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A GAUSS-GALERKIN FINITE ELEMENT METHOD FOR A CLASS OF SINGULAR DIFFUSION EQUATIONS IN TWO SPACE VARIABLES

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ABSTRACT

A GAUSS-GALERKIN FINITE ELEMENT METHOD FOR A CLASS OF SINGULAR DIFFUSION EQUATIONS IN TWO SPACE VARIABLES

By

Li Liu

In this dissertation we are concerned with a Gauss-Galerkin finite element method for a class of singular diffusion equations in two space variables. More specifically, we consider the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = -\frac{\partial (au)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2} - \frac{\partial (cu)}{\partial y} + \frac{1}{2}\frac{\partial^2 (d^2 u)}{\partial y^2} \quad (x, y, t) \in (0,1)^2 \times [0,T].$$

We are concerned with the case when the problem is "singular" in the x variable and "regular" in the y variable, i.e., the coefficient b^2 may vanish along $x = 0$ and $x = 1$, but c and d^2 are bounded away from zero for $y \in [0,1]$.

In the proposed Gauss-Galerkin finite element method, finite element discretization is made in the y variable and the Gauss-Galerkin method is used in the x variable because of the nature of the problem. Convergence of the finite element approximation is established first. Then we analyze the convergence of the resulting Gauss-Galerkin

approximation. Finally, by combining the above results, the convergence of the Gauss-Galerkin finite element approximation is obtained.

A number of test problems are studied. The numerical results show that the proposed Gauss-Galerkin finite element method is efficient in solving singular diffusion equations of the type considered here.

To my parents, my wife and my son

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CHAPTER 1

Introduction

1.1 Background and Motivation

In this dissertation we are concerned with a Gauss-Galerkin finite element method for a class of singular diffusion equations in two space variables. Let $\Omega = (0,1)^2$ be the unit square. We consider the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = -\frac{\partial (au)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2} - \frac{\partial (cu)}{\partial y} + \frac{1}{2}\frac{\partial^2 (d^2 u)}{\partial y^2}. \tag{1.1}$$

with boundary conditions

$$\mathrm{B}u = g \quad \text{on} \quad (0,T) \times \partial\Omega \tag{1.2}$$

and initial condition

$$u(x,y,0) = u_0(x,y) \quad \text{on} \quad \Omega \tag{1.3}$$

(1.1) describes the probability density $u = u(x,y,t)$ for a stochastic process governed by a set of two stochastic differential equations in (x,y) with the respective drifts

$a = a(x,t)$ and $c = c(y,t)$ and diffusions $b = b(x,t)$ and $d = d(y,t)$:

$$\begin{aligned} dX &= adt + bdW_1 \\ dY &= cdt + ddW_2 \\ P\{X(0) &= x, Y(0) = y\} = p(x,y) = \text{given} \end{aligned} \tag{1.4}$$

where $W_1 = W_1(t)$ and $W_2 = W_2(t)$ are independent standard Wiener processes. This is a special case of (2.19) with $n = 2$ and $b_{12} = b_{21} = 0$. We are concerned with the case when the problem is "singular" in the x variable and "regular" in the y variable, i.e., the coefficient b^2 may vanish along $x = 0$ and $x = 1$, but c and d^2 are bounded away from zero for $y \in [0,1]$.

We shall propose and analyze a numerical method called the "Gauss-Galerkin finite element method" to solve the initial-boundary value problems for the above Fokker-Planck equations. The method is a generalization of the one dimensional Gauss-Galerkin method which was originally proposed by Dawson[1] and further developed by Hajjafar, Salehi and Yen [4]. We shall briefly describe the one dimensional Gauss-Galerkin method here. More details can be found in [4]. Consider the stochastic differential equation

$$dX = a(X,t)dt + b(X,t)dW \tag{1.5}$$

for $X = X(t)$ with the initial condition

$$X(0) = X_0 \quad \text{given.} \tag{1.6}$$

The state space is assumed to be a finite interval which we take as $[0,1]$ and $W = W(t)$ above is the standard Wiener process. It is assumed that the coefficients a and b in (1.5) are continuous functions of X and t in $[0,1] \times [0,T]$ where $T > 0$ is a constant. The initial value X_0 in (1.6) is assumed to be a random variable so that the solution

$X(t)$ to (1.5) and (1.6) is a Markov process. It is known that if the law of the process $X(t)$ has a smooth density $u(x,t)$, then $u(x,t)$ satisfies the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = Lu \equiv -\frac{\partial (au)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2}, \tag{1.7}$$

$$u(x,0) = \text{given}, \tag{1.8}$$

plus boundary conditions at $x = 0$ and $x = 1$. The operator $L = L(t)$ is known as the forward Kolmogorov operator. (1.7) and (1.8) lead to a parabolic initial boundary value problem for $u(x,t)$. The formal adjoint $L^* = L^*(t)$ of L above is

$$L^* \equiv a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2}, \tag{1.9}$$

and is known as the backward Kolmogorov operator. For each $v = v(x)$ in some appropriate space V, we may multiply (1.7) by v, integrate with respect to x from 0 to 1 and obtain

$$\frac{d}{dt}(u,v) = (u, L^*v), \quad v \in V, \tag{1.10}$$

where $(\cdot,\cdot)$ stands for the L_2 inner product. We note that in order to lead to (1.10) from (1.7) it is necessary that all the boundary terms resulting from integrations by parts vanish. We are concerned with singular processes for which the coefficients a and b vanish at boundary $x = 0$ and $x = 1$. To solve (1.10), we replace $u(x,t)dx$ by

$$d\mu_n(x,t) = \sum_{k=1}^{n} a_k(t)\delta_{x_k(t)}dx \tag{1.11}$$

associated with an n-point discrete measure $\mu_n(t)$, where in (1.11) $x_k = x_k(t), k = 1, 2, \cdots, n$, are the nodes and the a_k's are the corresponding weights. We also let $\{v_i(x)\}, i = 1, 2, \cdots, 2n$, be a set of linearly independent functions in V. Substituting

$\mu_n(x,t)$ and each of $v_i(x)$ into (1.10) yieds

$$\frac{d}{dt}\sum_{k=1}^{n} a_k(t)v_i(x_k(t)) = \sum_{k=1}^{n} a_k(t)L^* v_i(x_k(t)), i = 1, 2, \cdots, 2n. \tag{1.12}$$

This is a system of 2n ordinary differential equations for the n nodes and the n weights as functions of t. A convergence analysis as n tends to infinity is made in [4]. An algorithm for computational purpose is also given in [4] along with numerial results of several test problems. The results show that the Gauss-Galerkin method gives excellent results and is efficient in capturing the singularities at the boundary for large t.

As there seems to exist no straightforward way to generalize the one dimensional Gauss-Galerkin method to a two dimensional one (there is no direct Gauss quadrature formula in two dimensions), Huang and Yen [5] made a modest generalization by discretizing the partial differential equations by the finite difference method in the y direction and then solving the resulting systems of partial differential equations in x and t by the Gauss-Galerkin method. The results showed that the method in [5] is indeed capable of treating singular parabolic partial differential equations.

In our Gauss-Galerkin finite element method, finite element method is made of the y variable while one dimensional Gauss-Galerkin method is used for the x variable. The convergence of the finite element approximation is established first based on the so called "energy" type estimates. Then, using the theories of measures and moments, the Gauss-Galerkin approximation for the semi-discrete equations is shown to converge when one dimensional Gauss-Galerkin approximation is made in x direction. Finally, Combining the above results, the convergence of Gauss-Galerkin finite element approximation is established.

We use piecewise linear finite elements in our numerical computations. we first test a problem where the exact solution is known. By comparing the numerical solution with the exact solution, we find that the approximation is very efficient and accurate, even when only a few finite elements and only a few Gauss-Galerkin nodes are used. For our model problem I Case I and model problem II, we compare the numerical solutions between our Gauass-Galerkin finite element method and Gauss-Galerkin finite difference method [5]. Those two methods produce almost identical numerical solutions. Whereas in [5] it is shown that the Gauss-Galerkin finite difference method is superior to the traditional two dimensional finite difference method in achieving high accuracy for solving this type of singular Fokker-Planck equations, Our results suggest that the Guass-Galerkin finite element method is at least as efficient and accurate as the Gauss-Galerkin finite difference method. We also study the dependence of the solutions upon parameters in the singularities and the dependency of the solutions upon lower order terms in our Fokker-Planck equations by applying our method to several problems. The numerical solutions obtained show that the proposed method is indeed capable in handling initial-boundary problems for singular diffusion equations of the type considered here.

1.2 Organization of the Dissertation

This dissertation is organized as follows. Chapter 2 contains some basic material related to stochastic processes and Fokker-Planck equations. Chapter 3 discusses the problem formulation and some mathematical properties. We set up the Fokker-Planck equations with a set of boundary conditions. Then we obtain a weak form in the y variable for fixed x and t. We also obtain several energy estimates. Chapter 4 establishes the Gauss-Galerkin finite element approximation and includes two parts. One is the one dimensional finite element approximation in the y variable while the other the one dimensional Gauss-Galerkin approximation in the x variable. In Chapter 5 we study the convergence of the Gauss-Galerkin finite element approximation. First, we establish the convergence of the semi-discrete finite element approximation in y. We then prove the convergence of the Gauss-Galerkin approximation in x. Finally, by combining the above results, the proof of the convergence of the Gauss-Galerkin finite element approximation is completed. In Chapter 6 we present our numerial results for problems involving several partial differential equations and discuss such numerical results. Chapter 7 contains conclusions and further discussions.

CHAPTER 2

Preliminaries

2.1 Markov Process

Let $T \subset [0, \infty)$ and $\mathcal{B}_{\mathbf{R}^n}$ be the Borel $\sigma-$algebra of $\mathbb{R}^n$. Consider a stochastic process $\vec{X}(t) = (X_1(t), X_2(t), \cdots, X_n(t))^T, t \in T$, defined on a probability space $(\Omega, \mathcal{F}, P)$. For all s and t in T such that

$$0 \leq s < t < +\infty, \tag{2.1}$$

for all $\vec{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ and for all B in $\mathcal{B}_{\mathbf{R}^n}$, we denote the conditional probability of the event $\{\vec{X}(t) \in B\}$ given $\vec{X}(s) = \vec{x}$ as $\mathrm{P}(s, \vec{x}; t, B)$:

$$\mathrm{P}(s, \vec{x}; t, B) = \mathrm{P}\{\vec{X}(t) \in B | \vec{X}(s) = \vec{x}\} = \int_{\vec{y} \in B} \mathrm{P}(s, \vec{x}; t, d\vec{y}). \tag{2.2}$$

If the probability $\mathrm{P}(s, \vec{x}; t, d\vec{y})$ on $(\mathbb{R}^n, \mathcal{B}_{\mathbf{R}^n})$ has a density with respect to the Lebesgue measure $d\vec{y}$, we denote this density as $p(s, \vec{x}; t, \vec{y})$ such that

$$\mathrm{P}(s, \vec{x}; t, d\vec{y}) = p(s, \vec{x}; t, \vec{y}) d\vec{y} \tag{2.3}$$

A stochastic process $\{\vec{X}(t), t \in T\}$ is called a *Markov process* if for any finite m and any collection of times

$$0 \le u_1 < u_2 < \cdots < u_m < s < t \tag{2.4}$$

in T, we have for all $\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^m$ and $\vec{x}$ in $\mathbb{R}^n$:

$$\begin{aligned} &\mathrm{P}\{\vec{X}(t) \in B | \vec{X}(u_1) = \vec{x}^1, \vec{X}(u_2) = \vec{x}^2, \cdots, \vec{X}(u_m) = \vec{x}^m, \vec{X}(s) = \vec{x}\} \\ &= \mathrm{P}\{\vec{X}(t) \in B | \vec{X}(s) = \vec{x}\} = \mathrm{P}(s, \vec{x}; t, B) \end{aligned} \tag{2.5}$$

The Markov property (2.5) implies that

$$\mathrm{P}(s, \vec{x}; t, B) = \int_{\mathbf{R}^n} \mathrm{P}(s, \vec{x}; u, d\vec{y}) \mathrm{P}(u, \vec{y}; t, B) \tag{2.6}$$

Equation (2.6) is called the *Chapman – Kolmogorov equation* for Markov processes.

2.2 Transition Probability

For all s and all t such that $0 \le s < t < \infty$ and for all B in $\mathcal{B}_{\mathbf{R}^n}$, $\mathrm{P}(s, \vec{x}; t, B)$ is called a transition probability if

(a) Probability $B \mapsto \mathrm{P}(s, \vec{x}; t, B)$ is defined for all fixed s, t and $\vec{x}$

(b) For all fixed s, t and B, the mapping $\vec{x} \mapsto \mathrm{P}(s, \vec{x}; t, B)$ is measurable from $\mathbb{R}^n$ into $[0, 1]$, and

(c) For all fixed $s, t, \vec{x}$ and B, the conditional probability P satisfies the Chapman-Kolmogorov equation (2.6).

2.3 Diffusion Process

Markov process $\{\vec{X}(t), t \in T\}$ is called a diffusion process if

(a) For all $\epsilon > 0$, $x \in \mathbb{R}^n$ and $0 \le t < t+h$ we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} \mathrm{P}(t, \vec{x}; t+h, d\vec{y}) = 0, \tag{2.7}$$

where $h \downarrow 0$ means $h \to 0$ with positive values, i.e., $h > 0$.

(b) There exists a function $\vec{a}(\vec{x}, t) = (a_1(\vec{x}, t), a_2(\vec{x}, t), \cdots, a_n(\vec{x}, t))^T$ such that for all $\epsilon > 0, \vec{x} \in \mathbb{R}^n, j \in \{1, 2, \cdots, n\}$ and $0 \le t < t+h$ we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} (y_j - x_j)\mathrm{P}(t, \vec{x}; t+h, d\vec{y}) = a_j(\vec{x}, t), \tag{2.8}$$

where $\vec{a}$ is called the drift vector and if $n = 1$ the drift coefficient.

(c) There exists a matrix function $b(\vec{x}, t) = \{b_{jk}(\vec{x}, t)\}_{j,k=1}^n$ such that for all $\epsilon > 0, \vec{x} \in \mathbb{R}^n, j, k \in \{1, 2, \cdots, n\}$ and $0 \le t < t+h$ we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} (y_j - x_j)(y_k - x_k)\mathrm{P}(t, \vec{x}; t+h, d\vec{y}) = b_{jk}(\vec{x}, t), \tag{2.9}$$

where for all $\vec{x}$ and t, the matrix $b(\vec{x}, t)$ is symmetric and positive but not necessarily positive-definite. The matrix $b(\vec{x}, t)$ is called the diffusion matrix and if $n = 1$ the diffusion coefficient.

2.4 Fokker-Planck Equation for a Diffusion Process

Let $\vec{X}(t) = (X_1(t), X_2(t), \cdots, X_n(t))^T, t \in T$ be a *Markov process*. We assume that the process $\vec{X}(t)$ satisfies the following three hypotheses (Schuss [9] and Soize [11]).

Hypothesis 2.4.1 *The transition probability* $\mathrm{P}(s, \vec{x}; t, d\vec{y})$ *on* $(\mathbb{R}^n, \mathcal{B}_{\mathbf{R}^n})$ *has a density* $p(s, \vec{x}; t, \vec{y})$ *with respect to the Lebesgue measure* $d\vec{y}$ *on* $\mathbb{R}^n$ *such that*

$$\mathrm{P}(s, \vec{x}; t, d\vec{y}) = p(s, \vec{x}; t, \vec{y})d\vec{y} \tag{2.10}$$

$$\lim_{h \downarrow 0} \mathrm{P}(s, \vec{x}; t+h, B) = \mathrm{I}_B(\vec{x}), \quad \forall B \in \mathcal{B}_{\mathbf{R}^n}. \tag{2.11}$$

Hypothesis 2.4.2 *The markov process* $\vec{X}$ *is a diffusion process. Consequently, the process* $\vec{X}$ *has almost surely continuous trajectories and equations (2.7), (2.8) and (2.9) hold with (2.10). We assume in addition that equations (2.7), (2.8) and (2.9) are satisfied uniformly with respect to* $\vec{x}$. *For all* $j, k \in \{1, 2, \cdots, n\}$ *and* $0 \le t < t+h$, *we then have*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} p(t, \vec{x}; t+h, \vec{y})d\vec{y} = 0, \tag{2.12}$$

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} (y_j - x_j)p(t, \vec{x}; t+h, \vec{y})d\vec{y} = a_j(\vec{x}, t), \tag{2.13}$$

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\|\vec{y}-\vec{x}\| \ge \epsilon} (y_j - x_j)(y_k - x_k)p(t, \vec{x}; t+h, \vec{y})d\vec{y} = b_{jk}(\vec{x}, t), \tag{2.14}$$

where $\vec{a}(\vec{x}, t)$ *is the drift vector and* $b(\vec{x}, t)$ *the diffusion matrix.*

Hypothesis 2.4.3 *Functions* $\vec{a}(\vec{x}, t)$, $b(\vec{x}, t)$, $\dfrac{\partial}{\partial t} p(s, \vec{x}; t, \vec{y})$, $\dfrac{\partial}{\partial y_j}(a_j(\vec{y}, t)p(s, \vec{x}; t, \vec{y}))$, $\forall j \in \{1, 2, \cdots, n\}$, $\dfrac{\partial^2}{\partial y_j \partial y_k}(b_{jk}(\vec{y}, t)p(s, \vec{x}; t, \vec{y}))$, $\forall j, k \in \{1, 2, \cdots, n\}$, *exist and are continuous.*

Theorem 2.4.1 *(Fokker-Planck equation for the transition probability). Under the hypotheses 2.4.1, 2.4.2 and 2.4.3 and for all fixed* $s \geq 0$ *and* $\vec{x} \in \mathbb{R}^n$, $p(s, \vec{x}; t, \vec{y})$ *of the diffusion process* $\vec{X}(t)$ *satisfies the Fokker-Planck equation (or the forward Kolmogorov equation)*

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial y_j}(a_j(\vec{y}, t)p(s, \vec{x}; t, \vec{y})) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2}{\partial y_j \partial y_k}(b_{jk}(\vec{y}, t)p(s, \vec{x}; t, \vec{y})). \qquad (2.15)$$

Theorem 2.4.2 *(Fokker-Planck equation for the probability density function) Let* $\vec{X}(t)$ *be a diffusion process satisfying hypothesis 2.4.1, 2.4.2 and 2.4.3, and such that*

$$\vec{X}(0) = \vec{X}_0 \quad a.s., \qquad (2.16)$$

where $\vec{X}_0$ *is a random variable with probability law*

$$\mathrm{P}_{\vec{X}_0}(d\vec{y}) = p_{\vec{X}_0}(\vec{y})d\vec{y}, \qquad (2.17)$$

where $p_{\vec{X}_0}$ *is a given continuous probability density function on* $\mathbb{R}^n$ *with respect to the Lebesgue measure* $d\vec{y}$*. Then, for all fixed* $t \in T$*, the probability law* $\mathrm{P}_{\vec{X}(t)}(t, d\vec{y})$ *of* $\vec{X}(t)$ *is written as*

$$\mathrm{P}_{\vec{X}(t)}(t, d\vec{y}) = p_{\vec{X}}(t, \vec{y})d\vec{y}, \qquad (2.18)$$

where the probability density function $p_{\vec{X}}(t, \vec{y})$ *on* $\mathbb{R}^n$ *with respect to* $d\vec{y}$ *satisfies the Fokker-Planck equation*

$$\frac{\partial p_{\vec{X}}}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial y_j}(a_j(\vec{y}, t)p_{\vec{X}}(t, \vec{y})) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2}{\partial y_j \partial y_k}(b_{jk}(\vec{y}, t)p_{\vec{X}}(t, \vec{y})). \qquad (2.19)$$

with the initial condition

$$p_{\vec{X}}(0, \vec{y}) = p_{\vec{X}_0}(y), \quad \forall \vec{y} \in \mathbb{R}^n. \tag{2.20}$$

2.5 Stochastic Differential Equation

We denote by $\mathrm{L}_\omega^p[\alpha, \beta], (1 \leq p < \infty)$, the class of all non-anticipative stochastic processes $f(t)$ satisfying:

$$\mathrm{P}\{\int_\alpha^\beta |f(t)|^p dt < \infty\} = 1.$$

We denote by $\mathrm{M}_\omega^p[\alpha, \beta], (1 \leq p < \infty)$, the subset of $\mathrm{L}_\omega^p[\alpha, \beta]$ consisting of all stochastic processes f(t) with

$$\mathrm{E}\{\int_\alpha^\beta |f(t)|^p dt\} < \infty.$$

The stochastic differential equation

$$d\vec{X}(t) = \vec{a}(\vec{X}(t), t)dt + b(\vec{X}(t), t)d\vec{W}(t), \tag{2.21}$$

is defined by Itô integral equation

$$\vec{X}(t) = \vec{X}(0) + \int_0^t \vec{a}(\vec{X}(s), s)ds + \int_0^t b(\vec{X}(s), s)d\vec{W}(s), \tag{2.22}$$

where $\vec{W}(t) = (W_1(t), W_2(t), \cdots, W_n(t))^T$ is a vector of independent Brownian motion.

2.6 Existence and Uniqueness

Let $|\vec{x}|^2 = \sum_{j=1}^{n} x_j^2$ and $|b|^2 = \sum_{j,k=1}^{n} |b_{jk}|^2$. We consider the stochastic differential equation

$$d\vec{X}(t) = \vec{a}(\vec{X}(t), t)dt + b(\vec{X}(t), t)d\vec{W}(t), \tag{2.23}$$

$$\vec{X}(0) = \vec{X}_0 \quad a.s. \tag{2.24}$$

Theorem 2.6.1 *Suppose $\vec{a}(\vec{x}, t) \in \mathrm{L}^1_\omega[\alpha, \beta]$ and $b(\vec{x}, t) \in \mathrm{L}^2_\omega[\alpha, \beta]$ are measurable in $(\vec{x}, t) \in \mathbb{R}^n \times [0, T]$ and there exist constants K and K_* such that*

$$|\vec{a}(\vec{x}, t)| \le K(1 + |\vec{x}|), \quad |b(\vec{x}, t)| \le K(1 + |\vec{x}|), \tag{2.25}$$

and

$$|\vec{a}(\vec{x}, t) - \vec{a}(\bar{\vec{x}}, t)| \le K_*|\vec{x} - \bar{\vec{x}}|, \quad |\vec{b}(\vec{x}, t) - \vec{b}(\bar{\vec{x}}, t)| \le K_*|\vec{x} - \bar{\vec{x}}| \tag{2.26}$$

for $\vec{x} \in \mathbb{R}^n$ and $0 \le t \le T$. Let $\vec{X}_0$ be any n-dimensional random variable independent of $\sigma -$ algebra $\mathcal{F}(\vec{W}(t), 0 \le t \le T)$ such that $\mathrm{E}|\vec{X}_0|^2 < \infty$. Then there exists a unique solution $\vec{X}(t)$ of (2.23) and (2.24) in $\mathrm{M}^2_\omega[\alpha, \beta]$ (Friedman[3]).

CHAPTER 3

Problem Formulation and Mathematical Properties

3.1 Initial-Boundary Value Problems and Their Weak Formulation

Let $\Omega = (0,1)^2$ be the unit square, $[0,T]$, $T > 0$ be a time interval of interest, $g = g(x,y,t)$ and $u_0 = u_0(x,y) \geq 0$ be given data. We consider the following Fokker-Planck equation

$$\frac{\partial u}{\partial t} = -\frac{\partial (au)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2} - \frac{\partial (cu)}{\partial y} + \frac{1}{2}\frac{\partial^2 (d^2 u)}{\partial y^2} \quad \text{in} \quad (0,T) \times \Omega \tag{3.1}$$

with boundary conditions

$$\mathrm{B}_i u = g_i \quad \text{on} \quad (0,T) \times \partial\Omega \quad (i = 1,2,3,4) \tag{3.2}$$

and initial condition

$$u(x,y,0) = u_0(x,y) \quad \text{on} \quad \Omega \tag{3.3}$$

(3.1) describes the probability density $u = u(x, y, t)$ for a stochastic process governed by a set of two stochastic differential equations in (x, y) with the respective drifts $a = a(x, t)$ and $c = c(y, t)$ and diffusions $b = b(x, t)$ and $d = d(y, t)$:

$$\begin{aligned} dX &= adt + bdW_1 \\ dY &= cdt + ddW_2 \\ P\{X(0) &= x, Y(0) = y\} = p(x, y) = \text{given} \end{aligned} \tag{3.4}$$

where $W_1 = W_1(t)$ and $W_2 = W_2(t)$ are independent standard Wiener processes. We note that in the more general case with drift vector $\vec{a} = (a_1, a_2)^T$ and diffusion matrix $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ that depend on x, y and t, the stochastic differential equations are given by the following:

$$\begin{aligned} dX &= a_1 dt + b_{11} dW_1 + b_{12} dW_2 \\ dY &= a_2 dt + b_{21} dW_1 + b_{22} dW_2 \\ P\{X(0) &= x, Y(0) = y\} = p(x, y) = \text{given.} \end{aligned} \tag{3.5}$$

The probability density function $p(x, y, t)$ is governed by the following (Schuss [9]):

$$\frac{\partial p}{\partial t} = -\frac{\partial(a_1 p)}{\partial x} - \frac{\partial(a_2 p)}{\partial y} + \frac{1}{2}\frac{\partial^2(a_{11} p)}{\partial x^2} + \frac{1}{2}\frac{\partial^2(a_{12} p)}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2(a_{21} p)}{\partial y \partial x} + \frac{1}{2}\frac{\partial^2(a_{22} p)}{\partial y^2} \tag{3.6}$$

where

$$a_{ij} = (bb^T)_{ij} \qquad i, j = 1, 2.$$

In the particular case, $a_1 = a_1(x, t)$, $a_2 = a_2(y, t)$, $b_{11} = b_{11}(x, t)$, $b_{22} = b_{22}(y, t)$ and $b_{12} = b_{21} = 0$, the above equation (3.6) reduces to (3.1) with p being replaced by u, a_1 being replaced by a, a_2 being replaced by c, b_{11} being replaced by b and b_{22} being replaced by d. Henceforth, we shall be primarily concerned with (3.1).

We assume that a, b, c and d are smooth functions with uniformly bounded derivatives of all orders. Many significant results in solving parabolic problems and efficient numerical methods are obtained by Luskin and Rannacher [6], Quarteroni and Valli [8], Strang and Fix [13], Thomee [14], [15], [16], Stoer and Bulirsch [12]. In this dissertation, we shall be concerned with the case when the problem is "singular". More specifically, we are concerned with the case when the problem is "singular" in the x variable and "regular" in the y variable, i.e., the coefficient b^2 may vanish along $x = 0$ and $x = 1$, but c and d^2 are bounded away from zero for $y \in [0, 1]$.

For boundary conditions in the y-direction we consider along $y = 0$ the Dirichlet condition

$$B_3 u = u(x, 0, t) = 0 \quad \text{along} \quad y = 0 \tag{3.7}$$

or the Neumann condition

$$B_3 u = \frac{\partial u(x, 0, t)}{\partial y} = 0 \quad \text{along} \quad y = 0 \tag{3.8}$$

Similarly along $y = 1$ we consider the Dirichlet condition

$$B_4 u = u(x, 1, t) = 0 \quad \text{along} \quad y = 1 \tag{3.9}$$

or the Neumann condition

$$B_4 u = \frac{\partial u(x, 1, t)}{\partial y} = 0 \quad \text{along} \quad y = 1 \tag{3.10}$$

The boundary conditions $B_1 u = 0$ and $B_2 u = 0$ depend on the type of singularity one has at $x = 0$ and $x = 1$. Suppose that

$$b^2(x, t) = O(x^p) \quad as \quad x \to 0 \tag{3.11}$$

$$b^2(x,t) = O((1-x)^q) \quad as \quad x \to 1 \tag{3.12}$$

where p and q are constants satisfying $p \geq 1$ and $q \geq 1$. The appropriate boundary conditions, according to Feller [2] in his study of one-dimensional singular stochastic problems on $(0,T) \times (0,1)$, Keller and Voronka [7], are

$$\begin{cases} B_1 u = u(0,y,t) = 0 & \text{if} \quad p > 1 \\ B_1 u = \lim_{x\to 0} xu = 0 & \text{if} \quad p = 1 \quad \text{and} \quad a(0,y,t) = 0 \\ \quad or & \\ B_1 u = \lim_{x\to 0}\left\{ au - \frac{1}{2}\frac{\partial(b^2 u)}{\partial x}\right\} = 0 & \text{if} \quad p = 1 \quad \text{and} \quad a(0,y,t) \neq 0 \end{cases} \tag{3.13}$$

and

$$\begin{cases} B_2 u = u(1,y,t) = 0 & \text{if} \quad q > 1 \\ B_2 u = \lim_{x\to 1} (1-x)u = 0 & \text{if} \quad q = 1 \quad \text{and} \quad a(1,y,t) = 0 \\ \quad or & \\ B_2 u = \lim_{x\to 1}\left\{ au - \frac{1}{2}\frac{\partial(b^2 u)}{\partial x}\right\} = 0 & \text{if} \quad q = 1 \quad \text{and} \quad a(1,y,t) \neq 0 \end{cases} \tag{3.14}$$

We now consider the following initial-boundary problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f & \text{in} \quad (0,T) \times \Omega \\ u(x,0,t) = 0 & \text{on} \quad (0,T) \times (0,1) \\ u(x,1,t) = 0 & \text{on} \quad (0,T) \times (0,1) \\ \mathrm{B}_1 u(0,y,t) = 0 & \text{on} \quad (0,T) \times (0,1) \\ \mathrm{B}_2 u(1,y,t) = 0 & \text{on} \quad (0,T) \times (0,1) \\ u(x,y,0) = u_0 & \text{on} \quad \Omega \end{cases} \tag{3.15}$$

where

$$Lu = \frac{\partial(au)}{\partial x} - \frac{1}{2}\frac{\partial^2(b^2 u)}{\partial x^2} + \frac{\partial(cu)}{\partial y} - \frac{1}{2}\frac{\partial^2(d^2 u)}{\partial y^2} \tag{3.16}$$

It is seen that for simplicity in the subsequenct analysis we have assumed Dirichlet conditions at $y = 0$ and $y = 1$. Problems with Neumann conditions at $y = 0$ and $y = 1$ require only minor modifications. Also we have included a non homogeneous term $f = f(x, y, t)$ in the partial differential equation. Let $V = H_0^1(0,1)$. Multiplying both sides of the partial differential equation in (3.15) by $v = v(y) \in V$ and integrating over (0,1), we get

$$
\begin{aligned}
&\int_0^1 \frac{\partial u}{\partial t} v dy + \int_0^1 \frac{\partial (au)}{\partial x} v dy - \int_0^1 \frac{1}{2} \frac{\partial^2 (b^2 u)}{\partial x^2} v dy + \int_0^1 \frac{\partial (cu)}{\partial y} v dy \\
&- \int_0^1 \frac{1}{2} \frac{\partial^2 (d^2 u)}{\partial y^2} v dy = \int_0^1 f v dy.
\end{aligned} \tag{3.17}
$$

$$
\int_0^1 \frac{\partial u}{\partial t} v dy + \int_0^1 \frac{\partial (au)}{\partial x} v dy - \int_0^1 \frac{1}{2} \frac{\partial^2 (b^2 u)}{\partial x^2} v dy + \int_0^1 \frac{\partial c}{\partial y} u v dy + \int_0^1 c \frac{\partial u}{\partial y} v dy
$$

$$
+ \int_0^1 \frac{1}{2} \frac{\partial (d^2)}{\partial y} u \frac{\partial v}{\partial y} dy + \int_0^1 \frac{1}{2} d^2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dy - \frac{1}{2} \frac{\partial (d^2 u)}{\partial y} v \Big|_{y=0}^{y=1} = \int_0^1 f v dy.
$$

Since $v|_{y=0} = 0$, $v|_{y=1} = 0$, we get

$$
\begin{aligned}
&\int_0^1 \frac{\partial u}{\partial t} v dy + \int_0^1 \frac{\partial (au)}{\partial x} v dy - \int_0^1 \frac{1}{2} \frac{\partial^2 (b^2 u)}{\partial x^2} v dy + \int_0^1 \frac{\partial c}{\partial y} u v dy \\
&+ \int_0^1 c \frac{\partial u}{\partial y} v dy + \int_0^1 \frac{1}{2} \frac{\partial (d^2)}{\partial y} u \frac{\partial v}{\partial y} dy + \int_0^1 \frac{1}{2} d^2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dy = \int_0^1 f v dy.
\end{aligned} \tag{3.18}
$$

Let the bilinear form $a(\cdot, \cdot)$ be defined by

$$
\begin{aligned}
a(u, v) \; &= \int_0^1 \frac{\partial (au)}{\partial x} v dy - \int_0^1 \frac{1}{2} \frac{\partial^2 (b^2 u)}{\partial x^2} v dy + \int_0^1 \frac{\partial c}{\partial y} u v dy \\
&+ \int_0^1 c \frac{\partial u}{\partial y} v dy + \int_0^1 \frac{1}{2} \frac{\partial (d^2)}{\partial y} u \frac{\partial v}{\partial y} dy + \int_0^1 \frac{1}{2} d^2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dy.
\end{aligned} \tag{3.19}
$$

Or

$$a(u,v) = \left(\frac{\partial(au)}{\partial x}, v\right) - \left(\frac{1}{2}\frac{\partial^2(b^2u)}{\partial x^2}, v\right) + \left(\frac{\partial c}{\partial y}u, v\right)$$
$$+\left(c\frac{\partial u}{\partial y}, v\right) + \left(\frac{1}{2}\frac{\partial(d^2)}{\partial y}u, \frac{\partial v}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right), \tag{3.20}$$

where $(f,g) = \int_0^1 f(x,y,t)g(x,y,t)dy$ is the scalar product of f and g in $L^2(0,1)$ with respect to variable y. We also denote the norm of $f \in L^2(0,1)$ with respect to y by $\|f(x,\cdot,t)\|_0^2 = \int_0^1 |f(x,y,t)|^2 dy$

The weak formulation of (3.15) now reads as follows: Given f and u_0, find u such that

$$\begin{cases} (u_t, v) + a(u,v) = (f,v) \quad \forall v \in V \\ \\ u(x,y,0) = u_0. \end{cases} \tag{3.21}$$

We note here that if homogeneous Neumann conditions are posed,the weak formulation above is to be modified with the space $H_0^1(0,1)$ replaced by $H^1(0,1)$ as such Neumann conditions are "natural" boundary conditions.

