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# SPECIAL METRICS ON SYMPLECTIC MANIFOLDS

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# ABSTRACT

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The central idea of this work is to find geometric and topological consequences of the existence of special types of Riemannian metrics on compact symplectic manifolds. The first part of the thesis is devoted to a conjecture of Goldberg about Einstein metrics on symplectic manifolds and to some related questions coming from a natural variational problem. The main result of chapter three is the relation we find between almost Kähler metrics and Hermitian conformal classes. The final chapter deals with a conjecture about the Seiberg-Witten invariants and the orientations of compact 4-manifolds.

To my parents and my wife,

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# CHAPTER 0

# Introduction

The main idea of this work is to study geometric and topological consequences of the existence of special types of Riemannian metrics on compact symplectic manifolds. After an introductory part, the second chapter of the thesis is centered around a conjecture of Goldberg from 1969 which states that any Einstein metric compatible to a symplectic structure on a compact manifold is, in fact, a Kähler Einstein metric. Some new positive partial results are given. Related to the Goldberg conjecture, we also study critical metrics coming from a natural variational problem on symplectic manifolds. The main result of chapter three is the relation we find between almost Kähler metrics and Hermitian conformal classes. We show that on most compact complex surfaces which also admit symplectic forms, each Hermitian conformal class contains almost Kähler metrics. We also give results about the number of symplectic forms compatible to a given metric. As applications, we obtain alternative proofs for results of LeBrun on the Yamabe constants of Hermitian conformal classes and give some answers to a question of Blair about the isometries of almost Kähler metrics. The final chapter deals with a conjecture from the Seiberg-Witten theory stating that for any compact, orientable, simply connected 4-manifold, with one of the orientations all the invariants will vanish. We prove this conjecture for a large class of complex surfaces. Kähler Einstein metrics play an important role in the proof.

# CHAPTER 1

# Preliminaries

This chapter sets notations and presents some basic results which we are going to use throughout the thesis. Central is the notion of *almost Kähler structure* and the whole chapter presents definitions and properties related to this notion. Section 1.1 discusses the case of general dimension, the particular aspects of almost Kähler structures in dimension 4 being treated in Section 1.2.

## 1.1 Almost Kähler structures

An *almost Kähler structure* on a manifold $M^{2n}$ is a triple $(g, J, \omega)$ of a Riemannian metric $g$, a $g$-orthogonal almost complex structure $J$ and a symplectic form $\omega$ given by

$$\omega(X, Y) = g(X, JY). \tag{1.1}$$

Alternatively, an almost Kähler structure is an almost Hermitian structure $(g, J, \omega)$ whose fundamental 2-form $\omega$ defined by (1.1) is closed. A Riemannian metric which admits an almost Kähler structure will be called *almost Kähler metric*.

If a symplectic form $\omega$ is given on $M$, then there are many almost Kähler structures with fundamental form $\omega$. Let us denote by $\mathcal{AM}_\omega$ the set of *associated metrics to* $\omega$, that is, all Riemannian metrics $g$ on $M$ for which there exists an almost complex structure $J$, such that $(g, J, \omega)$ is an almost Kähler structure. The following two propositions describe some of the properties of the set of associated metrics to a symplectic form.

**Proposition 1.1:** *The space $\mathcal{AM}_\omega$ is a non-empty, contractible space.*

*Proof:* We will use a well known fact from linear algebra known as the polar decomposition: any $m \times m$ matrix $A \in GL(m, \mathbf{R})$ can be uniquely written as $A = F \cdot G$, with $F \in O(m)$ and $G \in S_+(m)$, where $O(m)$ denotes as usual the group of orthogonal $m \times m$ matrices and $S_+(m)$ denotes the group of symmetric, positive definite $m \times m$ matrices.

Remark that if $A$ is a skew-symmetric, non-singular matrix to start with, the matrices $F$ and $G$ from its decomposition satisfy $F^2 = -I_m$ and $FG = GF$. Indeed, from $A = -A^t$, it follows that $FGF = -G$, which can also be written as, $F^2F^tGF = -I_mG$. But now note that $F^2 \in O(m)$ and $F^tGF \in S_+(m)$ and by the uniqueness of the decomposition it follows that $F^2 = -I_m$ and $FG = GF$.

Fixing a Riemannian metric $k$ on a manifold $M^m$, the above statements immediately translate to statements about non-singular endomorphisms of the tangent bundle, where the properties of symmetry, skew-symmetry, positive definiteness are all understood with respect to the metric $k$.

Now let $(M^{2n}, \omega)$ be a symplectic manifold and choose $k$ an arbitrary Riemannian metric on $M$. The symplectic form $\omega$ and the metric $k$ induce an endomorphism $A$ of the tangent bundle at every point defined by $\omega(X,Y) = k(X, AY)$. This is clearly skew-symmetric and non-singular at every point since $\omega$ is so. Therefore $A$ can be uniquely written as $A = JG$, where $J \in \Gamma(End(TM))$ is an orthogonal endomorphism, and $G \in \Gamma(End(TM))$ is a symmetric and positive definite. Moreover

$J^2 = -I_{TM}$ and $JG = GJ$. Define a bilinear form $g$ on $TM$ by $g(X,Y) = k(X,GY)$. This is clearly a Riemannian metric and it is an associated metric to $\omega$ the corresponding almost Kähler structure being easily checked to be $(g, J, \omega)$.

This proves that $\mathcal{AM}_\omega$ is not empty. Remark that the above construction gives, in fact, a map, say $p_\omega$ from the space $\mathcal{M}$ of all Riemannian metrics to the space of associated metrics $\mathcal{AM}_\omega$. But $\mathcal{M}$ is contractible. Indeed, choosing $g_0 \in \mathcal{M}$, the map $F_t(g) = (1-t)g_0 + tg$ defines a contraction of $\mathcal{M}$ to $g_0$. The composition $p_\omega F_t$ defines a contraction of $\mathcal{AM}_\omega$ to a point. $\square$

**Proposition 1.2** *The space of associated metrics $\mathcal{AM}_\omega$ is an infinite dimensional Frechet manifold.*

The proof of this result could be found in [12].

We see that despite the fact that there are many almost Kähler structures associated with a given symplectic form, any two such almost Kähler structures have homotopic almost complex structures. In particular, the Chern classes $c_i$, for $i \in \{1, ..., n\}$, are invariants of the symplectic structure (in fact, of the homotopy class of the symplectic structure). Finally, let us also remark that the symplectic form determines a volume element $\sigma$ on $M^{2n}$, hence an orientation. All associated metrics to $\omega$ induce this volume element, namely:

$$\sigma_g = \sigma = \frac{1}{n!}\omega^n,$$

for all $g \in \mathcal{AM}_\omega$.

For a Riemannian manifold $(M, g)$ we denote by $\nabla, R, \rho, s$ the Levi-Civita connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively. Our conventions for the definitions of the curvature and the Ricci tensor are the following:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

$$\rho(X,Y) = tr(Z \longrightarrow R(Z,X)Y).$$

A Riemann metric $g$ on a manifold $M^m$ is said to be *Einstein* if the Ricci tensor $\rho$ is (at each point) a multiple of the metric. Equivalently,

$$\rho = \frac{1}{m} sg.$$

Note that the second Bianchi identity implies for $m > 2$ that the scalar curvature $s$ must be a constant. Einstein metrics will play a central role in this thesis.

The *curvature operator* acting on 2-forms, denoted by $\mathcal{R}$, is defined by

$$\mathcal{R}(\alpha)_{ij} = R_{ijab}\alpha_{ab},$$

for any $\alpha \in \Lambda^2 M$. We also denote by $( \, , \, )$ and $| \; |$ the local scalar product and norm induced by the metric on various types of tensor fields.

Let now $(M^{2n}, g, J, \omega)$ be an almost Hermitian manifold. A (2,0) tensor $D$ on $M$ is called *J-invariant* (or *J-Hermitian*) if it satisfies

$$D(JX, JY) = D(X,Y),$$

for any tangent vectors $X, Y$. We say that $D$ is *J-anti invariant* (or *J-anti Hermitian*) if it satisfies

$$D(JX, JY) = -D(X,Y).$$

For example, the metric $g$ and the fundamental form $\omega$ of any almost Hermitian manifold are, by definition $J$-invariant tensors. Also, the Ricci tensor of Kähler metrics is $J$-invariant. In general, for almost Hermitian metrics this last statement is no longer true and it will be useful to use the decomposition

$$\rho = \rho^{inv} + \rho^{anti},$$

where $\rho^{inv}$ is the $J$-invariant part of the Ricci tensor defined by

$$\rho^{inv}(X,Y) = \frac{1}{2}(\rho(X,Y) + \rho(JX,JY)),$$

and $\rho^{anti}$ is the $J$-anti invariant part given by

$$\rho^{anti}(X,Y) = \frac{1}{2}(\rho(X,Y) - \rho(JX,JY)).$$

The $\star$-*Ricci tensor*, $\rho^\star$ is an analog of the Ricci tensor, but involving also the almost complex structure:

$$\rho^\star(X,Y) = tr(Z \longrightarrow R(X,JZ)JY).$$

Alternatively, $\rho^\star$ is given by:

$$\rho^\star(X,JY) = \mathcal{R}(\omega)(X,Y).$$

The trace of $\rho^\star$ is called $\star$-*scalar curvature* and is denoted by $s^\star$. For Kähler metrics the $\star$-Ricci tensor coincides with the Ricci tensor. This is not true for general almost Hermitian manifolds. In fact, in this case the $\star$-Ricci tensor is not necessarily symmetric. We may remark though, that it satisfies the following identity:

$$\rho^\star(JX,JY) = \rho^\star(Y,X),$$

for any tangent vectors $X, Y$. This implies that the symmetric part, $\rho^{\star sym}$, is a $J$-

invariant tensor, whereas the skew-symmetric part, $\rho^{\star skew}$, is a $J$-anti-invariant tensor. Using the analogous condition on the $\star$-Ricci tensor, we can define *$\star$-Einstein metrics*. However, in this case it does not follow that the $\star$-scalar curvature is a constant, so we have two notions of $\star$-Einstein. An almost Hermitian metric is said to be:

*weakly $\star$-Einstein* if at every point the $\star$-Ricci tensor is a multiple of the metric (the $\star$-scalar curvature need not be constant);

*$\star$-Einstein* if the $\star$-Ricci tensor is a constant multiple of the metric (the $\star$-scalar curvature is constant).

For 2-forms on an almost Hermitian manifold the following pointwise, orthogonal decomposition is useful:

$$\Lambda^2 M = \mathbf{R}\omega \oplus \Lambda_0^{inv} M \oplus \Lambda^{anti} M,$$

where the factors denote respectively multiples of the fundamental form $\omega$, $J$-invariant 2-forms of zero trace, and $J$-anti-invariant 2-forms. For a 2-form $\gamma$, the components with respect to this decomposition are:

$$\gamma = \frac{1}{n}(\gamma, \omega)\omega + \gamma_0^{inv} + \gamma^{anti}.$$

$\gamma^{anti}$ denotes the $J$-anti-invariant part of $\gamma$. The $J$-invariant part is

$$\gamma^{inv} = \frac{1}{n}(\gamma, \omega)\omega + \gamma_0^{inv}.$$

If we complexify the tangent space and consider the usual decomposition of complex 2-forms in (2,0), (1,1) and (0,2) forms, it is easy to see that the $J$-invariant forms are real parts of (1,1) forms and $J$-anti-invariant forms are real parts of (2,0) forms (equivalently, of (0,2) forms).

By easy computation we have:

$$\gamma \wedge \omega^{n-1} = \frac{1}{n}(\gamma, \omega)\omega^n = (n-1)!\,(\gamma, \omega)\sigma; \tag{1.2}$$

$$\gamma \wedge \gamma \wedge \omega^{n-2} = (n-2)!\,\Big[\frac{n-1}{n}(\gamma, \omega)^2 - |\gamma_0^{inv}|^2 + |\gamma^{anti}|^2\Big]\sigma = \tag{1.3}$$

$$= (n-2)!\,[(\gamma, \omega)^2 - |\gamma^{inv}|^2 + |\gamma^{anti}|^2]\sigma.$$

Let us recall now some formulas specific to almost Kähler manifolds. Let $(M^{2n}, g, J, \omega)$ be an almost Kähler manifold and let $\nabla$ be the Levi-Civita connection of the metric $g$. The fact that $\omega$ is closed is equivalent to

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0, \tag{1.4}$$

for any vector fields $X, Y, Z$ on $M$. As a consequence of (1.4) we have

$$(\nabla_{JX} J)JY = -(\nabla_X J)Y, \tag{1.5}$$

which is known as the quasi-Kähler condition. In dimension 4, relations (1.4) and (1.5) are equivalent, but in higher dimensions there are examples of manifolds satisfying (1.5), but which are not almost Kähler (look at $S^6$ for instance, with the standard metric and the standard almost complex structure).

Taking trace in (1.5), we see that the co-differential of $\omega$ vanishes, hence $\omega$ is also a co-closed form. Therefore for any almost Kähler structure $(g, J, \omega)$, the symplectic form $\omega$ is harmonic with respect to the associated metric $g$.

In our study of almost Kähler manifolds two symmetric tensor fields appear quite often, so we give them names. We call $B$ and $D$ the global tensor fields whose local expressions are

$$B_{ij} = (\nabla_b J_{ik})(\nabla_b J_{jk}), \quad D_{ij} = (\nabla_i J_{sk})(\nabla_j J_{sk}). \tag{1.6}$$

Here and in many other places throughout this work, for local notations and computations we are using local $J$-basis, that is an orthonormal basis of the form $\{e_1, Je_1, ..., e_n, Je_n\}$. We adopt the summation convention of Einstein on repeated indices, but, as we work with orthonormal base, there is no need to raise and lower the indices. The tensor $D$ has a nice invariant description, $D(X,Y) = (\nabla_X J, \nabla_Y J)$, but we do not have a good invariant form for $B$. It can be easily seen from (1.5) that both $B$ and $D$ are $J$-invariant, symmetric tensor fields, with trace equal to $|\nabla J|^2$.

