJ) = tr(J^2pTpDp) = -tr(pTpDp)$$

$$tr(GTpDG) = tr(G^2pTpDp) = -tr(pTpDp)$$

$$tr(HTpDH) = tr(H^2pTpDp) = -tr(pTpDp).$$

So, $tr(pTpDp) = -3tr(pTpDp) = 0$. Thus, $tr(TD) = tr(pTpDp) = 0$. This implies that

$$\int_M tr(TD)dV_g = 0.$$

Since D was an arbitrary element of $T_g\mathcal{A}$, we have that $\int_M tr(TD)dV_g = 0$ for any $D \in T_g\mathcal{A}$. This proves the lemma.

Let $g \in \mathcal{A}, D \in T_g\mathcal{A}$. Then it is easy to see that for any $(1,1)$-tensor S on M skew-symmetric with respect to g, we have $tr(DS) = 0$. Thus, we may modify the above lemma as follows:

Lemma 2.3.2 Let $g \in \mathcal{A}$, and T be any $(1,1)$-tensor on M, Then:

$$\int_M tr(TD)dV_g = 0 \ \forall \ D \in T_g\mathcal{A},$$

if and only if

$$p(sym(T^*))p = -G(sym(T^*))G.$$

We will now review a fairly easy example of how we use this lemma to characterize the critical associated metrics of a particular functional.

Theorem 2.3.3 Let M be a complex complex contact manifold; $\mathcal{A}$ its space of associated metrics. Then $g \in \mathcal{A}$ is critical for the functional $A(G) = \int_M \tau_g dV_g$ if and only if

$$pQp - J'QJ' = -GQG + HQH.$$

Proof: Let g_t be a path of metrics in $\mathcal{A}$ with $g = g_0$ and $\frac{d}{dt}(g_t)|_{t=0} = D$, i.e. for any $X, Y \in TM$,

$$\frac{d}{dt}(g_t(X,Y))|_{t=0} = D(X,Y).$$

Then, as proven in [Bl5],

$$\frac{d}{dt}(A)|_{t=0} = -\int_M tr(QD)dV_g.$$

Thus, g is critical for A if and only if $0 = \int_M tr(QD)dV_g$ for any $D \in T_g\mathcal{A}$, which, by Lemma 2.3.1, is equivalent to:

$$pQp - J'QJ' = -GQG + HQH.$$

2.4 Ricci Curvature of $\mathcal{V}$

This section and the next will be spent analyzing two specific Riemannian functionals. Both of these functionals can be thought of as complex analogues of the Riemannian functional, $g \mapsto \int_M Ric(\xi)dV_g$, on the space of associated metrics of a compact, real contact manifold, where ξ is the characteristic vector field (or Reeb vector field) of the contact structure [Bl1].

Let U be a unit vertical vector field on an open domain $\mathcal{O} \subset M$. As usual, set $V = -JU$. Suppose $U', V' = -JU'$ are also unit vertical vector fields with the same domain. Then, there exist real functions on $\mathcal{O}, a$ and b such that:

1) $a^2 + b^2 = 1$;

2) $U' = aU - bV$;

3) $V' = bU + aV$.

Then, letting R_{ij} be the components with respect to any basis of $T\mathcal{O}$ of the Ricci operator,

$$\begin{aligned} Ric(U') + Ric(V') &= R_{ij}U'^iU'^j + R_{ij}V'^iV'^j \\ &= R_{ij}(aU - bV)^i(aU - bV)^j + R_{ij}(bU + aV)^i(bU + aV)^j \\ &= R_{ij}(a^2U^iU^j - abU^iV^j - abV^iU^j + b^2V^iV^j \\ &\qquad + b^2U^iU^j + abU^iV^j + abV^iU^j + b^2V^iV^j) \\ &= R_{ij}(U^iU^j + V^iV^j) \\ &= Ric(U) + Ric(V). \end{aligned}$$

Thus, if we define $Ric(\mathcal{V})$ locally by:

$$Ric(\mathcal{V}) = Ric(U) + Ric(V),$$

$Ric(\mathcal{V})$ is a globally-defined Riemannian function on M, called the *Ricci curvature of* $\mathcal{V}$.

Proposition 2.4.1 Locally, $Ric(\mathcal{V}) = -4d\sigma(U, V) + 8n - tr(h_U^2) - tr(h_{JU}^2)$.

Proof: Let U be a unit vertical vector field. Then, by Proposition 1.6.1,

$$pR_{XU}U = X - h_U^2 X - p(\nabla_U h_U)X - \sigma(U)h_{JU},$$

for any horizontal vector field X. Thus,

$$Ric(U) = g(R_{VU}U, V) + tr(p) - tr(h_U^2) - tr(p(\nabla_U h_U)p) - \sigma(U)tr(h_{JU})$$

$$= g(R_{VU}U, V) + 4n - tr(h_U^2) - tr(p(\nabla_U h_U)p).$$

Now, h_U is symmetric with respect to g. So, there exists an orthonormal basis $\{X_1, \ldots, X_{4n}\}$ of $\mathcal{H}$ such that $h_U X_j = \lambda X_j$ for each j. Now, $tr(h_U) = 0$; so, $\sum_{j=1}^{4n} \lambda_j = 0$. Then

$$g((\nabla_U h_U)X_j, X_j) = g(\nabla_U(h_U X_j), X_j) - g(h_U(\nabla_U X_j), X_j)$$

$$= g(\nabla_U(\lambda_j X_j), X_j) - g(\nabla_U X_j, h_U X_j)$$

$$= (U\lambda_j)g(X_j, X_j) + \lambda_j g(\nabla_U X_j, X_j) - \lambda_j g(\nabla_U X_j, X_j)$$

$$= U\lambda_j.$$

So, $tr(p(\nabla_U h_U)p) = \sum_{j=1}^{4n}(U\lambda_j) = U(\sum_{j=1}^{4n}\lambda_j) = 0$. Furthermore,

$$2d\sigma(U, V) = U\sigma(V) - V\sigma(U) - \sigma([U, V])$$

$$= Ug(\nabla_V U, V) - Vg(\nabla_U U, V) - g(\nabla_{[U,V]}U, V)$$

$$= g(\nabla_U \nabla_V U, V) + g(\nabla_V U, \nabla_U V) - g(\nabla_V \nabla_U U, V)$$

$$- g(\nabla_U U, \nabla_V V) - g(\nabla_{[U,V]}U, V)$$

$$= g(\nabla_U \nabla_V U, V) - g(\nabla_V \nabla_U U, V) - g(\nabla_{[U,V]}U, V)$$

$$= g(R_{UV}U, V).$$

So, $g(R_{VU}U, V) = -2d\sigma(U, V)$. Thus,

$$Ric(\mathcal{V}) = Ric(U) + Ric(V)$$

$$= Ric(U) + Ric(JU)$$

$$= -4d\sigma(U, V) + 8n - tr(h_U^2) - tr(h_{JU}^2).$$

Set $I : \mathcal{A} \to R$ by $I(g) = \int_M Ric(\mathcal{V})dV_g$. We now seek the critical points of I. By the above proposition, we know:

$$I(g) = -4\int_M d\sigma(U, V)dV_g + 8nVol(M) - \int_M (tr(h_U{}^2) + tr(h_{JU}{}^2))dV_g.$$

Recall that the definitions of U, V, and σ do not depend on the given associated metric, so that $\int_M (d\sigma(U,V))dV_g$ does not depend on the particular associated metric. Thus, I can be written in the form

$$I(g) = \text{constant} + \int_M (tr(h_U{}^2) + tr(h_{JU}{}^2))dV_g,$$

so that any projectable metric, i.e. one for which $h_U \equiv 0$ for any vertical vector field U, is not only a critical metric, but a maximum as well, since $tr(A^2) \geq 0$ for any g-symmetric linear transformation A. However, these might not be the only critical metrics of I.

Theorem 2.4.2 Let M be a complex contact manifold with space of associated metrics $\mathcal{A}$. Then $g \in \mathcal{A}$ is critical for the Riemannian functional I if and only if its structure tensors satisfy:

$$(\nabla_U h_U)^s + (\nabla_{JU} h_{JU})^s = -\sigma(U)h^s_{JU} + \sigma(JU)h^s_U + 4k_U h^d_U,$$

for each unit vertical vector field U.

Proof: Let g_t be a path in $\mathcal{A}$ with $g_0 = g$. Define $D \in T_g\mathcal{A}$ by:

$$D_{jk} = \frac{d}{dt}(g_{tjk})|_{t=0}.$$

Also, we define tensor fields:

$$D_{jk}{}^l = \frac{1}{2}(\nabla_j D_k{}^l + \nabla_k D_j{}^l - \nabla^l D_{jk}),$$
$$D_{jkl}{}^m = \nabla_j D_{kl}{}^m - \nabla_k D_{jl}{}^m.$$

Here ∇ is the Levi-Civita connection of g. Then, it is known [Bl5] that:

$$D_{jk}{}^l = \frac{d}{dt}(\Gamma_{jk}{}^l)|_{t=0},$$
$$D_{jkl}{}^m = \frac{d}{dt}(R_{jkl}{}^m)|_{t=0}.$$

where $\Gamma_{jk}{}^l$ are the Christoffel symbols of g and $R_{jkl}{}^m$ are the components of the Riemannian curvature of g.

Fix a unit vertical vector field U; denote its corresponding local complex almost contact structure by $\{G, H, U, V, u, v, g, \mathcal{O}\}$. Then, denoting by *div* any object which is a divergence,

$$\begin{aligned}
\frac{d}{dt}(Ric_t(U))|_{t=0} &= \frac{d}{dt}(R_{ijk}{}^{i}U^jU^k)|_{t=0} \\
&= D_{ijk}{}^{i}U^jU^k \\
&= (\nabla_i D_{jk}{}^{i})U^jU^k - (\nabla_j D_{ik}{}^{i})U^jU^k \\
&= \text{div} - D_{jk}{}^{i}\nabla_i(U^jU^k) + D_{ik}{}^{i}\nabla_j(U^jU^k) \\
&= \text{div} - \frac{1}{2}(\nabla_j D_k{}^{i} + \nabla_k D_j{}^{i} - \nabla^i D_{jk})\nabla_i(U^jU^k) \\
&\quad + \frac{1}{2}(\nabla_i D_k{}^{i} + \nabla_k D_i{}^{i} - \nabla^i D_{ik})\nabla_j(U^jU^k).
\end{aligned}$$

Now, $tr(D) = 0$. So, $\nabla_k D_i{}^{i} = 0$. Also, $\nabla^i D_{ik} = \nabla^i D_{ki} = \nabla_i D_k{}^{i}$. Thus,

$$\begin{aligned}
D_{ijk}{}^{i}U^jU^k &= \text{div} - \frac{1}{2}(\nabla_j D_k{}^{i} + \nabla_k D_j{}^{i} - \nabla^i D_{jk})\nabla_i(U^jU^k) \\
&= \text{div} - (\nabla_j D_k{}^{i})\nabla_i(U^jU^k) + \frac{1}{2}(\nabla^i D_{jk})\nabla_i(U^jU^k) \\
&= \text{div} + D_k{}^{i}\nabla_j\nabla_i(U^jU^k) - \frac{1}{2}D_{jk}\nabla^i\nabla_i(U^jU^k).
\end{aligned}$$

We set:

$$P_1 = D_k{}^{i}\nabla_j\nabla_i(U^jU^k),$$

$$P_2 = D_{jk}\nabla^i\nabla_i(U^jU^k).$$

First, we will consider P_1. Using the facts that $\mathcal{V}$ is totally geodesic and that $D = pDp$, we find

$$\begin{aligned}
P_1 &= D_k{}^{i}\nabla_j\nabla_i(U^jU^k) \\
&= D_k{}^{i}\nabla_j((\nabla_i U^j)U^k + U^j(\nabla_i U^k)) \\
&= D_k{}^{i}\nabla_j((\sigma_i V^j - G_i{}^{j} + (h_U)_i{}^{j})U^k + U^j(\sigma_i V^k - G_i{}^{k} + (h_U)_i{}^{k})) \\
&= D_k{}^{i}\nabla_j(\sigma_i V^jU^k) - D_k{}^{i}\nabla_j(G_i{}^{j}U^k) + D_k{}^{i}\nabla_j((h_U)_i{}^{j}U^k) \\
&\quad + D_k{}^{i}\nabla_j(\sigma_i U^jV^k) - D_k{}^{i}\nabla_j(U^jG_i{}^{k}) + D_k{}^{i}\nabla_j(U^j(h_U)_i{}^{k})
\end{aligned}$$

$$= D_k{}^i \sigma_i V^j (\nabla_j U^k) - D_k{}^i G_i{}^j (\nabla_j U^k) + D_k{}^i (h_U)_i{}^j (\nabla_j U^k)$$

$$+ D_k{}^i \sigma_i U^j (\nabla_j V^k) - D_k{}^i (\nabla_j U^j) G_i{}^k - D_k{}^i U^j (\nabla_j G_i{}^k)$$

$$+ D_k{}^i (\nabla_j U^j)(h_U)_i{}^k + D_k{}^i U^j (\nabla_j (h_U)_i{}^k)$$

$$= D_k{}^i \sigma_i (\nabla_V U^k) - D_k{}^i G_i{}^j (\nabla_j U^k) + D_k{}^i (h_U)_i{}^j (\nabla_j U^k)$$

$$+ D_k{}^i \sigma_i (\nabla_U V^k) - D_k{}^i (\nabla_j U^j) G_i{}^k - D_k{}^i (\nabla_U G_i{}^k)$$

$$+ D_k{}^i (\nabla_j U^j)(h_U)_i{}^k + D_k{}^i (\nabla_U h_U)_i{}^k$$

$$= -D_k{}^i G_i{}^j (\nabla_j U^k) + D_k{}^i (h_U)_i{}^j (\nabla_j U^k) - D_k{}^i (\nabla_j U^j) G_i{}^k$$

$$- D_k{}^i (\nabla_U G_i{}^k) + D_k{}^i (\nabla_j U^j)(h_U)_i{}^k + D_k{}^i (\nabla_U h_U)_i{}^k,$$

$$= -D_k{}^i G_i{}^j (\sigma_i V^k - G_j{}^k + (h_u)_j{}^k)$$

$$+ D_k{}^i (h_U)_i{}^j (\sigma_j V^k - G_j{}^k + (h_U)_j{}^k)$$

$$- D_k{}^i (\sigma_j V^j - G_j{}^j + (h_U)_j{}^j) G_i{}^k - D_k{}^i (\nabla_U G)_i{}^k$$

$$+ D_k{}^i (\sigma_j V^j - G_j{}^j + (h_U)_j{}^j)(h_U)_i{}^k + D_k{}^i (\nabla_U h_U)_i{}^k$$

$$= D_k{}^i G_i{}^j G_j{}^k - D_k{}^i G_i{}^j (h_U)_j{}^k - D_k{}^i (h_U)_i{}^j G_j{}^k$$

$$+ D_k{}^i (h_U)_i{}^j (h_U)_j{}^k - \sigma(V) D_k{}^i G_i{}^k - \sigma(U) D_k{}^i H_i{}^k$$

$$+ D_k{}^i \sigma(V)(h_U)_i{}^k + D_k{}^i (\nabla_U h_U)_i{}^k$$

$$= \underbrace{-tr(Dp)}_{=-tr(D)=0} - tr(Dh_U G) \underbrace{-tr(DGh_U)}_{=tr(Dh_U G)} + tr(Dh_U{}^2)$$

$$- \sigma(V) \underbrace{tr(DG)}_{=0} - \sigma(U) \underbrace{tr(DH)}_{=0} + \sigma(V) tr(Dh_U) + tr(D(\nabla_U h_U))$$

$$= tr(Dh_U{}^2) + \sigma(V) tr(Dh_U) + tr(D(\nabla_U h_U)).$$

Now, we will consider P_2.

