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COMPOSITION OPERATORS ON THE DIRICHLET SPACE

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ABSTRACT

COMPOSITION OPERATORS ON THE DIRICHLET SPACE

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We examine some properties of functions belonging to, and composition operators acting upon, the Dirichlet and Dirichlet-type spaces of analytic functions on $\mathbf{D}$. Every function in one of these spaces has boundary values on all of $\partial\mathbf{D}$ except perhaps on a set of capacity zero. We show that when C_φ is Hilbert-Schmidt, φ may have boundary values of unit modulus only on a set of capacity zero (the converse, of course, does not generally hold). This result is an immediate consequence of an appreciably more descriptive integral condition, which shows that $\mid \varphi(e^{it}) \mid$ cannot be "too big, too often" if C_φ is Hilbert-Schmidt.

The space $\mathcal{D}_0$ denotes the Dirichlet space modulo the constant functions. We determine the spectrum of each composition operator C_φ on $\mathcal{D}_0$ which is induced by a linear fractional map φ taking $\mathbf{D}$ into itself. The spectrum of the corresponding composition operator on the Dirichlet space is essentially the same.

To my parents

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TABLE OF CONTENTS

1 Introduction **1**
1.1 Preliminary lemma 3

2 The Spaces $\mathcal{D}_0$ and $\mathcal{D}_\pi$ **5**
2.1 Unitary composition operator theorem 6
2.2 Reproducing kernel lemma for $\mathcal{D}_0$ 7

3 The Spectra of Composition Operators on $\mathcal{D}_0$ Induced by Linear Fractional Transformations **8**
3.1 Elliptic automorphism theorem 9
3.2 Lemma relating $\|F\|_\pi$ and $\hat{F}$ for functions F in $H^2(\mathbf{\Pi}^+)$ 12
3.3 Parabolic automorphism theorem 14
3.4 Hyperbolic automorphism theorem 17
3.5 A Cauchy lemma 21
3.6 Holomorphic semigroup lemma 22
3.7 Parabolic non-automorphism theorem 24
3.8 Hyperbolic non-automorphism theorem (no interior fixed point) . . . 29
3.9 Hilbert space lemma 33
3.10 Hyperbolic non-automorphism theorem (two fixed points in $\overline{\mathbf{D}}$) 34
3.11 A well-known theorem concerning compact iterates 44

4 Hilbert-Schmidt Composition Operators and Capacity **47**
4.1 Hilbert-Schmidt operator theorem 47
4.2 Capacity equivalence lemma 50
4.3 Kahane and Salem's theorem 51
4.4 Lemma (weak capacitary inequality) 53
4.5 Factorization lemma 54
4.6 Radial limit theorem 56
4.7 Corollary of the proof of the radial limit theorem 59
4.8 Lebesgue point theorem 59
4.9 Lemma 60
4.10 The main theorem 62
4.11 Corollary of the main theorem 64

BIBLIOGRAPHY **66**

CHAPTER 1

Introduction

Here and henceforth, φ will denote a non-constant analytic function which maps the unit disk $\mathbf{D}$ into itself. The induced *composition operator* C_φ is defined for each $f \in H(\mathbf{D})$ by

$$C_\varphi(f) = f \circ \varphi.$$

Thus C_φ is linear and has range in $H(\mathbf{D})$. In this thesis, the primary concern is on those composition operators which are continuous on the Dirichlet space. The Dirichlet space, denoted by $\mathcal{D}$, consists of all $f \in H(\mathbf{D})$ for which

$$\int_{\mathbf{D}} |f'|^2 \, dA < \infty,$$

where A is the Lebesgue area measure. $\mathcal{D}$ is a Hilbert space with inner product defined for f and g in $\mathcal{D}$ by

$$< f, g > \doteq f(0)\overline{g(0)} + \frac{1}{\pi} \int_{\mathbf{D}} f' \, \overline{g'} \, dA,$$

and the induced norm

$$\|f\|_{\mathcal{D}}^2 \doteq | f(0) |^2 + \frac{1}{\pi} \int_{\mathbf{D}} | f' |^2 \, dA.$$

If f is univalent, then $\int_{\mathbf{D}} |f'|^2\, dA$ is precisely the area of $f(\mathbf{D})$. In general, $\int_{\mathbf{D}} |f'|^2\, dA$ still yields the area of the image of f on $\mathbf{D}$ if one takes multiplicities into account. This area interpretation of the $\mathcal{D}$-norm offers a constructive way to view the space. In Lemma 1.1 below, we prove the well known relation:

$$\frac{1}{\pi}\int_{\mathbf{D}} |f'|^2\, dA = \sum_{n=1}^{\infty} n \mid \hat{f}(n) \mid^2,$$

where $\hat{f}(n)$ denotes the n^{th} Taylor coefficient of f. This provides an alternative formula for the norm:

$$\|f\|_{\mathcal{D}}^2 = \mid f(0) \mid^2 + \sum_{n=1}^{\infty} n \mid \hat{f}(n) \mid^2 .$$

Contrasting the formulae for $\| \ \|_{\mathcal{D}}$ and $\| \ \|_2$, one might expect greater regularity of the functions in $\mathcal{D}$ than of the functions in the Hardy space H^2. This does turn out to be the situation, and it is reflected in the theorems of Chapter 4. Briefly stated, capacity tends to play the role in $\mathcal{D}$ (and the $\mathcal{D}_\alpha$ spaces defined below) that Lebesgue measure plays in H^2.

The $\mathcal{D}_\alpha$ space, $\alpha \in (0,1)$, consists of all $f \in H^1(\mathbf{D})$ for which

$$\sum_{n=1}^{\infty} n^\alpha \mid \hat{f}(n) \mid^2 < \infty.$$

It is normed by

$$\|f\|_{\mathcal{D}_\alpha}^2 = \mid f(0) \mid^2 + \sum_{n=1}^{\infty} n^\alpha \mid \hat{f}(n) \mid^2 .$$

The $\mathcal{D}_\alpha$ function spaces are "larger" than the Dirichlet space, "smaller" than H^2, and tend to have "intermediate" regularity.

1.1 Lemma.

If $f \in H(\mathbf{D})$, then

$$\frac{1}{\pi}\int_{\mathbf{D}} |f'|^2 dA = \sum_{n=1}^{\infty} n \mid \hat{f}(n) \mid^2 .$$

In particular, f is a member of $\mathcal{D}$ if and only if either side, and hence each side, of the equation is finite.

PROOF.

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbf{D}} \mid f' \mid^2 \, dA &= \frac{1}{\pi}\int_{\mathbf{D}} \mid \sum_{n=1}^{\infty} n\hat{f}(n) z^{n-1} \mid^2 \, dA(z) \\
&= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \mid \sum_{n=1}^{\infty} n\hat{f}(n) r^{n-1} e^{(n-1)\theta} \mid^2 r \, d\theta \, dr \\
&= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \mid \sum_{n=1}^{\infty} n\hat{f}(n) r^{n-\frac{1}{2}} e^{(n-1)\theta} \mid^2 \, d\theta \, dr \\
&= \frac{1}{\pi}\int_0^1 2\pi \sum_{n=1}^{\infty} \mid n\hat{f}(n) r^{n-\frac{1}{2}} \mid^2 \, dr \\
&= 2\sum_{n=1}^{\infty} n^2 \mid \hat{f}(n) \mid^2 \int_0^1 r^{2n-1} \, dr \\
&= 2\sum_{n=1}^{\infty} n^2 \mid \hat{f}(n) \mid^2 \frac{1}{2n} \\
&= \sum_{n=1}^{\infty} n \mid \hat{f}(n) \mid^2 .
\end{aligned}$$

///

In his pioneering 1968 paper which examined composition operators and inner functions [7], Nordgren determined the spectrum of C_φ as an operator on the Hardy space H^2 when φ is an automorphism of $\mathbf{D}$. Cowen has proven many elegant spectral theorems, mostly for H^2 and larger spaces (see [2]). Here, we are interested in determining the spectrum of the composition operators C_φ on $\mathcal{D}$ which are induced by a linear fractional map φ. In Chapter 2 we define $\mathcal{D}_0$ to be $\mathcal{D}$ modulo the constant functions, and we show that the (induced) operator C_φ is unitary on $\mathcal{D}_0$ when φ is an automorphism. This suggests the space $\mathcal{D}_0$ as a good starting point, and in Chapter 3 we determine the spectrum of the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ when φ is a linear fractional transformation. We conclude that chapter by observing that the operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ has essentially the same spectrum as the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$. The only difference is that the point 1 is not automatically a spectral value in the $\mathcal{D}_0$ case—since the constant functions are identified with the zero element of $\mathcal{D}_0$. The eigenfunctions of an operator C_φ on H^2 are often rather abundant, however, just as often they fail to lie in the Dirichlet space. It is due to this, mainly, that the proofs given here are distinguished from those of the corresponding H^2 results. Again, spectral results concerning H^2 appear in Cowen's work [2] (see also MacCluer's and Cowen's book [1, Chapter 7]). In particular, we embrace Cowen's use of a semigroup in case φ is a parabolic non-automorphism (the idea for which he attributes to R. P. Kaufman [2]), as well as his resourceful application of the invariance of the reproducing kernels under C_φ^*.

In Chapter 4 the main theorem, Theorem 4.10, is a generalization of the following well-known result ([9, p.32]).

If $C_\varphi : H^2 \to H^2$ is compact, then the Lebesgue measure of the set $\{e^{it} : |\varphi(e^{it})| = 1\}$ is zero.

Theorem 4.10 shows that if C_φ is *Hilbert-Schmidt* on $\mathcal{D}$ or $\mathcal{D}_\alpha$, then the *capacity* of the set upon which φ has unit modulus ($\{e^{it} : |\varphi(e^{it})| = 1\}$) is zero.

CHAPTER 2

The Spaces $\mathcal{D}_0$ and $\mathcal{D}_\pi$

Let $\mathcal{C}$ denote the class of constant functions in $\mathcal{D}$. Let $\mathcal{D}_0$ denote the Hilbert space $\mathcal{D}/\mathcal{C}$ with the norm and inner product that it inherits from $\mathcal{D}$. That is,

$$\|[f]\|^2_{\mathcal{D}_0} \doteq \frac{1}{\pi}\int_{\mathbf{D}} \mid f' \mid^2 \, dA \quad \text{for } f \in [f] \in \mathcal{D}_0,$$

and

$$< [f], [g] >_{\mathcal{D}_0} \doteq \frac{1}{\pi}\int_{\mathbf{D}} f' \, \overline{g'} \, dA \quad \text{for } f \in [f] \in \mathcal{D}_0, \;\; g \in [g] \in \mathcal{D}_0.$$

These definitions do not depend on the representatives chosen and are thus well-defined.

Let φ be an analytic self-map of the unit disk for which $C_\varphi : \mathcal{D} \to \mathcal{D}$ is continuous (or equivalently, by the Closed Graph Theorem, merely well-defined as a mapping). For any representatives f and g of $[f] \in \mathcal{D}_0$, we have

$$f \circ \varphi - g \circ \varphi \in \mathcal{C}.$$

This shows that the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ induced by $C_\varphi : \mathcal{D} \to \mathcal{D}$ is well-defined.

The following theorem is a simple consequence of how the $\mathcal{D}_0$ norm neatly translates composition by φ into a change of variables.

2.1 Theorem.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is an automorphism, then $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is an isometric isomorphism (i.e. C_φ is unitary).

PROOF.

For any $f \in \mathcal{D}_0$,

$$\begin{aligned}
\|C_\varphi(f)\|^2_{\mathcal{D}_0} &= \frac{1}{\pi}\int_{\mathbf{D}} |\,(f\circ\varphi)'\,|^2 \, dA \\
&= \frac{1}{\pi}\int_{\mathbf{D}} |\, f'(\varphi(z))\,|^2 \,|\,\varphi'(z)\,|^2 \, dA(z) \\
&= \frac{1}{\pi}\int_{\varphi(\mathbf{D})} |\, f'(z)\,|^2 \, dA(z) \\
&= \frac{1}{\pi}\int_{\mathbf{D}} |\, f'(z)\,|^2 \, dA(z) \\
&= \|f\|^2_{\mathcal{D}_0}.
\end{aligned}$$

Moreover, φ^{-1} is also a disk automorphism and $(C_\varphi)^{-1} = C_{\varphi^{-1}}$. ///

Similar reasoning shows, more generally, that $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ satisfies $\|C_\varphi\| \leq \sqrt{n}$ whenever $\varphi : \mathbf{D} \to \mathbf{D}$ is at most n-valent.

Let $\mathcal{D}_\pi$ denote the space of equivalence classes of analytic functions, defined on the upper half-plane $\mathbf{\Pi}^+$, which is analogous to $\mathcal{D}/\mathcal{C}$. More precisely,

$$[F] \doteq \{F(z) + c \in H(\mathbf{\Pi}^+) : c \in \mathbf{C}\} \in \mathcal{D}_\pi$$

if

$$\|[F]\|^2_\pi \doteq \frac{1}{\pi}\int_{\mathbf{\Pi}^+} |\, F'\,|^2 \, dA < \infty.$$

The situation here is the same as on the disk—some analytic functions $\psi : \mathbf{\Pi}^+ \to \mathbf{\Pi}^+$ induce well-defined operators $C_\psi : \mathcal{D}_\pi \to \mathcal{D}_\pi$. We will see that many of the composition operators $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ we consider are similar to simpler composition operators on $\mathcal{D}_\pi$. In the sequel, we will only consider composition operators C_ψ on $\mathcal{D}_\pi$ where $\psi : \mathbf{\Pi}^+ \to \mathbf{\Pi}^+$ is a translation or multiplication by a positive scalar. In these cases, it is very easy to see that $C_\psi : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is a bounded operator. To simplify notation in the sequel, we will identify any member of $\mathcal{D}_0$ or $\mathcal{D}_\pi$ with any (and often a particular one) of its representatives. The statement and proof of the following lemma illustrates this usage.

2.2 Lemma.

For $w \in \mathbf{D}$, the functions

$$K_w(z) \doteq \log\left(\frac{1}{1-\overline{w}z}\right) = \sum_{n=1}^{\infty} \frac{\overline{w}^n}{n} z^n \qquad (z \in \mathbf{D})$$

are reproducing kernels for $\mathcal{D}_0$.

PROOF.