For any almost Kähler manifold the following relation is due to Koto ([29]):

$$\rho^{*sym} = \rho^{inv} + \frac{1}{2}B. \tag{1.7}$$

This formula implies the relation between the scalar curvatures:

$$s^{\star} = s + \frac{1}{2}|\nabla J|^2. \tag{1.8}$$

Therefore, we see that an almost Kähler structure is Kähler if and only if $s^{\star} = s$.

Besides the Levi-Civita connection $\nabla$, on an almost Kähler manifold $(M, g, J, \omega)$ it is useful to consider the *first canonical connection* $\nabla^0$, defined by Lichnerowicz in [38] to be

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}J(\nabla_X J)(Y). \tag{1.9}$$

Since $\nabla^0$ preserves $J$, its Ricci form $\gamma$ represents $2\pi c_1$. Using relation (1.9) it is not hard to obtain the expression for $\gamma$ in terms of the Levi-Civita connection $\nabla$:

$$\gamma(X,Y) = \rho^\star(X,JY) - \frac{1}{4}D(X,JY). \tag{1.10}$$

Using (1.2), (1.10) and (1.8), we get the expression of the first Chern number in terms of curvature computed by Blair in [14]:

$$2\pi c_1 \cup [\omega]^{n-1} = (n-1)!\int_M (\gamma,\omega)\sigma = (n-1)!\int_M \frac{s^\star + s}{4}\sigma \tag{1.11}$$

The right hand-side of this equality which a priori seems to depend on the metric, turns out to be a symplectic invariant.

Relations (1.8) and (1.11) imply the basic scalar curvature inequality for almost Kähler metrics:

$$\int_M s d\sigma \leq \frac{4\pi}{(n-1)!} c_1 \cup [\omega]^{n-1}, \tag{1.12}$$

with equality if and only if the metric is Kähler.

From (1.3), (1.10) and (1.8)

$$4\pi^2 c_1^2 \cup [\omega]^{n-2} = \int_M \gamma \wedge \gamma \wedge \omega^{n-2} = \tag{1.13}$$

$$= (n-2)!\int_M \Big[\frac{(s+s^\star)^2}{16} + \frac{1}{2}|\rho^{\star skew}|^2 - \frac{1}{2}|\rho^{\star sym}|^2 - \frac{1}{32}|D|^2 + \frac{1}{4}(\rho^\star, D)\Big] d\sigma.$$

A short computation making use of Koto's formula (1.7) gives:

$$-\frac{1}{2}|\rho^{\star sym}|^2 - \frac{1}{32}|D|^2 + \frac{1}{4}(\rho^\star, D) =$$

$$= -\frac{1}{2}|\rho^{inv}|^2 - \frac{1}{2}(\rho^{inv}, B) + \frac{1}{4}(\rho^{inv}, D) - \frac{1}{32}|2B - D|^2.$$

Therefore, replacing in (1.13) we get:

$$\frac{4\pi^2}{(n-2)!}c_1^2 \cup [\omega]^{n-2} = \int_M \Big[\frac{(s+s^\star)^2}{16} + \frac{1}{2}|\rho^{\star skew}|^2 - \tag{1.14}$$

$$-\frac{1}{2}|\rho^{inv}|^2 - \frac{1}{2}(\rho, B) + \frac{1}{4}(\rho, D) - \frac{1}{32}|2B - D|^2\Big]\sigma.$$

Next we give an integral formula which holds on any compact almost Kähler manifold. It was derived by Sekigawa in [42] and will play an important role in the next section. Sekigawa used the connections $\nabla$, $\nabla^0$ and the Chern-Weil homomorphism to obtain two different representatives, $\mu(\nabla)$ and $\mu(\nabla^0)$, for the first Pontrjagin class $p_1(M)$. The 4-form $\mu(\nabla^0) - \mu(\nabla)$ is then an exact form and hence by Stokes Theorem,

$$\int_M (\mu(\nabla^0) - \mu(\nabla)) \wedge \omega^{n-2} = 0.$$

After an extensive calculation, Sekigawa obtains from the above relation the following:

**Proposition 1.3 (Sekigawa, [42])** *For any almost Kähler manifold $(M^{2n}, g, J, \omega)$ the following integral formula holds:*

$$\int_M \Big[\frac{1}{4}f_1 + 2\sum(\nabla_i\rho_{bj} - \nabla_j\rho_{bi})(\nabla_b J_{ik})J_{jk} + \tag{1.15}$$

$$+2(\rho, B) + \frac{1}{4n}f_2 + \frac{1}{2n}|\nabla J|^4 + 4|\rho^{\star skew}|^2\Big]\sigma = 0,$$

*where*

$f_1 = \sum\left(\mathcal{R}(e^i \wedge e^j - Je^i \wedge Je^j), e^a \wedge e^b - Je^a \wedge Je^b\right)^2$ and $f_2 = \sum(\lambda_i - \lambda_j)^2$, $\lambda_1 = \lambda_{n+1} \leq \dots \leq \lambda_n = \lambda_{2n}$ *being the eigenvalues of the tensor $B$.*

## 1.2 4-dimensional almost Kähler structures

There are a few special features of the dimension 4. It is well known that the Hodge operator of a Riemannian 4-manifold $(M, g)$ satisfies $\star^2 = id$ acting on 2-forms. Therefore we have the splitting of the bundle of 2-forms

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

into self-dual 2-forms and anti-self-dual 2-forms, corresponding to the $+1$ and $-1$-eigenspaces of $\star$. It is well known that a 4-dimensional Riemannian manifold $(M, g)$ is Einstein if and only if the curvature operator satisfies:

$$(\mathcal{R}\alpha, \beta) = 0, \quad \forall \alpha \in \Lambda^2_+ M, \beta \in \Lambda^2_- M.$$

Basic topological invariants of 4-manifolds can be nicely described in terms of various parts of the curvature of a Riemannian metric $g$. Thus, if $\sigma(M)$ is the signature and $\chi(M)$ is the Euler class of a compact 4-manifold $M$, then

$$\sigma(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2)\sigma, \tag{1.16}$$

$$2\chi(M) = \frac{1}{4\pi^2} \int_M (\frac{s^2}{6} - \frac{|\rho|^2}{2} + |W^+|^2 + |W^-|^2)\sigma. \tag{1.17}$$

Now let $(M^4, g, J, \omega)$ be a compact, 4-dimensional almost Hermitian manifold. We have the following equalities due to Hirzebruch and Wu relating topological invariants of the manifold with the Chern classes induced by the almost complex structure:

$$c_1^2(M) = 3\sigma(M) + 2\chi(M), \quad c_2(M) = \chi(M), \quad p_1(M) = 3\sigma(M). \tag{1.18}$$

The decomposition into self-dual and anti-self-dual 2-forms is very nicely related with the decomposition induced by $J$:

$$\Lambda^2_+ M = \mathbf{R}\omega \oplus \Lambda^{anti} M, \quad \Lambda^2_- M = \Lambda^{inv}_0 M. \tag{1.19}$$

The behavior of the curvature operator with respect to decomposition (1.19) characterizes some interesting geometrical conditions on almost Hermitian 4-manifolds. The following can be proven by easy computations:

(i) $\rho^{anti} = 0$ if and only if $\big(\mathcal{R}(\Lambda^{anti} M), \Lambda^{inv}_0 M\big) = 0$;

(ii) $\rho^{\star skew} = 0$ if and only if $\big(\mathcal{R}(\Lambda^{anti} M), \mathbf{R}\omega\big) = 0$.

Tricerri and Vanhecke [47] have also shown the following relations on an almost Hermitian manifold of dimension 4:

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W) \Leftrightarrow (\rho^{anti} = 0, \rho^{\star skew} = 0); \tag{1.20}$$

$$\rho^{inv} - \rho^{\star sym} = \frac{1}{4}(s - s^\star) g. \tag{1.21}$$

Now let us go to almost Kähler 4-manifolds. First of all, the definition can be given in a different way. If $(M^4, g)$ is a Riemannian 4-manifold, there is a bijective correspondence between (oriented) orthogonal almost complex structures and self-dual forms of pointwise constant length $\sqrt{2}$. Because of this, a harmonic, self-dual form $\omega$, with $|\omega| = \sqrt{2}$, induces an almost Kähler structure $(g, J, \omega)$. By virtue of this equivalent definition in this dimension, when the metric is fixed, we will very often just refer to the form when thinking of the almost Kähler structure.

Let us recall now a definition due to Gray [26]. For every point $p \in M$, he defines $\mathcal{D}_p := \{X \in T_p M | \nabla_X J = 0\}$ and calls $\mathcal{D}$ the *Kähler nullity distribution* of the almost Hermitian manifold $M$. Note that $\mathcal{D}$ need not be a distribution in the usual sense since the dimension might vary with the point.

On a 4-dimensional almost Kähler manifold it follows from relation (1.5) that $\nabla_X\omega$ is a $J$-anti-invariant form for any vector $X$. Also, $\nabla_X\omega = 0$ if and only if $\nabla_{JX}\omega = 0$. Therefore, we conclude that $dim\mathcal{D}_p$ is an even number.

Now, since $M$ is 4-dimensional, the fibers of $\Lambda^{anti}M$ have dimension 2. Therefore, locally we can write

$$\nabla\omega = \alpha\otimes\Phi + \beta\otimes J\Phi,$$

where $\{\Phi, J\Phi\}$ is any (local) orthonormal frame of $\Lambda^{anti}M$, and $\alpha, \beta$ are (local) 1-forms. Hence, $\mathcal{D}_p$ contains the intersection of $ker\alpha_p$ and $ker\beta_p$. This proves that the Kähler nullity distribution $\mathcal{D}_p$ has dimension either 2 or 4. Hence, the $J$-invariant, symmetric tensor field $D$, previously defined in (1.6)

$$D(X,Y) = (\nabla_X J, \nabla_Y J) = 2(\nabla_X\omega, \nabla_Y\omega),$$

has at every point a double eigenvalue $\lambda_1 = 0$, and a double eigenvalue $\lambda_2 = \frac{1}{2}|\nabla J|^2$. In particular, in dimension 4,

$$|D|^2 = \frac{1}{2}|\nabla J|^4.$$

The tensor $B$, we also defined in (1.6) has an even simpler form in this dimension. Indeed, if we combine relation (1.21) with the Koto's formula (1.7), we see that for a 4-dimensional almost Kähler manifold we have

$$B = \frac{1}{4}|\nabla J|^2 g.$$

With these facts in mind, the formula (1.14) specializes in dimension 4 to:

$$c_1^2(M) = \frac{1}{16\pi^2}\int_M [s^2 - 2|\rho^{inv}|^2 + 2|\rho^{\star skew}|^2 + (\rho, D)]\sigma. \tag{1.22}$$

Also, the formula (1.15) of Proposition 1.3 has a nicer expression in dimension 4.

**Proposition 1.4 (Sekigawa, [42])** *For any 4-dimensional almost Kähler manifold* $(M^4, g, J, \omega)$ *the following integral formula holds:*

$$\int_M \Big[\frac{1}{4}f_1 + 2\sum(\nabla_i\rho_{bj} - \nabla_j\rho_{bi})(\nabla_b J_{ik})J_{jk} + \tag{1.23}$$

$$+\frac{1}{2}s|\nabla J|^2 + \frac{1}{4}|\nabla J|^4 + 4|\rho^{\star skew}|^2\Big]\sigma = 0,$$

*where*

$$f_1 = \sum\Big(\mathcal{R}(e^i \wedge e^j - Je^i \wedge Je^j), e^a \wedge e^b - Je^a \wedge Je^b\Big)^2.$$

# CHAPTER 2

# The Goldberg Conjecture and Related problems

In this chapter we will analyze some special associated metrics on compact symplectic manifolds. The first section deals with Einstein associated metrics. It is a conjecture of Goldberg [25] that any compact almost Kähler Einstein manifold is in fact Kähler Einstein. We give a few new positive partial results to this conjecture. In section 2.2 we consider critical associated metrics coming from a variational problem studied by Blair and Ianus [15]. Some parallel results to those mentioned in 2.1 are obtained here for critical metrics satisfying some additional assumptions. Some of the results presented in this chapter have been published in [17], [18].

## 2.1 The Goldberg Conjecture

A long-standing problem on almost Kähler manifolds is the following conjecture formulated by Goldberg in 1969 [25]:

**Conjecture (Goldberg, [25]):** *The almost complex structure of a compact, almost Kähler Einstein manifold is integrable, hence the manifold is, in fact Kähler Einstein.*

Important progress was made by K. Sekigawa who proved the following partial result (see [42]):

**Theorem 2.1 (Sekigawa):** *The conjecture is true if we additionally assume that the scalar curvature is non-negative.*

*Proof:* For an Einstein, almost Kähler manifold, the integral formula of Proposition 1.3 becomes:

$$\int_M [\frac{1}{4}f_1 + \frac{s}{n}|\nabla J|^2 + \frac{1}{4n}f + \frac{1}{2n}|\nabla J|^4 + 4|\rho^{\star skew}|^2]d\sigma = 0. \tag{2.1}$$

Since both functions $f_5$ and $f$ are positive, we get the inequality

$$\int_M [\frac{s}{n}|\nabla J|^2 + \frac{1}{2n}|\nabla J|^4]d\sigma \leq 0. \tag{2.2}$$

If $s \geq 0$, the above inequality implies $\nabla J = 0$, hence the metric is Kähler. □

Replacing the Riemannian condition on the scalar curvature with the natural symplectic condition, the result still holds.