$$P_2 = D_{jk}\nabla^i\nabla_i(U^jU^k)$$

$$= D_{jk}\nabla^i((\nabla_iU^j)U^k + U^j(\nabla_iU^k))$$

$$= 2D_{jk}\nabla^i((\nabla_iU^j)U^k)$$

$$= 2D_{jk}(\sigma_iV^jU^k - G_i{}^jU^k + (h_U)_i{}^jU^k$$

$$= -2D_{jk}G_i{}^j(\nabla^iU^k) + 2D_{jk}(h_U)_i{}^j(\nabla^iU^k)$$

$$= -2D_{jk}G_i{}^j(\sigma^iV^k - G^{ik} + (h_U)^{ik}) + 2D_{jk}(h_U)_i{}^j(\sigma^iV^k - G^{ik} + (h_U)^{ik})$$

$$= 2D_{jk}G_i{}^jG^{ik} - 2D_{jk}G_i{}^j(h_U)^{ik} - 2D_{jk}(h_U)_i{}^jG^{ik} + 2D_{jk}(h_U)_i{}^j(h_U)^{ik}$$

$$= -2D_j{}^kG_i{}^jG_k{}^i - 2D_j{}^kG_i{}^j(h_U)_k{}^i + 2D_j{}^k(h_U)_i{}^jG_k{}^i + 2D_j{}^k(h_U)_i{}^j(h_U)_k{}^i$$

$$= 2tr(Dp) - 2tr(DG(h_U)) + 2tr(D(h_U)G) + 2tr(D(h_U)^2)$$

$$= -4tr(DG(h_U)) + 2tr(D(h_U)^2).$$

Therefore,

$$D_{ijk}{}^iU^jU^k = div + P_1 - \frac{1}{2}P_2$$

$$= \mathrm{div} + tr(D(h_U)^2) + \sigma(V)tr(Dh_U) + tr(D(\nabla_Uh_U))$$

$$+ 2tr(DG(h_U)) - tr(D(h_U)^2)$$

$$= \mathrm{div} + tr[D((\nabla_Uh_U) + \sigma(V)h_U + 2Gh_U)]$$

$$= \mathrm{div} + tr[D((\nabla_Uh_U) - \sigma_U(JU)h_U - 2k_Uh_U)]$$

So,

$$\frac{d}{dt}(Ric(U))|_{t=0} = \text{div} + tr[D((\nabla_U h_U) - \sigma_U(JU)h_U - 2k_U h_U)],$$

for every unit vertical vector field U.

Thus, for a fixed unit vertical vector field U with $V = -JU$,

$$\begin{aligned}\frac{d}{dt}(Ric(V))|_{t=0} &= \frac{d}{dt}(Ric(JU))|_{t=0} \\ &= \text{div} + tr[D((\nabla_{JU} h_{JU}) + \sigma_{JU}(U)h_{JU} - 2k_{JU} h_{JU}))] \\ &= \text{div} + tr[D((\nabla_{JU} h_{JU}) + \sigma_U(U)h_{JU} - 2k_{JU} h_{JU}))].\end{aligned}$$

Therefore, locally,

$$\begin{aligned}\frac{d}{dt}(Ric(\mathcal{V}))|_{t=0} = \text{div} + tr[D((\nabla_U h_U) + (\nabla_{JU} h_{JU}) \\ - \sigma_U(JU)h_U + \sigma_{JU}(U)h_{JU} - 2k_U h_U - 2k_{JU} h_{JU})].\end{aligned}$$

So,

$$\begin{aligned}\frac{d}{dt}(I(g_t))|_{t=0} &= \frac{d}{dt}\left(\int_M Ric(\mathcal{V})dV_{g_t}\right)|_{t=0} \\ &= \int_M \frac{d}{dt}(Ric(\mathcal{V}))|_{t=0} dV_g \\ &= \int_M tr[D((\nabla_U h_U) + (\nabla_{JU} h_{JU}) - \sigma_U(JU)h_U \\ &\qquad + \sigma_{JU}(U)h_{JU} - 2k_U h_U - 2k_{JU} h_{JU})]dV_g.\end{aligned}$$

Set

$$T_U = p(\nabla_U h_U)p + p(\nabla_{JU} h_{JU})p - \sigma_U(JU)h_U + \sigma_{JU}(U)h_{JU} - 2k_U h_U - 2k_{JU} h_{JU}.$$

Then, by Lemma 2.3.1, g is critical for I if and only if $T_U^s = -GT_U^s G$ for every unit vertical vector field U with corresponding complex almost contact structure $\{G, H, U, V, u, v, g, \mathcal{O}\}$.

Now, $k_U^s = 0 = k_{JU}^s$ and $k_{JU} h_{JU}{}^d = Jk_U Jh_U{}^d = k_U h_U{}^d$. Also, recall that $k_U h_U = -h_U k_U$. Thus,

$$T_U^s = p(\nabla_U h_U)^s p + p(\nabla_{JU} h_{JU})^s p - \sigma_U(JU)h_U{}^s + \sigma_{JU}(U)h_{JU}{}^s - 4k_U h_U{}^d.$$

Now,

$$G(\nabla_U h_U)G = \nabla_U(Gh_UG) - (\nabla_U G)h_UG - Gh_U(\nabla_U G)$$

$$= \nabla_U h_U - \sigma(U)Hh_UG - \sigma(U)Gh_UH$$

$$= \nabla_U h_U + \sigma(U)HGh_U + \sigma(U)h_UGH$$

$$= \nabla_U h_U + \sigma(U)Jh_U - \sigma(U)h_UJ$$

$$= \nabla_U h_U + 2\sigma(U)Jh_U{}^d.$$

So, $G(\nabla_U h_U)^s G = (\nabla_U h_U)^s$, since $Jh_U{}^d$ anti-commutes with J. Thus, for any unit vertical vector field U, we have: $k_U(\nabla_U h_U)^s k_U = (\nabla_U h_U)^s$. So, for any vertical vector field U, we also have:

$$k_{JU}(\nabla_{JU}h_{JU})^s k_{JU} = (\nabla_{JU}h_{JU})^s.$$

But

$$k_{JU}(\nabla_{JU}h_{JU})^s k_{JU}$$

$$= Jk_U(\nabla_{JU}h_{JU})^s Jk_U$$

$$= -J^2k_U(\nabla_{JU}h_{JU})^s k_U$$

$$= k_U(\nabla_{JU}h_{JU})^s k_U$$

$$= G(\nabla_{JU}h_{JU})^s G.$$

So, for any unit vertical vector field U, we have:

$$(\nabla_U h_U)^s = G(\nabla_U h_U)^s G;$$

$$(\nabla_{JU}h_{JU})^s = G(\nabla_{JU}h_{JU})^s G.$$

Furthermore, we already know that $h_U{}^s, h_{JU}{}^s, k_Uh_U{}^d$ all anti-commute with G. So, we have already that T^s anti-commutes with G. Thus, $T^s = -GT^sG$ if and only if $T^s = 0$. This proves the theorem.

This first corollary is obvious from the results of the above theorem, since the critical condition of I is obviously true, if $h \equiv 0$.

Corollary 2.4.3 Let $g \in \mathcal{A}$. If g is projectable, then g is critical for the Riemannian functional I.

Corollary 2.4.4 Suppose $g \in \mathcal{A}$ is Kaehler and critical for I, then g is projectable.

Proof: Let U be a unit vertical vector field. Since g is Kaehler, we know that $h_U = h_U^d$ by Lemma 1.4.6. So, we know:

$$(\nabla_U h_U{}^d)^s + (\nabla_{JU} h_{JU}{}^d)^s = 4h_U{}^d k_U,$$

by Theorem 2.4.2. But, Proposition 1.7.3 tells us that:

$$(\nabla_U h_U{}^d)^s = \frac{1}{2}[\nabla_{JU} J, h_U{}^d] = 0,$$
$$(\nabla_{JU} h_{JU}{}^d)S = -\frac{1}{2}[\nabla_U J, h_{JU}{}^d] = 0.$$

So, we know that $4h_U{}^d k_U = 0$. Since k_U is non-singular on $\mathcal{H}$, we know that $h_U^d = 0$, and thus g is projectable.

2.5 $*$-Ricci Curvature of $\mathcal{V}$

We will now discuss another functional on $\mathcal{A}$ which may be viewed as a complex analogue of the Ricci curvature of ξ on real contact manifolds. This is the so-called $*$-Ricci curvature of $\mathcal{V}$.

In general, for any vector fields X, Y, we define the $*$-Ricci curvature of X, Y to be:

$$Ric^*(X,Y) = -tr[Z \mapsto JR(X,Z)JY].$$

Let U be any unit vertical vector field on M. Set $V = -JU$. Then:

$$\begin{aligned} Ric^*(U,U) &= -tr[Z \mapsto JR(U,Z)JU] \\ &= -U^k U^p J_p{}^m J_l{}^j R_{kjl}{}^m. \\ &= U^k V^m J_l{}^j R_{kjl}{}^m. \end{aligned}$$

Let $Z, X, Y \in TM$ be vector fields. Then:

$$-g(JR(X,Z)JY, Z) = g(R(X,Z)JY, JZ)$$

Let $\underline{E} = \{E_1, \ldots, E_{4n+2}\}$ be any orthonormal J-basis of TM. Then

$$\begin{aligned} Ric^*(X,Y) &= -tr[Z \mapsto JR(X,Z)JY] \\ &= -\sum_{j=1}^{4n+2} g(JR(X,E_j)JY, E_j) \\ &= \sum_{j=1}^{4n+2} g(R(X,E_j)JY, JE_j) \\ &= \sum_{j=1}^{4n+2} g(R(X,JE_j)JY, JJE_j) \\ &= -\sum_{j=1}^{4n+2} g(R(X,JE_j)JY, E_j) \\ &= -\sum_{j=1}^{4n+2} g(R(JY,E_j)X, JE_j) \\ &= \sum_{j=1}^{4n+2} g(R(JY,E_j)JJX, JE_j) \\ &= Ric^*(JY,JX). \end{aligned}$$

So, we have:

$$Ric^*(U,U) = Ric^*(JU,JU),$$

$$Ric^*(U,JU) = Ric^*(JJU,JU) = -Ric^*(U,JU) = 0.$$

Thus, if a, b are local real functions on the domain of U such that $a^2 + b^2 = 1$, we have:

$$Ric^*(aU + bJU, aU + bJU) = Ric^*(U,U).$$

So, the function $Ric^*(\mathcal{V})$ given locally by:

$$Ric^*(\mathcal{V}) = Ric^*(U,U),$$

for every unit vertical vector field U, is a globally defined function. We call $Ric^*(\mathcal{V})$ the **-Ricci curvature* of the vertical subbundle. We define the functional $I^* : \mathcal{A} \to R$ by:

$$I^*(g) = \int_M Ric^*(\mathcal{V}) dV_g.$$

Theorem 2.5.1 Let $g \in \mathcal{A}$. Then g is critical for I^* if and only if, for all unit vertical vector fields U,

$$h_U{}^d(\nabla_U J) = 0.$$

Before we give the proof of this theorem, we will prove the following necessary lemma.

Lemma 2.5.2 Let $g \in \mathcal{A}, D \in T_g\mathcal{A}$. For any unit vertical vector field $U, tr(Dh_U J) = 0$.

Proof: $tr(Dh_U J) = tr(DJh_U)$, since J commutes with D. So, $tr(Dh_U J) = tr(Dh_U{}^s J)$. But, $tr(Dh_U{}^s J) = -tr(G^2 Dh_U{}^s J) = tr(GDh_U{}^s JG) = tr(G^2 Dh_U{}^s J) = -tr(Dh_U{}^s J)$, since G anti-commutes with $D, h_U{}^s$, and J. Thus, $tr(Dh_U J) = 0$.

Proof (of Theorem 2.5.1): Let $g \in \mathcal{A}$. Let U be a unit vertical vector field with corresponding local complex almost contact structure $\{G, H, U, V, u, v, g, \mathcal{O}\}$. Suppose $t \mapsto g_t$

is a path in $\mathcal{A}$ with $g_0 = g$ and tangent vector $D \in T_g\mathcal{A}$, i.e. $D_{jk} = \frac{d}{dt}((g_t)_{jk})|_{t=0}$. Using the same notation as in the proof of Theorem 2.4.2, we have:

$$
\begin{aligned}
\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} &= \frac{d}{dt}(U^k V^m J_l{}^j R_{kjm}{}^l)|_{t=0} \\
&= (\nabla_k D_{jm}{}^l)U^k V^m J_l{}^j - (\nabla_j D_{km}{}^l)U^k V^m J_l{}^j \\
&= div - D_{jm}{}^l \nabla_k (U^k V^m J_l{}^j) + D_{km}{}^l \nabla_j (U^k V^m J_l{}^j) \\
&= div - \frac{1}{2}(\nabla_j D_m{}^l + \nabla_m D_j{}^l - \nabla^l D_{mj})\nabla_k (U^k V^m J_l{}^j) \\
&\quad + \frac{1}{2}(\nabla_k D_m{}^l + \nabla_m D_k{}^l - \nabla^l D_{mk})\nabla_j (U^k V^m J_l{}^j) \\
&= div - \frac{1}{2}(\nabla_j D_m{}^l)\nabla_k (U^k V^m J_l{}^j) - \frac{1}{2}(\nabla_m D_j{}^l)\nabla_k (U^k V^m J_l{}^j) \\
&\quad + \frac{1}{2}(\nabla^l D_{mj})\nabla_k (U^k V^m J_l{}^j) \\
&\quad + \frac{1}{2}(\nabla_k D_m{}^l)\nabla_j (U^k V^m J_l{}^j) + \frac{1}{2}(\nabla_m D_k{}^l)\nabla_j (U^k V^m J_l{}^j) \\
&\quad - \frac{1}{2}(\nabla^l D_{mk})\nabla_j (U^k V^m J_l{}^j).
\end{aligned}
$$