3.2 Mathematical Analysis of Initial-Boundary Value Problem

Let the bilinear form $b(\cdot,\cdot)$ be defined by

$$b(u,v) = a(u,v) - a(v,u) \tag{3.22}$$

Taking $v = u$ in (3.20), we have

$$
\begin{aligned}
a(u,u) &= \left(\frac{\partial(au)}{\partial x}, u\right) - \left(\frac{1}{2}\frac{\partial^2(b^2u)}{\partial x^2}, u\right) + \left(\frac{\partial c}{\partial y}u, u\right) + \left(c\frac{\partial u}{\partial y}, u\right) \\
&\quad + \left(\frac{1}{2}\frac{\partial(d^2)}{\partial y}u, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right)
\end{aligned}
\tag{3.23}
$$

Taking $v = u_t$ in (3.22), we obtain

$$
\begin{aligned}
b(u,u_t) &= a(u,u_t) - a(u_t,u) \\
&= \left\{\left(\frac{\partial(au)}{\partial x}, u_t\right) - \left(\frac{1}{2}\frac{\partial^2(b^2u)}{\partial x^2}, u_t\right) + \left(\frac{\partial c}{\partial y}u, u_t\right)\right. \\
&\quad \left. + \left(c\frac{\partial u}{\partial y}, u_t\right) + \left(\frac{1}{2}\frac{\partial(d^2)}{\partial y}u, \frac{\partial u_t}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u}{\partial y}, \frac{\partial u_t}{\partial y}\right)\right\} \\
&\quad - \left\{\left(\frac{\partial(au_t)}{\partial x}, u\right) - \left(\frac{1}{2}\frac{\partial^2(b^2u_t)}{\partial x^2}, u\right) + \left(\frac{\partial c}{\partial y}u_t, u\right)\right. \\
&\quad \left. + \left(c\frac{\partial u_t}{\partial y}, u\right) + \left(\frac{1}{2}\frac{\partial(d^2)}{\partial y}u_t, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u_t}{\partial y}, \frac{\partial u}{\partial y}\right)\right\}.
\end{aligned}
\tag{3.24}
$$

Letting

$$
\begin{aligned}
a_t(u,u) &= \left(\frac{\partial(a_tu)}{\partial x}, u\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)_tu)}{\partial x^2}, u\right) + \left(\frac{\partial c_t}{\partial y}u, u\right) \\
&\quad + \left(c_t\frac{\partial u}{\partial y}, u\right) + \left(\frac{1}{2}\frac{\partial(d^2)_t}{\partial y}u, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}(d^2)_t\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right)
\end{aligned}
\tag{3.25}
$$

we have

$$
\frac{d}{dt}a(u,u) = \left(\frac{\partial(a_tu)}{\partial x}, u\right) + \left(\frac{\partial(au_t)}{\partial x}, u\right) + \left(\frac{\partial(au)}{\partial x}, u_t\right)
$$

$$
\begin{aligned}
&- \left(\frac{1}{2}\frac{\partial^2((b^2)_t u)}{\partial x^2}, u\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)u_t)}{\partial x^2}, u\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)u)}{\partial x^2}, u_t\right) \\
&+ \left(\frac{\partial c_t}{\partial y}u, u\right) + \left(\frac{\partial c}{\partial y}u_t, u\right) + \left(\frac{\partial c}{\partial y}u, u_t\right) \\
&+ \left(c_t\frac{\partial u}{\partial y}, u\right) + \left(c\frac{\partial u_t}{\partial y}, u\right) + \left(c\frac{\partial u}{\partial y}, u_t\right) \\
&+ \left(\frac{1}{2}\frac{\partial (d^2)_t}{\partial y}u, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u_t, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u, \frac{\partial u_t}{\partial y}\right) \\
&+ \left(\frac{1}{2}(d^2)_t\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u_t}{\partial y}, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u}{\partial y}, \frac{\partial u_t}{\partial y}\right) \\
&= \left\{\left(\frac{\partial(a_t u)}{\partial x}, u\right) - \frac{1}{2}\left(\frac{\partial^2((b^2)_t u)}{\partial x^2}, u\right) + \left(\frac{\partial c_t}{\partial y}u, u\right) + \left(c_t\frac{\partial u}{\partial y}, u\right)\right. \\
&+ \left.\left(\frac{1}{2}\frac{\partial (d^2)_t}{\partial y}u, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}(d^2)_t\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right)\right\} \\
&+ \left\{\left(\frac{\partial(a u_t)}{\partial x}, u\right) - \left(\frac{1}{2}\frac{\partial^2((b^2) u_t)}{\partial x^2}, u\right) + \left(\frac{\partial c}{\partial y}u_t, u\right) + \left(c\frac{\partial u_t}{\partial y}, u\right)\right. \\
&+ \left.\left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u_t, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u_t}{\partial y}, \frac{\partial u}{\partial y}\right)\right\} \\
&+ \left\{\left(\frac{\partial(a u)}{\partial x}, u_t\right) - \left(\frac{1}{2}\frac{\partial^2((b^2) u)}{\partial x^2}, u_t\right) + \left(\frac{\partial c}{\partial y}u, u_t\right) + \left(c\frac{\partial u}{\partial y}, u_t\right)\right. \\
&+ \left.\left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u, \frac{\partial u_t}{\partial y}\right) + \left(\frac{1}{2}d^2\frac{\partial u}{\partial y}, \frac{\partial u_t}{\partial y}\right)\right\} \\
&= \left\{\left(\frac{\partial(a_t u)}{\partial x}, u\right) - \frac{1}{2}\left(\frac{\partial^2((b^2)_t u)}{\partial x^2}, u\right) + \left(\frac{\partial c_t}{\partial y}u, u\right) + \left(c_t\frac{\partial u}{\partial y}, u\right)\right.
\end{aligned}
$$

$$+ \quad \left(\frac{1}{2}\frac{\partial (d^2)_t}{\partial y}u, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}(d^2)_t \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right)\Bigg\}$$

$$+ \quad 2\Bigg\{\left(\frac{\partial (au)}{\partial x}, u_t\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)u)}{\partial x^2}, u_t\right) + \left(\frac{\partial c}{\partial y}u, u_t\right) + \left(c\frac{\partial u}{\partial y}, u_t\right)$$

$$+ \quad \left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u, \frac{\partial u_t}{\partial y}\right) + \left(\frac{1}{2}d^2 \frac{\partial u}{\partial y}, \frac{\partial u_t}{\partial y}\right)\Bigg\}$$

$$+ \quad \Bigg\{\left(\frac{\partial (au_t)}{\partial x}, u\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)u_t)}{\partial x^2}, u\right) + \left(\frac{\partial c}{\partial y}u_t, u\right) + \left(c\frac{\partial u_t}{\partial y}, u\right)$$

$$+ \quad \left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u_t, \frac{\partial u}{\partial y}\right) + \left(\frac{1}{2}d^2 \frac{\partial u_t}{\partial y}, \frac{\partial u}{\partial y}\right)\Bigg\}$$

$$- \quad \Bigg\{\left(\frac{\partial (au)}{\partial x}, u_t\right) - \left(\frac{1}{2}\frac{\partial^2((b^2)u)}{\partial x^2}, u_t\right) + \left(\frac{\partial c}{\partial y}u, u_t\right) + \left(c\frac{\partial u}{\partial y}, u_t\right)$$

$$+ \quad \left(\frac{1}{2}\frac{\partial (d^2)}{\partial y}u, \frac{\partial u_t}{\partial y}\right) + \left(\frac{1}{2}d^2 \frac{\partial u}{\partial y}, \frac{\partial u_t}{\partial y}\right)\Bigg\}.$$

So

$$\frac{d}{dt}a(u,u) = a_t(u,u) + 2a(u,u_t) - b(u,u_t). \tag{3.26}$$

Thus we have

$$a(u,u_t) = \frac{1}{2}\frac{d}{dt}a(u,u) - \frac{1}{2}a_t(u,u) + \frac{1}{2}b(u,u_t) \tag{3.27}$$

We now ssume that $a(\cdot,\cdot)$ is continuous and coercive, i.e., there is a positive constant α, independent of t, such that

$$a(v,v) \geq \alpha\|v\|_1^2, \quad \forall v \in V. \tag{3.28}$$

Theorem 3.2.1 *If $u_0 \in L^2(0,1), f \in L^2(0,1)$, then for almost every $x \in (0,1)$, the* *energy estimate*

$$\max_{0\le t\le T} ||u(x,\cdot,t)||_0^2 + \alpha \int_0^T ||u(x,\cdot,t)||_1^2 dt \le ||u_0(x,\cdot)||_0^2 + \frac{1}{\alpha}\int_0^T ||f(x,\cdot,t)||_0^2 dt \tag{3.29}$$

holds.

Proof. By (3.21), we have

$$(u_t, v) + a(u,v) = (f,v)$$

Taking $v = u$ and using (3.28), we get

$$\begin{aligned}
&(u_t, u) + a(u,u) = (f,u)\\
&\frac{1}{2}\frac{d}{dt}(u,u) + a(u,u) = (f,u)\\
&\frac{1}{2}\frac{d}{dt}||u(x,\cdot,t)||_0^2 + a(u,u) = (f,u)\\
&\frac{1}{2}\frac{d}{dt}||u(x,\cdot,t)||_0^2 + \alpha||u(x,\cdot,t)||_1^2 \le \frac{1}{2}\frac{d}{dt}||u(x,\cdot,t)||_0^2 + a(u,u)\\
&\le |(f,u)| \le \frac{1}{2\alpha}||f(x,\cdot,t)||_0^2 + \frac{\alpha}{2}||u(x,\cdot,t)||_0^2\\
&\le \frac{1}{2\alpha}||f(x,\cdot,t)||_0^2 + \frac{\alpha}{2}||u(x,\cdot,t)||_1^2
\end{aligned}$$

Integrating over $(0,\tau), \tau \in (0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau \frac{d}{dt}||u(x,\cdot,t)||_0^2 dt + \alpha\int_0^\tau ||u(x,\cdot,t)||_1^2 dt\\
&\quad\le \frac{1}{2\alpha}\int_0^\tau ||f(x,\cdot,t)||_0^2 dt + \frac{\alpha}{2}\int_0^\tau ||u(x,\cdot,t)||_1^2 dt\\
&\frac{1}{2}||u(x,\cdot,\tau)||_0^2 - \frac{1}{2}||u(x,\cdot,0)||_0^2 + \frac{\alpha}{2}\int_0^\tau ||u(x,\cdot,t)||_1^2 dt \le \frac{1}{2\alpha}\int_0^\tau ||f(x,\cdot,t)||_0^2 dt\\
&||u(x,\cdot,\tau)||_0^2 + \alpha\int_0^\tau ||u(x,\cdot,t)||_1^2 dt \le ||u_0(x,\cdot)||_0^2 + \frac{1}{\alpha}\int_0^\tau ||f(x,\cdot,t)||_0^2 dt
\end{aligned}$$

So,

$$\max_{0\le t\le T} ||u(x,\cdot,t)||_0^2 + \alpha \int_0^T ||u(x,\cdot,t)||_1^2 dt \le ||u_0(x,\cdot)||_0^2 + \frac{1}{\alpha}\int_0^T ||f(x,\cdot,t)||_0^2 dt \qquad \square$$

From now on, we shall assume that

$$\begin{aligned} |a(u,v)| &\le c\|u(x,\cdot,t)\|_1\|v(x,\cdot,t)\|_1 \\ |a_t(u,v)| &\le c\|u(x,\cdot,t)\|_1\|v(x,\cdot,t)\|_1 \\ |b(u,v)| &\le c\|u(x,\cdot,t)\|_1\|v(x,\cdot,t)\|_0 \end{aligned} \tag{3.30}$$

Theorem 3.2.2 *If $u_0 \in H_0^1(0,1), f \in L^2(0,1)$, then for almost every $x \in (0,1)$ the energy estimate*

$$\begin{aligned} &\sup_{t\in(0,T)} \|u(x,\cdot,t)\|_1^2 + \frac{1}{\alpha}\int_0^T \|\frac{\partial u(x,\cdot,t)}{\partial t}\|_0^2 dt \\ &\le C_\alpha\left\{\|u_0(x,\cdot)\|_1^2 + \int_0^T \|f(x,\cdot,t)\|_0^2 dt\right\} \end{aligned} \tag{3.31}$$

holds, where $C_\alpha > 0$ is a constant independent of T.

Proof. By (3.21),

$$(u_t, v) + a(u,v) = (f,v).$$

Taking $v = u_t$, we obtain

$$\begin{aligned} (u_t,u_t) + a(u,u_t) &= (f,u_t) \\ \|u_t\|_0^2 + a(u,u_t) &= (f,u_t). \end{aligned}$$

By (3.27), we obtain

$$\|u_t\|_0^2 + \frac{1}{2}\frac{d}{dt}a(u,u) - \frac{1}{2}a_t(u,u) + \frac{1}{2}b(u,u_t) = (f,u_t).$$

So,

$$\begin{aligned} ||u_t||_0^2 + \frac{1}{2}\frac{d}{dt}a(u,u) &= \frac{1}{2}a_t(u,u) - \frac{1}{2}b(u,u_t) + (f,u_t) \\ &\le \frac{c}{2}||u||_1^2 + \frac{c}{2}||u||_1||u_t||_0 + |(f,u_t)|. \end{aligned}$$

Integrating over $(0,t), t \in (0,T]$, we obtain

$$\begin{aligned} &\int_0^t ||u_s||_0^2 ds + \frac{1}{2}\int_0^t \frac{d}{ds}a(u,u)ds \\ &\le \frac{c}{2}\int_0^t ||u||_1^2 ds + \frac{c}{2}\int_0^t ||u||_1||u_s||_0 ds + \int_0^t |(f,u_s)|ds \\ &\le c\int_0^t ||u||_1^2 ds + c^2\int_0^t ||u||_1^2 ds + \frac{1}{4}\int_0^t ||u_s||_0^2 ds + \frac{1}{4}\int_0^t ||u_s||_0^2 ds + \int_0^t ||f||_0^2 ds. \end{aligned}$$

Hence,

$$\frac{1}{2}\int_0^t ||u_s||_0^2 ds + \frac{1}{2}\int_0^t \frac{d}{ds}a(u,u)ds \quad \le \quad c(1+c)\int_0^t ||u||_1^2 ds + \int_0^t ||f||_0^2 ds.$$

By (3.29), we have

$$\int_0^t ||u||_1^2 ds \le \frac{1}{\alpha}||u_0||_0^2 + \frac{1}{\alpha^2}\int_0^T ||f||_0^2 ds.$$

So,

$$\begin{aligned} &\frac{1}{2}\int_0^t ||u_s||_0^2 ds + \frac{1}{2}a(u(t),u(t)) - \frac{1}{2}a(u(0),u(0)) \\ &\le \frac{c(1+c)}{\alpha}||u_0||_0^2 + \left(1 + \frac{c(1+c)}{\alpha^2}\right)\int_0^T ||f||_0^2 ds \\ &\frac{1}{2}\int_0^t ||u_s||_0^2 ds + \frac{1}{2}a(u(t),u(t)) \\ &\le \frac{1}{2}a(u(0),u(0)) + \frac{c(1+c)}{\alpha}||u_0||_0^2 + \left(1 + \frac{c(1+c)}{\alpha^2}\right)\int_0^T ||f||_0^2 ds. \end{aligned}$$

By assumption of (3.28) and (3.30), we have

$$a(u(t),u(t)) \geq \alpha||u||_1^2, \quad a(u(0),u(0)) \leq c||u_0||_1^2 \quad \text{and} \quad ||u_0||_0^2 \leq ||u_0||_1^2.$$

Therefore

$$\frac{1}{2}\int_0^t ||u_s||_0^2 ds + \frac{\alpha}{2}||u||_1^2 \leq \frac{c}{2}||u_0||_1^2 + \frac{c(1+c)}{\alpha}||u_0||_1^2 + \left(1+\frac{c(1+c)}{\alpha^2}\right)\int_0^T ||f||_0^2 ds$$

$$||u||_1^2 + \frac{1}{\alpha}\int_0^t ||u_s||_0^2 ds \leq \frac{2c}{\alpha}\left(\frac{1}{2}+\frac{1+c}{\alpha}\right)||u_0||_1^2 + \frac{2}{\alpha}\left(1+\frac{c(1+c)}{\alpha^2}\right)\int_0^T ||f||_0^2 ds$$

$$\sup_{t\in(0,T)} ||u||_1^2 + \frac{1}{\alpha}\int_0^T ||\frac{\partial u}{\partial t}||_0^2 dt \leq C_\alpha\left\{||u_0||_1^2 + \int_0^T ||f||_0^2 ds\right\} \quad \square$$

A simple consequence of Theorem (3.2.2) is given by

Corollary 3.2.1 *Assume that for almost every $x \in (0,1)$ and $t \in [0,T]$, u in Theorem 3.2.2 satisfies*

$$||u(x,\cdot,t)||_2^2 \leq C(||Lu(x,\cdot,t)||_0^2 + ||u(x,\cdot,t)||_1^2). \tag{3.32}$$

Then for almost every $x \in (0,1)$ u satisfies the estimate

$$\begin{aligned}\max_{t\in[0,T]} ||u(x,\cdot,t)||_1^2 + \frac{1}{\alpha}\int_0^T \left\{\left\|\frac{\partial u(x,\cdot,t)}{\partial t}\right\|_0^2 + ||u(x,\cdot,t)||_2^2\right\} dt \\ \leq C_\alpha\left\{||u_0(x,\cdot)||_1^2 + \int_0^T ||f(x,\cdot,t)||_0^2 dt\right\}\end{aligned} \tag{3.33}$$

Proof. Estimate (3.33) follows at once from (3.32), (3.31) and (3.29), since $Lu = f - \dfrac{\partial u}{\partial t}$.

CHAPTER 4

Gauss-Galerkin Finite Element Method

4.1 Finite Element Approximation in the y Variable

Let $V_h \subset H_0^1(0,1)$ be a finite dimensional subspace and $\{\phi_l(y)\}_{l=0}^{N_y}$ be base functions of V_h. By (3.18), we have

$$\begin{aligned}&\int_0^1 \frac{\partial u}{\partial t} v dy + \int_0^1 \frac{\partial (au)}{\partial x} v dy - \int_0^1 \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2} v dy + \int_0^1 \frac{\partial c}{\partial y} uv dy \\ &+ \int_0^1 c\frac{\partial u}{\partial y} v dy + \int_0^1 \frac{1}{2}\frac{\partial (d^2)}{\partial y} u \frac{\partial v}{\partial y} dy + \int_0^1 \frac{1}{2} d^2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} dy = \int_0^1 f v dy.\end{aligned} \tag{4.1}$$

Taking $v = \phi_l(y), l = 0, 1, \cdots, N_y$, we obtain

$$\begin{aligned}&\int_0^1 \frac{\partial u}{\partial t} \phi_l(y) dy + \int_0^1 \frac{\partial (au)}{\partial x} \phi_l(y) dy - \int_0^1 \frac{1}{2}\frac{\partial^2 (b^2 u)}{\partial x^2} \phi_l(y) dy \\ &+ \int_0^1 \frac{\partial c}{\partial y} u\phi_l(y) dy + \int_0^1 c\frac{\partial u}{\partial y} \phi_l(y) dy + \int_0^1 \frac{1}{2}\frac{\partial (d^2)}{\partial y} u\phi_l'(y) dy \\ &+ \int_0^1 \frac{1}{2} d^2 \frac{\partial u}{\partial y}\phi_l'(y) dy = \int_0^1 f\phi_l(y) dy.\end{aligned} \tag{4.2}$$

We approximate $u(x, y, t)$ by

$$u(x,y,t) \approx \sum_{j=0}^{N_y} \alpha_j(x,t)\phi_j(y). \tag{4.3}$$

From (4.2), we obtain

$$\begin{aligned}
&\sum_{j=0}^{N_y} \frac{\partial \alpha_j(x,t)}{\partial t} \int_0^1 \phi_l(y)\phi_j(y)dy = -\sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x} \phi_l(y)\phi_j(y)dy \\
&+\frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2} \phi_l(y)\phi_j(y)dy - \sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y}\phi_l(y)\phi_j(y)dy \\
&-\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 c\phi_l(y)\phi_j'(y)dy - \frac{1}{2}\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial(d^2)}{\partial y}\phi_l'(y)\phi_j(y)dy \\
&-\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 d^2\phi_l'(y)\phi_j'(y)dy + \int_0^1 f\phi_l(y)dy,
\end{aligned} \tag{4.4}$$

or,

$$\begin{aligned}
&\begin{pmatrix} \phi(l,0) & \phi(l,1) & \cdots & \phi(l,N_y) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \alpha_0(x,t)}{\partial t} \\ \dfrac{\partial \alpha_1(x,t)}{\partial t} \\ \vdots \\ \dfrac{\partial \alpha_{N_y}(x,t)}{\partial t} \end{pmatrix} \\
&= -\sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x} \phi_l(y)\phi_j(y)dy \\
&+\frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2} \phi_l(y)\phi_j(y)dy - \sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y}\phi_l(y)\phi_j(y)dy \\
&-\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 c\phi_l(y)\phi_j'(y)dy - \frac{1}{2}\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial(d^2)}{\partial y}\phi_l'(y)\phi_j(y)dy \\
&-\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 d^2\phi_l'(y)\phi_j'(y)dy + \int_0^1 f\phi_l(y)dy,
\end{aligned} \tag{4.5}$$

where $\phi(l,j) = \int_0^1 \phi_l(y)\phi_j(y)dy$. Thus

$$\begin{pmatrix} \phi(0,0) & \phi(0,1) & \cdots & \phi(0,N_y) \\ \phi(1,0) & \phi(1,1) & \cdots & \phi(1,N_y) \\ \vdots & \vdots & \cdots & \vdots \\ \phi(N_y,0) & \phi(N_y,1) & \cdots & \phi(N_y,N_y) \end{pmatrix} \begin{pmatrix} \frac{\partial \alpha_0(x,t)}{\partial t} \\ \frac{\partial \alpha_1(x,t)}{\partial t} \\ \cdots \\ \frac{\partial \alpha_{N_y}(x,t)}{\partial t} \end{pmatrix}$$

$$= - \begin{pmatrix} \sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x}\phi_0(y)\phi_j(y)dy \\ \sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x}\phi_1(y)\phi_j(y)dy \\ \vdots \\ \sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x}\phi_{N_y}(y)\phi_j(y)dy \end{pmatrix} \tag{4.6}$$

$$+ \begin{pmatrix} \frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2}\phi_0(y)\phi_j(y)dy \\ \frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2}\phi_1(y)\phi_j(y)dy \\ \vdots \\ \frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2}\phi_{N_y}(y)\phi_j(y)dy \end{pmatrix}$$

$$
- \begin{pmatrix}
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_0(y)\phi_j(y)dy \\
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_1(y)\phi_j(y)dy \\
\vdots \\
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_{N_y}(y)\phi_j(y)dy
\end{pmatrix}
- \begin{pmatrix}
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 c\phi_0(y)\phi_j'(y)dy \\
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 c\phi_1(y)\phi_j'(y)dy \\
\vdots \\
\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 c\phi_{N_y}(y)\phi_j'(y)dy
\end{pmatrix}
$$

$$
- \begin{pmatrix}
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 \frac{\partial(d^2)}{\partial y} \phi_0'(y)\phi_j(y)dy \\
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 \frac{\partial(d^2)}{\partial y} \phi_1'(y)\phi_j(y)dy \\
\vdots \\
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 \frac{\partial(d^2)}{\partial y} \phi_{N_y}'(y)\phi_j(y)dy
\end{pmatrix}
- \begin{pmatrix}
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 d^2\phi_0'(y)\phi_j'(y)dy \\
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 d^2\phi_1'(y)\phi_j'(y)dy \\
\vdots \\
\frac{1}{2}\sum_{j=0}^{N_y} \alpha_j \int_0^1 d^2\phi_{N_y}'(y)\phi_j'(y)dy
\end{pmatrix}
$$

$$
+ \begin{pmatrix}
\int_0^1 f\phi_0(y)dy \\
\int_0^1 f\phi_0(y)dy \\
\vdots \\
\int_0^1 f\phi_0(y)dy
\end{pmatrix}
$$

And

$$\begin{pmatrix} \dfrac{\partial \alpha_0(x,t)}{\partial t} \\ \\ \dfrac{\partial \alpha_1(x,t)}{\partial t} \\ \vdots \\ \dfrac{\partial \alpha_{N_y}(x,t)}{\partial t} \end{pmatrix} = -\Phi^{-1} \begin{pmatrix} \displaystyle\sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x} \phi_0(y)\phi_j(y)dy \\ \\ \displaystyle\sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x} \phi_1(y)\phi_j(y)dy \\ \vdots \\ \displaystyle\sum_{j=0}^{N_y} \int_0^1 \frac{\partial(a\alpha_j(x,t))}{\partial x} \phi_{N_y}(y)\phi_j(y)dy \end{pmatrix}$$

$$+\Phi^{-1} \begin{pmatrix} \displaystyle\frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2} \phi_0(y)\phi_j(y)dy \\ \\ \displaystyle\frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2} \phi_1(y)\phi_j(y)dy \\ \vdots \\ \displaystyle\frac{1}{2}\sum_{j=0}^{N_y} \int_0^1 \frac{\partial^2(b^2\alpha_j(x,t))}{\partial x^2} \phi_{N_y}(y)\phi_j(y)dy \end{pmatrix} \tag{4.7}$$

$$-\Phi^{-1} \begin{pmatrix} \displaystyle\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_0(y)\phi_j(y)dy \\ \\ \displaystyle\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_1(y)\phi_j(y)dy \\ \vdots \\ \displaystyle\sum_{j=0}^{N_y} \alpha_j(x,t) \int_0^1 \frac{\partial c}{\partial y} \phi_{N_y}(y)\phi_j(y)dy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\alpha_j(x,t)\int_0^1 c\phi_0(y)\phi_j'(y)dy \\ \\ \sum_{j=0}^{N_y}\alpha_j(x,t)\int_0^1 c\phi_1(y)\phi_j'(y)dy \\ \vdots \\ \sum_{j=0}^{N_y}\alpha_j(x,t)\int_0^1 c\phi_{N_y}(y)\phi_j'(y)dy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 \frac{\partial(d^2)}{\partial y}\phi_0'(y)\phi_j(y)dy \\ \\ \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 \frac{\partial(d^2)}{\partial y}\phi_1'(y)\phi_j(y)dy \\ \vdots \\ \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 \frac{\partial(d^2)}{\partial y}\phi_{N_y}'(y)\phi_j(y)dy \end{pmatrix} - \Phi^{-1}\begin{pmatrix} \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 d^2\phi_0'(y)\phi_j'(y)dy \\ \\ \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 d^2\phi_1'(y)\phi_j'(y)dy \\ \vdots \\ \frac{1}{2}\sum_{j=0}^{N_y}\alpha_j\int_0^1 d^2\phi_{N_y}'(y)\phi_j'(y)dy \end{pmatrix}$$

$$+\Phi^{-1}\begin{pmatrix} \int_0^1 f\phi_0(y)dy \\ \\ \int_0^1 f\phi_0(y)dy \\ \vdots \\ \int_0^1 f\phi_0(y)dy \end{pmatrix}.$$

Define

$$\vec{\alpha}(x,t) = (\alpha_0(x,t), \alpha_1(x,t), \cdots, \alpha_{N_y}(x,t))^T \tag{4.8}$$

and let L_1 be defined by (4.7), we can write (4.7) as

$$\frac{\partial \vec{\alpha}(x,t)}{\partial t} = L_1 \vec{\alpha}(x,t). \tag{4.9}$$

Using (4.3), we obtain

$$\begin{aligned}
&\int_0^1 u(x,y,0)\phi_l(y)dy \\
&= \int_0^1 \sum_{j=0}^{N_y} \alpha_j(x,0)\phi_j(y)\phi_l(y)dy \\
&= \sum_{j=0}^{N_y} \alpha_j(x,0) \int_0^1 \phi_l(y)\phi_j(y)dy \\
&= \sum_{j=0}^{N_y} \alpha_j(x,0)\phi(l,j).
\end{aligned} \tag{4.10}$$

That is,

$$\int_0^1 u(x,y,0)\phi_l(y)dy = \begin{pmatrix} \phi(l,0) & \phi(l,1) & \cdots & \phi(l,N_y) \end{pmatrix} \begin{pmatrix} \alpha_0(x,0) \\ \alpha_1(x,0) \\ \vdots \\ \alpha_{N_y}(x,0) \end{pmatrix}$$

$$\begin{pmatrix} \int_0^1 u(x,y,0)\phi_0(y)dy \\ \int_0^1 u(x,y,0)\phi_1(y)dy \\ \vdots \\ \int_0^1 u(x,y,0)\phi_{N_y}(y)dy \end{pmatrix} = \begin{pmatrix} \phi(0,0) & \phi(0,1) & \cdots & \phi(0,N_y) \\ \phi(1,0) & \phi(1,1) & \cdots & \phi(1,N_y) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(N_y,0) & \phi(N_y,1) & \cdots & \phi(N_y,N_y) \end{pmatrix} \begin{pmatrix} \alpha_0(x,0) \\ \alpha_1(x,0) \\ \vdots \\ \alpha_{N_y}(x,0) \end{pmatrix} \tag{4.11}$$

$$\begin{pmatrix} \alpha_0(x,0) \\ \alpha_1(x,0) \\ \vdots \\ \alpha_{N_y}(x,0) \end{pmatrix} = \Phi^{-1} \begin{pmatrix} \int_0^1 u(x,y,0)\phi_0(y)dy \\ \int_0^1 u(x,y,0)\phi_1(y)dy \\ \vdots \\ \int_0^1 u(x,y,0)\phi_{N_y}(y)dy \end{pmatrix}. \tag{4.12}$$

4.2 Gauss-Galerkin Approximation in the x Variable

We approximate $\alpha_j(x,t)dx$ by

$$d\mu_j^n(x,t) = \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij})(t)dx, \quad j = 0, 1, \cdots, N_y, \tag{4.13}$$

i.e., each $\alpha_j(x,t)$ is associated with an n-point discrete Dirac-delta function, or "measure", $x_{ij}(t), 1 \le i \le n$, are the n "nodes" and $\omega_{ij}(t), 1 \le i \le n$, are the corresponding "weights". We choose $x^k, 0 \le k \le 2n-1$ as test functions.

Multiplying by $x^k, k = 0, 1, \cdots, 2n-1$, and integrating over $[0,1]$ in (4.2) we obtain

$$\int_0^1\int_0^1 \frac{\partial u}{\partial t}\phi_l(y)x^k dxdy + \int_0^1\int_0^1 \frac{\partial(au)}{\partial x}\phi_l(y)x^k dxdy - \int_0^1\int_0^1 \frac{1}{2}\frac{\partial^2(b^2u)}{\partial x^2}\phi_l(y)x^k dxdy$$
$$+ \int_0^1\int_0^1 \frac{\partial c}{\partial y}u\phi_l(y)x^k dxdy + \int_0^1\int_0^1 c\frac{\partial u}{\partial y}\phi_l(y)x^k dxdy + \int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}u\phi_l'(y)x^k dxdy$$
$$+ \int_0^1\int_0^1 \frac{1}{2}d^2\frac{\partial u}{\partial y}\phi_l'(y)x^k dxdy = \int_0^1\int_0^1 f\phi_l(y)x^k dxdy \tag{4.14}$$

Using integration by parts, we obtain

$$\int_0^1\int_0^1 \frac{\partial u}{\partial t}\phi_l(y) + \int_0^1\int_0^1 \frac{\partial c}{\partial y}u\phi_l(y)dxdy + \int_0^1\int_0^1 c\frac{\partial u}{\partial y}\phi_l(y)dxdy$$

$$+\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}u\phi_l'(y)dxdy + \int_0^1\int_0^1 \frac{1}{2}d^2\frac{\partial u}{\partial y}\phi_l'(y)dxdy \tag{4.15}$$

$$+\int_0^1 \left\{au - \frac{1}{2}\frac{\partial(b^2u)}{\partial x}\right\}\Bigg|_{x=0}^{1}\phi_l(y)dy = \int_0^1\int_0^1 f\phi_l(y)dxdy \quad (k=0).$$

and

$$\int_0^1\int_0^1 \frac{\partial u}{\partial t}\phi_l x^k dxdy - \int_0^1\int_0^1 kaux^{k-1}\phi_l dxdy - \int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2ux^{k-2}\phi_l dxdy$$

$$+\int_0^1\int_0^1 \frac{\partial c}{\partial y}u\phi_l(y)x^k dxdy + \int_0^1\int_0^1 c\frac{\partial u}{\partial y}\phi_l(y)x^k dxdy + \int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}u\phi_l'(y)x^k dxdy$$

$$+\int_0^1\int_0^1 \frac{1}{2}d^2\frac{\partial u}{\partial y}\phi_l'(y)x^k dxdy + \int_0^1 \left\{aux^k - \frac{1}{2}\frac{\partial(b^2u)}{\partial x}x^k + kb^2ux^{k-1}\right\}\Bigg|_{x=0}^{1}\phi_l(y)dy$$

$$= \int_0^1\int_0^1 f\phi_l(y)x^k dxdy \quad (k \geq 1). \tag{4.16}$$

By the assumed boundary condition (3.13) and (3.14), the boundary terms above drop out in all except the special case when $p = q = 1$ and $a = 0$ at $x = 0$ and $x = 1$. We shall assume in the subsequent analysis that we are not in this special case. Problems in which

special case occurs will be handled separately. (See, e.g., the test problem in Section 6.2).

Thus, we have

$$\int_0^1\int_0^1 \frac{\partial u}{\partial t}\phi_l dxdy + \int_0^1\int_0^1 \frac{\partial c}{\partial y}u\phi_l dxdy + \int_0^1\int_0^1 c\frac{\partial u}{\partial y}\phi_l dxdy$$

$$+\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}u\phi_l' dxdy + \int_0^1\int_0^1 \frac{1}{2}d^2\frac{\partial u}{\partial y}\phi_l' dxdy = \int_0^1\int_0^1 f\phi_l dxdy \quad (k=0) \tag{4.17}$$

and

$$\int_0^1\int_0^1 \frac{\partial u}{\partial t}\phi_l x^k dxdy - \int_0^1\int_0^1 kaux^{k-1}\phi_l dxdy - \int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2ux^{k-2}\phi_l dxdy$$

$$+\int_0^1\int_0^1 \frac{\partial c}{\partial y}u\phi_l x^k dxdy + \int_0^1\int_0^1 c\frac{\partial u}{\partial y}\phi_l x^k dxdy + \int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}u\phi_l' x^k dxdy$$

$$+\int_0^1\int_0^1 \frac{1}{2}d^2\frac{\partial u}{\partial y}\phi_l'(y)x^k dxdy = \int_0^1\int_0^1 f\phi_l(y)x^k dxdy \quad (k\geq 1). \tag{4.18}$$

We get the following by using (4.3), (4.18) and (4.17),

$$\sum_{j=0}^{N_y}\frac{d}{dt}\int_0^1 \alpha_j(x,t)x^k dx\int_0^1 \phi_l(y)\phi_j(y)dy$$

$$=\sum_{j=0}^{N_y}\int_0^1\int_0^1 ka\alpha_j x^{k-1}\phi_l\phi_j dxdy + \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j x^{k-2}\phi_l\phi_j dxdy$$

$$-\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_l(y)\phi_j(y)x^k dxdy - \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_l(y)\phi_j'(y)x^k dxdy$$

$$-\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j\phi_l'\phi_j x^k dxdy - \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j\phi_l'\phi_j' x^k dxdy$$

$$+\int_0^1\int_0^1 f\phi_l(y)x^k dxdy \quad (k\geq 0). \tag{4.19}$$

From (4.19) we obtain

$$\begin{pmatrix} \phi(l,0) & \phi(l,1) & \cdots & \phi(l,N_y) \end{pmatrix} \begin{pmatrix} \frac{d}{dt}\int_0^1 \alpha_0(x,t)x^k dx \\ \\ \frac{d}{dt}\int_0^1 \alpha_1(x,t)x^k dx \\ \vdots \\ \frac{d}{dt}\int_0^1 \alpha_{N_y}(x,t)x^k dx \end{pmatrix}$$

$$= \sum_{j=0}^{N_y}\int_0^1\int_0^1 ka\alpha_j x^{k-1}\phi_l\phi_j dxdy + \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j x^{k-2}\phi_l\phi_j dxdy$$

$$-\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_l(y)\phi_j(y)x^k dxdy - \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_l(y)\phi_j'(y)x^k dxdy$$

$$-\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j\phi_l'(y)\phi_j(y)x^k dxdy - \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j\phi_l'(y)\phi_j'(y)x^k dxdy$$

$$+\int_0^1\int_0^1 f\phi_l(y)x^k dxdy.$$

or,

$$\begin{pmatrix} \phi(0,0) & \phi(0,1) & \cdots & \phi(0,N_y) \\ \\ \phi(1,0) & \phi(1,1) & \cdots & \phi(1,N_y) \\ \vdots & \vdots & \cdots & \vdots \\ \phi(N_y,0) & \phi(N_y,1) & \cdots & \phi(N_y,N_y) \end{pmatrix} \begin{pmatrix} \frac{d}{dt}\int_0^1 \alpha_0(x,t)x^k dx \\ \\ \frac{d}{dt}\int_0^1 \alpha_1(x,t)x^k dx \\ \vdots \\ \frac{d}{dt}\int_0^1 \alpha_{N_y}(x,t)x^k dx \end{pmatrix} \tag{4.20}$$

$$= \begin{pmatrix} \sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_0(y)\phi_j(y)dxdy \\ \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_1(y)\phi_j(y)dxdy \\ \vdots \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_{N_y}(y)\phi_j(y)dxdy \end{pmatrix}$$

$$+ \begin{pmatrix} \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_0(y)\phi_j(y)dxdy \\ \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_1(y)\phi_j(y)dxdy \\ \vdots \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_{N_y}(y)\phi_j(y)dxdy \end{pmatrix}$$

$$- \begin{pmatrix} \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_0(y)\phi_j(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_1(y)\phi_j(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_{N_y}(y)\phi_j(y)x^k dxdy \end{pmatrix}$$

$$-\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_0(y)\phi_j'(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_1(y)\phi_j'(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_{N_y}(y)\phi_j'(y)x^k dxdy \end{pmatrix}$$

$$-\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_0'(y)\phi_j(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_1'(y)\phi_j(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_{N_y}'(y)\phi_j(y)x^k dxdy \end{pmatrix}$$

$$-\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j\phi_0'\phi_j'x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j\phi_1'\phi_j'x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j\phi_{N_y}'\phi_j'x^k dxdy \end{pmatrix} + \begin{pmatrix} \int_0^1\int_0^1 f\phi_0(y)x^k dxdy \\ \\ \int_0^1\int_0^1 f\phi_1(y)x^k dxdy \\ \vdots \\ \int_0^1\int_0^1 f\phi_{N_y}(y)x^k dxdy \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \dfrac{d}{dt}\displaystyle\int_0^1 \alpha_0(x,t)x^k dx \\ \\ \dfrac{d}{dt}\displaystyle\int_0^1 \alpha_1(x,t)x^k dx \\ \vdots \\ \dfrac{d}{dt}\displaystyle\int_0^1 \alpha_{N_y}(x,t)x^k dx \end{pmatrix}$$

$$= \Phi^{-1} \begin{pmatrix} \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_0(y)\phi_j(y)dxdy \\ \\ \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_1(y)\phi_j(y)dxdy \\ \vdots \\ \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 ka\alpha_j(x,t)x^{k-1}\phi_{N_y}(y)\phi_j(y)dxdy \end{pmatrix} \tag{4.21}$$

$$+\Phi^{-1} \begin{pmatrix} \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_0(y)\phi_j(y)dxdy \\ \\ \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_1(y)\phi_j(y)dxdy \\ \vdots \\ \displaystyle\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}k(k-1)b^2\alpha_j(x,t)x^{k-2}\phi_{N_y}(y)\phi_j(y)dxdy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_0(y)\phi_j(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_1(y)\phi_j(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\alpha_j(x,t)\phi_{N_y}(y)\phi_j(y)x^k dxdy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_0(y)\phi_j'(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_1(y)\phi_j'(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 c\alpha_j(x,t)\phi_{N_y}(y)\phi_j'(y)x^k dxdy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} -\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_0'(y)\phi_j(y)x^k dxdy \\ \\ -\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_1'(y)\phi_j(y)x^k dxdy \\ \vdots \\ -\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\alpha_j(x,t)\phi_{N_y}'(y)\phi_j(y)x^k dxdy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j(x,t)\phi_0'(y)\phi_j'(y)x^k dxdy \\ \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j(x,t)\phi_1'(y)\phi_j'(y)x^k dxdy \\ \vdots \\ \sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\alpha_j(x,t)\phi_{N_y}'(y)\phi_j'(y)x^k dxdy \end{pmatrix}$$

$$+\Phi^{-1}\begin{pmatrix} \int_0^1\int_0^1 f\phi_0(y)x^k dxdy \\ \\ \int_0^1\int_0^1 f\phi_1(y)x^k dxdy \\ \vdots \\ \int_0^1\int_0^1 f\phi_{N_y}(y)x^k dxdy \end{pmatrix}.$$

where

$$\Phi = \begin{pmatrix} \phi(0,0) & \phi(0,1) & \cdots & \phi(0,N_y) \\ \phi(1,0) & \phi(1,1) & \cdots & \phi(1,N_y) \\ \vdots & \vdots & \vdots & \vdots \\ \phi(N_y,0) & \phi(N_y,1) & \cdots & \phi(N_y,N_y) \end{pmatrix}$$

and

$$\Phi^{-1} = \begin{pmatrix} \tilde{\phi}(0,0) & \tilde{\phi}(0,1) & \cdots & \tilde{\phi}(0,N_y) \\ \tilde{\phi}(1,0) & \tilde{\phi}(1,1) & \cdots & \tilde{\phi}(1,N_y) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\phi}(N_y,0) & \tilde{\phi}(N_y,1) & \cdots & \tilde{\phi}(N_y,N_y) \end{pmatrix}.$$