**Theorem 2.2:** *Let $(M^{2n}, \omega)$ be a compact symplectic manifold.*

*(a) If the first Chern number $c_1(M) \cup [\omega]^{n-1}$ is non-negative, then any associated Einstein metric is a Kähler Einstein metric.*

*(b) If $c_1(M) \cup [\omega]^{n-1} < 0$, then the scalar curvature of any Einstein associated metric must satisfy*

$$2c_1(M) \cup [\omega]^{n-1} \leq \frac{(n-1)!}{4\pi} sVol(M) \leq c_1(M) \cup [\omega]^{n-1}. \tag{2.3}$$

**Proof:** (a) First, by eventually scaling the symplectic form, we can assume that the total volume of $M$ is 1. For an Einstein associated metric we have the inequality (2.2). With our assumption on the total volume and since $s$ is a constant, we can rewrite this as:

$$\int_M |\nabla J|^4\sigma \leq -2s\int_M |\nabla J|^2\sigma, \tag{2.4}$$

so by Cauchy-Schwarz inequality we get:

$$(\int_M |\nabla J|^2\sigma)(2s + \int_M |\nabla J|^2\sigma) \leq 0. \tag{2.5}$$

Our assumption on the sign of the symplectic invariant $c_1(M) \cup [\omega]^{n-1}$ is expressed in terms of the scalar curvature by (1.11):

$$0 \leq 4\pi c_1(M) \cup [\omega]^{n-1} = (n-1)!\int_M (s + \frac{1}{4}|\nabla J|^2)\sigma.$$

This and (2.5) imply

$$\int_M |\nabla J|^2\sigma = 0,$$

hence $\nabla J = 0$ so the metric is Kähler .

(b) For any associated metric, the inequality

$$\frac{(n-1)!}{4\pi}\int_M s\sigma \leq c_1(M) \cup [\omega]^{n-1},$$

was proved in (1.12), with equality if and only if the metric is Kähler. If the metric is not Kähler, from (2.5) we obtain

$$2s + \int_M |\nabla J|^2\sigma \leq 0,$$

and using (1.11), we see that this is exactly the first inequality of (2.3). □

We see that the basic scalar curvature inequality (1.12) plays an important role. Here is a lemma giving an estimate for the square of the scalar curvature of an Einstein almost Kähler metric in terms of another symplectic invariant involving the square of the first Chern class $c_1^2$.

**Lemma 2.1:** *Suppose g is an Einstein associated metric on a compact symplectic manifold* $(M^{2n},\omega)$. *Then the following inequality holds:*

$$c_1^2 \cup [\omega]^{n-2} \leq \frac{(n-1)!}{16n\pi^2}\int_M s^2 d\sigma. \tag{2.6}$$

*Equality holds if and only if the metric is Kähler Einstein.*

*Proof:* If the scalar curvature is non-negative, by Theorem 2.1 of Sekigawa the metric is in fact Kähler Einstein. It is easy to see from (1.14) that in this case we have equality in (2.6). So it is enough to assume that we have an Einstein associated metric of negative scalar curvature. In this case, from (1.14) we obtain

$$\frac{4\pi^2}{(n-2)!}c_1^2 \cup [\omega]^{n-2} =$$

$$= \int_M \Big[\frac{(s+s^\star)^2}{16} + \frac{1}{2}|\rho^{\star skew}|^2 - \frac{1}{4n}s^2 - \frac{1}{8n}s|\nabla J|^2 - \frac{1}{32}|2B-D|^2\Big] d\sigma.$$

From the integral formula (2.1) we get the inequality:

$$\int_M \frac{1}{2}|\rho^{\star skew}|^2 d\sigma \leq \int_M -\frac{1}{8n}s|\nabla J|^2 d\sigma.$$

Hence, also making use of (1.8) we get:

$$\frac{4\pi^2}{(n-2)!}c_1^2 \cup [\omega]^{n-2} \leq \tag{2.7}$$

$$\leq \int_M \Big[\frac{n-1}{4n}s^2 + \frac{1}{8}s|\nabla J|^2 + \frac{1}{64}|\nabla J|^4 - \frac{1}{4n}s|\nabla J|^2 - \frac{1}{32}|2B-D|^2\Big] d\sigma.$$

We now distinguish two cases. If $n = 2$ then

$$|2B - D|^2 = |\frac{|\nabla J|^2}{2} g - D|^2 = |D|^2 = \frac{|\nabla J|^4}{2},$$

and we see that the inequality (2.7) is exactly (2.6) for $n = 2$. If $n \geq 3$, we see that the integral formula (2.1) also gives the inequality:

$$\int_M \Big[\frac{1}{32} s|\nabla J|^2 + \frac{1}{64}|\nabla J|^4\Big] d\sigma \leq 0.$$

Since $s < 0$, using this last relation it is not hard to see that we obtain the inequality claimed in (2.6). Equality holds if and only if $\nabla J = 0$. Therefore the lemma is proved. □

Lemma 2.1 has interesting consequences in dimension 4. In this case, it says that an Einstein associated metric on a symplectic manifold satisfies

$$c_1^2(M) \leq \frac{1}{32\pi^2} \int_M s^2 d\sigma, \tag{2.8}$$

with equality if and only if the metric is Kähler Einstein.

**Theorem 2.3:** *Let $(M, \omega)$ be a compact symplectic 4-manifold which admits an Einstein associated metric. Then the Miyaoka - Yau inequality*

$$c_1^2(M) \leq 3c_2(M)$$

*is satisfied.*

*Proof:* Using the inequality (2.8) combined with relations (1.16), (1.17) and (1.18), we get right away

$$c_1^2(M) \leq 3(2\chi(M) - 3\tau(M)).$$

From the Hirzebruch-Wu equalities (1.18) we see that the above relation gives us the result. □

The proof of the Theorem 2.3 immediately implies the following

**Corollary 2.1:** *For compact symplectic 4-manifolds* $(M, \omega)$ *which satisfy* $c_1^2(M) = 3c_2(M)$ *the Goldberg conjecture is true.*

**Remark:** Using Seiberg-Witten invariants, C. LeBrun obtained stronger statements than Theorem 2.3 and inequality (2.8) in dimension 4. We will indicate in Chapter 4, how these stronger results are proved.

**Proposition 2.1:** *Let* $M$ *be a compact, symplectic, 4-dimensional manifold with* $c_1 = \frac{1}{8\pi}\lambda[\omega]$, *for* $\lambda \in \mathbf{R}$. *Suppose there exists a weakly* $\star$*-Einstein associated metric with the property that there exist three real numbers* $a, b, c$ ($a \geq 0, b \geq 0$, *but* $a^2 + b^2 \neq 0$) *such that*

$$as + bs^\star = c.$$

*Then this must be a Kähler metric.*

*Proof:* Without loss of generality we can assume that the total volume of $M$ is 1. Let $\gamma$ the representative of $2\pi c_1$ given in (1.10). From $c_1 = \frac{1}{8\pi}\lambda[\omega]$, it follows that there exists a 1-form $\alpha$ such that

$$8\pi\gamma = \lambda\omega + d\alpha.$$

Since $\omega$ is harmonic, it is orthogonal on exact forms, so from the previous relation

$$\lambda = \frac{1}{2}\int_M (s + s^\star)\sigma.$$

Therefore we get

$$c_1^2(M) = \frac{1}{32\pi^2}\Big(\int_M \frac{s + s^\star}{2}\sigma\Big)^2.$$

On the other hand, from (1.22)

$$c_1^2(M) = \frac{1}{16\pi^2}\int_M [s^2 - 2|\rho^{inv}|^2 + 2|\rho^{\star skew}|^2 + (\rho, D)]\sigma.$$

Comparing the two expressions for $c_1^2(M)$ we have

$$\frac{1}{2}(\int_M \frac{s^\star + s}{2}\sigma)^2 = \int_M [s^2 - 2|\rho^{inv}|^2 + 2|\rho^{\star skew}|^2 + (\rho, D)]\sigma. \tag{2.9}$$

The weak $\star$-Einstein condition is equivalent to $\rho^{\star skew} = 0$ and $\rho^{\star sym} = \frac{1}{4}s^\star g$. From (1.21) we deduce also that $\rho^{inv} = \frac{1}{4}sg$. With these relations, (2.9) becomes

$$(\int_M \frac{s^\star + s}{2}\sigma)^2 = \int_M ss^\star\sigma,$$

which we can rewrite as

$$(\int_M \frac{s^\star - s}{2}\sigma)^2 = \int_M ss^\star\sigma - (\int_M s\sigma)(\int_M s^\star\sigma). \tag{2.10}$$

Suppose that $as + bs^\star = c$, with $a$, $b$, $c$ constants, $a, b \geq 0$, but not both 0. Without loss of generality we can assume $b \neq 0$. Then

$$\int_M ss^\star\sigma - (\int_M s\sigma)(\int_M s^\star\sigma) = \frac{a}{b}[(\int_M s\sigma)^2 - \int_M s^2\sigma].$$

By Cauchy-Schwarz inequality we see that this last expression is non-positive. Hence the right-hand side of (2.10) is non-positive, whereas the left-hand side is non-negative. This implies that $s^\star = s$ and by (1.8), the metric is Kähler. □

Without any effort, we have the following corollaries:

**Corollary 2.2:** *Let $M$ be a compact, symplectic, 4-dimensional manifold with $c_1 = \frac{1}{8\pi}\lambda[\omega]$, for $\lambda \in \mathbf{R}$. Any $\star$-Einstein associated metric is a Kähler metric.*

**Corollary 2.3:** *Let $M$ be a compact, symplectic, 4-dimensional manifold with $c_1 = \frac{1}{8\pi}\lambda[\omega]$, for $\lambda \in \mathbf{R}$. Any Einstein, weakly $\star$-Einstein associated metric is a Kähler metric.*

We would like to end this section with some more remarks related to the Goldberg conjecture in dimension 4. We will refer to results which are included in the later sections of this thesis since they have more significance there, but we would like to point out here their consequences to the Goldberg conjecture. Given a compact, oriented 4-manifold $M$ with an Einstein metric $g$, a counter-example to the Goldberg conjecture would be provided by a self-dual, harmonic 2-form $\omega$ which has constant length $\sqrt{2}$, but it is not parallel. From Proposition 3.5, we see that we cannot find such a form if the metric $g$ is a Kähler Einstein metric on $M$. Moreover, using the Seiberg-Witten equations, LeBrun [34] showed that any Riemannian metric on a compact symplectic 4-manifold $M$ satisfies the inequality (2.8):

$$c_1^2(M) \leq \frac{1}{32\pi^2}\int_M s^2 d\sigma,$$

with equality if and only if the metric is Kähler Einstein. This implies the following

**Proposition 2.2:** *Let $M$ be a compact, oriented 4-manifold admitting Einstein metrics. If a connected component of the moduli space of Einstein metrics contains a Kähler Einstein metric, then all the metrics from that connected component are Kähler Einstein.*

*Proof:* Note first that if $g_t$ is a path of Einstein metrics of equal volume on a compact manifold, then the scalar curvature does not change $s_t = s_0$. ([9], [10]). The result then follows applying the above mentioned result of LeBrun. □

Thus, a counter-example to the Goldberg conjecture will not be obtained in dimension 4 by deforming a little bit a given Kähler Einstein metric. Another attempt could be to consider both orientations of 4-manifolds with Einstein metrics. For instance, let $(M, g)$ be a compact Kähler surface and consider $(\bar{M}, g)$. As a consequence of Theorem 4.1 we see that this also fails to give counter-examples to the Goldberg conjecture if the signature of $M$ satisfies $\sigma(M) \leq 0$. We do not know yet what happens in the case of positive signature, but in the proof of Theorem 4.2 we give some potentially interesting examples for this problem.

## 2.2 Critical associated metrics

A more recent, but parallel problem to the Goldberg conjecture arose from the work of D. Blair and S. Ianus, [15]. They studied variational problems in the set $\mathcal{AM}_\omega$ of all associated Riemannian metrics to a symplectic manifold, hoping to characterize Kähler metrics by a variational method. For the integral of the scalar curvature functional restricted to $\mathcal{AM}_\omega$, Blair and Ianus showed that the critical metrics are those for which the Ricci tensor is $J$-invariant. Kähler metrics are among the critical metrics and, in fact, from (1.12), we see that they are absolute maxima for this functional. It is therefore interesting to see whether almost Kähler metrics with $J$-invariant Ricci tensor have to be in fact Kähler. We will refer to this question as the "Blair-Ianus question" as it originates from [15], and by critical metrics we will understand for the rest of this section almost Kähler metrics with $J$-invariant Ricci tensor. If true, the question of Blair and Ianus would be a stronger statement than the Goldberg conjecture. However, a series of examples of strictly almost Kähler manifolds with $J$-invariant Ricci tensor were given in dimensions $4k + 2$ (see [16], [2]), showing that not all critical metrics are Kähler in general. These examples are obtained on twistor spaces of quaternionic Kähler manifolds. In spite of this, it is still interesting to see under what additional assumptions the question of Blair and Ianus has a positive answer, in view of possible analogies with the Goldberg conjecture. For the rest of this section we will consider critical metrics trying to prove parallel results to some of those from the previous section.