Let $w \in \mathbf{D}$. Then

$$\sum_{n=1}^{\infty} n|\hat{K}_w(n)|^2 = \sum_{n=1}^{\infty} n \left|\frac{\overline{w}^n}{n}\right|^2 = \sum_{n=1}^{\infty} \frac{|w|^{2n}}{n} = \log\frac{1}{1-|w|^2},$$

so $K_w \in \mathcal{D}_0$. Choose the representative f of $[f] \in \mathcal{D}_0$ with $f(0) = 0$. Then

$$< f, K_w >_{\mathcal{D}_0} = \sum_{n=1}^{\infty} n\, \hat{f}(n)\, \overline{\hat{K}_w(n)} = \sum_{n=1}^{\infty} n\, \hat{f}(n) \overline{\left(\frac{\overline{w}^n}{n}\right)} = \sum_{n=1}^{\infty} \hat{f}(n)\, w^n,$$

so $< f, K_w >_{\mathcal{D}_0} = f(w)$. ///

CHAPTER 3

The Spectra of Composition Operators on $\mathcal{D}_0$ Induced by Linear Fractional Transformations

In this chapter, we shall determine the spectrum of each composition operator C_φ on $\mathcal{D}_0$ induced by a linear fractional map φ which takes the unit disk into itself. The Remark at the end of this chapter shows that the spectrum of a composition operator $C_\varphi : \mathcal{D} \rightarrow \mathcal{D}$ is essentially the same as that of $C_\varphi : \mathcal{D}_0 \rightarrow \mathcal{D}_0$. The only difference is that the point 1 is not automatically a spectral value in the $\mathcal{D}_0$ case—since the constant functions are identified with the zero element of $\mathcal{D}_0$. For determining the spectrum of composition operators on $\mathcal{D}$, this is an important reduction. We presume that the reader is familiar with the following ideas:

- The elliptic maps are those which are similar to a rotation of the disk.
- The parabolic maps are those which are similar to a translation in a half plane.
- The hyperbolic maps are those which are similar to a positive dilation in a disk or a half plane.

We furnish some explicit examples of these mappings below. See [9], for instance, for more detailed information on the fundamental characteristics of the linear fractional transformations.

Some Sample Linear Fractional Transformations of D

Define μ by $\mu(z) \doteq \frac{i(1+z)}{1-z}$. μ is a linear fractional transformation which maps $\mathbf{D}$ onto the upper half plane. It has inverse $\mu^{-1}(w) = \frac{w-i}{w+i}$. We use the formulae for μ and μ^{-1} below.

- $\varphi(z) \doteq iz$, a rotation, is an elliptic automorphism of $\mathbf{D}$.
- $\varphi(z) \doteq \mu^{-1}(\mu(z)+1) = \frac{(1-2i)z-1}{z-(1+2i)}$ is a parabolic automorphism of $\mathbf{D}$.
- $\varphi(z) \doteq \mu^{-1}((.5)\,\mu(z)) = \frac{3z-1}{3-z}$ is a hyperbolic automorphism of $\mathbf{D}$.
- $\varphi(z) \doteq \mu^{-1}(\mu(z)+i) = \frac{1+z}{3-z}$ is a parabolic non-automorphism of $\mathbf{D}$.
- $\varphi(z) \doteq \frac{1+z}{2}$ is a hyperbolic non-automorphism of $\mathbf{D}$ having fixed point $1 \in \partial\mathbf{D}$ and no fixed point in $\mathbf{D}$.
- $\varphi(z) \doteq \frac{z}{2-z}$ is a hyperbolic non-automorphism of $\mathbf{D}$ having fixed point $1 \in \partial\mathbf{D}$ and fixed point $0 \in \mathbf{D}$.
- $\varphi(z) \doteq \frac{z}{2}$ is a hyperbolic non-automorphism of $\mathbf{D}$ having fixed point $0 \in \mathbf{D}$ and no fixed point in $\partial\mathbf{D}$. $\varphi(z) \doteq \frac{iz}{2}$ is loxodromic.

The theorems of this chapter yield the spectrum of the operator C_φ whenever φ is a linear fractional transformation.

3.1 Theorem.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is an elliptic automorphism, then the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ has spectrum equal to the closure of the set $\{\varphi'(a)^n : n = 1, 2, 3, \ldots\}$, where a is the point of $\mathbf{D}$ fixed by φ.

Remark.

This shows that the spectrum of C_φ is either the entire unit circle $\mathbf{T}$ or the set of k^{th} roots of unity, for some integer k.

PROOF.

By Theorem 2.1, C_φ is unitary and so $\sigma(C_\varphi) \subseteq \mathbf{T}$. There is a linear fractional map μ taking $\mathbf{D}$ onto $\mathbf{D}$ and a number λ, of modulus one, so that $\varphi = \mu^{-1}(\lambda\,\mu)$. The relation $\mu \circ \varphi = \lambda\,\mu$ easily implies that $\lambda = \varphi'(a)$, where a is the interior fixed point of φ. We have

$$C_\varphi = C_\mu \circ C_{\lambda z} \circ C_{\mu^{-1}} = C_\mu \circ C_{\lambda z} \circ (C_\mu)^{-1},$$

and it follows that $\sigma(C_\varphi) = \sigma(C_{\lambda z})$. Therefore to determine $\sigma(C_\varphi)$, we may as well assume that φ is the map

$$\varphi(z) = \lambda z \quad (z \in \mathbf{D}).$$

Then $C_\varphi(z^n) = \lambda^n z^n$ for each $n \in \mathbf{N}$. Hence

$$E \doteq \overline{\{\lambda^n : \; n = 1, 2, 3, \ldots\}} \subseteq \sigma(C_\varphi),$$

since the spectrum itself is closed. It is only left to show that there are no points besides those of E lying in the spectrum. If $E \neq \mathbf{T}$, then the following claim addresses this point.

Claim: $(C_\varphi - \xi I) : \mathcal{D}_0 \to \mathcal{D}_0$ is invertible for every $\xi \in \mathbf{T} \backslash E$.

Let $\xi \in \mathbf{T} \backslash E$ and set $d = \text{dist}\,(\xi, E)$. Then $\delta > 0$. It suffices to show that $C_\varphi - \xi I$ is both surjective and bounded from below on $\mathcal{D}_0$. Let $f \in \mathcal{D}_0$. Define

$$g(z) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{\lambda^n - \xi} z^n \qquad (z \in \mathbf{D}).$$

Then

$$| \hat{g}(n) | = \left| \frac{\hat{f}(n)}{\lambda^n - \xi} \right| \leq \frac{| \hat{f}(n) |}{d} \tag{3.1}$$

for each n, and so Lemma 1.1 shows that $g \in \mathcal{D}_0$. For each $n \in \mathbf{N}$,

$$\begin{aligned} ((C_\varphi - \xi I)g)\hat{}(n) &= C_\varphi(g)\hat{}(n) - \xi \hat{g}(n) \\ &= (g(\lambda z))\hat{}(n) - \xi \hat{g}(n) \\ &= \lambda^n \hat{g}(n) - \xi \hat{g}(n) \\ &= (\lambda^n - \xi) \frac{\hat{f}(n)}{\lambda^n - \xi} \\ &= \hat{f}(n). \end{aligned}$$

Hence $(C_\varphi - \xi I)g = f$, and along with (3.1), this shows that $C_\varphi - \xi I$ is surjective and bounded from below. ///

We regard $H^2(\mathbf{\Pi}^+)$ as the set of all $f \in H(\mathbf{\Pi}^+)$ for which

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} | f(x + iy) |^2 \, dx < \infty.$$

Our principle interest in $H^2(\mathbf{\Pi}^+)$ rests in Lemma 3.2 below, which allows us use Fourier analysis to study of some of the functions in $\mathcal{D}_\pi$. By the Paley-Wiener theorem, every $f \in H^2(\mathbf{\Pi}^+)$ can be expressed in the form

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t) e^{itz}\, dt \qquad (z \in \mathbf{\Pi}^+) \tag{3.2}$$

for some $g \in L^2([0, \infty))$ (see [8, p.372]). We define the $L^2(\mathbf{R})$ norm by the formula

$$\|g\|_2^2 \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty |g(t)|^2\, dt.$$

When f is defined by (3.2) on $\mathbf{R}$, Plancherel's theorem shows that $\hat{f} = g$ (in $L^2(\mathbf{R})$). In this case, $\hat{f}(x) = 0$ for *a.e.* $x < 0$.

3.2 Lemma.

For $f \in H^2(\mathbf{\Pi}^+)$,

$$\int_{\mathbf{\Pi}^+} |\, f' \,|^2 \, \frac{dA}{\pi} = \frac{1}{2\pi} \int_0^\infty t \,|\, \hat{f}(t) \,|^2 \, dt$$

In particular, such a function f *is a member of* $\mathcal{D}_\pi$ *if and only if either side, and hence each side, of the equation is finite.*

PROOF.

Let $f \in H^2(\mathbf{\Pi}^+)$. Then $\hat{f} \in L^2$ and by our remarks above,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t) e^{itz}\, dt \qquad (z \in \mathbf{\Pi}^+).$$

Then for each $y > 0$,

$$f'(z) = \frac{\partial}{\partial x} f(x + iy) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t) ite^{-yt} e^{ixt}\, dt.$$

By Plancherel's theorem, for each $y > 0$,

$$\int_{-\infty}^{\infty} | f'(x+iy) |^2 \, dx = \int_{-\infty}^{\infty} | \hat{f}(t) i t e^{-yt} |^2 \, dt.$$

Then since $\hat{f}(x) = 0$ for *a.e.* $x < 0$,

$$\begin{aligned}
\int_{\Pi^+} | f' |^2 \, dA &= \int_0^{\infty} \int_{-\infty}^{\infty} | f'(x+iy) |^2 \, dx \, dy \\
&= \int_0^{\infty} \int_{-\infty}^{\infty} | \hat{f}(t) i t e^{-yt} |^2 \, dt \, dy \\
&= \int_0^{\infty} | \hat{f}(t) |^2 \, t^2 \int_0^{\infty} e^{-2yt} \, dy \, dt \\
&= \int_0^{\infty} | \hat{f}(t) |^2 \, t^2 \frac{1}{2t} \, dt \\
&= \frac{1}{2} \int_0^{\infty} | \hat{f}(t) |^2 \, t \, dt.
\end{aligned}$$

Division by π gives the desired result. ///

Remark.

The relationship

$$\int_{\Pi^+} | f' |^2 \, \frac{dA}{\pi} = \frac{1}{2\pi} \int_0^{\infty} t \, | \hat{f}(t) |^2 \, dt,$$

which we have shown to be valid for all $f \in H^2(\Pi^+) \cap \mathcal{D}_\pi$, holds more generally. To each $F \in \mathcal{D}_\pi$, there corresponds a function S in $L^2([0,\infty), t \, \frac{dt}{2\pi})$ with

$$\|F\|^2_{\mathcal{D}_\pi} = \frac{1}{2\pi} \int_0^{\infty} t \, | S(t) |^2 \, dt.$$

Moreover, this correspondence is surjective as well as isometric. We omit the proof since we do not have a current need for this generalization. The relatively simple proof will be included in an article which is in preparation for publication.

3.3 Theorem.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is a parabolic automorphism, then the spectrum of the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is $\mathbf{T}$.

PROOF.

By Theorem 2.1, $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is unitary and so $\sigma(C_\varphi) \subseteq \mathbf{T}$. It remains only to prove the other inclusion.

There is a linear fractional map μ taking $\mathbf{D}$ onto $\mathbf{\Pi}^+$ and a real number α so that $\varphi = \mu^{-1} \circ \tau \circ \mu$, where $\tau(w) \doteq w - \alpha$. Note that the operators $C_{\mu^{-1}} : \mathcal{D}_0 \to \mathcal{D}_\pi$ and $C_\mu : \mathcal{D}_\pi \to \mathcal{D}_0$ are unitary. Moreover,

$$C_\varphi = C_\mu \circ C_\tau \circ C_{\mu^{-1}} = C_\mu \circ C_\tau \circ (C_\mu^{-1})$$

and it follows that $\sigma(C_\varphi) = \sigma(C_\tau)$. Therefore the proof of the theorem will be complete upon establishing that $\mathbf{T} \subseteq \sigma(C_\tau)$. Fix any point $e^{i\beta} \in \mathbf{T}$, where $\beta \in \mathbf{R}$. We will show that $(C_\tau - e^{i\beta}) : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is not bounded from below. Choose $k \in \mathbf{R}$ so that

$$2\pi k\alpha = 0 \ (mod\ 2\pi) \ \text{ and } \ 2\pi k \geq \beta/\alpha.$$

For $1 > c > 0$, set

$$[a, b] = [-\beta/\alpha + 2\pi k, -\beta/\alpha + 2\pi k + c] \subset [0, \infty)$$

(although c will be used as an indexing parameter in this proof, we will refrain from subscripting a and b). Define $F_c \in H^2(\mathbf{\Pi}^+)$ by

$$F_c(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \chi_{[a,b]}(t)\, e^{izt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{izt}\, dt.$$

Then we have $\hat{F}_c = \chi_{[a,b]}$.

Claim: $F_c \in \mathcal{D}_\pi$ and $\|F_c\|_\pi^2 = (C_1\, c + c^2)/(4\pi)$ for a constant C_1 which does not depend on c.

$F_c \in H^2(\mathbf{\Pi}^+)$ since $\hat{F}_c \in L^2([0,\infty))$ ([8, p. 372]). Thus application of Lemma 3.2 shows that

$$\begin{aligned}
\|\hat{F}\|_\pi^2 &= \frac{1}{2\pi} \int_0^\infty \mid \hat{F}_c(t) \mid^2 \; t\, dt \\
&= \frac{1}{2\pi} \int_a^b t\, dt \\
&= (b^2 - a^2)/(4\pi) \\
&= [(-\beta/\alpha + 2\pi k + c)^2 - (-\beta/\alpha + 2\pi k)^2]/(4\pi) \\
&= (2(-\beta/\alpha + 2\pi k)c + c^2)/(4\pi) \\
&= (C_1\, c + c^2)/(4\pi).
\end{aligned}$$

For $F \in H^2(\mathbf{\Pi}^+)$,

$$(C_\tau(F(x)))\hat{}\,(t) = (F(x - \alpha))\hat{}\,(t) = e^{-i\alpha t}\hat{F}(t),$$

and from this we see that

$$((C_\tau - e^{i\beta})F_c)\hat{}\,(t) \quad = (e^{-i\alpha t} - e^{i\beta})\hat{F}_c(t)$$

$$= (e^{-i\alpha t} - e^{i\beta})\chi_{[a,b]}(t).$$

As $(C_\tau - e^{i\beta})F_c \in H^2(\Pi^+)$, Lemma 3.2 implies that

$$\begin{aligned} \|(C_\tau - e^{i\beta})F_c\|_\pi^2 &= \frac{1}{2\pi}\int_0^\infty \mid (e^{-i\alpha t} - e^{i\beta})\chi_{[a,b]}(t) \mid^2 \, t\,dt \\ &= \frac{1}{2\pi}\int_a^b \mid e^{-i\alpha t} - e^{i\beta} \mid^2 \, t\,dt \\ &= \frac{1}{2\pi}\int_0^{b-a} \mid e^{-i\alpha(t+a)} - e^{i\beta} \mid^2 \, (t+a)\,dt \\ &= \frac{1}{2\pi}\int_0^c \mid e^{-i\alpha(t-\beta/\alpha+2\pi k)} - e^{i\beta} \mid^2 \, (t+a)\,dt \qquad (3.3) \end{aligned}$$

since $b - a = c$. As $2\pi k\alpha = 0$ $(mod\ 2\pi)$, the quantity in (3.3) equals

$$\frac{1}{2\pi}\int_0^c \mid e^{-i\alpha t} - 1 \mid^2 \, (t+a)\,dt. \qquad (3.4)$$

For small values of c, the factor in the integrand, $\mid e^{-i\alpha t} - 1 \mid$, satisfies

$$\mid e^{-i\alpha t} - 1 \mid \asymp t.$$

Hence, there exists $\delta > 0$ and a constant C_2 so that

$$\mid e^{-i\alpha t} - 1 \mid^2 \, (t+a) \le C_2\, t^2$$

when $t \in [0,\delta]$. This shows that for each $c \in (0,\delta)$, the quantity in (3.4) does not exceed $C_2\, c^3$. Therefore, when $c < \delta$,

$$\|(C_\tau - e^{i\beta})F_c\|_\pi^2 \le C_2\, c^3.$$

By this result and the Claim, for $c < \delta$, we have

$$\frac{\|(C_\tau - e^{i\beta})F_c\|_\pi^2}{\|F_c\|_\pi^2} \leq \frac{C_2\, c^3}{(C_1\, c + c^2)/(4\pi)};$$

and the right hand side tends to 0 as $c \to 0$. Thus the operator $(C_\tau - e^{i\beta}) : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is not bounded from below, and so $e^{i\beta} \in \sigma(C_\tau)$. By the freedom with which we chose β, it follows that $\mathbf{T} \subseteq \sigma(C_\tau)$. ///

3.4 Theorem.

If $C_\varphi : \mathbf{D} \to \mathbf{D}$ is a hyperbolic automorphism, then the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ has spectrum $\mathbf{T}$.