Let L_1^* be the formal adjoint operator of L_1 given by (4.21), we can write (4.21) as

$$\frac{d}{dt}(\vec{\alpha}(x,t),x^k) = \int_0^1 (\vec{\alpha}(x,t), L_1^*(x^k)). \qquad (4.22)$$

From (4.13) and (4.21), we have

$$
\begin{pmatrix}
\dfrac{d}{dt}\displaystyle\int_0^1 \sum_{i=1}^{n} \omega_{i0}(t)\delta(x - x_{i0}(t))x^k dx \\
\dfrac{d}{dt}\displaystyle\int_0^1 \sum_{i=1}^{n} \omega_{i1}(t)\delta(x - x_{i1}(t))x^k dx \\
\vdots \\
\dfrac{d}{dt}\displaystyle\int_0^1 \sum_{i=1}^{n} \omega_{iN_y}(t)\delta(x - x_{iN_y}(t))x^k dx
\end{pmatrix}
$$

$$
= \Phi^{-1}
\begin{pmatrix}
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-1}\phi_0(y)\phi_j(y)dxdy \\
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-1}\phi_1(y)\phi_j(y)dxdy \\
\vdots \\
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 ka \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-1}\phi_{N_y}(y)\phi_j(y)dxdy
\end{pmatrix}
$$

$$
+\Phi^{-1}
\begin{pmatrix}
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2 \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-2}\phi_0(y)\phi_j(y)dxdy \\
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2 \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-2}\phi_1(y)\phi_j(y)dxdy \\
\vdots \\
\displaystyle\sum_{j=0}^{N_y} \int_0^1 \int_0^1 \frac{1}{2}k(k-1)b^2 \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))x^{k-2}\phi_{N_y}(y)\phi_j(y)dxdy
\end{pmatrix}
$$

$$
-\Phi^{-1}\begin{pmatrix}
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_0(y)\phi_j(y)x^k dxdy \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_1(y)\phi_j(y)x^k dxdy \\
\vdots \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{\partial c}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_{N_y}(y)\phi_j(y)x^k dxdy
\end{pmatrix}
$$

$$
-\Phi^{-1}\begin{pmatrix}
\sum_{j=0}^{N_y}\int_0^1\int_0^1 c\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_0(y)\phi_j'(y)x^k dxdy \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 c\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_1(y)\phi_j'(y)x^k dxdy \\
\vdots \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 c\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_{N_y}(y)\phi_j'(y)x^k dxdy
\end{pmatrix}
$$

$$
-\Phi^{-1}\begin{pmatrix}
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_0'(y)\phi_j(y)x^k dxdy \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_1'(y)\phi_j(y)x^k dxdy \\
\vdots \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}\frac{\partial(d^2)}{\partial y}\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_{N_y}'(y)\phi_j(y)x^k dxdy
\end{pmatrix}
$$

$$
-\Phi^{-1}\begin{pmatrix}
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_0'(y)\phi_j'(y)x^k dxdy \\
\\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_1'(y)\phi_j'(y)x^k dxdy \\
\vdots \\
\sum_{j=0}^{N_y}\int_0^1\int_0^1 \frac{1}{2}d^2\sum_{i=1}^{n}\omega_{ij}(t)\delta(x-x_{ij}(t))\phi_{N_y}'(y)\phi_j'(y)x^k dxdy
\end{pmatrix}
$$

$$
+\Phi^{-1}\begin{pmatrix}
\int_0^1\int_0^1 f\phi_0(y)x^k dxdy \\
\\
\int_0^1\int_0^1 f\phi_1(y)x^k dxdy \\
\vdots \\
\int_0^1\int_0^1 f\phi_{N_y}(y)x^k dxdy
\end{pmatrix}.
$$

$$
\begin{pmatrix}
\frac{d}{dt}\sum_{i=1}^{n}\omega_{i0}(t)x_{i0}^k(t) \\
\\
\frac{d}{dt}\sum_{i=1}^{n}\omega_{i1}(t)x_{i1}^k(t) \\
\vdots \\
\frac{d}{dt}\sum_{i=1}^{n}\omega_{in_y}(t)x_{iN_y}^k(t)
\end{pmatrix}
$$

$$
= \Phi^{-1} \begin{pmatrix} \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-1}(t) \int_0^1 ka(x_{ij}(t), y, t) \phi_0(y) \phi_j(y) dy \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-1}(t) \int_0^1 ka(x_{ij}(t), y, t) \phi_1(y) \phi_j(y) dy \\ \vdots \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-1}(t) \int_0^1 ka(x_{ij}(t), y, t) \phi_{N_y}(y) \phi_j(y) dy \end{pmatrix}
$$

$$
+ \Phi^{-1} \begin{pmatrix} \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-2}(t) \int_0^1 \frac{1}{2} k(k-1) b^2(x_{ij}(t), y, t) \phi_0(y) \phi_j(y) dy \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-2}(t) \int_0^1 \frac{1}{2} k(k-1) b^2(x_{ij}(t), y, t) \phi_1(y) \phi_j(y) dy \\ \vdots \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k-2}(t) \int_0^1 \frac{1}{2} k(k-1) b^2(x_{ij}(t), y, t) \phi_{N_y}(y) \phi_j(y) dy \end{pmatrix} \quad (4.23)
$$

$$
- \Phi^{-1} \begin{pmatrix} \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k}(t) \int_0^1 \frac{\partial c(x_{ij}(t), y, t)}{\partial y} \phi_0(y) \phi_j(y) dy \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k}(t) \int_0^1 \frac{\partial c(x_{ij}(t), y, t)}{\partial y} \phi_1(y) \phi_j(y) dy \\ \vdots \\ \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t) x_{ij}^{k}(t) \int_0^1 \frac{\partial c(x_{ij}(t), y, t)}{\partial y} \phi_{N_y}(y) \phi_j(y) dy \end{pmatrix}
$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 c(x_{ij}(t),y,t)\phi_0(y)\phi_j'(y)dy \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 c(x_{ij}(t),y,t)\phi_1(y)\phi_j'(y)dy \\ \vdots \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 c(x_{ij}(t),y,t)\phi_{N_y}(y)\phi_j'(y)dy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}\frac{\partial(d^2(x_{ij}(t),y,t))}{\partial y}\phi_0'(y)\phi_j(y)dy \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}\frac{\partial(d^2(x_{ij}(t),y,t))}{\partial y}\phi_1'(y)\phi_j(y)dy \\ \vdots \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}\frac{\partial(d^2(x_{ij}(t),y,t))}{\partial y}\phi_{N_y}'(y)\phi_j(y)dy \end{pmatrix}$$

$$-\Phi^{-1}\begin{pmatrix} \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}d^2\phi_0'\phi_j'dy \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}d^2\phi_1'\phi_j'dy \\ \vdots \\ \sum_{j=0}^{N_y}\sum_{i=1}^{n}\omega_{ij}(t)x_{ij}^k(t)\int_0^1 \frac{1}{2}d^2\phi_{N_y}'\phi_j'dy \end{pmatrix} + \Phi^{-1}\begin{pmatrix} \int_0^1\int_0^1 f\phi_0(y)x^k dxdy \\ \int_0^1\int_0^1 f\phi_1(y)x^k dxdy \\ \vdots \\ \int_0^1\int_0^1 f\phi_{N_y}(y)x^k dxdy \end{pmatrix}.$$

This is a system of $2n \times (N_y + 1)$ ordinary differential equations for the $n(N_y + 1)$ nodes and $n(N_y + 1)$ weights. Next, we shall pose the initial conditions for solving equation (4.23). From (4.13), we have

$$\int_0^1 \alpha_j(x,0)x^k dx = \int_0^1 \sum_{i=1}^{n} \omega_{ij}(0)\delta(x - x_{ij}(0))x^k dx = \sum_{i=1}^{n} \omega_{ij}(0)x_{ij}^k(0) \tag{4.24}$$

$$\begin{pmatrix} \int_0^1 \alpha_0(x,0)x^k dx \\ \int_0^1 \alpha_1(x,0)x^k dx \\ \vdots \\ \int_0^1 \alpha_{N_y}(x,0)x^k dx \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} \omega_{i0}(0)x_{i0}^k(0) \\ \sum_{i=1}^{n} \omega_{i1}(0)x_{i1}^k(0) \\ \vdots \\ \sum_{i=1}^{n} \omega_{im}(0)x_{iN_y}^k(0) \end{pmatrix} \tag{4.25}$$

By (4.3) and (4.13), we obtain

$$u(x,y,t) \approx \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(t)\delta(x - x_{ij}(t))\phi_j(y). \tag{4.26}$$

Therefore,

$$\begin{aligned}
&\int_0^1 \int_0^1 u(x,y,0)x^k \phi_l(y) dx dy \\
&= \int_0^1 \int_0^1 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \omega_{ij}(0)\delta(x - x_{ij}(0))\phi_j(y)x^k \phi_l(y) dx dy \\
&= \sum_{j=0}^{N_y} \sum_{i=1}^{n} \int_0^1 \omega_{ij}(0)\delta(x - x_{ij}(0))x^k dx \int_0^1 \phi_l(y)\phi_j(y) dy \\
&= \sum_{j=0}^{N_y} \left(\sum_{i=1}^{n} \omega_{ij}(0)x_{ij}^k(0) \right) \int_0^1 \phi_l(y)\phi_j(y) dy \\
&= \sum_{j=0}^{N_y} \left(\sum_{i=1}^{n} \omega_{ij}(0)x_{ij}^k(0) \right) \phi(l,j).
\end{aligned} \tag{4.27}$$

$$\int_0^1 \int_0^1 u(x,y,0)x^k \phi_l(y)\,dx\,dy$$

$$= \begin{pmatrix} \phi(l,0) & \phi(l,1) & \cdots & \phi(l,N_y) \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} \omega_{i0}(0) x_{i0}^k(0) \\ \sum_{i=1}^{n} \omega_{i1}(0) x_{i1}^k(0) \\ \vdots \\ \sum_{i=1}^{n} \omega_{iN_y}(0) x_{iN_y}^k(0) \end{pmatrix}.$$

or,

$$\begin{pmatrix} \int_0^1 \int_0^1 u(x,y,0)x^k \phi_0(y)\,dx\,dy \\ \int_0^1 \int_0^1 u(x,y,0)x^k \phi_1(y)\,dx\,dy \\ \vdots \\ \int_0^1 \int_0^1 u(x,y,0)x^k \phi_{N_y}(y)\,dx\,dy \end{pmatrix}$$

$$= \begin{pmatrix} \phi(0,0) & \phi(0,1) & \cdots & \phi(0,N_y) \\ \phi(1,0) & \phi(1,1) & \cdots & \phi(1,N_y) \\ \vdots & \vdots & \cdots & \vdots \\ \phi(N_y,0) & \phi(N_y,1) & \cdots & \phi(N_y,N_y) \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} \omega_{i0}(0) x_{i0}^k(0) \\ \sum_{i=1}^{n} \omega_{i1}(0) x_{i1}^k(0) \\ \vdots \\ \sum_{i=1}^{n} \omega_{iN_y}(0) x_{iN_y}^k(0) \end{pmatrix} \tag{4.28}$$

$$= \Phi \begin{pmatrix} \sum_{i=1}^{n} \omega_{i0}(0) x_{i0}^k(0) \\ \sum_{i=1}^{n} \omega_{i1}(0) x_{i1}^k(0) \\ \vdots \\ \sum_{i=1}^{n} \omega_{iN_y}(0) x_{iN_y}^k(0) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \sum_{i=1}^{n} \omega_{i0}(0) x_{i0}^{k}(0) \\ \sum_{i=1}^{n} \omega_{i1}(0) x_{i1}^{k}(0) \\ \vdots \\ \sum_{i=1}^{n} \omega_{iN_y}(0) x_{iN_y}^{k}(0) \end{pmatrix} = \Phi^{-1} \begin{pmatrix} \int_0^1 \int_0^1 u(x,y,0) x^k \phi_0(y) dx dy \\ \int_0^1 \int_0^1 u(x,y,0) x^k \phi_1(y) dx dy \\ \vdots \\ \int_0^1 \int_0^1 u(x,y,0) x^k \phi_{N_y}(y) dx dy \end{pmatrix}. \tag{4.29}$$

CHAPTER 5

Convergence Results

5.1 Convergence of Semi-Discrete Finite Element Approximation in y

Let $V_h \in H_0^1(0,1)$ be a finite dimensional subspace as defined in Section 4.1. Given $f \in L^2(0,1)$ and $u_{0,h} \in V_h$, a suitable approximation of the initial datum $u_0 \in L^2([0,1])$, for each $t \in [0,T]$, we attempt to find $u_h(t) \in V_h$ such that

$$\begin{cases} \left(\dfrac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad t \in (0,T), \\ \\ u_h(0) = u_{0,h}. \end{cases} \tag{5.1}$$

Lemma 5.1.1 *For fixed x and t, let π_h be the linear interpolant of u, that is, $\pi_h \in V_h$, $\pi_h(y_j) = u(y_j)$, $0 \leq j \leq N_y$. Then*

$$||u' - \pi_h'||_0 \leq h||u''||_0 \tag{5.2}$$

Proof. For any $y \in (y_j, y_{j+1})$,

$$
\begin{aligned}
u'(y) - \pi_h'(y) &= u'(y) - \frac{u(y_{j+1}) - u(y_j)}{h} \\
&= \frac{1}{h}\int_{y_j}^{y_{j+1}} u'(y)dz - \frac{1}{h}\int_{y_j}^{y_{j+1}} u'(z)dz \\
&= \frac{1}{h}\int_{y_j}^{y_{j+1}} (u'(y) - u'(z))dz \\
&= \frac{1}{h}\int_{y_j}^{y_{j+1}} \int_z^y u''(w)dwdz \\
\|u' - \pi_h'\|_0^2 &= \int_0^1 (u'(y) - \pi_h'(y))^2 dy \\
&= \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} (u'(y) - \pi_h'(y))^2 dy \\
&= \frac{1}{h^2} \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} \left\{ \int_{y_j}^{y_{j+1}} \int_z^y u''(w)dwdz \right\}^2 dy \\
&\leq \frac{1}{h^2} \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} \left\{ \int_{y_j}^{y_{j+1}} \int_z^y |u''(w)|^2 dwdz \right\} \left\{ \int_{y_j}^{y_{j+1}} \int_z^y dwdz \right\} dy \\
&\leq \frac{1}{h^2} \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} \left\{ \int_{y_j}^{y_{j+1}} \int_z^y |u''(w)|^2 dwdz \right\} h^2 dy \\
&\leq \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} \left\{ \int_{y_j}^{y_{j+1}} \int_{y_j}^{y_{j+1}} |u''(w)|^2 dwdz \right\} dy \\
&= h^2 \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} |u''(w)|^2 dw \\
&= h^2 \int_0^1 |u''(w)|^2 dw \\
&= h^2 \|u''\|_0^2.
\end{aligned}
$$

Lemma 5.1.2 *Let π_h be the linear interpolant of u, that is, $\pi_h \in V_h, \pi_h(y_j) = u(y_j), \forall j$. Then*

$$\|u - \pi_h\|_0 \leq h^2 \|u''\|_0. \tag{5.3}$$

Proof. For $y \in (y_j, y_{j+1})$, since $u(y_j) = \pi_h(y_j)$, we have

$$\begin{aligned}
u(y) - \pi_h(y) &= \int_{y_j}^{y} (u'(z) - \pi_h'(z))dz \\
|u(y) - \pi_h(y)|^2 &= \left(\int_{y_j}^{y} (u'(z) - \pi_h'(z))dz\right)^2 \\
&\leq \int_{y_j}^{y} |u'(z) - \pi_h'(z)|^2 dz \int_{y_j}^{y} dz \\
&= (y - y_j) \int_{y_j}^{y_{j+1}} |u'(z) - \pi_h'(z)|^2 dz \\
\|u - \pi_h\|_0^2 &= \int_0^1 |u(y) - \pi_h(y)|^2 dy \\
&= \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} |u(y) - \pi_h(y)|^2 dy \\
&\leq \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} (y - y_j) \int_{y_j}^{y_{j+1}} |u'(z) - \pi_h'(z)|^2 dz dy \\
&= \frac{h^2}{2} \sum_{j=0}^{N_y-1} \int_{y_j}^{y_{j+1}} |u'(z) - \pi_h'(z)|^2 dz \\
&= \frac{h^2}{2} \int_0^1 |u'(z) - \pi_h'(z)|^2 dz \\
&= \frac{h^2}{2} \|u' - \pi_h'\|_0^2.
\end{aligned}$$

Inequality (5.3) follows at once from (5.2).

Theorem 5.1.1 *If* $u_0 \in H_0^1(0,1)$, $f \in L^2(0,1)$ *and* u *satisfies the following condition,*

$$\|u(x,\cdot,t)\|_2^2 \leq C(\|Lu(x,\cdot,t)\|_0^2 + \|u(x,\cdot,t)\|_1^2), \tag{5.4}$$

then the energy estimate

$$\begin{aligned}
&\max_{0\leq t\leq T} \|u(t) - u_h(t)\|_0^2 + \alpha \int_0^T \|u(t) - u_h(t)\|_1^2 \\
&\leq \|u_0 - u_{0,h}\|_0^2 + C_{\alpha,\gamma} h^2 \left(\|u_{0,h}\|_1^2 + \|u_0\|_1^2 + \int_0^T \|f\|_0^2 dt\right)
\end{aligned} \tag{5.5}$$

holds; where $C_{\alpha,\gamma} > 0$ is a constant independent of h.

Proof. For each $t \in (0, T]$ define $e_h(t) = u(t) - u_h(t)$. Taking $v = v_h$ in (3.21), we get the weak form

$$(u_t, v_h) + a(u, v_h) = (f, v_h) \tag{5.6}$$

Subtracting (5.1) from (5.6), we obtain

$$\left(\frac{\partial(u - u_h)}{\partial t}, v_h\right) + a(u - u_h, v_h) = 0,$$

$$\left(\frac{\partial e_h(t)}{\partial t}, v_h\right) + a(e_h(t), v_h) = 0. \tag{5.7}$$

For almost any fixed t, choose $v_h = u_h(t) - w_h$, $w_h \in V_h$ in (5.7). For each $\epsilon > 0$ and for almost any $t \in [0, T]$ we find

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(e_h(t), e_h(t)) + a(e_h(t), e_h(t)) \\
= \;& \frac{1}{2}\left(\frac{\partial e_h(t)}{\partial t}, e_h(t)\right) + \frac{1}{2}\left(e_h(t), \frac{\partial e_h(t)}{\partial t}\right) + a(e_h(t), e_h(t)) \\
= \;& \left(\frac{\partial e_h(t)}{\partial t}, e_h(t)\right) + a(e_h(t), e_h(t)) \\
= \;& \left(\frac{\partial e_h(t)}{\partial t}, u - u_h(t)\right) + a(e_h(t), u - u_h(t)) \\
= \;& \left(\frac{\partial e_h(t)}{\partial t}, u - w_h + w_h - u_h(t)\right) + a(e_h(t), u - w_h + w_h - u_h(t)) \\
= \;& \left(\frac{\partial e_h(t)}{\partial t}, u - w_h\right) + a(e_h(t), u - w_h) + \left(\frac{\partial e_h(t)}{\partial t}, w_h - u_h\right) + a(e_h(t), w_h - u_h).
\end{aligned}$$

By (5.7)

$$\left(\frac{\partial e_h(t)}{\partial t}, w_h - u_h\right) + a(e_h(t), w_h - u_h) = 0.$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial}{\partial t}(e_h(t), e_h(t)) + a(e_h(t), e_h(t)) \\
= & \left(\frac{\partial e_h(t)}{\partial t}, u - w_h\right) + a(e_h(t), u - w_h) \\
\leq & \left\|\frac{\partial e_h(t)}{\partial t}\right\|_0 \|u - w_h\|_0 + c\|e_h(t)\|_1 \|u(t) - w_h\|_1 \\
\leq & \left\|\frac{\partial e_h}{\partial t}\right\|_0 \|u - w_h\|_0 + \frac{c^2}{4\epsilon}\|u - w_h\|_1^2 + \epsilon\|e_h(t)\|_1^2.
\end{aligned}
$$

Let $w_h = \pi_h(u(t))$ be the linear interpolant of u, for $t \in [0, T]$. By Lemma 5.1.1 and Lemma 5.1.2, we have

$$\|u(t) - \pi_h(u(t))\|_0 \leq \gamma h^2 \|u(t)\|_2 \tag{5.8}$$

and

$$\|u(t) - \pi_h(u(t))\|_1^2 \leq \gamma^2 h^2 \|u(t)\|_2^2. \tag{5.9}$$

We thus obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(e_h(t), e_h(t)) + a(e_h(t), e_h(t)) \leq \left\|\frac{\partial e_h}{\partial t}\right\|_0 \gamma h^2 \|u\|_2 + \frac{c^2}{4\epsilon}\gamma^2 h^2 \|u(t)\|_2^2 + \epsilon\|e_h(t)\|_1^2. \tag{5.10}$$

Using (3.28), we have $a(e_h(t), e_h(t)) \geq \alpha\|e_h(t)\|_1^2$. If we choose $\epsilon = \frac{\alpha}{2}$, then

$$
\begin{aligned}
& \frac{1}{2}\frac{\partial}{\partial t}(e_h(t), e_h(t)) + \alpha\|e_h(t)\|_1^2 \\
\leq & \frac{1}{2}\frac{\partial}{\partial t}(e_h(t), e_h(t)) + a(e_h(t), e_h(t)) \\
\leq & \left\|\frac{\partial e_h}{\partial t}\right\|_0 \gamma h^2 \|u\|_2 + \frac{c^2}{2\alpha}\gamma^2 h^2 \|u(t)|_2^2 + \frac{\alpha}{2}\|e_h(t)\|_1^2.
\end{aligned} \tag{5.11}
$$

So

$$\frac{1}{2}\frac{d}{dt}\|e_h(t)\|_0^2 + \frac{\alpha}{2}\|e_h(t)\|_1^2 \leq \left\|\frac{\partial e_h}{\partial t}\right\|_0 \gamma h^2 \|u\|_2 + \frac{c^2}{2\alpha}\gamma^2 h^2 \|u(t)\|_2^2. \tag{5.12}$$

Integrating over $(0,t)$ for $t \in (0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{d\tau}\|e_h(\tau)\|_0^2 d\tau + \frac{\alpha}{2}\int_0^t \|e_h(\tau)\|_1^2 d\tau \\
\leq\ & \int_0^t \left\|\frac{\partial e_h(\tau)}{\partial \tau}\right\|_0 \gamma h^2 \|u(\tau)\|_2 d\tau + \frac{c^2}{2\alpha}\gamma^2 h^2 \int_0^T \|u(\tau)\|_2^2 d\tau \\
\leq\ & \gamma h^2 \int_0^t \left(\left\|\frac{\partial e_h(\tau)}{\partial \tau}\right\|_0^2 + \|u(\tau)\|_2^2\right) d\tau + \gamma^2 h^2 \frac{c^2}{2\alpha}\int_0^T \|u(\tau)\|_2^2 d\tau \\
=\ & \gamma h^2 \int_0^T \left(\left\|\frac{\partial e_h(\tau)}{\partial \tau}\right\|_0^2 + \left(1+\frac{\gamma c^2}{2\alpha}\right)\|u(\tau)\|_2^2\right) d\tau \\
\leq\ & \frac{1}{2} C_{\alpha,\gamma} h^2 \int_0^T \left(\left\|\frac{\partial e_h(\tau)}{\partial \tau}\right\|_0^2 + \|u(\tau)\|_2^2\right) d\tau.
\end{aligned}$$

So

$$\frac{1}{2}\|e_h(t)\|_0^2 - \frac{1}{2}\|e_h(0)\|_0^2 + \frac{\alpha}{2}\int_0^t \|e_h(\tau)\|_1^2 d\tau \leq \frac{1}{2} C_{\alpha,\gamma} h^2 \left(\int_0^T \left\|\frac{\partial e_h(\tau)}{\partial \tau}\right\|_0^2 + \|u(\tau)\|_2^2\right) d\tau.$$

Since

$$\|e_h(0)\|_0^2 = \|u_0 - u_{h,0}\|_0^2$$

and

$$\left\|\frac{\partial e_h(t)}{\partial t}\right\|_0^2 = \left\|\frac{\partial u(t)}{\partial t} - \frac{\partial u_h(t)}{\partial t}\right\|_0^2 \leq \left\|\frac{\partial u(t)}{\partial t}\right\|_0^2 + \left\|\frac{\partial u_h(t)}{\partial t}\right\|_0^2,$$

it follows that

$$\|e_h(t)\|_0^2 + \alpha\int_0^t \|e_h(\tau)\|_1^2 d\tau \leq \|u_0 - u_{h,0}\|_0^2 + C_{\alpha,\gamma} h^2 \int_0^T \left(\left\|\frac{\partial u}{\partial t}\right\|_0^2 + \left\|\frac{\partial u_h}{\partial t}\right\|_0^2 + \|u\|_2^2\right) dt.$$

As in (3.31), we have

$$\int_0^t \left\|\frac{\partial u_h}{\partial t}\right\|_0^2 d\tau \leq C_\alpha \left\{\|u_{0,h}\|_1^2 + \int_0^t \|f\|_0^2 ds\right\}.$$

Therefore,

$$\begin{aligned}&\|e_h(t)\|_0^2 + \alpha \int_0^t \|e_h(\tau)\|_1^2 d\tau \\ &\leq \|u_0 - u_{h,0}\|_0^2 + C_{\alpha,\gamma} h^2 \int_0^T \left(\left\| \frac{\partial u(t)}{\partial t} \right\|_0^2 + \|u_{0,h}\|_1^2 + \int_0^t \|f\|_0^2 + \|u(t)\|_2^2 \right) dt.\end{aligned}$$

Using (3.29), (3.31) and (3.32), we obtain

$$\|e_h(t)\|_0^2 + \alpha \int_0^t \|e_h(\tau)\|_1^2 d\tau \leq \|u_0 - u_{h,0}\|_0^2 + C_{\alpha,\gamma} h^2 \left\{ \|u_0\|_1^2 + \|u_{0,h}\|_1^2 + \int_0^t \|f\|_0^2 dt \right\}.$$

So

$$\begin{aligned}&\max_{0 \leq t \leq T} \|u(t) - u_h(t)\|_0^2 + \alpha \int_0^T \|u(t) - u_h(t)\|_1^2 \\ &\leq \|u_0 - u_{0,h}\|_0^2 + C_{\alpha,\gamma} \left(\|u_{0,h}\|_1^2 + \|u_0\|_1^2 + \int_0^T \|f\|_0^2 ds \right)\end{aligned}$$

5.2 Convergence of the Gauss-Galerkin Approximations in x

In Section 4.2, we replaced $\alpha_j(x,t)dx$ by an n-point dirac-delta function, i.e.,

$$d\mu_j^n(x,t) = \sum_{i=1}^n \omega_{ij}(t)\delta(x - x_{ij}(t))dx \tag{5.13}$$

Now we define moments of the discrete measures by

$$m_{j,n}^l(t) = \int_0^1 x^l d\mu_j^n(x,t), \quad t \in [0,T]. \tag{5.14}$$

Substituting (5.13) into (5.14), we obtain

$$m_{j,n}^l(t) = \sum_{i=1}^n \omega_{ij}(t) x_{ij}^l(t) \tag{5.15}$$

Define

$$d\vec{\mu}^n(x,t) = (d\mu_0^n(x,t), d\mu_1^n(x,t), \cdots, d\mu_{N_y}^n(x,t))^T \tag{5.16}$$

and

$$\vec{m}_n^l(t) = (m_{0,n}^l(t), m_{1,n}^l(t), \cdots, m_{N_y,n}^l(t))^T. \tag{5.17}$$

For given $u(x,y,0)$, as in (4.29), we can calculate $\vec{m}_n^l(0)$ as follows:

$$\vec{m}_n^l(0) = \begin{pmatrix} m_{0,n}^l(0) \\ m_{1,n}^l(0) \\ \vdots \\ m_{N_y,n}^l(0) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \omega_{i0}(0) x_{i0}^l(0) \\ \sum_{i=1}^n \omega_{i1}(0) x_{i1}^l(0) \\ \vdots \\ \sum_{i=1}^n \omega_{im}(0) x_{iN_y}^l(0) \end{pmatrix} = \Phi^{-1} \begin{pmatrix} \int_0^1 \int_0^1 u_0 x^l \phi_0 dxdy \\ \int_0^1 \int_0^1 u_0 x^l \phi_1 dxdy \\ \vdots \\ \int_0^1 \int_0^1 u_0 x^l \phi_{N_y} dxdy \end{pmatrix}. \tag{5.18}$$

We assume that

$$\begin{aligned} &|a(x,y,t)| \le a_1 x + a_0 \\ &|b^2(x,y,t)| \le b_2 x^2 + b_1 x + b_0 \\ &|c(x,y,t)| \le c_0 \\ &\left|\frac{\partial c(x,y,t)}{\partial y}\right| \le c_1 \\ &\left|\frac{\partial (d^2(x,y,t))}{\partial y}\right| \le d_1 \\ &|d^2(x,y,t)| \le d_0. \end{aligned} \tag{5.19}$$

Lemma 5.2.1 *Let l be any positive integer. The set $\left\{\vec{m}_n^l(t) : n \ge \frac{1}{2}(l+1)\right\}$ is uniformly bounded and equicontinuous in $t \in [0,T]$ for each l.*

Proof. We have by (4.23)

$$
\begin{aligned}
&\frac{d}{dt}m^l_{p,n}(t)\\
&=\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-1}(t)\int_0^1 la(x_{ij}(t),y,t)\phi_s(y)\phi_j(y)dy\right)\\
&+\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-2}(t)\int_0^1 \frac{1}{2}l(l-1)b^2(x_{ij}(t),y,t)\phi_s(y)\phi_j(y)dy\right)\\
&-\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 \frac{\partial c(x_{ij}(t),y,t)}{\partial y}\phi_s(y)\phi_j(y)dy\right)\\
&-\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 c(x_{ij}(t),y,t)\phi_s(y)\phi_j'(y)dy\right) \qquad (5.20)\\
&-\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 \frac{1}{2}\frac{\partial(d^2(x_{ij}(t),y,t))}{\partial y}\phi_s'(y)\phi_j(y)dy\right)\\
&-\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 \frac{1}{2}d^2(x_{ij}(t),y,t)\phi_s'(y)\phi_j'(y)dy\right)\\
&+\sum_{s=0}^{N_y}\tilde{\phi}(p,s)\int_0^1\int_0^1 f\phi_s(y)x^k dxdy.
\end{aligned}
$$

Since $|\phi_j(y)|\le 1$, $|\phi_j'(y)|\le 1$ and

$$
\begin{aligned}
&\int_0^1\int_0^1 |f\phi_s(y)x^k|dxdy \le \int_0^1\int_0^1 |f|dxdy\\
&\le \left\{\int_0^1\int_0^1 |f|^2dxdy\right\}^{\frac{1}{2}}\left\{\int_0^1\int_0^1 dxdy\right\}^{\frac{1}{2}} = \|f(t)\|_0,
\end{aligned}
\qquad (5.21)
$$

we have

$$
\begin{aligned}
&\frac{d}{dt} m^l_{p,n}(t) \\
&\le \rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-1}(t)\int_0^1 l|(a_1 x_{ij}(t)+a_0)\phi_s(y)\phi_j(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-2}(t)\int_0^1 \frac{1}{2}l(l-1)|(b_2 x_{ij}^2(t)+b_1 x_{ij}(t)+b_0)\phi_s(y)\phi_j(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 c_1|\phi_s(y)\phi_j(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 c_0|\phi_s(y)\phi_j'(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 \frac{1}{2}d_1|\phi_s'(y)\phi_j(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\int_0^1 \frac{1}{2}d_0|\phi_s'(y)\phi_j'(y)|dy\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\int_0^1\int_0^1 |f\phi_s(y)x^k|dxdy \\
&\le \rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-1}(t)l(a_1 x_{ij}(t)+a_0)\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l-2}(t)\frac{1}{2}l(l-1)(b_2 x_{ij}^2(t)+b_1 x_{ij}(t)+b_0)\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)c_1\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)c_0\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\frac{1}{2}d_1\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\sum_{j=0}^{N_y}\sum_{i=1}^{n}\left(\omega_{ij}(t)x_{ij}^{l}(t)\frac{1}{2}d_0\right) \\
&+\rho_1 \sum_{s=0}^{N_y}\|f(t)\|_0
\end{aligned}
\tag{5.22}
$$

where $\rho_1 = \max\limits_{0 \le s,p \le N_y} |\widetilde{\phi}(p,s)|$. So we have

$$\begin{aligned}
&\frac{d}{dt} m^l_{p,n}(t) \\
&\le (N_y+1) l \rho_1 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^{l-1}(t) (a_1 x_{ij}(t) + a_0) \Big) \\
&+ \frac{1}{2} (N_y+1) l(l-1) \rho_1 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^{l-2}(t) (b_2 x_{ij}^2(t) + b_1 x_{ij}(t) + b_0) \Big) \\
&+ (N_y+1) \rho_1 c_1 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^l(t) \Big) \\
&+ (N_y+1) \rho_1 c_0 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^l(t) \Big) \\
&+ \frac{1}{2} (N_y+1) \rho_1 d_1 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^l(t) \Big) \\
&+ \frac{1}{2} (N_y+1) \rho_1 d_0 \sum_{j=0}^{N_y} \sum_{i=1}^{n} \Big(\omega_{ij}(t) x_{ij}^l(t) \Big) \\
&+ (N_y+1) \rho_1 \|f(t)\|_0 \\
&\le (N_y+1) l \rho_1 \sum_{j=0}^{N_y} \Big(a_1 m^l_{j,n}(t) + a_0 m^{l-1}_{j,n}(t) \Big) \\
&+ \frac{1}{2} (N_y+1) l(l-1) \rho_1 \sum_{j=0}^{N_y} \Big(b_2 m^l_{j,n}(t) + b_1 m^{l-1}_{j,n}(t) + b_0 m^{l-2}_{j,n}(t) \Big) \\
&+ (N_y+1) \rho_1 (c_1 + c_0 + \frac{1}{2} d_2 + \frac{1}{2} d_1 + \frac{1}{2} d_0) \sum_{j=0}^{N_y} m^l_{j,n}(t) \\
&+ (N_y+1) \rho_1 \|f(t)\|_0 \\
&= \eta_1 \sum_{j=0}^{N_y} m^l_{j,n}(t) + \eta_2 \sum_{j=0}^{N_y} m^{l-1}_{j,n}(t) + \eta_3 \sum_{j=0}^{N_y} m^{l-2}_{j,n}(t) + \eta_4 \|f(t)\|_0 \\
&= \vec{\eta}_1^T \vec{m}^l_n(t) + \vec{\eta}_2^T \vec{m}^{l-1}_n(t) + \vec{\eta}_3^T \vec{m}^{l-2}_n(t) + \eta_4 \|f(t)\|_0,
\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
\eta_1 &= (N_y+1)\rho_1\left(la_1+\frac{1}{2}l(l-1)b_2+c_1+c_0+\frac{1}{2}d_2+\frac{1}{2}d_1+\frac{1}{2}d_0\right)\\
\eta_2 &= (N_y+1)l\rho_1\left(a_0+\frac{1}{2}(l-1)b_1\right)\\
\eta_3 &= \frac{1}{2}(N_y+1)l(l-1)\rho_1 b_0\\
\vec{\eta}_i^T &= (\eta_i,\eta_i,\cdots,\eta_i),\quad i=1,2,3\\
\eta_4 &= (N_y+1)\rho_1.
\end{aligned}\tag{5.24}$$

We define K_i by

$$K_i=\begin{pmatrix}\vec{\eta}_i^T\\ \vec{\eta}_i^T\\ \vdots\\ \vec{\eta}_i^T\end{pmatrix}_{(N_y+1)\times(N_y+1)} \qquad i=1,2,3 \qquad \vec{\eta}_4=\begin{pmatrix}\eta_4\\ \eta_4\\ \vdots\\ \eta_4\end{pmatrix}_{(N_y+1)\times 1}\tag{5.25}$$

Then we have the following differential inequality

$$\frac{d}{dt}\vec{m}_n^l(t)\le K_1\vec{m}_n^l(t)+K_2\vec{m}_n^{l-1}(t)+K_3\vec{m}_n^{l-2}(t)+\|f(t)\|_0\vec{\eta}_4.\tag{5.26}$$

Define

$$\begin{cases}\dfrac{d}{dt}\vec{m}^l(t)=K_1\vec{m}^l(t)+K_2\vec{m}^{l-1}(t)+K_3\vec{m}^{l-2}(t)+\|f(t)\|_0\vec{\eta}_4\\[2ex] m_j^l(0)=m_{j,n}^l(0),\quad 0\le j\le N_y,\end{cases}\tag{5.27}$$

where

$$\vec{m}^{-2}(t)=\vec{m}^{-1}(t)=0.$$

Defining $\vec{M}_n^l(t) = \vec{m}_n^l(t) - \vec{m}^l(t)$ and subtracting (5.27) from (5.26) we obtain

$$\begin{cases} \dfrac{d}{dt}\vec{M}_n^l(t) \le K_1\vec{M}_n^l(t) + K_2\vec{M}_n^{l-1}(t) + K_3\vec{M}_n^{l-2}(t) \\ \\ \vec{M}_n^l(0) = 0, \quad 0 \le j \le N_y. \end{cases} \tag{5.28}$$