**Theorem 2.4:** *Let $(M^{2n}, g, J, \omega)$ be a compact almost Kähler manifold with J-invariant Ricci tensor.*

*A. If $2n \geq 6$ and there exist a function $\lambda \geq 0$ on $M$ such that*

$$\lambda g(X,X) \leq \rho(X,X) \leq 2\lambda g(X,X)$$

*for any $X \in TM$, then the almost complex structure is integrable, that is, $M$ is a Kähler manifold.*

*B. If $2n = 4$ and the Ricci tensor is non-negative definite when restricted to the Kähler nullity distribution $\mathcal{D} := \{X \in TM | \nabla_X J = 0\}$ then $M$ is a Kähler manifold.*

*Proof:* We will use identity (1.15) of Proposition 1.3. Integration by parts and equation (1.5) give

$$\int_M [(\nabla_i \rho_{bj} - \nabla_j \rho_{bi})(\nabla_b J_{ik}) J_{jk}]\sigma =$$

$$= \int_M [\rho_{bi} \nabla_j (\nabla_b J_{ik}) J_{jk} - \rho_{bj} \nabla_i ((\nabla_b J_{ik}) J_{jk})]\sigma = 2 \int_M \rho_{bi} \nabla_j (\nabla_b J_{ik}) J_{jk} \sigma.$$

To get the second equality we used the fact that

$$(\nabla_b J_{ik}) J_{jk} = -J_{ik} (\nabla_b J_{jk}),$$

and therefore

$$\nabla_i ((\nabla_b J_{ik}) J_{jk}) = -J_{ik} \nabla_i (\nabla_b J_{jk}).$$

On any almost Kähler manifold we have the following identity which can be easily obtained after a straightforward computation making use of (1.4) and (1.5):

$$\nabla_j \nabla_{\bar{b}} J_{\bar{i}k} = -\nabla_j \nabla_b J_{ik} - (\nabla_j J_{bs})(\nabla_s J_{\bar{i}k}) - (\nabla_{\bar{b}} J_{sk})(\nabla_j J_{is}).$$

Since $\rho$ is $J$-invariant, using the formula above and, repeatedly, the quasi-Kähler condition (1.5) we obtain :

$$2\rho_{bi}(\nabla_j\nabla_b J_{ik})J_{jk} = \rho_{bi}(\nabla_k J_{bs})(\nabla_s J_{ik}) - \rho_{bi}(\nabla_b J_{ks})(\nabla_k J_{is}) = (\rho, B) - (\rho, D).$$

Replacing in (1.15) and neglecting a few terms, we get the inequality:

$$0 \leq -2\int_M [2(\rho, B) - (\rho, D)]\sigma - \int_M \frac{|\nabla J|^4}{2n}\sigma. \tag{2.11}$$

Now, if the eigenvalues of $\rho$ are $0 \leq \lambda_1 \leq ... \leq \lambda_n$ then, using the hypothesis $2\lambda_1 \geq \lambda_n$ we get:

$$2(\rho, B) \geq 2\lambda_1|\nabla J|^2 \geq \lambda_n|\nabla J|^2 \geq (\rho, D). \tag{2.12}$$

This and inequality (2.11) imply $|\nabla J| = 0$, thus $J$ is parallel so we proved A.

To prove B, note that for almost Kähler 4-manifolds it was shown that $B = \frac{1}{4}|\nabla J|^2 g$ and that $D$ has a double eigenvalue 0 and another double eigenvalue $\frac{|\nabla J|^2}{2}$. Assume there exists a point $p \in M$ such that $\nabla J(p) \neq 0$. Then in a neighborhood of $p$, $dim\mathcal{D} = 2$. In this neighborhood choose $\{e_1, Je_1, e_2, Je_2\}$ a local $J$-orthonormal basis which diagonalizes $D$. We can assume that $\{e_1, Je_1\}$ correspond to the eigenvalue 0 of $D$ so $\{e_1, Je_1\}$ generate $\mathcal{D}$ in this neighborhood. Then

$$2(\rho, B) - (\rho, D) = \frac{1}{2}s|\nabla J|^2 - \rho(e_2, e_2)|\nabla J|^2 = \rho(e_1, e_1)|\nabla J|^2 \geq 0,$$

where the last equality is obtained from the hypothesis that $\rho$ is non-negative definite on $\mathcal{D}$. Using again (2.11) we get $\nabla J = 0$, hence the conclusion for B. □

Next we prove a result analogous to Corollary 2.3.

**Theorem 2.5:** *Let $(M^4, \omega)$ be a compact, 4-dimensional symplectic manifold with $H^2(M; \mathbf{R}) = \mathbf{R}$. A critical associated metric $g$ with its $\star$-Ricci tensor symmetric, is a*

*Kähler metric.*

We need first the following technical result:

**Lemma 2.2:** *On an almost Kähler manifold $(M, g, J, \omega)$ let $B$ be a symmetric, J-invariant tensor field of type (2,0), satisfying $\delta B = 0$ . Then the 2-form $\beta(X, Y) = B(X, JY)$ is co-closed (i.e. $\delta\beta = 0$).*

*Proof of Lemma 2.2:* We work in a local $J$-basis $\{e_1, Je_1, ..., e_n, Je_n\}$. Locally, $\beta$ is given by $\beta_{ij} = B_{is}J_{js}$. Since $\nabla_i B_{is} = 0$, we have

$$\nabla_i \beta_{ij} = B_{is} \nabla_i J_{js}.$$

Using the J-invariance of $B$ and the quasi-Kähler condition (1.5) we see that

$$B_{is}\nabla_i J_{js} = B_{\bar{i}\bar{s}} \nabla_{\bar{i}} J_{j\bar{s}} = B_{is} \nabla_{\bar{i}} J_{j\bar{s}} = -B_{is} \nabla_i J_{js}.$$

Hence $\nabla_i \beta_{ij} = B_{is} \nabla_i J_{js} = 0$ which concludes the proof. □

*Proof of Theorem 2.5:* From the fact that $\rho$ is $J$-invariant it follows that $\alpha$ defined by $\alpha(X, Y) = \rho(X, JY)$ is a $J$-invariant 2-form on $M$. We show first that $\alpha$ is closed. The tensor field $B = \rho - \frac{1}{2}sg$ is symmetric, $J$-invariant by hypothesis and, from the second Bianchi identity, also satisfies $\delta B = 0$ . Applying the Lemma, the corresponding 2-form, $\beta = \alpha - \frac{1}{2}s\omega$ is therefore co-closed. Therefore $\star\beta$ is closed. Using the decomposition (1.19),

$$\star\beta = \star(\alpha - \frac{1}{4}s\omega - \frac{1}{4}s\omega) = -\alpha + \frac{1}{4}s\omega - \frac{1}{4}s\omega = -\alpha.$$

Thus $\alpha$ is closed and induces the cohomology class $[\alpha]$.

Since $H^2(M; \mathbf{R}) = \mathbf{R}$, there exist real numbers $<, \mu$ such that $c_1(M) = \frac{1}{8\pi}\lambda[\omega]$ and $[\alpha] = \mu[\omega]$. Hence $8\pi\gamma - \lambda\omega$ and $\alpha - \mu\omega$ are exact, so orthogonal to $\omega$ with respect to the global inner product defined on forms. From this we get

$$\lambda = \frac{1}{2}\int_M (s + s^\star)\sigma, \quad \mu = \frac{1}{4}\int_M s\sigma.$$

Under the assumptions, the 2-form $\beta = \alpha - \frac{1}{2}s\omega$ is co-closed, and hence orthogonal to $8\pi\gamma - \lambda\omega$ and $\alpha - \mu\omega$. Expressing this, we get

$$\int_M [-4|\rho|^2 + 2s^2 + (\rho, D)]\sigma = \frac{1}{2}\Big(\int_M s\sigma\Big)\Big(\int_M (s^\star + s)\sigma\Big); \tag{2.13}$$

$$\int_M |\rho|^2\sigma = \frac{1}{2}\int_M s^2\sigma - \frac{1}{4}\Big(\int_M s\sigma\Big)^2. \tag{2.14}$$

Relations (2.13) and (2.14) combined give

$$\int_M (\rho, D)\sigma = \Big(\int_M s\sigma\Big)\Big(\int_M \frac{s^\star - s}{2}\sigma\Big). \tag{2.15}$$

Using now (2.14) and (2.15) in (2.9), which holds under our assumptions, we obtain

$$\int_M 4|\rho^{\star skew}|^2\sigma = \Big(\int_M \frac{s^\star - s}{2}\sigma\Big)^2 = \Big(\int_M \frac{|\nabla J|^2}{4}\sigma\Big)^2. \tag{2.16}$$

By assumption the $\star$-Ricci tensor is symmetric, so (2.16) gives $\nabla J = 0$ completing the proof.□

# CHAPTER 3

# Hermitian conformal classes and almost Kähler structures on 4-manifolds

In this chapter we show that for most compact complex surfaces which also admit symplectic forms, each Hermitian conformal class contains almost Kähler metrics. We also give results about the number of almost Kähler structures a given metric can be part of. As applications, we obtain alternative proofs for results of LeBrun on the Yamabe constants of Hermitian conformal classes and give some answers to a question of Blair about the isometries of almost Kähler metrics. We end with a section discussing the difference between the set of Kähler forms and the set of symplectic forms on a given manifold. Sections 3.1 to 3.4 of this chapter are part of a joint work with Vestislav Apostolov. With slight modifications the reader could find them in [3].

## 3.1 Two Problems

Let us consider a 4-manifold $M$ which admits symplectic structures. Given a symplectic form, $\omega$, the space of associated metrics to $\omega$ can be also defined in this dimension by

$$\mathcal{AM}_\omega = \{g|\ (i)\ \ \omega \in \Lambda^+ M,\ (ii)\ \ |\omega|_g = \sqrt{2}\}.$$

Note that there can be no two elements of $\mathcal{AM}_\omega$ in the same conformal class of metrics. Giving up condition *(ii)*, define the space of conformal associated metrics to $\omega$ by,

$$\mathcal{CAM}_\omega = \{g|\ \omega \in \Lambda^+ M\}.$$

Indeed, it is easily seen that

$$\mathcal{CAM}_\omega = \mathcal{C}^\infty_+(M) \cdot \mathcal{AM}_\omega,$$

where $\mathcal{C}^\infty_+(M)$ denotes the space of smooth, positive functions on $M$.
If $\mathcal{S}$ denotes the set of all symplectic forms on $M$, then the space of all almost Kähler metrics and the space of all conformal almost Kähler metrics are, respectively:

$$\mathcal{AK} = \cup_{\omega\in\mathcal{S}}\mathcal{AM}_\omega,\ \ \mathcal{CAK} = \cup_{\omega\in\mathcal{S}}\mathcal{CAM}_\omega.$$

The following easy proposition motivates the questions we are addressing in this chapter.

**Proposition 3.1:** *Let $M$ be a closed, oriented 4-manifold, admitting symplectic structures.*
*(a) If $\omega$ and $\omega'$ are distinct, but cohomologus symplectic forms on $M$, then $\mathcal{CAM}_\omega \cap \mathcal{CAM}_{\omega'} = \emptyset$.*

*(b) Let $g$ be a Riemannian metric on $M$. There exists a finite dimensional vector subspace $V$ of $C^\infty(M)$, such that for any $f \in C^\infty_+(M)$ with $f^2 \notin V$, the metric $g' = fg$ is not an almost Kähler metric.*

*Proof:* (a) Assume there exists a metric $g \in \mathcal{CAM}_\omega \cap \mathcal{CAM}_{\omega'}$. Then $\omega$ and $\omega'$ are both harmonic with respect to $g$. But by Hodge decomposition theorem there is a unique harmonic representative in a given cohomology class. This contradicts the assumption $\omega \neq \omega'$.

(b) Let us recall that in dimension 4, harmonic 2-forms are invariant to conformal changes of metric, as also invariant is the splitting into self-dual and anti self-dual forms. Given the metric $g$, assume that the metric $g' = fg$ is an almost Kähler metric. This is equivalent to the existence of a self-dual, harmonic 2-form, $\omega'$, with

$$\frac{1}{f^2}|\omega'|^2_g = |\omega'|^2_{g'} = 2.$$

Let $\alpha_1, ..., \alpha_k$ form an orthogonal basis for the space of self-dual, harmonic 2-forms with respect to the global inner product induced by the metric $g$. As $\omega'$ is self-dual, harmonic with respect to $g$ as well, it must be a linear combination of the $\alpha_i$'s:

$$\omega' = a_1\alpha_1 + ... + a_k\alpha_k,$$

for some constants $a_1, ..., a_k$.

It follows that

$$2f^2 = \sum a_i a_j f_{ij},$$

where $f_{ij}$ are the smooth functions given by the pointwise $g$-scalar product of $\alpha_i$ and $\alpha_j$, $f_{ij} = (\alpha_i, \alpha_j)_g$. Taking $V$ to be the space generated by the $f_{ij}$'s, the conclusion follows. □

A short way of rephrasing part (b) of the Proposition 3.1 is that in a given conformal class most of the metrics are not almost Kähler. As for (a), it leads to some questions. First, one may ask under what conditions two symplectic forms share a same associated metric. As we saw, this is not possible if the forms are cohomologus. From a metric point of view, this can be restated as follows:

**Problem 1:** When does a Riemannian 4-manifold $(M, g)$ carry two almost Kähler structures $(g, J_1, \omega_1)$, $(g, J_2, \omega_2)$ with $\omega_1 \neq \pm\omega_2$ ?

From a symplectic form $\omega$, many others can be obtained by deforming the given one with "small" closed 2-forms. As symplectic forms in the same cohomology class have all disjoint sets of conformal associated metrics, it looks that "many" conformal classes contain almost Kähler metrics. It is natural to ask the following problem.

**Problem 2:** Find conformal classes which do not admit almost Kähler metrics.

## 3.2 Main Result

We give some answers to Problems 1 and 2 for compact complex surfaces where we consider the space $\mathcal{H}$ of all Hermitian metrics. First of all, it makes sense to consider only compact, complex surfaces which also admit symplectic structures. Note that any closed complex surface with $b_1$ even admits Kähler structures, hence, in particular, symplectic structures. In the case $b_1$ odd, the situation is more delicate and has been settled only recently (see [11, 24]).

**Proposition 3.2: (O. Biquard [11])** *The only complex surfaces with $b_1$ odd that also admit symplectic structures are primary Kodaira surfaces and blow-ups of these.*

Here is the main result of this chapter.