Now, since D is symmetric and J is skew-symmetric with respect to g, we know:

$$(\nabla_m D_j{}^l)\nabla_k (U^k V^m J_l{}^j) = 0.$$

Also,

$$
\begin{aligned}
(\nabla^l D_{mj})\nabla_k (U^k V^m J_l{}^j) &= (\nabla_l D_{mj})\nabla_k (U^k V^m J^{lj}) \\
&= -(\nabla_l D_{mj})\nabla_k (U^k V^m J^{jl}) \\
&= -(\nabla_j D_{ml})\nabla_k (U^k V^m J^{lj}) \\
&= -(\nabla_j D_m{}^l)\nabla_k (U^k V^m J_l{}^j).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} &= div - (\nabla_j D_m{}^l)\nabla_k (U^k V^m J_l{}^j) + \frac{1}{2}(\nabla_k D_m{}^l)\nabla_j (U^k V^m J_l{}^j) \\
&\quad + \frac{1}{2}(\nabla_m D_k{}^l)\nabla_j (U^k V^m J_l{}^j) - \frac{1}{2}(\nabla^l D_{mk})\nabla_j (U^k V^m J_l{}^j) \\
&= div + D_m{}^l \nabla_j \nabla_k (U^k V^m J_l{}^j) - \frac{1}{2} D_m{}^l \nabla_k \nabla_j (U^k V^m J_l{}^j) \\
&\quad - \frac{1}{2} D_k{}^l \nabla_m \nabla_j (U^k V^m J_l{}^j) + \frac{1}{2} D_{mk} \nabla^l \nabla_j (U^k V^m J_l{}^j).
\end{aligned}
$$

Set

$$Q_1 = D_m{}^l \nabla_j \nabla_k (U^k V^m J_l{}^j);$$

$$Q_2 = -\frac{1}{2} D_m{}^l \nabla_k \nabla_j (U^k V^m J_l{}^j);$$

$$Q_3 = -\frac{1}{2} D_k{}^l \nabla_m \nabla_j (U^k V^m J_l{}^j);$$

$$Q_4 = \frac{1}{2} D_{mk} \nabla^l \nabla_j (U^k V^m J_l{}^j).$$

So, $\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} = Q_1 + Q_2 + Q_3 + Q_4$. We now will analyze each of these terms.

$$\begin{aligned}
Q_1 &= D_m{}^l \nabla_j \nabla_k (U^k V^m J_l{}^j) \\
&= D_m{}^l \nabla_j ((\nabla_k U^k) V^m J_l{}^j + U^k (\nabla_k V^m) J_l{}^j + U^k V^m (\nabla_k J_l{}^j)) \\
&= D_m{}^l \nabla_j (\sigma(V) V^m J_l{}^j - U^k \sigma_k U^m J_l{}^j + V^m (\nabla_U J)_l{}^j) \\
&= D_m{}^l \nabla_j (\sigma(V) V^m J_l{}^j - \sigma(U) U^m J_l{}^j + V^m (\nabla_U J)_l{}^j) \\
&= D_m{}^l (\sigma(V)(\nabla_j V^m) J_l{}^j - \sigma(U)(\nabla_j U^m) J_l{}^j + (\nabla_j V^m)(\nabla_U J)_l{}^j) \\
&= D_m{}^l (\sigma(V)(-\sigma_i U^m - H_j{}^m + (h_V)_j{}^m) J_l{}^j \\
&\qquad - \sigma(U)(\sigma_j V^m - G_j{}^m + (h_U)_j{}^m) J_l{}^j \\
&\qquad\quad + (-\sigma_j U^m - H_j{}^m + (h_V)_j{}^m)(\nabla_U J)_l{}^j) \\
&= D_m{}^l (\sigma(V)(G_l{}^m + (h_V J)_l{}^m) + \sigma(U)(H_l{}^m + (h_U J)_l{}^m) \\
&\qquad - (H \nabla_U J)_l{}^m) + (h_V \nabla_U J)_l{}^m) \\
&= \sigma(V) D_m{}^l G_l{}^m + \sigma(V) D_m{}^l (h_V J)_l{}^m + \sigma(U) D_m{}^l H_l{}^m + \sigma(U) D_m{}^l (h_U J)_l{}^m \\
&\qquad - D_m{}^l (H \nabla_U J)_l{}^m + D_m{}^l (h_V \nabla_U J)_l{}^m \\
&= \sigma(V) tr(D h_V J) - \sigma(U) tr(D h_U J) - tr(DH(\nabla_U J)) + tr(D h_V (\nabla_U J)) \\
&= -tr(DH(\nabla_U J)) + tr(D h_V (\nabla_U J)).
\end{aligned}$$

In order to analyze Q_2, Q_3, and Q_4, we will need the following identity.

$$\begin{aligned}
\nabla_j (U^k V^m J_l{}^j) &= (\nabla_j U^k) V^m J_l{}^j + U^k (\nabla_j V^m) J_l{}^j + U^k V^m (\nabla_j J_l{}^j) \\
&= (\sigma_j V^k - G_j{}^k + (h_U)_j{}^k) V^m J_l{}^j \\
&\qquad + U^k (-\sigma_j U^m - H_j{}^m + (h_V)_j{}^m) J_l{}^j - U^k V^m (\delta J)_l
\end{aligned}$$

$$= \sigma_j V^k V^m J_l{}^j - G_j{}^k V^m J_l{}^j + (h_U)_j{}^k V^m J_l{}^j$$

$$- \sigma_j U^k U^m J_l{}^j - U^k H_j{}^m J_l{}^j + U^k (h_V)_j{}^m J_l{}^j - U^k V^m (\delta J)_l$$

$$= \sigma_j V^k V^m J_l{}^j - H_l{}^k V^m + (h_U J)_l{}^k V^m$$

$$- \sigma_j U^k U^m J_l{}^j + U^k G_l{}^m + U^k (h_V J)_l{}^m - U^k V^m (\delta J)_l$$

$$= (\sigma \circ J)_l V^k V^m - (\sigma \circ J)_l U^k U^m - H_l{}^k V^m + (h_U J)_l{}^k V^m$$

$$+ U^k G_l{}^m + U^k (h_V J)_l{}^m - U^k V^m (\delta J)_l.$$

$$Q_2 = -\frac{1}{2} D_m{}^l \nabla_k \nabla_j (U^k V^m J_l{}^j)$$

$$= -\frac{1}{2} D_m{}^l \nabla_k ((\sigma \circ J)_l V^k V^m - (\sigma \circ J)_l U^k U^m$$

$$- H_l{}^k V^m + (h_U J)_l{}^k V^m$$

$$+ U^k G_l{}^m + U^k (h_V J)_l{}^m - U^k V^m (\delta J)_l)$$

$$= -\frac{1}{2} D_m{}^l ((\sigma \circ J)_l V^k (\nabla_k V^m) - (\sigma \circ J)_l U^k (\nabla_k U^m)$$

$$+ (\nabla_k U^k) G_l{}^m + U^k (\nabla_k G_l{}^m)$$

$$+ (\nabla_k U^k)(h_V J)_l{}^m + U^k \nabla_k (h_V J)_l{}^m$$

$$- (\nabla_k V^m) H_l{}^k + (\nabla_k V^m)(h_U J)_l{}^k$$

$$- U^k (\nabla_k V^m)(\delta J)_l$$

$$= -\frac{1}{2} D_m{}^l ((\nabla_k U^k) G_l{}^m + U^k (\nabla_k G_l{}^m)$$

$$+ (\nabla_k U^k)(h_V J)_l{}^m + U^k \nabla_k (h_V J)_l{}^m$$

$$- (\nabla_k V^m) H_l{}^k + (\nabla_k V^m)(h_U J)_l{}^k)$$

$$= -\frac{1}{2} D_m{}^l (\sigma(V) G_l{}^m + \sigma(U) H_l{}^m$$

$$+ \sigma(V)(h_V J)_l{}^m + \nabla_U (h_V J)_l{}^m$$

$$- (-H_k{}^m + (h_V)_k{}^m) H_l{}^k + (-H_k{}^m + (h_V)_k{}^m)(h_U J)_l{}^k)$$

$$= -\frac{1}{2} D_m{}^l (\sigma(V)(h_V J)_l{}^m + \nabla_U (h_V J)_l{}^m$$

$$- (h_V)_k{}^m H_l{}^k - H_k{}^m (h_U J)_l{}^k + (h_V)_k{}^m (h_U J)_l{}^k)$$

Hence,

$$
\begin{aligned}
Q_2 &= -\frac{1}{2}\sigma(v)tr(Dh_V J) - \frac{1}{2}tr(D\nabla_U(h_V J)) + \frac{1}{2}tr(Dh_V H) - \frac{1}{2}tr(Dh_U G) - \frac{1}{2}tr(Dh_V h_U J) \\
&= -\frac{1}{2}tr(D\nabla_U(h_V J)) + \frac{1}{2}tr(Dh_V H) - \frac{1}{2}tr(Dh_U G) - \frac{1}{2}tr(Dh_V h_U J),
\end{aligned}
$$

by use of Lemma 2.5.1.

Also,

$$
\begin{aligned}
Q_3 &= -\frac{1}{2}D_k{}^l\nabla_m\nabla_j(U^k V^m J_l{}^j) \\
&= -\frac{1}{2}D_k{}^l\nabla_m((\sigma\circ J)_l V^k V^m - (\sigma\circ J)_l U^k U^m + U^k G_l{}^m + U^k(h_V J)_l{}^m \\
&\qquad - V^m H_l{}^k + V^m(h_U J)_l{}^k - U^k V^m(\delta J)_l) \\
&= -\frac{1}{2}D_k{}^l((\sigma\circ J)_l(\nabla_m V^k)V^m - (\sigma\circ J)_l(\nabla_m U^k)U^m \\
&\qquad + (\nabla_m U^k)G_l{}^m + (\nabla_m U^k)(h_V J)_l{}^m \\
&\qquad - (\nabla_m V^m)H_l{}^k - V^m(\nabla_m H_l{}^k) \\
&\qquad + (\nabla_m V^m)(h_U J)_l{}^k + V^m(\nabla_m(h_U J)_l{}^k) - U^k V^m(\delta J)_l) \\
&= -\frac{1}{2}D_k{}^l((\nabla_m U^k)G_l{}^m + (\nabla_m U^k)(h_V J)_l{}^m \\
&\qquad - (\nabla_m V^m)H_l{}^k - V^m(\nabla_m H_l{}^k) \\
&\qquad + (\nabla_m V^m)(h_U J)_l{}^k + V^m(\nabla_m(h_U J)_l{}^k) \\
&= -\frac{1}{2}D_k{}^l((\sigma_m V^k - G_m{}^k + (h_U)_m{}^k)G_l{}^m \\
&\qquad + (\sigma_m V^k - G_m{}^k + (h_U)_m{}^k)(h_V J)_l{}^m \\
&\qquad + \sigma(U)H_l{}^k - (\nabla_V H_l{}^k) \\
&\qquad - \sigma(U)(h_U J)_l{}^k + \nabla_V(h_U J)_l{}^k) \\
&= -\frac{1}{2}D_k{}^l((-G_m{}^k + (h_U)_m{}^k)G_l{}^m \\
&\qquad + (-G_m{}^k + (h_U)_m{}^k)(h_V J)_l{}^m \\
&\qquad + \sigma(U)H_l{}^k - (\nabla_V H_l{}^k) \\
&\qquad - \sigma(U)(h_U J)_l{}^k + \nabla_V(h_U J)_l{}^k)
\end{aligned}
$$

$$= -\frac{1}{2}D_k{}^l(p_l{}^k + (h_U G)_l{}^k - (G h_V J)_l{}^k + (h_U h_V J)_l{}^k$$

$$+ \sigma(U)H_l{}^k + \sigma(U)G_l{}^k - \sigma(U)(h_U J)_l{}^k + \nabla_V(h_U J)_l{}^k)$$

$$= -\frac{1}{2}tr(D) - \frac{1}{2}tr(D h_U G) + \frac{1}{2}tr(D G h_V J) - \frac{1}{2}tr(D h_U h_V J)$$

$$- \frac{1}{2}\sigma(U)tr(DH) - \frac{1}{2}\sigma(U)tr(DG) + \sigma(U)\frac{1}{2}tr(D h_U J) - \frac{1}{2}tr(D\nabla_V(h_U J))$$

$$= -\frac{1}{2}tr(D h_U G) + \frac{1}{2}tr(D G h_V J) - \frac{1}{2}tr(D h_U h_V J)$$

$$- \frac{1}{2}tr(D\nabla_V(h_U J))$$

Thus,

$$Q_3 = -\frac{1}{2}tr(D h_U G) - \frac{1}{2}tr(D G J h_V) - \frac{1}{2}tr(D h_U h_V J)$$

$$- \frac{1}{2}tr(D\nabla_V(h_U J))$$

$$= -\frac{1}{2}tr(D h_U G) - \frac{1}{2}tr(D H h_V) - \frac{1}{2}tr(D h_U h_V J)$$

$$- \frac{1}{2}tr(D\nabla_V(h_U J))$$

$$= -\frac{1}{2}tr(D h_U G) + \frac{1}{2}tr(D h_V H) - \frac{1}{2}tr(D h_U h_V J)$$

$$- \frac{1}{2}tr(D\nabla_V(h_U J))$$

Finally,

$$Q_4 = \frac{1}{2}D_{mk}\nabla^l\nabla_j(U^k V^m J_l{}^j)$$

$$= \frac{1}{2}D_{mk}\nabla^l((\sigma\circ J)_l V^k V^m - (\sigma\circ J)_l U^k U^m$$

$$+ U^k G_l{}^m + U^k(h_V J)_l{}^m$$

$$- V^m H_l{}^k + V^m(h_U J)_l{}^k - U^k V^m(\delta J)_l)$$

$$= \frac{1}{2}D_{mk}\nabla^l(U^k G_l{}^m + U^k(h_V J)_l{}^m$$

$$- V^m H_l{}^k + V^m(h_U J)_l{}^k)$$

$$= \frac{1}{2}D_{mk}((\nabla^l U^k)G_l{}^m + (\nabla^l U^k)(h_V J)_l{}^m$$

$$- (\nabla^l V^m)H_l{}^k + (\nabla^l V^m)(h_U J)_l{}^k)$$

$$= \frac{1}{2} D_{mk}((\sigma^l V^k - G^{lk} + (h_U)^{lk}) G_l{}^m$$
$$+ (\sigma^l V^k - G^{lk} + (h_U)^{lk})(h_V J)_l{}^m$$
$$- (-\sigma^l U^m - H^{lm} + (h_V)^{lm}) H_l{}^k$$
$$+ (-\sigma^l U^m - H^{lm} + (h_V)^{lm})(h_U J)_l{}^k)$$
$$= \frac{1}{2} D_{mk}(-G^{lk} G_l{}^m + (h_U)^{lk} G_l{}^m$$
$$- G^{lk}(h_V J)_l{}^m + (h_U)^{lk}(h_V J)_l{}^m$$
$$+ H^{lm} H_l{}^k - (h_V)^{lm} H_l{}^k$$
$$- H^{lm}(h_U J)_l{}^k + (h_V)^{lm}(h_U J)_l{}^k)$$
$$= \frac{1}{2} D_{mk}(G^{kl} G_l{}^m + (h_U)^{kl} G_l{}^m$$
$$+ G^{kl}(h_V J)_l{}^m + (h_U)^{kl}(h_V J)_l{}^m)$$
$$+ \frac{1}{2} D_{km}(-H^{ml} H_l{}^k - (h_V)^{ml} H_l{}^k$$
$$+ H^{ml}(h_U J)_l{}^k + (h_V)^{ml}(h_U J)_l{}^k)$$
$$= \frac{1}{2} tr(GGD) + \frac{1}{2} tr(G h_U D) + \frac{1}{2} tr(h_V J G D) + \frac{1}{2} tr(h_V J h_U D)$$
$$- \frac{1}{2} tr(HHD) - \frac{1}{2} tr(H h_V D) + \frac{1}{2} tr(h_U J H D) + \frac{1}{2} tr(h_U J h_V D).$$