PROOF.

By Theorem 2.1, $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is unitary and so $\sigma(C_\tau) \subseteq \mathbf{T}$. It therefore suffices to establish the reverse inclusion. There is a linear fractional map μ taking $\mathbf{D}$ onto $\mathbf{\Pi}^+$ and a positive number $\lambda, \lambda \neq 1$, so that $\varphi = \mu^{-1} \circ \tau \circ \mu$, where $\tau(w) \doteq \lambda w$. As in the previous theorem, by similarity, we have

$$\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) = \sigma(C_\tau : \mathcal{D}_\pi \to \mathcal{D}_\pi).$$

Fix $e^{i\beta} \in \mathbf{T}$ for any real number β. We will show that the operator $(C_\tau - e^{i\beta} I) : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is not bounded from below. Define

$$g(t) = \frac{1}{t} e^{-i\beta \log_\lambda t} \qquad (t > 0).$$

Claim 1: $\frac{1}{\lambda}\, g(t/\lambda) = e^{i\beta} g(t)$ for all $t > 0$.

For any $t > 0$,

$$\begin{aligned}\frac{1}{\lambda}\, g(t/\lambda) &= \frac{1}{\lambda}(t/\lambda)^{-1} e^{-i\beta \log_\lambda (t/\lambda)} \\ &= \frac{1}{t} e^{-i\beta((\log_\lambda t)-1)} \\ &= \frac{1}{t} e^{i\beta} e^{-i\beta \log_\lambda t} \\ &= e^{i\beta} g(t).\end{aligned}$$

For each value of c with $0 < c < k \doteq \min(\lambda, 1/\lambda)$ (which ensures that $[c,1] \cap [\lambda c, \lambda] = \emptyset$), define $F_c \in H^2(\mathbf{\Pi}^+)$ by

$$F_c(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t)\, \chi_{[c,1]}(t)\, e^{izt}\, dt \qquad (z \in \mathbf{\Pi}^+).$$

Then we have $\hat{F}_c = g\, \chi_{[c,1]}$. Denote by C_τ the composition operator on $H(\mathbf{\Pi}^+)$ induced by τ.

Claim 2: $C_\tau(F_c) \in H^2(\mathbf{\Pi}^+)$.

For any $y > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} \mid [C_\tau(F_c)](x+iy) \mid^2 \, dx &= \int_{-\infty}^{\infty} \mid F_c(\lambda x + i\lambda y) \mid^2 \, dx \\ &= \frac{1}{\lambda} \int_{-\infty}^{\infty} \mid F_c(x + i\lambda y) \mid^2 \, dx. \qquad (3.5)\end{aligned}$$

Since $F_c \in H^2(\mathbf{\Pi}^+)$, the quantity in (3.5) is bounded by a constant which does not depend on the value of y ($y > 0$). This proves Claim 2.

Observe that

$$\begin{aligned}
(C_\tau(F_c) - e^{i\beta}F_c)\hat{}(t) &= (F_c(\lambda x))\hat{}(t) - e^{i\beta}\hat{F}_c(t) \\
&= \frac{1}{\lambda}\hat{F}_c(t/\lambda) - e^{i\beta}\hat{F}_c(t) \\
&= \frac{1}{\lambda}g(t/\lambda)\chi_{[c,1]}(t/\lambda) - e^{i\beta}g(t)\chi_{[c,1]}(t) \\
&= \frac{1}{\lambda}g(t/\lambda)\chi_{[c\lambda,\lambda]}(t) - e^{i\beta}g(t)\chi_{[c,1]}(t) \\
&= e^{i\beta}g(t)\chi_{[c\lambda,\lambda]}(t) - e^{i\beta}g(t)\chi_{[c,1]}(t)
\end{aligned}$$

by Claim 1. This function certainly vanishes for $t \in [c\lambda, \lambda] \cap [c, 1]$. Moreover, if $\lambda > 1$ then it has support given by the union of the disjoint intervals $[c, c\lambda] \cup [1, \lambda]$. If $\lambda < 1$ then it has support given by the union of the disjoint intervals $[c\lambda, c] \cup [\lambda, 1]$. Using Claim 2, we that that $C_\tau(F_c) - e^{i\beta}F_c \in H^2(\mathbf{\Pi}^+)$, so by Lemma 3.2

$$\|(C_\tau - e^{i\beta})F_c\|_\pi^2 = \frac{1}{2\pi}\int_0^\infty |\ ((C_\tau - e^{i\beta})F_c)\hat{}(t)\ |^2\ t\,dt. \tag{3.6}$$

As $|\ g(t)\ | = \frac{1}{t}$ for $t > 0$, by the observations above, the quantity in (3.6) equals

$$\frac{1}{2\pi}\left|\int_c^{c\lambda}\frac{1}{t}\,dt\right| + \frac{1}{2\pi}\left|\int_1^{\lambda}\frac{1}{t}\,dt\right| = |\ \ln\lambda\ |/\pi.$$

Hence for all $c \in (0, k)$,

$$\|(C_\tau - e^{i\beta})F_c\|_\pi^2 = |\ \ln\lambda\ |/\pi.$$

However, for each $c \in (0, k)$,

$$\begin{aligned}
\|F_c\|_\pi^2 &= \frac{1}{2\pi}\int_0^1 |\ \hat{F}_c(t)\ |^2\ t\,dt \\
&= \frac{1}{2\pi}\int_0^\infty |\ g(t)\chi_{[c,1]}(t)\ |^2\ t\,dt
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi}\int_c^1 \mid g(t) \mid^2 t\,dt \\ &= \frac{1}{2\pi}\int_c^1 \mid \frac{1}{t} \mid^2 t\,dt \\ &= -\frac{1}{2\pi}\ln c. \end{aligned}$$

Therefore, for $c \in (0, k)$

$$\frac{\|(C_\tau - e^{i\beta})F_c\|_\pi^2}{\|F_c\|_\pi^2} = \frac{-2 \mid \ln \lambda \mid}{\ln c},$$

and the right hand side tends to 0 as $c \rightarrow 0$. Thus, the operator $(C_\tau - e^{i\beta} I) : \mathcal{D}_\pi \rightarrow \mathcal{D}_\pi$ is not bounded from below, and so we obtain

$$\mathbf{T} \subseteq \sigma(C_\varphi : \mathcal{D}_0 \rightarrow \mathcal{D}_0).$$

The reverse inclusion was established earlier. ///

Towards The Case Where φ Is A Parabolic Non-automorphism

In [2] (see also [1, Chapter 7]) Cowen constructs, rather generally, a *holomorphic semigroup of operators* $\{C_t\}_{t\in\Lambda}$ on H^2 and exhibits a spiral-like set or segment E_t for which $\sigma(C_t) \subseteq E_t$. Following his method, we construct a holomorphic semigroup of operators on $\mathcal{D}_0$ which we utilize in an analogous way in Theorem 3.7. That theorem deals with the case where φ is a parabolic non-automorphism. The following lemma, which will be used in the construction of the semigroup, follows from a slightly more general version which appears in [4].

3.5 Lemma.

Let $f \in H(\mathbf{\Pi}^+)$ and let $S \subset \mathbf{\Pi}^+$ be a compact subset. Then there exists a number M satisfying

$$\left| \frac{1}{\alpha - \beta} \left[\frac{1}{\alpha} (f(\zeta + \alpha) - f(\zeta)) - \frac{1}{\beta} (f(\zeta + \beta) - f(\zeta)) \right] \right| \leq M$$

whenever three different numbers ζ, $\zeta + \alpha$, and $\zeta + \beta$ lie in S.

PROOF.

Let C be a closed path in $\mathbf{\Pi}^+$, having index 1 on S, which satisfies dist$(C^*, S) > 0$. By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\tau)\, d\tau}{\tau - z} \qquad (z \in S).$$

Using this representation for each of the four occurrences of f in the expression we are taking the absolute value of, the expression becomes

$$\frac{1}{2\pi i} \int_C \frac{f(\tau)\, d\tau}{(\tau - (\zeta + \alpha))(\tau - (\zeta + \beta))}.$$

Observe that the modulus of the integrand above, and hence the integral, is uniformly bounded for all choices of ζ, α, and β satisfying the hypothesis. This proves the lemma. ///

Let $\mu : \mathbf{D} \rightarrow \mathbf{\Pi}^+$ be an analytic, bijective mapping. For $w \in \mathbf{\Pi}^+$, define $\varphi_w : \mathbf{D} \rightarrow \mathbf{D}$ by

$$\varphi_w(z) = \mu^{-1}(\mu(z) + w).$$

Write C_w for C_{φ_w}, the composition operator on $\mathcal{D}_0$ induced by φ_w.

3.6 Lemma.

$\{C_w\}_{w \in \Pi^+}$ is a holomorphic semigroup of operators on $\mathcal{D}_0$. This means that :

(a) $C_{w_1} C_{w_2} = C_{w_1+w_2} \qquad (w_1, w_2 \in \Pi^+)$.

(b) *$w \mapsto C_w$ is a continuous map into the space of operators on $\mathcal{D}_0$* $\quad (w \in \Pi^+)$.

(c) *For any $\Lambda \in B(\mathcal{D}_0, \mathcal{D}_0)^*$, the function $w \mapsto \Lambda(C_w)$ lies in $H(\Pi^+)$.*

PROOF.

It is trivial to verify that (a) holds, so we prove (b) and (c).

Claim: For each f and g in $\mathcal{D}_0$, $< C_w f, g >$ is an analytic function of w, $(w \in \Pi^+)$.

Let f and g be given. Denoting the reproducing kernel at the point $p \neq 0$ by K_p (see Lemma 2.2), we see that

$$\begin{aligned} < C_w f, K_p > &= [C_w f](p) \\ &= f(\varphi_w(p)) \\ &= f(\mu^{-1}(\mu(p) + w)). \end{aligned}$$

Therefore $< C_w f, K_p >$ is an analytic function of w, since f and σ^{-1} are analytic functions. As the linear span of the set $\{K_p : p \in \mathbf{D} \backslash \{0\} \}$ is dense in $\mathcal{D}_0$, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in this linear span with $g_n \to g$ in $\mathcal{D}_0$. The observations above then imply that $< C_w f, g_n >$ is analytic in $w \in \Pi^+$, for each $n \in \mathbf{N}$. As $g_n \to g$, there exists a constant M, with $M \geq \|g_n\|_{\mathcal{D}_0}$ for all n. Then by the Cauchy-Schwartz inequality,

$$\begin{aligned} |<C_w f, g_n>| &\leq \|C_w f\|_{\mathcal{D}_0} \|g_n\|_{\mathcal{D}_0} \\ &\leq M\|f\|_{\mathcal{D}_0} \end{aligned}$$

and so $\{<C_w f, g_n>\}_{n=1}^{\infty}$ is a normal family. One easily shows that it has $<C_w f, g>$ as a limit point, in the topology of uniform convergence, proving the Claim.

Fix $\zeta \in \mathbf{\Pi}^+$, and choose r so that $B(\zeta, r) \subset \mathbf{\Pi}^+$. For every α and β in $B(0,r)\backslash\{0\}$ with $\alpha \neq \beta$, define the operator $U(\alpha,\beta) : \mathcal{D}_0 \rightarrow \mathcal{D}_0$ by

$$U(\alpha,\beta) = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha}(C_{\zeta+\alpha} - C_\zeta) - \frac{1}{\beta}(C_{\zeta+\beta} - C_\zeta)\right].$$

Let $f, g \in \mathcal{D}_0$. By the Claim,

$$h(w) \doteq <C_w f, g> \in H(\mathbf{\Pi}^+).$$

Thus by Lemma 3.5, there exists M such that for any α and β as above

$$M \geq \left|\frac{1}{\alpha-\beta}\left[\frac{1}{\alpha}(h(\zeta+\alpha) - h(\zeta)) - \frac{1}{\beta}(h(\zeta+\beta) - h(\zeta))\right]\right|. \qquad (3.7)$$

As $h(w) = <C_w f, g>$, (3.7) may be written

$$M \geq |<U(\alpha,\beta)f, g>|.$$

By applying the Uniform Boundedness Principle, twice, there exists a constant M_2 satisfying

$$\|U(\alpha,\beta)\| \leq M_2$$

for all α and β as above. Equivalently,

$$\|\frac{1}{\alpha}(C_{\zeta+\alpha} - C_\zeta) - \frac{1}{\beta}(C_{\zeta+\beta} - C_\zeta)\| \leq M_2 \mid \alpha - \beta \mid . \tag{3.8}$$

Define $\Gamma(\gamma) = \frac{1}{\gamma}(C_{\zeta+\gamma} - C_\zeta)$ for $\gamma \in B(0,r)\backslash\{0\}$. Inequality (3.8) shows that $\Gamma(\gamma)$ is uniformly Cauchy, in the operator norm, as $\gamma \to 0$ in $\mathbf{C}$. Therefore the following limit exists:

$$C'_\zeta \doteq \lim_{\gamma \to 0} \Gamma(\gamma) = \lim_{\gamma \to 0} \frac{1}{\gamma}(C_{\zeta+\gamma} - C_\zeta).$$

This shows (b), that the mapping $w \mapsto C_w$ is continuous at ζ (hence on $\mathbf{\Pi}^+$). This also implies (c), for let $\Lambda \in B(\mathcal{D}_0, \mathcal{D}_0)^*$. Then

$$\begin{aligned} \lim_{h \to 0} \frac{\Lambda(C_{\zeta+h}) - \Lambda(C_\zeta)}{h} &= \lim_{h \to 0} \Lambda\left(\frac{C_{\zeta+h} - C_\zeta}{h}\right) \\ &= \Lambda\left(\lim_{h \to 0} \frac{C_{\zeta+h} - C_\zeta}{h}\right) \\ &= \Lambda(C'_\zeta) \end{aligned}$$

and so $w \mapsto \Lambda(C_w)$ is analytic at $w = \zeta$, hence on $\mathbf{\Pi}^+$. ///

Following Cowen's work on H^2, we prove the following theorem.