**Theorem 3.1:** *Let $(M, J)$ be a compact complex surface which also admits symplectic structures.*

*(a) If $b_1$ is even then $\mathcal{H} \subset \mathcal{CAK}$. Moreover:*

*(a1) Assume that $g$ is a Kähler, non-hyper-Kähler metric on $M$, with Kähler form $\omega$. Then $\omega$ and $-\omega$ are the only almost Kähler structures compatible to $g$;*

*(a2) Assume that $g$ is a non-Kähler, conformally-Kähler metric on $M$.*
*If $c_1 \neq 0$, one of the following two situations occurs: $g$ has exactly two $S^1$ families of associated almost Kähler structures, or $g$ is not an almost Kähler metric.*
*If $c_1 = 0$, one of the following three situations occurs: $g$ has exactly two $S^1$ families of associated almost Kähler structures, $g$ has exactly one $S^1$ family of associated almost Kähler structures, or $g$ is not an almost Kähler metric.*

*(b) If $b_1$ is odd, there are two cases:*

*(b1) If $(M, J)$ is minimal then $\mathcal{H} \subset \mathcal{CAK}$. In this case, each metric $g \in \mathcal{H} \cap \mathcal{AK}$ has exactly one $S^1$ family of almost Kähler structures associated.*

*(b2) If $(M, J)$ is not minimal then $\mathcal{H} \cap \mathcal{CAK} = \emptyset$.*

Regarding Problem 1, we see that Kähler, non-hyper-Kähler metrics have an essentially unique compatible almost Kähler structure. However, as we see in (a2), there are examples of Hermitian metrics having a whole family of compatible almost Kähler structures. As $c_1 \neq 0$, these examples are not hyper-Kähler.

As an immediate consequence of (b2), here is an answer to Problem 2.

**Corollary 3.1:** *On blow-ups of primary Kodaira surfaces, Hermitian conformal classes do not contain almost Kähler metrics.*

The proof of the Theorem 3.1 relies on a series of propositions which we give below.

Recall that the Lee form $\theta$ of an almost Hermitian 4-manifold $(M, g, J)$ is defined by $dF = \theta \wedge F$, or equivalently $\theta = J\delta F$, where $F$ denotes the Kähler form of $(g, J)$,

$\delta$ is the co-differential operator defined by $g$, and $J$ acts by duality on 1-forms. It easily seen that $d\theta$ is a conformal invariant, that is, it depends on the conformal class of $g$ and not on the metric itself. It is also known that Hermitian metrics with $d\theta = 0$ correspond to locally conformal Kähler metrics and the Hermitian metrics with $\theta = 0$ are, in fact, Kähler metrics.

A Hermitian metric such that the Lie form is co-closed, i.e. $\delta\theta = 0$, is called by Gauduchon a *standard Hermitian metric*. He proves in [21] the existence of standard metrics in each Hermitian conformal class (in any dimension) and its uniqueness modulo a homothety. In some sense, the standard Hermitian metric is the "closest" to a Kähler metric in its conformal class.

The first result we need is due to Gauduchon. For completeness, we give a proof, slightly different than the original argument in [23].

**Proposition 3.3: (Gauduchon [23])** *On a compact complex surface $M$, endowed with a standard Hermitian metric $g$, the trace of a harmonic, self-dual form is a constant.*

*Proof:* Let $(M, g, J, F)$ be the standard Hermitian structure on $M$. Any self-dual form $\alpha \in \Lambda^+ M$ can be uniquely written as:

$$\alpha = aF + \beta + \overline{\beta}, \tag{3.1}$$

with $a \in \mathcal{C}^\infty(M)$ and $\beta \in \Lambda^{2,0}M$. We have to prove that if $\alpha$ is also (co)closed then $a$ is a constant.

Taking the divergence of both sides of (3.1), it follows

$$0 = Jda + aJ\theta + \delta(\beta + \overline{\beta}).$$

Applying $J$ to the above relation, we get:

$$da = -a\theta + J\delta(\beta + \overline{\beta}).$$

Taking inner product of both sides with $da$ and integrating over the manifold implies

$$\int_M |da|^2 d\mu = -\int_M \frac{1}{2}(\theta, d(a^2))d\mu + \int_M (J\delta(\beta + \overline{\beta}), da)d\mu =$$

$$= -\int_M \frac{1}{2}(\delta\theta, a^2)d\mu - \int_M ((\beta + \overline{\beta}), dJda)d\mu = 0,$$

since $\delta\theta = 0$ and $dJda \in \Lambda^{1,1}M$. Therefore $da = 0$, so $a$ is a constant. □

**Corollary 3.2:** *Let $(M, g, J, F)$ be a Hermitian surface. Then any harmonic, self-dual form $\omega$ is either the real part of a holomorphic (2,0) form or is non-degenerate everywhere on $M$.*

This result already gives the relations between the spaces $\mathcal{H}$ and $\mathcal{CAK}$ stated in the Theorem 3.1 at (a), (b1) and (b2). The next propositions deal with the number of compatible almost Kähler structures that various Hermitian metrics can have.

**Lemma 3.1:** *Let $(M, J)$ be a complex manifold with $c_1 \neq 0$, equipped with a standard Hermitian metric $g$ (which may be Kähler), and let $F$ be the fundamental form. Suppose $\alpha_1, \alpha_2$ are two harmonic self-dual 2-forms which satisfy $\alpha_1^2 = \alpha_2^2$. Then the traces of these forms (which are necessarily constants) are equal up to sign:*

$$(\alpha_1, F) = \pm(\alpha_2, F).$$

*Proof:* Using decomposition (1.19), the 2-forms $\alpha_1, \alpha_2$ can be written uniquely as:

$$\alpha_1 = a_1 F + \beta_1 + \overline{\beta}_1,$$

$$\alpha_2 = a_2 F + \beta_2 + \overline{\beta}_2,$$

where $\beta_1, \beta_2$ are (2,0) forms and $a_1, a_2$ are constants (by Proposition 3).

Now $\alpha_1^2 = \alpha_2^2$ is equivalent to

$$(a_1^2 - a_2^2)F^2 = 2(\beta_2 \wedge \overline{\beta}_2 - \beta_1 \wedge \overline{\beta}_1) = Re((\beta_2 - \beta_1) \wedge (\overline{\beta}_2 + \overline{\beta}_1)).$$

By the assumption $c_1 \neq 0$, it follows that the form $\beta_2 - \beta_1$ must vanish at some point on $M$. From the above equality, as $F^2$ is a volume form on $M$ and $a_1, a_2$ are constants, it follows $a_1^2 - a_2^2 = 0$. □

**Proposition 3.4:** *Let $g$ be a standard Hermitian metric on a complex surface $(M, J)$ with $c_1 \neq 0$, and let $F$ be the fundamental form. Denote by $\omega$ the unique self-dual, harmonic form which has trace equal to 1 and is orthogonal to the space of holomorphic (2,0) forms with respect to the cup product. Suppose $\alpha$ is a harmonic, self-dual form such that $\alpha^2 = \omega^2$ everywhere on $M$. Then $\alpha = \pm\omega$.*

*Proof:* By Lemma 3.1, $trace\alpha = \pm trace\omega = \pm 1$. Assume $trace\alpha = trace\omega = 1$. In this case $\alpha$ can be written as

$$\alpha = \omega + Re(\beta),$$

where $\beta$ is a holomorphic (2,0) form. From $\alpha^2 = \omega^2$, it follows the relation

$$2\omega \wedge Re(\beta) + Re(\beta)^2 = 0,$$

everywhere on $M$. Integrating this relation on $M$, the first term vanishes because of the choice of $\omega$. Therefore we get $Re(\beta) = 0$, but as $\beta$ is a (2,0) form this implies $\beta = 0$. Therefore we proved $\alpha = \omega$. Similarly, if $trace\alpha = -trace\omega = -1$, it follows that $\alpha = -\omega$. □

**Proposition 3.5:** *Let $g$ be a Kähler metric on $M$ with Kähler form $\omega$. Then either $g$ is a hyper-Kähler metric, or $\pm\omega$ are the only almost Kähler structures compatible to $g$. Moreover, for a hyper-Kähler metric $g$, all self-dual, harmonic forms of constant length are parallel.*

*Proof:* The last claim, about self-dual, harmonic forms of constant length on a compact hyper-Kähler manifold, is probably well-known and could be obtain in many different ways. For example, it follows from Theorem 2.1 of Sekigawa, since a hyper-Kähler metric is Ricci flat. It remains to show the first part of the proposition. If $c_1 \neq 0$ the conclusion follows immediately from the Proposition 3.4. Our argument below covers all cases.

Assume there exists another harmonic, self-dual form $\omega' \neq \pm\omega$, inducing same volume form as $\omega$. Then $\omega'$ is uniquely written as

$$\omega' = a\omega + \eta,$$

where $a$ is a constant and $\eta$ is a smooth section of the canonical bundle. From $\omega'^2 = \omega^2$ we deduce

$$a^2 + \frac{|\eta|^2}{2} = 1,$$

hence $|a| \leq 1$. If $|a| = 1$, then $\eta = 0$, therefore $\omega' = \pm\omega$. If $|a| < 1$, we show that the metric $g$ is in fact hyper-Kähler. Indeed, $\omega_1 = \frac{1}{\sqrt{1-a^2}}\eta$ is a self-dual harmonic 2-form of length $\sqrt{2}$, pointwise orthogonal to $\omega$, so it induces another almost Kähler structure on $M$, $(g, J_1, \omega_1)$, with $J$ and $J_1$ anti-commuting. Since $J$ is parallel with respect to the Levi-Civita connection of $g$, it follows that $(g, J_2 = J \circ J_1)$ is another almost Kähler structure, with $J_2$ anti-commuting with both $J$ and $J_1$. Now, using an observation of Hitchin ([27], Lemma 6.8) that any triple of anti-commuting almost Kähler structures $(g, J, J_1, J_2)$ defines a hyper-Kähler structure, we complete the proof. □

**Proposition 3.6:** *Assume that $g$ is a non-Kähler, conformally-Kähler metric on a compact complex surface $(M, J)$.*
*If $c_1 \neq 0$, one of the following two situations occurs: $g$ has exactly two $S^1$ families of associated almost Kähler structures, or $g$ is not an almost Kähler metric.*
*If $c_1 = 0$, one of the following three situations occurs: $g$ has exactly two $S^1$ families of associated almost Kähler structures, $g$ has exactly one $S^1$ family of associated almost Kähler structures, or $g$ is not an almost Kähler metric.*

*Proof:* First we will consider the case $c_1 \neq 0$. Let $g = fg'$, where $f \in C^\infty_+$ and $g'$ is a Kähler metric on $(M, J)$ with Kähler form $F$. Let us assume also that $(g, J, \omega)$ is an almost Kähler structure. Then $\omega$ is a $g$-harmonic, self-dual form, of $g$-length $\sqrt{2}$ at every point on $M$. As $g'$ is a conformal metric to $g$, the form $\omega$ is harmonic and self-dual with respect to $g'$ as well. Hence there exists a constant $a \neq 0$ and a holomorphic (2,0) form $\beta$ such that

$$\omega = aF + Re(\beta).$$

But in this case, note that the forms

$$\omega_t^+ = aF + cos(2\pi t)Re(\beta) + sin(2\pi t)Im(\beta),$$

$$\omega_t^- = -aF + cos(2\pi t)Re(\beta) + sin(2\pi t)Im(\beta),$$

are also harmonic, self-dual and of length $\sqrt{2}$ with respect to the metric $g$, for any $t \in$

$[0,1]$. Therefore $g$ has at least two $S^1$-families of almost Kähler structures compatible to $g$.

Suppose now that $(g, J', \omega')$ is some almost Kähler structure compatible to $g$ and we would like to show that it must be one of the almost Kähler structures described by the the two $S^1$-families above. With the same reasoning as above

$$\omega' = a'F + Re(\beta'),$$

where $a'$ is a non-zero constant and $\beta'$ is a holomorphic (2,0) form. Since $\omega^2 = \omega'^2$, by Lemma 1 we get $a' = \pm a$. Let us assume $a' = a$, the argument being similar in the other case. Now $\omega^2 = \omega'^2$ implies $Re(\beta)^2 = Re(\beta')^2$, which is equivalent to

$$Re(\beta - \beta') \wedge Re(\beta + \beta') = 0.$$

This means that at every point on $M$, the form $Re(\beta + \beta')$ is collinear to $Im(\beta - \beta')$. As both $Re(\beta + \beta')$ and $Im(\beta - \beta')$ are closed, we must have

$$Re(\beta + \beta') = \lambda Im(\beta - \beta'),$$

for $\lambda$ a constant on $M$. The above relation implies

$$\beta' = \frac{\lambda^2 - 1}{\lambda^2 + 1}\beta - \frac{2\lambda}{\lambda^2 + 1} i\beta,$$

or, further,

$$Re(\beta') = \frac{\lambda^2 - 1}{\lambda^2 + 1} Re(\beta) + \frac{2\lambda}{\lambda^2 + 1} Im(\beta).$$

It is easy to see now that $\omega'$ is in fact one of the forms in the family $\omega_t^+$.

Next, let us consider the case $c_1 = 0$. By Kodaira's classification Theorem we distinguish two sub-cases.

(i) $(M, J)$ is a hyperelliptic surface or an Enriques surface. For these $b^+ = 1$, hence, by Proposition 3.1, there are no non-Kähler, globally conformal Kähler almost Kähler metrics.

(ii) $(M, J)$ is a complex torus or a K3 surface. For these $b_+ = 3$ and they have hyper-Kähler metrics. Let us first remark that if $g'$ is such a metric, then all self-dual, harmonic forms with respect to $g'$ have constant length. Therefore there is no non-Kähler, almost Kähler metric which is conformal to a hyper-Kähler metric. However, a complex torus or a K3 surface do have Kähler metrics other then the hyper-Kähler ones. Choose one such metric and denote it again by $g'$, the corresponding Kähler form being $\omega'$. Suppose that $g = fg'$ is a non-Kähler, conformally Kähler metric which has an almost Kähler structure $\omega$. Then we have

$$\omega = a\omega' + Re(\beta),$$

where $a$ is a real constant and $\beta$ is a holomorphic (2,0) form. In fact, $\beta$ is everywhere nondegenerate, so it is a holomorphic symplectic form on $M$.