So,

$$Q_4 = \frac{1}{2} tr(G h_U D) - \frac{1}{2} tr(h_V H D) + \frac{1}{2} tr(h_V J h_U D)$$
$$- \frac{1}{2} tr(H h_V D) + \frac{1}{2} tr(h_U G D) + \frac{1}{2} tr(h_U J h_V D)$$
$$= \frac{1}{2} tr(h_V J h_U D) + \frac{1}{2} tr(h_U J h_V D)$$
$$= \frac{1}{2} tr(D h_V J h_U) + \frac{1}{2} tr(D h_U J h_V)$$

Thus,

$$\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} = div + Q_1 + Q_2 + Q_3 + Q_4$$
$$= div - tr(DH(\nabla_U J)) + tr(D h_V(\nabla_U J)) - \frac{1}{2} tr(D\nabla_U(h_V J)) + \frac{1}{2} tr(D h_V H)$$
$$- \frac{1}{2} tr(D h_U G) - \frac{1}{2} tr(D h_V h_U J) - \frac{1}{2} tr(D h_U G) + \frac{1}{2} tr(D h_V H)$$
$$- \frac{1}{2} tr(D h_U h_V J) - \frac{1}{2} tr(D\nabla_V(h_U J)) + \frac{1}{2} tr(D h_V J h_U) + \frac{1}{2} tr(D h_U J h_V)$$

So,

$$\begin{aligned}\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} = div &- tr(DH(\nabla_U J)) + tr(Dh_V(\nabla_U J)) \\ &- \frac{1}{2}tr(D(\nabla_U h_V)J)) - \frac{1}{2}tr(Dh_V(\nabla_U J)) \\ &+ \frac{1}{2}tr(Dh_V H) - \frac{1}{2}tr(Dh_U G) \\ &- \frac{1}{2}tr(D(\nabla_V h_U)J) - \frac{1}{2}tr(Dh_U(\nabla_V J)) \\ &+ \frac{1}{2}tr(Dh_V H) - \frac{1}{2}tr(Dh_U G) \\ &- \frac{1}{2}tr(Dh_V h_U J) - \frac{1}{2}tr(Dh_U h_V J) \\ &+ \frac{1}{2}tr(Dh_V J h_U) + \frac{1}{2}tr(Dh_U J h_V) \\ = div &- tr(DH(\nabla_U J)) + \frac{1}{2}tr(Dh_V(\nabla_U J)) - \frac{1}{2}tr(Dh_U(\nabla_V J)) \\ &- \frac{1}{2}tr(D(\nabla_U h_V)J) - \frac{1}{2}tr(D(\nabla_V h_U)J) - tr(Dh_U G) + tr(Dh_V H) \\ &- \frac{1}{2}tr(Dh_V h_U J) - \frac{1}{2}tr(Dh_U h_V J) + \frac{1}{2}tr(Dh_V J h_U) + \frac{1}{2}tr(Dh_U J h_V)\end{aligned}$$

Now,

$$\begin{aligned}tr(Dh_V h_U J) &= tr(D(h_V h_U)^s J) \\ &= tr(D{h_V}^s {h_U}^s J) + tr(D{h_V}^d {h_U}^d J) \\ &= tr(D{h_V}^s {h_U}^s J) + tr(DJ{h_V}^d {h_U}^d) \\ &= tr(D{h_V}^s {h_U}^s J) + tr(D({h_U}^d)^2).\end{aligned}$$

But,

$$\begin{aligned}tr(D({h_U}^d)^2) &= -tr(G^2 D({h_U}^d)^2) \\ &= tr(GD({h_U}^d)^2 G) \\ &\quad \text{(since } DG = -GD\text{)} \\ &= tr(G^2 D({h_U}^d)^2) \\ &= -tr(D({h_U}^d)^2).\end{aligned}$$

So, $tr(D({h_U}^d)^2) = 0$. Thus, $tr(Dh_V h_U J) = tr(D{h_V}^s {h_U}^s J)$. Similarly, we find:

$$tr(Dh_U h_V J) = tr(D{h_U}^s {h_V}^s J)$$

$$tr(Dh_V J h_U) = tr(D{h_V}^s J {h_U}^s) = tr(D{h_V}^s {h_U}^s J)$$

$$tr(Dh_U J h_V) = tr(D{h_U}^s J {h_V}^s) = tr(D{h_U}^s {h_V}^s J)$$

Using these facts, we get:

$$\frac{d}{dt}(Ric^*(\mathcal{V}))|_{t=0} = div - tr(DH(\nabla_U J)) + \frac{1}{2}tr(Dh_V(\nabla_U J)) - \frac{1}{2}tr(Dh_U(\nabla_V J))$$
$$- \frac{1}{2}tr(D(\nabla_U h_V)J) - \frac{1}{2}tr(D(\nabla_V h_U)J)$$
$$- tr(Dh_U G) + tr(Dh_V H)$$
$$= div - tr[D(H - \frac{1}{2}h_V - \frac{1}{2}h_U J)(\nabla_U J)] - \frac{1}{2}tr[D((\nabla_U h_V)J + (\nabla_V h_U)J)]$$
$$+ tr[D(-h_U G + h_V H)].$$

Set

$$T = p(H + \frac{1}{2}h_{JU} - \frac{1}{2}h_U J)(\nabla_U J)p - \frac{1}{2}p(\nabla_U h_{JU})Jp - \frac{1}{2}p(\nabla_{JU} h_U)Jp + h_U G + h_{JU} H.$$

Now, since ${h_{JU}}^d = J{h_U}^d$, we know that ${h_U}^d G = -{h_{JU}}^d H$. So, $(h_U G)^s = {h_U}^d G = -{h_{JU}}^d H = -(h_{JU} H)^s$. Thus,

$$T^s = p(H + \frac{1}{2}{h_{JU}}^d - \frac{1}{2}{h_U}^d J)(\nabla_U J)p$$
$$- \frac{1}{2}p(\nabla_U h_{JU})^s Jp - \frac{1}{2}p(\nabla_{JU} h_U)^s Jp$$
$$= p(H + J{h_U}^d)(\nabla_U J)p - \frac{1}{2}p(\nabla_U h_{JU})^s Jp - \frac{1}{2}p(\nabla_{JU} h_U)^s Jp$$
$$= H(\nabla_U J)p + J{h_U}^d(\nabla_U J)p - \frac{1}{2}p(\nabla_U h_{JU})^s Jp - \frac{1}{2}p(\nabla_{JU} h_U)^s Jp.$$

Since $G(\nabla_U J) = -(\nabla_U J)G$ and $J(\nabla_U J) = -(\nabla_U J)J$, we know that $H(\nabla_U J) = (\nabla_U J)H$. Thus, $H(\nabla_U J)$ is a g-symmetric operator. However, since $\nabla_U h_{JU}$ and $\nabla_{JU} h_U$ are symmetric and J is skew-symmetric with respect to g, we know that the operators $(\nabla_{JU} h_U)^s J$ and $(\nabla_U h_{JU})^s J$ are skew-symmetric with respect to g. Thus,

$$sym(T^s) = H(\nabla_U J)p + sym(Jh_U^d(\nabla_U J)p)$$
$$= H(\nabla_U J) + sym({h_{JU}}^d(\nabla_U J)p)$$
$$= H(\nabla_U J)p + \frac{1}{2}p[{h_{JU}}^d, \nabla_U J]p.$$

Now,

$$-GH(\nabla_U J)G = -H(\nabla_U J)G^2 = H(\nabla_U J).$$

And

$$-G[{h_{JU}}^d, \nabla_U J]G = [{h_{JU}}^d, \nabla_U J]G^2 = -[{h_{JU}}^d, \nabla_U J].$$

Thus,

$$-G(sym(T^{s}))G = -G(H(\nabla_U J) + \frac{1}{2}[h_{JU}{}^d, \nabla_U J])G$$
$$= H(\nabla_U J) - \frac{1}{2}[h_{JU}{}^d, \nabla_U J].$$

So, $sym(T^{s}) = -G(sym(T^{s}))G$ if and only if $[h_{JU}{}^d, \nabla_U J] = 0$. Since $h_{JU}{}^d = Jh_U{}^d$, we see that $[h_{JU}{}^d, \nabla_U J] = 0$ if and only if $h_U{}^d(\nabla_U J) = -(\nabla_U J)h_U{}^d$. However, by Propostion 1.7.4, we know that $h_U{}^d$ and $\nabla_U J$ commute. Therefore, g is critical for I^{*} if and only if $h_U{}^d(\nabla_U J) = 0$. This proves the theorem.

Corollary 2.5.2 Let $g \in \mathcal{A}$. If g is projectable, then g is critical for I^{*}.

Corollary 2.5.3 Let $g \in \mathcal{A}$. If g is Kaehler, then g is critical for I^{*}.

Corollary 2.5.4 Let M be a compact complex contact manifold with $dim_C M = 3$. Then $g \in \mathcal{A}$ is critical for I^{*} if and only if, at each point $m \in M$, either $h_U^d = 0$ for every $U \in \mathcal{V}_m$ or $\nabla_U J = 0$ for every $U \in \mathcal{V}_m$, i.e. locally either $h_U^d = 0$ for any vertical vector U or $\nabla_U J$ for any vertical vector U.

Proof: Let $g \in \mathcal{A}$. Let U be a unit vertical vector. Let $\{G, H, U, V, u, v, g, \mathcal{O}\}$ be the local complex almost contact structure corresponding to U. Suppose $X \in \mathcal{H}$ is a unit eigenvector of h_U with eigenvector λ. Then $h_U GX = -Gh_U X = -\lambda GX$. So, GX is also an eigenvector of h_U. Furthermore, since X is an eigenvector of h_U and h_U is symmetric, we know $h_U(JX), h_U(HX) \in span\{JX, HX\}$. Also, $g(h_U HX, HX) = g(h_U GJX, GJX) = -g(Gh_U JX, GJX) = -g(h_U JX, JX)$.

Let $\underline{X} = \{X, GX, JX, HX\}$ so that $\underline{X}$ is an orthonormal frame of $\mathcal{H}$. Then, with respect to $\underline{X}$,

$$h_U = \left(\begin{array}{cc|cc} \lambda & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -\lambda & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \mu & \nu \\ & & \nu & -\mu \end{array}\right).$$

With respect to $\underline{X}$, we also have:

$$J = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \quad G = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

So,

$$Jh_UJ = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{array}\right)\left(\begin{array}{cc|cc} \lambda & 0 & \multicolumn{2}{c}{0} \\ 0 & -\lambda & & \\ \hline \multicolumn{2}{c|}{0} & \mu & \nu \\ & & \nu & -\mu \end{array}\right)\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} -\mu & \nu & \multicolumn{2}{c}{0} \\ \nu & \mu & & \\ \hline \multicolumn{2}{c|}{0} & -\lambda & 0 \\ & & 0 & \lambda \end{array}\right).$$

Thus,

$$h_U{}^d = \frac{1}{2}(h_U + Jh_UJ)$$

$$= \frac{1}{2}\left(\begin{array}{cc|cc} \lambda-\mu & \nu & \multicolumn{2}{c}{0} \\ \nu & -\lambda+\mu & & \\ \hline \multicolumn{2}{c|}{0} & -\lambda+\mu & \nu \\ & & \nu & \lambda-\mu \end{array}\right).$$

Furthermore, using Proposition 1.4.6,

$$h_{JU}{}^d = Jh_U{}^d$$

$$= \frac{1}{2}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \lambda-\mu & -\nu \\ & & \nu & \lambda-\mu \\ \hline \lambda-\mu & \nu & \multicolumn{2}{c}{0} \\ -\nu & \lambda-\mu & & \end{array}\right).$$

Now, suppose

$$E = \begin{pmatrix} A & {}^tB \\ B & D \end{pmatrix}$$

is a 4×4 matrix such that A, B, and D are 2×2 matrices with

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$$

$$D = \begin{pmatrix} d_1 & d_2 \\ d_2 & d_3 \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Let

$$J = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{array}\right).$$

Then

$$JEJ = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{array}\right)\begin{pmatrix} A & {}^tB \\ B & D \end{pmatrix}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} -d_1 & d_2 & b_1 & -b_2 \\ d_2 & -d_3 & -b_3 & b_4 \\ \hline b_1 & -b_3 & -a_1 & a_2 \\ -b_2 & b_4 & a_2 & -a_3 \end{array}\right)$$

So,

$$E^d = \frac{1}{2}(E + JEJ) = \frac{1}{2}\left(\begin{array}{cc|cc} a_1 - d_1 & a_2 + d_2 & 2b_1 & 0 \\ a_2 + d_2 & a_3 - d_3 & 0 & 2b_4 \\ \hline 2b_1 & 0 & -a_1 + d_1 & a_2 + d_2 \\ 0 & 2b_4 & a_2 + d_2 & -a_3 + d_3 \end{array}\right).$$

In particular, suppose E is the matrix representation of h_{JU} with respect to $\underline{X}$. Comparing the above matrix with our previous representation of $h_{JU}{}^d$, we see that $\nu = 0$. Therefore, JX and HX are also eigenvectors of h_U with respective eigenvalues μ and $-\mu$. Furthermore, this allows us to get a better description of h_U and $h_U{}^d$:

$$h_U = \left(\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ \hline 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \end{array}\right),$$

$$h_U{}^d = \frac{1}{2}\left(\begin{array}{cc|cc} \lambda - \mu & 0 & 0 & 0 \\ 0 & -(\lambda - \mu) & 0 & 0 \\ \hline 0 & 0 & -(\lambda - \mu) & 0 \\ 0 & 0 & 0 & \lambda - \mu \end{array}\right).$$

Now, with respect to $\underline{X}$, $\nabla_U J$ can be written as a skew-symmetric matrix:

$$\nabla_U J = \begin{pmatrix} 0 & a & c & d \\ -a & 0 & e & f \\ -c & -e & 0 & b \\ -d & -f & -b & 0 \end{pmatrix}.$$

Furthermore, $(\nabla_U J)J = -J(\nabla_U J)$; and

$$(\nabla_U J)J = \begin{pmatrix} 0 & a & c & d \\ -a & 0 & e & f \\ -c & -e & 0 & b \\ -d & -f & -b & 0 \end{pmatrix} \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$= \begin{pmatrix} c & -d & 0 & a \\ e & -f & a & 0 \\ 0 & -b & c & -e \\ -b & 0 & d & -f \end{pmatrix}$$

$$-J(\nabla_U J) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 & a & c & d \\ -a & 0 & e & f \\ -c & -e & 0 & b \\ -d & -f & -b & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -c & -e & 0 & b \\ d & f & b & 0 \\ 0 & -a & -c & -d \\ -a & 0 & e & f \end{pmatrix}.$$

