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CURVATURE AND NORMALITY OF COMPLEX CONTACT MANIFOLDS

By

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ABSTRACT

CURVATURE AND NORMALITY OF COMPLEX CONTACT MANIFOLDS

By

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In the first part of this thesis, we define complex contact metric structures and introduce a notion of normality for complex contact metric manifolds. In terms of the covariant derivatives of the structure tensors, we give a necessary and sufficient condition for a complex contact metric manifold to be normal. Then we define the GH-sectional curvature for normal complex contact metric manifolds, and classify those with constant GH-sectional curvature $+1$. We also define $\mathcal{H}$-homothetic deformations and use them to get examples of normal complex contact metric manifolds with constant GH-sectional curvature c with $c > -3$.

In the second part, we show that complex contact metric manifolds with $R(X,Y)\mathcal{V} = 0$ are given locally by $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$ under a certain assumption. We give a complex contact metric structure on $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$ with $R(X,Y)\mathcal{V} = 0$.

To my parents, sisters and to my husband

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Contents

Chapter 0

Introduction

The theory of complex contact manifolds started with the papers of Kobayashi [12] and Boothby [4], [5], in late 1950's and early 1960's, shortly after the celebrated Boothby-Wang fibration in real contact geometry [6]. It did not receive as much attention as the theory of real contact geometry. In 1965, Wolf studied homogeneous complex contact manifolds [17]. Recently, more examples are appearing in the literature, especially twistor spaces over quaternionic Kähler manifolds (e.g. [13], [14], [15], [16], [18]). Other examples include the odd dimensional complex projective spaces [9], and the complex Heisenberg group [1].

In the 1970's and early 1980's there was a development of the Rieamannian theory of complex contact manifolds by Ishihara and Konishi [8], [9], [10]. However, their notion of normality as it appears in [9] seems too strong, since it does not include the complex Heisenberg group and it forces the structure to be Kähler. In the first chapter of this thesis, we introduce a slightly different notion of normality which includes the complex Heisenberg group. The main theorem of the first chapter states the necessary and sufficient conditions, in terms of the covariant derivatives of the structure tensors, for a complex contact manifold to be normal.

In Chapter 2, following the corresponding theory of real contact geometry, we define the GH-sectional curvature for normal complex contact manifolds and we classify those with constant GH-sectional curvature $+1$. Then we define $\mathcal{H}$-homothetic deformations and show that they preserve normality. Here we note that Ishihara-Konishi's notion of normality is not preserved under $\mathcal{H}$-homothetic deformations.

In Chapter 3, we study complex contact manifolds for which the vertical plane is annihilated by the curvature. We show that those manifolds are given locally by $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$. We also give the complex contact metric structure on $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$.

Chapter 1

Complex contact metric structures and normality

1.1 Basic definitions

Definition 1.1 *Let M be a complex manifold with* $\dim_{\mathbf{C}} M = 2n+1$ *and let J denote the complex structure on M. M is a complex contact manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M, such that*

1) *on each $\mathcal{O}_\alpha$, there is a holomorphic 1-form ω_α with $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ everywhere, and*

2) *if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ then there is a non-vanishing holomorphic function $\lambda_{\alpha\beta}$ in $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ such that*

$$\omega_\alpha = \lambda_{\alpha\beta}\,\omega_\beta \text{ in } \mathcal{O}_\alpha \cap \mathcal{O}_\beta.$$

On each $\mathcal{O}_\alpha$, we define $\mathcal{H}_\alpha = \{X \in T\mathcal{O}_\alpha \,|\, \omega_\alpha(X) = 0\}$. Since $\lambda_{\alpha\beta}$'s are nonvanishing, $\mathcal{H}_\alpha = \mathcal{H}_\beta$ on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$. So $\mathcal{H} = \cup \mathcal{H}_\alpha$ is a well-defined, holomorphic, non-integrable subbundle on M, called *the horizontal subbundle.*

Definition 1.2 *Let M be a complex manifold with* $\dim_{\mathbf{C}} M = 2n+1$*, complex structure J and hermitian metric g. M is called a complex almost contact metric manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M such that*

1) *in each* $\mathcal{O}_\alpha$, *there are* 1*-forms* u_α *and* $v_\alpha = u_\alpha J$, $(1,1)$ *tensors* G_α *and* $H_\alpha = G_\alpha J$, *unit vector fields* U_α *and* $V_\alpha = -JU_\alpha$ *such that*

$$H_\alpha^2 = G_\alpha^2 = -\mathrm{Id} + u_\alpha \otimes U_\alpha + v_\alpha \otimes V_\alpha$$

$$g(G_\alpha X, Y) = -g(X, G_\alpha Y)$$

$$g(U_\alpha, X) = u_\alpha(X)$$

$$G_\alpha J = -JG_\alpha$$

$$G_\alpha U_\alpha = 0$$

$$u_\alpha(U_\alpha) = 1,$$

2) *if* $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ *then there are functions* a, b *on* $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ *such that*

$$u_\beta = au_\alpha - bv_\alpha$$

$$v_\beta = bu_\alpha + av_\alpha$$

$$G_\beta = aG_\alpha - bH_\alpha$$

$$H_\beta = bG_\alpha + aH_\alpha$$

$$a^2 + b^2 = 1.$$

As a result of this definition, on a complex almost contact metric manifold M, the following identities hold (cf. [9]):

$$H_\alpha G_\alpha = -G_\alpha H_\alpha = J + u_\alpha \otimes V_\alpha - v_\alpha \otimes U_\alpha$$

$$JH_\alpha = -H_\alpha J = G_\alpha$$

$$g(H_\alpha X, Y) = -g(X, H_\alpha Y)$$

$$G_\alpha V_\alpha = H_\alpha U_\alpha = H_\alpha V_\alpha = 0$$

$$u_\alpha G_\alpha = v_\alpha G_\alpha = u_\alpha H_\alpha = v_\alpha H_\alpha = 0$$

$$JV_\alpha = U_\alpha, \;\; g(U_\alpha, V_\alpha) = 0.$$

Let $(M,\{\omega_\alpha\})$ be a complex contact manifold. We can find a non-vanishing, complex-valued function multiple π_α of ω_α such that on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, $\pi_\alpha = h_{\alpha\beta}\pi_\beta$ with $h_{\alpha\beta}: \mathcal{O}_\alpha \cap \mathcal{O}_\beta \to \mathbf{S}^1$. Let $\pi_\alpha = u_\alpha - iv_\alpha$. Then $v_\alpha = u_\alpha J$ since ω_α is holomorphic.

From now on, we will suppress the subscripts if $\mathcal{O}_\alpha$ is understood.

Locally, we can define a vector field U by $du(U,X) = 0$ for all X in $\mathcal{H}$ and $u(U) = 1$, $v(U) = 0$. Then we have a global subbundle $\mathcal{V}$ locally spanned by U and $V = -JU$ with $TM = \mathcal{H} \oplus \mathcal{V}$. We call $\mathcal{V}$ the *vertical subbundle* of the contact structure. Here we note that we can find a local $(1,1)$ tensor G such that $(u,v,U,V,G,H = GJ,g)$ form a complex almost contact metric structure on M (cf. [10]).

Definition 1.3 *Let $(M,\{\omega\})$ be a complex contact manifold with the complex structure J and hermitian metric g. We call (M,u,v,U,V,g) a complex contact metric manifold if*

1) *there is a local $(1,1)$ tensor G such that $(u,v,U,V,G,H = GJ,g)$ is a complex almost contact metric structure on M, and*

2) *$g(X,GY) = du(X,Y)$ and $g(X,HY) = dv(X,Y)$ for all X,Y in $\mathcal{H}$.*

In his thesis [7], Foreman shows the existence of complex contact metric structures on complex contact manifolds.

We will assume that the subbundle $\mathcal{V}$ is integrable. Since every known example of a complex contact manifold has an integrable vertical subbundle, this is a reasonable assumption for our work. From now on, we will work with a complex contact metric manifold M with structure tensors (u,v,U,V,G,H,g) and complex structure J.

Define 2-forms $\hat{G}$ and $\hat{H}$ on M by

$$\hat{G}(X,Y) = g(X,GY), \quad \hat{H}(X,Y) = g(X,HY).$$

Then for horizontal vector fields X, Y,

$$\hat{G}(X,Y) = du(X,Y), \ \hat{H}(X,Y) = dv(X,Y).$$

In general, we have

$$\hat{G} = du - \sigma \wedge v, \tag{1.1}$$

$$\hat{H} = dv + \sigma \wedge u. \tag{1.2}$$

where $\sigma(X) = g(\nabla_X U, V)$ (cf. [7]).

Let p denote the projection map $p : TM \to \mathcal{H}$.

In real contact geometry, there is a symmetric operator $h = \frac{1}{2}\pounds_\xi\phi$,where ξ is the characteristic vector field and ϕ is the structure tensor of the real contact metric structure, which plays an important role. Here, $\pounds$ denotes the Lie differetiation. In particular, on a real contact metric manifold we have

$$\nabla_X \xi = -\phi X - \phi h X$$

Cf. [3].

Similarly, we define symmetric operators $h_U, h_V\colon TM \to \mathcal{H}$ as follows:

$$h_U = \tfrac{1}{2} sym(\pounds_U G) \circ p$$
$$h_V = \tfrac{1}{2} sym(\pounds_V H) \circ p$$

where sym denotes the symmetrization. Then we have

$$h_U G = -G h_U, \ h_V H = -H h_V,$$
$$h_U(U) = h_U(V) = h_V(U) = h_V(V) = 0,$$

and

$$\nabla_X U = -GX - Gh_U X + \sigma(X)V, \tag{1.3}$$

$$\nabla_X V = -HX - Hh_V X - \sigma(X)U, \tag{1.4}$$

where ∇ is the Levi-Civita connection of g (cf. [7]).

Hence

$$\nabla_U U = \sigma(U)V, \nabla_V U = \sigma(V)V, \nabla_U V = -\sigma(U)U, \nabla_V V = -\sigma(V)U. \tag{1.5}$$

It can be seen easily by a direct computation that

$$(\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) = 3d\hat{G}(X,Y,Z),$$

and

$$(\nabla_X \hat{H})(Y,Z) + (\nabla_Y \hat{H})(Z,X) + (\nabla_Z \hat{H})(X,Y) = 3d\hat{H}(X,Y,Z).$$

Then, using equations (1.1) and (1.2) we get

$$\begin{aligned} &(\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) \\ &= -v(X)\Omega(Y,Z) - v(Y)\Omega(Z,X) - v(Z)\Omega(X,Y) \\ &\quad +\sigma(X)g(Y,HZ) + \sigma(Y)g(Z,HX) + \sigma(Z)g(X,HY), \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &(\nabla_X \hat{H})(Y,Z) + (\nabla_Y \hat{H})(Z,X) + (\nabla_Z \hat{H})(X,Y) \\ &= u(X)\Omega(Y,Z) + u(Y)\Omega(Z,X) + u(Z)\Omega(X,Y) \\ &\quad -\sigma(X)g(Y,GZ) - \sigma(Y)g(Z,GX) - \sigma(Z)g(X,GY), \end{aligned} \tag{1.7}$$

where $\Omega = d\sigma$.

Lemma 1.4 *$\nabla_U G = \sigma(U)H$, and $\nabla_V H = -\sigma(V)G$.*

Proof: By equations (1.6) and (1.3) we get

$$\begin{aligned}
(\nabla_U \hat{G})(X,Y) &= -(\nabla_X \hat{G})(Y,U) - (\nabla_Y \hat{G})(U,X) + v(X)\Omega(U,Y) \\
&\quad + v(Y)\Omega(X,U) + \sigma(U)g(X,HY) \\
&= -g(\nabla_X U, GY) + g(\nabla_Y U, GX) + v(X)\Omega(U,Y) \\
&\quad + v(Y)\Omega(X,U) + \sigma(U)g(X,HY) \\
&= g(GX + Gh_U X, GY) - g(GY + Gh_U Y, GX) + v(X)\Omega(U,Y) \\
&\quad + v(Y)\Omega(X,U) + \sigma(U)g(X,HY) \\
&= v(X)\Omega(U,Y) + v(Y)\Omega(X,U) + \sigma(U)g(X,HY). \qquad (1.8)
\end{aligned}$$

If X and Y are horizontal then

$$(\nabla_U \hat{G})(X,Y) = \sigma(U)g(X,HY).$$

On the other hand by (1.5)

$$(\nabla_U \hat{G})(U,Y) = -g(\nabla_U U, GY) = 0,$$

and

$$(\nabla_U \hat{G})(V,Y) = -g(\nabla_U V, GY) = 0.$$

So, $(\nabla_U G)Y = \sigma(U)HY$ for any Y.

Similarly, using (1.7) and (1.4) we get

$$(\nabla_V \hat{H})(X,Y) = u(X)\Omega(Y,V) + u(Y)\Omega(V,X) - \sigma(V)g(X,GY). \qquad (1.9)$$

Again by (1.5) $(\nabla_V \hat{H})(U,Y) = (\nabla_V \hat{H})(V,Y) = 0$. So, $(\nabla_V H)Y = -\sigma(V)GY$. □

Now, if we use Lemma 1.4 in equations (1.8) and (1.9) we get

$$\Omega(U,X) = v(X)\Omega(U,V), \qquad (1.10)$$

and

$$\Omega(V,X) = -u(X)\Omega(U,V). \qquad (1.11)$$

1.2 Normality on complex contact metric manifolds

Let M be a complex contact metric manifold. Ishihara and Konishi [9] defined $(1,2)$ tensors S and T on a complex almost contact manifolds as follows:

$$\begin{aligned} S(X,Y) &= [G,G](X,Y) + 2v(Y)HX - 2v(X)HY + 2g(X,GY)U \\ &\quad -2g(X,HY)V - \sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY \\ &\quad -\sigma(Y)GHX \\ T(X,Y) &= [H,H](X,Y) + 2u(Y)GX - 2u(X)GY + 2g(X,HY)V \\ &\quad -2g(X,GY)U + \sigma(HX)GY - \sigma(HY)GX - \sigma(X)HGY \\ &\quad +\sigma(Y)HGX \end{aligned}$$

where

$$[G,G](X,Y) = (\nabla_{GX}G)Y - (\nabla_{GY}G)X - G(\nabla_X G)Y + G(\nabla_Y G)X$$

is the Nijenhuis torsion of G. In [9], they introduced the notion of normality which is the vanishing of the two tensors S and T. One of their results is that if M is normal then it is Kähler. This result suggests that Ishihara-Konishi's notion of normality is too strong. Here we will give a somewhat weaker definition.

Definition 1.5 *A complex contact metric manifold M is normal if*

1) $S(X,Y) = T(X,Y) = 0$ *for all* X, Y *in* $\mathcal{H}$, *and*

2) $S(U,X) = T(V,X) = 0$ *for all* X.

In real contact geometry, normality implies the vanishing of the operator h. The following proposition is the analogous result for complex contact geometry.

Proposition 1.6 *If M is normal, then $h_u = h_v = 0$.*

Proof: Since M is normal,

$$\begin{aligned}
0 &= S(GX,U) \\
&= [G,G](GX,U) - \sigma(U)GHGX \\
&= (\nabla_{G^2X}G)U - G(\nabla_{GX}G)U + G(\nabla_U G)GX - \sigma(U)HX \\
&= -G\nabla_{G^2X}U + G^2\nabla_{GX}U + G\nabla_U G^2X - G^2\nabla_U GX - \sigma(U)HX \\
&= G\nabla_X U - u(X)G\nabla_U U - v(X)G\nabla_V U - \nabla_{GX}U + u(\nabla_{GX}U)U \\
&\quad +v(\nabla_{GX}U)V - G\nabla_U X + u(X)G\nabla_U U + v(X)G\nabla_U V + \nabla_U GX \\
&\quad -u(\nabla_U GX)U - v(\nabla_U GX)V - \sigma(U)HX.
\end{aligned}$$

By (1.5) $G(\nabla_V U) = G(\nabla_U V) = G(\nabla_U U) = 0$. Also

$$u(\nabla_U GX) = g(\nabla_U GX, U) = -g(\nabla_U U, GX) = 0$$

$$v(\nabla_U GX) = g(\nabla_U GX, V) = -g(\nabla_U V, GX) = 0.$$

Again using (1.3)

$$\nabla_{GX}U = -G^2X - Gh_U GX + \sigma(GX)V.$$

So $u(\nabla_{GX}U) = 0$, $v(\nabla_{GX}U) = \sigma(GX)$. Hence

$$\begin{aligned}
0 &= G(-GX - Gh_U X + \sigma(X)V) + G^2X - G^2h_U X - \sigma(GX)V \\
&\quad +\sigma(GX)V + (\nabla_U G)X - \sigma(U)HX \\
&= 2h_U X + \sigma(U)HX - \sigma(U)HX \\
&= 2h_U X.
\end{aligned}$$

Therefore $h_U = 0$. Similarly, using $T(HX,V) = 0$ and Lemma 1.4 we get $h_V = 0$. □

By the above proposition, on a normal complex contact metric manifold we have

$$\nabla_X U = -GX + \sigma(X)V \tag{1.12}$$

and

$$\nabla_X V = -HX - \sigma(X)U. \qquad (1.13)$$

In the next proposition, we give the necessary and sufficient conditions, in terms of ∇G and ∇H, for M to be normal. Again, compare with the condition for a real contact metric manifold to be normal.