Now we have two possibilities: if $a = 0$, then the metric $g$ has one $S^1$ family of almost Kähler structures given by

$$\omega_t = cos(2\pi t)Re(\beta) + sin(2\pi t)Im(\beta);$$

if $a \neq 0$, then the metric $g$ has two $S^1$ families of almost Kähler structures given by

$$\omega_t^{\pm} = \pm a\omega' + cos(2\pi t)Re(\beta) + sin(2\pi t)Im(\beta).$$

In either case, if $g$ had other almost Kähler structures, it would follow that $\beta$ has constant length with respect to $g'$, which is a contradiction to the fact that $g'$ is not hyper-Kähler. □

We finally put together the above results to prove Theorem 3.1.

*Proof of Theorem 3.1:* Let us denote by $p_g$ the geometric genus of the complex surface $(M, J)$, i.e. the complex dimension of the space of holomorphic (2,0) forms. It is well known that $b_+ = 2p_g$ when $b_1$ is odd and $b_+ = 2p_g + 1$ when $b_1$ is even.

Let us consider first the case $b_1$ odd. By Proposition 3.2 of O. Biquard, the only compact complex surfaces that also admit symplectic structures are primary Kodaira surfaces (case of (b1)) and blow-ups of these (case of (b2)). For the primary Kodaira surfaces it is also known that they do admit holomorphic symplectic structures, that this, there exists a nowhere vanishing holomorphic (2,0) form. Denote such a form $\beta$ and consider now a Hermitian metric $g$. The real form $\omega = Re(\beta)$ is the real part of a holomorphic (2,0) form on $M$ hence it is a harmonic, self-dual form for the metric $g$. As $\omega$ is also non-degenerate, there is a conformal metric to $g$ such that $\omega$ and the new metric define an almost Kähler structure. We hence proved $\mathcal{H} \subset \mathcal{CAK}$ for primary Kodaira surfaces. Note also that if $g$ is Hermitian, then any almost Kähler structure, say $\omega$, has to be the real part of a holomorphic $(2,0)$ form since $b_+ = 2p_g$. Hence $\omega = Re(\beta)$, but then

$$\omega_t = cos(2\pi t)Re(\beta) + sin(2\pi t)Im(\beta)$$

is a whole $S^1$ family of almost Kähler structures compatible to the metric $g$. Finally, since for a primary Kodaira surface $b_+ = 2$, from Proposition 3.1 it follows that each Hermitian, almost Kähler metric has exactly one $S^1$ family of compatible almost Kähler forms.

To prove (b2) note first that if $(M, J)$ is a blow-up of a primary Kodaira surface, then $c_1 \neq 0$ in this case. Let $g$ be a Hermitian metric and let $\omega$ be a real, self-dual, harmonic form with respect to $g$. As above, since $b_+ = 2p_g$, it follows that $\omega = Re(\beta)$, where $\beta$ is a holomorphic (2,0) form. Since $c_1 \neq 0$, $\beta$ must vanish at some point on $M$

and so does $\omega$. Therefore, for any Hermitian metric there are no harmonic, self-dual, everywhere non-degenerate forms. This proves (b2).

Let us now consider the case $b_1$ even. In this case $b_+ = 2p_g + 1$, so for any Hermitian metric $g$, the space of real parts of holomorphic (2,0) forms is strictly contained in the space of all self-dual, harmonic forms. Let $\omega$ denote the (unique) self-dual, harmonic form which has trace equal to 1 and is orthogonal, with respect to the cup product, to the space of real parts of holomorphic (2,0) forms. This form is non-degenerate everywhere on $M$ and hence for a conformal metric to $g$ this form will define an almost Kähler structure. The statements from (a1) and (a2) follow form Propositions 3.5 and 3.6, respectively. □

**Remark 3.1:** It would be nice to complete (a) in the Theorem 3.1 with a statement about the possible number of almost Kähler structure compatible to an arbitrary Hermitian metric (non-Kähler and not conformally Kähler). Proposition 3.4 shows that there are some Hermitian, non-Kähler metrics with a unique, up to sign, almost Kähler structure. However, we do not know a complete answer to this problem yet.

## 3.3 Yamabe and fundamental constants of Hermitian surfaces

The *Yamabe constant*, $Y(c)$, of the conformal class $c$ on a compact 4-manifold $M$ is defined to be

$$Y(c) = inf_{g \in c}\Big[\frac{\int_M s_g dV_g}{\sqrt{\int_M d\mu_g}}\Big],$$

where $s_g$ is the scalar curvature of the Riemannian metric $g$ and $d\mu_g$ denotes it volume form. It was proved by R. Shoen [43] that each conformal class $c$ contains metrics of constant scalar curvature. These realize the infimum in the above definition and, for this reason, scalar curvature metrics are also referred as *Yamabe metrics.* We shall say that $(M, c)$ is of *positive (resp. zero or negative) type* if $Y(c)$ is positive (resp. zero or negative).

It is a remarkable fact that the existence of metrics with positive scalar curvature on a compact 4-manifold leads to important informations about the differentiable structure of the manifold. In particular, all Seiberg-Witten invariants must vanish. This was successfully used by C. LeBrun to prove that on a compact complex surface $(M, J)$ with even first Betti number the existence of conformal classes (not necessarily compatible with $J$) of positive type forces $(M, J)$ to have negative Kodaira dimension, i.e to be either a rational surface, or a blow up of a ruled surface [33]. Considering only the conformal classes of Hermitian metrics, LeBrun's result was previously observed by several other authors [50, 48, 4]. The main idea dealing with Hermitian conformal classes is to use the Gauduchon's vanishing theorem, as it is explained below.

Let $(M, J)$ be a compact complex surface and let $c$ be a conformal class of Hermitian metrics on $M$. For any metric $g \in c$ we denote by $u_g$ the *Hermitian scalar curvature* of $(g, J)$, which is defined to be the trace of the Ricci form of the Chern connection $\nabla^c$, i.e. we have

$$u_g = 2 < R^c(F), F >_g$$

where $R^c$ is the curvature of $\nabla^c$ and $F$, as usually, is the Kähler form of $(g, J)$. Using the relation between the Chern connection $\nabla^c$ and the Riemannian connection $\nabla$, given by (c.f. [23, 48])

$$\nabla^c_X Y = \nabla_X Y - \frac{1}{2}\theta(Y)X - \frac{1}{2}\theta(JX)JY + \frac{1}{2}g(X, Y)\theta,$$

on can easily see (c.f.[23]) that $u_g$ and $s_g$ are related by

$$u_g = s_g - \delta\theta + \frac{1}{2}|\theta|_g^2. \tag{3.2}$$

The eccentricity function $f_0(g)$ of a metric $g$ in $c$ is the positive function determined by the property $g = \frac{1}{f_0(g)} g_0$, where $g_0$ is the standard metric of Gauduchon on $c$ giving $M$ a total volume 1 (different normalization than [7]). Note that a metric $g$ is standard if and only if the corresponding function $f_0$ is a positive constant.

The *fundamental constant* $C(M, J, g)$ of a compact Hermitian surface we will define to be (compare with [7]):

$$C(M, J, g) = \int_M f_0(g) u_g d\mu_g.$$

Note that $C(M, J, g)$ does not depend on the choice of $g \in c$ and is a conformal invariant of $c$ equal to $C(M, J, g_0) = \int_M u_{g_0} d\mu_{g_0}$, so we can denote it just as $C(M, J, c)$. It follows from (3.2) that $\int_M s_{g_0} d\mu_{g_0} \leq C(M, J, c)$ which gives the estimate

$$Y(c) \leq C(M, J, c), \tag{3.3}$$

with equality in (3.3) if and only if $g_0$ is a Yamabe-Kähler metric:

The fundamental constant $C(M, J, c)$ is closely related to the complex geometry of $(M, J)$ in view of the following vanishing theorems of Gauduchon [22]. Denote by $P_m$ (resp.$Q_m$) the dimension of the space of holomorphic sections of $K^{\otimes m}$ (resp. of $K^{-\otimes m}$). Then we have:

(a) $C(M, J, c) > 0 \Longrightarrow P_m = 0, \forall m > 0$;

(b) $C(M, J, c) < 0 \Longrightarrow Q_m = 0, \forall m > 0$;

(c) $C(M, J, c) = 0 \Longrightarrow P_m = Q_m$ and $P_m \in 1, 0, \forall m > 0$.

In particular, for any positive conformal class $c$, the estimate (3.3) gives $C(M, J, c) >$

0, hence such a surface has to be of negative Kodaira dimension.

It is clear that except for the case when $P_m = Q_m = 0, \forall m > 0$ (some surfaces of negative Kodaira dimension), the sign of $C(M, J, c)$ is independent of $c$ (see [7]). We also note that the existence of a Hermitian conformal class $c$ with $C(M, J, c) = 0$ does imply the existence of a metric $g \in c$ of vanishing Hermitian scalar curvature $u_g$ (see [7], Corollary 1.9), hence the Ricci form $R^c(F)$ (which represents up to multiplication with $\frac{1}{2\pi}$ the first real Chern class of $(M, J)$) is anti-self-dual. In particular, we have $c_1^2 \leq 0$ with equality if and only if $c_1 = 0$. So, on any complex surface $(M, J)$ satisfying $2\chi(M) + 3\sigma(M) > 0$ (or $2\chi(M) + 3\sigma(M) = 0$ and $c_1 \neq 0$), the sign of $C(M, J, c)$ is also independent on the Hermitian conformal class $c$.

On the other hand, for a compact almost Kähler manifold $(M, g, J, \omega)$ we have another estimate for the Yamabe constant, coming from the basic scalar curvature inequality proved in (1.12).

$$\int_M s_g d\mu_g \leq 4\pi c_1 \cdot [\omega],$$

with equality if and only if the structure is Kähler. It follows that

$$Y(c) \leq 4\sqrt{2}\pi \frac{c_1 \cdot [\omega]}{\sqrt{[\omega] \cdot [\omega]}}, \tag{3.4}$$

with equality if and only if $g$ is a Yamabe-Kähler metric.

Now we shall use Theorem 3.1 to compare (3.3) and (3.4) on some Hermitian surfaces. We start with the following proposition, due to LeBrun in a more general setting [35]:

**Proposition 3.7:** *Let $(M, g, J, F)$ be a Hermitian surface with $b_1$ even and let $\omega$ be a harmonic, self-dual form on $M$ of non-negative trace. Then the following inequality* *holds:*

$$\int s\frac{|\omega|}{\sqrt{2}}d\mu \leq 4\pi c_1 \cdot [\omega],$$

*where $s$ is the scalar curvature, $d\mu$ is the volume form and $|\cdot|$ is the pointwise norm determined by the metric $g$.*

*Proof:* According to Corollary 3.2, we have two cases to consider.

Case 1: The form $\omega$ is non-degenerate everywhere on $M$.

Denote by $u$ the (strictly) positive function given by $\omega^2 = u^4F^2$, or, equivalently $\sqrt{2}u^2 = |\omega|$. The metric $g' = u^2g$ is an associated metric for the symplectic form $\omega$. The almost complex structure induced by $g'$ and $\omega$ is homotopic to $J$, hence it has the same real first Chern class as $J$. Using (19), we get:

$$\int s_{g'}d\mu_{g'} \leq 4\pi c_1 \cdot [\omega] \tag{3.5}$$

Standard formulas for a conformal change of metric $g' = u^2g$ give

$$s_{g'} = u^{-2}s_g + 6u^{-3}\Delta_g u,$$

$$d\mu_{g'} = u^4 d\mu_g.$$

From these we obtain

$$\int s_{g'}d\mu_{g'} = \int s_g\frac{|\omega|_g}{\sqrt{2}}d\mu_g + 6\int |du|_g^2 d\mu_g \geq \int s_g\frac{|\omega|_g}{\sqrt{2}}d\mu_g, \tag{3.6}$$

and the proof is finished for the Case 1.

Case 2: The form $\omega$ is the real part of a holomorphic (2,0) form.

In this case we have $c_1 \cdot [\omega] = 0$, since on a complex surface $c_1$ can be represented by a (1,1) form (the Ricci form of a Hermitian connection). Consider $\omega_0$ a harmonic, self-dual form, nowhere degenerate on $M$ and denote

$$\omega_t = \omega_0 + t\omega,$$

for $t > 0$. Then $\omega_t$ are non-degenerate, harmonic self-dual forms for any $t$, so we can apply Case 1 to them. It follows

$$\int s\frac{|\omega_t|}{\sqrt{2}}d\mu \leq 4\pi c_1 \cdot [\omega_t].$$

Taking into account that $c_1 \cdot [\omega] = 0$, this becomes

$$\int s\frac{|\omega_0 + t\omega|}{\sqrt{2}}d\mu \leq 4\pi c_1 \cdot [\omega_0],$$

and, after dividing by $t$,

$$\int \frac{s}{\sqrt{2}}(\frac{|\omega_0|^2}{t^2} + \frac{2 < \omega, \omega_0 >}{t} + |\omega|^2)^{\frac{1}{2}} d\mu_g \leq \frac{4\pi}{t}c_1 \cdot [\omega_0].$$

Taking the limit $t \longrightarrow \infty$, we obtain the conclusion in this case too. □

**Remark 3.2:** A more careful application of relation (3.6) implies the inequality

$$\int s_g|\omega|_g d\mu_g + 6\int |d(|\omega|^{1/2})|_g^2 d\mu_g \leq 4\pi\sqrt{2}c_1 \cdot [\omega],$$

for any Hermitian metric $g$ and any harmonic, self-dual form $\omega$ of non-negative trace. As a consequence, we see that on a scalar-flat Hermitian surface with $b_1$ even, all holomorphic (2,0) forms have constant length.