So, we have that $a = b, c = f = 0, d = e$; or

$$\nabla_U J = \begin{pmatrix} 0 & a & 0 & d \\ -a & 0 & d & 0 \\ 0 & -d & 0 & a \\ -d & 0 & -a & 0 \end{pmatrix}.$$

Furthermore, we also know that $G(\nabla_U J) = -(\nabla_U J)G$.

$$G(\nabla_U J) = \left(\begin{array}{cc|cc} 0 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & 1 & 0 \end{array}\right) \begin{pmatrix} 0 & a & 0 & d \\ -a & 0 & d & 0 \\ 0 & -d & 0 & a \\ -d & 0 & -a & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a & 0 & -d & 0 \\ 0 & a & 0 & d \\ d & 0 & a & 0 \\ 0 & -d & 0 & a \end{pmatrix}.$$

$$-(\nabla_U J)G = \begin{pmatrix} 0 & a & 0 & d \\ -a & 0 & d & 0 \\ 0 & -d & 0 & a \\ -d & 0 & -a & 0 \end{pmatrix} \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & -1 & 0 \end{array}\right)$$
$$= \begin{pmatrix} -a & 0 & -d & 0 \\ 0 & -a & 0 & d \\ d & 0 & -a & 0 \\ 0 & -d & 0 & -a \end{pmatrix}.$$

So, we know that $a = 0$ or that

$$\nabla_U J = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & d \\ & & d & 0 \\ \hline 0 & -d & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -d & 0 & & \end{array}\right).$$

Now,

$$h_U{}^d(\nabla_U J) = \frac{1}{2}\left(\begin{array}{cc|cc} \lambda-\mu & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -(\lambda-\mu) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -(\lambda-\mu) & 0 \\ & & 0 & \lambda-\mu \end{array}\right)\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & d \\ & & d & 0 \\ \hline 0 & -d & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -d & 0 & & \end{array}\right)$$
$$= d(\lambda-\mu)\left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -1 & 0 \\ & & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \end{array}\right)$$
$$= d(\lambda-\mu)\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & -1 & 0 \\ \hline 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \end{array}\right)$$
$$= (\nabla_U J)h_U{}^d.$$

Thus, at each point of M, $h_U{}^d(\nabla_U J) = 0$ for every vertical vector U if and only if either $d = 0$ or $\lambda = \mu$. Since g is critical for I^* if and only if $h_U{}^d(\nabla_U J) = 0$ for every vertical vector U, this proves the corollary.

Another interesting corollary arises from Theorem 2.5.1, when we consider complex contact manifolds which have a global complex contact structure. Before we continue with the corollary, we need the following results concerning global contact structures.

Proposition 2.5.5 Suppose ω is a global complex contact form on M. Then there a normalized contact structure on M such that $\sigma = 0$.

Proof: Set $\pi = \omega$ with $\omega = u - iv$. Locally, by the complex version of Darboux's theorem (c.f. [LeB]), there is a local complex coordinate system $(z_0, \ldots, z_{2n})$ such that

$$\omega = dz_0 + z_1 dz_2 + \ldots + z_{2n-1} dz_{2n}$$

with

$$\begin{aligned} u &= Re\ \omega \\ &= dx_0 + x_1 dx_2 + \ldots + x_{2n-1} dx_{2n} + y_1 dy_2 + \ldots + y_{2n-1} dy_{2n}; \\ v &= -Im\ \omega \\ &= -dy_0 - x_1 dy_2 - \ldots - x_{2n-1} dy_{2n} - y_1 dx_2 - \ldots - y_{2n-1} dy_{2n}. \end{aligned}$$

Thus,

$$du = dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n} + dy_1 \wedge dy_2 + \ldots + dy_{2n-1} \wedge dy_{2n};$$

$$dv = -dx_1 \wedge dy_2 - \ldots - dx_{2n-1} \wedge dx_{2n} - dy_1 \wedge dx_2 - \ldots - dy_{2n-1} \wedge dx_{2n}.$$

In particular, we know $U = \frac{\partial}{\partial x_0}; V = -\frac{\partial}{\partial y_0}$. Furthermore, $du(-\frac{\partial}{\partial y_0}, X) = 0$ for all $X \in TM$. Since $\hat{G} = du - \sigma \wedge v$, we know $\sigma(X) = 0$ for all $X \in TM$. This proves the proposition.

Corollary 2.5.6 Suppose M is a compact complex contact manifold with a global complex contact structure. Then I^* is constant on $\mathcal{A}$.

Proof: Let $g \in \mathcal{A}$. Let be a unit vertical vector field U with corresponding complex almost contact structure $\{G, H, U, V, u, v, g\}$. By the above proposition,we can choose such a structure with $\sigma = 0$.

Using Proposition 1.6.3, we get:

$$0 = 2J' + (\nabla_U J)G - 2Hh_U{}^d.$$

By composing the right side by G, we get

$$0 = -2H - \nabla_U J + 2Jh_U{}^d.$$

However, both H and $(\nabla_U J)$ are skew-symmetric operators with respect to g; whereas $Jh_U{}^d = h_{JU}{}^d$ is symmetric with respect to g. Therefore, we have

$$h_U{}^d = 0;$$

$$\nabla_U J = -2H.$$

In particular, since $h_U{}^d = 0$, g satisfies the critical condition of Theorem 2.5.1. Now, g is any associated metric on M. Therefore, we know that I^* is constant on $\mathcal{A}$. This completes the proof of the corollary.

Chapter Three

TWISTOR SPACES OVER QUATERNIONIC-KÄHLER MANIFOLDS

In this chapter, we apply a few of the results of the previous two chapters to a particular class of complex contact manifolds: the twistor spaces of a quaternionic-Kähler manifold with positive scalar curvature and dimension $4n \geq 8$. In section 1, we discuss the complex contact structure of twistor spaces and establish some basic facts. In section 2, we verify that the Salamon-Bérard-Bergery metric is associated to the complex contact structure. In section 3, we apply the work for I^* to the space of all associated metrics to a twistor space.

3.1 Quaternionic-Kähler Manifolds and Twistor Spaces

Recall that a Riemannian manifold $(\tilde{M}, \tilde{g})$ with dim $\tilde{M} \geq 8$ is called *quaternionic-Kähler*, if the holomony group of $\tilde{M}$ is contained in $Sp(n) \cdot Sp(1)$. This means that there exists a 3-dimensional subbbundle $E \subset End(T\tilde{M})$ such that locally there exists a basis of $E, \{A, B, C\}$, with:

1) $A^2 = B^2 = C^2 = -id.$

2) $AB = -BA = C.$

3) $\tilde{\nabla}_X A, \tilde{\nabla}_X B, \tilde{\nabla}_X C \in span\{A, B, C\}$ for any vector field X on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{g}$. We call each of these local frames a *local quaternionic-Kähler frame* of $\tilde{M}$.

From this point on, we will assume that $(\tilde{M}, \tilde{g})$ is a quaternionic-Kähler manifold with an open atlas $\tilde{\mathcal{U}} = \{\tilde{O}\}$ such that, on each $\tilde{O} \in \mathcal{U}$, there exists a quaternionic-Kähler frame $\{A, B, C\}$. One of the most important properties of quaternionic-Kähler manifolds is given by the following Theorem due to Alekseevskii (See [Be]):

Theorem 3.1.1 $\tilde{g}$ is an Einstein metric on $\tilde{M}$.

Now, $\tilde{g}$ induces a bundle metric g' on E by making each local quaternionic-Kähler frame of $\tilde{M}$ an orthnormal frame of E. Let $\tilde{\rho} : E \to \tilde{M}$ be the natural projection. Set $\tilde{\mathcal{V}} = ker(\tilde{\rho})_*$. So, E is an R^3-bundle over $\tilde{M}$ with a bundle metric, which reduces to the Euclidean metric on R^3 when restricted to the vertical fibres. In other words, if, over $\tilde{m} \in \tilde{\mathcal{O}} \in \mathcal{U}$, $S_1 = a_1A + b_1B + c_1C, S_2 = a_2A + b_2B + c_2C \in (\tilde{\rho})^{-1}(\tilde{m})$, then we have:

$$g'(S_1, S_2) = a_1a_2 + b_1b_2 + c_1c_2.$$

Let M be the S^2-bundle in E with respect to g', i.e. locally

$$M = \{xA + yB + zC \in E : x^2 + y^2 + z^2 = 1\}.$$

We call M the twistor space of $\tilde{M}$. Let $\rho : M \to \tilde{M}$. be the natural projection. Set $\mathcal{V}' = ker\ \rho_*$. Since we can view $M \subset E$ fibre-wise as simply $S^2 \subset R^3$, for $F = xA+yB+zC \in \rho^{-1}(\tilde{\mathcal{O}})$ with $\tilde{\mathcal{O}} \in \mathcal{U}$ and $x^2 + y^2 + z^2 = 1$, we can make the idenfication:

$$\mathcal{V}'_F = \{X \in E : X \perp F\} = \{aA + bB + cC \in E : xa + yb + zc = 0\}.$$

Let $\hat{g}$ denote g' restricted on $\mathcal{V}'$, using this identification. Now, since g' is simply the Euclidean metric on R^3, we see that $\hat{g}$ is the standard metric on S^2. Furthermore, the orientation of the vertical fibres of M is preserved by $\hat{g}$; so we see that we have a well-defined complex structure $\hat{J}$ on $\mathcal{V}'$ given by the natural complex structure of S^2.

Now, $\tilde{\nabla}$ on E induces a splitting $TE = \tilde{\mathcal{V}} \oplus \mathcal{H}'$. This splitting is given, on M, by $TM = \mathcal{V}' \oplus \mathcal{H}$, where $\mathcal{H} = \mathcal{H}'|_M$. Let $\alpha : TM = \mathcal{V}' \oplus \mathcal{H} \to \mathcal{V}'$ be the induced projection.

We give M an almost complex structure J as follows: Let $S \in M$ with $\rho(S) = \tilde{m}$.

1) Suppose $W \in \mathcal{V}'_S$. Then we let $JW = \hat{J}W$.

2) Suppose $X \in \mathcal{H}_S$. Let $(hor)_S : T_{\tilde{m}}\tilde{M} \to \mathcal{H}_S$ be the horizontal lift function at $\tilde{m}$ to $S \in \rho^{-1}(\tilde{m})$, i.e. for $Y \in T_{\tilde{m}}\tilde{M}$, $(hor)_SY$ is the unique vector in $\mathcal{H}_S$ such that

$\rho_*((hor)_S)Y) = Y$. Recall that S is, in fact, an almost complex structure on $T_{\tilde{m}}\tilde{M}$. Then we define:

$$JX = ((hor)_S \circ S \circ \rho_*)X.$$

Finally, we define a metric on M by:

$$g(X,Y) = \hat{g}(\alpha X, \alpha Y) + \tilde{g}(\rho_* X, \rho_* Y),$$

for each $X, Y \in TM$. Note that, by its definition, g is projectable, i.e. $h_U = 0$ for every vertical vector U. This metric is called the ***Salamon-Bérard-Bergery metric*** on M corresponding to $\tilde{M}$. Using all of this notation and contruction, we have the following important results due to Salamon and Bérard-Bergery [Sa],[B-B]:

Theorem 3.1.2

1) J is an integrable complex structure on M; the vertical fibers of the projection $\rho : M \to \tilde{M}$ are compact complex curves of genus 0.

2) If M has nonzero Ricci curvature, then $\alpha : TM \to \mathcal{V}'$ is a contact form on M, i.e.

$$\alpha \wedge (d^{\hat{\nabla}}\alpha)^n \neq 0 \text{ on } M,$$

where $\hat{\nabla}$ is the bundle connection on $\mathcal{V}'$ with respect to $\hat{g}$.

3) If $\tilde{g}$ has positive Ricci curvature on $\tilde{M}$, then the Salamon-Bérard-Bergery metric g is Kähler-Einstein with positive Ricci curvature such that the vertical fibres of ρ are totally geodesic.

From this point on, we will assume that $\tilde{M}$ is compact with positive Ricci curvature.

Let $\tau : \mathcal{V}' \to M$ be the natural bundle projection. Recall that $\hat{g}$ is the Salamon-Bérard-Bergery metric restricted to $\mathcal{V}'$; and $\hat{\nabla}$ is the bundle connection on $\mathcal{V}'$ induced from the bundle metric $\hat{g}$. So, $\hat{\nabla} = \alpha\nabla$. Again, $\hat{J}$ is the restriction of J on $\mathcal{V}'$. Thus, $\alpha \circ J = \hat{J} \circ \alpha$.

Let $\mathcal{U} = \{\mathcal{O}\}$ be an open atlas of M with local trivializations of $\mathcal{V}'$ $\Phi : \tau^{-1}(\mathcal{O}) \to \mathcal{O} \times C$ with $\Phi(v) = (\tau(v), \phi(v))$ for all $v \in \tau^{-1}(\mathcal{O})$ such that:

1) $\hat{g}(v_1, v_2) = <\phi(v_1), \phi(v_2)>$ on $\tau^{-1}(\mathcal{O})$, where $<,>$ is the Euclidean inner product on R^2.

2) $\phi(\hat{J}v) = i\phi(v)$ $\forall v \in \tau^{-1}(\mathcal{O})$.

Set $\pi = \phi \circ \alpha$. Let u and v be the real 1-forms on $\mathcal{O}$ such that $\pi = u - iv$. Now, $\pi(JX) = \phi(\alpha(JX)) = \phi(\hat{J}(\alpha(X))) = i\phi\alpha X = i\pi(X)$ for any vector X. So, if we were to extend π to be a complex 1-form on $T^C\mathcal{O}$, then we would have $\pi(X + iJX) = 0$ for all $X \in T^C\mathcal{O}$. Thus, $v = u \circ J$.

Since for $\mathcal{O}, \mathcal{O}' \in \mathcal{U}$ with respective trivializations $\Phi = (\tau, \phi), \Phi' = (\tau, \phi')$ we know that $\phi = h\phi'$ for some function $h : \mathcal{O} \cap \mathcal{O}' \to S^1$, we have $\pi = h\pi'$ for the same h. Thus, $\underline{\pi} = \{\pi\}$ is a normalized contact structure on M corresponding to α. Using $\underline{\pi}$, we construct the contact line subbundle $\mathcal{V}$. Thus, ostensibly we have two vertical subbundles: $\mathcal{V}'$ and $\mathcal{V}$. We now will show that, in fact, $\mathcal{V}' = \mathcal{V}$.

Let $\mathcal{O} \in \mathcal{U}$ with trivialization $\Phi = (\tau, \phi), \pi = \phi \circ \alpha$. For each $x \in \mathcal{O}$, let $e_1(x) = \Phi^{-1}(x, 1), e_2(x) = \Phi^{-1}(x, -i)$, so that $\underline{e} = \{e_1, e_2\}$ is a local orthonormal basis of $\mathcal{V}'$ with respect to $\hat{g}$. Also, $e_2 = -Je_1$.