Proposition 1.7 *Let M be a complex contact metric manifold. M is normal if and only if*

$$\text{(I)} \qquad g((\nabla_X G)Y, Z) = \sigma(X)g(HY, Z) + v(X)\Omega(GZ, GY) - 2v(X)g(HGY, Z) \\ -u(Y)g(X, Z) - v(Y)g(JX, Z) + u(Z)g(X, Y) - v(Z)g(X, JY)$$

and

$$\text{(II)} \qquad g((\nabla_X H)Y, Z) = -\sigma(X)g(GY, Z) + u(X)\Omega(HZ, HY) - 2u(X)g(HGY, Z) \\ +u(Y)g(JX, Z) - v(Y)g(X, Z) + u(Z)g(X, JY) + v(Z)g(X, Y).$$

Proof: Suppose that M is normal. For arbitrary vector fields X and Y, we can write

$$X = X' + u(X)U + v(X)V, \;\; Y = Y' + u(Y)U + v(Y)V$$

where X' and Y' are in $\mathcal{H}$. Then $GX = GX'$, $GY = GY'$ and

$$\begin{aligned} S(X, Y) &= \nabla_{GX}GY - G\nabla_{GX}Y - \nabla_{GY}GX + G\nabla_{GY}X - G\nabla_X GY \\ &\quad +G^2\nabla_X Y + G\nabla_Y GX - G^2\nabla_Y X + 2v(Y)HX - 2v(X)HY \\ &\quad +2g(X, GY)U - 2g(X, HY)V - \sigma(GX)HY + \sigma(GY)HX \\ &\quad +\sigma(X)GHY - \sigma(Y)GHX \\ &= \nabla_{GX'}GY' - G\nabla_{GX'}Y' - u(Y)G\nabla_{GX'}U - v(Y)G\nabla_{GX'}V - \nabla_{GY'}GX' \end{aligned}$$

$$
\begin{aligned}
&+G\nabla_{GY'}X' + u(X)G\nabla_{GY'}U + v(X)G\nabla_{GY'}V - G\nabla_{X'}GY' \\
&-u(X)G\nabla_U GY' - v(X)G\nabla_V GY' + G^2(\nabla_{X'}Y' + u(Y)\nabla_{X'}U \\
&+v(Y)\nabla_{X'}V + u(X)\nabla_U Y + v(X)\nabla_V Y) + G\nabla_{Y'}GX' + u(Y)G\nabla_U GX' \\
&+v(Y)G\nabla_V GX' - G^2(\nabla_{Y'}X' + u(X)\nabla_{Y'}U + v(X)\nabla_{Y'}V + u(Y)\nabla_U X \\
&+v(Y)\nabla_V X) + 2v(Y)HX - 2v(X)HY + 2g(X', GY')U - 2g(X', HY')V \\
&-\sigma(GX')HY' + \sigma(GY')HX' + \sigma(X')GHY' + u(X)\sigma(U)GHY \\
&+v(X)\sigma(V)GHY - \sigma(Y')GHX' - u(Y)\sigma(U)GHX - v(Y)\sigma(V)GHX \\
= \;& S(X', Y') - u(Y)G(-G^2X + \sigma(GX)V) - v(Y)G(-HGX - \sigma(GX)U) \\
&+u(X)G(-G^2Y + \sigma(GY)V) + v(X)G(-HGY - \sigma(GY)U) \\
&-u(X)G\nabla_U GY - v(X)G\nabla_V GY + u(Y)G^2(-GX' + \sigma(X')V) \\
&+v(Y)G^2(-HX' - \sigma(X')U) + u(X)G^2\nabla_U Y + v(X)G^2\nabla_V Y \\
&+u(Y)G\nabla_U GX + v(Y)G\nabla_V GX - u(X)G^2(-GY' + \sigma(Y')V) \\
&-v(X)G^2(-HY' - \sigma(Y')U) - u(Y)G^2\nabla_U X - v(Y)G^2\nabla_V X \\
&+2v(Y)HX - 2v(X)HY + u(X)\sigma(U)GHY + v(X)\sigma(V)GHY \\
&-u(Y)\sigma(U)GHX - v(Y)\sigma(V)GHX.
\end{aligned}
$$

Since M is normal, $S(X', Y') = 0$. So

$$
\begin{aligned}
S(X,Y) \;=\;& -u(Y)GX + v(Y)HX + u(X)GY - v(X)HY - u(X)G(\nabla_U G)Y \\
&-v(X)G(\nabla_V G)Y + u(Y)GX + v(Y)HX + u(Y)G(\nabla_U G)X \\
&+v(Y)G(\nabla_V G)X - u(X)GY - v(X)HY + 2v(Y)HX - 2v(X)HY \\
&+u(X)\sigma(U)GHY + v(X)\sigma(V)GHY - u(Y)\sigma(U)GHX \\
&-v(Y)\sigma(V)GHX \\
=\;& 4v(Y)HX - 4v(X)HY - u(X)G(\nabla_U G)Y - v(X)G(\nabla_V G)Y \\
&+u(Y)G(\nabla_U G)X + v(Y)G(\nabla_V G)X + u(X)\sigma(U)GHY
\end{aligned}
$$

$$+v(X)\sigma(V)GHY - u(Y)\sigma(U)GHX - v(Y)\sigma(V)GHX.$$

From (1.6) and (1.11) we get

$$\begin{aligned}(\nabla_V\hat{G})(X,Y) &= -(\nabla_X\hat{G})(Y,V) - (\nabla_Y\hat{G})(V,X) - v(X)u(Y)\Omega(U,V)\\ &\quad +v(Y)u(X)\Omega(U,V) - \Omega(X,Y) + \sigma(V)g(X,HY)\\ &= g(HX,GY) - g(HY,GX) + 2u\wedge v(X,Y)\Omega(U,V)\\ &\quad -\Omega(X,Y) + \sigma(V)g(X,HY).\end{aligned}$$

Thus

$$\begin{aligned}(\nabla_V\hat{G})(X,Y) = 2g(X,GHY) + 2u\wedge v(X,Y)\Omega(U,V)\\ -\Omega(X,Y) + \sigma(V)g(X,HY).\end{aligned} \tag{1.14}$$

Now, using equation (1.14) and Lemma 1.4, for any vector field Z we have

$$\begin{aligned}g(S(X,Y),Z) &= 4v(Y)g(HX,Z) - 4v(X)g(HY,Z) + u(X)(\nabla_U\hat{G})(GZ,Y)\\ &\quad +v(X)(\nabla_V\hat{G})(GZ,Y) - u(Y)(\nabla_U\hat{G})(GZ,X)\\ &\quad -v(Y)(\nabla_V\hat{G})(GZ,X) + u(X)\sigma(U)g(GHY,Z)\\ &\quad +v(X)\sigma(V)g(GHY,Z) - u(Y)\sigma(U)g(GHX,Z)\\ &\quad -v(Y)\sigma(V)g(GHX,Z)\\ &= 4v(Y)g(HX,Z) - 4v(X)g(HY,Z) + u(X)\sigma(U)g(GZ,HY)\\ &\quad +v(X)(2g(GZ,GHY) - \Omega(GZ,Y) + \sigma(V)g(GZ,HY))\\ &\quad -u(Y)\sigma(U)g(GZ,HX) - v(Y)(2g(GZ,GHX) - \Omega(GZ,X)\\ &\quad +\sigma(V)g(GZ,HX)) + u(X)\sigma(U)g(GHY,Z)\\ &\quad +v(X)\sigma(V)g(GHY,Z) - u(Y)\sigma(U)g(GHX,Z)\\ &\quad -v(Y)\sigma(V)g(GHX,Z).\end{aligned}$$

Therefore

$$g(S(X,Y),Z) = 2v(Y)g(HX,Z) - 2v(X)g(HY,Z)$$

$$-v(X)\Omega(GZ,Y)+v(Y)\Omega(GZ,X) \qquad (1.15)$$

If we take $Y=V$ and GX instead of X in (1.15), we get

$$g(S(GX,V),Z)=2g(HGX,Z)+\Omega(GZ,GX) \qquad (1.16)$$

On the other hand,

$$\begin{aligned}
S(GX,V) &= -G\nabla_{G^2X}V+G^2\nabla_{GX}V+G\nabla_V G^2X \\
&\quad -G^2\nabla_V GX+2HGX-\sigma(V)HX \\
&= -G(-HG^2X-\sigma(G^2X)U)+G^2(-HGX-\sigma(GX)U) \\
&\quad +G(-\nabla_V X+u(X)\nabla_V U+v(X)\nabla_V V+V(u(X))U \\
&\quad +V(v(X))V)+\nabla_V GX-u(\nabla_V GX)U \\
&\quad -v(\nabla_V GX)V+2HGX-\sigma(V)HX.
\end{aligned}$$

By (1.3), $\nabla_V U=\sigma(V)V$. So $0=-g(\nabla_V U,GX)=g(\nabla_V GX,U)=u(\nabla_V GX)$. Similarly, using (1.4) we get $v(\nabla_V GX)=0$. When we substitute these in $S(GX,V)$ we get

$$S(GX,V)=4HGX+(\nabla_V G)X-\sigma(V)HX$$

Hence,

$$\begin{aligned}
g(S(GX,V),Z) &= 4g(HGX,Z)+(\nabla_V\hat{G})(Z,X)-\sigma(V)g(HX,Z) \\
&= 4g(HGX,Z)-(\nabla_Z\hat{G})(X,V)-(\nabla_X\hat{G})(V,Z) \\
&\quad +v(X)u(Z)\Omega(U,V)-\Omega(Z,X)-v(Z)u(X)\Omega(U,V) \\
&\quad +\sigma(V)g(Z,HX)-\sigma(V)g(HX,Z) \\
&= 4g(HGX,Z)-g(\nabla_Z V,GX)+g(\nabla_X V,GZ) \\
&\quad -2u\wedge v(X,Z)\Omega(U,V)+\Omega(X,Z) \\
&= 4g(HGX,Z)+g(HZ,GX)-g(HX,GZ)
\end{aligned}$$

$$-2u \wedge v(X,Z)\Omega(U,V) + \Omega(X,Z)$$
$$= 2g(HGX,Z) - 2u \wedge v(X,Z)\Omega(U,V) + \Omega(X,Z)$$

Combining with (1.16), we get

$$\Omega(GZ,GX) = \Omega(X,Z) - 2u \wedge v(X,Z)\Omega(U,V) \tag{1.17}$$

Applying the above process to $T(X,Y)$ we get

$$\begin{aligned} g(T(X,Y),Z) &= 2u(Y)g(GX,Z) - 2u(X)g(GY,Z) \\ &\quad + u(X)\Omega(HZ,Y) - u(Y)\Omega(HZ,X) \end{aligned} \tag{1.18}$$

and

$$\Omega(HZ,HX) = \Omega(X,Z) - 2u \wedge v(X,Z)\Omega(U,V) \tag{1.19}$$

Combining (1.17) with (1.19) gives

$$\Omega(GZ,GX) = \Omega(HZ,HX). \tag{1.20}$$

Equation (1.20) implies

$$\Omega(G^2Z,G^2X) = \Omega(HGZ,HGX).$$

If we compute the left-hand side and the right-hand side seperately using (1.10) and (1.11), we get

$$\Omega(G^2Z,G^2X) = \Omega(Z,X) + (u(X)v(Z) - v(X)u(Z))\Omega(U,V),$$

and

$$\Omega(HGZ,HGX) = \Omega(JZ,JX) + (u(X)v(Z) - u(Z)v(X))\Omega(U,V).$$

Therefore

$$\Omega(Z,X) = \Omega(JZ,JX). \tag{1.21}$$

Replacing X with GX in (1.17) we get

$$\begin{aligned}\Omega(GX, Z) &= \Omega(GZ, G^2X) \\ &= -\Omega(GZ, X) + u(X)\Omega(GZ, U) + v(X)\Omega(GZ, V).\end{aligned}$$

Equations (1.10) and (1.11) imply $\Omega(GZ, U) = \Omega(GZ, V) = 0$. Hence

$$\Omega(GX, Z) = \Omega(X, GZ). \tag{1.22}$$

Similarly, replacing X with HX in (1.19) we get

$$\Omega(HX, Z) = \Omega(X, HZ). \tag{1.23}$$

Finally, replacing X with JX in (1.21) we get

$$\Omega(JX, Z) = -\Omega(X, JZ). \tag{1.24}$$

We now want to compute $S(X, Y)$ in a different way. First, we can rewrite $G(\nabla_X G)Y$ as follows:

$$\begin{aligned}G(\nabla_X G)Y &= G\nabla_X GY - G^2\nabla_X Y \\ &= \nabla_X G^2 Y - (\nabla_X G)GY + \nabla_X Y - u(\nabla_X Y)U - v(\nabla_X Y)V \\ &= -\nabla_X Y + u(Y)\nabla_X U + X(u(Y))U + v(Y)\nabla_X V + X(v(Y))V \\ &\quad -(\nabla_X G)GY + g(\nabla_X U, Y)U - X(u(Y))U + g(\nabla_X V, Y)V \\ &\quad -X(v(Y))V + \nabla_X Y \\ &= u(Y)(-GX + \sigma(X)V) + v(Y)(-HX - \sigma(X)U) - (\nabla_X G)GY \\ &\quad +g(-GX + \sigma(X)V, Y)U + g(-HX - \sigma(X)U, Y)V \\ &= -u(Y)GX + \sigma(X)u(Y)V - v(Y)HX - \sigma(X)v(Y)U - (\nabla_X G)GY \\ &\quad -g(GX, Y)U + \sigma(X)v(Y)U - g(HX, Y)V - \sigma(X)u(Y)V.\end{aligned}$$

It follows that

$$G(\nabla_X G)Y = -u(Y)GX - v(Y)HX - (\nabla_X G)GY + g(X,GY)U + g(X,HY)V. \tag{1.25}$$

Now let us substitute (1.25) in $S(X,Y)$ to get

$$\begin{aligned} S(X,Y) &= (\nabla_{GX}G)Y - (\nabla_{GY}G)X + (\nabla_X G)GY - (\nabla_Y G)GX \\ &\quad +u(Y)GX + 3v(Y)HX - u(X)GY - 3v(X)HY - 4g(X,HY)V \\ &\quad -\sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY - \sigma(Y)GHX. \end{aligned}$$

Taking the inner product with Z and using equations (1.6), (1.22) and (1.25) gives

$$\begin{aligned} g(S(X,Y),Z) &= 2g((\nabla_Z G)Y, GX) + 2v(Z)\Omega(X,GY) - v(Y)\Omega(X,GZ) \\ &\quad +v(X)\Omega(Y,GZ) + 2\sigma(Z)g(Y,HGX) + 2u(Y)g(GX,Z) \\ &\quad +4v(Y)g(HX,Z) - 2v(X)g(HY,Z) - 4v(Z)g(X,HY). \end{aligned}$$

If we combine the above equation with equation (1.15) we get

$$\begin{aligned} &2g((\nabla_Z G)Y, GX) + 2v(Z)\Omega(X,GY) + 2\sigma(Z)g(Y,HGX) \\ &\quad +2u(Y)g(GX,Z) + 2v(Y)g(HX,Z) - 4v(Z)g(X,HY) = 0. \end{aligned}$$

In order to get the equation we want, we replace X with GX which gives

$$\begin{aligned} &2g((\nabla_Z G)X, Y) + 2v(Z)\Omega(GX,GY) + 2\sigma(Z)g(X,HY) - 2u(Y)g(X,Z) \\ &\quad -2v(Y)g(X,JZ) - 2v(Y)u(Z)v(X) + 4v(Z)g(X,GHY) \\ &\quad +2u(X)g(Z,Y) - 2v(X)g(Z,JY) + 2v(X)v(Y)u(Z) = 0. \end{aligned}$$

Now equation (I) follows.

Applying the same process to $T(X,Y)$ we can easily see that equation (II) also holds.

Conversely, suppose that formulas (I) and (II) hold. To show that M is normal, first let us check $S(X,U)$. Since formula (I) holds,

$$\begin{aligned} g(S(X,U),Y) &= g((\nabla_U G)GY,X) + g((\nabla_{GX}G)U,Y) + g((\nabla_X G)U,GY) \\ &\quad -\sigma(U)g(GHX,Y) \\ &= \sigma(U)g(HGY,X) - g(GX,Y) - g(X,GY) - \sigma(U)g(GHX,Y) \\ &= 0. \end{aligned}$$

Therefore $S(X,U)=0$. Similarly, $T(X,V)=0$.

Now let X and Y be two vector fields in $\mathcal{H}$. Making use of the fact that $u(X) = v(X) = u(Y) = v(Y) = 0$ and applying formula (I), we get

$$\begin{aligned} g(S(X,Y),Z) &= g((\nabla_{GX}G)Y,Z) + g((\nabla_{GY}G)Z,X) + g((\nabla_X G)Y,GZ) \\ &\quad +g((\nabla_Y G)GZ,X) + 2u(Z)g(X,GY) - 2v(Z)g(X,HY) \\ &\quad -\sigma(GX)g(HY,Z) + \sigma(GY)g(HX,Z) + \sigma(X)g(GHY,Z) \\ &\quad -\sigma(Y)g(GHX,Z) \\ &= \sigma(GX)g(HY,Z) + u(Z)g(GX,Y) - v(Z)g(GX,JY) \\ &\quad +\sigma(GY)g(HZ,X) - u(Z)g(GY,X) - v(Z)g(JGY,X) \\ &\quad +\sigma(X)g(HY,GZ) + \sigma(Y)g(HGZ,X) + 2u(Z)g(X,GY) \\ &\quad -2v(Z)g(X,HY) - \sigma(GX)g(HY,Z) + \sigma(GY)g(HX,Z) \\ &\quad +\sigma(X)g(GHY,Z) - \sigma(Y)g(GHX,Z) \\ &= 0. \end{aligned}$$

Therefore $S(X,Y)=0$.

In a similar way, we can also show that $T(X,Y)=0$. Therefore M is normal. □

At the moment, normality appers to be a local notion since the tensors S and T were defined locally. Our next step is to show that normality is, in fact, a global notion. Towards this end let us define a third tensor W as follows:

$$\begin{aligned} W(X,Y) &= [G,H](X,Y) + \frac{1}{2}(\sigma(GX)GY - \sigma(HX)HY - \sigma(GY)GX \\ &\quad +\sigma(HY)HX) - u(Y)HX - v(Y)GX + u(X)HY \\ &\quad +v(X)GY + 2g(X,GY)V + 2g(X,HY)U \end{aligned}$$

where $[G,H](X,Y) = \frac{1}{2}([GX,HY]+[HX,GY]-G[HX,Y]-H[GX,Y]-G[X,HY]-H[X,GY])$.