3.7 Theorem.

Let $\varphi : \mathbf{D} \to \mathbf{D}$ be a parabolic non-automorphism. Then the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ has spectrum

$$E \doteq \{e^{iw_0 t} : t \in [0, \infty)\} \cup \{0\}$$

for some number $w_0 \in \mathbf{\Pi}^+$.

PROOF.

There is a linear fractional map μ taking $\mathbf{D}$ onto $\mathbf{\Pi}^+$ and a non-real number $w_0 = x_0 + i\,y_0 \in \mathbf{\Pi}^+$ so that $\varphi = \mu^{-1} \circ \tau \circ \mu$, where $\tau(w) \doteq w + w_0$.

Claim: $\sigma(C_\varphi) \subseteq E$.

For $w \in \mathbf{\Pi}^+$, define $\varphi_w : \mathbf{D} \to \mathbf{D}$ by

$$\varphi_w(z) = \mu^{-1}(\mu(z) + w) \qquad (z \in \mathbf{D}).$$

Write C_w for C_{φ_w}, the composition operator on $\mathcal{D}_0$ induced by φ_w. Then $\varphi = \varphi_{w_0}$. By Lemma 3.6, $\{C_w\}_{w \in \mathbf{\Pi}^+}$ is a holomorphic semigroup of operators. Let $\mathcal{A}$ be the norm-closed algebra of operators generated by

$$\{I\} \cup \bigcup_{w \in \mathbf{\Pi}^+} C_w.$$

As $\mathcal{A}$ is a commutative Banach algebra with identity, we know that ([8, Theorem 18.17])

$$\sigma_{\mathcal{A}}(C_w) = \{\Lambda(C_w) : \ \Lambda \text{ is a multiplicative linear functional on } \mathcal{A}\}, \qquad (3.9)$$

where $\sigma_{\mathcal{A}}(C_w)$ denotes the spectrum of C_w with respect to invertibility in $\mathcal{A}$. That is, $\beta \in \sigma_{\mathcal{A}}(C_w)$ if $(C_w - \beta I)$ has no inverse contained in the set $\mathcal{A}$. Let Λ be a multiplicative linear functional on $\mathcal{A}$. Define the function λ by $\lambda(w) = \Lambda(C_w)$ for $w \in \mathbf{\Pi}^+$. $\lambda \in H(\mathbf{\Pi}^+)$ since $\{C_w\}_{w \in \mathbf{\Pi}^+}$ is a holomorphic semigroup. Since Λ is multiplicative,

$||\Lambda|| = 1$ and for all $w_1,\ w_2 \in \mathbf{\Pi}^+$,

$$\begin{aligned} \lambda(w_1 + w_2) &= \Lambda(C_{w_1+w_2}) \\ &= \Lambda(C_{w_1} \circ C_{w_2}) \\ &= \Lambda(C_{w_1})\, \Lambda(C_{w_1}) \\ &= \lambda(w_1)\, \lambda(w_2). \end{aligned}$$

Therefore

$$\lambda \equiv 0 \ \text{ or } \ \lambda(w) = e^{\beta w} \tag{3.10}$$

for some $\beta \in \mathbf{C}$. In the latter case, we have

$$\begin{aligned} |\, e^{\beta w}\, | &= \lim_{n\to\infty} |\, e^{\beta n w}\, |^{\frac{1}{n}} \\ &= \lim_{n\to\infty} |\, \lambda(w)^n\, |^{\frac{1}{n}} \\ &= \lim_{n\to\infty} |\, \Lambda(C_w)^n\, |^{\frac{1}{n}} \\ &= \lim_{n\to\infty} |\, \Lambda(C_w^n)\, |^{\frac{1}{n}}\ . \end{aligned}$$

Therefore, since $||\Lambda|| = 1$,

$$|\, e^{\beta w}\, | \leq \lim_{n\to\infty} ||C_w^n||^{\frac{1}{n}}\ .$$

The right hand side is of course a familiar formula for the spectral radius of C_w, and so we obtain

$$|\, e^{\beta w}\, | \leq 1 \quad (w \in \mathbf{\Pi}^+). \tag{3.11}$$

This implies that $\beta \in \{it\ :\ t \in [0,\infty)\}$, and so by (3.9) and (3.10),

$$\sigma_{\mathcal{A}}(C_w) \subseteq \{e^{iwt} : \ t \in [0,\infty)\} \ \cup\ \{0\}. \tag{3.12}$$

If $C_w - \lambda I$ has no inverse in $B(\mathcal{D}_0, \mathcal{D}_0)$, then it also fails to have an inverse in the smaller class $\mathcal{A}$; hence

$$\sigma(C_w) \subseteq \sigma_{\mathcal{A}}(C_w). \tag{3.13}$$

Since $\varphi(z) = \varphi_{w_0}(z)$, by (3.12) and (3.13),

$$\sigma(C_\varphi) \subseteq \{e^{iw_0 t} : t \in [0, \infty)\} \cup \{0\} = E.$$

Claim: $E \subseteq \sigma(C_\varphi)$.

As $C_{\mu^{-1}} : \mathcal{D}_0 \to \mathcal{D}_\pi$ and $C_\mu : \mathcal{D}_\pi \to \mathcal{D}_0$ are unitary, and since

$$C_\varphi = C_\mu \circ C_\tau \circ C_{\mu^{-1}} = C_\mu \circ C_\tau \circ C_\mu^{-1},$$

it follows that $\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) = \sigma(C_\tau : \mathcal{D}_\pi \to \mathcal{D}_\pi)$. It therefore suffices to show that $E \subseteq \sigma(C_\tau)$. Let $\lambda = e^{iw_0 t_0} \in E$, for any $t_0 \in [0, \infty)$. For $c > 0$, define $F_c \in H^2(\mathbf{\Pi}^+)$ by

$$F_c(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[t_0, t_0+c]}(t)\, e^{izt}\, dt \qquad (z \in \mathbf{\Pi}^+).$$

Then we have $\hat{F}_c = \chi_{[t_0, t_0+c]}$. By Plancherel's theorem,

$$\begin{aligned}
(C_\tau F_c)\hat{}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(x + w_0) e^{-isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}_c(t) e^{i(x+w_0)t}\, dt\, e^{-isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}_c(t) e^{iw_0 t} e^{ixt}\, dt\, e^{-isx}\, dx \\
&= \hat{F}_c(s) e^{iw_0 s} \qquad (m - a.e.).
\end{aligned}$$

Therefore,

$$\begin{aligned}(C_\tau F_c - \lambda F_c)\hat{}(s) &= (e^{iw_0 s} - e^{iw_0 t_0})\hat{F}_c(s) \\ &= (e^{iw_0 s} - e^{iw_0 t_0})\chi_{[t_0,t_0+c]}(s).\end{aligned}$$

As $C_\tau F_c - \lambda F_c \in H^2(\mathbf{\Pi}^+)$, by Lemma 3.2,

$$\begin{aligned}\|C_\tau F_c - \lambda F_c\|_\pi^2 &= \frac{1}{2\pi}\int_0^\infty |\, (C_\tau F_c - \lambda F_c)\hat{}(t)\, |^2\, t\, dt \\ &= \frac{1}{2\pi}\int_0^\infty |\, (e^{iw_0 t} - e^{iw_0 t_0})\chi_{[t_0,t_0+c]}(t)\, |^2\, t\, dt \\ &= \frac{1}{2\pi}\int_{t_0}^{t_0+c} |\, (e^{iw_0 t} - e^{iw_0 t_0}\, |^2\, t\, dt. \end{aligned} \tag{3.14}$$

Define $k(t) = e^{iw_0 t}$ for $t \in [t_0, t_0 + c]$. Then the quantity in (3.14) becomes

$$\frac{1}{2\pi}\int_{t_0}^{t_0+c} |\, k(t) - k(t_0)\, |^2\, t\, dt. \tag{3.15}$$

Note that $k(t) - k(t_0) \approx k'(t_0)(t - t_0)$ when t is near t_0. Therefore, there exists a constant K and $\delta > 0$ so that

$$|\, k(t) - k(t_0)\, |^2 \le K^2\, |\, t - t_0\, |$$

whenever $|\, t - t_0\, | < \delta$. Then for $c < \delta$, the quantity in (3.15) is at most

$$\begin{aligned}\frac{K^2}{2\pi}\int_{t_0}^{t_0+c}(t - t_0)^2 t\, dt &= \frac{K^2}{2\pi}\int_0^c t^2(t + t_0)\, dt \\ &\le \frac{K^2}{2\pi}(t_0 + c)\int_0^c t^2\, dt \\ &\le C_1 c^3\end{aligned}$$

for a constant C_1 independent of c $(0 < c < \delta)$. This shows that

$$\|C_\tau F_c - \lambda F_c\|_\pi^2 \leq C_1\, c^3$$

whenever $c < \delta$. On the other hand,

$$\begin{aligned}
\|F_c\|_\pi^2 &= \frac{1}{2\pi}\int_0^\infty |\, \hat{F}_c(t)\, |^2\, t\, dt \\
&= \frac{1}{2\pi}\int_{t_0}^{t_0+c} t\, dt \\
&= \frac{1}{4\pi}((t_0+c)^2 - t_0^2) \\
&= \frac{1}{4\pi}(2c\, t_0 + c^2) \\
&= C_2\, c + c^2/(4\pi)
\end{aligned}$$

for all $c > 0$. Therefore for all $c \in (0, \delta)$,

$$\frac{\|C_\tau F_c - \lambda F_c\|_\pi^2}{\|F_c\|_\pi^2} \leq \frac{C_1\, c^3}{C_2\, c + c^2/(4\pi)};$$

and the right hand side tends to 0 as $c \to 0$. Thus $C_\tau - \lambda I : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is not bounded from below. Therefore $\lambda \in \sigma(C_\tau)$, and hence

$$\{e^{iw_0 t} :\ t \in [0, \infty)\ \} \subseteq \sigma(C_\tau).$$

Since the spectrum is closed, this implies that $E \subseteq \sigma(C_\tau)$. This completes the proof of the theorem. ///

3.8 Theorem.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is hyperbolic with precisely one fixed point on $\partial\mathbf{D}$ and no interior fixed point, then $\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) = \overline{\mathbf{D}}$.

PROOF.

There exists a linear fractional map μ and a positive number λ, with $\lambda \neq 1$, so that $\varphi = \mu^{-1} \circ \tau \circ \mu$, where $\tau(z) \doteq \lambda z$. If $\lambda > 1$, define $\rho(z) = \frac{1}{\mu(z)}$; then $\rho^{-1}(w) = \mu^{-1}(1/w)$, and

$$\begin{aligned} \varphi(z) &= \mu^{-1}(\lambda\,\mu(z)) \\ &= \mu^{-1}\left(\frac{1}{\frac{1}{\lambda}\frac{1}{\mu(z)}}\right) \\ &= \rho^{-1}\left(\frac{1}{\lambda}\,\rho(z)\right). \end{aligned}$$

Therefore, we may further assume that $\lambda \in (0,1)$.

Claim: $\mu(\mathbf{D})$ is a circle with the point 0 on its boundary.

As μ is a linear fractional map, $\mu(\mathbf{D})$ is either a half-plane or a circle. Suppose first that $\mu(\mathbf{D})$ is a half-plane. Then there exists a point $c \in \partial\mathbf{D}$ at which μ is singular. Then $\varphi(c) = \mu^{-1} \circ \tau \circ \mu(c) = c$ and so c is the boundary fixed point of φ. If $0 \in \overline{\mu(\mathbf{D})}$, then $\mu^{-1}(0)$ is another fixed point of φ (contrary to our hypothesis). Thus $0 \notin \overline{\mu(\mathbf{D})}$. But then

$$\tau \circ \mu(\mathbf{D}) = \lambda\mu(\mathbf{D}) \not\subseteq \overline{\mu(\mathbf{D})},$$

and this implies that φ is not a self map of the disk. Therefore $\mu(\mathbf{D})$ must be a circle.

Reasoning as above, if $0 \in \mu(\mathbf{D})$ we obtain an interior fixed point for φ; if $0 \notin \overline{\mu(\mathbf{D})}$, then φ is not a self-map of the disk. Each of these conclusions is contrary to the hypothesis, and so $0 \in \partial(\mu(\mathbf{D}))$, completing the proof of the Claim.

Set $P = \mu(\mathbf{D})$, and denote by $\mathcal{D}_P$ the space of functions analytic on P which is analogous to $\mathcal{D}_\pi$, i.e.

$$\mathcal{D}_P \doteq \{[F] : F \in H(P), \; \|[F]\|_P^2 \doteq \frac{1}{\pi}\int_P |F'|^2 \, dA < \infty\}.$$

It is easy to see that $C_{\mu^{-1}} : \mathcal{D}_0 \to \mathcal{D}_P$ and $C_\mu : \mathcal{D}_P \to \mathcal{D}_0$ are unitary operators. Moreover, since

$$C_\varphi = C_\mu \circ C_\tau \circ C_{\mu^{-1}} = C_\mu \circ C_\tau \circ C_\mu^{-1},$$

$C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ and $C_\tau : \mathcal{D}_P \to \mathcal{D}_P$ share the same eigenvalues. As φ is univalent, $\|C_\varphi\| \leq 1$ and so $\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) \subseteq \overline{\mathbf{D}}$. We will show that each $\beta \in \mathbf{D}\backslash\{0\}$ is an eigenvalue of C_τ.