There exists $\vec{R}^l(t) \le 0$ such that

$$\begin{cases} \dfrac{d}{dt}\vec{M}_n^l(t) = K_1\vec{M}_n^l(t) + K_2\vec{M}_n^{l-1}(t) + K_3\vec{M}_n^{l-2}(t) + \vec{R}^l(t) \\ \\ \vec{M}_n^l(0) = 0, \quad 0 \le j \le N_y. \end{cases} \tag{5.29}$$

The solution of (5.29) is

$$\begin{aligned} \vec{M}_n^l(t) &= e^{K_1 t}\vec{M}_n^l(0) + \int_0^t e^{K_1(t-\tau)}\left\{K_2\vec{M}_n^{l-1}(\tau) + K_3\vec{M}_n^{l-2}(\tau) + \vec{R}^l(\tau)\right\}d\tau \\ &= \int_0^t e^{K_1(t-\tau)}\left\{K_2\vec{M}_n^{l-1}(\tau) + K_3\vec{M}_n^{l-2}(\tau) + \vec{R}^l(\tau)\right\}d\tau \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\vec{M}_n^1(t) = \int_0^t e^{K_1(t-\tau)}\left\{\vec{R}^1(\tau)d\tau\right\} \le 0 \\ &\vec{M}_n^2(t) = \int_0^t e^{K_1(t-\tau)}\left\{K_2\vec{M}_n^1(\tau) + \vec{R}^2(\tau)d\tau\right\} \le 0 \\ &\vdots \end{aligned}$$

By induction, $\vec{M}_n^{2n-1}(t) \le 0$. Therefore,

$$\vec{m}_n^l(t) \le \vec{m}^l(t), \quad l = 0, 1, 2, \cdots, 2n-1 \tag{5.31}$$

Define

$$\vec{M}^l(t) = \begin{pmatrix} \vec{m}^l(t) \\ \vec{m}^{l-1}(t) \\ \vdots \\ \vec{m}^1(t) \\ \vec{m}^0(t) \end{pmatrix}_{(l+1)(N_y+1)\times 1} \qquad \vec{B}^l = \begin{pmatrix} \vec{\eta}_4 \\ \vec{\eta}_4 \\ \vdots \\ \vec{\eta}_4 \\ \vec{\eta}_4 \end{pmatrix}_{(l+1)(N_y+1)\times 1} \tag{5.32}$$

$$A = \begin{pmatrix} K_1 & K_2 & K_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & K_1 & K_2 & K_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K_1 & K_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & K_1 & K_2 & K_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & K_1 & K_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & K_1 \end{pmatrix}. \tag{5.33}$$

We have

$$\begin{cases} \dfrac{d}{dt}\vec{M}^l(t) = A\vec{M}^l(t) + \|f(t)\|_0\vec{B}^l \\ \\ m_j^k(0) = m_{j,n}^k(0), \quad 0 \le j \le m, \quad 0 \le k \le l. \end{cases} \tag{5.34}$$

The solution of (5.34) is

$$\vec{M}^l(t) = e^{At}\vec{M}^l(0) + \int_0^t e^{A(t-\tau)}\|f(\tau)\|_0\vec{B}^l d\tau. \tag{5.35}$$

Thus,

$$\begin{aligned} &\|\vec{m}^l(t)\|_\infty \le \|\vec{M}^l(t)\|_\infty \\ &\le e^{\|A\|t}\|\vec{M}^l(0)\|_\infty + \int_0^t e^{\|A\|(t-\tau)}\|f(\tau)\|_0\|\vec{B}^l\|_\infty d\tau. \end{aligned} \tag{5.36}$$

By (5.25) and (5.32), we have

$$\|\vec{B}^l\|_\infty = \|\vec{\eta}_4\|_\infty = \eta_4. \tag{5.37}$$

By (5.32) and (5.27) we have

$$\|\vec{M}^l(0)\|_\infty = \max_{0\le k\le l} \|\vec{m}^l(0)\|_\infty = \max_{0\le k\le l} \|\vec{m}^l_n(0)\|_\infty. \tag{5.38}$$

By (5.18) we obtain

$$\begin{aligned}
&\|\vec{m}^l_n(0)\|_\infty \le \|\Phi^{-1}\| \max_{0\le s\le N_y} \left|\int_0^1\int_0^1 u(x,y,0)x^l\phi_s(y)dxdy\right| \\
&\le \|\Phi^{-1}\| \int_0^1\int_0^1 |u(x,y,0)|dxdy \\
&\le \|\Phi^{-1}\| \left\{\int_0^1\int_0^1 |u(x,y,0)|^2dxdy\right\}^{\frac{1}{2}} \left\{\int_0^1\int_0^1 dxdy\right\}^{\frac{1}{2}} \\
&= \|\Phi^{-1}\|\|u_0\|_0.
\end{aligned} \tag{5.39}$$

Substituting (5.37), (5.38) and (5.39) into (5.36), we have

$$\begin{aligned}
&\|\vec{m}^l(t)\|_\infty \le e^{\|A\|T}\|\Phi^{-1}\|\|u_0\|_0 + \eta_4 \left\{\int_0^t e^{2\|A\|(t-\tau)}d\tau\right\}^{\frac{1}{2}} \left\{\int_0^t \|f(\tau)\|_0^2 d\tau\right\}^{\frac{1}{2}} \\
&= e^{\|A\|T}\|\Phi^{-1}\|\|u_0\|_0 + \eta_4 \left\{\frac{1}{2\|A\|}\left(e^{2\|A\|t}-1\right)\right\}^{\frac{1}{2}} \left\{\int_0^t \|f(\tau)\|_0^2 d\tau\right\}^{\frac{1}{2}} \\
&\le e^{\|A\|T}\|\Phi^{-1}\|\|u_0\|_0 + \eta_4 \left\{\frac{1}{2\|A\|}\left(e^{2\|A\|T}-1\right)\right\}^{\frac{1}{2}} \left\{\int_0^t \|f(\tau)\|_0^2 d\tau\right\}^{\frac{1}{2}} = C(l,N_y),
\end{aligned} \tag{5.40}$$

where

$$C(l,N_y) = e^{\|A\|T}\|\Phi^{-1}\|\|u_0\|_0 + \eta_4 \left\{\frac{1}{2\|A\|}\left(e^{2\|A\|T}-1\right)\right\}^{\frac{1}{2}} \left\{\int_0^t \|f(\tau)\|_0^2 d\tau\right\}^{\frac{1}{2}} \tag{5.41}$$

So, for each given l, $\{\vec{m}^l_n(t) : n \ge \frac{1}{2}(l+1)\}$ is uniformly bounded.

Now, we consider the equicontinuity of $\{\vec{m}^l_n(t) : n \ge \frac{1}{2}(l+1)\}$ in $[0,T]$. By the mean-value theorem,

$$|m_{j,n}^l(t_2) - m_{j,n}^l(t_1)| = \left|\frac{d}{dt}m_{j,n}^l(\xi)\right| |t_2 - t_1|, \quad t_1, t_2 \in [0, T], \quad \xi \in (t_1, t_2). \tag{5.42}$$

Following the same process leading to (5.26), we also obtain

$$\begin{aligned}
&\left\|\frac{d}{dt}\vec{m}_n^l(t)\right\|_\infty \\
&\le \|K_1\|\|\vec{m}_n^l(t)\|_\infty + \|K_2\|\|\vec{m}_n^{l-1}(t)\|_\infty + \|K_3\|\|\vec{m}_n^{l-2}(t)\|_\infty + \|f(t)\|_0\eta_4 \\
&\le (\|K_1\| + \|K_2\| + \|K_3\|)C(l, N_y) + \max_{0\le t\le T} \|f(t)\|_0\eta_4 \\
&\le \tilde{C}(l, N_y), \quad n \ge \tfrac{1}{2}(l+1).
\end{aligned} \tag{5.43}$$

Therefore,

$$|m_{j,n}^l(t_2) - m_{j,n}^l(t_1)| \le \tilde{C}(l, N_y)|t_2 - t_1|, \quad t_1, t_2 \in [0, T]. \tag{5.44}$$

This proves the equicontinuity of $\{\vec{m}_n^l(t) : n \ge \frac{1}{2}(l+1)\}$ in $[0, T]$. □

Lemma 5.2.2 *Given N_y, there exists a sequence $k_n, k_n \to \infty$, and a sequence of functions $\left\{\vec{m}_*^l(t)\right\}$ such that for every fixed integer l, we have*

$$\lim_{k_n\to\infty} \vec{m}_{k_n}^l(t) = \vec{m}_*^l(t), \quad \forall t \in [0, T]. \tag{5.45}$$

Proof. For each l and j, using the Ascoli-Arzela theorem [17] and Lemma 5.2.1, we get a subsequence of $\left\{m_{j,n}^l(t) : n \ge \frac{1}{2}(l+1), t \in [0, T]\right\}$ that converges uniformly to a limit which we denote by $m_{j,*}^l(t)$. Taking intersections of these subsequences successively with respect to j and applying a diagonal selection principle with respect to l, we obtain a sequence k_n, such that

$$\lim_{k_n \to \infty} \vec{m}^l_{k_n}(t) = \vec{m}^l_*(t), \quad \forall t \in [0,T] \quad \square$$

Given a sequence $\{m_n\}_{n=0}^{\infty}$, define the following differences

$$\begin{aligned} \triangle^0 m_n &= m_n, \\ \triangle^k m_n &= \triangle^{k-1} m_n - \triangle^{k-1} m_{n+1}, \quad k = 1, 2, \cdots. \end{aligned} \tag{5.46}$$

We have the following result concerning with the classical moment problem [10].

Theorem 5.2.1 *A necessary and sufficient condition for the existence of a solution of the Hausdorff moment problem, i.e., the existence of a unique nonnegative measure* μ *satisfying*

$$m_l = \int_0^1 x^l d\mu, l = 0, 1, 2, \cdots,$$

is that

$$\triangle^k m_l \geq 0, \quad k, l = 0, 1, 2, \cdots.$$

Lemma 5.2.3 *Given* N_y*, for each* $0 \leq j \leq N_y$ *and for any* $t \in [0,T]$*, the elements of the sequence* $\left\{m^l_{j,*}(t)\right\}_{l=0}^{\infty}$ *are the moments of a nonnegative measure, i.e., there exists a nonnegative measure* $dP_{j,*}(x,t)$*, such that for each* $l \geq 0$ *and each* $0 \leq j \leq N_y$*, we have*

$$\int_0^1 x^l dP_{j,*}(x,t) = m^l_{j,*}(t), \quad \forall t \in [0,T]. \tag{5.47}$$

Proof. For given j and n, $\left\{m_{j,n}^l(t), l \geq 0\right\}$ are moments of the measure $d\mu_j^n(t)$. By Theorem 5.2.1, the related differences satisfy

$$0 \leq \Delta^i \vec{m}_n^l(t), \quad \forall i = 0, 1, 2, \cdots, \text{ and } \quad l \leq 2n-1.$$

By Lemma 5.2.2, for any l, we have

$$\lim_{k_n \to \infty} \vec{m}_{k_n}^l(t) = \vec{m}_*^l(t), \quad \forall t \in [0, T].$$

Hence

$$0 \leq \Delta^i \vec{m}_*^l(t), \quad \forall i, l = 0, 1, 2, \cdots.$$

Thus, for each j, there exists a nonnegative measure $dP_{j,*}(x,t)$ such that

$$m_{j,*}^l(t) = \int_0^1 x^l dP_{j,*}(x,t). \quad \square$$

Let $d\vec{P}_*(x,t) = (dP_{0,*}(x,t), dP_{1,*}(x,t), \cdots, dP_{m,*}(x,t))^T$. We have the following corollary.

Corollary 5.2.1 *Given N_y, for any $f \in C[0,1]$, we have*

$$\lim_{k_n \to \infty} \int_0^1 f(x) d\vec{\mu}^{k_n}(x,t) = \int_0^1 f(x) d\vec{P}_*(x,t), \quad \forall t \in [0, T]. \tag{5.48}$$

Proof. For every fixed integer $l \geq 0$, by Lemma 5.2.2 and Lemma 5.2.3 we have

$$\lim_{k_n \to \infty} \int_0^1 x^l d\vec{\mu}^{k_n}(x,t) = \lim_{k_n \to \infty} \vec{m}_{k_n}^l(t) = \vec{m}_*^l(t) = \int_0^1 x^l d\vec{P}_*(x,t), \quad \forall t \in [0, T]$$

By the well-known Weierstrass approximation theorem, $\left\{x^l\right\}_{l=0}^{\infty}$ is dense in $C[0,T]$. So for any continuous function f,

$$\lim_{k_n \to \infty} \int_0^1 f(x) d\vec{\mu}^{k_n}(x,t) = \int_0^1 f(x) d\vec{P}_*(x,t), \quad \forall t \in [0,T] \quad \square$$

Using (5.15) and definition of L_1^*, we could write (4.22) as follows,

$$\frac{d}{dt}\vec{m}_n^l(t) = \int_0^1 L_1^*(x^l) d\vec{\mu}^n(t) \tag{5.49}$$

Lemma 5.2.4 *Given N_y, for each l, we have*

$$\vec{m}_*^l(t) - \vec{m}_*^l(0) = \int_0^t \int_0^1 L_1^*(x^l) d\vec{P}_*(x,s) ds = \int_0^t \int_0^1 L_1^*(x^l) \vec{p}_*(x,s) dx ds. \tag{5.50}$$

Proof. Integrating (5.49) over $(0,t)$, we get

$$\vec{m}_{k_n}^l(t) - \vec{m}_{k_n}^l(0) = \int_0^t \int_0^1 L_1^*(x^l) d\vec{\mu}^{k_n}(s) ds$$

Similar to the proof of inequality (5.26), we have

$$\int_0^1 L_1^*(x^l) d\vec{\mu}^{k_n}(s) \leq K_1 \vec{m}_{k_n}^l(t) + K_2 \vec{m}_{k_n}^{l-1}(t) + K_3 \vec{m}_{k_n}^{l-2}(t) + \|f(t)\|_0 \vec{\eta}_4$$

By the Lebesgue dominated convergence theorem, Lemma 5.2.2 and Corollary 5.2.1, we obtain as $k_n \to \infty$,

$$\vec{m}_*^l(t) - \vec{m}_*^l(0) = \int_0^t \int_0^1 L_1^*(x^l) d\vec{P}_*(x,s) ds = \int_0^t \int_0^1 L_1^*(x^l) \vec{p}_*(x,s) dx \quad \square$$

Theorem 5.2.2 *Under all the previous assumptions, given N_y, for any function $f \in$*

$C[0,1]$, *we have*

$$\lim_{n\to\infty}\int_0^1 f(x)d\vec{\mu}_j^n(x,t) = \int_0^1 f(x)\vec{\alpha}_j(x,t)dx \tag{5.51}$$

Proof. From Lemma 5.2.4, for any l, we have

$$(x^l, \vec{p}_*(x,t)) - (x^l, \vec{p}_*(x,0)) = \int_0^t \int_0^1 L_1^*(x^l)\vec{p}_*(x,s)dxds$$

Since $\{x^l\}$ is dense in $C[0,1]$, we see that $\{\vec{p}_*(x,t)\}$ are the weak solution of (4.7). Since (4.7) has a unique solution, we conclude that the limit of $\{\vec{\mu}^{k_n}\}$ is independent of the choice of subsequences. It follows that the whole sequence is convergent. This completes the proof of (5.51) . $\square$

Theorem 5.2.3 *Under all the previous assumptions, for any continuous functions* $g(x)$ *and* $h(y)$ *in* $[0,1]$, *we have*

$$\lim_{N_y\to\infty}\lim_{n\to\infty}\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)d\mu_j^n(t)dy = \int_0^1\int_0^1 u(x,y,t)h(y)g(x)dxdy. \tag{5.52}$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)d\mu_j^n(t)dy - \int_0^1\int_0^1 u(x,y,t)h(y)g(x)dxdy \\
&= \left\{\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)d\mu_j^n(t)dy - \sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)\alpha_j(x,t)dxdy\right\} \\
&+ \left\{\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)\alpha_j(x,t)dxdy - \int_0^1\int_0^1 u(x,y,t)h(y)g(x)dxdy\right\}.
\end{aligned} \tag{5.53}$$

For fixed N_y, using Theorem 5.2.2, we get

$$\lim_{n\to\infty}\left\{\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)d\mu_j^n(t)dy-\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)\alpha_j(x,t)dxdy\right\}$$

$$=\lim_{n\to\infty}\sum_{j=0}^{N_y}\int_0^1 h(y)\phi_j(y)\left\{\int_0^1 g(x)(d\mu_j^n(t)-\alpha_j(x,t)dx)\right\}dy=0.$$

Now, we estimate the second term in (5.53). Since

$$\begin{aligned}
&\left|\sum_{j=0}^{N_y}\int_0^1\int_0^1 h(y)g(x)\phi_j(y)\alpha_j(x,t)dxdy-\int_0^1\int_0^1 u(x,y,t)h(y)g(x)dxdy\right|\\
&=\left|\int_0^1 g(x)\int_0^1 h(y)\left\{\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right\}dydx\right|\\
&\le\int_0^1 g(x)\int_0^1 h(y)\left|\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right|dydx\\
&\le\int_0^1 g(x)\left\{\int_0^1|h(y)|^2dy\right\}^{\frac{1}{2}}\left\{\int_0^1\left|\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right|^2dy\right\}^{\frac{1}{2}}dx\\
&=\left\{\int_0^1|h(y)|^2dy\right\}^{\frac{1}{2}}\int_0^1 g(x)\left\{\int_0^1\left|\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right|^2dy\right\}^{\frac{1}{2}}dx\\
&\le\left\{\int_0^1|h(y)|^2dy\right\}^{\frac{1}{2}}\left\{\int_0^1|g(x)|^2dx\right\}^{\frac{1}{2}}\int_0^1\int_0^1\left|\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right|^2dydx.
\end{aligned}\tag{5.54}$$

In Theorem 5.1.1, by choosing $u_{0,h}=u_0$, we get

$$\max_{0\le t\le T}\|u(t)-u_h(t)\|_0^2\le C_{\alpha,\gamma}h^2(\|u_0\|_1^2+\int_0^T\|f\|_0^2dt).\tag{5.55}$$

We then obtain

$$\int_0^1\int_0^1\left|\sum_{j=0}^{N_y}\phi_j(y)\alpha_j(x,t)-u(x,y,t)\right|^2dydx\le C_{\alpha,\gamma}h^2\int_0^1(\|u_0\|_1^2+\int_0^T\|f\|_0^2dt)dx.\tag{5.56}$$

Therefore,

$$\lim_{N_y \to \infty} \left\{ \sum_{j=0}^{N_y} \int_0^1 \int_0^1 h(y)g(x)\phi_j(y)\alpha_j(x,t)dxdy - \int_0^1 \int_0^1 u(x,y,t)h(y)g(x)dxdy \right\} = 0$$

This proves (5.52). □

CHAPTER 6

Numerical Results

In this chapter, we shall consider the following partial differential equations

$$u_t = -(au)_x + \frac{1}{2}(b^2u)_{xx} + u_{yy}. \tag{6.1}$$

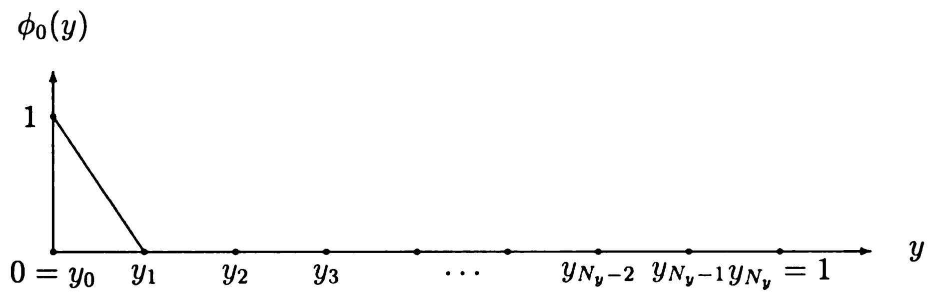
(6.1) models a family of singular processes with singularities at $x = 0$ and $x = 1$. We shall use the Gauss-Galerkin finite element method to find the solutions. We use the finite element method in the y-direction with piece-wise linear finite element space and the Gauss-Galerkin method in the x-direction to solve the proposed problems. Our numerical results will show that the Gauss-Galerkin finite element method is a efficient method dealing with a variety of such singular problems.

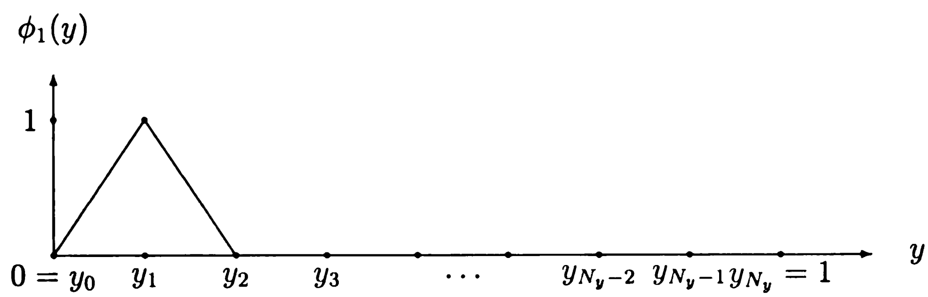
6.1 Piecewise Linear Finite Element Space

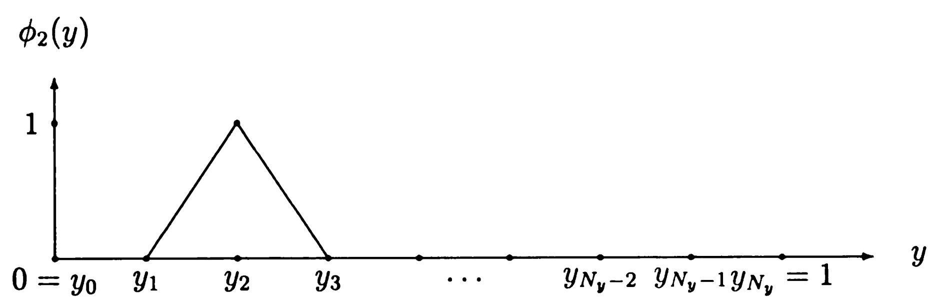
We choose grid points $\{y_j\}_{j=0}^{N_y}$ by dividing [0,1] into N_y equal subintervals with $h = \frac{1}{N_y}, y_j = jh, j = 0, 1, \cdots, N_y$, such that

$$0 = y_0 < y_1 < y_2 < ... < y_j < ... < y_{N_y} = 1.$$

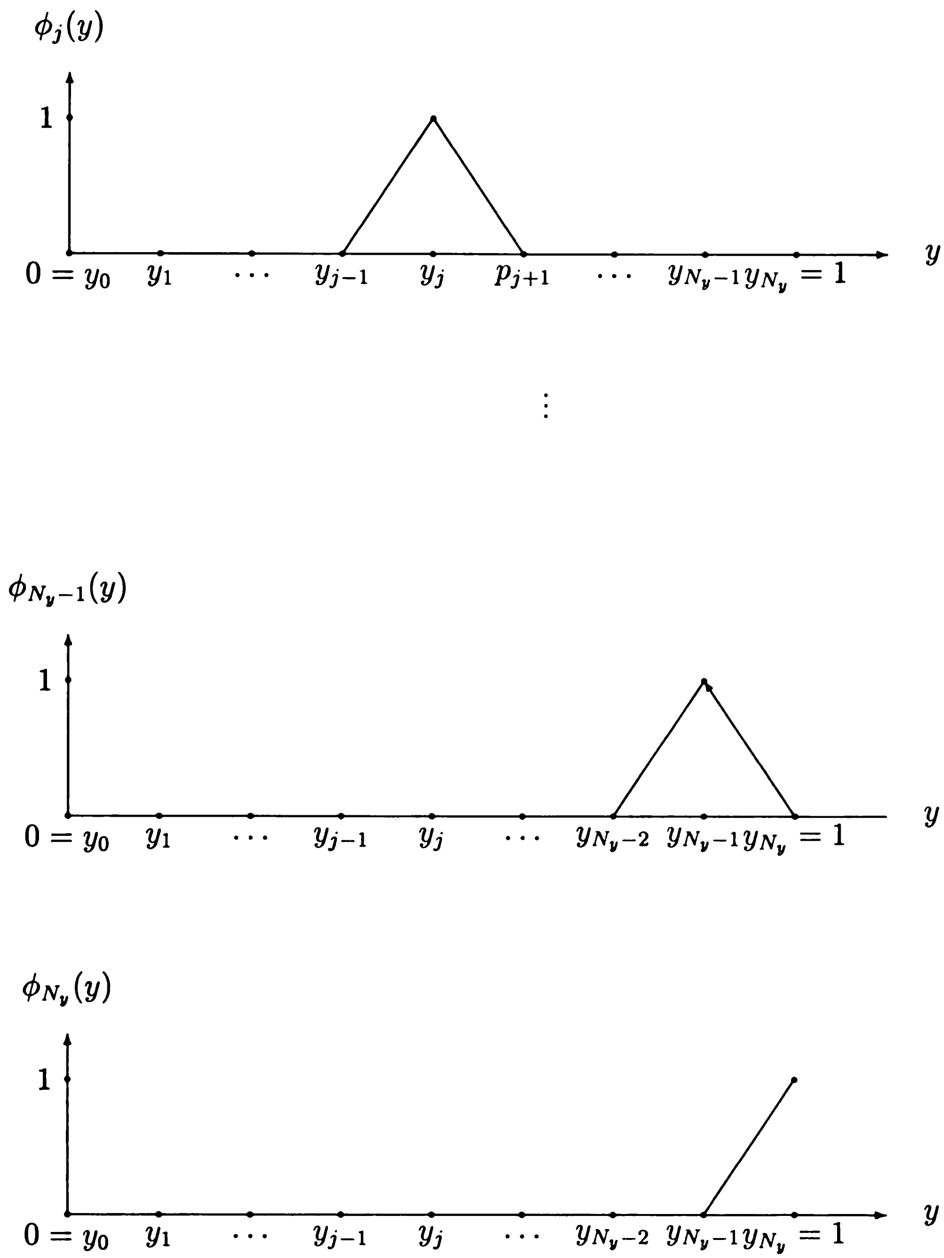
Let $V_h \subset H^1(0,1)$ be a finite dimensional subspace and $\phi_j(y)$ be base functions of V_h sketched as follows:



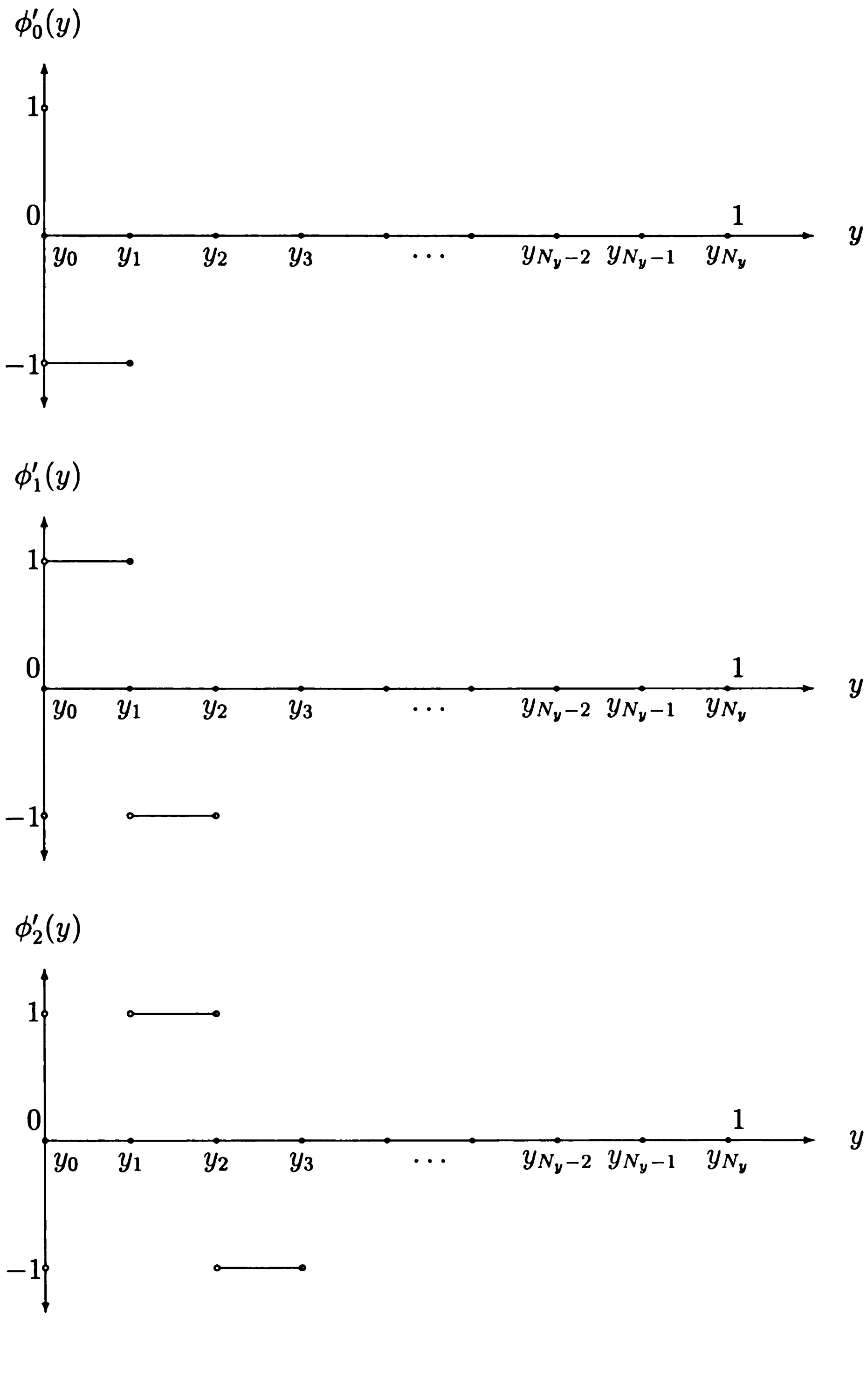




$\vdots$

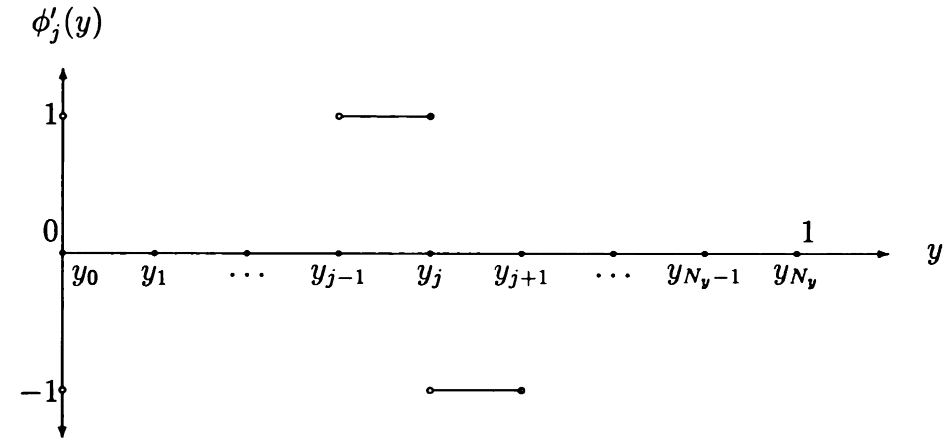


$\phi_j(y)$
1
$0 = y_0$ y_1 $\cdots$ y_{j-1} y_j p_{j+1} $\cdots$ y_{N_y-1} $y_{N_y} = 1$
y
$\vdots$
$\phi_{N_y-1}(y)$
1
$0 = y_0$ y_1 $\cdots$ y_{j-1} y_j $\cdots$ y_{N_y-2} y_{N_y-1} $y_{N_y} = 1$
y
$\phi_{N_y}(y)$
1
$0 = y_0$ y_1 $\cdots$ y_{j-1} y_j $\cdots$ y_{N_y-2} y_{N_y-1} $y_{N_y} = 1$
y

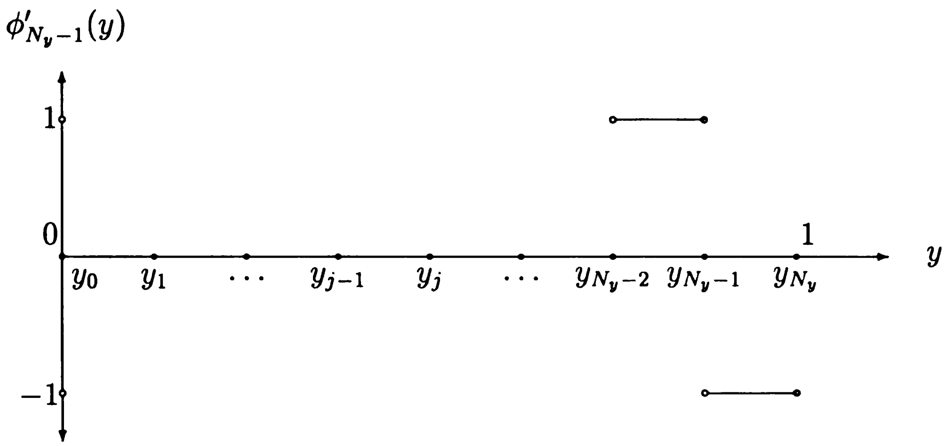


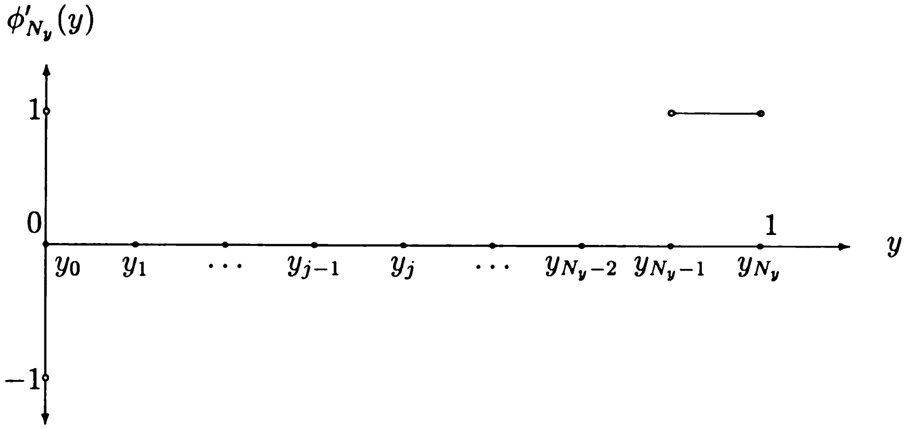
$\phi_0'(y)$
1
0
1
y
y_0 y_1 y_2 y_3 $\cdots$ y_{N_y-2} y_{N_y-1} y_{N_y}
-1
$\phi_1'(y)$
1
0
1
y
y_0 y_1 y_2 y_3 $\cdots$ y_{N_y-2} y_{N_y-1} y_{N_y}
-1
$\phi_2'(y)$
1
0
1
y
y_0 y_1 y_2 y_3 $\cdots$ y_{N_y-2} y_{N_y-1} y_{N_y}
-1

$\vdots$


$\phi'_j(y)$
1
0
1
y
y_0 y_1 ⋯ y_{j-1} y_j y_{j+1} ⋯ y_{N_y-1} y_{N_y}
−1

$\vdots$


$\phi'_{N_y-1}(y)$
1
0
1
y
y_0 y_1 ⋯ y_{j-1} y_j ⋯ y_{N_y-2} y_{N_y-1} y_{N_y}
−1



$$\phi_0(y) = \begin{cases} -\frac{y-h}{h}, & \text{when } 0 \le y < y_1 \\ 0, & \text{when } y_1 \le y \le 1 \end{cases}$$

$$\phi_1(y) = \begin{cases} \frac{y}{h}, & \text{when } 0 \le y < y_1 \\ -\frac{y-y_2}{h}, & \text{when } y_1 \le y < y_2 \\ 0, & \text{when } y_2 \le y \le 1 \end{cases}$$

$$\phi_2(y) = \begin{cases} 0, & \text{when } 0 \le y < y_1 \\ \frac{y-y_1}{h}, & \text{when } y_1 \le y < y_2 \\ -\frac{y-y_3}{h}, & \text{when } y_2 \le y < y_3 \\ 0, & \text{when } y_3 \le y \le 1 \end{cases}$$

$$\vdots$$

$$\phi_{j-1}(y) = \begin{cases} 0, & \text{when } 0 \le y < y_{j-2} \\ \frac{y-y_{j-2}}{h}, & \text{when } y_{j-2} \le y < y_{j-1} \\ -\frac{y-y_j}{h}, & \text{when } y_{j-1} \le y < y_j \\ 0, & \text{when } y_j \le y \le 1 \end{cases}$$

$$\phi_{j}(y) = \begin{cases} 0, & \text{when } 0 \le y < y_{j-1} \\ \frac{y-y_{j-1}}{h}, & \text{when } y_{j-1} \le y < y_{j} \\ -\frac{y-y_{j+1}}{h}, & \text{when } y_{j} \le y < y_{j+1} \\ 0, & \text{when } y_{j+1} \le y \le 1 \end{cases}$$

$$\phi_{j+1}(y) = \begin{cases} 0, & \text{when } 0 \le y < y_{j} \\ \frac{y-y_{j}}{h}, & \text{when } y_{j} \le y < y_{j+1} \\ -\frac{y-y_{j+2}}{h}, & \text{when } y_{j+1} \le y < y_{j+2} \\ 0, & \text{when } y_{j+2} \le y \le 1 \end{cases}$$

$$\vdots$$

$$\phi_{N_y-1}(y) = \begin{cases} 0, & \text{when } 0 \le y < y_{N_y-2} \\ \frac{y-y_{N_y-2}}{h}, & \text{when } y_{N_y-2} \le y < y_{N_y-1} \\ -\frac{y-1}{h}, & \text{when } y_{N_y-1} \le y \le 1 \end{cases}$$

$$\phi_{N_Y}(y) = \begin{cases} 0, & \text{when } 0 \le y < y_{N_y-1} \\ \frac{y-y_{N_y-1}}{h}, & \text{when } y_{N_y-1} \le y \le 1 \end{cases}$$

We have

$$\phi_0'(y) = \begin{cases} -\frac{1}{h}, & \text{when } 0 < y < y_1 \\ 0, & \text{when } y_1 < y < 1 \end{cases}$$

$$\phi_1'(y) = \begin{cases} \frac{1}{h}, & \text{when } 0 < y < y_1 \\ -\frac{1}{h}, & \text{when } y_1 < y < y_2 \\ 0, & \text{when } y_2 < y < 1 \end{cases}$$

$$\phi_2'(y) = \begin{cases} 0, & \text{when } 0 < y < y_1 \\ \frac{1}{h}, & \text{when } y_1 < y < y_2 \\ -\frac{1}{h}, & \text{when } y_2 < y < y_3 \\ 0, & \text{when } y_3 < y < 1 \end{cases}$$

$$\vdots$$

$$\phi_{j-1}'(y) = \begin{cases} 0, & \text{when } 0 < y < y_{j-2} \\ \frac{1}{h}, & \text{when } y_{j-2} < y < y_{j-1} \\ -\frac{1}{h}, & \text{when } y_{j-1} < y < y_j \\ 0, & \text{when } y_j < y < 1 \end{cases}$$

$$\phi_j'(y) = \begin{cases} 0, & \text{when } 0 < y < y_{j-1} \\ \frac{1}{h}, & \text{when } y_{j-1} < y < y_j \\ -\frac{1}{h}, & \text{when } y_j < y < y_{j+1} \\ 0, & \text{when } y_{j+1} < y < 1 \end{cases}$$

$$\phi_{j+1}'(y) = \begin{cases} 0, & \text{when } 0 < y < y_j \\ \frac{1}{h}, & \text{when } y_j < y < y_{j+1} \\ -\frac{1}{h}, & \text{when } y_{j+1} < y < y_{j+2} \\ 0, & \text{when } y_{j+2} < y < 1 \end{cases}$$

$$\vdots$$

$$\phi_{N_y-1}'(y) = \begin{cases} 0, & \text{when } 0 < y < y_{N_y-2} \\ \frac{1}{h}, & \text{when } y_{N_y-2} < y < y_{N_y-1} \\ -\frac{1}{h}, & \text{when } y_{N_y-1} < y < 1 \end{cases}$$

$$\phi_{N_y}'(y) = \begin{cases} 0, & \text{when } 0 < y < y_{N_y-1} \\ \frac{1}{h}, & \text{when } y_{N_y-1} < y < 1 \end{cases}$$

6.2 A Test Problem

In this section, we shall consider the following partial differential equation:

$$u_t = (x(1-x)u)_{xx} + u_{yy} \tag{6.2}$$

with the boundary conditions

$$\lim_{x\to 0} xu = \lim_{x\to 1}(1-x)u = 0 \qquad \text{on} \quad (0,T)\times(0,1) \tag{6.3}$$

$$u_y(x,0,t) = u_y(x,1,t) = 0 \qquad \text{on} \quad (0,T)\times(0,1) \tag{6.4}$$

and initial condition

$$u(x,y,0) = 1-\cos(\pi y) \quad \text{on} \quad \Omega. \tag{6.5}$$

As we shall see below, the boundary terms corresponding to (4.16) do not drop out. We shall show how to handle such terms by interpreting them as fluxes across the boundaries at $x = 0$ and $x = 1$ and the moments of such fluxes. By redefining a new "total" U including that due to the fluxes and new "total" moments M^i, a new system for such U and M^i results. Such a new system can readily be analyzed by the method in Chapter 5 and convergence results easily follow.