If M is normal, in other words if

$$\begin{aligned} &S(U,X) = T(V,X) = 0 \quad \text{for all } X \text{, and} \\ &S(X,Y) = T(X,Y) = 0 \quad \text{for all } X \text{ and } Y \text{ in } \mathcal{H}, \end{aligned}$$

then equations (I) and (II) hold. Then using (I) and (II), we get

$$\begin{aligned} g([G,H](X,Y),Z) &= \frac{1}{2}(\sigma(HX)g(HY,Z) - \sigma(GX)g(GY,Z) - 4u(Z)g(X,HY) \\ &\quad -4v(Z)g(X,GY) + \sigma(GY)g(GX,Z) - \sigma(HY)g(HX,Z) \\ &\quad +u(X)\Omega(GZ,Y) - v(X)\Omega(HZ,Y) + v(Y)\Omega(HZ,X) \\ &\quad -u(Y)\Omega(GZ,X)). \end{aligned}$$

Hence for X, Y in $\mathcal{H}$

$$\begin{aligned} W(X,Y) &= \frac{1}{2}(\sigma(HX)HY - \sigma(GX)GY - 4g(X,HY)U - 4g(X,GY)V \\ &\quad +\sigma(GY)GX - \sigma(HY)HX + \sigma(GX)GY - \sigma(HX)HY - \sigma(GY)GX \\ &\quad +\sigma(HY)HX) + 2g(X,GY)V + 2g(X,HY)U \\ &= 0. \end{aligned}$$

Now we want to check the normality condition on an overlap $\mathcal{O} \cap \mathcal{O}'$. On the open set $\mathcal{O}$, we have tensors u, v, G, H, S, T and W. On $\mathcal{O}'$, we have u', v', G', H', S', T'. Since M is a contact metric manifold, there are functions a and b on $\mathcal{O} \cap \mathcal{O}'$ such that

$$u' = au - bv$$

$$v' = bu + av$$

$$G' = aG - bH$$

$$H' = bG + aH$$

$$a^2 + b^2 = 1.$$

Lemma 1.8 *$S' = a^2S + b^2T - 2abW$ and $T' = b^2S + a^2T + 2abW$.*

Proof: First of all $U' = aU - bV$ and $V' = bU + aV$.Using this fact we can compute σ' as follows:

$$\begin{aligned}
\sigma'(X) &= g(\nabla_X U', V') \\
&= g(\nabla_X(aU - bV), bU + aV) \\
&= ga\nabla_X U - b\nabla_X V + X(a)U - X(b)V, bU + aV) \\
&= a^2 g(\nabla_X U, V) - b^2 g(\nabla_X V, U) + bX(a) - aX(b) \\
&= \sigma(X) + bX(a) - aX(b).
\end{aligned}$$

Note that $aX(a) + bX(b) = 0$ for any X since $a^2 + b^2 = 1$. Also $G'H' = GH$. Now let us compute $S'(X,Y)$ using what we have so far and grouping terms under a^2, b^2 and ab:

$$\begin{aligned}
S'(X,Y) = \; & a^2[(\nabla_{GX}G)Y - (\nabla_{GY}G)X - G(\nabla_X G)Y + G(\nabla_Y G)X \\
& +2v(Y)HX - 2v(X)HY + 2g(X,GY)U - 2g(X,HY)V \\
& -\sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY - \sigma(Y)GHX] \\
& +b^2[(\nabla_{HX}H)Y - (\nabla_{HY}H)X - H(\nabla_X H)Y + H(\nabla_Y H)X \\
& +2u(Y)GX - 2u(X)GY + 2g(X,HY)V - 2g(X,GY)U \\
& +\sigma(HX)GY - \sigma(HY)GX - \sigma(X)HGY + \sigma(Y)HGX]
\end{aligned}$$

$$
\begin{aligned}
&-ab[(\nabla_{GX}H)Y + (\nabla_{HX}G)Y - (\nabla_{GY}H)X - (\nabla_{HY}G)X \\
&-G(\nabla_X H)Y - H(\nabla_X G)Y + G(\nabla_Y H)X + H(\nabla_Y G)X \\
&-2u(Y)HX - 2v(Y)GX + 2u(X)HY + 2v(X)GY \\
&+4g(X,GY)V + 4g(X,HY)U + \sigma(GX)GY - \sigma(HX)HY \\
&-\sigma(GY)GX + \sigma(HY)HX] + [aGX(a) - bHX(a) - ab^2GX(a) \\
&+b^3HX(a) + a^2bGX(b) - ab^2HX(b)]GY + [-aGX(b) \\
&+bHX(b) - a^2bGX(a) + ab^2HX(a) + a^3GX(b) - a^2bHX(b)]HY \\
&+[-aGY(a) + bHY(a) + ab^2GY(a) - b^3HY(a) - a^2bGY(b) \\
&+ab^2HY(b)]GX + [aGY(b) - bHY(b) + a^2bGY(a) - ab^2HY(a) \\
&-a^3GY(b) + a^2bHY(b)]HX + [aX(b) - bX(a) + bX(a) \\
&-aX(b))GHY + (-aY(b) + bY(a) - bY(a) + aY(b)]GHX \\
= \;& a^2S(X,Y) + b^2T(X,Y) - 2abW(X,Y).
\end{aligned}
$$

Applying the same process to $T'(X,Y)$ we get

$$T'(X,Y) = b^2S(X,Y) + a^2T(X,Y) + 2abW(X,Y).$$

The proof of the lemma is complete. □

Now assume that $S(X,Y) = T(X,Y) = 0$ for all horizontal X and Y, and $S(U,X) = T(V,X) = 0$ for all X. Then, as we checked above, $W(X,Y) = 0$ for all horizontal X and Y. Therefore, $S'(X,Y) = T'(X,Y) = 0$ by the above lemma.

For an arbitrary vector field X, let us apply the above lemma to $S'(U',X)$ to get

$$
\begin{aligned}
S'(U',X) &= a^2S(U',X) + b^2T(U',X) - 2abW(U',X) \\
&= a^3S(U,X) - a^2bS(V,X) + ab^2T(U,X) - b^3T(V,X) - 2a^2bW(U,X) \\
&\quad +2ab^2W(V,X) \\
&= -a^2b[-(\nabla_{GX}G)V - G(\nabla_V G)X + G(\nabla_X G)V - 2HX + \sigma(V)GHX]
\end{aligned}
$$

$$+ab^2[-(\nabla_{HX}H)U - H(\nabla_U H)X + H(\nabla_X H)U - 2GX - \sigma(U)HGX]$$

$$-a^2b[-(\nabla_{GX}H)U - (\nabla_{HX}G)U - G(\nabla_U H)X + G(\nabla_X H)U$$

$$-H(\nabla_U G)X + H(\nabla_X G)U + 2HX] + ab^2[-(\nabla_{GX}H)V$$

$$-(\nabla_{HX}G)V - G(\nabla_V H)X + G(\nabla_X H)V - H(\nabla_V G)X$$

$$+H(\nabla_X G)V + 2GX]$$

$$= -a^2b[G(\nabla_{GX}V - G(\nabla_V G)X - G^2\nabla_X V - 2HX + \sigma(V)GHX]$$

$$+ab^2[H(\nabla_{HX}U - H(\nabla_U H)X - H^2\nabla_X U - 2GX - \sigma(U)HGX]$$

$$-a^2b[2HX + H(\nabla_{GX}U + G(\nabla_{HX}U - G(\nabla_U H)X - GH\nabla_X U$$

$$-H(\nabla_U G)X - HG\nabla_X U] + ab^2[2GX + H\nabla_{GX}V + G\nabla_{HX}V$$

$$-G(\nabla_V H)X - GH\nabla_X V - H(\nabla_V G)X - HG\nabla_X V]$$

$$= -a^2b[-G(\nabla_V G)X - 4HX + \sigma(V)GHX] + ab^2[-H(\nabla_U H)X$$

$$-4GX - \sigma(U)HGX] - a^2b[4HX - G(\nabla_U H)X - H(\nabla_U G)X]$$

$$+ab^2[4GX - G(\nabla_V H)X - H(\nabla_V G)X]$$

$$= a^2b[G(\nabla_V G)X + \sigma(V)HGX + G(\nabla_U H)X + H(\nabla_U G)X]$$

$$-ab^2[H(\nabla_U H)X + \sigma(U)HGX + G(\nabla_V H)X + H(\nabla_V G)X].$$

Now, taking the inner product with Y and using equations (I) and (II) gives

$$g(S'(U',X),Y) = -a^2b[g((\nabla_V G)X, GY) - \sigma(V)g(HGX,Y) + g((\nabla_U H)X, GY)$$

$$+g((\nabla_U G)X, HY)] + ab^2[g((\nabla_U H)X, HY) - \sigma(U)g(HGX,Y)$$

$$+g((\nabla_V H)X, GY) + g((\nabla_V G)X, HY)]$$

$$= -a^2b[\sigma(V)g(HX,GY) + \Omega(G^2Y, GX) - 2gHGX, GY)$$

$$-\sigma(V)g(HGX,Y) - \sigma(U)g(GX,GY) - \Omega(HGY,HX)$$

$$-2g(GHX,GY) + \sigma(U)g(HX,HY)] + ab^2[-\sigma(U)g(GX,HY)$$

$$-\Omega(H^2Y,HX) - 2g(GHX,HY) - \sigma(U)G(HGX,Y)$$

$$-\sigma(V)g(GX,GY)+\sigma(V)g(HX,HY)+\Omega(GHY,GX)$$
$$-2g(HGX,HY)]$$
$$= -a^2b[-\Omega(GY,X)+\Omega(GY,X))+ab^2(\Omega(HY,X)-\Omega(HY,X)]$$
$$= 0.$$

Therefore $S'(U',X)=0$.

Similarly we can show that $T'(V',X)=0$.

Therefore normality conditions agree on the overlaps. So the notion of normality is global.

We now give an expression for $\nabla_X J$. Recall that on a complex contact manifold we have $H=GJ=-JG, V=-JU, U=JV$. Also, using Proposition 1.6 gives

$$(\nabla_X J)U = HX+\sigma(X)U - J(-GX+\sigma(X)V)=0$$

and

$$(\nabla_X J)V = -GX+\sigma(X)V - J(-HX-\sigma(X)U)=0.$$

Then we can write

$$(\nabla_X H)GY = (\nabla_X J)Y - J(\nabla_X G)GY.$$

Taking the inner product with Z and applying equations (I) and (II) gives

$$\text{(III)} \qquad g((\nabla_X J)Y,Z) = u(X)(\Omega(Z,GY)-2g(HY,Z))+v(X)(\Omega(Z,HY) + 2g(GY,Z)).$$

1.3 Some basic facts on normal complex contact metric manifolds

In this section, we will establish some basic formulas for a normal complex contact metric manifold M with structure tensors u,v,U,V,G,H,J,g. First, we will consider the curvature of the vertical plane, $g(R(U,V)V,U)$. Using Proposition 1.6,

$$\begin{aligned} R(U,V)V &= \nabla_U(-\sigma(V)U) - \nabla_V(-\sigma(U)U) + \sigma([U,V])U \\ &= -U(\sigma(V))U - \sigma(V)\sigma(U)V + V(\sigma(U))U + \sigma(U)\sigma(V)V \\ &\quad +\sigma([U,V])U \\ &= -2\Omega(U,V)U. \end{aligned}$$

Therefore

$$g(R(U,V)V,U) = -2\Omega(U,V). \tag{1.26}$$

Now let X and Y be two horizontal vector fields. Then using Proposition 1.6,

$$\begin{aligned} R(X,Y)U &= \nabla_X(-GY + \sigma(Y)V) - \nabla_Y(-GX + \sigma(X)V) + G[X,Y] - \sigma([X,Y])V \\ &= -\nabla_X GY + X(\sigma(Y))V + \sigma(Y)\nabla_X V + \nabla_Y GX - Y(\sigma(X))V \\ &\quad -\sigma(X)\nabla_Y V + G\nabla_X Y - G\nabla_Y X - \sigma([X,Y])V \\ &= -(\nabla_X G)Y + (\nabla_Y G)X + 2\Omega(X,Y)V - \sigma(Y)HX + \sigma(X)HY. \end{aligned}$$

By equation (I) we know that

$$(\nabla_X G)Y = \sigma(X)HY + g(X,Y)U + g(JX,Y)V.$$

If we substitute this in $R(X,Y)U$ we get

$$R(X,Y)U = 2(g(X,JY) + \Omega(X,Y))V. \tag{1.27}$$

Similarly, using Proposition 1.6 we have

$$R(X,Y)V = -2(g(X,JY) + \Omega(X,Y))U. \tag{1.28}$$

Now, let us compute $R(X,U)U$ for horizontal X, using Proposition 1.6:

$$R(X,U)U = \nabla_X(\sigma(U)V) - \nabla_U(-GX + \sigma(X)V) + G[X,U] - \sigma([X,U])V$$

$$= X(\sigma(U))V + \sigma(U)(-HX - \sigma(X)U) + \nabla_U GX$$
$$-U(\sigma(X))V + \sigma(X)\sigma(U)U + G\nabla_X U - G\nabla_U X - \sigma([X,U])V$$
$$= 2\Omega(X,U)V - \sigma(U)HX + (\nabla_U G)X + X.$$

Since X is horizontal, $\Omega(X,U) = 0$ by (1.10), and $(\nabla_U G)X = \sigma(U)HX$ by equation (I). Therefore

$$R(X,U)U = X. \tag{1.29}$$

Similarly,

$$R(X,V)V = X. \tag{1.30}$$

Again, for a horizontal vector field X we will compute $R(X,U)V$ and $R(X,V)U$ using Proposition 1.6 as follows:

$$\begin{aligned} R(X,U)V &= \nabla_X(-\sigma(U)U) - \nabla_U(-HX - \sigma(X)U) + H[X,U] + \sigma([X,U])U \\ &= -X(\sigma(U))U - \sigma(U)(-GX + \sigma(X)V) + \nabla_U HX + U(\sigma(X))U \\ &\quad +\sigma(X)\sigma(U)V + H\nabla_X U - H\nabla_U X + \sigma([X,U])U \\ &= \sigma(U)GX + (\nabla_U H)X - JX. \end{aligned} \tag{1.31}$$

Similarly,

$$R(X,V)U = -\sigma(V)HX + (\nabla_V G)X + JX. \tag{1.32}$$

Now we want to define a new tensor P_G as follows: For a $(1,1)$ tensor G, let

$$P_G(X,Y,Z,W) = g(R(X,Y)GZ,W) + g(R(X,Y)Z,GW).$$

In this way we also have P_H and P_J.

Our next step is to get an expression for P_G free of the curvature tensor R. By a direct computation, it is easy to see that we can write

$$P_G(X,Y,Z,W) = -(\nabla_X\nabla_Y\hat{G} - \nabla_Y\nabla_X\hat{G} - \nabla_{[X,Y]}\hat{G})(Z,W).$$

For horizontal vector fields X, Y, Z and W, if we compute the right hand side of the above equation using (I), we get:

$$\begin{aligned} P_G(X,Y,Z,W) &= 2g(HZ,W)\Omega(X,Y) - 2g(HX,Y)\Omega(Z,W) \\ &\quad +4g(HX,Y)g(JZ,W) + g(GX,Z)g(Y,W) \\ &\quad +g(HX,Z)g(JY,W) - g(GX,W)g(Y,Z) \\ &\quad -g(HX,W)g(JY,Z) - g(GY,Z)g(X,W) \\ &\quad -g(HY,Z)g(JX,W) + g(GY,W)g(X,Z) \\ &\quad +g(HY,W)g(JX,Z). \end{aligned} \tag{1.33}$$

In the same way, we can show that

$$\begin{aligned} P_H(X,Y,Z,W) &= -2g(GZ,W)\Omega(X,Y) + 2g(GX,Y)\Omega(Z,W) \\ &\quad -4g(GX,Y)g(JZ,W) + g(HX,Z)g(Y,W) \\ &\quad -g(GX,Z)g(JY,W) - g(HX,W)g(Y,Z) \\ &\quad +g(GX,W)g(JY,Z) - g(HY,Z)g(X,W) \\ &\quad +g(GY,Z)g(JX,W) + g(HY,W)g(X,Z) \\ &\quad -g(GY,W)g(JX,Z). \end{aligned} \tag{1.34}$$

Since $JX = HGX = -GHX$ for horizontal X,

$$\begin{aligned} P_J(X,Y,Z,W) &= g(R(X,Y)HGZ,W) - g(R(X,Y)Z,GHW) \\ &= P_H(X,Y,GZ,W) - P_G(X,Y,Z,HW) \\ &= 2g(GX,Y)\Omega(GZ,W) + 2g(HX,Y)\Omega(HZ,W) \\ &\quad +4g(GX,Y)g(HZ,W) - 4g(HX,Y)g(GZ,W). \end{aligned} \tag{1.35}$$

Lemma 1.9 *For horizontal vector fields X, Y, Z and W, the curvature tensor satisfies the following equations:*

(i) $g(R(GX,GY)GZ,GW) = g(R(X,Y)Z,W) - 2g(JZ,W)\Omega(X,Y)$
$$+ 2g(HX,Y)\Omega(GZ,W) + 2g(JX,Y)\Omega(Z,W) - 2g(HZ,W)\Omega(GX,Y),$$

(ii) $g(R(HX,HY)HZ,HW) = g(R(X,Y)Z,W) - 2g(JZ,W)\Omega(X,Y)$
$$- 2g(GX,Y)\Omega(HZ,W) + 2g(JX,Y)\Omega(Z,W) + 2g(GZ,W)\Omega(HX,Y).$$

Proof: By the definition of P_G, the left hand side of (i) is equal to

$$g(R(X,Y)Z,W) + P_G(Z,W,X,GY) + P_G(GX,GY,Z,GW).$$

Equation (1.33) gives

$$P_G(Z,W,X,GY) + P_G(GX,GY,Z,GW) = 2g(JX,Y)\Omega(Z,W)$$
$$-2g(HZ,W)\Omega(GX,Y) - 2g(JZ,W)\Omega(X,Y) + 2g(HX,Y)\Omega(GZ,W).$$

Therefore equation (i) holds.

Similarly, using the definition of P_H and equation (1.34) we obtain (ii). □

Lemma 1.10 *The following equations hold for horizontal vector fields X, Y, Z and W:*

(i) $g(R(X,GX)Y,GY) = g(R(X,Y)X,Y) + g(R(X,GY)X,GY)$
$$+ 4g(JX,Y)\Omega(X,Y) - 4g(HX,Y)\Omega(GX,Y) - 2g(GX,Y)^2$$
$$- 4g(HX,Y)^2 - 2g(X,Y)^2 + 2g(X,X)g(Y,Y) - 4g(JX,Y)^2$$

(ii) $g(R(X,HX)Y,HY) = g(R(X,Y)X,Y) + g(R(X,HY)X,HY)$
$$+ 4g(JX,Y)\Omega(X,Y) + 4g(GX,Y)\Omega(HX,Y) - 2g(HX,Y)^2$$
$$- 4g(GX,Y)^2 - 2g(X,Y)^2 + 2g(X,X)g(Y,Y) - 4g(JX,Y)^2.$$

Proof: By Bianchi's first identity

$$g(R(X,GX)Y,GY) = -g(R(GX,Y)X,GY) - g(R(Y,X)GX,GY).$$

The definition of P_G implies

$$-g(R(GX,Y)X,GY) = g(R(X,GY)X,GY) - P_G(X,GY,X,Y)$$

and

$$-g(R(Y,X)GX,GY) = g(R(X,Y)X,Y) + P_G(X,Y,X,GY).$$

Using equation (1.33) we get

$$\begin{aligned} P_G(X,Y,X,GY) - P_G(X,GY,X,Y) &= 4g(JX,Y)\Omega(X,Y) \\ -4g(HX,Y)\Omega(GX,Y) &- 4g(HX,Y)^2 - 2g(X,Y)^2 \\ -4g(JX,Y)^2 &- 2g(GX,Y)^2 + 2g(X,X)g(Y,Y) \end{aligned}$$

which gives equation (i) and equation (ii) is obtained in the same way. □

Lemma 1.11 *If X is a horizontal vector field, then*

$$\begin{aligned} g(R(X,GX)GX,X) + g(R(X,HX)HX,X) + g(R(X,JX)JX,X) \\ = -6g(X,X)(\Omega(JX,X) + g(X,X)). \end{aligned}$$

Proof: Recall that $GX = -HJX$. Then by the definition of P_H

$$g(R(X,GX)GX,X) = g(R(X,GX)JX,GJX) - P_H(X,GX,JX,X).$$

By Lemma 1.10

$$\begin{aligned} g(R(X,GX)JX,GJX) &= -g(R(X,JX)JX,X) - g(R(X,HX)HX,X) \\ &\quad -4g(X,X)\Omega(JX,X) - 2g(X,X)^2. \end{aligned}$$

We can compute $P_H(X,GX,JX,X)$ using equation (1.34) to get

$$P_H(X,GX,JX,X) = 2g(X,X)\Omega(JX,X) + 4g(X,X)^2.$$

We get the lemma by joining the above equations. □

We can use the definition of P_G and equation (1.33) to see that the following formulas hold for a horizontal vector field X:

$$\begin{aligned} g(R(X,HX)JX,GX) &= -g(R(X,HX)HX,X) \\ &\quad -2g(X,X)\Omega(JX,X) - 4g(X,X)^2, \end{aligned} \tag{1.36}$$

$$\begin{aligned} g(R(X,JX)HX,GX) &= g(R(X,JX)JX,X) \\ &\quad +2g(X,X)\Omega(JX,X) - 2g(X,X)^2, \end{aligned} \tag{1.37}$$

$$g(R(GX,HX)HX,GX) = g(R(X,JX)JX,X), \tag{1.38}$$

$$g(R(GX,JX)JX,GX) = g(R(X,HX)HX,X). \tag{1.39}$$

Similarly, using the definition of P_J and equation (1.35) we get the following formulas for horizontal vector fields X, Y:

$$g(R(JX,JY)JY,JX) = g(R(X,Y)Y,X), \tag{1.40}$$

$$\begin{aligned} g(R(X,Y)JX,JY) &= g(R(X,Y)Y,X) + 2g(X,GY)\Omega(X,HY) \\ &\quad -2g(X,HY)\Omega(X,GY) + 4g(X,GY)^2 + 4g(X,HY)^2, \end{aligned} \tag{1.41}$$

$$g(R(Y,JX)JX,Y) = g(R(X,JY)JY,X), \tag{1.42}$$

$$\begin{aligned} g(R(X,JY)JX,Y) &= g(R(X,JY)JY,X) - 2g(X,HY)\Omega(X,GY) \\ &\quad +2g(X,GY)\Omega(X,HY) + 4g(X,HY)^2 + 4g(X,GY)^2. \end{aligned} \tag{1.43}$$

By Bianchi's 1st identity

$$g(R(X,JX)JY,Y) = -g(R(JX,JY)X,Y) - g(R(JY,X)JX,Y).$$

Substituting formulas (1.41) and (1.43) in the above equation we get

$$\begin{aligned} g(R(X,JX)JY,Y) &= g(R(X,Y)Y,X) + g(R(X,JY)JY,X) \\ &\quad +4(g(X,GY)\Omega(X,HY) - g(X,HY)\Omega(X,GY) \\ &\qquad +2g(X,GY)^2 + 2g(X,HY)^2 \end{aligned} \tag{1.44}$$

Chapter 2

Normal complex contact metric manifolds with constant GH-sectional curvature

2.1 GH-sectional curvature

Let M be a normal complex contact metric manifold with structure tensors u, v, U, V, G, H, J, g. For a horizontal vector field X, the plane section generated by X and $Y = aGX + bHX$,$a^2 + b^2 = 1$, is called a *GH-section* or an *$\mathcal{H}$-holomorphic section*. We define the GH-sectional curvature $\mathcal{GH}_{a,b}(X)$ as the curvature of a GH-section:

$$\mathcal{GH}_{a,b}(X) = K(X, aGX + bHX)$$

where $K(X,Y)$ is the curvature of the plane section generated by X and Y.