**Corollary 3.3:** *Under the same assumptions as Proposition 3.7, we also have the inequality:*

$$\int s^2 d\mu \geq 32\pi^2 (c_1^+)^2,$$

*where $c_1^+$ denotes the harmonic, self-dual part of $c_1$.*

*Proof:* Apply Proposition 3.6 to the harmonic, self-dual form $\omega$ which satisfies $\omega = -c_1^+$. We get

$$4\sqrt{2}\pi(c_1^+)^2 \leq \int -s|\omega|d\mu \leq \int |s||\omega|d\mu.$$

Schwarz inequality implies

$$4\sqrt{2}\pi(c_1^+)^2 \leq (\int s^2 d\mu)^{\frac{1}{2}}(\int |\omega|^2 d\mu)^{\frac{1}{2}}.$$

Since $\omega$ is the harmonic representative of the class $c_1^+$, we have

$$\int |\omega|^2 d\mu = (c_1^+)^2,$$

and the conclusion follows. □

As already mentioned, on a rational surface $(M, J)$ with $c_1^2 \geq 0$, the sign of $C(M, J, c)$ does not depend on the Hermitian conformal class $c$. Therefore it is always positive, since any rational surface admits a Kähler metric of positive total scalar curvature (cf. [50, 20]). With this observation and Proposition 3.7 in hand, we prove the following

**Proposition 3.8:** *Let $(M, J)$ be a rational surface with $c_1^2 \geq 0$. Then for any Hermitian conformal class $c$ on $M$ we have*

$$Y(c) \leq 4\pi\sqrt{2(c_1^+)^2} \leq C(M, J, c), \tag{3.7}$$

*where $c_1^+$ denotes the harmonic self-dual part of $c_1$. Moreover, equality in the right-hand side holds if and only if $c$ contains a Kähler metric, while equality in the left-hand* *side holds if and only if c contains a Yamabe-Kähler metric.*

*Proof:* Let $g \in c$ be an almost Kähler metric, with fundamental 2-form $\omega$ given by $\omega = F + Re(\alpha)$, where $F$ denotes the fundamental 2-form of the standard metric $g_0$ and $\alpha$ is a (2,0) form. The almost complex structure given by $g$ and $\omega$ is homotopic to the complex structure $J$ and hence they induce the same first Chern class, $c_1$. Denoting by $\gamma = R^c(F)$ the $(1,1)$-Ricci form of $(J, g_0)$, we have

$$\begin{aligned} \frac{c_1 \cdot [\omega]}{\sqrt{[\omega]\cdot[\omega]}} &= \frac{1}{2\pi}\frac{\int_M \gamma \wedge \omega}{\sqrt{\int_M \omega \wedge \omega}} = \\ &= \frac{1}{4\sqrt{2}\pi}\frac{\int_M u_{g_0} d\mu_{g_0}}{\sqrt{\int_M d\mu_{g_o} + \frac{1}{2}\int_M |Re(\alpha)|^2 d\mu_{g_0}}} \leq \frac{1}{4}C(M,J,c) \end{aligned} \tag{3.8}$$

with equality if and only if $Re(\alpha)$ vanishes, i.e. if and only if $g_0$ is a Kähler metric. On the other hand, since $b^+(M) = 1$ and $c_1 \cdot [\omega] > 0$ ([45, 32]), we have that $(c_1)^+ = \lambda\omega$, for some positive real constant $\lambda$. Hence

$$\frac{c_1.[\omega]}{\sqrt{[\omega].[\omega]}} = \lambda\sqrt{[\omega].[\omega]} = \sqrt{(c_1^+)^2},$$

which after a substitution in (3.8) completes the proof of the right-hand side inequality of (3.7). The other inequality is a consequence of Proposition 3.7 and the above observation. □

**Corollary 3.4:** *Let $(M, J)$ be as in Proposition 3.8. Then for any Hermitian conformal class c, the fundamental constant $C(M,J,c)$ satisfies*

$$C(M,J,c) \geq 4\pi\sqrt{2c_1^2}$$

*with equality if and only if c contains a Kähler metric and the first Chern class has a self-dual representative with respect to c.*

**Corollary 3.5:** *For any Hermitian conformal class c on* $\mathbf{CP}^2$ *the Yamabe constant* $Y(c)$ *and the fundamental constant* $C(M, J, c)$ *satisfy*

$$Y(c) \leq 12\sqrt{2}\pi \leq C(M, J, g),$$

*with equality in the right-hand side if and only if c contains a Kähler metric and with equality in the left-hand side if and only if c is conformally equivalent to the class of the Fubini-Study metric.*

*Proof:* Since $b^-(\mathbf{CP}^2) = 0$ we have that $4\pi\sqrt{2(c_1^+)^2} = 4\pi\sqrt{2c_1^2} = 12\sqrt{2}\pi$. The case of equality in the left hand side of the inequality follows from the observation that the only Kähler metric of constant scalar curvature on $\mathbf{CP}^2$ is the Fubini-Study metric. □

**Remark 3.3:** The inequality $Y(g) \leq 12\sqrt{2}\pi$ was proved by LeBrun in [35] for an arbitrary conformal class on $\mathbf{CP}^2$, investigating the "size" of the zero set of a self-dual form. As was noted there ([35], Corollary 3), this estimate can be used to give a simple proof the Poon's result of the uniqueness of the self-dual structure of positive type on $\mathbf{CP}^2$. Our Corollary 3.5, the fact that any Hermitian self-dual structure on $\mathbf{CP}^2$ is of positive type (see [4]) and LeBrun's arguments give a simple proof in the framework of Hermitian geometry of the following:

**Corollary 3.6** [4] *Any self-dual Hermitian conformal structure on* $\mathbf{CP}^2$ *is equivalent to the standard one.*

## 3.4 Conformal transformations of almost Kähler metrics on 4-manifolds

D. Blair asked in [13] the following question: *given a compact almost Kähler manifold* $(M^{2n}, g, J, \omega)$ *and* $\phi$ *an isometry of the almost Kähler metric, is* $\phi$ *necessarily a symplectomorphism (or anti-symplectomorphism)?*

This is a particular case of our Problem 1 and we use the results proven so far to give some answers in dimension 4. In fact, in our results $\phi$ will be a conformal transformation of the almost Kähler metric, i.e. the pull-back metric $\phi^\star g$ is conformal to $g$. We first remark that Blair's question has an affirmative answer for compact 4-manifolds with $b_+ = 1$, as an easy consequence of Proposition 3.1. From the same Proposition 3.1, our next partial positive result also follows easily.

**Proposition 3.9:** *Let* $(M^4, g, J, \omega)$ *be a compact almost Kähler manifold and let* $\phi$ *be a conformal transformation of* $g$*, homotopic to the identity inside the group of diffeomorphisms of* $M$*. Then* $\phi$ *is an automorphism of the almost Kähler structure* $(g, J, \omega)$.

*Proof:* By assumptions, $\phi^\star\omega$ is cohomologus to $\omega$ and $\phi^\star g$ is conformal to $g$. Since $\phi^\star g$ is an almost Kähler metric for the symplectic form $\phi^\star\omega$, it follows that $g \in \mathcal{CAM}_\omega \cap \mathcal{CAM}_{\phi^\star\omega}$. By Proposition 1 (a), this may hold only if $\phi^\star\omega = \omega$, so $\phi$ is a symplectomorphism. To conclude that $\phi$ is also an isometry just note that a symplectic form cannot have two distinct, conformal associated metrics.

**Remark 3.4:** Note that the above result is true in any dimensions if we assume $\phi$ to be an isometry in the identity component of the diffeomorphisms group. It can be considered as a slight generalization (in complex dimension 2) of the well-known results of Lichnerovicz [37] about the connected group of isometries of a compact Kähler manifold.

Hence Blair's question has an affirmative answer in this case.

The next result appears as a consequence of Theorem 3.1.

**Theorem 3.2:** *Let $(M^4, g, J, \omega)$ be a compact Kähler , non-hyper-Kähler surface. If $\phi$ is a conformal transformation of the Kähler metric then $\phi$ is a symplectomorphism or an anti-symplectomorphism.*

*Proof:* Let $\phi$ be a positive conformal isometry. Suppose that $\phi$ is not an isometry. Then $\phi^*$ would be an almost Kähler structure in the conformal class of $g$. Now, according to Theorem 3.1,(a2), we have that there is a whole $S^1$ family of almost Kähler structures with respect to the metric $\phi^* g$. Using $\phi^{-1}$, we can induce a $S^1$-family of almost Kähler structures with respect to $g$, which contradicts with Theorem 3.1, (a1). So, $\phi$ must be an isometry. We use now Theorem 3.1,(a1) one more time to complete the proof. □

**Remark 3.5:** The above result is closely related to Theorem 5.3 in [41].

Now we will give examples when Blair's question has a negative answer. However, all such examples that we know so far are very special (all have $c_1 = 0$, for instance). It might be possible that in most instances isometries of almost Kähler metrics do indeed preserve (up to sign) the symplectic form.

**Remark 3.6:** The conclusion of Theorem 3.2 is no longer true for $T^4 = (S^1)^4$. Take the standard metric and consider the Kähler form $\omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4$. Let $\phi$ be the diffeomorphism which acts as identity on the first and third components and switches the second and the fourth. This is an isometry of the metric, but is clearly not an $\pm$-symplectomorphism. Hence Blair's question has a negative answer for $T^4$. Fore some special K3 surfaces such isometries (with respect to a hyper-Kähler metric) have been shown to exist by Alekseevsky-Graev [1]. Non-Kähler examples of this type can be given on $T^4$ (see [5]) an on primary Kodaira surfaces which are

$T^2$-bundles over $T^2$.

**Remark 3.7:** It may really happen that an isometry of an almost Kähler metric is an anti-symplectomorphism, as the following example shows:

Let $M^4 = S^2 \times S^2$ with the standard product metric. This metric is Kahler with respect to the form $\omega = \omega_1 - \omega_2$, the diffeomorphism taking one factor into the other is an isometry, but it is an anti-symplectomorphism of the form $\omega$.

## 3.5 Kähler forms versus symplectic forms

Let $M$ be a compact manifold admitting Kähler structures. Let us denote by $\mathcal{K}$ the set of Kähler forms on $M$ and by $\mathcal{S}$ the set of symplectic forms on $M$. Obviously $\mathcal{K} \subseteq \mathcal{S}$ and we will be interested to detect differences between the two sets. The following lemma gives an invariant which distinguishes cohomology classes that can be represented by Kähler forms. It is due to Perrone in [40], but the proof we present here is shorter.

**Lemma 3.2:** *For any Kähler manifold $(M^{2n}, g, J, \omega)$, the following inequality holds:*

$$(c_1 \cup [\omega]^{n-1})^2 \geq (c_1^2 \cup [\omega]^{n-2})([\omega]^n), \tag{3.9}$$

*with equality if and only if $c_1 = \lambda[\omega]$, for $\lambda \in \mathbf{R}$.*

*Proof:* Consider the bilinear form

$$b([\alpha], [\beta]) = [\alpha] \cup [\beta] \cup [\omega]^{n-2},$$

defined on real (1,1) cohomology classes with values in $\mathbf{R}$. From relation (1.3) we see that this is a symmetric form of signature $(1, k-1)$, where $k = dim_{\mathbf{R}} \mathbf{H}^{1,1}$. Let $\gamma$ be the harmonic representative of $c_1$. It is a (1,1) form and since we are on a Kähler manifold, it decomposes further as

$$\gamma = a\omega + \xi,$$

where $a = \frac{1}{n!}c_1 \cup [\omega]^{n-1}$ is a constant and $\xi$ is a trace free, harmonic, (1,1) form. By relation (3), $b([\xi],[\xi]) \leq 0$, hence $b(c_1 - a[\omega], c_1 - a[\omega]) \leq 0$ and this implies the inequality from the statement. Equality holds if and only if $c_1 = a[\omega]$. □

Using the above Lemma, we will now show that starting with a certain Kähler form after small deformations in certain directions we will leave the space $\mathcal{K}$, but still remain in $\mathcal{S}$.

**Proposition 3.10:** *Let $(M^{2n}, g, J, \omega)$ be a Kähler manifold with $c_1 = \lambda[\omega]$, for $\lambda \in \mathbf{R}$, $\lambda \neq 0$. Assume that $\beta$ is a holomorphic (2,0) form on $M$. Then, for $t$ small, $t \neq 0$, the forms $\omega_t = \omega + tRe(\beta)$ are symplectic but not Kähler forms.*

*Proof:* Let us denote by

$$F(t) := (c_1 \cup [\omega_t]^{n-1})^2 - (c_1^2 \cup [\omega_t]^{n-2})([\omega_t]^n).$$

Since $c_1 = \lambda[\omega]$, we clearly have $F(0) = 0$ and an easy computation shows that $F'(0) = 0$ as well, for any complex dimension $n$. We will show that $F''(0) < 0$, hence $F(t) < 0$ for $t \neq 0$, small. In fact, if $n = 2$

$$F''(t) = -2(c_1^2)([Re(\beta)]^2) = -2\lambda^2([\omega]^2)([Re(\beta)]^2) < 0,$$

for any value of $t$, hence the result follows from the Lemma 3.2. If $n \geq 3$, after an elementary calculation which uses the assumption $c_1 = \lambda[\omega]$ we get

$$F''(0) = -2\lambda^2([Re(\beta)]^2 \cup [\omega]^{n-2})([\omega]^n).$$

By (), since $Re(\beta)$ is a $J$-anti-invariant 2-form, we see that $F''(0) < 0$. Thus, for $t \neq 0$, small, we get $F(t) < 0$, so by Lemma 3.2 the cohomology classes $[\omega_t]$ cannot contain any Kähler forms. On the other hand, $\omega_t$ are symplectic forms for small values of $t$, since the non-degeneracy is an open condition. (As a matter of fact it can be shown that $\omega_t$ are symplectic for any value of $t$.) □

The statement could be considerably strengthen in dimension 4.