Let (ω_{ij}) be the connection matrix with respect to $\underline{e}$ given by

$$\hat{\nabla}_X e_j = \omega_{j1}(X)e_1 + \omega_{j2}(X)e_2 \;\forall\; i, j = 1, 2.$$

Since $\underline{e}$ is orthonormal, we know $\omega_{11} = \omega_{22} = 0$ and $\omega_{12} = -\omega_{21}$. So,

$$\hat{\nabla}_X e_1 = \omega_{12}(X)e_2$$

$$\hat{\nabla}_X e_2 = -\omega_{12}(X)e_1.$$

Furthermore, $\alpha = u \otimes e_1 + v \otimes e_2$; and

$$u(e_1) = 1; v(e_1) = 0;$$

$$u(e_2) = 0; v(e_2) = 1.$$

Let $X, Y \in T\mathcal{O}$. Then:

$$\begin{aligned}
2(d^{\hat{\nabla}}\alpha)(X,Y) &= \hat{\nabla}_X(\alpha(Y)) - \hat{\nabla}_Y(\alpha(X)) - \alpha([X,Y]) \\
&= \hat{\nabla}_X(u(Y)e_1 + v(Y)e_1) - \hat{\nabla}_Y(u(X)e_1 + v(X)e_1) - u([X,Y])e_1 - v([X,Y])e_2 \\
&= X(u(Y))e_1 + u(Y)\hat{\nabla}_X e_1 + X(v(Y))e_2 + v(Y)\hat{\nabla}_X e_2 \\
&\quad - Y(u(X))e_1 - u(X)\hat{\nabla}_Y e_1 - Y(v(X))e_2 - v(X)\hat{\nabla}_Y e_2 \\
&\quad - u([X,Y])e_1 - v([X,Y])e_2 \\
&= (Xu(Y) - Yu(X) - u([X,Y]))e_1 + u(Y)\omega_{12}(X)e_2 - v(Y)\omega_{12}(X)e_1 \\
&\quad + (Xv(Y) - Yv(X) - v([X,Y]))e_2 - u(X)\omega_{12}(Y)e_2 + v(X)\omega_{12}(Y)e_1 \\
&= (2du(X,Y) + 2v \wedge \omega_{12}(X,Y))e_1 \\
&\quad + (2dv(X,Y) - 2u \wedge \omega_{12})(X,Y))e_2.
\end{aligned}$$

So,

$$\phi(d^{\hat{\nabla}}\alpha) = (du - \beta \wedge v) - i(dv + \beta \wedge u),$$

where $\beta = \omega_{12}$.

Now, since $\mathcal{V}'$ is a subbundle of $T\mathcal{O}$, we may apply these various forms to elements of $\mathcal{V}'$. Recall $\hat{\nabla}_X e_j = \alpha \nabla_X e_j$ $j = 1,2$. Let $X \in \mathcal{H}$, so that $\alpha(X) = 0$. Then, since $\mathcal{V}'$ is totally geodesic with respect to g.

$$\begin{aligned}
(d^{\hat{\nabla}}\alpha)(e_1, X) &= \frac{1}{2}(\hat{\nabla}_{e_1}(\alpha(X)) - \hat{\nabla}_X(\alpha(e_1)) - \alpha[e_1, X]) \\
&= \frac{1}{2}(-\alpha\nabla_X e_1 - \alpha(\nabla_{e_1}X - \nabla_X e_1)) \\
&= 0.
\end{aligned}$$

In particular,

$$(du - \beta \wedge v)(e_1, X) = Re(\phi(d^{\hat{\nabla}}\alpha))(e_1, X) = 0 \ \forall X \in \mathcal{H}.$$

Thus, $du(e_1, X) = 0$ for all $X \in \mathcal{H}$. This means that $e_1 = U$, since it satisfies the definition of U. Also, $e_2 = -Je_1 = -JU = V$. Therefore, on $\mathcal{O}, \mathcal{V}' = \mathcal{V}$. Since $\mathcal{O}$ was an arbitrary element of $\mathcal{U}$, we have: $\mathcal{V}' = \mathcal{V}$.

Now, we have:

$$\phi(d^{\hat{\nabla}}\alpha)(U,X) = 0,$$

$$\phi(d^{\hat{\nabla}}\alpha)(V,X) = 0,$$

for any horizontal vector X. Furthermore, we have:

$$\begin{aligned}(d^{\hat{\nabla}}\alpha)(e_1,e_2) &= \frac{1}{2}(\hat{\nabla}_{e_1}(\alpha(e_2)) - \hat{\nabla}_{e_2}(\alpha(e_1)) - \alpha([e_1,e_2])) \\ &= \frac{1}{2}(\hat{\nabla}_{e_1}e_2 - \hat{\nabla}_{e_2}e_1 - \alpha([e_1,e_2])) \\ &= \frac{1}{2}\alpha(\nabla_{e_1}e_2 - \nabla_{e_2}e_1 - [e_1,e_2]) \\ &= 0.\end{aligned}$$

So, $\phi(d^{\hat{\nabla}}\alpha)(U,V) = 0$, or

$$\phi(d^{\hat{\nabla}}\alpha)(X,Y) = \phi(d^{\hat{\nabla}}\alpha)(pX,pY)\ \forall X,Y \in T\mathcal{O}.$$

Furthermore, we have:

$$d\pi(pX,pY) = \phi(d^{\hat{\nabla}}\alpha)(pX,pY),$$

for any vectors X,Y by eqns. (). Thus, for any vectors X,Y,

$$\begin{aligned}\Omega(X,Y) &= d\pi(pX,pY) \\ &= \phi(d^{\hat{\nabla}}\alpha)(pX,pY) \\ &= \phi(d^{\hat{\nabla}}\alpha)(X,Y).\end{aligned}$$

So, $\Omega = \phi(d^{\hat{\nabla}}\alpha)$, and $\beta = \sigma$, where σ is the Ishihara-Konishi connection of the normalized contact structure $\underline{\pi}$.

In particular, this means that, when restricted to vertical vectors, the Ishihara-Konishi connection of M is simply the standard connection of S^2. In other words, if $F \in M$ and we let S^2_F be the vertical S^2-leaf through F, then, for $U,W \in \mathcal{V}_F$, we may identify U,W with vectors tangent to S^2, U^*, W^*, respectively, using the local trivialization

$$M \cong \mathcal{O} \times S^2 \hookrightarrow \mathcal{O} \times R^3 \cong E,$$

and $\sigma_U(W) = -g^*(\nabla^*_{W^*}U^*, J^*U^*)$, where g^* is the standard metric on S^2 with induced Levi-Civita connection ∇^* and J^* is the standard complex structure on S^2.

At this point, we review some important facts about the standard metric and complex structure on S^2.

Consider $S^2 = \{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\}$. Let $O = \{(x,y,z) \in S^2 : z \neq 1\}$. Define coordinates on O_+ by:

$$(s,t) = (\frac{x}{1-z}, \frac{y}{1-z}).$$

Then we find that:

$$\frac{\partial}{\partial s} = (1-z-x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + x(1-z)\frac{\partial}{\partial z},$$
$$\frac{\partial}{\partial t} = -xy\frac{\partial}{\partial x} + (1-z-y^2)\frac{\partial}{\partial y} + y(1-z)\frac{\partial}{\partial z}.$$

Then $|\frac{\partial}{\partial s}| = |\frac{\partial}{\partial t}| = 1-z$. Set $U = \frac{\frac{\partial}{\partial s}}{|\frac{\partial}{\partial s}|}, JU = \frac{\frac{\partial}{\partial t}}{|\frac{\partial}{\partial t}|}$. Then we find:

$$\hat{\nabla}_U(JU) = -tU; \hat{\nabla}_{JU}JU = sU.$$

So, if $\beta_W(X) = -\hat{g}(\hat{\nabla}_X W, JW)$ for all $X, W \in TS^2$, then we have:

$$\beta_U(U) = -t; \beta_U(JU) = s.$$

Now, if we set $O' = \{(x,y,z) \in S^2 : z \neq -1\}$, define coordinates on O' by:

$$(s',t') = (\frac{x}{1+z}, \frac{y}{1+z}),$$

and set

$$U' = \frac{\frac{\partial}{\partial s}'}{|\frac{\partial}{\partial s}'|}, JU' = \frac{\frac{\partial}{\partial t}'}{|\frac{\partial}{\partial t}'|},$$

then we find that

$$\beta_{U'}(U') = -t'; \beta_{U'}(JU') = s'.$$

Therefore, if (s,t) are the stereographic coordinates of any hemisphere of S^2 with U, JU the orthonormal vectors in the directions of $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$, respectively, then we have:

$$\beta_U(U) = t; \beta_U(JU) = -s.$$

Furthermore, if we take $s' = -s, t' = -t$ and $U' = \frac{\frac{\partial}{\partial s}'}{|\frac{\partial}{\partial s}'|}, V' = \frac{\frac{\partial}{\partial t}'}{|\frac{\partial}{\partial t}'|}$, then

$$\beta_{U'}(U') = t'; \beta_{U'}(JU') = -s'.$$

3.2 The Salamon-Bérard-Bergery Metric

For this section, we will use the same notation that we used in the previous section, however we will also assume that the quaternionc-Kähler metric $\tilde{g}$ on $\tilde{M}$ has been rescaled so that its scalar curvature is $\frac{n+2}{4n}$. We now would like to show that the corresponding Salamon-Berard-Bergery metric g on M is an associated metric.

Let $\mathcal{O} \in \mathcal{U}$. Using the same notation as the previous section, we already know: for $X \in \mathcal{H}, g(U,X) = g(e_1, X) = 0$ by definition of g. Also, for the same $X, u(X) = Re(\phi \circ \alpha)(X) = 0$. Also, $g(U,U) = g(e_1,e_1) = \tilde{g}(e_1,e_1) = 1 = u(U)$ by definition of e_1. And $g(U,V) = g(e_1,e_2) = 0 = u(V)$, by definition of e_2. This means that $u(Y) = g(U,Y)$ for all $Y \in \mathcal{H}$ or $Y \in \mathcal{V}$. Thus, $u(Y) = g(U,Y)$ for any vector in $T\mathcal{O}$. Also,

$$v(Y) = u(JY) = g(U,JY) = -g(JU,Y) = g(V,Y), \ \forall Y \in T\mathcal{O}.$$

Furthermore, g is Hermitian with respect to J (in fact, it is Kähler).

Now, on $\mathcal{O}$, define the local endomorphism $G : T\mathcal{O} \to T\mathcal{O}$ by:

$$g(X,GY) = Re(\Omega(X,Y)) = Re(\phi(d^{\nabla}\alpha)(X,Y)),$$

for all vectors $X, Y \in T\mathcal{O}$. Thus, we have automatically that G is skew-symmetric with respect to g. Furthermore,

$$GU = 0, GV = 0, u \circ G = 0.$$

Also, by the definition of $\{\Omega\}$, the endomorphisms $\{G\}$ transform on the intersections of elements of $\mathcal{U}$ correctly. Thus, in order for g to be an associated metric,we only need to show two things:

1) $G^2 = -p$.

2) $G \circ J = -J \circ G$.

In order to do this, we will need to specify a trivialization of $\mathcal{V}$. Let $\mathcal{U} = \{\tilde{\mathcal{O}}\}$ be the usual open atlas of $\tilde{M}$ such that, on each $\tilde{\mathcal{O}}$, there exists a local quaternionic-Kähler structure $\{A, B, C\} \subset M$. As usual, for $\tilde{\mathcal{O}} \in \tilde{\mathcal{U}}$, $\mathcal{O} = \rho^{-1}(\tilde{\mathcal{O}})$ so that $\mathcal{O} = \{xA + yB + zC \in E : x^2 + y^2 + z^2 = 1\}$. Set

$$\mathcal{O}' = \{xA + yB + zC \in \mathcal{O} : z \neq 1\},$$
$$\mathcal{O}'' = \{xA + yB + zC \in \mathcal{O} : z \neq -1\}.$$

Now, over $\tilde{\mathcal{O}}$, we have the usual trivialization:

$$M \cong \tilde{\mathcal{O}} \times S^2 \hookrightarrow \tilde{\mathcal{O}} \times R^3 \cong E.$$

So, for $F = xA + yB + zC$, we may make the following identification:

$$\mathcal{V}_F = \{aA + bB + cC \in E : ax + by + cz = 0\}$$

with the metric $\hat{g}$ of $\mathcal{V}_F$ induced from the bundle metric of E.

Using this identification, we define the following sections of $\mathcal{V}$ over M :

1) For $F = xA + yB + zC \in \mathcal{O}'$, set

$$U' = \frac{1}{2}\frac{(-x^2 + y^2 + (1-z)^2)}{1-z}A - \frac{xy}{1-z}B + xC,$$
$$V' = \frac{xy}{1-z}A - \frac{1}{2}\frac{(-x^2 + y^2 + (1-z)^2)}{1-z}B - yC.$$

2) For $F = xA + yB + zC \in \mathcal{O}''$,

$$\text{set } U'' = \frac{1}{2}\frac{(-x^2 + y^2 + (1+z)^2)}{1+z}A - \frac{xy}{1+z}B + xC,$$
$$V'' = \frac{xy}{1+z}A - \frac{1}{2}\frac{(-x^2 + y^2 + (1+z)^2)}{1+z}B - yC.$$

It is easy to verify that both $\{U', V'\}$ and $\{U'', V''\}$ are local orthonormal frames of $\mathcal{V}$ on $\mathcal{O}'$ and $\mathcal{O}''$, respectively.

Using these local frames, we define the following trivializations of $\mathcal{V}$:

1) $\phi' : \tau^{-1}(\mathcal{O}') \to \mathcal{O}' \times C$ is defined by:

$$\phi'(F, aA + bB + cC) = (F, k + il),$$

where $aA + bB + bC = kU' + lV'$, for all $F \in \mathcal{O}'$, $aA + bB + bC \in \mathcal{V}_F$.

2) $\phi'' : \tau^{-1}(\mathcal{O}'') \to \mathcal{O}'' \times C$ is defined by:

$$\phi''(F, aA + bB + cC) = (F, k + il),$$

where $aA + bB + bC = kU'' + lV''$, for all $F \in \mathcal{O}''$, $aA + bB + bC \in \mathcal{V}_F$.

Now, we need to use some results about quaternionic-Kähler manifolds, which we will briefly review. All of these results are to be found in [Be].

Using the Riemannian submersion $\rho : (M, g) \to (\tilde{M}, \tilde{g})$ and the contact form $\alpha : \mathcal{V} \oplus \mathcal{H} \to \mathcal{H}$, we define on M, $\mathbf{A} : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ by

$$\mathbf{A}_X Y = \frac{1}{2}\alpha([X, Y]), \ \forall X, Y \in \mathcal{H}.$$

$\mathbf{A}$ is called the second O'Neill tensor of the submersion ρ.

For $F \in M$ and $X, Y \in \mathcal{H}_F$, we get:

$$(\mathbf{A}_X Y)_F = -\frac{1}{2} R^{\tilde{\nabla}}(\rho_* X, \rho_* Y) F,$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$ and $R^{\tilde{\nabla}}$ is the Riemannian curvature of $\tilde{\nabla}$ extended to elements of $M \subset End(T\tilde{M})$ in the usual way. By Ricci's identity, we get:

$$(\mathbf{A}_X Y)_F = -\frac{1}{2}[\tilde{R}(\rho_* X, \rho_* Y), F],$$

where $\tilde{R}$ is the Riemannian curvature of $\tilde{g}$ so that $\tilde{R}(\rho_* X, \rho_* Y) \in Hom(T\tilde{M}, T\tilde{M})$ for $X, Y \in T_F M$.