Lemma 2.1 *$\mathcal{GH}_{a,b}(X)$ is independent of the choice of the numbers a and b if and only if $K(X,GX) = K(X,HX)$ and $g(R(X,GX)HX,X) = 0$.*

Proof: We can write the GH-sectional curvature as

$$\mathcal{GH}_{a,b}(X) = a^2K(X,GX) + b^2K(X,HX) + \frac{2ab}{g(X,X)^2}g(R(X,GX)HX,X).$$

If $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b, then taking $a = 1, b = 0$ gives $\mathcal{GH}_{a,b}(X) = K(X,GX)$ and taking $a = 0, b = 1$ gives $\mathcal{GH}_{a,b}(X) = K(X,HX)$. So

$K(X,GX) = K(X,HX)$ and $g(R(X,GX)HX,X) = 0$.

Conversely, if $K(X,GX) = K(X,HX) = K$ and $g(R(X,GX)HX,X) = 0$ then $\mathcal{GH}_{a,b}(X) = K$ and hence $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b.□

From now on we will assume that $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b, and denote it by $\mathcal{GH}(X)$.

As the next step, we want to write holomorphic curvature in terms of GH-sectional curvature. In order to do this, we are going to use the formulas from section 1.3.

Proposition 2.2 *For a horizontal vector field X,*

$$K(X,JX) = \frac{1}{2}(\mathcal{GH}(X+GX) + \mathcal{GH}(X-GX)) + 3.$$

Proof: Since $\mathcal{GH}(X)$ is independent of the choice of a and b, we can choose $a = 0$, $b = 1$. Then $\mathcal{GH}(X) = K(X,HX)$. So $\mathcal{GH}(X+GX) = K(X+GX, HX+JX)$ and $\mathcal{GH}(X-GX) = K(X-GX, HX-JX)$. By direct computation we get

$$\begin{aligned} & g(R(X+GX,HX+JX)HX+JX,X+GX) \\ &\quad = g(R(X,HX)HX,X) + g(R(X,JX)JX,X) \\ &\quad\quad + g(R(GX,HX)HX,GX) + g(R(GX,JX)JX,GX) \\ &\quad\quad + 2[g(R(X,HX)HX,GX) + g(R(X,HX)JX,X) \\ &\quad\quad + g(R(X,HX)JX,GX) + g(R(X,JX)HX,GX) \\ &\quad\quad + g(R(X,JX)JX,GX) + g(R(GX,HX)JX,GX)] \end{aligned}$$

and

$$\begin{aligned} & g(R(X-GX,HX-JX)HX-JX,X-GX) \\ &\quad = g(R(X,HX)HX,X) + g(R(X,JX)JX,X) \end{aligned}$$

$$+g(R(GX,HX)HX,GX)+g(R(GX,JX)JX,GX)$$
$$+2[-g(R(X,HX)HX,GX)-g(R(X,HX)JX,X)$$
$$+g(R(X,HX)JX,GX)+g(R(X,JX)HX,GX)$$
$$-g(R(X,JX)JX,GX)-g(R(GX,HX)JX,GX)].$$

If we add the two equations above we get

$$\begin{aligned}
&\mathcal{GH}(X+GX)+\mathcal{GH}(X-GX)\\
&\quad= \frac{1}{2g(X,X)^2}[g(R(X,HX)HX,X)+g(R(X,JX)JX,X)\\
&\quad\quad +g(R(GX,HX)HX,GX)+g(R(GX,JX)JX,GX)\\
&\quad\quad +2[g(R(X,HX)JX,GX)+g(R(X,JX)HX,GX)]].
\end{aligned}$$

Now, using formulas (1.36)-(1.39) we have

$$\mathcal{GH}(X+GX)+\mathcal{GH}(X-GX)=2K(X,JX)-6.$$

Therefore

$$K(X,JX)=\frac{1}{2}(\mathcal{GH}(X+GX)+\mathcal{GH}(X-GX))+3. \ \square$$

We now want to work with the assumption that the GH-sectional curvature is independent of the choice of the GH-section at each point. Let $\mathcal{GH}(X)=c$, where c does not depend on X. Then by the previous proposition

$$K(X,JX)=c+3.$$

Next we give an expression for the sectional curvature in terms of the holomorphic curvature.

Lemma 2.3 *For horizontal vector fields X and Y, we have*

$$\begin{aligned}
g(R(X,Y)Y,X) &= \frac{1}{32}[3Q(X+JY)+3Q(X-JY)-Q(X+Y)\\
&\quad -Q(X-Y)-4Q(X)-4Q(Y)]+\frac{3}{2}[g(X,HY)\Omega(X,GY)\\
&\quad -g(X,GY)\Omega(X,HY)-2g(X,GY)^2-2g(X,HY)^2].
\end{aligned}$$

where $Q(X) = g(R(X, JX)JX, X)$.

Proof: By direct computation

$$\begin{aligned} Q(X+JY) &= g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) + g(R(JX,JY)JY,JX) \\ &\quad + g(R(X,Y)Y,X) + 2[g(R(X,JX)JX,JY) - g(R(X,JX)Y,X) \\ &\quad - g(R(X,JX)Y,JY) - g(R(X,Y)JX,JY) + g(R(X,Y)Y,JY) \\ &\quad - g(R(JY,JX)Y,JY)] \end{aligned}$$

and

$$\begin{aligned} Q(X-JY) &= g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) + g(R(JX,JY)JY,JX) \\ &\quad + g(R(X,Y)Y,X) + 2[-g(R(X,JX)JX,JY) + g(R(X,JX)Y,X) \\ &\quad - g(R(X,JX)Y,JY) - g(R(X,Y)JX,JY) - g(R(X,Y)Y,JY) \\ &\quad + g(R(JY,JX)Y,JY)]. \end{aligned}$$

By combining the two equations above, we get

$$\begin{aligned} Q(X+JY) + Q(X-JY) &= 2[g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) \\ &\quad + g(R(JX,JY)JY,JX) + g(R(X,Y)Y,X)] \\ &\quad - 4[g(R(X,JX)Y,JY) + g(R(X,Y)JX,JY)]. \end{aligned}$$

Using the formulas (1.40), (1.41) and (1.44) we have

$$\begin{aligned} Q(X+JY) + Q(X-JY) &= 2[g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y)] \\ &\quad + 4[3g(R(X,Y)Y,X) + g(R(X,JY)JY,X)] \\ &\quad + 24[g(X,GY)\Omega(X,HY) - g(X,HY)\Omega(X,GY) \\ &\quad + 2g(X,GY)^2 + 2g(X,HY)^2]. \end{aligned}$$

Doing the same calculations for $Q(X+Y)+Q(X-Y)$ and using the formulas (1.42), (1.43) and (1.44), we get

$$\begin{aligned} Q(X+Y)+Q(X-Y) &= 2[g(R(X,JX)JX,X)+g(R(Y,JY)JY,Y)] \\ &\quad +4[3g(R(X,JY)JY,X)+g(R(X,Y)Y,X)] \\ &\quad +24[g(X,GY)\Omega(X,HY)-g(X,HY)\Omega(X,GY) \\ &\quad +2g(X,GY)^2+2g(X,HY)^2]. \end{aligned}$$

Finally, combining what we have so far

$$\begin{aligned} &3Q(X+JY)+3Q(X-JY)-Q(X+Y)-Q(X-Y)-4Q(X)-4Q(Y) \\ &\qquad = 32g(R(X,Y)Y,X)+48[g(X,GY)\Omega(X,HY) \\ &\qquad\quad -g(X,HY)\Omega(X,GY)+2g(X,GY)^2+2g(X,HY)^2], \end{aligned}$$

giving us the desired result. □

Since $K(X,JX)=c+3$ does not depend on X, from the above lemma we get

$$\begin{aligned} g(R(X,Y)Y,X) &= \frac{c+3}{4}[g(X,X)g(Y,Y)-g(X,Y)^2+3g(X,JY)^2] \\ &\quad +\frac{3}{2}[g(X,HY)\Omega(X,GY)-g(X,GY)\Omega(X,HY) \\ &\quad -2g(X,GY)^2-2g(X,HY)^2], \end{aligned} \tag{2.1}$$

for horizontal X and Y.

Now let X and Y be two arbitrary vector fields. We can write

$$X=Z+u(X)U+v(X)V,\ Y=W+u(Y)U+v(Y)V$$

where Z and W are in $\mathcal{H}$. Then using the formulas (1.26)-(1.32) and (2.1), we have

$$\begin{aligned} g(R(X,Y)Y,X) &= g(R(Z,W)W,Z)-2(u(X)u(Y)+v(X)v(Y))g(Z,W) \\ &\quad +(u(Y)^2+v(Y)^2)g(Z,Z)+(u(X)^2+v(X)^2)g(W,W) \end{aligned}$$

$$
\begin{aligned}
& -12u\wedge v(X,Y)g(Z,JW) - 12u\wedge v(X,Y)\Omega(Z,W) \\
& -8(u\wedge v(X,Y))^2\Omega(U,V) \qquad (2.2)\\
= \; & g(R(Z,W)W,Z) - 2(u(X)u(Y)+v(X)v(Y))g(X,Y) \\
& +(u(Y)^2+v(Y)^2)g(X,X) + (u(X)^2+v(X)^2)g(Y,Y) \\
& -12u\wedge v(X,Y)g(X,JY) - 12u\wedge v(X,Y)\Omega(X,Y) \\
& +16(u\wedge v(X,Y))^2(1+\Omega(U,V)) \\
= \; & \frac{c-1}{2}[u(X)u(Y)+v(X)v(Y)]g(X,Y) \\
& -\frac{c-1}{4}[(u(Y)^2+v(Y)^2)g(X,X) + (u(X)^2+v(X)^2)g(Y,Y)] \\
& -3(c+7)u\wedge v(X,Y)g(X,JY) \\
& +\frac{c+3}{4}[g(X,X)g(Y,Y) + 3g(X,JY)^2 - g(X,Y)^2] \\
& +\frac{3}{2}[g(X,HY)\Omega(X,GY) - g(X,GY)\Omega(X,HY) - 2g(X,GY)^2 \\
& -2g(X,HY)^2] + 4(c+7)(u\wedge v(X,Y))^2 - 12u\wedge v(X,Y)\Omega(X,Y) \\
& +16(u\wedge v(X,Y))^2\Omega(U,V). \qquad (2.3)
\end{aligned}
$$

In order to simplify the above equation somewhat, we need to examine the term $\Omega(X,Y)$. Since $\mathcal{GH}(X) = c+3$ does not depend on X,

$$g(R(X,GX)GX,X) = g(R(X,HX)HX,X) = cg(X,X)^2$$

and

$$g(R(X,JX)JX,X) = (c+3)g(X,X)^2.$$

Substituting these in Lemma 1.11 we get

$$\Omega(JX,X) = -\frac{c+3}{2}g(X,X) \qquad (2.4)$$

for horizontal X.

In order to compute $\Omega(JX, X)$ for an arbitrary vector field X, we can apply formula (2.4) to the horizontal component of X to get

$$\begin{aligned}\Omega(JX,X) &= -\frac{c+3}{2}g(X,X)+\frac{c+3}{2}(u(X)^2+v(X)^2)\\ &\quad +(u(X)^2+v(X)^2)\Omega(U,V). \qquad (2.5)\end{aligned}$$

Replacing X with $JX+Y$ in (2.5) we have

$$\Omega(X,Y)=\frac{c+3}{2}g(JX,Y)+u\wedge v(X,Y)(c+3+2\Omega(U,V)) \qquad (2.6)$$

Now if we substitute (2.6) in (2.3) we get a somewhat simpler expression for the sectional curvature as

$$\begin{aligned}g(R(X,Y)Y,X) &= \frac{c-1}{2}[u(X)u(Y)+v(X)v(Y)]g(X,Y)\\ &\quad -\frac{c-1}{4}[(u(Y)^2+v(Y)^2)g(X,X)+(u(X)^2+v(X)^2)g(Y,Y)]\\ &\quad +3(c-1)u\wedge v(X,Y)g(X,JY)\\ &\quad +\frac{c+3}{4}[g(X,X)g(Y,Y)+3g(X,JY)^2-g(X,Y)^2]\\ &\quad +3\frac{c-1}{4}[g(X,GY)^2+g(X,HY)^2]\\ &\quad -8(u\wedge v(X,Y))^2(c+1+\Omega(U,V). \qquad (2.7)\end{aligned}$$

Now to get an expression for the curvature tensor, we will use the following identity of [2]:

$$6g(R(X,Y)Z,W)=\frac{\partial^2}{\partial s\partial t}(B(X+sW,Y+tZ)-B(X+sZ,Y+tW))|_{s=0,t=0},$$

where $B(X,Y)=g(R(X,Y)Y,X)$.

If we compute the right hand side of the above identity using (2.7), we get the following expression for the curvature tensor:

$$R(X,Y)Z = \frac{c-1}{4}[(u(X)u(Z)+v(X)v(Z))Y-(u(Y)u(Z)+v(Y)v(Z))X$$

$$+4u\wedge v(X,Y)JZ+2u\wedge v(X,Z)JY+2u\wedge v(Z,Y)JX$$
$$+2g(X,GY)GZ+g(X,GZ)GY+g(Z,GY)GX+2g(X,HY)HZ$$
$$+g(X,HZ)HY+g(Z,HY)HX+[u(Y)g(X,Z)-u(X)g(Y,Z)$$
$$+v(X)g(Z,JY)+v(Y)g(X,JZ)+2v(Z)g(X,JY)]U$$
$$+[v(Y)g(X,Z)-v(X)g(Y,Z)-u(X)g(Z,JY)-u(Y)G(X,JZ)$$
$$-2u(Z)g(X,JY)]V]$$
$$+\frac{c+3}{4}[g(Y,Z)X-g(X,Z)Y+g(Z,JY)JX$$
$$+g(X,JZ)JY+2g(X,JY)JZ]$$
$$-\frac{4}{3}(c+1+\Omega(U,V))[(v(X)u\wedge v(Z,Y)+v(Y)u\wedge v(X,Z)$$
$$+2v(Z)u\wedge v(X,Y))U-(u(X)u\wedge v(Z,Y)+u(Y)u\wedge v(X,Z)$$
$$+2u(Z)u\wedge v(X,Y))V]. \tag{2.8}$$

Now we are ready to prove the following proposition.

Proposition 2.4 *Let M be a normal comlex contact metric manifold with complex dimension greater than or equal to* 5. *If the $GH-$sectional curvature is independent of the choice of the GH-section at each point, then it is constant on M.*

Proof: Suppose that the complex dimension of M is $2n+1$. If the GH-sectional curvature is independent of the choice of the GH-section at each point, then the curvature tensor has the form (2.8). Let us choose a local orthonormal basis of the form

$$\{X_i, GX_i, HX_i, JX_i, U, V | 1\leq i\leq n\}.$$

Then the Ricci tensor has the form

$$S(X,Y) = \sum_{i=1}^{n}[g(R(X_i,X)Y,X_i)+g(R(GX_i,X)Y,GX_i)+g(R(HX_i,X)Y,HX_i)$$

$$+g(R(JX_i,X)Y,JX_i)]+g(R(U,X)Y,U)+g(R(V,X)Y,V)$$

$$= ((n+2)c+3n+2)g(X,Y)$$

$$+(-(n+2)c+n-2-2\Omega(U,V))(u(X)u(Y)+v(X)v(Y)).$$

The scalar curvature S has the form

$$S = \sum_{i=1}^{n}[S(X_i,X_i)+S(GX_i,GX_i)+S(HX_i,HX_i)$$

$$+S(JX_i,JX_i)]+S(U,U)+S(V,V)$$

$$= 2(n+2)(2n-1)c+4n(3n+4)-4\Omega(U,V).$$

Since $\Omega = d\sigma$, $d\Omega = 0$. In particular, $d\Omega(U,V,X)=0$, which implies

$$X\Omega(U,V) = u(X)U\Omega(U,V)+v(X)V\Omega(U,V).$$

By Bianchi's identity,

$$2[\sum_{i=1}^{n}((\nabla_{X_i}S)(X,X_i)+(\nabla_{GX_i}S)(X,GX_i)+(\nabla_{HX_i}S)(X,HX_i)$$

$$+(\nabla_{JX_i}S)(X,JX_i))+(\nabla_U S)(X,U)+(\nabla_V S)(X,V)]-\nabla_X S=0.$$

Substituting the expressions for $S(X,Y)$ and S, the above equation gives

$$2(1-n)X(c)-(u(X)U(c)+v(X)V(c))=0.$$

If we let $X=U$, we get $U(c)=0$, and if we let $X=V$, we get $V(c)=0$. Therefore, $X(c)=0$ if n is different from 1. So c is constant on M when $n>1$. □

Definition 2.5 *A normal complex contact metric manifold M with constant GH-sectional curvature is called a complex contact space form.*

The following theorem is an easy consequence of Proposition 2.2 and Lemma 2.3.

Theorem 2.6 *Let M be a normal complex contact metric manifold. Then M has constant GH-sectional curvature c if and only if for horizontal X, the holomorphic sectional curvature of the plane generated by X and JX is $c+3$.*

This theorem gives rise to a natural question; is it possible for a normal complex contact metric manifold to have constant holomorphic sectional curvature? We answer this question by the following proposition.