The right hand side of (6.2) is the divergence of the vector field $((x(1-x)u)_x, u_y)$. Let $\mathrm{P}(x,y,t)$ express the flux across x in the positive direction and $\mathrm{Q}(x,y,t)$ express the flux across y in the positive direction at time t respectively. We have

$$\mathrm{P}(x,y,t) = -(x(1-x)u)_x \tag{6.6}$$

$$\mathrm{Q}(x,y,t) = -u_y. \tag{6.7}$$

$P(0,y,t), P(1,y,t), Q(x,0,t)$ and $Q(x,1,t)$ express the flux across $x = 0, x = 1, y = 0$ and $y = 1$ in the positive direction at time t respectively.

By (6.6), we have

$$P(x,y,t) = -(x(1-x)u)_x = -(1-x)u + xu - x(1-x)u_x. \tag{6.8}$$

So,

$$P(0,y,t) = -u(0,y,t) \quad \text{and} \quad P(1,y,t) = u(1,y,t). \tag{6.9}$$

Integrating (6.2) over Ω, integrating by parts and using (6.4) and (6.9) we have

$$\begin{aligned} \frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy &= \int_0^1\int_0^1((x(1-x)u)_{xx} + u_{yy})dxdy \\ &= \int_0^1 (x(1-x)u)_x|_{x=0}^1 dy + \int_0^1 u_y|_{y=0}^1 dx \\ &= \int_0^1 -P(x,y,t)|_{x=0}^1 dy \\ &= \int_0^1 -\{P(1,y,t) - P(0,y,t)\}dy \\ &= -\int_0^1 u(0,y,t)dy - \int_0^1 u(1,y,t)dy. \end{aligned} \tag{6.10}$$

Integrating (6.10) over $\tau \in (0,t)$, we have

$$\begin{aligned} &\textstyle\int_0^1\int_0^1 u(x,y,t)dxdy - \int_0^1\int_0^1 u(x,y,0)dxdy \\ &= -\textstyle\int_0^t\int_0^1 u(0,y,\tau)dyd\tau - \int_0^t\int_0^1 u(1,y,\tau)dyd\tau. \end{aligned} \tag{6.11}$$

So,

$$\begin{aligned} &\int_0^1\int_0^1 u(x,y,t)dxdy + \int_0^t\int_0^1 u(0,y,\tau)dyd\tau + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau \\ &= \int_0^1\int_0^1 u(x,y,0)dxdy. \end{aligned} \tag{6.12}$$

Multiplying by x and integrating (6.2) over Ω, integrating by parts and using (6.4) and (6.9), we have

$$\begin{aligned}\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)xdxdy &= \int_0^1\int_0^1((x(1-x)u)_{xx}x + u_{yy}x)dxdy\\ &= \int_0^1 (x(1-x)u)_x x|_{x=0}^1 dy + \int_0^1 u_y|_{y=0}^1 xdx\\ &= \int_0^1 -\mathrm{P}(x,y,t)x|_{x=0}^1 dy\\ &= \int_0^1 -\mathrm{P}(1,y,t)dy = -\int_0^1 u(1,y,t)dy.\end{aligned} \tag{6.13}$$

Integrating (6.13) over $\tau \in (0,t)$, we have

$$\int_0^1\int_0^1 u(x,y,t)xdxdy - \int_0^1\int_0^1 u(x,y,0)xdxdy = -\int_0^t\int_0^1 u(1,y,\tau)dyd\tau \tag{6.14}$$

So,

$$\int_0^1\int_0^1 u(x,y,t)xdxdy + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau = \int_0^1\int_0^1 u(x,y,0)xdxdy \tag{6.15}$$

where $\int_0^t\int_0^1 u(0,y,\tau)dyd\tau$ and $\int_0^t\int_0^1 u(1,y,\tau)dyd\tau$ represent the moments of the fluxes across the boundaries $x=0$ and $x=1$.

Multiplying by $x^k, k=2,3,\cdots,2n-1$, integrating (6.2) over Ω, integrating by parts and using (6.4) and (6.9), we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)x^k dxdy = \int_0^1\int_0^1((x(1-x)u)_{xx}x^k + u_{yy}x^k)dxdy\\ &= \int_0^1 (x(1-x)u)_x x^k|_{x=0}^1 dy - \int_0^1 x(1-x)ukx^{k-1}|_{x=0}^1 dy\\ &+\int_0^1\int_0^1 x(1-x)uk(k-1)x^{k-2}dxdy + \int_0^1 u_y|_{y=0}^1 xdx\\ &= \int_0^1 -\mathrm{P}(x,y,t)x^k|_{x=0}^1 dy + \int_0^1\int_0^1 x(1-x)uk(k-1)x^{k-2}dxdy\\ &= \int_0^1 -\mathrm{P}(1,y,t)dy + \int_0^1\int_0^1 x(1-x)uk(k-1)x^{k-2}dxdy\\ &= -\int_0^1 u(1,y,t)dy + \int_0^1\int_0^1 x(1-x)uk(k-1)x^{k-2}dxdy.\end{aligned} \tag{6.16}$$

Integrating (6.16) over $\tau \in (0,t)$, we have

$$\begin{aligned}&\int_0^1\int_0^1 u(x,y,t)x^k dxdy - \int_0^1\int_0^1 u(x,y,0)x^k dxdy\\ &= -\int_0^t\int_0^1 u(1,y,\tau)dyd\tau + \int_0^t\int_0^1\int_0^1 x(1-x)u(x,y,\tau)k(k-1)x^{k-2}dxdyd\tau.\end{aligned} \tag{6.17}$$

So,

$$\begin{aligned}&\int_0^1\int_0^1 u(x,y,t)x^k dxdy + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau\\ &= \int_0^1\int_0^1 u(x,y,0)x^k dxdy + \int_0^t\int_0^1\int_0^1 x(1-x)u(x,y,\tau)k(k-1)x^{k-2}dxdyd\tau\end{aligned} \tag{6.18}$$

On the other hand, we can rewrite (6.10), 6.13) and (6.16) as

$$\frac{d}{dt}\int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} dxdy = 0 \tag{6.19}$$

$$\frac{d}{dt}\int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} xdxdy = 0 \tag{6.20}$$

and, for $k = 2, 3, \cdots, 2n-1$,

$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} x^k dxdy\\ &= \int_0^1\int_0^1 x(1-x)\left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau\right.\\ &\qquad \left. + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} k(k-1)x^{k-2}dxdy.\end{aligned} \tag{6.21}$$

Integrating (6.19), (6.20) and (6.21) over $\tau \in (0,t)$, we have

$$\begin{aligned}&\int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} dxdy\\ &= \int_0^1\int_0^1 \left\{u(x,y,0) + \int_0^0 u(x,y,\tau)\delta(x)d\tau + \int_0^0 u(x,y,\tau)\delta(x-1)d\tau\right\} dxdy\end{aligned} \tag{6.22}$$

$$\int_0^1\int_0^1\left\{u(x,y,t)+\int_0^t u(x,y,\tau)\delta(x)d\tau+\int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\}xdxdy$$
$$=\int_0^1\int_0^1\left\{u(x,y,0)+\int_0^0 u(x,y,\tau)\delta(x)d\tau+\int_0^0 u(x,y,\tau)\delta(x-1)d\tau\right\}xdxdy \tag{6.23}$$

and, for $k=2,3,\cdots,2n-1$,

$$\int_0^1\int_0^1\left\{u(x,y,t)+\int_0^t u(x,y,\tau)\delta(x)d\tau+\int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\}x^kdxdy$$
$$=\int_0^1\int_0^1\left\{u(x,y,0)+\int_0^0 u(x,y,\tau)\delta(x)d\tau+\int_0^0 u(x,y,\tau)\delta(x-1)d\tau\right\}x^kdxdy$$
$$+\int_0^1\int_0^1\int_0^t x(1-x)\left\{u(x,y,\tau)+\int_0^\tau u(x,y,\tau')\delta(x)d\tau'\right.$$
$$\left.+\int_0^\tau u(x,y,\tau')\delta(x-1)d\tau'\right\}k(k-1)x^{k-2}d\tau dxdy. \tag{6.24}$$

We now let

$$U(x,y,t)=u(x,y,t)+\int_0^t u(x,y,\tau)\delta(x)d\tau+\int_0^t u(x,y,\tau)\delta(x-1)d\tau \tag{6.25}$$

We can the rewrite (6.19), (6.20) and (6.21) as

$$\frac{d}{dt}\int_0^1\int_0^1 U(x,y,t)dxdy=0 \tag{6.26}$$

$$\frac{d}{dt}\int_0^1\int_0^1 U(x,y,t)xdxdy=0 \tag{6.27}$$

and, for $k=2,3,\cdots,2n-1$,

$$\frac{d}{dt}\int_0^1\int_0^1 U(x,y,t)x^kdxdy=\int_0^1\int_0^1 x(1-x)U(x,y,t)k(k-1)x^{k-2}dxdy \tag{6.28}$$

We can also rewrite (6.22), (6.23) and (6.24) as

$$\int_0^1\int_0^1 U(x,y,t)dxdy=\int_0^1\int_0^1 U(x,y,0)dxdy \tag{6.29}$$

$$\int_0^1\int_0^1 U(x,y,t)xdxdy=\int_0^1\int_0^1 U(x,y,0)xdxdy \tag{6.30}$$

and, for $k = 2, 3, \cdots, 2n-1$,

$$\int_0^1\int_0^1 U(x,y,t)x^k dxdy = \int_0^1\int_0^1 U(x,y,0)x^k dxdy \\ +\int_0^1\int_0^1\int_0^t x(1-x)U(x,y,\tau)k(k-1)x^{k-2}d\tau dxdy. \tag{6.31}$$

Let us now calculate a few quantities:

(1) The exact solution of (6.2) with boundary conditions (6.3),(6.4) and initial condition (6.5) is

$$u(x,y,t) = e^{-2t} - \cos(\pi y)e^{-(\pi^2+2)t} \quad \text{on} \quad (0,T)\times\Omega \tag{6.32}$$

(2)

$$\int_0^1\int_0^1 u(x,y,0)x^k dxdy = \frac{1}{k+1}, \quad k = 0,1,2,\cdots,2n-1. \tag{6.33}$$

(3)

$$\int_0^1 u(0,y,t)dy = \int_0^1 u(1,y,t)dy = e^{-2t}. \tag{6.34}$$

(4)

$$\int_0^t\int_0^1 u(0,y,\tau)dyd\tau = \int_0^t\int_0^1 u(1,y,\tau)dyd\tau = \frac{1}{2}(1-e^{-2t}). \tag{6.35}$$

(5)

$$\int_0^1\int_0^1 u(x,y,t)x^k dxdy = \frac{1}{k+1}e^{-2t}, \quad k = 0,1,2,\cdots,2n-1. \tag{6.36}$$

We denote by $M^i(t)$ the ith total moment

$$M^i(t) = \int_0^1\int_0^1 U(x,y,t)x^i dxdy. \tag{6.37}$$

Then we have

(6)

$$\int_0^1\int_0^1 U(x,y,0)x^k dxdy = \int_0^1\int_0^1 u(x,y,0)x^k dxdy = \frac{1}{k+1},\ k=0,1,\cdots,2n-1. \tag{6.38}$$

(7)

$$\begin{aligned}
M^0(t) &= \int_0^1\int_0^1 U(x,y,t)dxdy \\
&= \int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} dxdy \\
&= \int_0^1\int_0^1 u(x,y,t)dxdy + \int_0^t\int_0^1 u(0,y,\tau)dyd\tau + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau \\
&= e^{-2t} + \frac{1}{2}(1-e^{-2t}) + \frac{1}{2}(1-e^{-2t}) = 1.
\end{aligned} \tag{6.39}$$

(8)

$$\begin{aligned}
M^1(t) &= \int_0^1\int_0^1 U(x,y,t)xdxdy \\
&= \int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} xdxdy \\
&= \int_0^1\int_0^1 u(x,y,t)xdxdy + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau \\
&= \frac{1}{2}e^{-2t} + \frac{1}{2}(1-e^{-2t}) = \frac{1}{2}.
\end{aligned} \tag{6.40}$$

(9) For $k = 2,\cdots,2n-1$,

$$\begin{aligned}
M^k(t) &= \int_0^1\int_0^1 U(x,y,t)x^k dxdy \\
&= \int_0^1\int_0^1 \left\{u(x,y,t) + \int_0^t u(x,y,\tau)\delta(x)d\tau + \int_0^t u(x,y,\tau)\delta(x-1)d\tau\right\} x^k dxdy \\
&= \int_0^1\int_0^1 u(x,y,t)x^k dxdy + \int_0^t\int_0^1 u(1,y,\tau)sy d\tau \\
&= \frac{1}{k+1}e^{-2t} + \frac{1}{2}(1-e^{-2t}).
\end{aligned} \tag{6.41}$$

For the steady state solution, we have

(10)

$$\int_0^1 u(0,y,\infty)dy = \int_0^1 u(1,y,\infty)dy = 0 \tag{6.42}$$

(11)

$$\int_0^\infty \int_0^1 u(0,y,\tau)dyd\tau = \int_0^\infty \int_0^1 u(1,y,\tau)dyd\tau = \frac{1}{2} \tag{6.43}$$

(12)

$$\int_0^1 \int_0^1 u(x,y,\infty)x^k dxdy = 0, \quad k = 0,1,2,\cdots,2n-1 \tag{6.44}$$

This implies that

$$u(x,y,\infty) = 0, \quad (x,y) \in \Omega \tag{6.45}$$

In this test problem, $x = 0$ and $x = 1$ are exit boundaries. For the steady state, u will "pile" at $x = 0$ and $x = 1$ with a Dirac-delta function singularity formed at $x = 0$ and $x = 1$.

(13)

$$M^0(\infty) = 1 \quad \text{and} \quad M^k(\infty) = \frac{1}{2}, \quad k = 1,2,\cdots,2n-1. \tag{6.46}$$

We apply the Gauss-Galerkin Finite Element method using finite element approximations in the y-direction and Gauss-Galerkin approximations in the x-direction. We divide $y \in [0,1]$ into four equal subintervals. Then we have five grid lines $y_j = jh, j = 0,1,2,3,4, h = 0.25$. For simplicity we use the Neumann boundary conditions to eliminate the unknowns along the grid lines $y = 0$ and $y = 1$ (though this is not necessary as the Neumann conditions are "natural".) In the x-direction we use both three nodes labeled by $*, +$ and x and five nodes labeled by $\diamond$, o, x, + and $*$, respectively, for each grid line $y = y_j$. We only show the numerical results along grid line $y = 0.5$. The results along grid lines $y = 0.25$ and $y = 0.75$ are similar. We use $*, +$ and x (or $\diamond$, o, x, + and $*$) to indicate the nodes moving to the boundary $x = 0$, interior point(s) and the boundary $x = 1$.

We difine by $m^i_{j,n}(t)$ the ith "moment" along the jth grid line $y = y_j$,

$$m^i_{j,n}(t) = \int_0^1 x^i d\mu^n_j(x,t) \tag{6.47}$$

and by $M^i_n(t)$ the ith "total moment"

$$M^i_n(t) = \sum_{j=0}^{N_y} \int_0^1 \int_0^1 x^i \phi_j(t) d\mu^n_j(x,t) dy \tag{6.48}$$

Figure 6.1 shows the movement of the three nodes as t increases. Table 6.1 shows the changes of the nodes $*, +$ and x as t increases. Table 6.2 shows the changes of the weights at the three nodes $*, +$ and x as t increases. Table 6.3 shows the changes of the ith "moment" $m^i_n(y,t)$ at $y = 0.5$ as t increases. Table 6.4 shows the changes of the "total moment " $M^i_n(t)$ as t increases.

Figure 6.2 shows the movement of the five nodes $\diamond$, o, x, $+$ and $*$ as t increases. Table 6.5 shows the changes of the five nodes $\diamond$, o, x, $+$ and $*$ as t increases. Table 6.6 shows the changes of the weights at the five nodes $\diamond$, o, x, $+$ and $*$ as t increases. Table 6.7 shows the changes of the ith "moment" $m^i_n(y,t)$ at $y = 0.5$ as t increases. Table 6.8 shows the changes of the "total moment " $M^i_n(t)$ as t increases. Table 6.9 shows the changes of the exact total moment $M^i(t)$ as t increases. Table 6.10 shows the changes of the errors between the total moment $M^i(t)$ and their approximation $M_n(t)$ as t increases. Table 6.11 shows the changes of the relative errors between the total moment $M^i(t)$ and their approximation $M_n(t)$ as t increases. We observe the following results:

(1) Table 6.8 shows that $M^0_n(t) = 1$ and $M^1_n(t) = 0.5$ for $t \geq 0$. This shows that the approximation of the 0th and 1st total moments equal the exact total moments.

(2) At $t = 1.5$, the solution reaches the steady state based on the tolerance chosen.

(3) The solution becomes uniform in y as t increases.

(4) The solution approaches zero in the interior and "piles" up at the boundary $x = 0$ and $x = 1$ as t increases.

(5) The weights for the interior nodes tend to zero as t increases.

(6) The weights at the other two nodes tend to 0.5 as t increases. They are the amounts of the fluxes leaking out from $x = 0$ and $x = 1$.

(7) Tables 6.10 and 6.11 show that the errors and relative errors between the exact total moments and their approximations are very small. So the approximation is very accurate.

(8) Table 6.4 and 6.8 shows that there is no significant difference for the approximations of the total moments between using three nodes and using five nodes in the x direction.

Table 6.1. Gauss-Galerkin Finite Element Method: Changes of the three nodes $*, +$ and x at $y = 0.5$ as t increases

time	x_1	x_2	x_3
0.00	0.1127016654	0.5	0.8872983346
0.05	0.08265386728	0.5	0.9173461327
0.10	0.06390525119	0.5	0.9360947488
0.15	0.05111110621	0.5	0.9488888938
0.20	0.04185349059	0.5	0.9581465094
0.25	0.03487139567	0.5	0.9651286043
0.30	0.02944063013	0.5	0.9705593699
0.35	0.02511492851	0.5	0.9748850715
0.40	0.02160400788	0.5	0.9783959921
0.45	0.0187107586	0.5	0.9812892414
0.50	0.01629656768	0.5	0.9837034323
0.55	0.01426107726	0.5	0.9857389227
0.60	0.01252981511	0.5	0.9874701849
0.65	0.01104633973	0.5	0.9889536603
0.70	0.009767086224	0.5	0.9902329138
0.75	0.008657887541	0.5	0.9913421125
0.80	0.007691567891	0.5	0.9923084321
0.85	0.006846241354	0.5	0.9931537586
0.90	0.006104085625	0.5	0.9938959144
0.95	0.005450442828	0.5	0.9945495572
1.00	0.004873149834	0.5	0.9951268502
1.05	0.004362032427	0.5	0.9956379676
1.10	0.003908518293	0.5	0.9960914817
1.15	0.003505337425	0.5	0.9964946626
1.20	0.003146287663	0.5	0.9968537123
1.25	0.002826049383	0.5	0.9971739506
1.30	0.002540037654	0.5	0.9974599623
1.35	0.002284283277	0.5	0.9977157167
1.40	0.002055336282	0.5	0.9979446637
1.45	0.001850187067	0.5	0.9981498129
1.50	0.001666201483	0.5	0.9983337985

Table 6.2. Gauss-Galerkin Finite Element Method: Changes of the weights at the three nodes $*$, $+$ and x at $y = 0.5$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.2777777778	0.4444444444	0.2777777778
0.05	0.2851871392	0.4296257215	0.2851871392
0.10	0.2992481303	0.4015037395	0.2992481303
0.15	0.3148991793	0.3702016414	0.3148991793
0.20	0.3304744604	0.3390510791	0.3304744604
0.25	0.3453345547	0.3093308906	0.3453345547
0.30	0.3592303075	0.281539385	0.3592303075
0.35	0.3720822962	0.2558354075	0.3720822962
0.40	0.3838908217	0.2322183565	0.3838908217
0.45	0.3946948176	0.2106103648	0.3946948176
0.50	0.4045515536	0.1908968929	0.4045515536
0.55	0.4135260497	0.1729479006	0.4135260497
0.60	0.4216853657	0.1566292687	0.4216853657
0.65	0.4290954774	0.1418090453	0.4290954774
0.70	0.4358195877	0.1283608245	0.4358195877
0.75	0.4419172578	0.1161654845	0.4419172578
0.80	0.4474440172	0.1051119655	0.4474440172
0.85	0.4524512591	0.09509748189	0.4524512591
0.90	0.4569863005	0.08602739892	0.4569863005
0.95	0.4610925419	0.07781491628	0.4610925419
1.00	0.4648096777	0.07038064451	0.4648096777
1.05	0.4681739356	0.06365212877	0.4681739356
1.10	0.4712183232	0.05756335362	0.4712183232
1.15	0.4739728751	0.05205424985	0.4739728751
1.20	0.4764648919	0.04707021628	0.4764648919
1.25	0.4787191679	0.0425616641	0.4787191679
1.30	0.480758206	0.0384835879	0.480758206
1.35	0.4826024173	0.03479516532	0.4826024173
1.40	0.4842703071	0.0314593858	0.4842703071
1.45	0.485778646	0.02844270799	0.485778646
1.50	0.4871426277	0.0257147447	0.4871426277

6.3 Dependence of Solution upon Parameters in the Singularities

In this section, we shall consider the following partial differential equations:

$$u_t = (x^p(1-x)^q u)_{xx} + u_{yy}. \tag{6.49}$$

The right hand side of (6.49) is the divergence of the vector field $((x^p(1-x)^q u)_x, u_y)$. Again, let $\mathrm{P}(x,y,t)$ express the flux across x in the positive direction and $\mathrm{Q}(x,y,t)$ express the flux across y in the positive direction at time t respectively, we have

$$\mathrm{P}(x,y,t) = -(x^p(1-x)^q u)_x \tag{6.50}$$

$$\mathrm{Q}(x,y,t) = -u_y \tag{6.51}$$

$\mathrm{P}(0,y,t), \mathrm{P}(1,y,t), \mathrm{Q}(x,0,t)$ and $\mathrm{Q}(x,1,t)$ express the fluxes across $x=0, x=1, y=0$ and $y=1$ in the positive direction at time t respectively.

Model I We consider the following initial-boundary problem with different parameters p and q:

$$u_t = (x^p(1-x)^q u)_{xx} + u_{yy} \quad \text{in} \quad (0,T)\times\Omega \tag{6.52}$$

with the boundary conditions

$$u(0,y,t) = u(1,y,t) = 0 \quad \text{on} \quad (0,T)\times(0,1), \tag{6.53}$$

$$u_y(x,0,t) = u_y(x,1,t) = 0 \quad \text{on} \quad (0,T)\times(0,1), \tag{6.54}$$

Table 6.3. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ with three nodes $*, +$ and x at $y = 0.5$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	1	0.5	0.333333	0.25	0.2	0.166667
0.05	1	0.5	0.349347	0.27402	0.228824	0.198693
0.10	1	0.5	0.363821	0.295732	0.254878	0.227642
0.15	1	0.5	0.376905	0.315358	0.278429	0.25381
0.20	1	0.5	0.388732	0.333098	0.299718	0.277464
0.25	1	0.5	0.399423	0.349134	0.318961	0.298845
0.30	1	0.5	0.409086	0.363629	0.336355	0.318172
0.35	1	0.5	0.417821	0.376731	0.352078	0.335642
0.40	1	0.5	0.425717	0.388575	0.36629	0.351433
0.45	1	0.5	0.432854	0.399281	0.379137	0.365707
0.50	1	0.5	0.439305	0.408958	0.390749	0.37861
0.55	1	0.5	0.445137	0.417705	0.401246	0.390273
0.60	1	0.5	0.450408	0.425612	0.410734	0.400816
0.65	1	0.5	0.455173	0.432759	0.419311	0.410345
0.70	1	0.5	0.45948	0.439219	0.427063	0.418959
0.75	1	0.5	0.463373	0.445059	0.434071	0.426745
0.80	1	0.5	0.466892	0.450338	0.440405	0.433784
0.85	1	0.5	0.470073	0.455109	0.446131	0.440146
0.90	1	0.5	0.472948	0.459422	0.451307	0.445896
0.95	1	0.5	0.475547	0.463321	0.455985	0.451095
1.00	1	0.5	0.477897	0.466845	0.460214	0.455793
1.05	1	0.5	0.48002	0.470031	0.464037	0.460041
1.10	1	0.5	0.48194	0.47291	0.467492	0.46388
1.15	1	0.5	0.483675	0.475513	0.470615	0.46735
1.20	1	0.5	0.485244	0.477866	0.473439	0.470487
1.25	1	0.5	0.486661	0.479992	0.475991	0.473323
1.30	1	0.5	0.487943	0.481915	0.478297	0.475886
1.35	1	0.5	0.489101	0.483652	0.480383	0.478203
1.40	1	0.5	0.490149	0.485223	0.482267	0.480297
1.45	1	0.5	0.491095	0.486643	0.483971	0.48219
1.50	1	0.5	0.491951	0.487926	0.485511	0.483901

Table 6.4. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ with three nodes $*$, $+$ and x at as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	1	0.5	0.333333	0.25	0.2	0.166667
0.05	1	0.5	0.349347	0.27402	0.228824	0.198693
0.10	1	0.5	0.363821	0.295732	0.254878	0.227642
0.15	1	0.5	0.376905	0.315358	0.278429	0.25381
0.20	1	0.5	0.388732	0.333098	0.299718	0.277464
0.25	1	0.5	0.399423	0.349134	0.318961	0.298845
0.30	1	0.5	0.409086	0.363629	0.336355	0.318172
0.35	1	0.5	0.417821	0.376731	0.352078	0.335642
0.40	1	0.5	0.425717	0.388575	0.36629	0.351433
0.45	1	0.5	0.432854	0.399281	0.379137	0.365707
0.50	1	0.5	0.439305	0.408958	0.390749	0.37861
0.55	1	0.5	0.445137	0.417705	0.401246	0.390273
0.60	1	0.5	0.450408	0.425612	0.410734	0.400816
0.65	1	0.5	0.455173	0.432759	0.419311	0.410345
0.70	1	0.5	0.45948	0.439219	0.427063	0.418959
0.75	1	0.5	0.463373	0.445059	0.434071	0.426745
0.80	1	0.5	0.466892	0.450338	0.440405	0.433784
0.85	1	0.5	0.470073	0.455109	0.446131	0.440146
0.90	1	0.5	0.472948	0.459422	0.451307	0.445896
0.95	1	0.5	0.475547	0.463321	0.455985	0.451095
1.00	1	0.5	0.477897	0.466845	0.460214	0.455793
1.05	1	0.5	0.48002	0.470031	0.464037	0.460041
1.10	1	0.5	0.48194	0.47291	0.467492	0.46388
1.15	1	0.5	0.483675	0.475513	0.470615	0.46735
1.20	1	0.5	0.485244	0.477866	0.473439	0.470487
1.25	1	0.5	0.486661	0.479992	0.475991	0.473323
1.30	1	0.5	0.487943	0.481915	0.478297	0.475886
1.35	1	0.5	0.489101	0.483652	0.480383	0.478203
1.40	1	0.5	0.490149	0.485223	0.482267	0.480297
1.45	1	0.5	0.491095	0.486643	0.483971	0.48219
1.50	1	0.5	0.491951	0.487926	0.485511	0.483901

Table 6.5. Gauss-Galerkin Finite Element Method: Changes of the five nodes ◇, o, x, + and * at $y = 0.5$ as t increases

time	x_1	x_2	x_3	x_4	x_5
0.00	0.046910077	0.2307653449	0.5	0.7692346551	0.953089923
0.05	0.023013041	0.2191331593	0.5	0.7808668407	0.976986958
0.10	0.014685321	0.2160106921	0.5	0.7839893079	0.985314678
0.15	0.010483349	0.2145721597	0.5	0.7854278403	0.989516650
0.20	0.992038063	0.7862514012	0.5	0.2137485988	0.0079619366
0.25	0.006288393	0.2132174718	0.5	0.7867825282	0.993711606
0.30	0.005101903	0.2128481441	0.5	0.7871518559	0.994898096
0.35	0.004220822	0.2125776662	0.5	0.7874223338	0.995779177
0.40	0.996456245	0.7876280313	0.5	0.2123719687	0.003543754
0.45	0.003009645	0.2122110044	0.5	0.7877889957	0.996990355
0.50	0.997420468	0.7879177936	0.5	0.2120822064	0.002579531
0.55	0.997772625	0.7880227035	0.5	0.2119772965	0.002227374
0.60	0.998064888	0.7881094017	0.5	0.2118905983	0.001935111
0.65	0.001689814	0.2118180894	0.5	0.7881819106	0.998310185
0.70	0.998518012	0.788243161	0.5	0.211756839	0.001481988
0.75	0.998695503	0.7882953389	0.5	0.2117046612	0.001304496
0.80	0.001151877	0.211659892	0.5	0.788340108	0.998848122
0.85	0.998980124	0.7883787575	0.5	0.2116212425	0.001019875
0.90	0.000905124	0.2115876985	0.5	0.7884123015	0.999094875
0.95	0.000804931	0.2115584512	0.5	0.7884415489	0.999195068
1.00	0.000717112	0.2115328475	0.5	0.7884671526	0.999282887
1.05	0.999360118	0.7884896454	0.5	0.2115103546	0.000639881
1.10	0.999428241	0.7885094665	0.5	0.2114905335	0.000571758
1.15	0.000511513	0.2114730193	0.5	0.7885269808	0.9994884863
1.20	0.999541887	0.7885424939	0.5	0.2114575061	0.000458112
1.25	0.999589318	0.7885562638	0.5	0.2114437362	0.000410681
1.30	0.999631524	0.7885685095	0.5	0.2114314905	0.000368475
1.35	0.999669141	0.788579418	0.5	0.211420582	0.000330858
1.40	0.000297284	0.2114108501	0.5	0.7885891499	0.999702715
1.45	0.000267278	0.2114021563	0.5	0.7885978436	0.999732721
1.50	0.000240431	0.2113943805	0.4999999998	0.7886056192	0.999759568

Table 6.6. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes ◇, o, x, + and ∗ at $y = 0.5$ as t increases

time	ω_1	ω_2	ω_3	ω_4	ω_5
0.00	0.118463442	0.239314335	0.284444444	0.239314335	0.11846344
0.05	0.139977042	0.225973313	0.268099287	0.225973313	0.13997704
0.10	0.170874697	0.206626649	0.244997305	0.206626649	0.17087469
0.15	0.200965839	0.187750113	0.222568094	0.187750113	0.20096583
0.20	0.228907253	0.170212913	0.201759666	0.170212913	0.228907253
0.25	0.254494478	0.154150051	0.182710939	0.154150051	0.254494478
0.30	0.277794365	0.139521743	0.165367781	0.139521743	0.277794365
0.35	0.298953252	0.126236970	0.149619555	0.126236970	0.298953252
0.40	0.318138876	0.114190794	0.135340657	0.114190794	0.318138876
0.45	0.335519514	0.103277744	0.122405482	0.103277744	0.335519514
0.50	0.351255856	0.093397020	0.110694245	0.0933970208	0.351255856
0.55	0.365497910	0.084454487	0.100095204	0.084454487	0.365497910
0.60	0.378384055	0.076363286	0.090505316	0.076363286	0.378384055
0.65	0.390041077	0.069043827	0.081830190	0.069043827	0.390041077
0.70	0.400584668	0.062423478	0.073983707	0.062423478	0.400584668
0.75	0.410120113	0.056436136	0.066887501	0.056436136	0.410120113
0.80	0.418743061	0.051021748	0.060470380	0.051021748	0.418743061
0.85	0.426540302	0.046125821	0.054667753	0.046125821	0.426540302
0.90	0.433590516	0.041698952	0.049421061	0.041698952	0.433590516
0.95	0.439964991	0.037696382	0.044677252	0.037696382	0.439964991
1.00	0.445728284	0.034077575	0.040388279	0.034077575	0.445728284
1.05	0.450938838	0.030805835	0.036510650	0.030805835	0.450938838
1.10	0.455649546	0.027847952	0.033005002	0.027847952	0.455649546
1.15	0.459908267	0.025173873	0.029835718	0.025173873	0.459908267
1.20	0.463758298	0.022756413	0.026970575	0.022756413	0.463758298
1.25	0.4672388082	0.020570978	0.024380426	0.020570978	0.467238808
1.30	0.47038522	0.018595325	0.022038909	0.018595325	0.47038522
1.35	0.473229572	0.016809337	0.019922181	0.016809337	0.473229572
1.40	0.475800836	0.015194822	0.01800868	0.015194822	0.475800836
1.45	0.478125214	0.013735330	0.016278911	0.013735330	0.478125214
1.50	0.480226393	0.012415985	0.014715243	0.012415985	0.480226393

Table 6.7. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ with five nodes ◇, o, x, + and $*$ at $y = 0.5$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
0.00	1	0.5	0.333	0.25	0.2	0.1667	0.1429	0.125	0.111	0.1
0.05	1	0.5	0.349	0.274	0.228	0.1987	0.1772	0.161	0.148	0.138
0.10	1	0.5	0.363	0.295	0.254	0.2276	0.2082	0.193	0.182	0.173
0.15	1	0.5	0.376	0.315	0.278	0.2538	0.2362	0.223	0.212	0.204
0.20	1	0.5	0.388	0.333	0.299	0.2775	0.2616	0.249	0.240	0.233
0.25	1	0.5	0.399	0.349	0.319	0.2988	0.2845	0.273	0.265	0.258
0.30	1	0.5	0.409	0.363	0.336	0.3182	0.3052	0.295	0.287	0.281
0.35	1	0.5	0.417	0.376	0.352	0.3356	0.3239	0.315	0.308	0.302
0.40	1	0.5	0.425	0.388	0.366	0.3514	0.3408	0.332	0.326	0.321
0.45	1	0.5	0.432	0.399	0.379	0.3657	0.3561	0.348	0.343	0.338
0.50	1	0.5	0.439	0.409	0.390	0.3786	0.3699	0.363	0.358	0.354
0.55	1	0.5	0.445	0.417	0.401	0.3903	0.3824	0.376	0.372	0.368
0.60	1	0.5	0.450	0.425	0.410	0.4008	0.3937	0.388	0.384	0.381
0.65	1	0.5	0.455	0.432	0.419	0.4103	0.4039	0.399	0.395	0.392
0.70	1	0.5	0.459	0.439	0.427	0.419	0.4132	0.408	0.405	0.4028
0.75	1	0.5	0.463	0.445	0.434	0.4267	0.4215	0.417	0.414	0.412
0.80	1	0.5	0.466	0.450	0.440	0.4338	0.4291	0.425	0.422	0.420
0.85	1	0.5	0.470	0.455	0.446	0.4401	0.4359	0.432	0.430	0.428
0.90	1	0.5	0.472	0.459	0.451	0.4459	0.442	0.439	0.436	0.435
0.95	1	0.5	0.475	0.463	0.456	0.4511	0.4476	0.445	0.442	0.441
1.00	1	0.5	0.477	0.466	0.460	0.4558	0.4526	0.450	0.448	0.447
1.05	1	0.5	0.48	0.47	0.464	0.46	0.4572	0.455	0.453	0.452
1.10	1	0.5	0.481	0.472	0.467	0.4639	0.4613	0.459	0.457	0.456
1.15	1	0.5	0.483	0.475	0.470	0.4674	0.465	0.463	0.461	0.460
1.20	1	0.5	0.485	0.477	0.473	0.4705	0.4684	0.466	0.465	0.464
1.25	1	0.5	0.486	0.48	0.476	0.4733	0.4714	0.47	0.468	0.468
1.30	1	0.5	0.487	0.481	0.478	0.4759	0.4742	0.472	0.471	0.471
1.35	1	0.5	0.489	0.483	0.480	0.4782	0.4766	0.475	0.474	0.473
1.40	1	0.5	0.490	0.485	0.482	0.4803	0.4789	0.477	0.477	0.476
1.45	1	0.5	0.491	0.486	0.484	0.4822	0.4809	0.48	0.479	0.478
1.50	1	0.5	0.492	0.487	0.485	0.4839	0.4828	0.481	0.481	0.480