Fix $\beta \in \mathbf{D}\backslash\{0\}$. Define the function F_β on P by

$$F_\beta(z) = z^{\frac{\ln\beta}{\ln\lambda}}.$$

Writing $\beta = |\beta| e^{\theta_1}$, we have

$$\begin{aligned} F_\beta(z) &= e^{(\ln z)(\frac{\ln\beta}{\ln\lambda})} \\ &= e^{(\ln z)(\frac{\ln|\beta| + i\theta_1}{\ln\lambda})}. \end{aligned}$$

Since the logarithm is analytic on P, so is F_β. Setting $c_1 = \frac{\ln\beta}{\ln\lambda}$, we have

$$F_\beta'(z) = c_1 \, z^{c_1 - 1} = \frac{c_1}{z} \, e^{c_1 \ln z}.$$

For each $z = re^{i\theta} \in P$,

$$\begin{aligned} |F_\beta'(re^{i\theta})| &= |\frac{c_1}{r} e^{c_1(\ln r + i\theta)}| \\ &= |\frac{c_1}{r} e^{(\ln|\beta| + i\theta_1)(\ln r + i\theta)/\ln\lambda}| \end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{c_1}{r}\right| \left|e^{(\ln|\beta|)(\ln r)} e^{-\theta_1 \theta}\right|^{\frac{1}{\ln \lambda}} \\
&\leq \frac{C_2}{r} \left|e^{(\ln|\beta|)(\ln r)}\right|^{\frac{1}{\ln \lambda}}
\end{aligned}
$$

for some constant C_2 which does not depend on z ($z \in P$). Thus

$$
|F'_\beta(re^{i\theta})| \leq \frac{C_2}{r} r^{\frac{\ln|\beta|}{\ln \lambda}} = C_2 r^\alpha
$$

where $\alpha > -1$, since $\frac{\ln|\beta|}{\ln \lambda} > 0$. Choose R large enough so that $P \subset B(0, R)$. We have

$$
\begin{aligned}
\int_P |F'_\beta|^2 \, dA &\leq C_2^2 \int_P |z|^{2\alpha} \, dA(z) \\
&\leq C_2^2 \int_{B(0,R)} |z|^{2\alpha} \, dA(z) \\
&\leq C_2^2 \, 2\pi \int_0^R r^{2\alpha+1} \, dr.
\end{aligned}
$$

Since $2\alpha + 2 = 2(\alpha + 1) > 0$, the latter integral is finite and so $F_\beta \in \mathcal{D}_P$. For each $z \in P$,

$$
\begin{aligned}
C_\tau(F_\beta)(z) &= (\lambda z)^{\frac{\ln \beta}{\ln \lambda}} \\
&= \lambda^{\frac{\ln \beta}{\ln \lambda}} z^{\frac{\ln \beta}{\ln \lambda}} \\
&= e^{(\ln \lambda)(\ln|\beta| + i\theta_1)/(\ln \lambda)} F_\beta(z) \\
&= e^{\ln|\beta| + i\theta_1} F_\beta(z) \\
&= |\beta| \, e^{i\theta_1} F_\beta(z) \\
&= \beta \, F_\beta(z).
\end{aligned}
$$

Thus $C_\tau(F_\beta) = \beta F_\beta$, and so β is an eigenvalue of C_τ. Because of the freedom with which we chose β, every point in $\mathbf{D} \backslash \{0\}$ is an eigenvalue. We observed that $\sigma(C_\varphi) = \sigma(C_\tau) \subseteq \overline{\mathbf{D}}$. Since the spectrum is a closed set, we conclude that $\sigma(C_\varphi) = \overline{\mathbf{D}}$. ///

For the convenience of the reader, we state the following lemma, which can be found for instance in [1, p. 270]. It will be helpful in the proof of Theorem 3.10, and again at the end of this chapter.

3.9 Hilbert Space Lemma

Suppose H is a Hilbert space with $H = K \oplus L$, where K is finite dimensional, and C is a bounded operator on H that leaves K or L invariant. If C has the matrix representation

$$C = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

with respect to this decomposition, then

$$\sigma(C) = \sigma(X) \cup \sigma(Z).$$

For $m \geq 2$, let L_m denote the subspace of $\mathcal{D}_0$ consisting of all $f \in \mathcal{D}_0$ for which

$$\hat{f}(1) = \hat{f}(2) = \cdots = \hat{f}(m-1) = 0.$$

We define L_1 to be $\mathcal{D}_0$. The reproducing kernels for L_m, denoted $K_{w,m}$, are defined for each $w \in \mathbf{D}$ by

$$K_{w,m}(z) = \sum_{n=m}^{\infty} \frac{\overline{w}^n}{n} z^n \qquad (z \in \mathbf{D}).$$

Suppose that C_φ is a bounded composition operator on $\mathcal{D}_0$ and that $\varphi(0) = 0$. Then the restriction of C_φ to L_m has its range contained in L_m. Let C_m^* denote the adjoint of the operator C_φ on L_m. A routine argument shows that the family of reproducing kernels $\{K_{w,m} : w \in \mathbf{D}\}$ is invariant under C_m^* and that, in particular,

$$C_m^*(K_{w,m}) = K_{\varphi(w),m}. \tag{3.16}$$

L_m has finite codimension in $\mathcal{D}_0$, so application of Lemma 3.9 ensures that

$$\sigma(C_\varphi : L_m \to L_m) \subseteq \sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0). \tag{3.17}$$

Following Cowen's proof of Theorem 7.30 in [1, p. 289], wherein he makes effective use of the H^2 analogues of (3.16) and (3.17), we are able to prove the following theorem.

3.10 Theorem.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is a hyperbolic map with an interior fixed point (necessarily attractive) and a boundary fixed point, then the spectrum of the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is $\overline{\mathbf{D}}$.

PROOF.

WLOG, we may assume that φ fixes the points 0 and 1. Hence, by our hypothesis

$$0 < \varphi'(0) < 1 < \varphi'(1).$$

Throughout the proof, m and J will always denote positive integers. In accordance with the remarks preceding the theorem, let C_m^* denote the adjoint of the operator $C_\varphi : L_m \to L_m$. Fix $\lambda \in \mathbf{D} \backslash \{0\}$. To see that λ is contained in $\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0)$, by (3.17), it is sufficient to show for some value of m, that λ is contained in $\sigma(C_\varphi : L_m \to L_m)$. This is our underlying goal in the remainder.

Since φ is a linear fractional transformation which fixes 0 and 1, it follows that φ is a homeomorphism of the interval $[0,1]$. For any point $x \in (0,1)$, consider the sequence $\{x_n\}_{n=-\infty}^{\infty}$ consisting of the forward and backward φ-iterates of x, i.e. $\{x_n\}_{n=-\infty}^{\infty}$ is the uniquely determined such sequence having $x_0 = x$, whose elements satisfy the family of relations

$$x_{n+1} = \varphi(x_n) \qquad (n \in \mathbf{Z}). \tag{3.18}$$

Let us pause to outline the rest of the proof. A primary tool in our argument is, from (3.16), that C_m^* is a forward shift of the sequence $\{K_{x_n,m}\}_{n=-\infty}^{\infty}$. It is not difficult to check, formally, that

$$\sum_{n=-\infty}^{\infty} \overline{\lambda}^{-n} K_{x_n,m}$$

is an eigenfunction of C_m^* corresponding to $\overline{\lambda}$. We shall see that for m sufficiently large, this is a convergent series. It is necessary, however, that the series not be zero—if it is to be an eigenfunction. We show in Claim 2, non-trivially, that a sequence of partial sums of the series is bounded away from zero. This lower bound, of course, also applies to the limit. In this way we obtain an eigenfunction for C_m^* corresponding to $\overline{\lambda}$, implying that $\lambda \in \sigma(C_\varphi : L_m \to L_m)$.

The homeomorphism of the interval $[0,1]$ described above, along with the Schwarz Lemma, provides that

$$0 < x_{n+1} < x_n < 1 \qquad (n \in \mathbf{Z}).$$

Since φ has no fixed points in (0,1), this implies that

$$\lim_{n\to\infty} x_{-n} = 1.$$

Indeed, since

$$\lim_{n\to\infty} \frac{1 - x_{-n}}{1 - x_{-n-1}} = \lim_{n\to\infty} \frac{\varphi(1) - \varphi(x_{-n-1})}{1 - x_{-n-1}} = \varphi'(1)$$

and $\varphi'(1) > 1$, we have

$$\sum_{n=0}^{\infty}(1 - x_{-n}) < \infty. \tag{3.19}$$

This shows that the backward iterates of x_0 tend to 1 quickly enough to be the zeros of a Blaschke product. Let s be a number satisfying $0 < s < \frac{1}{\varphi'(1)}$. Then $s < 1$, and there exists a number a in the interval $(.5, 1)$ such that

$$\frac{1 - x}{1 - \varphi(x)} \geq s \quad \text{whenever } 1 > x \geq a. \tag{3.20}$$

We now fix the sequence $\{x_n\}_{n=-\infty}^{\infty}$ determined by $x_0 = a$ and the relations given in (3.18). For any value of J, the backward φ-iterates $x_{-1}, x_{-2}, \ldots, x_{-J}$ lie in $(a, 1)$, and so by (3.20)

$$\begin{aligned} 1 - (x_{-J})^2 > 1 - x_{-J} &= \frac{1 - x_{-J}}{1 - x_{-J+1}} \frac{1 - x_{-J+1}}{1 - x_{-J+2}} \cdots \frac{1 - x_{-1}}{1 - x_0}(1 - x_0) \\ &= \frac{1 - x_{-J}}{1 - \varphi(x_{-J})} \frac{1 - x_{-J+1}}{1 - \varphi(x_{-J+1})} \cdots \frac{1 - x_{-1}}{1 - \varphi(x_{-1})}(1 - x_0) \\ &\geq s^J(1 - x_0). \end{aligned}$$

This inequality provides the entire means for the following claim.

Claim 1: There are constants M and J_0 so that

$$\|K_{x_{-J}}\|_{\mathcal{D}_0} \leq M\sqrt{J} \text{ whenever } J \geq J_0.$$

Since

$$\begin{aligned} \|K_{x_{-J}}\|^2_{\mathcal{D}_0} &= \log \frac{1}{1-(x_{-J})^2} \\ &\leq \log \frac{1}{s^J(1-x_0)} \\ &= J\,|\log s| - \log(1-x_0), \end{aligned}$$

we have

$$\|K_{x_{-J}}\|_{\mathcal{D}_0} \leq \sqrt{J\,|\log s| - \log(1-x_0)}. \tag{3.21}$$

Claim 1 follows from (3.21).

A simple application of the Schwarz Lemma yields a constant c in (0,1) which satisfies the condition:

$$|\varphi(z)| \leq c|z| \text{ when } (|z| \leq .5). \tag{3.22}$$

Set

$$N = \min\{n : x_n \leq .5\}.$$

Then $x_N \leq .5$, and $N > 0$ since $x_0 > .5$. By (3.22),

$$x_{N+k} \leq c^k \cdot x_N \qquad \text{for all } k \geq 0. \tag{3.23}$$

Fix a positive integer m_0 which satisfies $\frac{c^{m_0}}{|\lambda|} \leq .5$. For m and J, with $m \geq m_0$, define the functions $F_{J,m}$ by

$$F_{J,m} = \sum_{n=-J}^{\infty} \overline{\lambda}^{-n} K_{x_n,m}. \tag{3.24}$$

We will now show that the functions $F_{J,m}$ lie in L_m.

It suffices to show, for each $m \geq m_0$, that

$$\sum_{n=N}^{\infty} |\lambda|^{-n} \, \|K_{x_n,m}\|_{\mathcal{D}_0} < \infty.$$

Fix $m \geq m_0$. For each $n \geq N$, we have

$$\begin{aligned}
|\lambda|^{-n} \, \|K_{x_n,m}\|_{\mathcal{D}_0} &= |\lambda|^{-n} \sqrt{\sum_{k=m}^{\infty} \frac{1}{k} \, |x_n|^{2k}} \\
&\leq |\lambda|^{-n} \sqrt{\sum_{k=m}^{\infty} \frac{1}{k} \, |c^{n-N} x_N|^{2k}} \\
&\leq |\lambda|^{-n} \, c^{(n-N)m} \sqrt{\sum_{k=m}^{\infty} \frac{1}{k} \, |x_N|^{2k}} \\
&= c^{-Nm} \left(\frac{c^m}{|\lambda|} \right)^n \, \|K_{x_N,m}\|_{\mathcal{D}_0} \\
&= \text{Const} \left(\frac{c^m}{|\lambda|} \right)^n \\
&\leq \text{Const} \, (.5)^n.
\end{aligned}$$

Therefore, for each $n \geq N$,

$$|\lambda|^{-n} \, \|K_{x_n,m}\|_{\mathcal{D}_0} \leq \text{Const} \, (.5)^n,$$

and so the series for $F_{J,m}$ converges in L_m.

Claim 2: For some integer m_1, greater than or equal to m_0, there is a constant $\delta > 0$ so that

$$\|F_{J,m_1}\|_{\mathcal{D}_0} \geq \delta \quad \text{for all } J > 0.$$

The proof of this claim is of some length, and for the reader's reference, we note that it will be completed at statement (3.34). By (3.19),

$$\sum_{\substack{k \leq N-1 \\ k \neq 0}} (1 - x_k) \; < \; \infty. \tag{3.25}$$

We define the function f in $H^\infty(\mathbf{D})$ by the formula

$$f(z) \; = \; (1-z)^2 \cdot \prod_{\substack{k \leq N-1 \\ k \neq 0}} \alpha_{x_k}(z) \qquad (z \in \mathbf{D}),$$

where α_{x_k} denotes the familiar automorphism of $\mathbf{D}$ which transposes 0 and x_k. Fundamental theory concerning Blaschke products provides that

$$f(x_k) = 0 \iff 0 \neq k \leq N-1. \tag{3.26}$$

Certainly $|f| \leq 4$, and since $\{x_k : \; 0 \neq k \leq N-1\} \subseteq (.5, 1)$, it follows that $|f'|$ is bounded (this is essentially problem #18 from [8, p. 318]). We will now prove this. Since the product defining f converges uniformly on compact subsets of $\mathbf{D}$, the product rule for differentiation shows that, for any $z \in \mathbf{D}$,

$$|f'(z)| \; \leq \; 2|1-z| + |1-z|^2 \sum_{\substack{k \leq N-1 \\ k \neq 0}} |\alpha'_{x_k}(z)|.$$

Hence,

$$|f'(z)| \; \leq \; 4 + \sum_{\substack{k \leq N-1 \\ k \neq 0}} \left| \frac{1-z}{1-x_k z} \right|^2 (1 - x_k^2). \tag{3.27}$$

Write z as $z = x + iy$. Observe that for k satisfying $0 \neq k \leq N-1$,

$$\left| \frac{1-z}{1-x_k z} \right|^2 \; \leq \; \frac{(1-x)^2 + y^2}{(1-x_k x)^2 + (x_k y)^2}$$

$$
\begin{aligned}
&\leq \frac{(1-x)^2}{(1-x_k x)^2} + \frac{1}{(x_k)^2} \\
&\leq \left(\frac{1-x}{1-x_k x}\right)^2 + 4 \\
&< \frac{16}{9} + 4 \\
&< 6.
\end{aligned}
$$