Proposition 2.7 *Let M be a normal complex contact metric manifold. If M has constant holomorphic sectional curvature c, then $c = 4$ and M is Kähler.*

Proof: For an arbitrary unit vector field X, let $X = Z + u(X)U + v(X)V$, where Z is horizontal. If we take $Y = JX$, $W = JZ$ in equation 2.2, we get

$$\begin{aligned} g(R(X,JX)JX,X) &= g(R(Z,JZ)JZ,Z) + 6(u(X)^2 + v(X)^2)\Omega(X,JX) \\ &\quad -4(u(X)^2 + v(X)^2) \\ &\quad +4(u(X)^2 + v(X)^2)^2(1 + \Omega(U,V)). \end{aligned} \tag{2.9}$$

Since M has constant holomorphic curvature c,

$$g(R(X,JX)JX,X) = g(R(U,V)V,U) = c,$$

and

$$g(R(Z,JZ)JZ,Z) = g(Z,Z)^2 c.$$

Theorem 2.6 implies that $\mathcal{GH}(X) = c - 3$. Also by formula 2.6

$$\Omega(X,Y) = \frac{c}{2} g(JX,Y) + u \wedge v(X,Y)(c + 2\Omega(U,V)).$$

Since $g(R(U,V)V,U) = -2\Omega(U,V)$, $\Omega(U,V) = -\frac{c}{2}$. Therefore $\Omega(X,Y) = \frac{c}{2} g(JX,Y)$, and hence $\Omega(X,JX) = \frac{c}{2}$.

Since X is unit, $g(Z,Z) = 1 - u(X)^2 - v(X)^2$. Substituting these back in 2.9, we get

$$(c-4)(u(X)^2 + v(X)^2)(1 - u(X)^2 - v(X)^2) = 0.$$

We can choose X so that $u(X) \neq 0$, $v(X) \neq 0$ and $u(X)^2 + v(X)^2 \neq 1$. Then we must have $c = 4$. In this case, $\mathcal{GH}(X) = 1$ and $\Omega(U,V) = -2$.

Since M is normal, by equation (III)

$$\begin{aligned} g((\nabla_X J)Y, Z) &= u(X)\Omega(Z, GY) + v(X)\Omega(Z, HY) - 2u(X)g(HY, Z) \\ &\quad + 2v(X)g(GY, Z) \\ &= 2u(X)g(JZ, GY) + 2v(X)g(JZ, HY) - 2u(X)g(HY, Z) \\ &\quad + 2v(X)g(GY, Z) \\ &= 0. \end{aligned}$$

Hence M is Kähler. □

Theorem 2.8 *Let M be a normal complex contact metric manifold with constant GH-sectional curvature* 1 *and $\Omega(U,V) = -2$. Then M has a constant holomorphic sectional curvature* 4 *and it is Kähler. If, in addition, M is complete and simply connected, then M is isometric to $\mathbf{CP}^{2n+1}$ with the Fubini-Study metric of constant holomorphic curvature* 4.

Proof: Since $\mathcal{GH}(X) = 1$, $g(R(X,JX)JX,X) = 4g(X,X)^2$ for a horizontal vector field X by Theorem 2.6. Substituting $c = 1$ and $\Omega(U,V) = -2$ in (2.6), we get $\Omega(X,Y) = 2g(JX,Y)$. For an arbitrary unit vector field X, let $X = Z + u(X)U + v(X)V$, where Z is horizontal. Then $g(Z,Z) = 1 - u(X)^2 - v(X)^2$. Now, from (2.9) it follows that

$$g(R(X,JX)JX,X) = 4(1 - u(X)^2 - v(X)^2)^2 - 4(u(X)^2 + v(X)^2)$$

$$+12(u(X)^2 + v(X)^2) - 4(u(X)^2 + v(X)^2)^2$$

$$= 4.$$

Hence M has constant holomorphic curvature 4, and by Proposition 2.7 M is Kähler. □

2.2 Examples of normal complex contact metric manifolds

Our first example of a normal complex contact metric manifold is the complex Heisenberg group.The complex Heisenberg group is the closed subgroup $\mathbf{H_C}$ of $GL(3, \mathbf{C})$ given by

$$\left\{ \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid b_{12}, b_{13}, b_{23} \in \mathbf{C} \right\}$$

Blair defined the following complex contact metric structure on $\mathbf{H_C}$ in [1]. See also [11]. Let z_1, z_2, z_3 be the coordinates on $\mathbf{H_C} \simeq \mathbf{C}^3$, defined by $z_1(B) = b_{23}, z_2(B) = b_{12}, z_3(B) = b_{13}$ for B in $\mathbf{H_C}$. Then the hermitian metric (matrix)

$$g = \frac{1}{8} \left(\begin{array}{ccc|ccc} & & & 1+|z_2|^2 & 0 & -z_2 \\ & 0 & & 0 & 1 & 0 \\ & & & -\bar{z}_2 & 0 & 1 \\ \hline 1+|z_2|^2 & 0 & -\bar{z}_2 & & & \\ 0 & 1 & 0 & & 0 & \\ -z_2 & 0 & 1 & & & \end{array} \right)$$

is a left invariant metric on $\mathbf{H_C}$. Define a holomorphic 1-form $\theta = \frac{1}{2}(dz_3 - z_2 dz_1)$ and set $\theta = u - iv$ and $4\frac{\partial}{\partial z_3} = U + iV$.

Also define a (1-1) tensor

$$G = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & z_2 & 0 \\ \hline 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & \bar{z}_2 & 1 & & & \end{array}\right)$$

Then $(u, v, U, V, G, H = GJ, g)$ is a complex contact metric structure on $\mathbf{H_C}$. Blair also computed the covariant derivatives of G and H as

$$(\nabla_X G)Y = g(X,Y)U - u(Y)X - g(X,JY)V - v(Y)JX + 2v(X)GHY$$

and

$$(\nabla_X H)Y = g(X,Y)V - v(Y)X - g(X,JY)U + u(Y)JX - 2u(X)GHY.$$

In [1], the following are also listed:

$$g(\nabla_X U, V) = 0,$$

$$\nabla_X U = -GX,$$

$$\nabla_X V = -HX.$$

As a consequence of the first equality, we see that σ is identically zero. Therefore, by Proposition 1.7 this structure on $\mathbf{H_C}$ is normal.

The hermitian connection of g is also given in [1]. So we can establish the following curvature identities easily:

$$g(R(X,GX)GX,X) = g(R(X,HX)HX,X) = -3g(X,X)^2,$$
$$g(R(X,GX)HX,X) = 0.$$

Therefore, $\mathbf{H_C}$ has constant GH-sectional curvature -3.

Our second example is the odd dimensional complex projective space $\mathbf{CP}^{2n+1}$ with the standard Fubini-Study metric g of constant holomorphic curvature 4. It is established in [8] that $(\mathbf{CP}^{2n+1}(4), g)$ admits a normal complex contact metric structure via the Hopf fibering

$$\pi : \mathbf{S}^{4n+3} \to \mathbf{CP}^{2n+1}.$$

Since this structure has constant holomorphic curvature 4, $(\mathbf{CP}^{2n+1}(4), g)$ has constant GH-sectional curvature 1 by Theorem 2.6.

2.3 $\mathcal{H}$-homothetic deformations

The odd dimensional complex projective space with the Fubini-Study metric is an example of a normal complex contact metric manifold with constant GH-sectional curvature 1.To get other examples with constant GH- sectional curvature, we need to study the $\mathcal{H}$-homothetic deformations.

Let M be a normal complex contact metric manifold with sructure tensors (u, v, U, V, G, H, g). For a positive constant α, we define new tensors by $\tilde{u} = \alpha u$, $\tilde{v} = \alpha v$, $\tilde{U} = \frac{1}{\alpha}U$, $\tilde{V} = \frac{1}{\alpha}V$, $\tilde{G} = G$, $\tilde{H} = H$, $\tilde{g} = \alpha g + \alpha(\alpha - 1)(u \otimes u + v \otimes v)$. This change of structure is called an *$\mathcal{H}$-homothetic deformation.*

Proposition 2.9 *If (u, v, U, V, G, H, g) is a normal complex contact metric structure on (M, J), then $(\tilde{u}, \tilde{v}, \tilde{U}, \tilde{V}, \tilde{G}, \tilde{H}, \tilde{g})$ is also a normal complex contact metric structure on (M, J).*

Proof: Clearly, $\tilde{\omega} = \alpha\omega$ is a complex contact structure on M. Also, $\tilde{\mathcal{H}} = \mathcal{H}$, $d\tilde{u}(\tilde{U}, X) = du(U, X) = 0$ for all X in $\mathcal{H}$, $\tilde{u}(\tilde{U}) = u(U) = 1$ and $\tilde{v}(\tilde{U}) = 0$. Now, let us check the first condition of Definition 1.2:

$$\begin{aligned}\tilde{G}^2 &= G^2 = -Id + u \otimes U + v \otimes V \\ &= -Id + \alpha u \otimes \frac{1}{\alpha} U + \alpha v \otimes \frac{1}{\alpha} V \\ &= -Id + \tilde{u} \otimes \tilde{U} + \tilde{v} \otimes \tilde{V},\end{aligned}$$

$$\begin{aligned}\tilde{g}(\tilde{G}X, Y) &= \tilde{g}(GX, Y) = \alpha g(GX, Y) \\ &= -\alpha g(X, GY) = -\tilde{g}(X, GY) \\ &= -\tilde{g}(X, \tilde{G}Y),\end{aligned}$$

$$\begin{aligned}\tilde{g}(\tilde{U}, X) &= \frac{1}{\alpha}\tilde{g}(U, X) = g(U, X) + (\alpha - 1)u(X) \\ &= u(X) + (\alpha - 1)u(X) = \alpha u(X) \\ &= \tilde{u}(X),\end{aligned}$$

$$\tilde{G}J = GJ = -JG = -J\tilde{G},$$

$$\tilde{G}\tilde{U} = G\tilde{U} = \frac{1}{\alpha} GU = 0.$$

If $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, then there are functions a and b on $\mathcal{O} \cap \mathcal{O}'$ which satisfy the second condition of definition 1.2. Then

$$\tilde{u}' = \alpha u' = \alpha(au - bv) = a\tilde{u} - b\tilde{v}$$

$$\tilde{v}' = \alpha v' = \alpha(bu + av) = b\tilde{u} + a\tilde{v}$$

$$\tilde{G}' = G' = aG - bH = a\tilde{G} - b\tilde{H}$$

$$\tilde{H}' = H' = bG + aH = b\tilde{G} + a\tilde{H}$$

$$a^2 + b^2 = 1.$$

Therefore the first condition of definition 1.3 is satisfied.

For horizontal X and Y, $d\tilde{u}(X,Y) = \alpha du(X,Y) = \alpha g(X,GY) = \tilde{g}(X,GY)$ and $d\tilde{v}(X,Y) = \alpha dv(X,Y) = \alpha g(X,HY) = \tilde{g}(X,HY)$. So the second condition of definition 1.3 is also satisfied and hence $(\tilde{u},\tilde{v},\tilde{U},\tilde{V},\tilde{G},\tilde{H},\tilde{g})$ is a complex contact metric structure on (M,J).

To check for normality, first we need to see how the covariant derivative changes.

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y,Z) + Y\tilde{g}(X,Z) - Z\tilde{g}(X,Y) \\
&\quad +\tilde{g}([X,Y],Z) + \tilde{g}([Z,X],Y) + \tilde{g}([Z,Y],X) \\
&= X(\alpha g(Y,Z) + \alpha(\alpha-1)u(Y)u(Z) + \alpha(\alpha-1)v(Y)v(Z)) \\
&\quad +Y(\alpha g(X,Z) + \alpha(\alpha-1)u(X)u(Z) + \alpha(\alpha-1)v(X)v(Z)) \\
&\quad -Z(\alpha g(Y,X) + \alpha(\alpha-1)u(Y)u(X) + \alpha(\alpha-1)v(Y)v(X)) \\
&\quad +\alpha g([X,Y],Z) + \alpha(\alpha-1)u([X,Y])u(Z) + \alpha(\alpha-1)v([X,Y])v(Z) \\
&\quad +\alpha g([Z,X],Y) + \alpha(\alpha-1)u([Z,X])u(Y) + \alpha(\alpha-1)v([Z,X])v(Y) \\
&\quad +\alpha g([Z,Y],X) + \alpha(\alpha-1)u([Z,Y])u(X) + \alpha(\alpha-1)v([Z,Y])v(X) \\
&= 2\alpha g(\nabla_X Y, Z) + \alpha(\alpha-1)[2u(\nabla_X Y)u(Z) + u(Z)g(Y,\nabla_X U) \\
&\quad +u(Y)g(Z,\nabla_X U) + 2v(\nabla_X Y)v(Z) + v(Z)g(Y,\nabla_X V) \\
&\quad +v(Y)g(Z,\nabla_X V) + u(X)g(Z,\nabla_Y U) + u(Z)g(X,\nabla_Y U) \\
&\quad +v(X)g(Z,\nabla_Y V) + v(Z)g(X,\nabla_Y V) - u(X)g(Y,\nabla_Z U) \\
&\quad -u(Y)g(X,\nabla_Z U) - v(X)g(Y,\nabla_Z V) - v(Y)g(X,\nabla_Z V)] \\
&= 2\tilde{g}(\nabla_X Y, Z) + \alpha(\alpha-1)[-u(Z)g(Y,GX) + u(Z)\sigma(X)v(Y) \\
&\quad -u(Y)g(Z,GX) + u(Y)\sigma(X)v(Z) - v(Z)g(Y,HX) - v(Z)\sigma(X)u(Y) \\
&\quad -v(Y)g(Z,HX) - v(Y)\sigma(X)u(Z) - u(X)g(Z,GY) + u(X)\sigma(Y)v(Z) \\
&\quad -u(Z)g(X,GY) + u(Z)\sigma(Y)v(X) - v(X)g(Z,HY) - v(X)\sigma(Y)u(Z) \\
&\quad -v(Z)g(X,HY) - v(Z)\sigma(Y)u(X) + u(X)g(Y,GZ) - u(X)\sigma(Z)v(Y)
\end{aligned}$$

$$
\begin{aligned}
&+u(Y)g(X,GZ)-u(Y)\sigma(Z)v(X)+v(X)g(Y,HZ)+v(X)\sigma(Z)u(Y) \\
&+v(Y)g(X,HZ)+v(Y)\sigma(Z)u(X)] \\
= \;& 2\tilde{g}(\nabla_X Y,Z)-2\alpha(\alpha-1)[u(Y)g(GX,Z)+v(Y)g(HX,Z) \\
&+u(X)g(GY,Z)+v(X)g(HY,Z)] \\
= \;& 2\tilde{g}(\nabla_X Y,Z)-2(\alpha-1)[u(Y)\tilde{g}(GX,Z)+v(Y)\tilde{g}(HX,Z) \\
&+u(X)\tilde{g}(GY,Z)+v(X)\tilde{g}(HY,Z)].
\end{aligned}
$$

Therefore

$$
\tilde{\nabla}_X Y=\nabla_X Y+(1-\alpha)[u(Y)GX+v(Y)HX+u(X)GY+v(X)HY] \qquad (2.10)
$$

If we take $Y=U$ in (2.10) we get

$$
\tilde{\nabla}_X U=\nabla_X U+(1-\alpha)GX.
$$

Hence

$$
\begin{aligned}
\tilde{\sigma}(X) &= \tilde{g}(\tilde{\nabla}_X\tilde{U},\tilde{V}) \\
&= \frac{1}{\alpha^2}\tilde{g}(\tilde{\nabla}_X U,V) \\
&= \frac{1}{\alpha}g(\tilde{\nabla}_X U,V)+\frac{\alpha-1}{\alpha}v(\tilde{\nabla}_X U) \\
&= \frac{1}{\alpha}g(\nabla_X U,V)+\frac{\alpha-1}{\alpha}v(\nabla_X U) \\
&= g(\nabla_X U,V)=\sigma(X).
\end{aligned}
$$

Thus $\sigma=\tilde{\sigma}$. Then

$$
\begin{aligned}
\tilde{S}(X,Y) \;=\;& \tilde{\nabla}_{\tilde{G}X}\tilde{G}Y-\tilde{G}\tilde{\nabla}_{\tilde{G}X}Y-\tilde{\nabla}_{\tilde{G}Y}\tilde{G}X+\tilde{G}\tilde{\nabla}_{\tilde{G}Y}X \\
&-\tilde{G}\tilde{\nabla}_X\tilde{G}Y+\tilde{G}^2\tilde{\nabla}_X Y+\tilde{G}\tilde{\nabla}_Y\tilde{G}X-\tilde{G}^2\tilde{\nabla}_Y X \\
&+2\tilde{v}(Y)\tilde{H}X-2\tilde{v}(X)\tilde{H}Y+2\tilde{g}(X,\tilde{G}Y)\tilde{U}-2\tilde{g}(X,\tilde{H}Y)\tilde{V} \\
&-\tilde{\sigma}(\tilde{G}X)\tilde{H}Y+\tilde{\sigma}(\tilde{G}Y)\tilde{H}X+\tilde{\sigma}(X)\tilde{G}\tilde{H}Y-\tilde{\sigma}(Y)\tilde{G}\tilde{H}X
\end{aligned}
$$

$$
\begin{aligned}
&= \nabla_{GX}GY - G\nabla_{GX}Y - (1-\alpha)G(u(Y)G^2X + v(Y)HGX) - \nabla_{GY}GX \\
&\quad +G\nabla_{GY}X + (1-\alpha)G(u(X)G^2Y + v(X)HGY) - G\nabla_X GY \\
&\quad -(1-\alpha)G(u(X)G^2Y + v(X)HGY) + G^2\nabla_X Y \\
&\quad +(1-\alpha)G^2(u(X)GY + v(X)HY + u(Y)GX + v(Y)HX) + G\nabla_Y GX \\
&\quad +(1-\alpha)G(u(Y)G^2X + v(Y)HGX) - G^2\nabla_Y X \\
&\quad -(1-\alpha)G^2(u(X)GY + v(X)HY + u(Y)GX + v(Y)HX) + 2\alpha v(Y)HX \\
&\quad -2\alpha v(X)HY + 2g(X,GY)U - 2g(X,HY)V - \sigma(GX)HY + \sigma(GY)HX \\
&\quad +\sigma(X)GHY - \sigma(Y)GHX \\
&= S(X,Y) + 2(\alpha-1)(v(Y)HX - v(X)HY).
\end{aligned}
$$

Similarly we can show that

$$\tilde{T}(X,Y) = T(X,Y) + 2(\alpha-1)(u(Y)GX - u(X)GY).$$

Thus

$$\tilde{S}(\tilde{U},X) = \frac{1}{\alpha}\tilde{S}(U,X) = \frac{1}{\alpha}S(U,X) = 0,$$

and

$$\tilde{T}(\tilde{V},X) = \frac{1}{\alpha}\tilde{T}(V,X) = \frac{1}{\alpha}T(V,X) = 0.$$

If X and Y are horizontal, then

$$\tilde{S}(X,Y) = S(X,Y) = 0,$$

and

$$\tilde{T}(X,Y) = T(X,Y) = 0.$$

Therefore the deformed sructure is also normal. □

Now we want to see what happens to the GH-sectonal curvature under an $\mathcal{H}$-homothetic deformation. First we check how the sectional curvature changes.