Table 6.8. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ with five nodes ◇, o, x, + and * as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
0.00	1	0.5	0.333	0.25	0.2	0.1667	0.1429	0.125	0.1111	0.1
0.05	1	0.5	0.349	0.274	0.228	0.1987	0.1772	0.161	0.1485	0.1384
0.10	1	0.5	0.363	0.295	0.254	0.2276	0.2082	0.1936	0.1822	0.1732
0.15	1	0.5	0.376	0.315	0.278	0.2538	0.2362	0.223	0.2128	0.2046
0.20	1	0.5	0.388	0.333	0.299	0.2775	0.2616	0.2496	0.2404	0.233
0.25	1	0.5	0.399	0.349	0.319	0.2988	0.2845	0.2737	0.2653	0.2586
0.30	1	0.5	0.409	0.363	0.336	0.3182	0.3052	0.2954	0.2879	0.2818
0.35	1	0.5	0.417	0.376	0.352	0.3356	0.3239	0.3151	0.3082	0.3028
0.40	1	0.5	0.425	0.388	0.366	0.3514	0.3408	0.3329	0.3267	0.3217
0.45	1	0.5	0.432	0.399	0.379	0.3657	0.3561	0.3489	0.3433	0.3388
0.50	1	0.5	0.439	0.409	0.390	0.3786	0.3699	0.3634	0.3584	0.3543
0.55	1	0.5	0.445	0.417	0.401	0.3903	0.3824	0.3766	0.372	0.3683
0.60	1	0.5	0.450	0.425	0.410	0.4008	0.3937	0.3884	0.3843	0.381
0.65	1	0.5	0.455	0.432	0.419	0.4103	0.4039	0.3991	0.3954	0.3924
0.70	1	0.5	0.459	0.439	0.427	0.419	0.4132	0.4088	0.4055	0.4028
0.75	1	0.5	0.463	0.445	0.434	0.4267	0.4215	0.4176	0.4145	0.4121
0.80	1	0.5	0.466	0.450	0.440	0.4338	0.4291	0.4255	0.4227	0.4205
0.85	1	0.5	0.470	0.455	0.446	0.4401	0.4359	0.4327	0.4302	0.4282
0.90	1	0.5	0.472	0.459	0.451	0.4459	0.442	0.4391	0.4369	0.4351
0.95	1	0.5	0.475	0.463	0.456	0.4511	0.4476	0.445	0.4429	0.4413
1.00	1	0.5	0.477	0.466	0.460	0.4558	0.4526	0.4503	0.4484	0.447
1.05	1	0.5	0.48	0.47	0.464	0.46	0.4572	0.455	0.4534	0.452
1.10	1	0.5	0.481	0.472	0.467	0.4639	0.4613	0.4594	0.4579	0.4567
1.15	1	0.5	0.483	0.475	0.470	0.4674	0.465	0.4633	0.4619	0.4608
1.20	1	0.5	0.485	0.477	0.473	0.4705	0.4684	0.4668	0.4656	0.4646
1.25	1	0.5	0.486	0.48	0.476	0.4733	0.4714	0.47	0.4689	0.468
1.30	1	0.5	0.487	0.481	0.478	0.4759	0.4742	0.4729	0.4719	0.4711
1.35	1	0.5	0.489	0.483	0.480	0.4782	0.4766	0.4755	0.4746	0.4738
1.40	1	0.5	0.490	0.485	0.482	0.4803	0.4789	0.4778	0.477	0.4764
1.45	1	0.5	0.491	0.486	0.484	0.4822	0.4809	0.48	0.4792	0.4786
1.50	1	0.5	0.492	0.487	0.485	0.4839	0.4828	0.4819	0.4812	0.4807

Table 6.9. Gauss-Galerkin Finite Element Method: Changes of the exact total moment $M^i(t)$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
0.00	1	0.5	0.333	0.25	0.2	0.166	0.1429	0.125	0.1111	0.1
0.05	1	0.5	0.349	0.273	0.228	0.198	0.1768	0.1607	0.1481	0.1381
0.10	1	0.5	0.363	0.295	0.254	0.227	0.2076	0.193	0.1816	0.1725
0.15	1	0.5	0.376	0.314	0.277	0.253	0.2354	0.2222	0.2119	0.2037
0.20	1	0.5	0.388	0.332	0.298	0.276	0.2606	0.2486	0.2393	0.2319
0.25	1	0.5	0.398	0.348	0.318	0.297	0.2834	0.2726	0.2641	0.2574
0.30	1	0.5	0.408	0.362	0.335	0.317	0.304	0.2942	0.2866	0.2805
0.35	1	0.5	0.417	0.375	0.351	0.334	0.3226	0.3138	0.3069	0.3014
0.40	1	0.5	0.425	0.387	0.365	0.350	0.3395	0.3315	0.3253	0.3203
0.45	1	0.5	0.432	0.398	0.378	0.364	0.3548	0.3475	0.3419	0.3374
0.50	1	0.5	0.438	0.408	0.389	0.377	0.3686	0.362	0.3569	0.3528
0.55	1	0.5	0.444	0.416	0.400	0.389	0.3811	0.3752	0.3706	0.3669
0.60	1	0.5	0.449	0.424	0.409	0.399	0.3924	0.3871	0.3829	0.3795
0.65	1	0.5	0.454	0.431	0.418	0.409	0.4027	0.3978	0.394	0.391
0.70	1	0.5	0.458	0.438	0.426	0.417	0.4119	0.4075	0.4041	0.4014
0.75	1	0.5	0.462	0.444	0.433	0.425	0.4203	0.4163	0.4132	0.4107
0.80	1	0.5	0.466	0.449	0.439	0.432	0.4279	0.4243	0.4215	0.4192
0.85	1	0.5	0.469	0.454	0.445	0.439	0.4348	0.4315	0.429	0.4269
0.90	1	0.5	0.472	0.458	0.450	0.444	0.441	0.438	0.4357	0.4339
0.95	1	0.5	0.475	0.462	0.455	0.450	0.4466	0.4439	0.4418	0.4402
1.00	1	0.5	0.477	0.466	0.459	0.454	0.4517	0.4492	0.4474	0.4459
1.05	1	0.5	0.479	0.469	0.463	0.459	0.4563	0.4541	0.4524	0.451
1.10	1	0.5	0.481	0.472	0.466	0.463	0.4604	0.4584	0.4569	0.4557
1.15	1	0.5	0.483	0.474	0.469	0.466	0.4642	0.4624	0.461	0.4599
1.20	1	0.5	0.484	0.477	0.472	0.469	0.4676	0.466	0.4647	0.4637
1.25	1	0.5	0.486	0.479	0.475	0.472	0.4707	0.4692	0.4681	0.4672
1.30	1	0.5	0.487	0.481	0.477	0.475	0.4735	0.4721	0.4711	0.4703
1.35	1	0.5	0.488	0.483	0.479	0.477	0.476	0.4748	0.4739	0.4731
1.40	1	0.5	0.489	0.484	0.481	0.479	0.4783	0.4772	0.4764	0.4757
1.45	1	0.5	0.490	0.486	0.483	0.481	0.4803	0.4794	0.4786	0.478
1.50	1	0.5	0.491	0.487	0.485	0.483	0.4822	0.4813	0.4806	0.4801

Table 6.10. Gauss-Galerkin Finite Element Method: Changes of the errors between the exact total moment $M^i(t)$ and their approximation $M_n(t)$ as t increases

time	EM_0	EM_1	EM_2	EM_3	EM_4	EM_5
0.00	2.2e-16	5.6e-17	-5.6e-17	2.8e-17	5.6e-17	2.8e-17
0.05	1.1e-16	1.1e-16	-0.00015	-0.00023	-0.00027	-0.00031
0.10	4.4e-16	5e-16	-0.00028	-0.00041	-0.0005	-0.00055
0.15	4.4e-16	1.7e-16	-0.00037	-0.00056	-0.00067	-0.00075
0.20	7.8e-16	5e-16	-0.00045	-0.00068	-0.00081	-0.0009
0.25	4.4e-16	5e-16	-0.00051	-0.00077	-0.00092	-0.001
0.30	7.8e-16	6.7e-16	-0.00055	-0.00083	-0.001	-0.0011
0.35	5.6e-16	5.6e-16	-0.00059	-0.00088	-0.0011	-0.0012
0.40	8.9e-16	5.6e-16	-0.0006	-0.00091	-0.0011	-0.0012
0.45	1.1e-15	6.1e-16	-0.00062	-0.00092	-0.0011	-0.0012
0.50	1e-15	7.2e-16	-0.00062	-0.00093	-0.0011	-0.0012
0.55	1.6e-15	8.3e-16	-0.00062	-0.00092	-0.0011	-0.0012
0.60	1.6e-15	6.1e-16	-0.00061	-0.00091	-0.0011	-0.0012
0.65	1.4e-15	7.2e-16	-0.00059	-0.00089	-0.0011	-0.0012
0.70	1.8e-15	1.2e-15	-0.00058	-0.00087	-0.001	-0.0012
0.75	1.3e-15	1e-15	-0.00056	-0.00084	-0.001	-0.0011
0.80	1.7e-15	8.3e-16	-0.00054	-0.00081	-0.00097	-0.0011
0.85	2.1e-15	1.2e-15	-0.00052	-0.00078	-0.00094	-0.001
0.90	2.1e-15	8.9e-16	-0.0005	-0.00075	-0.0009	-0.001
0.95	2.3e-15	1.1e-15	-0.00048	-0.00071	-0.00086	-0.00095
1.00	2.6e-15	1.4e-15	-0.00045	-0.00068	-0.00081	-0.00091
1.05	2.6e-15	1.4e-15	-0.00043	-0.00064	-0.00077	-0.00086
1.10	2.7e-15	1.3e-15	-0.00041	-0.00061	-0.00073	-0.00081
1.15	2.7e-15	1.2e-15	-0.00039	-0.00058	-0.00069	-0.00077
1.20	3.3e-15	1.4e-15	-0.00036	-0.00055	-0.00065	-0.00073
1.25	3e-15	1.4e-15	-0.00034	-0.00051	-0.00062	-0.00068
1.30	3.2e-15	1.4e-15	-0.00032	-0.00048	-0.00058	-0.00064
1.35	3.6e-15	1.7e-15	-0.0003	-0.00045	-0.00054	-0.0006
1.40	3.8e-15	1.7e-15	-0.00028	-0.00043	-0.00051	-0.00057
1.45	3.8e-15	1.6e-15	-0.00027	-0.0004	-0.00048	-0.00053
1.50	4.1e-15	1.9e-15	-0.00025	-0.00037	-0.00045	-0.0005

Table 6.11. Gauss-Galerkin Finite Element Method: Changes of the relative errors between the exact total moment $M^i(t)$ and their approximation $M_n(t)$ as t increases

time	REM_0	REM_1	REM_2	REM_3	REM_4
0.00	1.6867e-16	1.6867e-16	1.6867e-16	1.6867e-16	1.6867e-16
0.05	-0.00027869	-0.00027869	-0.00027869	-0.00027869	-0.00027869
0.10	-0.00054912	-0.00054912	-0.00054912	-0.00054912	-0.00054912
0.15	-0.00079	-0.00079	-0.00079	-0.00079	-0.00079
0.20	-0.00099243	-0.00099243	-0.00099243	-0.00099243	-0.00099243
0.25	-0.0011547	-0.0011547	-0.0011547	-0.0011547	-0.0011547
0.30	-0.0012788	-0.0012788	-0.0012788	-0.0012788	-0.0012788
0.35	-0.0013688	-0.0013688	-0.0013688	-0.0013688	-0.0013688
0.40	-0.0014291	-0.0014291	-0.0014291	-0.0014291	-0.0014291
0.45	-0.0014642	-0.0014642	-0.0014642	-0.0014642	-0.0014642
0.50	-0.0014783	-0.0014783	-0.0014783	-0.0014783	-0.0014783
0.55	-0.0014753	-0.0014753	-0.0014753	-0.0014753	-0.0014753
0.60	-0.0014584	-0.0014584	-0.0014584	-0.0014584	-0.0014584
0.65	-0.0014305	-0.0014305	-0.0014305	-0.0014305	-0.0014305
0.70	-0.0013939	-0.0013939	-0.0013939	-0.0013939	-0.0013939
0.75	-0.0013508	-0.0013508	-0.0013508	-0.0013508	-0.0013508
0.80	-0.0013028	-0.0013028	-0.0013028	-0.0013028	-0.0013028
0.85	-0.0012513	-0.0012513	-0.0012513	-0.0012513	-0.0012513
0.90	-0.0011975	-0.0011975	-0.0011975	-0.0011975	-0.0011975
0.95	-0.0011424	-0.0011424	-0.0011424	-0.0011424	-0.0011424
1.00	-0.0010867	-0.0010867	-0.0010867	-0.0010867	-0.0010867
1.05	-0.0010312	-0.0010312	-0.0010312	-0.0010312	-0.0010312
1.10	-0.00097627	-0.00097627	-0.00097627	-0.00097627	-0.00097627
1.15	-0.00092237	-0.00092237	-0.00092237	-0.00092237	-0.00092237
1.20	-0.00086983	-0.00086983	-0.00086983	-0.00086983	-0.00086983
1.25	-0.00081888	-0.00081888	-0.00081888	-0.00081888	-0.00081888
1.30	-0.00076971	-0.00076971	-0.00076971	-0.00076971	-0.00076971
1.35	-0.00072245	-0.00072245	-0.00072245	-0.00072245	-0.00072245
1.40	-0.00067719	-0.00067719	-0.00067719	-0.00067719	-0.00067719
1.45	-0.00063398	-0.00063398	-0.00063398	-0.00063398	-0.00063398
1.50	-0.00059284	-0.00059284	-0.00059284	-0.00059284	-0.00059284

and initial conditions

$$u(x,y,0) = \pi \cos^2\left(\frac{\pi y}{2}\right)\sin(\pi x) \quad \text{on} \quad \Omega. \tag{6.55}$$

Case I $p = 2, q = 2$

This implies that both $x = 0$ and $x = 1$ are natural boundaries. In this example, for fixed y, one can show that the density $u(x,y,\infty) = 0$, $\forall x \in (0,1)$, is a steady state solution and escaping to the boundary $x = 0$ and $x = 1$ as t becomes arbitrary large and u will "pile" near boundary $x = 0$ and $x = 1$ with a Dirac-delta function singularity formed at $x = 0^+$ and $x = 1^-$. Our numerical result will show this.

We apply the Gauss-Galerkin Finite Element method using finite element approximations in the y-direction and Gauss-Galerkin approximations in the x-direction. We divided $y \in [0,1]$ into four equal subintervals. Then we get five grid lines $y_j = jh, j = 0, 1, \cdots, 4, h = 0.25$. In the x-direction we use three nodes labeled by $*$, $+$ and x for each grid line $y = y_j$. We only discuss the results along grid $y = 0.5$. The results along $y = 0.25$ and $y = 0.75$ are similar. We use $*$, $+$ and x to indicate the nodes moving to the boundary $x = 0$, interior point and the boundary $x = 1$. Figure 6.3 showes the movement of the three nodes as t increases. Table 6.12 shows the changes of nodes $*$, $+$ and x as t increases. Table 6.13 shows the changes of the weights at the three nodes $*$, $+$ and x as t increases. Table 6.14 shows the changes of the ith "moment" $m_n^i(y,t)$ at $y = 0.5$ as t increases. Table 6.15 shows the changes of the "total moment" $M_n^i(t)$ as t increases.

(1) Integrating (6.49) over Ω, integrating by parts and using boundary condition (6.53), (6.54) we have $\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy = \int_0^1\int_0^1((x^2(1-x)^2u)_{xx} + u_{yy})dxdy = 0$. This shows that the total probability is conserved. Therefore, the rate of change of the exact 0th total moment $\frac{dM^0(t)}{dt} = 0$. So, the exact 0th total moment $M^0(t) = M^0(0) = \int_0^1\int_0^1 u(x,y,0)dxdy = 1$.

(2) Multiplying (6.49) by x and integrating over Ω, integrating by parts and using boundary condition (6.53), (6.54) we have $\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)xdxdy = \int_0^1\int_0^1((x^2(1-x)^2u)_{xx}x + u_{yy}x)dxdy = 0$. This means the rate of change of the exact 1st total moment $\frac{dM^1(t)}{dt} = 0$. So, the exact 1st total moment $M^1(t) = M^1(0) = \int_0^1\int_0^1 u(x,y,0)xdxdy = 0.5$.

(3) Table 6.15 shows that $M_n^0(t) = 1 = M^0(t)$ for $t \geq 1$. This means the total mass = 1. Table 6.13 shows that is equal to the sum of the two weights at the boundaries $x = 0$ and $x = 1$.

(4) Table 6.15 shows that $M_n^1(t) = 0.5 = M^1(t)$ for $t \geq 1$. This shows that the approximation of the 1st total moment equals the exact 1st total moment.

(5) At $t = 30$, the solution reaches the steady state based on the tolerance chosen.

(6) The solutinon becomes uniform in y as t increases based on observations of the numerical results along the other y-grid lines.

(7) The solution approaches zero in the interior and "piles" up near the boundaries $x = 0$ and $x = 1$ as t increases. The steady state solution is $\frac{1}{2}(\delta(x) + \delta(x-1))$.

(8) The weight for the middle node tends to zero as t increases.

(9) The weights at the other two nodes tend to 0.5 as t increases.

(10) Table (6.15) shows that the ith "total moments" of the steady state solution are all about 0.5 for i from 1 to 5. Since the weight tends to zero for the interior node and 0.5 for the other two points which approach to $x = 0$ and $x = 1$, the contribution of u to the ith total moment is the contribution of these two weights to the ith total moment. The total moment contributed by the weight at $x = 0$ is zero and the total moment contributed by the weight at $x = 1$ is 0.5 which is equal to the weight at $x = 1$.

We now discuss how the movement of nodes depends on the parameters p and q. We know that the degree of singularities of (6.52) depend on the parameters p and q. If $p < q$(or $p > q$), the singularity at $x = 0$ will be less (or greater) than the singularity at $x = 1$. The movement of nodes to boundary $x = 0$ should be faster (or slower) than the movement of nodes to boundary $x = 1$. Following Examples shall show that. In the following Case II, Case III, and case IV, we use the same method as in Case I to get the numerical results.

Case II $p < q$

We take $p = 1$ and $q = 2$ as an example. The numerical results are in Figure 6.4 and Tables 6.16, 6.17, 6.18 and 6.19. We observe that the nodes move to the boundary $x = 0$ faster than to the boundary $x = 1$.

Case III $p > q$

We take $p = 2$ and $q = 1$ as an example. The numerical results are in Figure 6.5 and Tables 6.20, 6.21, 6.22 and 6.3. We observe that the nodes move to the boundary $x = 0$ more slowly than to the boundary $x = 1$.

Case IV Fixing p and changing q

Tables 6.24 6.25 6.26 and 6.27 show how the movement of nodes to boundary $x = 1$ depends on the parameter q. Nodes with smaller parameter q move to the boundary $x = 1$ faster than those with larger q.

Model II. Let $p = 1$ and $q = 1$. This implies that both $x = 0$ and $x = 1$ are exit boundaries. We consider the following initial -boundary problem:

$$u_t = (x(1-x)u)_{xx} + u_{yy} \quad \text{in} \quad (0,T) \times \Omega \tag{6.56}$$

with the boundary conditions

$$\lim_{x \to 0} xu(x,y,t) = \lim_{x \to 1} (1-x)u(x,y,t) = 0 \quad \text{on} \quad (0,T) \times (0,1), \tag{6.57}$$

$$u_y(x,0,t) = u_y(x,1,t) = 0 \quad \text{on} \quad (0,T) \times (0,1), \tag{6.58}$$

and initial conditions

$$u(x,y,0) = \pi \left[0.55 - 0.1\cos^2\left(\frac{\pi y}{2}\right)\right] x\sin(\pi x) \quad \text{on} \quad \Omega. \tag{6.59}$$

By (6.50),

$$\mathrm{P}(x,y,t) = -(x(1-x)u)_x = -(1-x)u + xu - x(1-x)u_x. \tag{6.60}$$

Using the boundary condition (6.57) we have

$$\mathrm{P}(0,y,t) = -u(0,y,t) \quad \text{and} \quad \mathrm{P}(1,y,t) = u(1,y,t). \tag{6.61}$$

Table 6.12. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 2$ and $q = 2$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
0.20	0.1569951415	0.5	0.8430048585
0.40	0.1387971313	0.5	0.8612028687
0.60	0.124213992	0.5	0.875786008
0.80	0.8875133494	0.5	0.1124866506
1.00	0.1028788815	0.5	0.8971211185
2.00	0.07248655953	0.5	0.9275134405
3.00	0.05589220564	0.5	0.9441077944
4.00	0.04523506173	0.5	0.9547649383
5.00	0.03777675115	0.5	0.9622232489
6.00	0.03227123193	0.5	0.9677287681
7.00	0.02805416464	0.5	0.9719458354
8.00	0.02473328072	0.5	0.9752667193
9.00	0.9779398506	0.5	0.02206014938
10.0	0.01986926287	0.5	0.9801307371
11.0	0.01804605226	0.5	0.9819539477
12.0	0.01650877966	0.5	0.9834912203
13.0	0.01519767562	0.5	0.9848023244
14.0	0.01406812019	0.5	0.9859318798
15.0	0.0130862042	0.5	0.9869137958
16.0	0.01222575257	0.5	0.9877742474
17.0	0.01146627805	0.5	0.988533722
18.0	0.01079154418	0.5	0.9892084558
19.0	0.01018853736	0.5	0.9898114626
20.0	0.009646719368	0.5	0.9903532806
21.0	0.009157475852	0.5	0.9908425241
22.0	0.008713703905	0.5	0.9912862961
23.0	0.008309499947	0.5	0.9916905001
24.0	0.007939920837	0.5	0.9920600792
25.0	0.00760079916	0.5	0.9923992008
26.0	0.007288599021	0.5	0.992711401
27.0	0.007000302461	0.5	0.9929996975
28.0	0.006733319251	0.5	0.9932666807
29.0	0.006485414691	0.5	0.9935145853
30.0	0.006254651387	0.5	0.9937453486

Table 6.13. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes $*$, $+$ and x at $y = 0.5$ with $p = 2$ and $q = 2$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.2279202041	0.5441595918	0.2279202041
0.20	0.2708900461	0.4582199078	0.2708900461
0.40	0.2993447133	0.4013105734	0.2993447133
0.60	0.3216170199	0.3567659601	0.3216170199
0.80	0.3402289554	0.3195420892	0.3402289554
1.00	0.356218263	0.287563474	0.356218263
2.00	0.4112542944	0.1774914111	0.4112542944
3.00	0.4421675514	0.1156648973	0.4421675514
4.00	0.4606419939	0.07871601222	0.4606419939
5.00	0.4721989331	0.05560213382	0.4721989331
6.00	0.4797123027	0.04057539469	0.4797123027
7.00	0.484765015	0.03046997007	0.484765015
8.00	0.4882673879	0.02346522416	0.4882673879
9.00	0.4907622074	0.01847558527	0.4907622074
10.0	0.4925835524	0.01483289517	0.4925835524
11.0	0.4939430195	0.0121139611	0.4939430195
12.0	0.4949781889	0.01004362224	0.4949781889
13.0	0.4957807059	0.008438588265	0.4957807059
14.0	0.4964129983	0.007174003459	0.4964129983
15.0	0.4969184809	0.006163038117	0.4969184809
16.0	0.49732793	0.005344140008	0.49732793
17.0	0.4976635516	0.004672896755	0.4976635516
18.0	0.4979416325	0.004116735089	0.4979416325
19.0	0.4981742991	0.003651401712	0.4981742991
20.0	0.4983707076	0.003258584867	0.4983707076
21.0	0.4985378601	0.002924279736	0.4985378601
22.0	0.4986811764	0.002637647259	0.4986811764
23.0	0.4988048972	0.002390205651	0.4988048972
24.0	0.4989123752	0.00217524967	0.4989123752
25.0	0.499006286	0.001987428079	0.499006286
26.0	0.4990887838	0.001822432463	0.4990887838
27.0	0.4991616173	0.001676765466	0.4991616173
28.0	0.4992262168	0.001547566318	0.4992262168
29.0	0.4992837609	0.001432478197	0.4992837609
30.0	0.4993352268	0.001329546435	0.4993352268

Table 6.14. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $p = 2$ and $q = 2$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.313742	0.220613	0.165612	0.129676
0.40	1	0.5	0.32811	0.242164	0.189855	0.154364
0.60	1	0.5	0.340834	0.261252	0.211579	0.176861
0.80	1	0.5	0.352182	0.278273	0.231117	0.197338
1.00	1	0.5	0.362355	0.293532	0.248751	0.215991
2.00	1	0.5	0.400328	0.350492	0.315467	0.287848
3.00	1	0.5	0.424419	0.386628	0.358529	0.335276
4.00	1	0.5	0.440532	0.410798	0.387702	0.367925
5.00	1	0.5	0.451771	0.427656	0.408265	0.391235
6.00	1	0.5	0.459894	0.43984	0.423259	0.408413
7.00	1	0.5	0.465946	0.448919	0.434518	0.421429
8.00	1	0.5	0.470578	0.455867	0.443191	0.431532
9.00	1	0.5	0.474206	0.461309	0.450024	0.439544
10.0	1	0.5	0.477106	0.465659	0.455513	0.446017
11.0	1	0.5	0.479466	0.469199	0.459999	0.451333
12.0	1	0.5	0.481416	0.472124	0.463721	0.455762
13.0	1	0.5	0.48305	0.474575	0.466849	0.459499
14.0	1	0.5	0.484436	0.476654	0.469511	0.462688
15.0	1	0.5	0.485624	0.478436	0.471799	0.465437
16.0	1	0.5	0.486652	0.479978	0.473784	0.467828
17.0	1	0.5	0.48755	0.481325	0.47552	0.469925
18.0	1	0.5	0.48834	0.48251	0.47705	0.471776
19.0	1	0.5	0.489039	0.483559	0.478408	0.473422
20.0	1	0.5	0.489663	0.484494	0.47962	0.474894
21.0	1	0.5	0.490222	0.485333	0.480709	0.476217
22.0	1	0.5	0.490726	0.486088	0.48169	0.477412
23.0	1	0.5	0.491182	0.486773	0.482581	0.478497
24.0	1	0.5	0.491596	0.487395	0.483391	0.479486
25.0	1	0.5	0.491975	0.487963	0.484131	0.48039
26.0	1	0.5	0.492322	0.488483	0.48481	0.481221
27.0	1	0.5	0.492641	0.488962	0.485435	0.481986
28.0	1	0.5	0.492935	0.489403	0.486012	0.482692
29.0	1	0.5	0.493208	0.489812	0.486546	0.483347
30.0	1	0.5	0.49346	0.490191	0.487042	0.483955

Table 6.15. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ with $p = 2$ and $q = 2$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.313742	0.220613	0.165612	0.129676
0.40	1	0.5	0.32811	0.242164	0.189855	0.154364
0.60	1	0.5	0.340834	0.261252	0.211579	0.176861
0.80	1	0.5	0.352182	0.278273	0.231117	0.197338
1.00	1	0.5	0.362355	0.293532	0.248751	0.215991
2.00	1	0.5	0.400328	0.350492	0.315467	0.287848
3.00	1	0.5	0.424419	0.386628	0.358529	0.335276
4.00	1	0.5	0.440532	0.410798	0.387702	0.367925
5.00	1	0.5	0.451771	0.427656	0.408265	0.391235
6.00	1	0.5	0.459894	0.43984	0.423259	0.408413
7.00	1	0.5	0.465946	0.448919	0.434518	0.421429
8.00	1	0.5	0.470578	0.455867	0.443191	0.431532
9.00	1	0.5	0.474206	0.461309	0.450024	0.439544
10.0	1	0.5	0.477106	0.465659	0.455513	0.446017
11.0	1	0.5	0.479466	0.469199	0.459999	0.451333
12.0	1	0.5	0.481416	0.472124	0.463721	0.455762
13.0	1	0.5	0.48305	0.474575	0.466849	0.459499
14.0	1	0.5	0.484436	0.476654	0.469511	0.462688
15.0	1	0.5	0.485624	0.478436	0.471799	0.465437
16.0	1	0.5	0.486652	0.479978	0.473784	0.467828
17.0	1	0.5	0.48755	0.481325	0.47552	0.469925
18.0	1	0.5	0.48834	0.48251	0.47705	0.471776
19.0	1	0.5	0.489039	0.483559	0.478408	0.473422
20.0	1	0.5	0.489663	0.484494	0.47962	0.474894
21.0	1	0.5	0.490222	0.485333	0.480709	0.476217
22.0	1	0.5	0.490726	0.486088	0.48169	0.477412
23.0	1	0.5	0.491182	0.486773	0.482581	0.478497
24.0	1	0.5	0.491596	0.487395	0.483391	0.479486
25.0	1	0.5	0.491975	0.487963	0.484131	0.48039
26.0	1	0.5	0.492322	0.488483	0.48481	0.481221
27.0	1	0.5	0.492641	0.488962	0.485435	0.481986
28.0	1	0.5	0.492935	0.489403	0.486012	0.482692
29.0	1	0.5	0.493208	0.489812	0.486546	0.483347
30.0	1	0.5	0.49346	0.490191	0.487042	0.483955

Table 6.16. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 2$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
0.20	0.04737482572	0.4409938921	0.8241783121
0.40	0.02403084443	0.4448604727	0.8476643036
0.60	0.01482958065	0.4550658882	0.8673940809
0.80	0.01008567793	0.4651212708	0.8830828805
1.00	0.007289118074	0.4740494633	0.8956991781
2.00	0.002250069707	0.5051335296	0.9335385097
3.00	0.0009863745097	0.524526958	0.9522693971
4.00	0.0005184071064	0.5382862795	0.9632779238
5.00	0.0003076309523	0.5484473913	0.9704182539
6.00	0.0001992931646	0.5560949596	0.9753690413
7.00	0.0001378318697	0.5619588701	0.9789751276
8.00	0.0001001977718	0.5665471543	0.9817040036
9.00	7.572609049e-05	0.5702100595	0.9838329178
10.0	5.902989083e-05	0.573188811	0.9855354713
11.0	4.718510394e-05	0.5756514259	0.9869252876
12.0	3.850700958e-05	0.5777170269	0.9880795345
13.0	3.197534068e-05	0.5794717514	0.9890522877
14.0	2.694566813e-05	0.5809791037	0.9898824725
15.0	2.299607804e-05	0.5822867591	0.9905987671
16.0	1.984166688e-05	0.5834311111	0.9912227338
17.0	1.728481074e-05	0.5844403654	0.9917708763
18.0	1.518516201e-05	0.5853366859	0.9922560286
19.0	1.344095633e-05	0.5861377084	0.9926883123
20.0	1.197703012e-05	0.5868576267	0.993075812
21.0	1.073694277e-05	0.5875079836	0.9934250589
22.0	9.677678823e-06	0.5880982562	0.9937413846
23.0	8.766011728e-06	0.5886362931	0.9940291821
24.0	7.975961631e-06	0.5891286474	0.9942921025
25.0	7.286987989e-06	0.5895808303	0.9945332042

Table 6.17. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 2$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.2279202041	0.5441595918	0.2279202041
0.20	0.208671766	0.4229852476	0.3683429864
0.40	0.2621331027	0.327113844	0.4107530533
0.60	0.3051459394	0.2600779851	0.4347760755
0.80	0.3380120395	0.2105440153	0.4514439451
1.00	0.3633954976	0.172785513	0.4638189894
2.00	0.4313189796	0.07435985001	0.4943211704
3.00	0.4583094813	0.03807751325	0.5036130054
4.00	0.4714473737	0.02208861808	0.5064640082
5.00	0.4787955491	0.0140616475	0.5071428034
6.00	0.4833461619	0.009598700584	0.5070551375
7.00	0.4863870524	0.00690942014	0.5067035274
8.00	0.488539708	0.005182539881	0.5062777521
9.00	0.490132959	0.004015963892	0.5058510771
10.0	0.4913542797	0.003194894872	0.5054508254
11.0	0.4923172628	0.002597202503	0.5050855347
12.0	0.493094267	0.002149729241	0.5047560038
13.0	0.4937333422	0.001806685886	0.5044599719
14.0	0.4942675308	0.001538322743	0.5041941464
15.0	0.4947202406	0.00132468135	0.503955078
16.0	0.4951084827	0.001151994663	0.5037395226
17.0	0.4954448965	0.001010532885	0.5035445707
18.0	0.4957390564	0.0008932708089	0.5033676728
19.0	0.4959983405	0.0007950388424	0.5032066207
20.0	0.4962285206	0.0007119678796	0.5030595115
21.0	0.4964341739	0.0006411177727	0.5029247083
22.0	0.4966189742	0.0005802234192	0.5028008024
23.0	0.4967859031	0.0005275179256	0.502686579
24.0	0.4969374041	0.0004816073189	0.5025809886
25.0	0.4970754986	0.0004413803845	0.502483121

Table 6.18. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $p = 1$ and $q = 2$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.332933	0.242511	0.185955	0.147129
0.40	1	0.5	0.360028	0.278982	0.22488	0.185462
0.60	1	0.5	0.381039	0.308246	0.257264	0.218551
0.80	1	0.5	0.397635	0.332077	0.284396	0.247027
1.00	1	0.5	0.410959	0.351707	0.307262	0.271535
2.00	1	0.5	0.449774	0.411751	0.380279	0.352931
3.00	1	0.5	0.467161	0.440382	0.417012	0.395875
4.00	1	0.5	0.476351	0.456138	0.437923	0.421054
5.00	1	0.5	0.481812	0.465774	0.451017	0.437138
6.00	1	0.5	0.485353	0.472153	0.459832	0.448121
7.00	1	0.5	0.487803	0.476637	0.466104	0.456017
8.00	1	0.5	0.489585	0.479937	0.470765	0.46193
9.00	1	0.5	0.490933	0.482456	0.474348	0.466503
10.0	1	0.5	0.491984	0.484435	0.47718	0.470135
11.0	1	0.5	0.492825	0.486027	0.479469	0.473083
12.0	1	0.5	0.493511	0.487334	0.481355	0.475518
13.0	1	0.5	0.494082	0.488424	0.482933	0.477563
14.0	1	0.5	0.494563	0.489347	0.484272	0.479301
15.0	1	0.5	0.494973	0.490136	0.485422	0.480796
16.0	1	0.5	0.495328	0.49082	0.486419	0.482095
17.0	1	0.5	0.495636	0.491417	0.487291	0.483233
18.0	1	0.5	0.495908	0.491943	0.48806	0.484238
19.0	1	0.5	0.496148	0.492409	0.488744	0.485132
20.0	1	0.5	0.496362	0.492826	0.489355	0.485932
21.0	1	0.5	0.496554	0.4932	0.489904	0.486652
22.0	1	0.5	0.496727	0.493537	0.490401	0.487303
23.0	1	0.5	0.496884	0.493843	0.490851	0.487895
24.0	1	0.5	0.497027	0.494122	0.491262	0.488435
25.0	1	0.5	0.497158	0.494378	0.491638	0.488929

Table 6.19. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ at $y = 0.5$ with $p = 1$ and $q = 2$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.332933	0.242511	0.185955	0.147129
0.40	1	0.5	0.360028	0.278982	0.22488	0.185462
0.60	1	0.5	0.381039	0.308246	0.257264	0.218551
0.80	1	0.5	0.397635	0.332077	0.284396	0.247027
1.00	1	0.5	0.410959	0.351707	0.307262	0.271535
2.00	1	0.5	0.449774	0.411751	0.380279	0.352931
3.00	1	0.5	0.467161	0.440382	0.417012	0.395875
4.00	1	0.5	0.476351	0.456138	0.437923	0.421054
5.00	1	0.5	0.481812	0.465774	0.451017	0.437138
6.00	1	0.5	0.485353	0.472153	0.459832	0.448121
7.00	1	0.5	0.487803	0.476637	0.466104	0.456017
8.00	1	0.5	0.489585	0.479937	0.470765	0.46193
9.00	1	0.5	0.490933	0.482456	0.474348	0.466503
10.0	1	0.5	0.491984	0.484435	0.47718	0.470135
11.0	1	0.5	0.492825	0.486027	0.479469	0.473083
12.0	1	0.5	0.493511	0.487334	0.481355	0.475518
13.0	1	0.5	0.494082	0.488424	0.482933	0.477563
14.0	1	0.5	0.494563	0.489347	0.484272	0.479301
15.0	1	0.5	0.494973	0.490136	0.485422	0.480796
16.0	1	0.5	0.495328	0.49082	0.486419	0.482095
17.0	1	0.5	0.495636	0.491417	0.487291	0.483233
18.0	1	0.5	0.495908	0.491943	0.48806	0.484238
19.0	1	0.5	0.496148	0.492409	0.488744	0.485132
20.0	1	0.5	0.496362	0.492826	0.489355	0.485932
21.0	1	0.5	0.496554	0.4932	0.489904	0.486652
22.0	1	0.5	0.496727	0.493537	0.490401	0.487303
23.0	1	0.5	0.496884	0.493843	0.490851	0.487895
24.0	1	0.5	0.497027	0.494122	0.491262	0.488435
25.0	1	0.5	0.497158	0.494378	0.491638	0.488929