Let $X, Y \in \tilde{\mathcal{O}}$. Let $\tilde{r}$ be the Ricci operator of $\tilde{g}$. Using the Einstein property of $\tilde{g}$ and the fact that the scalar curvature of $\tilde{g}$ is $\frac{n+2}{4n}$, we get:

$$\begin{aligned}
[\tilde{R}(\tilde{X}, \tilde{Y}), A] &= \frac{2}{n+2}(\tilde{r}(C\tilde{X}, \tilde{Y})B - \tilde{r}(B\tilde{X}, \tilde{Y})C) \\
&= 2(\tilde{g}(C\tilde{X}, \tilde{Y})B - \tilde{g}(B\tilde{X}, \tilde{Y})C), \\
[\tilde{R}(\tilde{X}, \tilde{Y}), B] &= \frac{2}{n+2}(-\tilde{r}(C\tilde{X}, \tilde{Y})A + \tilde{r}(A\tilde{X}, \tilde{Y})C) \\
&= 2(-\tilde{g}(C\tilde{X}, \tilde{Y})A + \tilde{g}(A\tilde{X}, \tilde{Y})C), \\
[\tilde{R}(\tilde{X}, \tilde{Y}), C] &= \frac{2}{n+2}(\tilde{r}(B\tilde{X}, \tilde{Y})A - \tilde{r}(A\tilde{X}, \tilde{Y})B) \\
&= 2(\tilde{g}(B\tilde{X}, \tilde{Y})A - \tilde{g}(A\tilde{X}, \tilde{Y})B),
\end{aligned}$$

for $\tilde{X}, \tilde{Y} \in T\tilde{\mathcal{O}}$.

Suppose $F = xA + yB + zC \in \mathcal{O}$ with $\rho(F) = \tilde{m}, x^2 + y^2 + z^2 = 1$. Let $X, Y \in \mathcal{H}_F$ with $\tilde{X} = \rho_* X, \tilde{Y} = \rho_* Y$. Then

$$\begin{aligned}
(d^{\hat{\nabla}}\alpha)(X,Y) &= \frac{1}{2}(\hat{\nabla}_X(\alpha(Y)) - \hat{\nabla}_Y(\alpha(X)) - \alpha([X,Y])) \\
&= -\frac{1}{2}\alpha([X,Y]) \\
&= -(\mathbf{A}_X Y)_F \\
&= \frac{1}{2}R^{\tilde{\nabla}}(\rho_* X, \rho_* Y)F \\
&= \frac{1}{2}[\tilde{R}(\tilde{X},\tilde{Y}), F] \\
&= \frac{1}{2}x[\tilde{R}(\tilde{X},\tilde{Y}), A] + \frac{1}{2}y[\tilde{R}(\tilde{X},\tilde{Y}), B] + \frac{1}{2}z[\tilde{R}(\tilde{X},\tilde{Y}), C]. \\
&= x(\tilde{g}(C\tilde{X},\tilde{Y})B - \tilde{g}(B\tilde{X},\tilde{Y})C) \\
&\quad + y(-\tilde{g}(C\tilde{X},\tilde{Y})A + \tilde{g}(A\tilde{X},\tilde{Y})C) \\
&\quad + z(\tilde{g}(B\tilde{X},\tilde{Y})A - \tilde{g}(A\tilde{X},\tilde{Y})B) \\
&= [-y\tilde{g}(C\tilde{X},\tilde{Y}) + z\tilde{g}(B\tilde{X},\tilde{Y})]A \\
&\quad + [x\tilde{g}(C\tilde{X},\tilde{Y}) - z\tilde{g}(A\tilde{X},\tilde{Y})]B \\
&\quad + [-x\tilde{g}(B\tilde{X},\tilde{Y}) + y\tilde{g}(A\tilde{X},\tilde{Y})]C.
\end{aligned}$$

Now, since basic vector fields span $\mathcal{H}$, we can assume that X and Y are basic vector fields on $\mathcal{O}$. Also, we can assume that $\tilde{X}$ and $\tilde{Y}$ are elements of an orthonormal A-B-C basis of $\tilde{M}$. Under these circumstances, we have only three non-trivial possibilities:

1) $\tilde{Y} = A\tilde{X}$,

2) $\tilde{Y} = B\tilde{X}$,

3) $\tilde{Y} = C\tilde{X}$.

Now, suppose $F = xA + yB + zC \in \mathcal{O}'$. Under each possibility, we wish to find a formula for $Re(\phi'(d^{\hat{\nabla}}\alpha)(X,Y))$. Using some linear algebra, we obtain the following results.

Case 1. Suppose $\tilde{Y} = A\tilde{X}$.

Then

$$(d^{\hat{\nabla}}\alpha)(X,Y) = -zB + yC$$
$$= \frac{xy}{1-z}U' - \frac{1-z-x^2}{1-z}V'.$$

So,

$$\phi'((d^{\hat{\nabla}}\alpha)(X,Y) = \frac{xy}{1-z} - i\frac{1-z-x^2}{1-z}.$$

Thus,

$$Re(\phi'((d^{\hat{\nabla}}\alpha)(X,Y)) = \frac{xy}{1-z}.$$

Case 2. Suppose $\tilde{Y} = B\tilde{X}$.

Then

$$(d^{\hat{\nabla}}\alpha)(X,Y) = zA - xC$$
$$= \frac{y^2+z-1}{1-z}U' + \frac{xy}{1-z}V'.$$

Thus,

$$Re(\phi'((d^{\hat{\nabla}}\alpha)(X,Y)) = \frac{y^2+z-1}{1-z}.$$

Case 3. Suppose $\tilde{Y} = C\tilde{X}$.

Then

$$(d^{\hat{\nabla}}\alpha)(X,Y) = -yA + xB$$
$$= -yU' - xV'.$$

So,

$$Re(\phi'((d^{\hat{\nabla}}\alpha)(X,Y)) = -y.$$

Now, we will denote the orthonormal A-B-C basis by $\underline{E} = \{E_1, \ldots, E_{4n}\}$ with $E_2 = AE_1, E_3 = BE_1, E_4 = CE_1$, etc. Denote the corresponding basic frame of $\mathcal{H}$ on $\mathcal{O}$ by $\underline{e} = \{e_1, \ldots, e_{4n}\}$ with $\rho_* e_j = E_j$ for each $j = 1, \ldots, 4n$. Since ρ is a Riemannian submersion, we know that $\underline{e}$ is also an orthonormal basis of $\mathcal{H}$ on $\mathcal{O}$.

So, if $[g]_{\underline{e}}$ is the matrix representation of g on $\mathcal{H}$ with respect to $\underline{e}$ and if $[\tilde{g}]_{\underline{E}}$ is the matrix representation of $\tilde{g}$ on $T\tilde{O}$ with respect to $\underline{E}$, then $[g]_{\underline{e}} = [\tilde{g}]_{\underline{E}} = I_{4n}$.

Now, on $\mathcal{O}'$, we let G' be the local endomorphism given by:

$$g(X, G'Y) = du(pX, pY) = Re(\phi'(d^{\nabla}\alpha)(X, Y)).$$

Let $[G']_{\underline{e}}, [Re(\phi'(d^{\nabla}\alpha))]_{\underline{e}}$ be the matrix representations with respect to $\underline{e}$. Since $[g]_{\underline{e}} = I_{4n}$, we have that $[G']_{\underline{e}} = [Re(\phi'(d^{\nabla}\alpha))]_{\underline{e}}$.

Using the previous linear algebra results, we know that, at $F = xA + yB + zC \in \mathcal{O}'$,

$$[G']_{\underline{e}} = \left(\begin{array}{c|c|c} \Phi' & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Phi' \end{array}\right),$$

where

$$\Phi' = \begin{pmatrix} 0 & \frac{xy}{1-z} & \frac{y^2+z-1}{1-z} & -y \\ -\frac{xy}{1-z} & 0 & -y & -\frac{y^2+z-1}{1-z} \\ -\frac{y^2+z-1}{1-z} & y & 0 & \frac{xy}{1-z} \\ y & \frac{y^2+z-1}{1-z} & -\frac{xy}{1-z} & 0 \end{pmatrix}.$$

Using the fact that $x^2 + y^2 + z^2 = 1$, it is easy to verify that $(\Phi')^2 = -I_4$. So, $[G']_{\underline{e}}^2 = I_{4n}$. Therefore, $(G')^2 = -p$. Similarly, if we define G'' on $\mathcal{O}''$ by

$$g(G''X, Y) = Re(\phi''(d^{\nabla}\alpha)(X, Y))$$

for each $X, Y \in T\mathcal{O}''$, we find that $(G'')^2 = -p$.

All that remains to show is that, on $\mathcal{O}', G' \circ J = -J \circ G'$, and that, on $\mathcal{O}'', G'' \circ J = -J \circ G''$. Clearly, we need only do this on $\mathcal{O}'$. The argument on $\mathcal{O}''$ will be exactly the same.

Let $F = xA + yB + zC \in \mathcal{O}'$ with $x^2 + y^2 + z^2 = 1$. Then, for $X \in \mathcal{H}_F, \tilde{X} = \rho_* X$ we have, by definition of J,

$$\begin{aligned} \rho_*(JX) &= (xA + yB + zC)\rho_* X \\ &= x(A\tilde{X}) + y(B\tilde{X}) + z(C\tilde{X}). \end{aligned}$$

Using the same bases $\underline{E}$ and $\underline{e}$ as before and letting $[J]_{\underline{e}}$ be the matrix representation of J with respect to $\underline{e}$, we find:

$$[J]_{\underline{e}} = \left(\begin{array}{c|c|c} \Psi' & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Psi' \end{array}\right),$$

where

$$\Psi' = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & z & 0 \end{pmatrix}.$$

Then,

$$[G']_{\underline{\varepsilon}}[J]_{\underline{\varepsilon}} = \left(\begin{array}{c|c|c} \Phi'\Psi' & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Phi'\Psi' \end{array}\right).$$

So, we need only verify that $\Phi'\Psi' = -\Psi'\Phi'$.

$$\Phi'\Psi' = \begin{pmatrix} 0 & \frac{xy}{1-z} & \frac{y^2+z-1}{1-z} & -y \\ -\frac{xy}{1-z} & 0 & -y & -\frac{y^2+z-1}{1-z} \\ -\frac{y^2+z-1}{1-z} & y & 0 & \frac{xy}{1-z} \\ y & \frac{y^2+z-1}{1-z} & -\frac{xy}{1-z} & 0 \end{pmatrix} \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & z & 0 \end{pmatrix}.$$

Now, except for the diagonal entries, it is clearly seen that $\Phi'\Psi'$ is a skew-symmetric matrix. Furthermore, the diagonal entries are given by:

$$\begin{aligned} x\frac{xy}{1-z} + y\frac{y^2+z-1}{1-z} - yz &= \frac{x^2y + y^3 + yz - y - yz(1-z)}{1-z} \\ &= \frac{y(x^2 + y^2 + z - 1 - z + z^2)}{1-z} \\ &= \frac{y(x^2 + y^2 + z^2 - 1)}{1-z} \\ &= 0. \end{aligned}$$

So, $\Phi'\Psi'$ is zero along its diagonal, that is, $\Phi'\Psi'$ is a skew-symmetric matrix. Thus, we have:

$$\begin{aligned} {}^t(\Phi'\Psi') &= -\Phi'\Psi', \\ {}^t(\Psi')^t(\Phi') &= -\Phi'\Psi', \\ \Psi'\Phi' &= -\Phi'\Psi', \\ \Phi'\Psi' &= -\Psi'\Phi'. \end{aligned}$$

So, $G' \circ J = J \circ G'$. Thus, g is an associated metric of the complex contact structure of M.

Since the fibration $p : (M, g) \to (\tilde{M}, \tilde{g})$ is a Riemannian submersion, we know that g is a projectable associated metric, that is, for any unit vertical vector field U, $h_U \equiv 0$.

We now use this fact to find that g is actually the only projectable associated metric on M. In order to do this, we need a few general facts about the space of associated metrics for any complex contact manifold. For the next three lemmas, we will assume that M is any complex contact manifold.

Lemma 3.2.1 Suppose $g \in \mathcal{A}$ with $h \equiv 0$. Let $D \in T_g\mathcal{A}$. Then $0 = p(\nabla_U e^D)p + e^D k_U - k_U e^D \ \forall U \in \mathcal{V}$ if and only if $p(\mathcal{L}_U e^D)p = 0 \ \forall U \in \mathcal{V}$.

Proof: Let X be a horizontal vector field and U a vertical vector field. Then:

$$\begin{aligned} 0 &= p(\nabla_U e^D)X + e^D k_U X - k_U e^D X \\ &= p\nabla_U(e^D X) - k_U e^D X - pe^D \nabla_U X + e^D k_U X \\ &= p\nabla_U(e^D X) - p\nabla_{e^D X} U - pe^D \nabla_U X + pe^D \nabla_X U \\ &= p\mathcal{L}_U(e^D X) - pe^D \mathcal{L}_U X \\ &= p(\mathcal{L}_U e^D)X. \end{aligned}$$

The converse is clear by reversing the order of the above equations.

Lemma 3.2.2 Suppose g is a projectable associated metric, $D \in T_g\mathcal{A}$. Also, suppose

$$p(\mathcal{L}_U e^D)p = 0 \ \forall U \in \mathcal{V}.$$

Then $U\lambda = 0$ for each vertical vector field U and for each eigenvalue λ of D.

Proof: Let $X \in \mathcal{H}$ such that $DX = \lambda X, |X| = 1$. Then:

$$\begin{aligned} p(\nabla_U e^D)X &= k_U e^D X - e^D k_U X \\ &= e^{\lambda} k_U X - e^{-\lambda} k_U X. \end{aligned}$$

So, $g((\nabla_U e^D)X, X) = 0$. But

$$\begin{aligned} 0 &= g((\nabla_U e^D)X, X) \\ &= g(\nabla_U(e^D X), X) - g(e^D \nabla_U X, X) \\ &= g(\nabla(e^{\lambda} X), X) - e^{\lambda} g(\nabla_U X, X) \\ &= (Ue^{\lambda})g(X, X) + e^{\lambda} g(\nabla_U X, X) - e^{\lambda} g(\nabla_U X, X) \\ &= (Ue^{\lambda}). \end{aligned}$$

So, $0 = Ue^{\lambda} = (U\lambda)e^{\lambda}$. So, $0 = U\lambda$, since e^{λ} is never zero.

Lemma 3.2.3 For $g \in \mathcal{A}$ a projectable associated metric and $D \in T_g\mathcal{A}$, we have $p(\mathcal{L}_U e^D)p = 0$ for every vertical vector field U if and only if $p(\mathcal{L}_U D)p = 0$ for every vertical vector field U.