We used above the facts that for each such k, $x_k > .5$ and $\left(\frac{1-x}{1-x_k x}\right)$ is a decreasing function on the interval [-1,1]. From (3.27), we obtain

$$
|f'(z)| \leq 4 + 6 \sum_{\substack{k \leq N-1 \\ k \neq 0}} (1 - x_k^2), \tag{3.28}
$$

and by (3.25), the right hand side of (3.28) is finite. This establishes the existence of a number B_1 satisfying

$$
|f'(z)| \leq B_1 \qquad (z \in \mathbf{D}).
$$

Since $|f| \leq 4$ and $|f'| \leq B_1$ it follows, in a straight forward manner, that

$$
\|fK_{x_0,m}\|_{\mathcal{D}_0} \leq (4 + B_1)\|K_{x_0,m}\|_{\mathcal{D}_0} \leq (4 + B_1)\|K_{x_0}\|_{\mathcal{D}_0} \quad \text{for all } m.
$$

We abbreviate this:

$$
\|fK_{x_0,m}\|_{\mathcal{D}_0} \leq B_2 \qquad (m \geq 1)
$$

and observe, then, that $f\,K_{x_0,m} \in L_m$ since the appropriate Taylor coefficients vanish. Since $f\,K_{x_0,m} \in L_m$,

$$
\begin{aligned}
\|F_{J,m}\|_{\mathcal{D}_0} &\geq |< fK_{x_0,m}, F_{J,m} >| / \|fK_{x_0,m}\|_{\mathcal{D}_0} \\
&\geq |\sum_{n=-K}^{\infty} \lambda^{-n} < fK_{x_0,m}, K_{x_n,m} >| / B_2
\end{aligned}
$$

$$= \left| \sum_{n=-K}^{\infty} \lambda^{-n} f(x_n) K_{x_0,m}(x_n) \right| / B_2. \tag{3.29}$$

We notice that if λ were a positive number, then by using 1 in place of f, the proof of Claim 2 would be done at (3.29). Considering (3.26), (3.29) becomes

$$\|F_{J,m}\|_{\mathcal{D}_0} \geq \frac{1}{B_2} \left| f(x_0)\, K_{x_0,m}(x_0) + \sum_{n=N}^{\infty} \lambda^{-n} f(x_n) K_{x_0,m}(x_n) \right|$$

and since $|f| \leq 4$,

$$\|F_{J,m}\|_{\mathcal{D}_0} \geq \frac{1}{B_2} \left(|f(x_0)|\, \|K_{x_0,m}\|^2_{\mathcal{D}_0} - 4 \sum_{n=N}^{\infty} |\lambda|^{-n} K_{x_0,m}(x_n) \right). \tag{3.30}$$

Because of the infinite set of zeros were were able to prescribe for the function f, the right hand side of (3.30) is independent of J. To prove Claim 2, it suffices to find just one value of m for which the right hand side of inequality (3.30) is positive. Observe that

$$\begin{aligned}
\sum_{n=N}^{\infty} |\lambda|^{-n} K_{x_0,m}(x_n) &= \sum_{n=N}^{\infty} |\lambda|^{-n} \sum_{k=m}^{\infty} \frac{x_0^k\, x_n^k}{k} \\
&\leq \sum_{n=N}^{\infty} |\lambda|^{-n} \sum_{k=m}^{\infty} \frac{x_0^k\, (c^{n-N}\, x_N)^k}{k} \\
&= \sum_{n=N}^{\infty} |\lambda|^{-n} \sum_{k=m}^{\infty} \frac{(c^{n-N}\, x_0\, x_N)^k}{k} \\
&= \sum_{k=m}^{\infty} \frac{(c^{-N}\, x_0\, x_N)^k}{k} \sum_{n=N}^{\infty} \left(\frac{c^k}{|\lambda|} \right)^n \\
&= \sum_{k=m}^{\infty} \frac{(c^{-N}\, x_0\, x_N)^k}{k} \frac{(\frac{c^k}{|\lambda|})^N}{(1 - \frac{c^k}{|\lambda|})} \qquad \left(\text{since } \frac{c^m}{|\lambda|} < .5\right) \\
&= \frac{1}{|\lambda|^N} \sum_{k=m}^{\infty} \frac{(c^{-N}\, x_0\, x_N)^k}{k} \frac{c^{kN}}{(1 - \frac{c^k}{|\lambda|})} \\
&= \frac{1}{|\lambda|^N} \sum_{k=m}^{\infty} \frac{(x_0\, x_N)^k}{k} \frac{1}{(1 - \frac{c^k}{|\lambda|})} \\
&\leq \frac{2}{|\lambda|^N} \sum_{k=m}^{\infty} \frac{(x_0/2)^k}{k} \qquad (\text{since } x_N \leq .5).
\end{aligned}$$

That is,

$$\sum_{n=N}^{\infty} |\lambda|^{-n} K_{x_0,m}(x_n) \le C \sum_{k=m}^{\infty} \frac{(x_0/2)^k}{k},$$

where C is a constant independent of m. Employing this estimate in (3.30), we obtain

$$\begin{aligned} \|F_{J,m}\|_{\mathcal{D}_0} &\ge \frac{1}{B_2}\left(|f(x_0)|\, \|K_{x_0,m}\|^2_{\mathcal{D}_0} - 4C \sum_{k=m}^{\infty} \frac{(x_0/2)^k}{k} \right) \\ &= \frac{1}{B_2}\left(|f(x_0)| \sum_{k=m}^{\infty} \frac{|x_0|^{2k}}{k} - 4C \sum_{k=m}^{\infty} \frac{(x_0/2)^k}{k} \right) \\ &= \frac{1}{B_2} \sum_{k=m}^{\infty} \frac{|f(x_0)|\,(x_0^k)^2 - 4C\,(x_0)^k/2^k}{k} \\ &= \frac{1}{B_2} \sum_{k=m}^{\infty} \frac{(x_0)^k\,[\,|f(x_0)|\,(x_0)^k - 4C/2^k\,]}{k}. \end{aligned}$$

That is,

$$\|F_{J,m}\|_{\mathcal{D}_0} \ge \frac{1}{B_2} \sum_{k=m}^{\infty} \frac{(x_0)^k\,[\,|f(x_0)|\,(x_0)^k - 4C/2^k\,]}{k}. \tag{3.31}$$

The comments proceeding inequality (3.30) explain why it suffices to show that the series in (3.31) is positive for just a single value of m, $m \ge m_0$. For this end, it is enough to verify that the condition

$$|f(x_0)|\,(x_0)^k - 4C/2^k > 0 \tag{3.32}$$

holds for all k sufficiently large. Condition (3.32) is equivalent to

$$(2x_0)^k - \frac{4C}{|f(x_0)|} > 0, \tag{3.33}$$

and since $x_0 > .5$ we have $(2x_0) > 1$. Hence Claim 2 is proven: for some $\delta > 0$,

$$\|F_{J,m_1}\|_{\mathcal{D}_0} \ge \delta \qquad (J > 0). \tag{3.34}$$

By Claim 1,

$$\sum_{n=-\infty}^{-J_0} \|\overline{\lambda}^{-n} K_{x_n,m_1}\|_{L_{m_1}} \leq \sum_{n=J_0}^{\infty} |\lambda|^n M \sqrt{n}$$
$$\leq M \sum_{n=J_0}^{\infty} \sqrt{|\lambda|^n n} \left(\sqrt{|\lambda|}\right)^n$$
$$\leq \text{Const} \sum_{n=J_0}^{\infty} \left(\sqrt{|\lambda|}\right)^n$$
$$< \infty.$$

Therefore,

$$F_{m_1} \doteq \sum_{n=-\infty}^{\infty} \overline{\lambda}^{-n} K_{x_n,m_1} = \lim_{J\to\infty} F_{J,m_1}$$

is well defined in L_{m_1}. Furthermore, $\|F_{m_1}\| \geq \delta$, by Claim 2. Now we may readily complete the proof of the theorem. Using (3.16),

$$\begin{aligned} C^*_{m_1}(F_{m_1}) &= C^*_{m_1}\left(\sum_{n=-\infty}^{\infty} \overline{\lambda}^{-n} K_{x_n,m_1}\right) \\ &= \sum_{n=-\infty}^{\infty} \overline{\lambda}^{-n} K_{x_{n+1},m_1} \\ &= \sum_{n=-\infty}^{\infty} \overline{\lambda}^{-n+1} K_{x_n,m_1} \\ &= \overline{\lambda} F_{m_1}. \end{aligned}$$

Therefore $\lambda \in \sigma(C_\varphi : L_{m_1} \to L_{m_1})$. The remarks made at the beginning of the proof provide, then, that $\lambda \in \sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0)$. By the way λ was chosen, we have

$$\mathbf{D}\backslash\{0\} \subseteq \sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0). \tag{3.35}$$

Since $||C_\varphi|| \leq 1$ bounds the spectral radius, and since the spectrum is a closed set, (3.35) implies that

$$\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) = \overline{\mathbf{D}}.$$

///

3.11 Theorem.

If the operator $C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0$ is continuous and $(C_\varphi)^n$ is compact for some n, then φ has an attractive fixed point $a \in \mathbf{D}$ and

$$\sigma(C_\varphi) \subseteq \{\varphi'(a)^k : k = 1, 2, 3, \ldots\} \cup \{0\}.$$

PROOF.

Fix n so that $(C_\varphi)^n$ is compact. Suppose that φ fixes no point in $\mathbf{D}$. By *The Grand Iteration Theorem* ([9, p. 78]) φ, and consequently φ_n (the composition of φ with itself n times), has a fixed point in $\partial\mathbf{D}$ at which the angular derivative exists. Thus $(C_\varphi)^n : H^2 \to H^2$ is not compact (a contradiction since $(C_\varphi)^n : \mathcal{D}_0 \to \mathcal{D}_0$ is compact). So φ, and consequently φ_n, has a fixed point in $\mathbf{D}$. We shall denote it by a. φ is certainly a non-automorphism, and so $|\varphi'(a)| < 1$. By König's theorem ([9, p. 93]),

$$\begin{aligned}\sigma(C_{\varphi_n}) &\subseteq \{\varphi_n'(a)^k : k = 1, 2, 3, \ldots\} \cup \{0\} \\ &= \{\varphi'(a)^{nk} : k = 1, 2, 3, \ldots\} \cup \{0\}.\end{aligned}$$

As $C_{\varphi_n} = (C_\varphi)^n$, the Spectral Mapping Theorem then implies that

$$\sigma(C_\varphi)^n \subseteq \{\varphi'(a)^{nk} : k = 1, 2, 3, \ldots\} \cup \{0\}. \tag{3.36}$$

Set

$$A_m = \{\varphi'(a)^k \lambda : \lambda^m = 1;\ k = 1, 2, 3, \ldots\} \cup \{0\},$$

for $m = n$ and $m = n + 1$. By (3.36), $\sigma(C_\varphi) \subseteq A_n$. Since $(C_\varphi)^{n+1}$ is also compact on $\mathcal{D}_0$, the same reasoning shows that

$$\sigma(C_\varphi) \subseteq A_{n+1}.$$

Hence

$$\begin{aligned} \sigma(C_\varphi) &\subseteq A_n \cap A_{n+1} \\ &= \{\varphi'(a)^k : k = 1, 2, 3, \ldots\} \cup \{0\}, \end{aligned}$$

which is the desired conclusion. ///

Remark.

If $\varphi : \mathbf{D} \to \mathbf{D}$ is a hyperbolic map with no boundary fixed point, or is a loxodromic map, then

$$\sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) = \{\varphi'(a)^n : n = 1, 2, 3, \ldots\} \cup \{0\} \tag{3.37}$$

where a denotes the point of $\mathbf{D}$ fixed by φ. Theorem 3.11 shows that the left hand side of (3.37) is contained in the right hand side, and it is not difficult to show that each of the non-zero members of the right hand side is an eigenvalue of C_φ. The spectrum is a closed set, and so (3.37) follows.

Furnished below is a summary of the spectra of composition operators on $\mathcal{D}_0$

induced by the linear fractional transformations, which are self-maps of $\mathbf{D}$. Where a appears below, it denotes the point of $\mathbf{D}$ fixed by φ.

- If φ is a parabolic or hyperbolic automorphism, then $\sigma(C_\varphi) = \mathbf{T}$.
- If φ is an elliptic automorphism, then $\sigma(C_\varphi) = \overline{\{\varphi'(a)^n : n = 1, 2, 3, \ldots\}} \subseteq \mathbf{T}$.
- If φ is a parabolic non-automorphism, then $\sigma(C_\varphi) = \{e^{iwt} : t \in [0, \infty)\} \cup \{0\}$ for some point $w \in \mathbf{\Pi}^+$.
- If φ is a hyperbolic non-automorphism without a fixed point in $\mathbf{D}$, then $\sigma(C_\varphi) = \overline{\mathbf{D}}$.
- If φ is hyperbolic with an interior and a boundary fixed point, then $\sigma(C_\varphi) = \overline{\mathbf{D}}$.
- If φ is a hyperbolic with no boundary fixed point, or is a loxodromic map, then $\sigma(C_\varphi) = \{\varphi'(a)^n : n = 1, 2, 3, \ldots\} \cup \{0\}$.

Remark.

Since an operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ leaves the constant functions fixed, upon writing $\mathcal{D} = \mathbf{C} \oplus \mathcal{D}_0$, Lemma 3.9 shows that

$$\sigma(C_\varphi : \mathcal{D} \to \mathcal{D}) = \sigma(C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0) \cup \{1\}.$$

In particular, all of the results listed above hold for $C_\varphi : \mathcal{D} \to \mathcal{D}$, if one merely includes the point $\{1\}$ in the last result.

CHAPTER 4

Hilbert-Schmidt Composition Operators and Capacity

Let H denote a Hilbert space. A linear operator $T : H \rightarrow H$ is said to be *Hilbert-Schmidt* if

$$\sum_{n=1}^{\infty} \|T(e_n)\|_H^2 < \infty$$

for an (or equivalently, any) orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H.

4.1 Theorem.

([9, p. 25]) *If $T : H \rightarrow H$ is Hilbert-Schmidt, then T is a compact operator.*

Outline of proof.

For $n \in \mathbf{N}$, define T_n on H so that $T_n(f)$ is the projection of $T(f)$ into

$$LS(\{T(e_1), T(e_2), \ldots, T(e_n)\}).$$

Hölder's inequality shows that $T_n \rightarrow T$ as $n \rightarrow \infty$. Since each T_n is a finite rank operator, T is therefore compact. ///

Denoting $\mathcal{D}$ by $\mathcal{D}_1$ here, we that $C_\varphi : \mathcal{D}_\alpha \rightarrow \mathcal{D}_\alpha$ is Hilbert-Schmidt for $\alpha \in (0, 1]$

provided

$$\sum_{n=1}^{\infty} \|C_\varphi \left(\frac{z^n}{n^{\alpha/2}}\right)\|^2_{\mathcal{D}_\alpha} = \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2_{\mathcal{D}_\alpha}}{n^\alpha} < \infty.$$

Shapiro proved, in Propositon 2.4 of [10], the following statement for any self map φ of $\mathbf{D}$.

$$C_\varphi \textit{ is Hilbert-Schmidt on } \mathcal{D} \iff \int_{\mathbf{D}} \frac{|\,\varphi'\,|^2}{(1-|\varphi|^2)^2}\, dA < \infty.$$

From this result, we see that C_φ is Hilbert-Schmidt on $\mathcal{D}$ provided that the image of φ on $\mathbf{D}$ has finite *hyperbolic area* (counting multiplicities).