For horizontal vector felds X and Y,

$$
\begin{aligned}
\tilde{R}(X,Y)Y &= \tilde{\nabla}_X\tilde{\nabla}_Y Y - \tilde{\nabla}_Y\tilde{\nabla}_X Y - \tilde{\nabla}_{[X,Y]}Y \\
&= \tilde{\nabla}_X\nabla_Y Y - \tilde{\nabla}_Y\nabla_X Y - \nabla_{[X,Y]}Y - (1-\alpha)(u([X,Y])GY + v([X,Y])HY) \\
&= \nabla_X\nabla_Y Y + (1-\alpha)(u(\nabla_Y Y)GX + v(\nabla_Y Y)HX) - \nabla_Y\nabla_X Y \\
&\quad -(1-\alpha)(u(\nabla_X Y)GY + v(\nabla_X Y)HY) - \nabla_{[X,Y]}Y \\
&\quad -(1-\alpha)(u([X,Y])GY + v([X,Y])HY).
\end{aligned}
$$

Since X and Y are horizontal and M is normal, we have

$$u(\nabla_X Y) = g(\nabla_X Y, U) = -g(\nabla_X U, Y) = g(GX, Y),$$

and

$$v(\nabla_X Y) = g(\nabla_X Y, V) = -g(\nabla_X V, Y) = g(HX, Y).$$

Hence, $u(\nabla_Y Y) = v(\nabla_Y Y) = 0$, $u([X,Y]) = 2g(GX,Y)$, $v([X,Y]) = 2g(HX,Y)$. Therefore

$$\tilde{R}(X,Y)Y = R(X,Y)Y + 3(1-\alpha)(g(X,GY)GY + g(X,HY)HY)$$

for X, Y in $\mathcal{H}$. So, for horizontal vector fields X and Y,

$$\tilde{g}(\tilde{R}(X,Y)Y,X) = \alpha g(R(X,Y)Y,X) + 3\alpha(1-\alpha)(g(X,GY)^2 + g(X,HY)^2).$$

Assume that the original structure on M has constant GH-sectional curvature c. Let X be a unit horizontal vector field with respect to the new structure on M. Let $Y = a\tilde{G}X + b\tilde{H}X$ with $a^2+b^2=1$. Then $GY = -aX - bJX$ and $HY = aJX - bX$. Thus

$$
\begin{aligned}
\tilde{g}(\tilde{R}(X,Y)Y,X) &= \alpha g(R(X,Y)Y,X) \\
&\quad +3\alpha(1-\alpha)(g(X,-aX-bJX)^2 + g(X,aJX-bX)^2)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha c g(X,X)^2 + 3\alpha(1-\alpha)(a^2 g(X,X)^2 + b^2 g(X,X)^2) \\
&= \alpha c \frac{1}{\alpha^2}\tilde{g}(X,X)^2 + 3\alpha(1-\alpha)\frac{1}{\alpha^2}\tilde{g}(X,X)^2 \\
&= \frac{c}{\alpha} + \frac{3(1-\alpha)}{\alpha} \\
&= \frac{c+3}{\alpha} - 3.
\end{aligned}
$$

Hence the new structure has constant GH-sectional curvature $\frac{c+3}{\alpha} - 3$.

Next, we want to see how the curvature of the vertical plane changes under an $\mathcal{H}$-homothetic deformation. We know that $\sigma = \tilde{\sigma}$. So, $\Omega = \tilde{\Omega}$.Hence

$$
\begin{aligned}
\tilde{g}(\tilde{R}(\tilde{U},\tilde{V})\tilde{V},\tilde{U}) &= -2\tilde{\Omega}(\tilde{U},\tilde{V}) \\
&= -\frac{2}{\alpha^2}\Omega(U,V) = \frac{1}{\alpha^2}g(R(U,V)V,U).
\end{aligned}
$$

In particular, if $c = 1$ and $\Omega(U,V) = -2$ then the new structure has constant GH-sectional curvature $\frac{4}{\alpha} - 3$ with $\tilde{\Omega}(\tilde{U},\tilde{V}) = -\frac{2}{\alpha^2}$. This observation gives us the following theorem.

Theorem 2.10 *In addition to its standard structure, complex projective space* $\mathbf{CP}^{2n+1}$ *also carries a normal complex contact metric structure with constant* GH*-sectional curvature* $\frac{4}{\alpha} - 3$ *and* $\Omega(U,V) = -\frac{2}{\alpha^2}$ *for every* α *greater than* 0.

With this theorem we get examples of normal complex contact metric manifolds with constant GH-sectional curvature $\tilde{c} > -3$. Conversely, as we state in the following theorem, every such manifold is $\mathcal{H}$-homothetic to a normal complex contact metric manifold with constant GH-sectional curvature $c = 1$.

Theorem 2.11 *A normal complex contact metric manifold with metric* $\tilde{g}$ *of constant* GH*-sectional curvature* $\tilde{c} > -3$ *is* $\mathcal{H}$*-homothetic to a normal complex contact metric manifold with metric* g *of constant* GH*-sectional curvature* $c = 1$. *Moreover, if*

$\Omega(\tilde{U},\tilde{V}) = -\frac{(\tilde{c}+3)^2}{8}$ *then the metric* g *is Kähler and has constant holomorphic curvature* 4.

Proof: Let M be a normal complex contact metric manifold with metric $\tilde{g}$ of constant GH-sectional curvature $\tilde{c} > -3$. Apply an $\mathcal{H}$-homothetic deformation to $(M,\tilde{g})$ with $\alpha = \frac{\tilde{c}+3}{4} > 0$. We know that the new structure is also a normal complex contact metric structure with constant GH-sectional curvature $c = \frac{\tilde{c}+3}{\alpha} - 3 = (\tilde{c}+3)\frac{4}{\tilde{c}+3} - 3 = 1$. Moreover, if $\Omega(\tilde{U},\tilde{V}) = -\frac{(\tilde{c}+3)^2}{8}$ then $\Omega(U,V) = \frac{1}{\alpha^2}\Omega(\tilde{U},\tilde{V}) = -\frac{16}{(\tilde{c}+3)^2}\frac{(\tilde{c}+3)^2}{8} = -2$. Then by Theorem 2.8, (M,g) is Kähler and has constant holomorphic curvature 4. □

Chapter 3

Complex contact metric structures with $R(X,Y)\mathcal{V}=0$

3.1 Preliminaries

Let M be a complex contact metric manifold with structure tensors (u, v, U, V, G, H, g). Recall from Section 1.1 that we can write the covariant derivatives of U and V as

$$\nabla_X U = -GX - Gh_U X + \sigma(X)V \tag{3.1}$$

$$\nabla_X V = -HX - Hh_V X - \sigma(X)U \tag{3.2}$$

where $\sigma(X) = g(\nabla_X U, V)$ and $h_U, h_V : TM \to \mathcal{H}$ are symmetric operators such that $h_U G = -Gh_U$, $h_V H = -Hh_V$. Again from Section 1.1 we have

$$\hat{G} = du - \sigma \wedge v \tag{3.3}$$

$$\hat{H} = dv + \sigma \wedge u \tag{3.4}$$

where $\hat{G}(X,Y) = g(X, GY)$ and $\hat{H}(X,Y) = g(X, HY)$. Also recall from Section 1.2 that

$$(\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) = -3d(\sigma \wedge v)(X,Y,Z). \tag{3.5}$$

Lemma 3.1 *The following equations hold for horizontal vector fields X, Y and Z:*

(i) $(\nabla_X \hat{G})(GY, Z) = (\nabla_X \hat{G})(Y, GZ)$

(ii) $(\nabla_X \hat{G})(GY, GZ) = -(\nabla_X \hat{G})(Y, Z)$

(iii) $(\nabla_X \hat{G})(Y, Z) + (\nabla_{GX} \hat{G})(GY, Z) = \frac{3}{2}[d(\sigma \wedge v)(X, GY, GZ) - d(\sigma \wedge v)(X, Y, Z)$ $- d(\sigma \wedge v)(GX, GY, Z) - d(\sigma \wedge v)(GX, Y, GZ)]$.

Proof: The first two parts of the lemma can be seen easily by a direct computation. In order to show (iii), let

$$
\begin{aligned}
A(X,Y,Z) &= (\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) \\
&\quad + (\nabla_{GX} \hat{G})(GY,Z) + (\nabla_{GY} \hat{G})(Z,GX) + (\nabla_Z \hat{G})(GX,GY) \\
&\quad + (\nabla_{GX} \hat{G})(Y,GZ) + (\nabla_Y \hat{G})(GZ,GX) + (\nabla_{GZ} \hat{G})(GX,Y) \\
&\quad - (\nabla_X \hat{G})(GY,GZ) - (\nabla_{GY} \hat{G})(GZ,X) - (\nabla_{GZ} \hat{G})(X,GY).
\end{aligned}
$$

By equation 3.5, $A(X, Y, Z)$ is equal to two times the left-hand side of (iii). On the other hand, if we apply (i) and (ii), we see that $A(X, Y, Z)$ is equal to two times the right-hand side of (iii). Therefore (iii) holds.□

In this chapter, we will consider the complex contact metric manifolds with $h_U = h_V$. So from now on we will assume that $h_U = h_V = h$. Then h is a symmetric operator which anti-commutes with G and H. We want to compute some curvature terms, using the above two lemmas.

Let X be a horizontal vector field. Then by (3.1) and (3.2)

$$
\begin{aligned}
R(U,X)U &= \nabla_U \nabla_X U - \nabla_X \nabla_U U - \nabla_{[U,X]} U \\
&= \nabla_U(-GX - GhX + \sigma(X)V) - \nabla_X(\sigma(U)V) + G[U,X] \\
&\quad + Gh[U,X] - \sigma([U,X])V \\
&= -\nabla_U GX - \nabla_U GhX + U(\sigma(X))V - \sigma(X)\sigma(U)U \\
&\quad - X(\sigma(U))V - \sigma(U)(-HX - HhX - \sigma(X)U) + G\nabla_U X
\end{aligned}
$$

$$\begin{aligned}
& -G\nabla_X U + Gh\nabla_U X - Gh\nabla_X U - \sigma([U,X])V \\
= \; & -(\nabla_U G)X - (\nabla_U G)hX - G\nabla_U hX + 2d\sigma(U,X)V \\
& +\sigma(U)H(X+hX) - G(-GX - GhX + \sigma(X)V) + G\nabla_U hX \\
& -G(\nabla_U h)X - Gh(-GX - GhX + \sigma(X)V) \\
= \; & -\sigma(U)HX - \sigma(U)HhX + 2d\sigma(U,X)V + \sigma(U)HX \\
& +\sigma(U)HhX - X - hX - G(\nabla_U h)X + hX + h^2X.
\end{aligned}$$

Hence

$$R(U,X)U = 2d\sigma(U,X)V - X - G(\nabla_U h)X + h^2X. \tag{3.6}$$

If we replace X with GX in (3.6) and apply G, we get

$$\begin{aligned}
GR(U,GX)U &= X - G^2(\nabla_U h)GX + Gh^2GX \\
&= X + (\nabla_U h)GX - u((\nabla_U h)GX)U \\
&\quad -v((\nabla_U h)GX)V - h^2X.
\end{aligned}$$

Hence

$$\begin{aligned}
R(U,X)U - GR(U,GX)U &= -2X + 2h^2X - G(\nabla_U h)X - (\nabla_U h)GX \\
&\quad +u((\nabla_U h)GX)U + v((\nabla_U h)GX)V + 2d\sigma(U,X)V.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
G(\nabla_U h)X + (\nabla_U h)GX &= G\nabla_U hX - Gh\nabla_U X + \nabla_U hGX - h\nabla_U GX \\
&= \nabla_U GhX - (\nabla_U G)hX + hG\nabla_U X - \nabla_U GhX \\
&\quad -h(\nabla_U G)X - hG\nabla_U X \\
&= -\sigma(U)HhX - \sigma(U)hHX \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned} u((\nabla_U h)GX) &= g((\nabla_U h)GX, U) \\ &= -g((\nabla_U h)U, GX) \\ &= g(\nabla_U U, hGX) \\ &= g(\sigma(U)V, hGX) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v((\nabla_U h)GX) &= g((\nabla_U h)GX, V) \\ &= -g((\nabla_U h)V, GX) \\ &= g(\nabla_U V, hGX) \\ &= g(-\sigma(U)U, hGX) \\ &= 0. \end{aligned}$$

Therefore

$$R(U,X)U - GR(U,GX)U = -2X + 2h^2X + 2d\sigma(U,X)V. \tag{3.7}$$

We now compute $u([X,Y]), v([X,Y])$ and $d(\sigma \wedge v)(X,Y,Z)$ for horizontal vector fields X, Y, Z. If X and Y are in $\mathcal{H}$, then by equation (3.3) we have

$$\begin{aligned} u([X,Y]) &= -2du(X,Y) \\ &= -2g(X,GY) - 2(\sigma \wedge v)(X,Y). \end{aligned}$$

Hence

$$u([X,Y]) = 2g(GX,Y). \tag{3.8}$$

Similarly,

$$v([X,Y]) = 2g(HX,Y). \tag{3.9}$$

If X, Y, Z are in $\mathcal{H}$ then

$$\begin{aligned} d(\sigma \wedge v)(X,Y,Z) &= (d\sigma \wedge v - \sigma \wedge dv)(X,Y,Z) \\ &= \frac{1}{3}(\sigma(X)dv(Z,Y) + \sigma(Y)dv(X,Z) + \sigma(Z)dv(Y,X) \\ &= \frac{1}{3}(\sigma(X)g(Z,HY) - \sigma(X)\sigma \wedge u(Z,Y) + \sigma(Y)g(X,HZ) \\ &\quad -\sigma(Y)\sigma \wedge u(X,Z) + \sigma(Z)g(Y,HX) - \sigma(Z)\sigma \wedge u(Y,X)). \end{aligned}$$

Therefore

$$d(\sigma \wedge v)(X,Y,Z) = \frac{1}{3}[\sigma(X)g(Z,HY) + \sigma(Y)g(X,HZ) + \sigma(Z)g(Y,HX)]. \quad (3.10)$$

3.2 Structures with $R(X,Y)\mathcal{V} = 0$

Let M be a complex contact manifold with $h_U = h_V = h$. For horizontal vector fields Y and Z, if we apply equation (3.1) to $R(Y,Z)U$, we get

$$\begin{aligned} R(Y,Z)U &= \nabla_Y\nabla_Z U - \nabla_Z\nabla_Y U - \nabla_{[Y,Z]}U \\ &= \nabla_Y(-GZ - GhZ + \sigma(Z)V) - \nabla_Z(-GY - GhY + \sigma(Y)V \\ &\quad +G[Y,Z] + Gh[Y,Z] - \sigma([Y,Z])V \\ &= -\nabla_Y GZ - \nabla_Y GhZ + Y(\sigma(Z))V + \sigma(Z)(-HY - HhY - \sigma(Y)U) \\ &\quad \nabla_Z GY + \nabla_Z GhY - Z(\sigma(Y))V - \sigma(Y)(-HZ - HhZ - \sigma(Z)U) \\ &\quad +G\nabla_Y Z - G\nabla_Z Y + Gh\nabla_Y Z - Gh\nabla_Z Y - \sigma([Y,Z])V \\ &= -(\nabla_Y G)Z - (\nabla_Y Gh)Z + 2d\sigma(Y,Z)V - \sigma(Z)H(Y + hY) \\ &\quad +(\nabla_Z G)Y + (\nabla_Z Gh)Y + \sigma(Y)H(Z + hZ) \end{aligned}$$

Then for X in $\mathcal{H}$,

$$\begin{aligned} g(R(Y,Z)U,X) &= (\nabla_Y\hat{G})(Z,X) - g((\nabla_Y Gh)Z,X) \\ &\quad -\sigma(Z)g(H(Y+hY),X) + (\nabla_Z\hat{G})(X,Y) \\ &\quad +g((\nabla_Z Gh)Y,X) + \sigma(Y)g(H(Z+hZ),X). \end{aligned}$$

Equation (3.5) implies

$$(\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) = -(\nabla_X \hat{G})(Y,Z) - 3d(\sigma \wedge v)(X,Y,Z).$$

Then

$$\begin{aligned} g(R(Y,Z)U,X) &= g((\nabla_X G)Y,Z) - g((\nabla_Y Gh)Z,X) + g((\nabla_Z Gh)Y,X) \\ &\quad -\sigma(Z)g(H(Y+hY),X) + \sigma(Y)g(H(Z+hZ),X) \\ &\quad -3d(\sigma \wedge v)(X,Y,Z). \qquad (3.11) \end{aligned}$$

Let

$$\begin{aligned} C(X,Y,Z) &= g(R(Y,Z)U,X) - g(R(GY,GZ)U,X) \\ &\quad +g(R(Y,GZ)U,GX) + g(R(GY,Z)U,GX) \end{aligned}$$

and

$$\begin{aligned} B(X,Y,Z) &= -g(X,(\nabla_Y G)hZ) + g(X,h(\nabla_Y G)Z) \\ &\quad +g(X,hG(\nabla_{GY} G)Z) + g(X,G(\nabla_{GY} G)hZ). \end{aligned}$$

Lemma 3.2

$$\begin{aligned} C(X,Y,Z) &= B(X,Y,Z) - B(X,Z,Y) - 2\sigma(Z)g(X,HhY) \\ &\quad +2\sigma(Y)g(X,HhZ) + 2\sigma(GY)g(X,JhZ) - 2\sigma(GZ)g(X,JhY). \end{aligned}$$

Proof: If we compute $C(X,Y,Z)$ using (3.11) and Lemma 3.1 we get

$$\begin{aligned} C(X,Y,Z) &= -g((\nabla_Y Gh)Z,X) + g((\nabla_Z Gh)Y,X) + g((\nabla_{GY} Gh)GZ,X) \\ &\quad -g((\nabla_{GZ} Gh)GY,X) - g((\nabla_Y Gh)GZ,GX) + g((\nabla_{GZ} Gh)Y,GX) \\ &\quad -g((\nabla_{GY} Gh)Z,GX) + g((\nabla_Z Gh)GY,GX) - 2\sigma(Z)g(HhY,X) \\ &\quad +2\sigma(Y)g(HhZ,X) - 2\sigma(GZ)g(JhY,X) + 2\sigma(GY)g(JhZ,X). \end{aligned}$$

Now, let us rewrite $B(X,Y,Z)$ as follows:

$$
\begin{aligned}
B(X,Y,Z) &= -g(X,\nabla_Y GhZ) + g(X,G\nabla_Y hZ) + g(X,h\nabla_Y GZ) \\
&\quad -g(X,hG\nabla_Y Z) + g(X,hG\nabla_{GY}GZ) - g(X,hG^2\nabla_{GY}Z) \\
&\quad +g(X,G\nabla_{GY}GhZ) - g(X,G^2\nabla_{GY}hZ) \\
&= -g(X,(\nabla_Y Gh)Z) - g(X,Gh\nabla_Y Z) + g(GX,\nabla_Y hG^2 Z) \\
&\quad +g(X,h\nabla_Y GZ) + g(X,Gh\nabla_Y Z) + g(X,hG\nabla_{GY}GZ) \\
&\quad +g(GX,Gh\nabla_{GY}Z) - g(GX,(\nabla_{GY}Gh)Z) - g(GX,Gh\nabla_{GY}Z) \\
&\quad -g(X,\nabla_{GY}hG^2 Z) \\
&= -g(X,(\nabla_Y Gh)Z) - g(GX,(\nabla_Y Gh)GZ) - g(GX,Gh\nabla_Y GZ) \\
&\quad +g(X,h\nabla_Y GZ) + g(X,hG\nabla_{GY}GZ) - g(GX,(\nabla_{GY}Gh)Z) \\
&\quad +g(X,(\nabla_{GY}Gh)GZ) + g(X,Gh\nabla_{GY}GZ) \\
&= -g(X,(\nabla_Y Gh)Z) - g(GX,(\nabla_Y Gh)GZ) - g(GX,(\nabla_{GY}Gh)Z) \\
&\quad +g(X,(\nabla_{GY}Gh)GZ).
\end{aligned}
$$

Combining the expressions we have for $C(X,Y,Z)$ and $B(X,Y,Z)$ gives us the lemma.