Proof: It is clear that, if $p(\mathcal{L}_U D)p = 0$ for a vertical vector field U, then $p(\mathcal{L}_U e^D)p = 0$. Now, suppose that $p(\mathcal{L}_U e^D)p = 0$. Let $X \in \mathcal{H}$ such that $DX = \lambda X$. Then:

$$\begin{aligned}
0 &= p(\mathcal{L}_U e^D)X \\
&= p\mathcal{L}_U(e^D X) - pe^D \mathcal{L}_U X \\
&= p\mathcal{L}_U(e^{\lambda} X) - pe^D \mathcal{L}_U X \\
&= (Ue^{\lambda})X + e^{\lambda} p\mathcal{L}_U X - e^D p\mathcal{L}_U X \\
&= e^{\lambda} p\mathcal{L}_U X - e^D p\mathcal{L}_U X,
\end{aligned}$$

by using Lemma 3.2.1. So, $pe^D \mathcal{L}_U X = e^{\lambda} p\mathcal{L}_U X$. Since every eigenvector of e^D with eigenvalue e^{λ} is an eigenvector of D with eigenvalue λ, $D\mathcal{L}_U X = \lambda \mathcal{L}_U X$. So,

$$\begin{aligned}
0 &= (U\lambda)X + \lambda p\mathcal{L}_U X - D\mathcal{L}_U X \\
&= p\mathcal{L}_U(\lambda X) - D(\mathcal{L}_U X) \\
&= p\mathcal{L}_U(DX) - D(\mathcal{L}_U X) \\
&= p(\mathcal{L}_U D)X.
\end{aligned}$$

Thus, $p(\mathcal{L}_U D)p = 0$. This proves the lemma.

Proposition 3.2.4 Let $g, g' \in \mathcal{A}$ such that g is projectable. Let $D \in T_g\mathcal{A}$ such that $g' = ge^D$. Then g' is projectable if and only if $p(\mathcal{L}_U D)p = 0$ for any vertical vector field U.

Proof: Let U be a vertical vector field. Then, by Proposition (2.2.1), we know:

$$\begin{aligned}
h'_U &= \frac{1}{2} pe^{-D}(\nabla_U e^D)p + \frac{1}{2}(k_U - e^{-D} k_U e^D) + \frac{1}{2}(h_U + e^{-D} h_U e^D) \\
&= \frac{1}{2} pe^{-D}(\nabla_U e^D)p + \frac{1}{2}(k_U - e^{-D} k_U e^D),
\end{aligned}$$

since g is projectable. So, g' is projectable if and only if

$$0 = p(\nabla_U e^D)p + e^D k_U - k_U e^D.$$

By Lemmas 3.2.1 and 3.2.3, g' is projectable if and only if $p(\mathcal{L}_U D)p = 0$ for any vertical vector field U.

We shall now apply this proposition to the twistor space over a quaternionic-Kähler manifold. Using the same notation as before, we let g be the Salamon-Berard-Bergery metric. If $D \in T_g\mathcal{A}$, then $p(\mathcal{L}_U D)p = 0$ if only if D "projects down" to a $(1,1)$-tensor on $\tilde{M}$, i.e. there exists a $(1,1)$-tensor $\tilde{D}$ on $\tilde{M}$ such that $\tilde{D} \circ \rho_* = \rho_* \circ D$. We will now use this fact to show the following theorem.

Theorem 3.2.5 The Salamon-Berard-Bergery metric is the only projectable associated metric of the twistor space over a compact quaternionic-Kähler manifold with dimension ≥ 8 and positive scalar curvature.

Proof: Let g be the Salamon-Berard-Bergery metric. Let $g' \in \mathcal{A}$. Then, by Propostion 2.1.3, there exists $D \in T_g\mathcal{A}$ such that $g' = ge^D$. By Proposition 3.2.4, g' is projectable if and only if $p(\mathcal{L}_U D)p = 0$ for any vertical vector field U.

Suppose $p(\mathcal{L}_U D)p = 0$ for any vertical vector field U. For $F \in M, x = \rho(F)$, let

$$L_F : T_x\tilde{M} \to T_F M$$

be the horizontal lift of the Riemannian submersion $\rho : (M, g) \to (\tilde{M}, \tilde{g})$, i.e.

$$\rho_* L_F = id|_{T_x\tilde{M}},$$

$$\alpha \circ L_F = 0.$$

Since D is projectable, there exists a $(1,1)$-tensor $\tilde{D}$ on $\tilde{M}$ such that D is the horizontal lift of $\tilde{D}$, i.e. on $T_F M$,

$$D = L_F \tilde{D} \rho_*.$$

This is actually another way of saying $\tilde{D} \circ \rho_* = \rho_* \circ D$. Or, $\tilde{D} = \rho_* D L_F$ on $T_x\tilde{M}$. Clearly, since D is g-symmetric, $\tilde{D}$ is $\tilde{g}$-symmetric.

Now, on $T_F M, J = L_F \circ F \circ \rho_*$, where we view $F \in End(T\tilde{M})$. Also, $DJ = JD$. So, we have that, for $F \in M$,

$$L_F \circ \tilde{D} \circ \rho_* \circ L_F \circ F \circ \rho_* = L_F \circ F \circ \rho_* \circ L_F \circ \tilde{D} \circ \rho_*.$$

But, $\rho_* \circ L_F = id|_{T\tilde{M}}$. So,

$$L_F \circ \tilde{D} \circ F \circ \rho_* = L_F \circ F \circ \tilde{D} \circ \rho_*.$$

By applying ρ_* on the left and L_F on the right, we get:

$$\tilde{D} \circ F = F \circ \tilde{D}.$$

Thus, $\tilde{D}$ commutes with each element of $M \subset End(T\tilde{M})$.

We also have that $DG = -GD$. So, for $X, Y \in \mathcal{H}$,

$$g(X, DGY) = -g(X, GDY)$$

$$g(DX, GY) = g(GX, DY)$$

$$du(DX, Y) = du(DY, X)$$

$$u([DX, Y]) = u([DY, X]).$$

Similarly, since $HD = -DH$, we have that, for $X, Y \in \mathcal{H}, v([DX, Y]) = v([DY, X])$. So, $\pi([DX, Y]) = \pi([DY, X])$. Thus, $\alpha([DX, Y]) = \alpha([DY, X]) = -\alpha([X, DY])$. This means that

$$\mathbf{A}_{DX} Y = -\mathbf{A}_X DY,$$

where $\mathbf{A}$ is the second O'Neill tensor defined previously in this section.

Recall that for $X_1, X_2 \in \mathcal{H}_F$,

$$\begin{aligned} \mathbf{A}_{X_1} X_2 &= -\frac{1}{2} R^{\tilde{\nabla}}(\rho_* X_1, \rho_* X_2) F \\ &= -\frac{1}{2}[\tilde{R}(\rho_* X_1, \rho_* X_2), F]. \end{aligned}$$

So, in our case, we have that for $X, Y \in \mathcal{H}_F$

$$-\frac{1}{2}[\tilde{R}(\rho_*(DX), \rho_* Y, F] = \frac{1}{2}[\tilde{R}(\rho_* X, \rho_*(DY), F].$$

Or, for $\tilde{X}, \tilde{Y} \in T_x\tilde{M}$,

$$-[\tilde{R}(\tilde{D}\tilde{X}, \tilde{Y}), F] = [\tilde{R}(\tilde{X}, \tilde{D}\tilde{Y}), F].$$

In particular, if $\{A, B, C\}$ is a local quaternionic-Kähler structure on $\tilde{M}$, we have:

$$-[\tilde{R}(\tilde{D}\tilde{X}, \tilde{Y}), A] = [\tilde{R}(\tilde{X}, \tilde{D}\tilde{Y}), A]$$

$$-2(\tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y})B - \tilde{g}(B\tilde{D}\tilde{X}, \tilde{Y})C) = 2(\tilde{g}(C\tilde{X}, \tilde{D}\tilde{Y})B - \tilde{g}(B\tilde{X}, \tilde{D}\tilde{Y})C)$$

$$-\tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y})B + \tilde{g}(B\tilde{D}\tilde{X}, \tilde{Y})C = \tilde{g}(C\tilde{X}, \tilde{D}\tilde{Y})B - \tilde{g}(B\tilde{X}, \tilde{D}\tilde{Y})C$$

$$-\tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y}) = \tilde{g}(C\tilde{X}, \tilde{D}\tilde{Y}),$$

since the endomorphisms $\{A, B, C\}$ are linearly independent. Now, recall that we have already established that $\tilde{D}$ is $\tilde{g}$-symmetric and that $\tilde{D}$ commutes with each element of M. Since $C \in M$, the last equation above gives us:

$$-\tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y}) = \tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y}).$$

Or

$$0 = \tilde{g}(C\tilde{D}\tilde{X}, \tilde{Y}).$$

By substituting $C\tilde{X}$ for $\tilde{X}$, we get that $\tilde{D} = 0$, which means that $D = 0$. Thus, $g' = g$. This proves the theorem.

3.3 Applications of Variational Methods on Twistor Spaces

We are now in the position to apply some of the results from Chapter Two to the space of associated metrics of the twistor space of a quaternionic-Kähler manifold of dimension $4n \geq 8$ with positive scalar curvature. Throughout this section, we will use the notation derived in the previous section.

Theorem 3.3.1 Let M be the twistor space of a compact quaternionic-Kähler manifold $\tilde{M}$ with positive scalar curvature and dimension $4n \geq 8$. Let $g' \in \mathcal{A}$. Then $g\prime$ is critical for I^* if and only $(h_U{}')^d = 0$ for every vertical vector U.

Proof: Let g be the Salamon-Bérard-Bergery metric on M. Let the structure tensors and connection of the Salamon-Bérard-Bergery metric be given by k, h, ∇, respectively. In particular, $h = 0$, and $\nabla J = 0$.

By Proposition 2.1.3, we know that there exists $D \in T_g\mathcal{A}$ such that $g' = ge^D$. By Proposition 2.2.1, we have:

$$(\nabla'_U J) = 2e^{-D}k_U J + pe^{-D}(\nabla_U e^D)^d J - e^{-D}(k_U e^D + e^D k_U)J,$$

$$k'_U = k_U e^D,$$

$$h'_U = \frac{1}{2}pe^{-D}(\nabla_U e^D) + \frac{1}{2}p(k_U - e^{-D}k_U e^D).$$

Hence,

$$(h'_U)^d = \frac{1}{2}pe^{-D}(\nabla_U e^D)^d + \frac{1}{2}p(k_U - e^{-D}k_U e^D).$$

Suppose that g' is a critical metric for I^*. Recall from Theorem (2.5.1) that the critical condition for I^* is given by:

$$(h'_U)^d(\nabla'_U J) = (\nabla'_U J)(h'_U)^d = 0,$$

for any unit vertical vector field U.

Fix a unit vertical vector field U with corresponding complex almost contact structure $\{G, H, U, V, u, v, g\mathcal{O}\}$. Let $\mathcal{H}_1 = Im((h'_U)^d); \mathcal{H}_0 = ker((h'_U)^d)$. Note that both $\mathcal{H}_0$ and $\mathcal{H}_1$ are preserved under the actions of J and $G = -k_U$.

Let $X \in \mathcal{H}_1$. Then $(\nabla'_U J)X = 0$. Now,

$$\nabla'_U J = pe^{-D}(\nabla_U e^D)^d J - (e^{-2D} - 2e^{-D} + I)k_U J.$$

So,

$$pe^{-D}(\nabla_U e^D)^d JX = (e^{-2D} - 2e^{-D} + I)k_U JX,$$

$$pe^{-D}(\nabla_U e^D)^d X = (e^{-2D} - 2e^{-D} + I)k_U X,$$

since $J\mathcal{H}_1 = \mathcal{H}_1$. Also,

$$(h'_U)^d X = \frac{1}{2}pe^{-D}(\nabla_U e^D)^d X - \frac{1}{2}(e^{-2D} - I)k_U X.$$

Combining these two equations, we get:

$$(h'_U)^d X = -(e^{-D} - I)k_U X.$$

Now, $(h'_U)^d k'_U = -k'_U(h'_U)^d$. Furthermore,

$$\begin{aligned}
(h'_U)^d k'_U X &= -(e^{-D} - I)k_U k_U e^D X \\
&= (e^{-D} - I)e^D X \\
&= X - e^D X, \\
-k'_U(h'_U)^d X &= k_U e^D(e^{-D} - I)k_U X \\
&= k_U(I - e^D)k_U X \\
&= k_U^2(I - e^{-D})X \\
&= -(I - e^{-D})X \\
&= -X + e^{-D}X.
\end{aligned}$$

So, $2X = (e^D + e^{-D})X$. Thus, $\mathcal{H}_1$ is an eigenspace of $(e^D + e^{-D})$ corresponding to the eigenvalue 2.

Now, D is diagonalizable, so we know that $(e^D + e^{-D})$ is diagonalizable such that every eigenvector of D is an eigenvector of $(e^D + e^{-D})$ and vice versa. Furthermore, we know that $(e^D + e^{-D})Y = \mu Y$ if and only $DY = \lambda Y$ with $\mu = e^\lambda + e^{-\lambda}$. So, since all elements of $\mathcal{H}_1$ are eigenvectors of $(e^D + e^{-D})$ with eigenvalue 2, for each $X \in \mathcal{H}_1, DX = \lambda X$ with $e^\lambda + e^{-\lambda} = 2$. Thus, $(e^\lambda - 1)^2 = 0$. Or, $\lambda = 0$. Therefore, $DX = 0 \ \forall X \in \mathcal{H}_1$.

Now, suppose $(h_U')^d \neq 0$. Then there exists $X \in \mathcal{H}_1, \lambda > 0$ such that $X \neq 0, (h_U')^d X = \lambda X$. Then

$$\begin{aligned}\lambda X &= (h_U')^d X \\ &= -(e^{-D} - I)k_U X \\ &= -k_U(e^D - I)X \\ &= -k_U(e^D X - X) \\ &= -k_U(X - X) \\ &= 0.\end{aligned}$$

This contradicts the fact that $\lambda X \neq 0$. So, $(h_U{}')^d = 0$. This proves the theorem.

Now, we use this result to give a characterization of the Salamon-Bérard-Bergery metric.

Theorem 3.3.2 Let M be the twistor space of a compact quaternionic-Kähler manifold $\tilde{M}$ with positive scalar curvature and dimension $4n \geq 8$. Let $g' \in \mathcal{A}$ with Levi-Civita connection ∇. Let J be the complex structure of M. Then the following conditions are equivalent:

1) g' is the Salamon-Bérard-Bergery metric.

2) g' is projectable.

3) g' is Kähler.

Proof: By Theorem 3.2.4, we already know that conditions 1 and 2 are equivalent. Also, we know that the Salamon-Bérard-Bergery metric is Kähler. Thus, we need only show that condition 3 implies condition 2 or 1.

Suppose g' is Kähler. In particular, this implies, by Lemma 1.4.6, that $(h_U')^s = 0$ for any vertical vector U. Also, we know that $\nabla_U J = 0$ for any vertical vector U. So, g' is critical for I^*. Thus, $(h_U{}')^d = 0$ for any vertical U. This means that g' is projectable, proving the theorem.

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