One of our principle interests here is in generalizing the following result whose proof is well-known ([9, p.32]).

If $C_\varphi : H^2 \to H^2$ is compact, then the Lebesgue measure of the set $\{e^{it} : |\varphi(e^{it})| = 1\}$ is zero.

For a self map φ of $\mathbf{D}$, set

$$E = E(\varphi) \doteq \{e^{it} : |\varphi(e^{it})| = 1\},$$

where the notation $\varphi(e^{it})$ refers to the radial limit, provided it exists, of φ at $e^{it} \in \mathbf{T}$. We show in Theorem 4.10 that if C_φ is Hilbert-Schmidt on $\mathcal{D}_\alpha$, then the capacity of E is zero. Here, the capacity function depends on the space. For example, when C_φ acts on the Dirichlet space, it follows that the (classical) *logarithmic capacity* of E is zero.

Kernels and Capacities

Here and in the sequel, we identify $t \in (0, 2\pi)$ with $e^{it} \in \mathbf{T}$. The *logarithmic kernel* and the *kernel of order* β $(0 < \beta < 1)$ are defined for $t \in (0, 2\pi)$ by

$$K_{log}(t) = \log\left(\frac{1}{|\sin\frac{t}{2}|}\right)$$

and

$$K_\beta(t) = \frac{1}{|\sin\frac{t}{2}|^\beta}$$

respectively. These functions are non-negative, even (when extended naturally), convex, and integrable on $(0, 2\pi)$. Furthermore (see [5, pp. 33,40]):

- $\hat{K}_{log}(n)$ and $\hat{K}_\beta(n)$ are positive for each $n \in \mathbf{Z}$.
- $\hat{K}_{log}(n) \asymp \frac{1}{n}$ as $n \to \infty$.
- $\hat{K}_\beta(n) \asymp \frac{1}{n^{1-\beta}}$ as $n \to \infty$.

Let K be one of the kernels above. Let $E \subseteq \mathbf{T}$ be a closed subset, and let $M^+(E)$ denote the class of positive measures supported by E. $L^2_+(\mathbf{T})$ will denote the subset of positive functions of $L^2(\mathbf{T})$. We define four different *capacities* of the set E with respect to the kernel K:

$$\begin{aligned}
c_{K,1}(E) &= \sup\{\|\mu\| : \mu \in M^+(E);\ \forall t \in E,\ K * \mu(t) \le 1\},\\
c_{K,2}(E) &= \sup\{\|\mu\| : \mu \in M^+(E);\ \|K * \mu\|_2^2 \le 1\},\\
C_{K,1}(E) &= \inf\{\|\mu\| : \mu \in M^+(E);\ \forall t \in E,\ K * \mu(t) \ge 1\},\\
C_{K,2}(E) &= \inf\{\|F\|_2^2 : F \in L^2_+(\mathbf{T});\ \forall t \in E,\ K * F(t) \ge 1\}.
\end{aligned}$$

If $E \subset \mathbf{T}$ is not closed, and C denotes one of the capacity functions above, define

$$C(E) = \sup_{F \subset E} C(F)$$

where the supremum is taken over all closed subsets F. By these definitions, each of these capacity functions is defined for every subset of $\mathbf{T}$ and is inner-regular. In each case, it is easy to see that

$$K_1 \leq K_2 \Rightarrow C_{K_1}(E) \geq C_{K_2}(E)$$

and

$$E_1 \subseteq E_2 \Rightarrow C_K(E_1) \leq C_K(E_2).$$

If $K = K_{log}$, we sometimes substitute "*log*" in place of "K" in the capacity notation. [5] is a good source of information on the capacity $c_{K,1}$ and its relationship with trigonometric series. [6] is a good source of information on capacity functions induced by potentials—including the ones which have been defined above.

In the sequel, some of our theorems will express results in terms of the *big-C* capacity functions $C_{K,2}$. The following theorem recognizes the equivalence of these capacities with the classical ones. By equivalence, we mean that they share the same null-sets.

4.2 Lemma (Capacity Equivalence).

For all subsets $E \subseteq \mathbf{T}$:

(a) $c_{\log,1}(E) = 0 \iff C_{K_{\frac{1}{2}},2}(E) = 0$

(b) $c_{K_{1-\alpha},1}(E) = 0 \iff C_{K_{1-\frac{\alpha}{2}},2}(E) = 0 \qquad (\alpha \in (0,1))$

The proof will use results from both [6] and [5]. The following identity is from [6, p. 273] and holds for all compact subsets $E \subseteq \mathbf{T}$:

$$c_{K,2}(E) = (C_{K,2}(E))^{\frac{1}{2}}. \tag{4.1}$$

Combining Theorems III and V from [5, pp. 37,40], we obtain the following lemma:

4.3 Lemma.

There exists $0 \neq \mu \in M^+(E)$ *satisfying*

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{|\hat{\mu}(n)|^2}{|n|} < \infty \quad \textit{or} \quad \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{|\hat{\mu}(n)|^2}{|n|^{1-\alpha}} < \infty$$

iff

$$c_{log,1}(E) > 0 \quad \textit{or} \quad c_{K_\alpha,1}(E) > 0, \ \textit{respectively.}$$

Proof of Lemma 4.2. For the purposes of this proof, define $K_0 = K_{log}$. Then statement (a) is statement (b) with $\alpha = 1$. Thus it suffices to prove (b) for arbitrary $\alpha \in (0, 1]$. Fix such a number α. Since these capacities are inner-regular, we may assume that $E \subseteq \mathbf{T}$ is a compact subset.

Claim 1: $C_{K_{1-\frac{\alpha}{2}},2}(E) > 0 \Rightarrow c_{K_{1-\alpha},1}(E) > 0.$

Suppose $C_{K_{1-\frac{\alpha}{2}},2}(E) > 0$. Then by (4.1), $c_{K_{1-\frac{\alpha}{2}},2}(E) > 0$. By the definition of $c_{K_{1-\frac{\alpha}{2}},2}$, there exists $0 \neq \mu \in M^+(E)$ satisfying

$$\|K_{1-\frac{\alpha}{2}} * \mu\|_2 < \infty.$$

Then we have

$$\begin{aligned}
\infty &> \|K_{1-\frac{\alpha}{2}} * \mu\|_2^2 \\
&= \frac{1}{2\pi}\int_{\mathbf{T}} \mid K_{1-\frac{\alpha}{2}} * \mu \mid^2 \, dm \\
&= \sum_{n=-\infty}^{\infty} \mid (K_{1-\frac{\alpha}{2}} * \mu)\hat{}(n) \mid^2 \\
&= \sum_{n=-\infty}^{\infty} \mid (K_{1-\frac{\alpha}{2}})\hat{}(n) \mid^2 \mid \hat{\mu}(n) \mid^2 .
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\infty &> \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left| \frac{1}{(|n|^{1-(1-\frac{\alpha}{2})})^2} \right| \mid \hat{\mu}(n) \mid^2 \\
&= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mid \hat{\mu}(n) \mid^2}{|n|^{\alpha}}.
\end{aligned}$$

Lemma 4.3 implies then that $c_{K_{1-\alpha},1}(E) > 0$, completing the proof of Claim 1.

Claim 2: $c_{K_{1-\alpha},1}(E) > 0 \Rightarrow C_{K_{1-\frac{\alpha}{2}},1}(E) > 0.$

Suppose $c_{K_{1-\alpha},1}(E) > 0$. Then by Lemma 4.3, there exists $0 \neq \mu \in M^+(E)$ satisfying

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mid \hat{\mu}(n) \mid^2}{\mid n \mid^{\alpha}} < \infty.$$

Hence

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mid \hat{\mu}(n) \mid^2}{|n^{\frac{\alpha}{2}}|^2} < \infty,$$

and this shows that $K_{1-\frac{\alpha}{2}} * \mu \in L^2(\mathbf{T})$. Considering the definition of $c_{K_{1-\frac{\alpha}{2}},2}$, this implies that $c_{K_{1-\frac{\alpha}{2}},2}(E) > 0$. Using the identity in (4.1), it follows that $C_{K_{1-\frac{\alpha}{2}},2}(E) > 0$. This completes the proof of Claim 2 which, along with Claim 1, completes the proof of the theorem. ///

By Lemma 4.2, if some property occurs *capacitarily almost everywhere* (i.e. except on a set of capacity zero) with respect to one of these capacities, then it occurs capacitarily almost everywhere with respect to the corresponding capacity (as indicated by Lemma 4.2). We frequently use the abbreviation

$$C_{K,j} - a.e.\ e^{it} \in \mathbf{T},$$

et. al., to mean capacitarily almost every member of $\mathbf{T}$ with respect to the capacity $C_{K,j}$ ($j = 1$ or 2).

4.4 Lemma (Weak Capacitary Inequality).

Let K be a kernel and $F \in L^2_+(\mathbf{T})$. For $a > 0$, set $E_a = \{e^{it} : K * F(e^{it}) \geq a\}$. Then

$$C_{K,2}(E_a) \leq \frac{\|F\|_2^2}{a^2}.$$

PROOF.

By definition, $C_{K,2}(E_a) = \inf\{\|F\|_2^2 : F \in L^2_+(\mathbf{T});\ \forall e^{it} \in E_a,\ K * F(e^{it}) \geq 1\}$. Therefore, since $K * \frac{F}{a}(e^{it}) \geq 1$ for each $e^{it} \in E_a$, we have

$$C_{K,2}(E_a) \leq \|F/a\|_2^2 = \frac{\|F\|_2^2}{a^2}.$$

///

Remark.

For $F \in L^2(\mathbf{T})$, $K * F$ is certainly defined pointwise wherever $K * |F|$ is finite. Lemma 4.4 then shows that $K * F(e^{it})$ is defined (and finite) for $C_{K,2} - a.e.\ e^{it} \in \mathbf{T}$.

4.5 Lemma.

Fix $\alpha \in (0,1]$ and define $K = K_{1-\frac{\alpha}{2}}$. Then for each $f \in \mathcal{D}_\alpha$ (where $\mathcal{D}_1 = \mathcal{D}$), there exists $F \in L^2(\mathbf{T})$ satisfying

$$f(z) = \sum_{n=0}^{\infty} (K * F)\hat{\ }(n)\, z^n \qquad (z \in \mathbf{D}); \tag{4.2}$$

moreover, for f and F associated in this way,

$$\|f\|^2_{\mathcal{D}_\alpha} \asymp \|F\|_2^2 \qquad (f \in \mathcal{D}_\alpha). \tag{4.3}$$

PROOF.

Recall that $\hat{K}(n) > 0$ for each $n \in \mathbf{Z}$, and that $\hat{K}(n) \asymp \frac{1}{n^{\frac{\alpha}{2}}}$ as $n \to \infty$. Let $f \in \mathcal{D}_\alpha$. Define a sequence $c = \{c_n\}_{n=-\infty}^{\infty}$ by

$$c_n = \frac{\hat{f}(n)}{\hat{K}(n)} \qquad (n \in \mathbf{Z}).$$

Then $|c_n|^2 \asymp n^\alpha \mid \hat{f}(n) \mid^2$ as $n \to \infty$. Since $f \in \mathcal{D}_\alpha$, it follows that $c \in l^2(n)$. By the Riesz-Fischer theorem, there exists $F \in L^2(\mathbf{T})$ with $\hat{F}(n) = c_n$ for all $n \in \mathbf{Z}$, and this gives (4.2). Since

$$n^\alpha (\hat{K}(n))^2 \asymp 1 \quad \text{for all } n \in \mathbf{Z} \setminus \{0\}$$

and

$$(\hat{K}(0))^2 \asymp 1 \quad \text{(trivially)},$$

the implicit pairs of constants associated with each of these statements can be chosen to be the same. Then for any $f \in \mathcal{D}_\alpha$, we have

$$\begin{aligned}
\|f\|^2_{\mathcal{D}_\alpha} &= |\ f(0)\ |^2 + \sum_{n=1}^{\infty} n^\alpha\ |\ \hat{f}(n)\ |^2 \\
&= |\ (K * F)\hat{\ }(0)\ |^2 + \sum_{n=1}^{\infty} n^\alpha\ |\ (K * F)\hat{\ }(n)\ |^2 \\
&= |\ \hat{K}(0)\ |^2 |\ \hat{F}(0)\ |^2 + \sum_{n=1}^{\infty} n^\alpha (\hat{K}(n))^2\ |\ \hat{F}(n)\ |^2 \\
&\asymp 1\ |\ \hat{F}(0)\ |^2 + \sum_{n=1}^{\infty} 1\ |\ \hat{F}(n)\ |^2 \\
&= \|F\|_2^2.
\end{aligned}$$

We note that the implicit constants associated with $\asymp$ here are the same as those we considered above (and are independent of $f \in \mathcal{D}_\alpha$). This yields (4.3). ///

The following two theorems, which are well known, help substantiate the statement make in the introduction that *capacity* tends to play the role in the Dirichlet and Dirichlet-type spaces that Lebesgue measure plays in H^2. They show that functions in these spaces have *boundary values* and *Lebesgue points* capacitarily almost everywhere.

4.6 Theorem.

Let $f \in \mathcal{D}_\alpha$ with $\alpha \in (0,1]$ (where $\mathcal{D}_1 = \mathcal{D}$). Set $K = K_{1-\frac{\alpha}{2}}$. Then the limit

$$f(e^{it}) \doteq \lim_{r\to 1} f(re^{it})$$

exists (and is finite) for $C_{K,2}$ – a.e. $e^{it} \in \mathbf{T}$.

Remark.

By Lemma 4.2, $C_{K,2}$ may be replaced above by $c_{log,1}$ if $f \in \mathcal{D}$, or by $c_{1-\alpha,1}$ if $f \in \mathcal{D}_\alpha$.

PROOF.