□

Now, we state and prove the main theorem of this chapter.

Theorem 3.3 *Let M be a complex contact metric manifold with $h_U = h_V$. If $R(X,Y)\mathcal{V} = 0$, then M is locally isometric to $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$.*

Proof: Let $h = h_U = h_V$. Since $R(X,Y)\mathcal{V} = 0$, in particular $R(X,Y)U = 0$ for all X, Y. Then by (3.7), $-X + h^2X + d\sigma(U,X)V = 0$. Hence, $h^2X = X$ for horizontal X. Therefore h has two non-zero eigenvalues, $+1$ and -1. Let $[+1]$ denote $+1$ eigenspace of h, and $[-1]$ denote -1 eigenspace of h. Recall that h anti-commutes with G and H. So, h commutes with J since $J = HG - u\otimes V + v\otimes u$ and $hU = hV = 0$. Hence, if X is in $[+1]$ (resp. $[-1]$) then GX, HX are in $[-1]$ (resp. $[+1]$) and JX is in $[+1]$ (resp. $[-1]$). Therefore $[+1]$ and $[-1]$ are $2n$ dimensional and $\mathcal{H} = [+1] \bigoplus [-1]$.

If X is in $[+1]$, then $hX = X$ and hence $\nabla_X U = -2GX + \sigma(X)V$, $\nabla_X V = -2HX - \sigma(X)U$. On the other hand if X is in $[-1]$ then $\nabla_X U = \sigma(X)V$, $\nabla_X V = -\sigma(X)U$.

By assumption, $R(X,Y)U = 0$ for all X, Y. So, $C(X,Y,Z) = 0$ for horizontal X, Y and Z. We will use this fact to compute $(\nabla_X G)Y$ for horizontal X and Y. We claim that

$$(\nabla_X \hat{G})(Y,Z) + \sigma(X)g(Z,HY) = 0.$$

To prove this claim let us compute the right-hand side of the equation in Lemma 3.2 in eight different cases.

Case 1*:* Suppose that X and Z are in $[+1]$, Y is in $[-1]$. Then

$$B(X,Y,Z) = -2(\nabla_{GY}\hat{G})(GX,Z).$$

By part *(iii)* of Lemma 3.1 and equation (3.10),

$$\begin{aligned}
-2(\nabla_{GY}\hat{G})(GX,Z) &= 2(\nabla_Y\hat{G})(X,Z) + 3[d(\sigma\wedge v)(Y,X,Z) + d(\sigma\wedge v)(GY,GX,Z) \\
&\quad + d(\sigma\wedge v)(GY,X,GZ) - d(\sigma\wedge v)(Y,GX,GZ)] \\
&= 2(\nabla_Y\hat{G})(X,Z) + \sigma(Y)g(Z,HX) + \sigma(X)g(Y,HZ) \\
&\quad + \sigma(Z)g(X,HY) + \sigma(GY)g(Z,JX) + \sigma(GX)g(Y,JZ) \\
&\quad - \sigma(Z)g(X,HY) + \sigma(GY)g(Z,JX) - \sigma(X)g(Y,HZ) \\
&\quad + \sigma(GZ)g(X,JY) + \sigma(Y)g(Z,HX) - \sigma(GX)g(Y,JZ) \\
&\quad - \sigma(GZ)g(X,JY) \\
&= 2(\nabla_Y\hat{G})(X,Z) + 2\sigma(Y)g(Z,HX) + 2\sigma(GY)g(Z,JX).
\end{aligned}$$

Also, $B(X,Z,Y) = 2(\nabla_Z \hat{G})(X,Y)$. Now, by Lemma 3.2,

$$\begin{aligned} 0 &= C(X,Y,Z) \\ &= 2(\nabla_Y \hat{G})(X,Z) + 2\sigma(Y)g(Z,HX) + 2\sigma(GY)g(Z,JX) \\ &\quad +2(\nabla_Z \hat{G})(Y,X) + 2\sigma(Z)g(X,HY) + 2\sigma(Y)g(X,HZ) \\ &\quad +2\sigma(GY)g(X,JZ) + 2\sigma(GZ)g(X,JY). \end{aligned}$$

Since Z is in $[+1]$, HZ is in $[-1]$ and hence $g(X,HZ) = 0$. Similarly, $g(X,JY) = 0$.

By equation (3.5)

$$\begin{aligned} (\nabla_Y \hat{G})(X,Z) + (\nabla_Z \hat{G})(Y,X) &= -(\nabla_X \hat{G})(Z,Y) + 3d(\sigma \wedge v)(X,Y,Z) \\ &= (\nabla_X \hat{G})(Y,Z) + \sigma(X)g(Z,HY) \\ &\quad +\sigma(Y)g(X,HZ) + \sigma(Z)g(Y,HX). \end{aligned}$$

Hence

$$0 = 2(\nabla_X \hat{G})(Y,Z) + 2\sigma(X)g(Z,HY).$$

Case 2*:* Supose that X and Z are in $[-1]$, Y is in $[+1]$. Following the same procedure as in *Case* 1, we have

$$\begin{aligned} B(X,Y,Z) &= -2g(X, G(\nabla_{GY} G)Z) \\ &= 2(\nabla_{GY} \hat{G})(GX,Z) \\ &= -2(\nabla_Y \hat{G})(X,Z) - 2\sigma(Y)g(Z,HX) - 2\sigma(GY)g(Z,JX) \end{aligned}$$

and

$$B(X,Z,Y) = -2(\nabla_Z \hat{G})(X,Y).$$

So by Lemma 3.2

$$0 = C(X,Y,Z)$$

$$
\begin{aligned}
&= -2(\nabla_Y \hat{G})(X,Z) - 2\sigma(GY)g(Z,JX) - 2(\nabla_Z \hat{G})(Y,X) \\
&\quad -2\sigma(Z)g(X,HY) - 2\sigma(Y)g(X,HZ) - 2\sigma(GY)g(X,JZ) - 2\sigma(GZ)g(X,JY) \\
&= -2(\nabla_X \hat{G})(Y,Z) - 2\sigma(X)g(Z,HY) - 2\sigma(Z)g(Y,HX) - 2\sigma(Z)g(X,HY) \\
&= -2(\nabla_X \hat{G})(Y,Z) - 2\sigma(X)g(Z,HY).
\end{aligned}
$$

Case 3*:* Supose that X and Y are in $[-1]$, Z is in $[+1]$. To get this case, we can interchange Y and Z in *Case* 2 which gives us the claim.

Case 4*:* Supose that X and Y are in $[-1]$, Z is in $[+1]$. Again, this case is obtained by interchanging Y and Z in *Case* 1.

Case 5*:* Suppose that X is in $[+1]$, Y and Z are in $[-1]$. Then

$$B(X,Y,Z) = 2(\nabla_Y \hat{G})(X,Z)$$

and

$$B(X,Z,Y) = 2(\nabla_Z \hat{G})(X,Y)$$

So

$$
\begin{aligned}
0 &= C(X,Y,Z) \\
&= 2(\nabla_Y \hat{G})(X,Z) + 2(\nabla_Z \hat{G})(Y,X) + 2\sigma(Z)g(X,HY) \\
&\quad -2\sigma(Y)g(X,HZ) - 2\sigma(GY)g(X,JZ) + +2\sigma(GZ)g(X,JY) \\
&= 2(\nabla_X \hat{G})(Y,Z) + 2\sigma(X)g(Z,HY) + 2\sigma(Y)g(X,HZ) \\
&\quad +2\sigma(Z)g(Y,HX) + 2\sigma(Z)g(X,HY) - 2\sigma(Y)g(X,HZ) \\
&= 2(\nabla_X \hat{G})(Y,Z) + 2\sigma(X)g(Z,HY).
\end{aligned}
$$

Case 6*:* Suppose that X is in $[-1]$, Y and Z are in $[+1]$. In this case, $C(X,Y,Z)$ turns out to be just the negative of its value in *Case* 5. So we get the same result.

Case 7: Suppose that X, Y and Z are in $[+1]$. Then

$$\begin{aligned} B(X,Y,Z) &= -2(\nabla_{GY}\hat{G})(GX,Z) \\ &= 2(\nabla_Y\hat{G})(X,Z) + 2\sigma(Y)g(Z,HX) + 2\sigma(GY)g(Z,JX) \end{aligned}$$

and

$$\begin{aligned} B(X,Z,Y) &= -2(\nabla_{GZ}\hat{G})(GX,Y) \\ &= 2(\nabla_Z\hat{G})(X,Y) + 2\sigma(Z)g(Y,HX) + 2\sigma(GZ)g(Y,JX). \end{aligned}$$

So

$$\begin{aligned} 0 &= C(X,Y,Z) \\ &= 2(\nabla_Y\hat{G})(X,Z) + 2\sigma(GY)g(Z,JX) + 2(\nabla_Z\hat{G})(Y,X) \\ &\quad -2\sigma(GZ)g(Y,JX) - 2\sigma(Z)g(X,HY) + 2\sigma(Y)g(X,HZ) \\ &\quad +2\sigma(GY)g(X,JZ) - 2\sigma(GZ)g(X,JY) \\ &= 2(\nabla_X\hat{G})(Y,Z) + 2\sigma(X)g(Z,HY) + 2\sigma(Y)g(X,HZ) + 2\sigma(Z)g(Y,HX) \\ &= 2(\nabla_X\hat{G})(Y,Z) + 2\sigma(X)g(Z,HY). \end{aligned}$$

Case 8: Suppose that X, Y and Z are in $[-1]$. This case gives the same result as *Case* 7 since $C(X,Y,Z)$ is just the negative of its value in *Case* 7.

Hence the claim is proved.

We can easily compute the vertical component of $(\nabla_X G)Y$ using equations (3.1) and (3.2) as follows:

$$\begin{aligned} g((\nabla_X G)Y,U) &= -g((\nabla_X G)U,Y) \\ &= -g(\nabla_X U, GY) \\ &= g(G(X+hX),GY) \\ &= g(X+hX,Y) \end{aligned}$$

and

$$\begin{aligned} g((\nabla_X G)Y, V) &= -g((\nabla_X G)V, Y) \\ &= -g(\nabla_X V, GY) \\ &= g(H(X + hX), GY) \\ &= g(J(X + hX), Y). \end{aligned}$$

Combining with the previous claim we have

$$(\nabla_X G)Y = \sigma(X)HY + g(X + hX, Y)U + g(J(X + hX), Y)V \tag{3.12}$$

for horizontal X and Y.

Now, we need to examine the values of $d\sigma$ on $[+1]$ and $[-1]$. Recall that $\Omega = d\sigma$. First, we write Ω in terms of ∇U and ∇V as follows:

$$\begin{aligned} 2\Omega(X, Y) &= 2d\sigma(X, Y) \\ &= X(\sigma(Y)) - Y(\sigma(X)) - \sigma([X, Y]) \\ &= Xg(\nabla_Y U, V) - Yg(\nabla_X U, V) - g(\nabla_{[X,Y]}U, V) \\ &= g(\nabla_x \nabla_Y U, V) + g(\nabla_Y U, \nabla_X V) - g(\nabla_Y \nabla_X U, V) \\ &\quad -g(\nabla_X U, \nabla_Y V) - g(\nabla_{[X,Y]}U, V) \\ &= g(R(X, Y)U, V) + g(\nabla_Y U, \nabla_X V) - g(\nabla_X U, \nabla_Y V) \\ &= g(\nabla_Y U, \nabla_X V) - g(\nabla_X U, \nabla_Y V). \end{aligned}$$

Then $2\Omega(U, V) = g(\nabla_V U, \nabla_U V) - g(\nabla_U U, \nabla_V V) = 0$.

If X or Y, say X, is in $[-1]$ then

$$2\Omega(X, Y) = -g(\nabla_Y U, \sigma(X)U) - g(\sigma(X)V, \nabla_Y V) = 0.$$

For arbitrary X,

$$2\Omega(X, U) = g(\sigma(U)V, \nabla_X V) + g(\nabla_X U, \sigma(U)U) = 0$$

and

$$2\Omega(X,V) = g(\sigma(V)V, \nabla_X V) + g(\nabla_X U, \sigma(V)U) = 0.$$

Therefore, if X or Y is in $[-1]\bigoplus \mathcal{V}$ then $\Omega(X,Y) = 0$.

If both X and Y are in $[+1]$, then

$$\begin{aligned}
\Omega(X,Y) &= \frac{1}{2}[g(-2GY + \sigma(Y)V, -2HX - \sigma(X)U) \\
&\quad -g(-2GX + \sigma(X)V, -2HY - \sigma(Y)U)] \\
&= 2[g(Y, JX) + g(JX, Y)] \\
&= 4g(Y, JX).
\end{aligned}$$

Now let X, Y and Z be in $[-1]$. Then, by equations (3.10), (3.11) and (3.12)

$$\begin{aligned}
0 &= g(R(X,Y)U, Z) \\
&= -g((\nabla_X Gh)Y, Z) + g((\nabla_Y Gh)X, Z) \\
&= g(\nabla_X GY, Z) + g(Gh\nabla_X Y, Z) - g(\nabla_Y GX, Z) - g(Gh\nabla_Y X, Z) \\
&= g((\nabla_X G)Y, Z) - g(\nabla_X Y, GZ) + g(G\nabla_X Y, Z) \\
&\quad -g((\nabla_Y G)X, Z) + g(\nabla_Y X, GZ) - g(G\nabla_Y X, Z) \\
&= \sigma(X)g(HY, Z) - g([X,Y], GZ) + g(G[X,Y], Z) - \sigma(Y)g(HX, Z) \\
&= -2g([X,Y], GZ).
\end{aligned}$$

Therefore $[X, Y]$ is in $[-1]\bigoplus \mathcal{V}$. Also,

$$\begin{aligned}
0 &= g(R(X,U)U, Y) \\
&= g(\nabla_X(\sigma(U)V) - \nabla_U(\sigma(X)V) + G[X,U] \\
&\quad +Gh[X,U] - \sigma([X,U])V, Y) \\
&= 2g(G[X,U], Y) \\
&= -2g([X,U], GY),
\end{aligned}$$

and

$$\begin{aligned}
0 &= g(R(X,V)U,Y) \\
&= g(\nabla_X(\sigma(V)V) - \nabla_V(\sigma(X)V) \\
&\quad +G[X,V] + Gh[X,V] - \sigma([X,V])V,Y) \\
&= 2g(G[X,V],Y) \\
&= -2g([X,V],GY).
\end{aligned}$$

So, $[X,U]$ and $[X,V]$ are in $[-1]\bigoplus\mathcal{V}$. We already know that $[U,V]$ is in $\mathcal{V}$ since $\mathcal{V}$ is integrable. Therefore $[-1]\bigoplus\mathcal{V}$ is integrable.

Next we want to show that $[-1]\bigoplus\mathcal{V}$ -integral submanifolds are totally geodesic.

Let X and Z be in $[+1]$ and Y in $[-1]$. Then

$$\begin{aligned}
0 &= g(R(X,Y)U,Z) \\
&= g(\nabla_X(\sigma(Y)V) - \nabla_Y(-2GX + \sigma(X)V) \\
&\quad +G[X,Y] + Gh[X,Y] - \sigma([X,Y])V,Z) \\
&= \sigma(Y)g(-2HX - \sigma(X)U,Z) + 2g(\nabla_Y GX,Z) \\
&= 2g(\nabla_Y GX,Z).
\end{aligned}$$

So, $\nabla_Y GX$ is in $[-1]\bigoplus\mathcal{V}$. Since Y is in $[-1]$, $\nabla_Y U = \sigma(Y)V$ and $\nabla_Y V = -\sigma(Y)U$ are in $\mathcal{V}$. So, $\nabla_U Y$ and $\nabla_V Y$ are in $[-1]\bigoplus\mathcal{V}$ since $[Y,U],[Y,V]$ are in $[-1]\bigoplus\mathcal{V}$. Therefore, $[-1]\bigoplus\mathcal{V}$-integral submanifolds are totally geodesic.

Now let X,Y and Z be in $[+1]$. Then, by equations (3.10), (3.11) and (3.12)

$$\begin{aligned}
0 &= g(R(X,Y)U,Z) \\
&= g((\nabla_Z G)X,Y) - g((\nabla_X Gh)Y,Z) + g((\nabla_Y Gh)X,Z) \\
&= \sigma(Z)g(HX,Y) - g(\nabla_X GY,Z) + g(Gh\nabla_X Y,Z)
\end{aligned}$$

$$
\begin{aligned}
&+g(\nabla_Y GX, Z) - g(Gh\nabla_Y X, Z) \\
=\;& -g((\nabla_X G)Y, Z) + g(\nabla_X Y, GZ) - g(G\nabla_X Y, Z) \\
&+g((\nabla_Y G)X, Z) - g(\nabla_Y X, GZ) + g(G\nabla_Y X, Z) \\
=\;& -\sigma(X)g(HY, Z) + 2g([X,Y], GZ) + \sigma(Y)g(HX, Z) \\
=\;& 2g([X,Y], GZ).
\end{aligned}
$$

Also, by equations (3.8) and (3.9)

$$g([X,Y], U) = u([X,Y]) = 2g(GX, Y) = 0,$$

and

$$g([X,Y], V) = v([X,Y]) = 2g(HX, Y) = 0.$$

Therefore $[X, Y]$ is in $[+1]$, and hence $[+1]$ is integrable.