Table 6.20. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 2$ and $q = 1$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
0.20	0.1758216879	0.5590061079	0.9526251743
0.40	0.1523356964	0.5551395273	0.9759691556
0.60	0.1326059191	0.5449341118	0.9851704193
0.80	0.1169171195	0.5348787292	0.9899143221
1.00	0.1043008219	0.5259505367	0.9927108819
2.00	0.06646149033	0.4948664704	0.9977499303
3.00	0.04773060294	0.475473042	0.9990136255
4.00	0.03672207624	0.4617137205	0.9994815929
5.00	0.0295817461	0.4515526087	0.999692369
6.00	0.02463095868	0.4439050404	0.9998007068
7.00	0.02102487236	0.4380411299	0.9998621681
8.00	0.01829599644	0.4334528457	0.9998998022
9.00	0.01616708221	0.4297899405	0.9999242739
10.0	0.01446452871	0.426811189	0.9999409701
11.0	0.01307471238	0.4243485741	0.9999528149
12.0	0.01192046553	0.4222829731	0.999961493
13.0	0.01094771232	0.4205282486	0.9999680247
14.0	0.01011752754	0.4190208963	0.9999730543
15.0	0.009401232879	0.4177132409	0.9999770039
16.0	0.008777266222	0.4165688889	0.9999801583
17.0	0.008229123669	0.4155596346	0.9999827152
18.0	0.007743971397	0.4146633141	0.9999848148
19.0	0.00731168767	0.4138622916	0.999986559
20.0	0.006924188018	0.4131423733	0.999988023
21.0	0.006574941083	0.4124920164	0.9999892631
22.0	0.006258615421	0.4119017438	0.9999903223
23.0	0.005970817947	0.4113637069	0.999991234
24.0	0.005707897548	0.4108713526	0.999992024
25.0	0.005466795779	0.4104191697	0.999992713
26.0	0.005244932026	0.4100024907	0.9999933173
27.0	0.005040114249	0.4096173381	0.9999938502
28.0	0.004850468907	0.4092603019	0.9999943223
29.0	0.004674385462	0.4089284431	0.9999947426
30.0	0.004510472031	0.4086192151	0.9999951184

Table 6.21. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes *, + and x at $y = 0.5$ with $p = 2$ and $q = 1$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.2279202041	0.5441595918	0.2279202041
0.20	0.3683429864	0.4229852476	0.208671766
0.40	0.4107530533	0.327113844	0.2621331027
0.60	0.4347760755	0.2600779851	0.3051459394
0.80	0.4514439451	0.2105440153	0.3380120395
1.00	0.4638189894	0.172785513	0.3633954976
2.00	0.4943211704	0.07435985001	0.4313189796
3.00	0.5036130054	0.03807751325	0.4583094813
4.00	0.5064640082	0.02208861808	0.4714473737
5.00	0.5071428034	0.0140616475	0.4787955491
6.00	0.5070551375	0.009598700584	0.4833461619
7.00	0.5067035274	0.00690942014	0.4863870524
8.00	0.5062777521	0.005182539881	0.488539708
9.00	0.5058510771	0.004015963892	0.490132959
10.0	0.5054508254	0.003194894872	0.4913542797
11.0	0.5050855347	0.002597202503	0.4923172628
12.0	0.5047560038	0.002149729241	0.493094267
13.0	0.5044599719	0.001806685886	0.4937333422
14.0	0.5041941464	0.001538322743	0.4942675308
15.0	0.503955078	0.00132468135	0.4947202406
16.0	0.5037395226	0.001151994663	0.4951084827
17.0	0.5035445707	0.001010532885	0.4954448965
18.0	0.5033676728	0.0008932708089	0.4957390564
19.0	0.5032066207	0.0007950388424	0.4959983405
20.0	0.5030595115	0.0007119678796	0.4962285206
21.0	0.5029247083	0.0006411177727	0.4964341739
22.0	0.5028008024	0.0005802234192	0.4966189742
23.0	0.502686579	0.0005275179256	0.4967859031
24.0	0.5025809886	0.0004816073189	0.4969374041
25.0	0.502483121	0.0004413803845	0.4970754986
26.0	0.5023921849	0.0004059428479	0.4972018722
27.0	0.5023074899	0.0003745686933	0.4973179415
28.0	0.5022284312	0.0003466637192	0.4974249051
29.0	0.5021544778	0.000321737949	0.4975237843
30.0	0.5020851616	0.0002993845278	0.4976154538

Table 6.22. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $p = 2$ and $q = 1$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.332933	0.256287	0.213507	0.186861
0.40	1	0.5	0.360028	0.301101	0.269119	0.249395
0.60	1	0.5	0.381039	0.33487	0.310512	0.295697
0.80	1	0.5	0.397635	0.360828	0.341898	0.330535
1.00	1	0.5	0.410959	0.381172	0.366192	0.357303
2.00	1	0.5	0.449774	0.437571	0.431919	0.428696
3.00	1	0.5	0.467161	0.461102	0.458453	0.456979
4.00	1	0.5	0.476351	0.472914	0.471475	0.47069
5.00	1	0.5	0.481812	0.479662	0.478792	0.478324
6.00	1	0.5	0.485353	0.483904	0.483334	0.48303
7.00	1	0.5	0.487803	0.486771	0.486373	0.486163
8.00	1	0.5	0.489585	0.488818	0.488527	0.488374
9.00	1	0.5	0.490933	0.490343	0.490122	0.490006
10.0	1	0.5	0.491984	0.491517	0.491344	0.491255
11.0	1	0.5	0.492825	0.492447	0.492309	0.492237
12.0	1	0.5	0.493511	0.4932	0.493087	0.493028
13.0	1	0.5	0.494082	0.493821	0.493727	0.493678
14.0	1	0.5	0.494563	0.494341	0.494262	0.494221
15.0	1	0.5	0.494973	0.494783	0.494715	0.49468
16.0	1	0.5	0.495328	0.495163	0.495104	0.495074
17.0	1	0.5	0.495636	0.495492	0.495441	0.495415
18.0	1	0.5	0.495908	0.49578	0.495735	0.495712
19.0	1	0.5	0.496148	0.496035	0.495995	0.495975
20.0	1	0.5	0.496362	0.496261	0.496225	0.496207
21.0	1	0.5	0.496554	0.496463	0.496431	0.496415
22.0	1	0.5	0.496727	0.496645	0.496616	0.496602
23.0	1	0.5	0.496884	0.49681	0.496784	0.49677
24.0	1	0.5	0.497027	0.496959	0.496935	0.496923
25.0	1	0.5	0.497158	0.497095	0.497074	0.497063
26.0	1	0.5	0.497277	0.49722	0.4972	0.49719
27.0	1	0.5	0.497387	0.497335	0.497316	0.497307
28.0	1	0.5	0.497489	0.49744	0.497423	0.497415
29.0	1	0.5	0.497583	0.497538	0.497522	0.497514
30.0	1	0.5	0.497671	0.497629	0.497614	0.497607

Table 6.23. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ at $y = 0.5$ with $p = 2$ and $q = 1$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
0.20	1	0.5	0.332933	0.256287	0.213507	0.186861
0.40	1	0.5	0.360028	0.301101	0.269119	0.249395
0.60	1	0.5	0.381039	0.33487	0.310512	0.295697
0.80	1	0.5	0.397635	0.360828	0.341898	0.330535
1.00	1	0.5	0.410959	0.381172	0.366192	0.357303
2.00	1	0.5	0.449774	0.437571	0.431919	0.428696
3.00	1	0.5	0.467161	0.461102	0.458453	0.456979
4.00	1	0.5	0.476351	0.472914	0.471475	0.47069
5.00	1	0.5	0.481812	0.479662	0.478792	0.478324
6.00	1	0.5	0.485353	0.483904	0.483334	0.48303
7.00	1	0.5	0.487803	0.486771	0.486373	0.486163
8.00	1	0.5	0.489585	0.488818	0.488527	0.488374
9.00	1	0.5	0.490933	0.490343	0.490122	0.490006
10.0	1	0.5	0.491984	0.491517	0.491344	0.491255
11.0	1	0.5	0.492825	0.492447	0.492309	0.492237
12.0	1	0.5	0.493511	0.4932	0.493087	0.493028
13.0	1	0.5	0.494082	0.493821	0.493727	0.493678
14.0	1	0.5	0.494563	0.494341	0.494262	0.494221
15.0	1	0.5	0.494973	0.494783	0.494715	0.49468
16.0	1	0.5	0.495328	0.495163	0.495104	0.495074
17.0	1	0.5	0.495636	0.495492	0.495441	0.495415
18.0	1	0.5	0.495908	0.49578	0.495735	0.495712
19.0	1	0.5	0.496148	0.496035	0.495995	0.495975
20.0	1	0.5	0.496362	0.496261	0.496225	0.496207
21.0	1	0.5	0.496554	0.496463	0.496431	0.496415
22.0	1	0.5	0.496727	0.496645	0.496616	0.496602
23.0	1	0.5	0.496884	0.49681	0.496784	0.49677
24.0	1	0.5	0.497027	0.496959	0.496935	0.496923
25.0	1	0.5	0.497158	0.497095	0.497074	0.497063
26.0	1	0.5	0.497277	0.49722	0.4972	0.49719
27.0	1	0.5	0.497387	0.497335	0.497316	0.497307
28.0	1	0.5	0.497489	0.49744	0.497423	0.497415
29.0	1	0.5	0.497583	0.497538	0.497522	0.497514
30.0	1	0.5	0.497671	0.497629	0.497614	0.497607

Integrating (6.56) over Ω, integrating by parts and using boundary condition (6.57), (6.58) we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)dxdy &= \int_0^1\int_0^1((x(1-x)u)_{xx}+u_{yy})dxdy \\
&= \int_0^1 (x(1-x)u)_x|_{x=0}^1 dy + \int_0^1 u_y|_{y=0}^1 dx \\
&= \int_0^1 -\mathrm{P}(x,y,t)|_{x=0}^1 dy \\
&= \int_0^1 -\{\mathrm{P}(1,y,t)-\mathrm{P}(1,y,t)\}dy \\
&= -\int_0^1 u(0,y,t)dy - \int_0^1 u(1,y,t)dy.
\end{aligned} \tag{6.62}$$

Integrating (6.62) over $\tau \in (0,t)$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 u(x,y,t)dxdy - \int_0^1\int_0^1 u(x,y,0)dxdy \\
&= -\int_0^t\int_0^1 u(0,y,\tau)dyd\tau - \int_0^t\int_0^1 u(1,y,\tau)dyd\tau
\end{aligned} \tag{6.63}$$

so,

$$\begin{aligned}
&\int_0^1\int_0^1 u(x,y,t)dxdy \\
&= \int_0^1\int_0^1 u(x,y,0)dxdy - \int_0^t\int_0^1 u(0,y,\tau)dyd\tau - \int_0^t\int_0^1 u(1,y,\tau)dyd\tau \\
&= 0.5 - \int_0^t\int_0^1 u(0,y,\tau)dyd\tau - \int_0^t\int_0^1 u(1,y,\tau)dydt\tau.
\end{aligned} \tag{6.64}$$

Multiplying by x and integrating (6.56) over Ω, integrating by parts and using boundary condition (6.57), (6.58) we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1\int_0^1 u(x,y,t)xdxdy &= \int_0^1\int_0^1((x(1-x)u)_{xx}x+u_{yy}x)dxdy \\
&= \int_0^1 (x(1-x)u)_x x|_{x=0}^1 dy + \int_0^1 u_y|_{y=0}^1 xdx \\
&= \int_0^1 -\mathrm{P}(x,y,t)x|_{x=0}^1 dy \\
&= \int_0^1 -\mathrm{P}(1,y,t)dy = -\int_0^1 u(1,y,t)dy.
\end{aligned} \tag{6.65}$$

Integrating (6.65) over $\tau \in (0,t)$, we have

$$\int_0^1\int_0^1 u(x,y,t)xdxdy - \int_0^1\int_0^1 u(x,y,0)xdxdy = -\int_0^t\int_0^1 u(1,y,\tau)dyd\tau. \quad (6.66)$$

Thus,

$$\begin{aligned}\int_0^1\int_0^1 u(x,y,t)xdxdy &= \int_0^1\int_0^1 u(x,y,0)xdxdy - \int_0^t\int_0^1 u(1,y,\tau)dyd\tau \\ &= 0.2974 - \int_0^t\int_0^1 u(1,y,\tau)dyd\tau.\end{aligned} \quad (6.67)$$

where $\int_0^t\int_0^1 u(0,y,\tau)dyd\tau$ and $\int_0^t\int_0^1 u(1,y,\tau)dyd\tau$ represent the fluxes across boundaries $x = 0$ and $x = 1$. In this example, $x = 0$ and $x = 1$ are exit boundaries. For fixed y, one can show that the density $u(x,y,\infty) = 0$, $\forall x \in (0,1)$, is a steady state solution and u will "pile" at $x = 0$ and $x = 1$ with a Dirac-delta function singularity formed at $x = 0$ and $x = 1$. So, for steady state solutin, $\int_0^1\int_0^1 u(x,y,t)dxdy = 0$ in the sense that the integration is only over the open region Ω. Therefore, by (6.64) and (6.67), we have

$$\int_0^t\int_0^1 u(0,y,\tau)dyd\tau + \int_0^t\int_0^1 u(1,y,\tau)dyd\tau = 0.5 \quad (6.68)$$

$$\int_0^t\int_0^1 u(1,y,\tau)dyd\tau = 0.2974. \quad (6.69)$$

Substituting (6.69) into (6.68), we obtain

$$\int_0^t\int_0^1 u(0,y,\tau)dyd\tau = 0.2026. \quad (6.70)$$

(6.69) and (6.70) show that the amounts of the fluxes leaking out of the boundaries $x = 0$ and $x = 1$ are 0.2026 and 0.2974 respectively. Our numerical result shall support this.

As in Model I, we apply the Gauss-Galerkin finite element method using finite element approximations in the y-direction and Gauss-Galerkin approximations in the x-direction. We divide $y \in [0, 1]$ into four equal subintervals. Then we have five grid lines $y_j = jh, j = 0, 1, \cdots, 4, h = 0.25$. In the x-direction we use three nodes labeled by $*, +$ and x for each grid line $y = y_j$. We only discuss the results along grid $y = 0.5$. The results along $y = 0.25$ and $y = 0.75$ are similar. Table 6.28 shows the changes of nodes $*, +$ and x as t increases. Table 6.29 shows the changes of the weights at the three nodes $*, +$ and x as t increases. Table 6.30 shows the changes of the ith "moment" $m_n^i(y, t)$ at $y = 0.5$ as t increases. Table 6.31 shows changes of the "total moment" $M_n^i(t)$ as t increases.

(1) Table 6.31 shows that $M_n^0(t) = 0.5$ and $M_n^1(t) = 0.297358$ for $t \geq 0$. This shows that the approximation of the 0th and 1st total moments equal the exact 0th and 1st total moments.

(2) At $t = 5$, the solution reaches the steady state based on the tolerance chosen.

(3) The solutinon becomes uniform in y as t increases.

(4) The solution approaches zero in the interior and "piles" up at the boundary $x = 0$ and $x = 1$ as t increases.

(5) The weights for the middle nodes tend to zero as t increases.

(6) The weights at the other two nodes tend to 0.202642 and 0.297357 respectively as t increases. They are the amounts of the fluxes leaking out from $x = 0$ and $x = 1$.

(7) The steady state solution shows that the ith "total moment" are all about 0.297358 for i from 1 to 5. Since the weight tends to zero for the interior node , 0.202642 for the node approaching to $x = 0$ and 0.297358 for the node approaching $x = 1$, the contribution of u to the total moments is that of those

two weights to the total moment. The total moment contributed by the weight at $x = 0$ is zero and the total moment contributed by the weight at $x = 1$ is 0.297358 which is equal to the weight at $x = 1$.

6.4 Dependence of Solution upon Lower Order Terms

We consider the following initial-boundary value problems:

$$u_t = -(au)_x + \frac{1}{2}(b^2u)_{xx} + u_{yy} \quad \text{in} \quad (0,T)\times\Omega \tag{6.71}$$

with the boundary conditions

$$\lim_{x\to 0}\left\{au - \frac{1}{2}\frac{\partial(b^2u)}{\partial x}\right\} = \lim_{x\to 1}\left\{au - \frac{1}{2}\frac{\partial(b^2u)}{\partial x}\right\} = 0 \quad \text{on} \quad (0,T)\times(0,1) \tag{6.72}$$

$$u_y(x,0,t) = u_y(x,1,t) = 0 \quad \text{on} \quad (0,T)\times(0,1) \tag{6.73}$$

and initial conditions

$$u(x,y,0) = \pi\cos^2\left(\frac{\pi y}{2}\right)\sin(\pi x) \quad \text{on} \quad \Omega \tag{6.74}$$

where

$$a(x) = sx(1-x)(h+(1-2h)x) - \mu x + \nu(1-x) \tag{6.75}$$

$$b^2(x) = \frac{1}{2}x(1-x) \tag{6.76}$$

Case I $s = 0, \mu = \nu = 0.375$

For this model, both $x = 0$ and $x = 1$ are entrance boundaries. The numerical solution is expressed in Figure 6.6, Tables 6.32, 6.33, 6.34, and 6.35. We observe the following facts:

(1) At $t = 5$, the solution reaches the steady state based on the tolerance chosen.

(2) Table 6.35 shows that $M_n^0(t) = 1, t \geq 0; M_n^1(t) = 0.5, t \geq 0; M_n^2(t) = 0.3125, t \geq 5; M_n^3(t) = 0.21875, t \geq 5; M_n^4(t) = 0.164062, t \geq 5, M_n^5(t) = 0.128906, t \geq 5.$

Table 6.24. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 1.5$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
1.00	0.007293214038	0.4974373111	0.9525687726
2.00	0.001729665784	0.5197840348	0.979205428
3.00	0.0005604709585	0.5350376191	0.9890444785
4.00	0.0002200237825	0.5471084894	0.9935435888
5.00	0.0001006076538	0.556561809	0.9958664928
6.00	5.20099078e-05	0.5637753157	0.9971781213
7.00	2.961587501e-05	0.5692445438	0.9979737196
8.00	1.81891456e-05	0.5734347156	0.9984853627
9.00	1.185755384e-05	0.5767034381	0.998830589
10.0	8.107449522e-06	0.5793039239	0.9990729698
11.0	5.762321927e-06	0.5814116194	0.999248879
12.0	4.228566618e-06	0.5831487323	0.9993801606
13.0	3.18718017e-06	0.584601736	0.9994804905
14.0	2.457345292e-06	0.5858329852	0.9995587449
15.0	1.931810867e-06	0.5868883096	0.9996208684
16.0	1.544429918e-06	0.5878020086	0.9996709522
17.0	1.253002659e-06	0.5886001862	0.999711881
18.0	1.029798929e-06	0.5893030166	0.9997457327
19.0	8.561160685e-07	0.5899263102	0.9997740324
20.0	7.190453803e-07	0.5904826164	0.999797919
21.0	6.094917483e-07	0.5909820126	0.9998182558
22.0	5.209274238e-07	0.5914326765	0.9998357065
23.0	4.485885282e-07	0.5918413091	0.9998507877
24.0	3.889454961e-07	0.5922134506	0.9998639063
25.0	3.3934703e-07	0.592553718	0.999875386
26.0	2.977763748e-07	0.5928659899	0.9998854866
27.0	2.626816346e-07	0.5931535445	0.9998944189
28.0	2.328558889e-07	0.593419172	0.9999023551
29.0	2.073512635e-07	0.5936652628	0.9999094371
30.0	1.854165287e-07	0.5938938743	0.9999157823

Table 6.25. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 2.5$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
1.00	0.006934697153	0.4474280303	0.8442717204
2.00	0.002390166716	0.4815722684	0.8828764173
3.00	0.001194973711	0.5020708575	0.9051789036
4.00	0.0007074098499	0.5162853797	0.919730071
5.00	0.0004640957196	0.5268758356	0.930018803
6.00	0.0003268038883	0.5350669994	0.9377028371
7.00	0.0002423600226	0.5415689932	0.9436744707
8.00	0.0001869473545	0.5468414933	0.9484583795
9.00	0.0001487047648	0.5511965095	0.9523838006
10.0	0.0001212280518	0.5548516757	0.9556679567
11.0	0.0001008288947	0.5579622653	0.9584600385
12.0	8.526779069e-05	0.5606414518	0.9608659536
13.0	7.312371114e-05	0.562973402	0.9629629983
14.0	6.346080839e-05	0.565021882	0.964808936
15.0	5.564286602e-05	0.5668360216	0.9664478215
16.0	4.922533953e-05	0.5684542522	0.9679138568
17.0	4.389009955e-05	0.5699070505	0.969234012
18.0	3.940465289e-05	0.5712188853	0.9704298507
19.0	3.559590347e-05	0.5724096224	0.9715188278
20.0	3.233282601e-05	0.5734955559	0.9725152292
21.0	2.951475796e-05	0.5744901743	0.9734308648
22.0	2.706332218e-05	0.5754047389	0.9742755861
23.0	2.491674961e-05	0.5762487247	0.975057679
24.0	2.302582074e-05	0.5770301612	0.9757841653
25.0	2.135091924e-05	0.5777558979	0.9764610369
26.0	1.98598627e-05	0.5784318137	0.9770934395
27.0	1.85262848e-05	0.579062983	0.977685818
28.0	1.732841432e-05	0.5796538091	0.9782420332
29.0	1.624814341e-05	0.5802081316	0.9787654554
30.0	1.527030942e-05	0.5807293142	0.9792590406

Table 6.26. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 3$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
1.00	0.006622257948	0.4211193674	0.8038513326
2.00	0.00233840896	0.457402078	0.837010961
3.00	0.001249800175	0.4785285189	0.8591502431
4.00	0.0007922273888	0.4924933102	0.8746602438
5.00	0.0005508072973	0.5028776844	0.8862115773
6.00	0.0004068231039	0.511067948	0.8952136158
7.00	0.0003138615836	0.5177367631	0.9024675006
8.00	0.0002502851745	0.5232846596	0.9084639723
9.00	0.0002048309658	0.5279787433	0.9135220107
10.0	0.0001711613512	0.5320071041	0.9178588005
11.0	0.0001454895413	0.5355063113	0.9216278052
12.0	0.000125440224	0.5385778075	0.9249408108
13.0	0.0001094620382	0.5412984505	0.9278813437
14.0	9.650658451e-05	0.5437275232	0.9305131806
15.0	8.584437752e-05	0.5459114949	0.932885936
16.0	7.695486689e-05	0.5478873207	0.9350388391
17.0	6.945832895e-05	0.54968477	0.9370033543
18.0	6.307235147e-05	0.5513281031	0.9388050395
19.0	5.758325838e-05	0.5528372972	0.9404648907
20.0	5.282688369e-05	0.5542289588	0.942000332
21.0	4.867535252e-05	0.5555170114	0.9434259562
22.0	4.502781372e-05	0.5567132207	0.9447540867
23.0	4.180382803e-05	0.5578276006	0.9459952095
24.0	3.893857494e-05	0.558868731	0.947158309
25.0	3.63793274e-05	0.5598440076	0.9482511324
26.0	3.408282432e-05	0.5607598416	0.9492803991
27.0	3.201328819e-05	0.5616218201	0.9502519686
28.0	3.014091267e-05	0.5624348355	0.9511709766
29.0	2.844069649e-05	0.5632031917	0.9520419453
30.0	2.68915359e-05	0.5639306908	0.9528688739

Table 6.27. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 4$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
1.00	0.006607377161	0.3792326904	0.7588257646
2.00	0.00220046939	0.4067958452	0.7706931329
3.00	0.001121610614	0.4347173604	0.7857362179
4.00	0.0007402373695	0.4511796931	0.7984336386
5.00	0.0005521756939	0.461159263	0.8088016673
6.00	0.0004366788498	0.4682746548	0.8174066602
7.00	0.0003569970948	0.4740401834	0.8247015148
8.00	0.0002986621627	0.4790028157	0.830998128
9.00	0.0002544180809	0.4833734662	0.8365121318
10.0	0.0002199843118	0.4872611973	0.8413976884
11.0	0.0001926105299	0.4907432172	0.8457687842
12.0	0.0001704486633	0.4938820927	0.8497120548
13.0	0.0001522201593	0.4967295525	0.8532948084
14.0	0.0001370190496	0.4993280139	0.8565702828
15.0	0.0001241892226	0.5017120656	0.8595812315
16.0	0.000113245834	0.5039099728	0.862362445
17.0	0.000103823944	0.505945009	0.8649425679
18.0	9.564427449e-05	0.5078365278	0.8673454335
19.0	8.848991239e-05	0.5096007849	0.8695910593
20.0	8.219013692e-05	0.5112515593	0.8716964022
21.0	7.660897502e-05	0.5128006243	0.8736759379
22.0	7.163695866e-05	0.5142581095	0.8755421114
23.0	6.718509269e-05	0.5156327856	0.8773056909
24.0	6.318037653e-05	0.5169322892	0.8789760491
25.0	5.956243628e-05	0.5181633045	0.8805613887
26.0	5.628096165e-05	0.5193317105	0.8820689248
27.0	5.329373437e-05	0.5204427018	0.8835050333
28.0	5.056509627e-05	0.5215008872	0.8848753732
29.0	4.806474791e-05	0.5225103727	0.8861849877
30.0	4.576679829e-05	0.5234748294	0.8874383881

Table 6.28. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 1$ as t increases

time	x_1	x_2	x_3
0.00	0.2502831702	0.5642793625	0.8499422355
0.10	0.1380694362	0.5448216696	0.9220406469
0.20	0.08464881217	0.5296369026	0.9516064912
0.30	0.05599679895	0.5196505098	0.9672745255
0.40	0.03910072152	0.5131443416	0.9766478603
0.50	0.02837399926	0.5088757108	0.9827179892
0.60	0.0211687011	0.5060494594	0.986880513
0.70	0.01611729371	0.5041619245	0.9898584227
0.80	0.01245896946	0.5028910896	0.9920570017
0.90	0.00974278762	0.502028702	0.993718743
1.00	0.007686885634	0.5014387613	0.9949969915
2.00	0.0008961392785	0.5000843021	0.9993932054
3.00	0.0001169158509	0.5000095131	0.9999203926
4.00	1.547189458e-05	0.5000012352	0.9999894575
5.00	2.051288952e-06	0.5000001633	0.9999986021
6.00	2.720307154e-07	0.5000000222	0.9999998146
7.00	3.607640753e-08	0.4999999934	0.9999999754
8.00	4.784437624e-09	0.5000000303	0.9999999967
9.00	6.345100001e-10	0.5000000345	0.9999999996
10.0	8.415590447e-11	0.5000064601	0.9999999999
11.0	1.116406967e-11	0.5000429575	1
12.0	1.48370205e-12	0.5002227197	1
13.0	1.962874308e-13	0.4991273235	1

Table 6.29. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes $*$, $+$ and x at $y = 0.5$ with $p = 1$ and $q = 1$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.08130670793	0.2760498056	0.1426434865
0.10	0.09270514293	0.2411981715	0.1660966856
0.20	0.1069472399	0.2031589108	0.1898938493
0.30	0.1213260558	0.1691551282	0.209518816
0.40	0.1345030966	0.1399413413	0.2255555621
0.50	0.146028531	0.1153124435	0.2386590255
0.60	0.1558557559	0.09477524188	0.2493690023
0.70	0.1641108891	0.07776472834	0.2581243826
0.80	0.1709822653	0.06373497565	0.265282759
0.90	0.1766687044	0.05219544718	0.2711358484
1.00	0.1813567149	0.04272149528	0.2759217898
2.00	0.1997940823	0.005698205879	0.2945077118
3.00	0.2022642862	0.0007561888181	0.296979525
4.00	0.2025922205	0.0001002939788	0.2973074855
5.00	0.2026357167	1.330109301e-05	0.2973509822
6.00	0.2026414853	1.763987685e-06	0.2973567507
7.00	0.2026422503	2.339393116e-07	0.2973575157
8.00	0.2026423518	3.102492757e-08	0.2973576172
9.00	0.2026423652	4.114513505e-09	0.2973576307
10.0	0.202642367	5.456614616e-10	0.2973576324
11.0	0.2026423672	7.236511499e-11	0.2973576327
12.0	0.2026423673	9.595827944e-12	0.2973576327
13.0	0.2026423673	1.272212315e-12	0.2973576327

Table 6.30. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $p = 1$ and $q = 1$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	0.5	0.297358	0.196036	0.138456	0.102747	0.0791428
0.10	0.5	0.297358	0.214571	0.169451	0.141335	0.122274
0.20	0.5	0.297358	0.229715	0.193886	0.17171	0.15665
0.30	0.5	0.297358	0.242089	0.213373	0.195746	0.183817
0.40	0.5	0.297358	0.252199	0.229037	0.214917	0.2054
0.50	0.5	0.297358	0.26046	0.241697	0.230316	0.222672
0.60	0.5	0.297358	0.267209	0.251966	0.242753	0.23658
0.70	0.5	0.297358	0.272724	0.260316	0.252836	0.247831
0.80	0.5	0.297358	0.27723	0.267118	0.26103	0.256963
0.90	0.5	0.297358	0.280912	0.272663	0.267703	0.264391
1.00	0.5	0.297358	0.28392	0.277188	0.273142	0.270443
2.00	0.5	0.297358	0.295576	0.294685	0.29415	0.293793
3.00	0.5	0.297358	0.297121	0.297003	0.296932	0.296885
4.00	0.5	0.297358	0.297326	0.297311	0.297301	0.297295
5.00	0.5	0.297358	0.297353	0.297351	0.29735	0.297349
6.00	0.5	0.297358	0.297357	0.297357	0.297357	0.297357
7.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297357
8.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
9.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
10.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
11.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
12.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
13.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358

Table 6.31. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ at $y = 0.5$ with $p = 1$ and $q = 1$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	0.5	0.297358	0.196036	0.138456	0.102747	0.0791428
0.10	0.5	0.297358	0.214571	0.169451	0.141335	0.122274
0.20	0.5	0.297358	0.229715	0.193886	0.17171	0.15665
0.30	0.5	0.297358	0.242089	0.213373	0.195746	0.183817
0.40	0.5	0.297358	0.252199	0.229037	0.214917	0.2054
0.50	0.5	0.297358	0.26046	0.241697	0.230316	0.222672
0.60	0.5	0.297358	0.267209	0.251966	0.242753	0.23658
0.70	0.5	0.297358	0.272724	0.260316	0.252836	0.247831
0.80	0.5	0.297358	0.27723	0.267118	0.26103	0.256963
0.90	0.5	0.297358	0.280912	0.272663	0.267703	0.264391
1.00	0.5	0.297358	0.28392	0.277188	0.273142	0.270443
2.00	0.5	0.297358	0.295576	0.294685	0.29415	0.293793
3.00	0.5	0.297358	0.297121	0.297003	0.296932	0.296885
4.00	0.5	0.297358	0.297326	0.297311	0.297301	0.297295
5.00	0.5	0.297358	0.297353	0.297351	0.29735	0.297349
6.00	0.5	0.297358	0.297357	0.297357	0.297357	0.297357
7.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297357
8.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
9.00	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
10.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
11.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
12.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358
13.0	0.5	0.297358	0.297358	0.297358	0.297358	0.297358

This agrees with the solution

$$g(x,y) = \frac{x^{4\nu-1}(1-x)^{4\mu-1}}{\mathrm{B}(4\mu,4\nu)} \tag{6.77}$$

where B(., .) is the beta function. It is concave down.

(3) The solution becomes uniform in y as t increases.

(4) Since $x = 0$ and $x = 1$ are entrance boundaries, they can not be reached from the interior. The nodes tend to 0.14647, 0.5 and 0.85355.

(5) The weight for the middle node tends to 0.5 as t increases.

(6) The weights at the other two nodes tend to 0.25 as t increases.

Case II $s = 0, \mu = \nu = 0.25$

In this case, we obtain similar numerical results. The steady state solution agrees with (6.77) again. It is a straight line.

Case III $s = 2, h = 0.5, \mu = \nu = 0$

For this model, both $x = 0$ and $x = 1$ are entrance boundaries. The numerical solution is shown in Figure 6.7, Tables 6.36, 6.37, 6.38 and 6.39. We observe the following facts:

(1) At $t = 30$, the solution reaches the steady state based on the tolerance chosen.

(2) Table 6.39 shows that $M_n^0(t) = 1, M_n^1(t) = M_n^2(t) = M_n^3(t) = M_n^4(t) = M_n^5(t) =$ 0.830323.

(3) The solutinon becomes uniform in x as t increases.

(4) Since $x = 0$ and $x = 1$ are exit boundaries, Table 6.36 shows that the solution tends to zero at the interior node and "piles" up at $x = 0^+$ and $x = 1^-$ as t increases.

(5) The weights for the middle nodes tend to zero as t increases.

(6) The weights at the $x = 0$ is 0.169677 and 0.830322 at $x = 1$as t increases.