Fix f, K and α as in the statement of the theorem. By Lemma 4.5, there exists $F \in L^2(\mathbf{T})$ satisfying

$$f(z) = \sum_{n=0}^{\infty} (K * F)\hat{\ }(n)\, z^n \qquad (z \in \mathbf{D}).$$

By the Remark following Lemma 4.4, $K * F(e^{it})$ is defined for $C_{K,2}$ – *a.e.* $e^{it} \in \mathbf{T}$. Define Ω for all such points by

$$\Omega(e^{it}) = \limsup_{r\to 1^-} |f(re^{it}) - K * F(e^{it})|.$$

Hence

$$\Omega(e^{it}) = \limsup_{r\to 1^-} |P_r * K * F(e^{it}) - K * F(e^{it})|. \tag{4.4}$$

Let $\epsilon > 0$. For $h \in C(\mathbf{T})$, define

$$g = g(h) \doteq F - h \in L^2(\mathbf{T}).$$

Then $F = g + h$ and for all $e^{it} \in \mathbf{T}$,

$$P_r * K * h(e^{it}) \to K * h(e^{it}) \quad \text{as } r \to 1^- .$$

Therefore, (4.4) becomes

$$\begin{aligned} \Omega(e^{it}) &= \limsup_{r\to 1^-} |P_r * K * g(e^{it}) - K * g(e^{it})| \\ &\leq \limsup_{r\to 1^-} |P_r * K * g(e^{it})| + |K * g(e^{it})|. \end{aligned}$$

Observe that

$$\begin{aligned} P_r * K * g(e^{it}) &= K * P_r * g(e^{it}) \\ &= \frac{1}{2\pi} \int_0^{2\pi} K(t-\theta)\, P_r * g(e^{i\theta})\, d\theta, \end{aligned}$$

so

$$\begin{aligned} \limsup_{r\to 1^-} |P_r * K * g(e^{it})| &\leq \frac{1}{2\pi} \int_0^{2\pi} K(t-\theta)\, M_{rad}(P[g])(e^{i\theta})\, d\theta \\ &= K * M_{rad}(P[g])(e^{it}), \end{aligned}$$

where M_{rad} denotes the radial maximal function. Therefore,

$$\Omega(e^{it}) \leq K * M_{rad}(P[g])(e^{it}) + |K * g(e^{it})|.$$

We denote the Hardy-Littlewood maximal function by M_{HL}. By Theorem 11.20 of [8],

$$\Omega(e^{it}) \leq K * M_{HL}(g)(e^{it}) + |K * g(e^{it})|,$$

and this easily implies that

$$\Omega(e^{it}) \leq 2\, K * M_{HL}(g)(e^{it}). \tag{4.5}$$

By inequality (4.5),

$$\{e^{it} : \Omega(e^{it}) > \epsilon\} \subseteq \{e^{it} : K * M_{HL}(g)(e^{it}) > \epsilon/2\}.$$

Hence by Lemma 4.4,

$$\begin{aligned} C_{K,2}(\{e^{it} : \Omega(e^{it}) > \epsilon\}) &\leq C_{K,2}(\{e^{it} : K * M_{HL}(g)(e^{it}) > \epsilon/2\}) \\ &\leq (2/\epsilon)^2 \, \|M_{HL}(g)\|_2^2 \\ &\leq (2/\epsilon)^2 \, \|M_{HL}\|^2 \, \|g\|_2^2. \end{aligned} \tag{4.6}$$

Recall that $g = F - h$ where h was as arbitrary continuous function on $\mathbf{T}$. Since the continuous functions are dense in $L^2(\mathbf{T})$, we may choose $g = g(h)$ and h so that $\|g\|_2^2$ is as small as we please. Therefore, the inequality above may be improved:

$$C_{K,2}(\{e^{it} : \Omega(e^{it}) > \epsilon\}) = 0. \tag{4.7}$$

Note that (4.7) holds for each $\epsilon > 0$. Using the σ-subadditivity of $C_{K,2}$, we obtain

$$C_{K,2}(\{e^{it} : \Omega(e^{it}) > 0\}) = 0.$$

Considering the definition of Ω, this completes the proof of the theorem. ///

4.7 Corollary of the proof.

For f, K and F as in Lemma 4.5,

$$f(e^{it}) = K * F(e^{it}) \quad \text{for } C_{K,2} - a.e.\ e^{it} \in \mathbf{T}.$$

4.8 Theorem.

Let $f \in \mathcal{D}_\alpha$ with $\alpha \in (0,1]$ (where $\mathcal{D}_1 = \mathcal{D}$). Set $K = K_{1-\frac{\alpha}{2}}$. Then $C_{K,2} - a.e.\ e^{it} \in \mathbf{T}$ is a Lebesgue point of f.

The proof of this theorem is analogous to that of Theorem 4.6.

Outline of proof. We identify t with $e^{it} \in \mathbf{T}$. Let f, g and h be as in the proof of Theorem 4.6. Then $K * F = f$ ($C_{K,2} - a.e.$) and $K * h$ is continuous. Define Ω by

$$\Omega(e^{it}) = \limsup_{r\to 0} \frac{1}{2r} \int_{t-r}^{t+r} |f(\theta) - f(t)|\, d\theta$$

for each t where $f(t)$ is defined ($C_{K,2} - a.e.$ by Theorem 4.6). Hence for $C_{K,2} - a.e.\ e^{it} \in \mathbf{T}$,

$$\Omega(e^{it}) = \limsup_{r\to 0} \frac{1}{2r} \int_{t-r}^{t+r} |K * g(\theta) - K * g(t)|\, d\theta \;\leq\; M_{HL}(K * g)(t) + |K * g(t)|.$$

Note that

$$\begin{aligned} M_{HL}(K * g)(t) &= \sup_{r>0} \left(\frac{1}{2r} \int_{t-r}^{t+r} |K * g(y)|\, dy \right) \\ &\leq \sup_{r>0} \left(\frac{1}{2r} \int_{t-r}^{t+r} \frac{1}{2\pi} \int_{-\pi}^{\pi} K(s)\, |g(y-s)|\, ds\, dy \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K(s)\left(\sup_{r>0}\frac{1}{2r}\int_{t-r}^{t+r}|g(y-s)|\,dy\right)ds \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K(s)\,M_{HL}(g)(t-s)\,ds \\
&= K * M_{HL}(g)(t).
\end{aligned}$$

Hence we see that for $C_{K,2} - a.e.\ e^{it} \in \mathbf{T}$,

$$\begin{aligned}
\Omega(e^{it}) &\leq K * M_{HL}(g)(t) + |K * g(t)| \\
&\leq 2\,K * M_{HL}(g)(t).
\end{aligned}$$

The remainder of the proof is the same as that of Theorem 4.6. ///

4.9 Lemma.

If $\alpha < 2$ and h is the function defined by

$$h(x) = \sum_{n=1}^{\infty} n^{1-\alpha}\, x^n,$$

then

$$h(x) \asymp \left(\frac{1}{1-x}\right)^{2-\alpha} \qquad \text{as } x \to 1^- .$$

PROOF.

If $\alpha = 1$, very little analysis is required to obtain the result. Therefore assume $1 \neq \alpha < 2$ and define $g(x) = \left(\frac{1}{1-x}\right)^{2-\alpha}$. Then $g(0) = 1$ and, for all $n \geq 1$,

$$g^{(n)}(0) = (2-\alpha)(3-\alpha)\cdots(n+1-\alpha). \tag{4.8}$$

Observe that $\hat{g}(n) > 0$ for all $n \geq 0$, and that $\hat{h}(n) > 0$ for all $n \geq 1$ (here $\hat{g}(n)$ and $\hat{h}(n)$ denote the n^{th} Taylor series coefficients of g and h). It suffices then, to show that $\hat{h}(n) \asymp \hat{g}(n)$ as $n \to \infty$. By (4.8), for $n \geq 1$,

$$\begin{aligned}\frac{\hat{h}(n)}{\hat{g}(n)} &= \frac{n!\, n^{1-\alpha}}{g^{(n)}(0)} \\ &= \frac{n!\, n^{1-\alpha}}{(2-\alpha)(3-\alpha)\cdots(n+1-\alpha)} \\ &= (1-\alpha)\left[\frac{n!\, n^{1-\alpha}}{(1-\alpha)(1-\alpha+1)(1-\alpha+2)\cdots(1-\alpha+n)}\right]. \end{aligned} \tag{4.9}$$

Consider the following formula, due to Gauss [11, p. 312]:

$$\lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \Gamma(z) \qquad (z \in \mathbf{C}).$$

Its application to (4.9) shows that

$$\lim_{n\to\infty} \frac{\hat{h}(n)}{\hat{g}(n)} = (1-\alpha)\,\Gamma(1-\alpha).$$

In particular, this limit exists and is non-zero; this shows that $\hat{h}(n) \asymp \hat{g}(n)$ as $n \to \infty$.

///

In Theorem 3.1 of [10], Shapiro gives a short direct proof that the condition

$$\int_0^{2\pi} \frac{dt}{1 - |\varphi(e^{it})|} < \infty \tag{4.10}$$

is both necessary and sufficient for a self map φ of $\mathbf{D}$ to induce a Hilbert-Schmidt composition operator on H^2. When C_φ is Hilbert-Schmidt on H^2, (4.10) evidently implies that

$$\frac{m(\{e^{it} : |\varphi(e^{it})| \geq \xi\})}{1-\xi} \to 0, \quad \text{as } \xi \to 1^-. \tag{4.11}$$

Shapiro's proof that (4.10) is necessary for Hilbert-Schmidt composition operators on

H^2 provides orientation for the proof of the following theorem concerning the Dirichlet and Dirichlet-type spaces. Hansson's Inequality (cited below), and Theorem 4.6 and its corollary are important ingredients of its proof. One will find it interesting to compare (4.11), satisfied when C_φ is Hilbert-Schmidt on H^2, with Corollary 4.11 concerning Hilbert-Schmidt composition operators on $\mathcal{D}$ and $\mathcal{D}_\alpha$.

4.10 Theorem.

Fix $\alpha \in (0,1]$ and define $K = K_{1-\frac{\alpha}{2}}$. If the composition operator $C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is Hilbert-Schmidt (where $\mathcal{D}_1 = \mathcal{D}$), then

$$\int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\}) \frac{d\xi}{(1-\xi)^{2-\alpha}} < \infty.$$

PROOF.

φ^n is in $\mathcal{D}_\alpha$ for each $n \in \mathbf{N}$. By the Corollary of the proof of Theorem 4.6, for each $n \in \mathbf{N}$ there exists $F_n \in L^2(\mathbf{T})$ with $\varphi^n(e^{it}) = K * F_n(e^{it})$ (for $C_{K,2}$ – *a.e.* $e^{it} \in \mathbf{T}$). By Lemma 4.5, there exists a constant B satisfying

$$B\|\varphi^n\| \geq \|F_n\|_2^2$$

for each n. By Hansson's inequality [3, Theorem 2.4, p. 93], there exists a constant A satisfying

$$A\|F_n\|_2^2 \geq \int_0^\infty C_{K,2}(\{e^{it} : K * |F_n|(e^{it}) \geq \lambda\})\, d(\lambda^2)$$

for $n \in \mathbf{N}$. Thus for each positive integer n,

$$\begin{aligned}
AB\,\|\varphi^n\|^2_{\mathcal{D}_\alpha} &\geq \int_0^\infty C_{K,2}(\{e^{it} : K * |F_n|(e^{it}) \geq \lambda\})\, d(\lambda^2) \\
&\geq \int_0^1 C_{K,2}(\{e^{it} : |K * F_n(e^{it})| \geq \lambda\})\, d(\lambda^2) \\
&= \int_0^1 C_{K,2}(\{e^{it} : |\varphi^n(e^{it})| \geq \lambda\})\, d(\lambda^2) \\
&= \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \lambda^{1/n}\})\, d(\lambda^2) \\
&= \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\})\, d(\xi^{2n}) \\
&= \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\})\, 2n\,\xi^{2n-1} d(\xi) \\
&\geq \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\})\, 2n\,\xi^{2n} d(\xi).
\end{aligned}$$

Since C_φ is Hilbert-Schmidt, we have

$$\sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2_{\mathcal{D}_\alpha}}{n^\alpha} < \infty;$$

thus summing over the preceding inequality gives

$$\begin{aligned}
\infty &> AB \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2_{\mathcal{D}_\alpha}}{n^\alpha} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\})\, 2n\,\xi^{2n} d(\xi) \\
&= 2\int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\}) \sum_{n=1}^{\infty} n^{1-\alpha}\,\xi^{2n}\, d\xi. \qquad (4.12)
\end{aligned}$$

The only possible singularity of the integrand in (4.12) occurs at $\xi = 1$, so Lemma 4.9 implies that

$$\infty > \int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\}) \frac{1}{(1-\xi^2)^{2-\alpha}}\, d\xi.$$

This quickly yields the result stated in the theorem:

$$\int_0^1 C_{K,2}(\{e^{it} : |\varphi(e^{it})| \geq \xi\}) \frac{d\xi}{(1-\xi)^{2-\alpha}} < \infty.$$

///

4.11 Corollary.

Fix $\alpha \in (0,1]$ and define $K = K_{1-\frac{\alpha}{2}}$. For a self map φ of the disk, define the capacitary distribution function

$$g(t) \doteq C_{K,2}(\{e^{is} : |\varphi(e^{is})| \geq t\}).$$

If $C_\varphi : \mathcal{D} \to \mathcal{D}$ is Hilbert-Schmidt, then there exists a constant M satisfying

$$g(t) \leq \frac{M}{\log \frac{1}{1-t}} \qquad (0 < t < 1).$$

If $C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is Hilbert-Schmidt and $\alpha \in (0,1)$, then there exists a constant M satisfying

$$g(t) \leq M\,(1-t)^{1-\alpha} \qquad (0 \leq t < 1).$$

PROOF.

By Theorem 4.10, there exists a number M_1 satisfying

$$M_1 \geq \int_0^t g(\zeta) \frac{d\zeta}{(1-\zeta)^{2-\alpha}} \qquad \text{for all } t \in [0,1). \tag{4.13}$$

If $\alpha = 1$, then define $h(\zeta) = \log \frac{1}{1-\zeta}$; if $\alpha \neq 1$, then define $h(\zeta) = \frac{1}{1-\alpha}(1-\zeta)^{\alpha-1}$. Then

$$h'(\zeta) = \frac{1}{(1-\zeta)^{2-\alpha}}.$$

Fix $t \in [0,1)$. Inequality (4.13) and integration by parts shows that

$$\begin{aligned} M_1 &\geq g(\zeta)\,h(\zeta)|_0^t \; - \int_0^t h(\zeta)\,dg(\zeta) \\ &= g(t)\,h(t) - g(0)\,h(0) \; + \int_0^t h(\zeta)\,|dg(\zeta)|. \end{aligned}$$

Hence, $M_2 \doteq M_1 + g(0)\,h(0) \geq g(t)\,h(t)$, and we obtain

$$M_2 \geq g(t)\,h(t).$$

If $\alpha = 1$, then this gives the desired result with $M = M_2$. If $\alpha \neq 1$, then this gives the desired result with $M = (1-\alpha)M_2$. ///

Remark.

From Theorem 4.10 (or Corollary 4.11), and Lemma 4.2, we see that

$$c_{log,1}(\{e^{it} : |\varphi(e^{it})| = 1\}) = 0$$

or

$$c_{1-\alpha,1}(\{e^{it} : |\varphi(e^{it})| = 1\}) = 0$$

whenever $C_\varphi : \mathcal{D} \to \mathcal{D}$ or $C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is Hilbert-Schmidt, respectively.

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