To show that $[+1]$-integral submanifolds are totally geodesic, let X, Z be in $[-1]$ and Y be in $[+1]$. Then again by equations (3.10), (3.11) and (3.12)

$$
\begin{aligned}
0 \;=\;& g(R(X,Y)U, Z) \\
=\;& g((\nabla_Z G)X, Y) - g((\nabla_X Gh)Y, Z) + g((\nabla_Y Gh)X, Z) \\
&+2\sigma(X)g(HY, Z) - \sigma(Z)g(Y, HX) - \sigma(X)g(Z, HY) \\
=\;& \sigma(Z)g(HX, Y) - g(\nabla_X GY, Z) + g(Gh\nabla_X Y, Z) \\
&-g(\nabla_Y GX, Z) - g(Gh\nabla_Y X, Z) - \sigma(Z)g(Y, HX) \\
&+\sigma(X)g(Z, HY) \\
=\;& -g((\nabla_X G)Y, Z) + g(\nabla_X Y, GZ) + g(G\nabla_X Y, Z) \\
&-g((\nabla_Y G)X, Z) + g(\nabla_Y X, GZ) - g(G\nabla_Y X, Z) \\
&+\sigma(X)g(Z, HY) \\
=\;& -\sigma(X)g(HY, Z) - \sigma(Y)g(HX, Z) + 2g(\nabla_Y X, GZ) \\
&+\sigma(X)g(Z, HY)
\end{aligned}
$$

$$= \quad -2g(\nabla_Y GZ, X).$$

Also,

$$g(\nabla_Y GZ, U) = -g(\nabla_Y U, GZ) = 2g(GY, GZ) = 0,$$

and

$$g(\nabla_Y GZ, V) = -g(\nabla_Y V, GZ) = 2g(HY, GZ) = 0.$$

So, $\nabla_Y GZ$ is in $[+1]$ and hence $[+1]$-integral submanifolds are totally geodesic.

Now, we want to show that $[-1] \oplus \mathcal{V}$-integral submanifolds are flat. We can choose coordinates $u^1, u^2, ..., u^{4n+2}$ such that $\{\frac{\partial}{\partial u^k}\}_{k=1}^{2n+2}$ span $[-1] \oplus \mathcal{V}$. Then, choose functions f_i^j such that

$$X_i = \frac{\partial}{\partial u^{2n+2+i}} + \sum_{j=1}^{2n+2} f_i^j \frac{\partial}{\partial u^j}$$

are in $[+1]$ for $i = 1, ..., 2n$. Then, for $k = 1, ..., 2n+2$, $[\frac{\partial}{\partial u^k}, X_i] = [\frac{\partial}{\partial u^k}, \sum_{j=1}^{2n+2} f_i^j \frac{\partial}{\partial u^j}]$ are in $[-1] \oplus \mathcal{V}$. So,

$$\nabla_{[\frac{\partial}{\partial u^k}, X_i]} U = \sigma([\frac{\partial}{\partial u^k}, X_i])V.$$

On the other hand, since $R(\frac{\partial}{\partial u^k}, X_i)U = 0$,

$$\begin{aligned}
\nabla_{[\frac{\partial}{\partial u^k}, X_i]} U &= \nabla_{\frac{\partial}{\partial u^k}} \nabla_{X_i} U - \nabla_{X_i} \nabla_{\frac{\partial}{\partial u^k}} U \\
&= \nabla_{\frac{\partial}{\partial u^k}} (-2GX_i + \sigma(X_i)V) - \nabla_{X_i} (\sigma(\frac{\partial}{\partial u^k})V) \\
&= -2\nabla_{\frac{\partial}{\partial u^k}} GX_i + \frac{\partial}{\partial u^k}(\sigma(X_i))V - \sigma(X_i)\sigma(\frac{\partial}{\partial u^k})U \\
&\quad -X_i(\sigma(\frac{\partial}{\partial u^k}))V - \sigma(\frac{\partial}{\partial u^k})(-2HX_i - \sigma(X_i)U) \\
&= -2\nabla_{\frac{\partial}{\partial u^k}} GX_i + 2\Omega(\frac{\partial}{\partial u^k}, X_i)V + \sigma([\frac{\partial}{\partial u^k}, X_i])V + 2\sigma(\frac{\partial}{\partial u^k})HX_i.
\end{aligned}$$

Since $\frac{\partial}{\partial u^k}$ is in $[-1] \oplus \mathcal{V}$, $\Omega(\frac{\partial}{\partial u^k}, X_i) = 0$. So,

$$\nabla_{\frac{\partial}{\partial u^k}} GX_i = \sigma(\frac{\partial}{\partial u^k})HX_i.$$

Hence, $\nabla_Y GX_i = \sigma(Y)HX_i$ for Y in $[-1] \oplus \mathcal{V}$. Therefore, there is a basis $\{e_i\}_{i=1}^{2n+2}$ of $[-1] \oplus \mathcal{V}$ such that $\nabla_Y e_i = -\sigma(Y)Je_i$ for Y in $[-1] \oplus \mathcal{V}$.

If $Z = \sum_{i=1}^{2n+2} z_i e_i$ then

$$\begin{aligned}
\nabla_Y Z &= \sum_{i=1}^{2n+2} \nabla_Y (z_i e_i) \\
&= \sum_{i=1}^{2n+2} (Y(z_i)e_i - z_i\sigma(Y)Je_i) \\
&= \sum_{i=1}^{2n+2} Y(z_i)e_i - \sigma(Y)JZ.
\end{aligned}$$

Then, using the fact that $\Omega(Y, Z) = 0$, we get

$$\begin{aligned}
R(Y,Z)Z &= \nabla_Y \sum_{i=1}^{2n+2} (Z(z_i)e_i - z_i\sigma(Z)Je_i) - \nabla_Z \sum_{i=1}^{2n+2} (Y(z_i)e_i - z_i\sigma(Y)Je_i) \\
&\quad - \sum_{i=1}^{2n+2} ([Y,Z](z_i)e_i - z_i\sigma([Y,Z])Je_i) \\
&= \sum_{i=1}^{2n+2} [Y(Z(z_i))e_i - Z(z_i)\sigma(Y)Je_i - Y(z_i)\sigma(Z)Je_i \\
&\quad - z_iY(\sigma(Z))Je_i - z_i\sigma(Z)\sigma(Y)e_i - Z(Y(z_i))e_i + Y(z_i)\sigma(Z)Je_i \\
&\quad + Z(z_i)\sigma(Y)Je_i + z_iZ(\sigma(Y))Je_i + z_i\sigma(Y)\sigma(Z)e_i \\
&\quad - [Y,Z](z_i)e_i + z_i\sigma([Y,Z])Je_i] \\
&= -2\sum_{i=1}^{2n+2} z_i\Omega(Y,Z)Je_i \\
&= 0.
\end{aligned}$$

Therefore the $[-1] \oplus \mathcal{V}$-integral submanifolds are flat.

Now let X be a unit vector in $[+1]$. We are going to show that $K(X, JX) = 16$. To do this, we are going to compute $g(R(X, JX)HX, GX)$ in two different ways. First, by a direct computation and (3.12) we have

$$\begin{aligned}
g(\nabla_X\nabla_{JX}GJX, GX) &= g(\nabla_X(\nabla_{JX}G)JX, GX) + g(\nabla_X G\nabla_{JX}JX, GX) \\
&= g(\nabla_X(\sigma(JX)HJX + 2U), GX) + g((\nabla_X G)\nabla_{JX}JX, GX) \\
&\quad + g(\nabla_X\nabla_{JX}JX, X)
\end{aligned}$$

$$\begin{aligned} &= -X(\sigma(JX)) - \sigma(JX)g(\nabla_X GX, GX) - 4g(GX, GX) \\ &\quad + g(\sigma(X)H\nabla_{JX}JX, GX) + g(\nabla_X\nabla_{JX}JX, X) \\ &= -X(\sigma(JX)) - 4 + g(\nabla_X\nabla_{JX}JX, X). \end{aligned}$$

Similarly,

$$\begin{aligned} g(\nabla_{JX}\nabla_X GJX, GX) &= g(\nabla_{JX}(\nabla_X G)JX, GX) + g(\nabla_{JX}G\nabla_X JX, GX) \\ &= g(\nabla_{JX}(\sigma(X)HJX + 2V), GX) + g((\nabla_{JX}G)\nabla_X JX, GX) \\ &\quad + g(\nabla_{JX}\nabla_X JX, X) \\ &= -JX(\sigma(X)) - \sigma(X)g(\nabla_{JX}GX, GX) - 4g(HJX, GX) \\ &\quad + g(\sigma(JX)H\nabla_X JX, GX) + g(\nabla_{JX}\nabla_X JX, X) \\ &= -JX(\sigma(X)) + 4 + g(\nabla_{JX}\nabla_X JX, X). \end{aligned}$$

Also

$$\begin{aligned} g(\nabla_{[X,JX]}GJX, GX) &= g((\nabla_{[X,JX]}G)JX, GX) + g(\nabla_{[X,JX]}JX, X) \\ &= g(\sigma([X, JX])HJX, GX) + g(\nabla_{[X,JX]}JX, X) \\ &= -\sigma([X, JX]) + g(\nabla_{[X,JX]}JX, X). \end{aligned}$$

Therefore

$$\begin{aligned} g(R(X, JX)HX, GX) &= g(R(X, JX)JX, X) - 8 - 2\Omega(X, JX) \\ &= g(R(X, JX)JX, X) - 8 - 8g(JX, JX) \\ &= g(R(X, JX)JX, X) - 16. \end{aligned}$$

On the other hand, $g(R(X, JX)HX, GX) = g(R(HX, GX)X, JX)$. Since X is in $[+1]$, we can write $X = GX'$ where X' is in $[-1]$. Then $X' = \sum_{i=1}^{2n+2} x_i e_i$. Using this and (3.12) we get

$$g(\nabla_{HX}\nabla_{GX}X, JX) = g(\nabla_{HX}\nabla_{GX}GX', JX)$$

$$= g(\nabla_{HX}((\nabla_{GX}G)X' + G\nabla_{GX}X'), JX)$$

$$= g(\nabla_{HX}(\sigma(GX)HX' + 2g(GX,X')U - 2g(HX,X')V), JX)$$
$$+g(\nabla_{HX}(\sum_{i=1}^{2n+2} GX(x_i)Ge_i - \sigma(GX)HX'), JX)$$

$$= g(\nabla_{HX}(-2U + \sum_{i=1}^{2n+2} GX(x_i)Ge_i), JX)$$

$$= \sum_{i=1}^{2n+2}(HX(GX(x_i))g(Ge_i, JX) + GX(x_i)g((\nabla_{HX}G)e_i, JX)$$
$$+GX(x_i)g(G\nabla_{HX}e_i, JX))$$
$$= \sum_{i=1}^{2n+2}(HX(GX(x_i))g(Ge_i, JX) + GX(x_i)g(\sigma(HX)He_i, JX)$$
$$-GX(x_i)g(\sigma(HX)He_i, JX))$$
$$= \sum_{i=1}^{2n+2} HX(GX(x_i))g(Ge_i, JX).$$

A similar calculation gives

$$g(\nabla_{GX}\nabla_{HX}X, JX) = \sum_{i=1}^{2n+2} GX(HX(x_i))g(Ge_i, JX),$$

and

$$g(\nabla_{[GX,HX]}X, JX) = \sum_{i=1}^{2n+2} [GX, HX](x_i)g(Ge_i, JX).$$

Therefore $g(R(HX, GX)X, JX) = 0$ and hence $K(X, JX) = 16$. □

3.3 A complex contact metric structure on the manifold $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$

Let $[t_0, \ldots, t_n]$ be the homogeneous coordinates on $\mathbf{CP}^n$ and let $\mathcal{U}_i = \{t_i \neq 0\}$. On $\mathcal{U}_i$ there are coordinates $w_j = \frac{t_j}{t_i}$, $j = 0, \ldots, n$, $j \neq i$. Let $\mathcal{O}_i = \mathbf{C}^{n+1} \times \mathcal{U}_i$ and let $\{z_0, \ldots, z_n\}$ be the coordinates on $\mathbf{C}^{n+1}$. Define a holomorphic 1-form ω_i on $\mathcal{O}_i$ as

$$\omega_i = \frac{1}{t_i}\sum_{k=0}^{n} t_k dz_k.$$

Then $\omega_i \wedge (d\omega_i)^n \neq 0$ on $\mathcal{O}_i$ and $\omega_j = \frac{t_i}{t_j}\omega_i$ on $\mathcal{O}_i \cap \mathcal{O}_j$. Thus $\{\omega_i\}_{i=0}^n$ is a complex contact structure on $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$.

For computational purposes, let us consider $\mathcal{O}_0$ with $\omega_0 = dz_0 + \sum_{k=1}^n w_k dz_k$. The product metric is given by the matrix

$$g = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & g_1 & 0 \\ & & 0 & g_2 \\ \hline g_1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & g_2^T & & \end{array}\right)$$

where $g_1 = \frac{1}{8} I_{n+1}$ and

$$(g_2)_{ij} = \frac{(1 + \sum_{k=1}^n |w_k|^2)\delta_{ij} - \overline{w}_i w_j}{8(1 + \sum_{k=1}^n |w_k|^2)^2}.$$

Here I_{n+1} is the $(n+1) \times (n+1)$ identity matrix. Let $f_0 = 1 + \sum_{k=1}^n |w_k|^2$. Define real 1-forms u_0, v_0 as

$$u_0 = \frac{1}{4\sqrt{f_0}}(dz_0 + d\overline{z}_0 + \sum_{k=1}^n (w_k dz_k + \overline{w}_k d\overline{z}_k)),$$

$$v_0 = \frac{i}{4\sqrt{f_0}}(dz_0 - d\overline{z}_0 + \sum_{k=1}^n (w_k dz_k - \overline{w}_k d\overline{z}_k)).$$

Set

$$U_0 = \frac{2}{\sqrt{f_0}}(\partial z_0 + \partial \overline{z}_0 + \sum_{k=1}^n (\overline{w}_k \partial z_k + w_k \partial \overline{z}_k))$$

and

$$V_0 = \frac{-2i}{\sqrt{f_0}}(\partial z_0 - \partial \overline{z}_0 + \sum_{k=1}^n (\overline{w}_k \partial z_k - w_k \partial \overline{z}_k)).$$

Then $\omega_0 = 2\sqrt{f_0}(u_0 - iv_0)$, $du_0(U_0, X) = 0$ for all X in $\mathcal{H}$, $u_0(U_0) = 1$, $v_0(U_0) = 0$ and $g(U_0, X) = u_0(X)$ for all X.

Let

$$G_0 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & G_1 \\ & & G_2 & 0 \\ \hline 0 & \overline{G}_1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \overline{G}_2 & 0 & & \end{array}\right)$$

where

$$G_1 = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ |w_1|^2 - f_0 & \overline{w}_1 w_2 & \cdots & \overline{w}_1 w_n \\ w_1 \overline{w}_2 & |w_2|^2 - f_0 & \cdots & \overline{w}_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_1 \overline{w}_n & w_2 \overline{w}_n & \cdots & |w_n|^2 - f_0 \end{pmatrix}$$

and

$$G_2 = f_0^2 \left(\begin{array}{c|c} -w_1 & \\ -w_2 & \\ \vdots & I_n \\ -w_n & \end{array} \right)$$

Then $G_0^2 = -Id + u_0 \otimes U_0 + v_0 \otimes V_0$, $G_0 J = -J G_0$, $G_0 U_0 = 0$, $g(G_0 X, Y) = -g(X, G_0 Y)$ and $g(X, G_0 Y) = du_0(X, Y)$, $g(X, H_0 Y) = dv_0(X, Y)$ for all X, Y in $\mathcal{H}$, where $H_0 = G_0 J$.

To check the second condition of Definition 1.2, let us check $\mathcal{O}_0 \cap \mathcal{O}_1$ as an example. We have

$$f_0 = 1 + \sum_{k=1}^{n} |w_k|^2$$

on $\mathcal{O}_0$ and

$$f_1 = 1 + |w_0|^2 + \sum_{k=2}^{n} |w_k|^2$$

on $\mathcal{O}_1$ so that $\frac{f_0}{f_1} = \frac{|t_1|^2}{|t_0|^2}$. Set $a - ib = \sqrt{\frac{f_0}{f_1}} \frac{t_0}{t_1}$ on $\mathcal{O}_0 \cap \mathcal{O}_1$. Then $a^2 + b^2 = 1$ and

$$u_1 = a u_0 - b v_0 \qquad G_1 = a G_0 - b H_0$$

$$v_1 = b u_0 + a v_0 \qquad H_1 = b G_0 + a H_0$$

where (u_1, v_1, G_1, H_1) are the structure tensors on $\mathcal{O}_1$. Therefore $(u_k, v_k, U_k, V_k, G_k, H_k, g)$ with the open cover $\{\mathcal{O}_k\}_{k=0}^{n}$ is a complex contact metric structure on $\mathbf{C}^{n+1} \times \mathbf{CP}^n(16)$.

We give the Levi-Civita connection of g below. We abbreviate $\frac{\partial}{\partial w_k}$ by ∂w_k. We list only the non-zero terms and we do not repeat terms with commutativity or conjugation.

$$\nabla_{\partial w_k} \partial w_k = \frac{-2\overline{w}_k}{f} \partial w_k,$$
$$\nabla_{\partial w_k} \partial w_j = -\frac{1}{f}(\overline{w}_j \partial w_k + \overline{w}_k \partial w_j).$$

Using the formulas for the connection given above, we can compute the covariant derivatives of U and V as follows:

$$\begin{aligned}
\nabla_X U &= -f^{\frac{3}{2}}[\sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k)(\partial z_0 + \partial \overline{z}_0) \\
&\quad + \sum_{j=1}^{n}((\overline{w}_j \sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k) - 2f\overline{b}_j)\partial z_j \\
&\quad + (w_j \sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k) - 2f b_j)\partial \overline{z}_j)],
\end{aligned}$$

$$\begin{aligned}
\nabla_X V &= if^{\frac{3}{2}}[\sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k)(\partial z_0 - \partial \overline{z}_0) \\
&\quad + \sum_{j=1}^{n}((\overline{w}_j \sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k) - 2f\overline{b}_j)\partial z_j \\
&\quad - (w_j \sum_{k=1}^{n}(b_k \overline{w}_k + \overline{b}_k w_k) - 2f b_j)\partial \overline{z}_j)]
\end{aligned}$$

where

$$X = \sum_{k=0}^{n}(a_k \partial z_k + \overline{a}_k \partial \overline{z}_k) + \sum_{k=1}^{n}(b_k \partial w_k + \overline{b}_k \partial \overline{w}_k).$$

Then we can compute the 1-form σ and the symmetric operator $h = h_U = h_V$.

$$\sigma = \frac{i}{4f}\sum_{k=1}^{n}(w_k d\overline{w}_k - \overline{w}_k dw_k),$$

$$\begin{aligned} hX &= \frac{1}{f}[(\sum_{k=1}^{n} w_k a_k - a_0 \sum_{k=1}^{n} |w_k|^2)\partial z_0 + (\sum_{k=1}^{n} \overline{w}_k \overline{a}_k - \overline{a}_0 \sum_{k=1}^{n} |w_k|^2)\partial \overline{z}_0 \\ &+ \sum_{j=1}^{n}((\overline{w}_j a_0 - f a_j + \overline{w}_j \sum_{k=1}^{n} w_k a_k)\partial z_j + (w_j \overline{a}_0 - f\overline{a}_j + w_j \sum_{k=1}^{n} \overline{w}_k \overline{a}_k)\partial \overline{z}_j)] \\ &+ \sum_{j=1}^{n}(b_j \partial w_j + \overline{b}_j \partial \overline{w}_j). \end{aligned}$$

As for the curvature, we can see by a direct computation that $R(X,Y)\mathcal{V} = 0$, for every X, Y. Then,

$$(\nabla_X G)Y = \sigma(X)HY + g(X + hX, Y)U + g(J(X + hX), Y)V$$

and

$$(\nabla_X H)Y = -\sigma(X)GY - g(J(X + hX), Y)U + g(X + hX, Y)V$$

for X, Y in $\mathcal{H}$.

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