**Proposition 3.11:** *Let $(M, g, J, \omega)$ be a Kähler surface with $c_1^2 > 0$. Assume that $\beta$ is a holomorphic (2,0) form on $M$. Then the forms $\omega_t = \omega + tRe(\beta)$ are symplectic for any value of $t \in \mathbf{R}$, but for $|t|$ sufficiently large they cannot be Kähler forms. Moreover, if the the Kähler surface satisfies $c_1 = \lambda[\omega]$, for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, then the forms $\omega_t$ are not Kähler for any value $t \neq 0$.*

*Proof:* Consider again the function

$$F(t) := (c_1 \cup [\omega_t])^2 - (c_1^2)([\omega_t]^2).$$

Note that

$$\omega_t^2 = (\omega + tRe(\beta))^2 = \omega^2 + t^2 Re(\beta)^2 = (1 + \frac{t^2}{2}|Re(\beta)|^2)\omega^2.$$

Thus $\omega_t$ are non-degenerate for any value of $t$, hence symplectic, and $[\omega_t]^2 > [\omega]^2$ for any $t \neq 0$. Since $c_1$ is a (1,1) class and $\beta$ is (2,0), $c_1 \cup [\omega_t] = c_1 \cup [\omega]$. Thus

$$F(t) = (c_1 \cup [\omega])^2 - (c_1^2)([\omega]^2) - \frac{(c_1^2)([Re(\beta)]^2)}{2}t^2,$$

so for $|t|$ big $F(t) < 0$.

If $c_1 = \lambda[\omega]$, we see that

$$F(t) = -\frac{(c_1^2)([Re(\beta)]^2)}{2}t^2,$$

so $F(t) < 0$ for any $t \neq 0$. □

**Proposition 3.12:** *Let $(M, J)$ be a minimal complex surface of general type, with $b_+ > 1$ and no two-torsion classes in $\mathbf{H}^2(X, \mathbf{Z})$. Consider $\omega$ to be a Kähler form on $M$ and $\beta$ a holomorphic (2,0) form on $M$. Let $\alpha_t$ be the line joining the cohomology classes $[\omega]$ and $[Re(\beta)]$, $\alpha_t = (1-t)[\omega] + t[Re(\beta)]$. Then the cohomology class $\alpha_0$ contains a Kähler form; for t sufficiently big, but $t \neq 1$, $\alpha_t$ contains a symplectic form, but does not contain any Kähler form; $\alpha_1$ cannot contain any symplectic form.*

*Proof:* The first two claims follow immediately from the hypothesis and the Proposition 3.11. It only remains to prove that $\alpha_1$ does not contain any symplectic form. Let us assume that $\omega_1$ is a symplectic form with $[\omega_1] = [\alpha_1]$. First let us remark that the canonical bundle $K_1$ induced by $\omega_1$ must be isomorphic to $\pm K$, where $K$ is the canonical bundle induced by the Kähler form $\omega$. This follows from two results of Seiberg-Witten theory. By a theorem of Taubes [44], the *spin$_c$* structure induced by $K_1$ has non-vanishing Seiberg-Witten invariant. But for minimal complex surfaces of general type Theorem 7.4.1 of [39] says that the only *spin$_c$* structures with non-zero invariants are those induced by $\pm K$, where $K$ is the canonical bundle. Hence $K_1 = \pm K$. But then

$$K_1 \cdot [\omega_1] = \pm K \cdot \alpha = K \cdot [Re(\beta)] = 0,$$

since $K$ is a (1,1) class. By another result of Taubes from [45], this can happen if and only if $K_1$ is trivial, hence $K$ is trivial. But for minimal complex surfaces it is known that $c_1^2(M) = K^2 > 0$. □

# CHAPTER 4

# Seiberg-Witten Invariants when Reversing Orientation

A conjecture formulated within the Donaldson's theory, but easily adapted to the Seiberg-Witten context states that each compact, orientable, simply-connected 4-manifold has with one of the orientations all the invariants equal to zero. In this chapter we give an affirmative answer to this conjecture for a large class of complex surfaces. The author has proved the conjecture for complex surfaces of negative signature admitting a Kähler Einstein metric. The same result was obtained independently by N. Leung and recently, D. Kotschick proved a more general theorem. We will state the theorem of Kotschick and indicate how the proof goes in general, but we will treat in detail the case considered by the author.

## 4.1 Statement of the result

Let $X$ be a closed, oriented 4-manifold and let $\bar{X}$ denote the manifold $X$ with the reversed orientation. Denote by $\chi(X)$ the Euler characteristic and by $\sigma(X)$ the signature of $X$. The following conjecture is known about the Seiberg-Witten (Donaldson) invariants and the orientation:

**Conjecture:** *For a compact, orientable, simply-connected 4-manifold $X$, all Seiberg-Witten invariants vanish either on $X$ or on $\bar{X}$.*

Although some work has been done (see [31]), within the frame of Donaldson's theory the conjecture remained wide open. Translated to Seiberg-Witten invariants, an affirmative answer to the conjecture has been recently given for a large class of complex 4-manifolds. The author has proved [19] the statement of the conjecture for complex surfaces with negative signature which admit Kähler Einstein metrics. A similar result has been also obtained independently by N. Leung, [36]. Recently, D. Kotschick, [30] obtained a more general theorem whose statement we give below:

**Theorem 4.1:** (Kotschick, [30]) *Let $X$ be a complex surface of general type and assume that $\bar{X}$ admits a non-zero Seiberg-Witten invariant (of any degree). Then $X$ has ample canonical bundle, $c_1^2(X)$ is even and the signature $\sigma(X)$ is non-negative. Moreover, $X$ has zero signature if and only if it is uniformized by the polydisk.*

This result implies that the above conjecture is true for any complex surface of general type $X$ satisfying one of the following conditions:

(i) $c_1^2(X)$ is odd;

(ii) canonical bundle is not ample;

(iii) $\sigma(X) < 0$.

## 4.2 Proof of Theorem 4.1 and Examples

Let us first remark that the result does not use simply connectedness. However, trying to extend the conjecture for **all** complex surfaces of general type, ignoring the assumption of simple connectivity does not work. Signature zero examples are easy to obtain, as there exist Kähler surfaces with orientation reversing diffeomorphisms. We will show that positive signature examples also exist. First we start by giving the proof of the particular case of the conjecture obtained by the author, but also indicating how Kotschick obtains the more general case (for details see [30]). Our main purpose is to highlight the role played by the signature.

*Sketch of Proof:* The conclusion that $c_1^2(X)$ must be even comes from the fact that the dimension of the moduli space for the $spin_c$ structure with non-zero invariant on $\bar{X}$ is even. Then Kotschick argues that the canonical bundle is ample, by showing that there are no embedded holomorphic spheres of self-intersection -1 or -2 in $X$. Indeed, if $X$ contains an embedded sphere of negative self-intersection, non-trivial in homology, in $\bar{X}$ it becomes sphere of positive self-intersection and this would imply that all invariants of $\bar{X}$ vanish.

When the canonical bundle is ample, by Yau's solution to the Calabi conjecture [49], it follows that $X$ admits a Kähler-Einstein metric $g$ (this is the case treated in [19] and [36]). Rescaling this metric, we may assume that $Vol_g(X) = Vol_g(\bar{X}) = 1$.

Because $g$ is a Kähler-Einstein metric on $X$, we have

$$c_1^2(X) = \frac{s^2}{32\pi^2}. \tag{4.1}$$

Denote by $L$ the determinant line bundle of the $spin_c$ structure on $\bar{X}$ with non-zero Seiberg-Witten invariant. A theorem of LeBrun [34] implies that

$$c_1(L)^2(\bar{X}) \leq \frac{s^2}{32\pi^2}. \tag{4.2}$$

On the other hand, from the dimension formula of the Seiberg-Witten moduli space,

$$c_1(L)^2(\bar{X}) \geq 3\sigma(\bar{X}) + 2\chi(\bar{X}) = -3\sigma(X) + 2\chi(X) = \tag{4.3}$$

$$= -6\sigma(X) + 3\sigma(X) + 2\chi(X) = -6\sigma(X) + c_1^2(X) = -6\sigma(X) + \frac{s^2}{32\pi^2},$$

where in the last equality we used (4.1).

Relations (4.2) and (4.3) imply that $\sigma(X) \geq 0$. For the equality case, the reader is referred to [36]. We will just say that $\sigma(X) = 0$ implies equality in (4.2) and this equality holds if and only if there exists a Kähler-Einstein structure $(g, \bar{J}, \bar{\omega})$ on $\bar{X}$ as well. But then a holonomy argument implies that $X$ is covered by the product of two disks. □

Next we show that the conclusion about the signature in the theorem of Kotschick is sharp. We achieve this by giving examples of bi-symplectic 4-manifolds $X$ (i.e. both $X$ and $\bar{X}$ are symplectic) and invoking the following important result of Taubes:

**Theorem:** (Taubes, [44]) *Let $(X, \omega)$ be a closed, symplectic 4-manifold with $b_+ \geq 2$. Then the Seiberg-Witten invariant of the canonical class is equal to* $\pm 1$.

For zero signature, the simplest examples are products of two Riemann surfaces, $X = \Sigma_1 \times \Sigma_2$. If $\omega_i$ is a volume form on $\Sigma_i$, $i = 1, 2$, then $\omega = \omega_1 + \omega_2$ and $\bar{\omega} = \omega_1 - \omega_2$ are symplectic forms on $X$ inducing opposite orientations. If we take $\Sigma_1, \Sigma_2$ with the genus of each at least 2, then $X = \Sigma_1 \times \Sigma_2$ is a complex surface of general type. If $\omega_1, \omega_2$ are volume forms corresponding to hyperbolic metrics on each surface, then the product metric on $X$ is Kähler-Einstein metric compatible with both $\omega$ and $\bar{\omega}$.

Now let us consider the case of positive signature.

**Theorem 4.2:** *There are examples of complex surfaces of general type having non-zero Seiberg-Witten invariants with both orientations.*

*Proof:* Let us remark that the product examples of bi-symplectic 4-manifolds that we discussed above belong to a larger class of manifolds admitting symplectic structures with both orientations. "Almost" all locally trivial fibre bundles $F \to X^4 \to \Sigma$, where $F$ and $\Sigma$ are closed Riemann surfaces, admit bi-symplectic structure. To see this we just have to repeat Thurston's construction of symplectic forms [46].

The only restriction is $[F] \neq 0$ in $\mathbf{H}_2(X, \mathbf{R})$. If this is satisfied, Thurston shows that there exists $\alpha$, closed 2-form on $X$, which restricts to a symplectic form on each fiber, $F_x, x \in X$. Taking $\sigma$ a symplectic form on the base $\Sigma$, for $\epsilon > 0$ small enough, $\omega = \pi^\star\sigma + \epsilon\alpha$ is a symplectic form on $X$, where $\pi$ is the projection $\pi : X \to \Sigma$. The induced volume form is

$$\omega \wedge \omega = \epsilon\pi^\star\sigma \wedge \alpha + \epsilon^2\alpha \wedge \alpha.$$

But then, for $\epsilon$ possibly smaller, $\bar{\omega} = \pi^\star\sigma - \epsilon\alpha$ is also a symplectic form on $X$ and

$$\bar{\omega} \wedge \bar{\omega} = -\epsilon\pi^\star\sigma \wedge \alpha + \epsilon^2\alpha \wedge \alpha$$

gives the opposite orientation.

Many other examples of bi-symplectic 4-manifolds with signature zero can be obtained in this way. For instance, if we take $F \to X^4 \to \Sigma$ to be a holomorphic fibre bundle, then it is shown easily that the signature of the total space must be zero.

However, the signature of the total space of a fibre bundle is not always zero. Independently, Kodaira [28] and Atiyah [6] constructed a class of examples of non-zero signature. In fact, with one of the orientations, the total space $X$ is a complex surface of general type, and with this orientation the signature is positive. Here is a short description of the examples. Take $R$ to be a Riemann surface which has a fixed point free holomorphic involution denoted by $\tau$ (any surface of odd genus has fixed point free holomorphic involutions). Let $C$ be the cover of $R$ corresponding to the homomorphism

$$\pi_1(R) \longrightarrow \mathbf{H}_1(R;\mathbf{Z}) \longrightarrow \mathbf{H}_1(R;\mathbf{Z}_2),$$

and let $f : C \longrightarrow R$ be the covering map. In $C \times R$ consider the divisor $\Gamma = \Delta \cup \Delta'$, where $\Delta = graph(f)$, $\Delta' = graph(\tau \circ f)$. From the way the covering $f$ was chosen, $\Gamma$ induces an even class in $\mathbf{H}_2(C \times R, \mathbf{Z})$. Denote by $X$ the 2-fold cover of $C \times R$

branched over $\Gamma$. Note that $X$ fibers over both $C$ and $R$, but $X$ is not a holomorphic fibre bundle. As for the signature of $X$, using the general formula for the signature of branched covers, we get

$$\sigma(X) = 2\sigma(C \times R) - \frac{1}{2}\Gamma \cdot \Gamma,$$

where $\Gamma \cdot \Gamma$ is the self-intersection of the branch locus in $C \times R$. Since $\sigma(C \times R) = 0$ and

$$\Gamma \cdot \Gamma = \Delta \cdot \Delta + \Delta' \cdot \Delta' = 2\Delta \cdot \Delta = 2\chi(C) < 0,$$

it follows that $\sigma(X) > 0$. □

It is worth remarking that the existence of symplectic forms inducing both orientations may be used in this case to show that the canonical bundle of $X$ is ample, therefore $X$ admits a Kähler-Einstein metric.

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