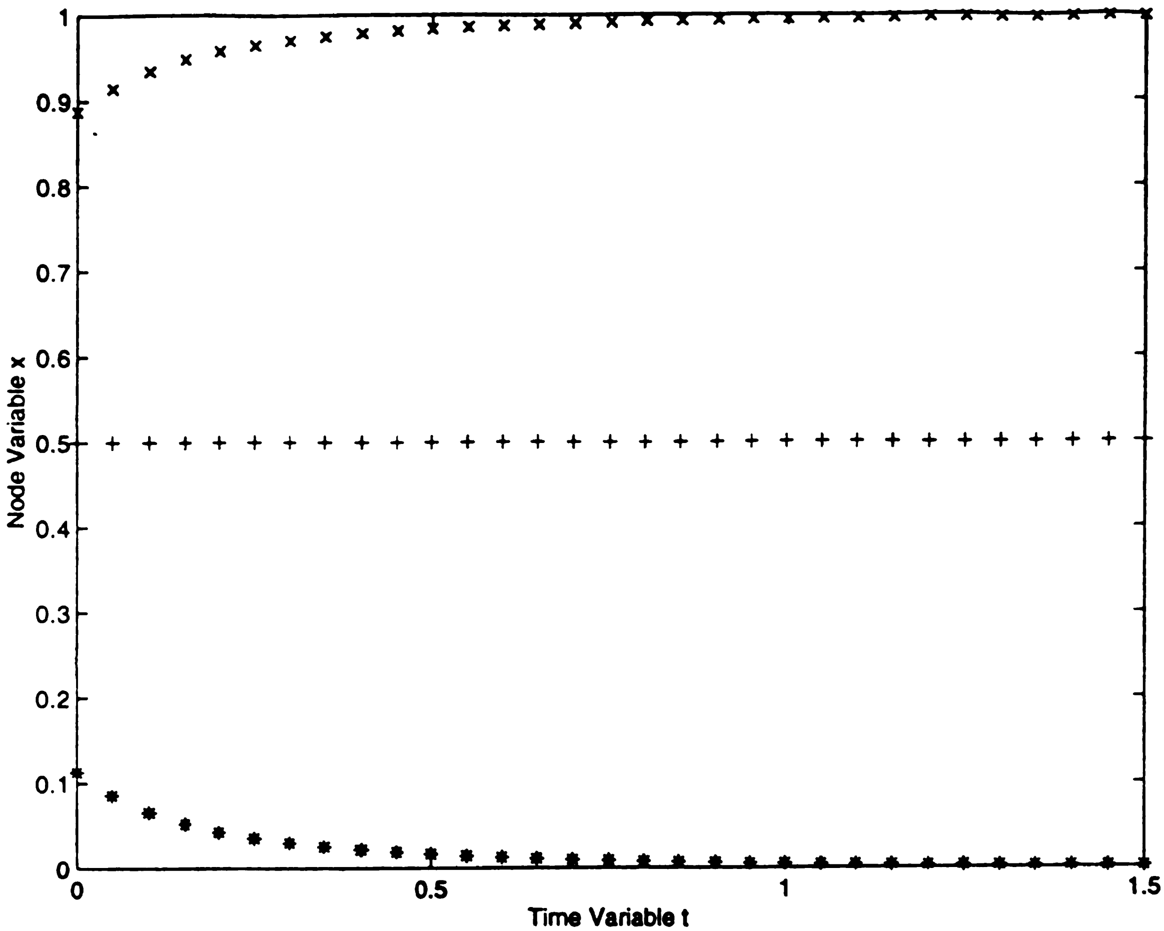


Figure 6.1. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x as t increases

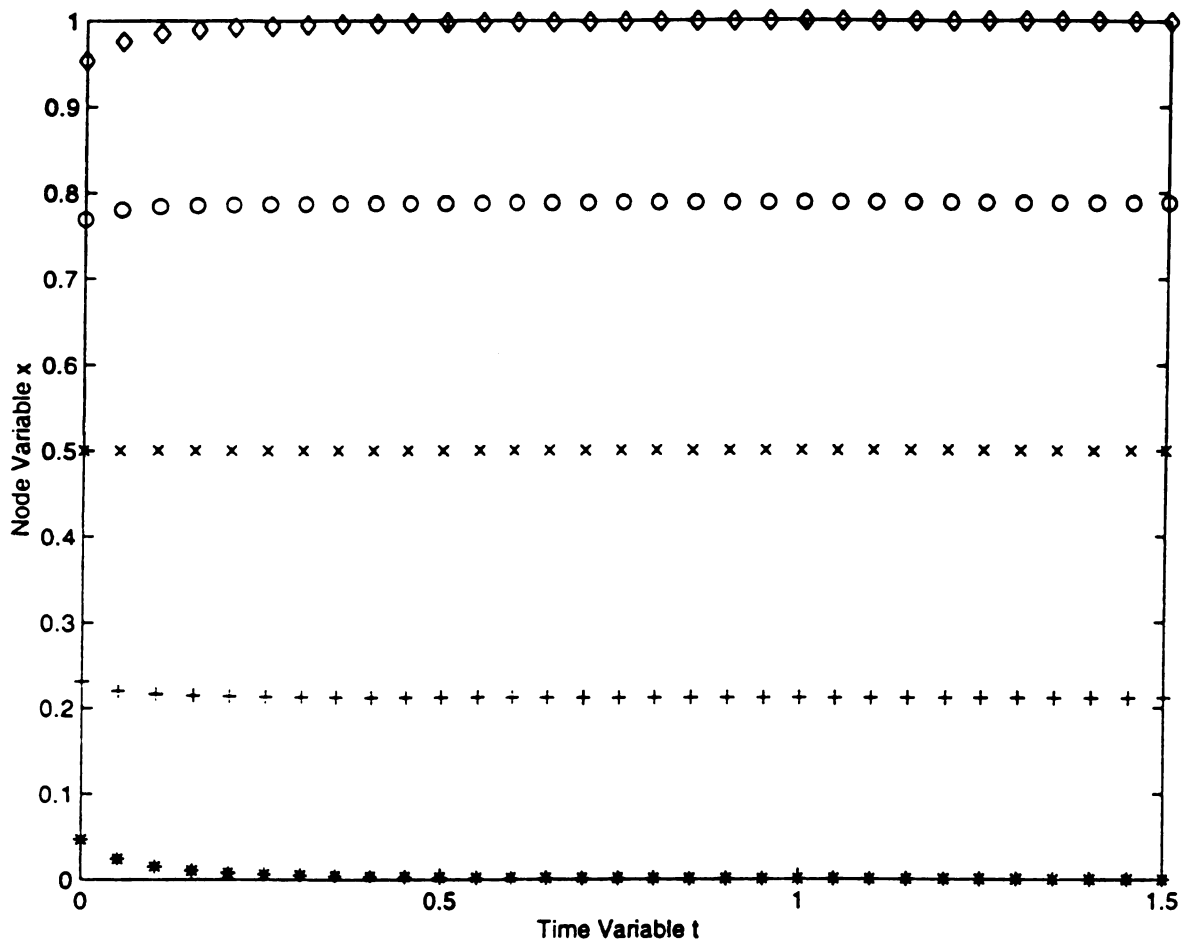


Figure 6.2. Gauss-Galerkin Finite Element Method: Movement of the nodes $\diamond$, o, x, + and $*$ as t increases

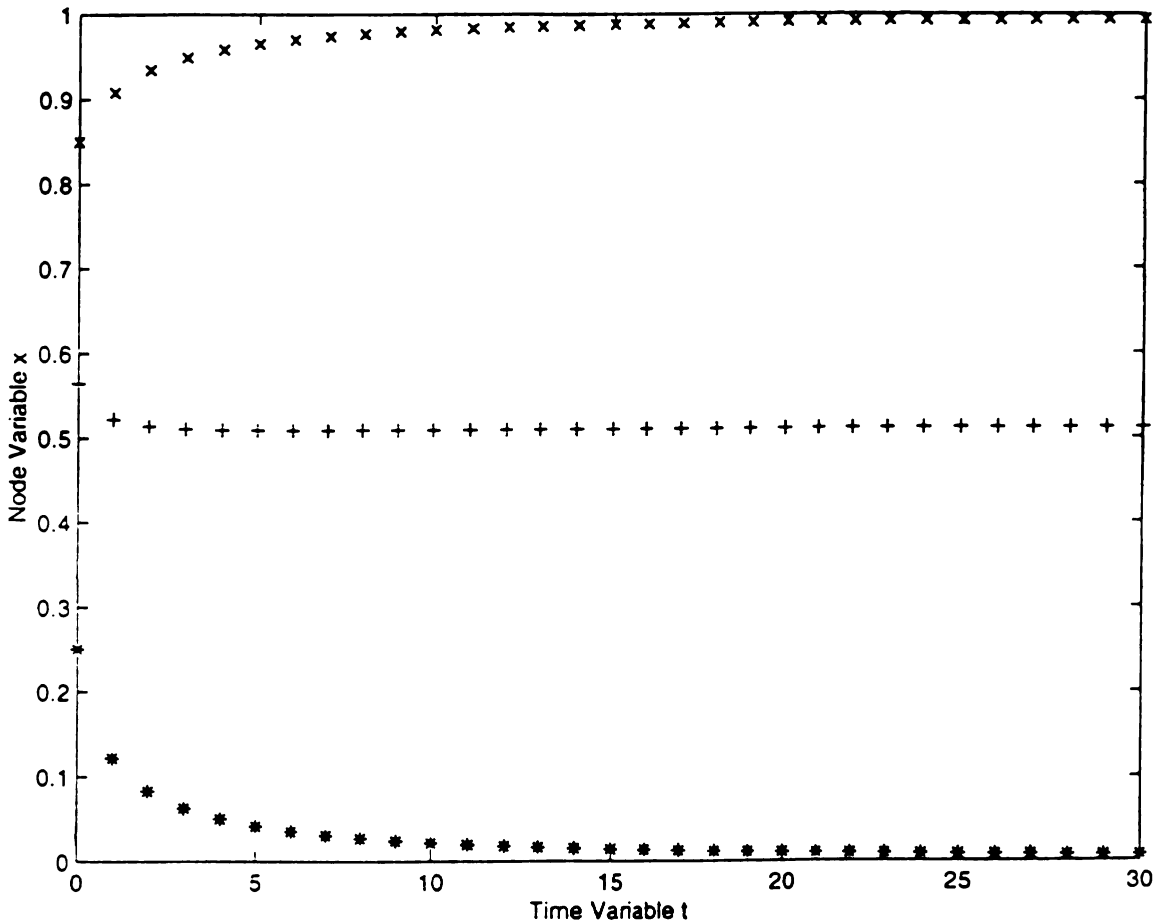


Figure 6.3. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x with $p = 2$ and $q = 2$ as t increases

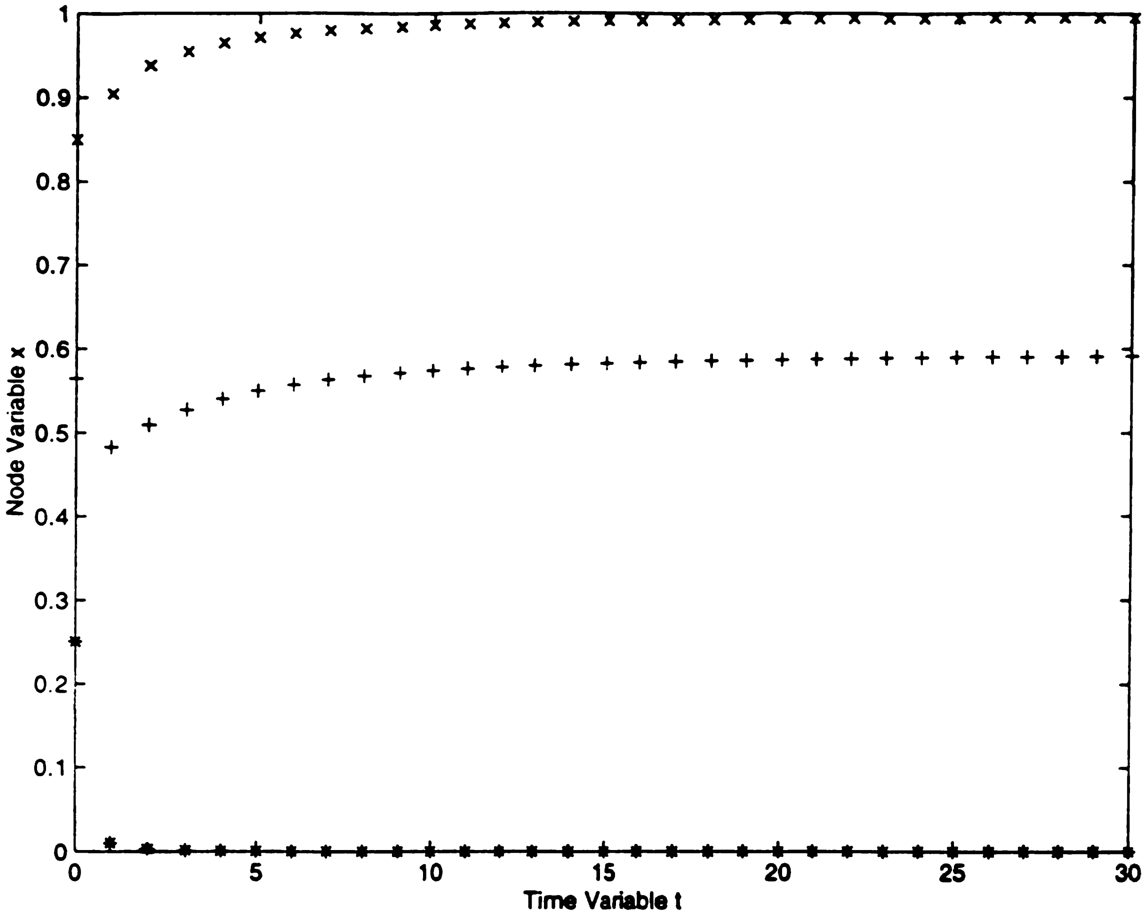


Figure 6.4. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x with $p = 1$ and $q = 2$ as t increases

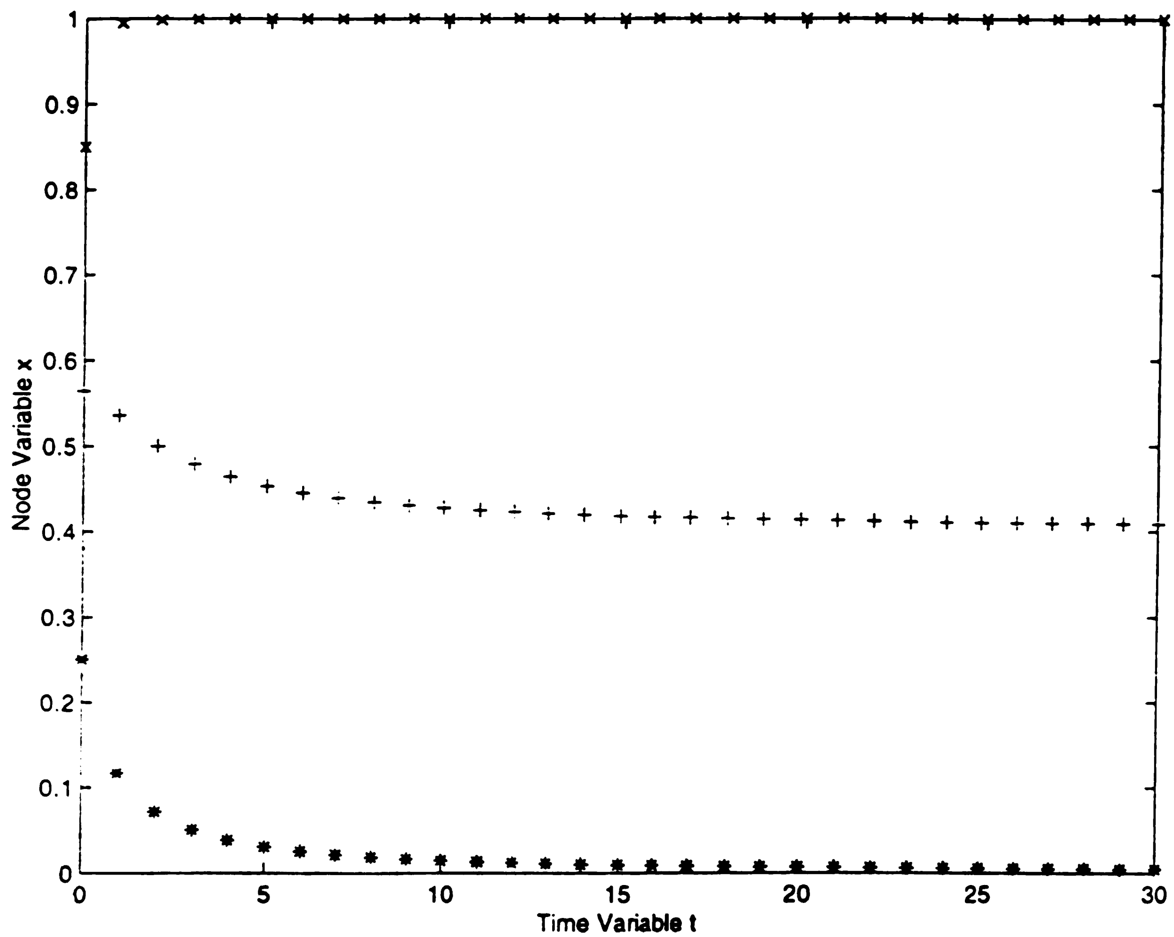


Figure 6.5. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x with $p = 2$ and $q = 1$ as t increases

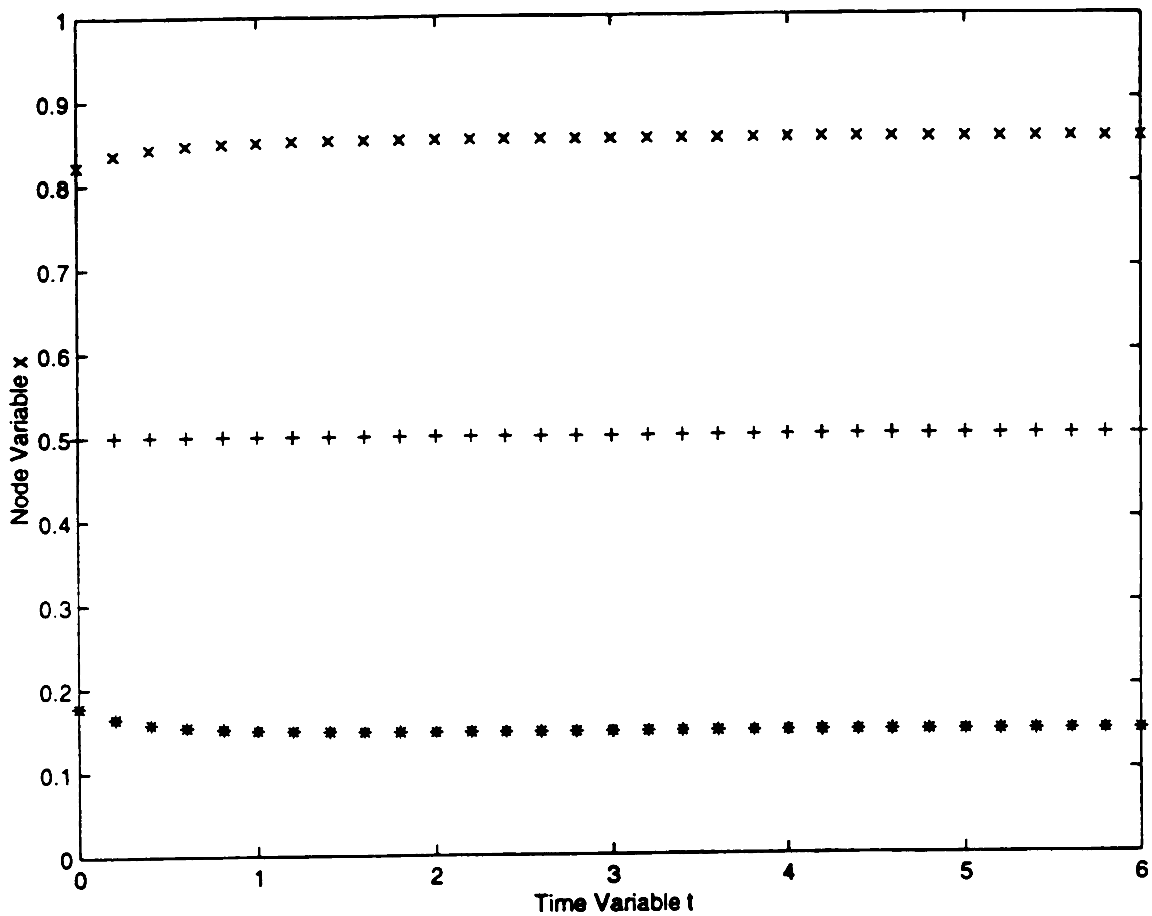


Figure 6.6. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x at $y = 0.5$ with $s = 0, \mu = 0.375, \nu = 0.375$ as t increases

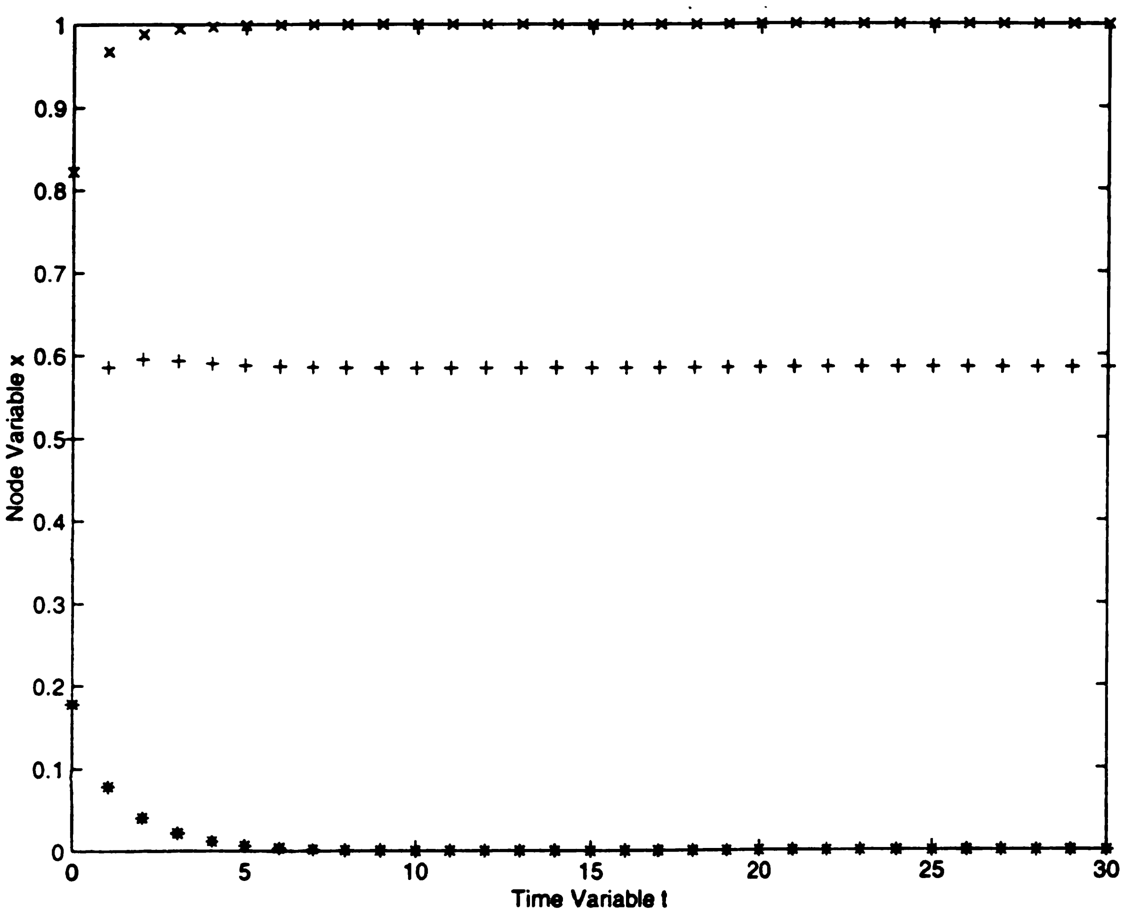


Figure 6.7. Gauss-Galerkin Finite Element Method: Movement of the nodes $*$, $+$ and x at $y = 0.5$ with $s = 2, h = 0.5, \mu = 0, \nu = 0$ as t increases

Table 6.32. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $s = 0, \mu = 0.375, \nu = 0.375$ as t increases

time	x_1	x_2	x_3
0.0	0.1776790146	0.5	0.8223209854
0.2	0.1642962137	0.5	0.8357037863
0.4	0.1574239668	0.5	0.8425760332
0.6	0.1534135247	0.5	0.8465864753
0.8	0.1509533372	0.5	0.8490466628
1.0	0.1493957345	0.5	0.8506042655
1.2	0.1483902464	0.5	0.8516097536
1.4	0.147733336	0.5	0.852266664
1.6	0.1473009068	0.5	0.8526990932
1.8	0.1470148679	0.5	0.8529851321
2.0	0.1468250667	0.5	0.8531749333
2.2	0.1466988646	0.5	0.8533011354
2.4	0.1466148369	0.5	0.8533851631
2.6	0.1465588394	0.5	0.8534411606
2.8	0.1465214994	0.5	0.8534785006
3.0	0.1464965907	0.5	0.8535034093
3.2	0.1464799703	0.5	0.8535200297
3.4	0.1464688782	0.5	0.8535311218
3.6	0.1464614749	0.5	0.8535385251
3.8	0.1464565331	0.5	0.8535434669
4.0	0.1464532343	0.5	0.8535467657
4.2	0.1464510321	0.5	0.8535489679
4.4	0.146449562	0.5	0.853550438
4.6	0.1464485805	0.5	0.8535514195
4.8	0.1464479253	0.5	0.8535520747
5.0	0.1464474879	0.5	0.8535525121
5.2	0.1464471959	0.5	0.8535528041
5.4	0.146447001	0.5	0.853552999
5.6	0.1464468708	0.5	0.8535531292
5.8	0.1464467839	0.5	0.8535532161
6.0	0.1464467259	0.5	0.8535532741

Table 6.33. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes $*$, $+$ and x at $y = 0.5$ with $s = 0, \mu = 0.375, \nu = 0.375$ as t increases

time	ω_1	ω_2	ω_3
0.0	0.2279202041	0.5441595918	0.2279202041
0.2	0.2325481017	0.5349037966	0.2325481017
0.4	0.2375344137	0.5249311726	0.2375344137
0.6	0.2413979521	0.5172040959	0.2413979521
0.8	0.2441527197	0.5116945607	0.2441527197
1.0	0.2460550664	0.5078898671	0.2460550664
1.2	0.2473494077	0.5053011846	0.2473494077
1.4	0.2482233103	0.5035533794	0.2482233103
1.6	0.2488108053	0.5023783893	0.2488108053
1.8	0.2492047529	0.5015904942	0.2492047529
2.0	0.2494685031	0.5010629938	0.2494685031
2.2	0.2496449112	0.5007101777	0.2496449112
2.4	0.249762826	0.500474348	0.249762826
2.6	0.2498416103	0.5003167795	0.2498416103
2.8	0.2498942353	0.5002115294	0.2498942353
3.0	0.2499293807	0.5001412386	0.2499293807
3.2	0.2499528495	0.5000943009	0.2499528495
3.4	0.24996852	0.50006296	0.24996852
3.6	0.2499789828	0.5000420344	0.2499789828
3.8	0.2499859684	0.5000280633	0.2499859684
4.0	0.2499906322	0.5000187356	0.2499906322
4.2	0.2499937459	0.5000125082	0.2499937459
4.4	0.2499958247	0.5000083506	0.2499958247
4.6	0.2499972125	0.500005575	0.2499972125
4.8	0.249998139	0.5000037219	0.249998139
5.0	0.2499987576	0.5000024848	0.2499987576
5.2	0.2499991706	0.5000016589	0.2499991706
5.4	0.2499994463	0.5000011075	0.2499994463
5.6	0.2499996303	0.5000007394	0.2499996303
5.8	0.2499997532	0.5000004936	0.2499997532
6.0	0.2499998352	0.5000003295	0.2499998352

Table 6.34. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $s = 0, \mu = 0.375, \nu = 0.375$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.0	1	0.5	0.297358	0.196036	0.138456	0.102747
0.2	1	0.5	0.302391	0.203586	0.146988	0.111492
0.4	1	0.5	0.305751	0.208627	0.152669	0.117295
0.6	1	0.5	0.307994	0.211992	0.156458	0.121159
0.8	1	0.5	0.309492	0.214238	0.158986	0.123735
1.0	1	0.5	0.310492	0.215738	0.160674	0.125454
1.2	1	0.5	0.311159	0.216739	0.1618	0.126602
1.4	1	0.5	0.311605	0.217407	0.162552	0.127368
1.6	1	0.5	0.311902	0.217854	0.163054	0.127879
1.8	1	0.5	0.312101	0.218152	0.163389	0.128221
2.0	1	0.5	0.312234	0.218351	0.163613	0.128449
2.2	1	0.5	0.312322	0.218483	0.163762	0.128601
2.4	1	0.5	0.312381	0.218572	0.163862	0.128702
2.6	1	0.5	0.312421	0.218631	0.163929	0.12877
2.8	1	0.5	0.312447	0.218671	0.163973	0.128815
3.0	1	0.5	0.312465	0.218697	0.164003	0.128846
3.2	1	0.5	0.312476	0.218715	0.164023	0.128866
3.4	1	0.5	0.312484	0.218726	0.164036	0.128879
3.6	1	0.5	0.312489	0.218734	0.164045	0.128888
3.8	1	0.5	0.312493	0.218739	0.164051	0.128894
4.0	1	0.5	0.312495	0.218743	0.164055	0.128898
4.2	1	0.5	0.312497	0.218745	0.164057	0.128901
4.4	1	0.5	0.312498	0.218747	0.164059	0.128903
4.6	1	0.5	0.312499	0.218748	0.16406	0.128904
4.8	1	0.5	0.312499	0.218749	0.164061	0.128905
5.0	1	0.5	0.312499	0.218749	0.164061	0.128905
5.2	1	0.5	0.3125	0.218749	0.164062	0.128906
5.4	1	0.5	0.3125	0.21875	0.164062	0.128906
5.6	1	0.5	0.3125	0.21875	0.164062	0.128906
5.8	1	0.5	0.3125	0.21875	0.164062	0.128906
6.0	1	0.5	0.3125	0.21875	0.164062	0.128906

Table 6.35. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ at y=0.5 with $s = 0, \mu = 0.375, \nu = 0.375$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.0	1	0.5	0.297358	0.196036	0.138456	0.102747
0.2	1	0.5	0.302391	0.203586	0.146988	0.111492
0.4	1	0.5	0.305751	0.208627	0.152669	0.117295
0.6	1	0.5	0.307994	0.211992	0.156458	0.121159
0.8	1	0.5	0.309492	0.214238	0.158986	0.123735
1.0	1	0.5	0.310492	0.215738	0.160674	0.125454
1.2	1	0.5	0.311159	0.216739	0.1618	0.126602
1.4	1	0.5	0.311605	0.217407	0.162552	0.127368
1.6	1	0.5	0.311902	0.217854	0.163054	0.127879
1.8	1	0.5	0.312101	0.218152	0.163389	0.128221
2.0	1	0.5	0.312234	0.218351	0.163613	0.128449
2.2	1	0.5	0.312322	0.218483	0.163762	0.128601
2.4	1	0.5	0.312381	0.218572	0.163862	0.128702
2.6	1	0.5	0.312421	0.218631	0.163929	0.12877
2.8	1	0.5	0.312447	0.218671	0.163973	0.128815
3.0	1	0.5	0.312465	0.218697	0.164003	0.128846
3.2	1	0.5	0.312476	0.218715	0.164023	0.128866
3.4	1	0.5	0.312484	0.218726	0.164036	0.128879
3.6	1	0.5	0.312489	0.218734	0.164045	0.128888
3.8	1	0.5	0.312493	0.218739	0.164051	0.128894
4.0	1	0.5	0.312495	0.218743	0.164055	0.128898
4.2	1	0.5	0.312497	0.218745	0.164057	0.128901
4.4	1	0.5	0.312498	0.218747	0.164059	0.128903
4.6	1	0.5	0.312499	0.218748	0.16406	0.128904
4.8	1	0.5	0.312499	0.218749	0.164061	0.128905
5.0	1	0.5	0.312499	0.218749	0.164061	0.128905
5.2	1	0.5	0.3125	0.218749	0.164062	0.128906
5.4	1	0.5	0.3125	0.21875	0.164062	0.128906
5.6	1	0.5	0.3125	0.21875	0.164062	0.128906
5.8	1	0.5	0.3125	0.21875	0.164062	0.128906
6.0	1	0.5	0.3125	0.21875	0.164062	0.128906

Table 6.36. Gauss-Galerkin Finite Element Method: Changes of the nodes $*$, $+$ and x at $y = 0.5$ with $s = 2, h = 0.5, \mu = 0, \nu = 0$ as t increases

time	x_1	x_2	x_3
0.00	0.1776790146	0.5	0.8223209854
1.00	0.07823484538	0.5855926881	0.9672094934
2.00	0.04055785446	0.5952942733	0.9882813902
3.00	0.02175701743	0.5935687484	0.994844609
4.00	0.01176627035	0.5905610497	0.9975060185
5.00	0.006369293956	0.5882085524	0.9987326259
6.00	0.003445438578	0.5866681196	0.9993384161
7.00	0.001862301052	0.58573549	0.999649455
8.00	0.001006005904	0.5851946307	0.9998127142
9.00	0.0005432371344	0.5848889872	0.9998994783
10.0	0.0002932798893	0.5847190463	0.9999459106
11.0	0.0001583142674	0.5846255329	0.9999708549
12.0	8.54529272e-05	0.5845744192	0.9999842838
13.0	4.612291437e-05	0.5845466024	0.9999915218
14.0	2.489414189e-05	0.5845315072	0.9999954253
15.0	1.343607637e-05	0.584523331	0.9999975313
16.0	7.251786085e-06	0.5845189076	0.9999986677
17.0	3.913956526e-06	0.5845165166	0.999999281
18.0	2.112448808e-06	0.5845152244	0.9999996119
19.0	1.140134147e-06	0.5845145278	0.9999997906
20.0	6.153546255e-07	0.5845141503	0.999999887
21.0	3.321198762e-07	0.5845139491	0.999999939
22.0	1.792520631e-07	0.5845138348	0.9999999671
23.0	9.674608925e-08	0.5845137629	0.9999999822
24.0	5.221588395e-08	0.5845137595	0.9999999904
25.0	2.81820044e-08	0.5845137242	0.9999999948
26.0	1.521042681e-08	0.5845137523	0.9999999972
27.0	8.209376512e-09	0.5845136238	0.9999999985
28.0	4.430779077e-09	0.5845135372	0.9999999992
29.0	2.391380427e-09	0.5845133642	0.9999999996
30.0	1.290684337e-09	0.5845139703	0.9999999998

Table 6.37. Gauss-Galerkin Finite Element Method: Changes of the weights at nodes $*$, $+$ and x at $y = 0.5$ with $s = 2, h = 0.5, \mu = 0, \nu = 0$ as t increases

time	ω_1	ω_2	ω_3
0.00	0.2279202041	0.5441595918	0.2279202041
1.00	0.203454735	0.3434686623	0.4530766027
2.00	0.1877795851	0.1832292954	0.6289911195
3.00	0.1795033149	0.09603249769	0.7244641874
4.00	0.1750216309	0.05062793336	0.7743504358
5.00	0.1725788113	0.02690733567	0.800513853
6.00	0.1712492829	0.01438787971	0.8143628374
7.00	0.1705276659	0.007723698604	0.8217486355
8.00	0.1701368564	0.004155979967	0.8257071636
9.00	0.169925507	0.002239281609	0.8278352114
10.0	0.1698113081	0.001207464186	0.8289812277
11.0	0.1697496338	0.0006513627194	0.8295990035
12.0	0.1697163352	0.0003514569213	0.8299322079
13.0	0.1696983599	0.0001896601995	0.8301119799
14.0	0.1696886572	0.0001023552587	0.8302089876
15.0	0.1696834201	5.524084473e-05	0.830261339
16.0	0.1696805935	2.981393018e-05	0.8302895925
17.0	0.1696790679	1.609099471e-05	0.8303048411
18.0	0.1696782445	8.684585931e-06	0.8303130709
19.0	0.1696778001	4.687234992e-06	0.8303175127
20.0	0.1696775602	2.52979357e-06	0.83031991
21.0	0.1696774308	1.365380974e-06	0.8303212039
22.0	0.1696773609	7.369242155e-07	0.8303219022
23.0	0.1696773232	3.977332705e-07	0.8303222791
24.0	0.1696773028	2.146649293e-07	0.8303224825
25.0	0.1696772918	1.158591251e-07	0.8303225923
26.0	0.1696772859	6.253159069e-08	0.8303226516
27.0	0.1696772827	3.374960315e-08	0.8303226835
28.0	0.169677281	1.821536086e-08	0.8303227008
29.0	0.1696772801	9.831214716e-09	0.8303227101
30.0	0.1696772795	5.306111562e-09	0.8303227151

Table 6.38. Gauss-Galerkin Finite Element Method: Changes of the "moment" $m_n^i(y,t)$ at $y = 0.5$ with $s = 2, h = 0.5, \mu = 0, \nu = 0$ as t increases

time	m_0	m_1	m_2	m_3	m_4	m_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
1.00	1	0.65527	0.542878	0.479022	0.436907	0.40716
2.00	1	0.738312	0.679577	0.645803	0.623033	0.606688
3.00	1	0.781637	0.750933	0.733402	0.72156	0.713057
4.00	1	0.804377	0.788174	0.778999	0.772813	0.768379
5.00	1	0.816426	0.807803	0.80295	0.799684	0.797349
6.00	1	0.822855	0.81824	0.815653	0.813914	0.812672
7.00	1	0.826302	0.823823	0.822437	0.821506	0.820842
8.00	1	0.828156	0.826821	0.826076	0.825576	0.825219
9.00	1	0.829154	0.828435	0.828034	0.827764	0.827572
10.0	1	0.829692	0.829304	0.829088	0.828943	0.82884
11.0	1	0.829983	0.829773	0.829657	0.829578	0.829523
12.0	1	0.830139	0.830026	0.829963	0.829921	0.829891
13.0	1	0.830224	0.830163	0.830129	0.830106	0.83009
14.0	1	0.830269	0.830236	0.830218	0.830206	0.830197
15.0	1	0.830294	0.830276	0.830266	0.83026	0.830255
16.0	1	0.830307	0.830298	0.830292	0.830289	0.830286
17.0	1	0.830314	0.830309	0.830306	0.830304	0.830303
18.0	1	0.830318	0.830315	0.830314	0.830313	0.830312
19.0	1	0.83032	0.830319	0.830318	0.830317	0.830317
20.0	1	0.830321	0.830321	0.83032	0.83032	0.83032
21.0	1	0.830322	0.830322	0.830321	0.830321	0.830321
22.0	1	0.830322	0.830322	0.830322	0.830322	0.830322
23.0	1	0.830323	0.830322	0.830322	0.830322	0.830322
24.0	1	0.830323	0.830323	0.830323	0.830322	0.830322
25.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
26.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
27.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
28.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
29.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
30.0	1	0.830323	0.830323	0.830323	0.830323	0.830323

Table 6.39. Gauss-Galerkin Finite Element Method: Changes of the "total moment" $M_n^i(t)$ at y=0.5 with $s = 2, h = 0.5, \mu = 0, \nu = 0$ as t increases

time	M_0	M_1	M_2	M_3	M_4	M_5
0.00	1	0.5	0.297358	0.196036	0.138456	0.102747
1.00	1	0.65527	0.542878	0.479022	0.436907	0.40716
2.00	1	0.738312	0.679577	0.645803	0.623033	0.606688
3.00	1	0.781637	0.750933	0.733402	0.72156	0.713057
4.00	1	0.804377	0.788174	0.778999	0.772813	0.768379
5.00	1	0.816426	0.807803	0.80295	0.799684	0.797349
6.00	1	0.822855	0.81824	0.815653	0.813914	0.812672
7.00	1	0.826302	0.823823	0.822437	0.821506	0.820842
8.00	1	0.828156	0.826821	0.826076	0.825576	0.825219
9.00	1	0.829154	0.828435	0.828034	0.827764	0.827572
10.0	1	0.829692	0.829304	0.829088	0.828943	0.82884
11.0	1	0.829983	0.829773	0.829657	0.829578	0.829523
12.0	1	0.830139	0.830026	0.829963	0.829921	0.829891
13.0	1	0.830224	0.830163	0.830129	0.830106	0.83009
14.0	1	0.830269	0.830236	0.830218	0.830206	0.830197
15.0	1	0.830294	0.830276	0.830266	0.83026	0.830255
16.0	1	0.830307	0.830298	0.830292	0.830289	0.830286
17.0	1	0.830314	0.830309	0.830306	0.830304	0.830303
18.0	1	0.830318	0.830315	0.830314	0.830313	0.830312
19.0	1	0.83032	0.830319	0.830318	0.830317	0.830317
20.0	1	0.830321	0.830321	0.83032	0.83032	0.83032
21.0	1	0.830322	0.830322	0.830321	0.830321	0.830321
22.0	1	0.830322	0.830322	0.830322	0.830322	0.830322
23.0	1	0.830323	0.830322	0.830322	0.830322	0.830322
24.0	1	0.830323	0.830323	0.830323	0.830322	0.830322
25.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
26.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
27.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
28.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
29.0	1	0.830323	0.830323	0.830323	0.830323	0.830323
30.0	1	0.830323	0.830323	0.830323	0.830323	0.830323

CHAPTER 7

Conclusions and Discussions

In the previous chapters, we proposed and studied a Gauss-Galerkin finite element method for solving a class of singular diffusion equations. The theoretic analysis is very general and therefore the results can be applied to a wide range of singular diffusion equations in two variables. In our examination of the test problems, even though we only use very few base functions in finite element approximation for y variable and very few nodes in Gauss-Galerkin approximation for x variable, the numerical approximation seems very efficient and accurate. In the proof of the convergence of Gauss-Galerkin finite method, we considered a set of boundary conditions under which the boudary terms drop out. Nevertheless, as our test problem in studing Section 6.2 shows, we can apply the proposed method even in those case where the boundary terms do not all drop out (there are net fluxes across the boundaries at $x = 0$ and $x = 1$).

A number of important and interesting issues remain to be studied in the future. We state some of them here. How do we modify our Gauss-Galerkin method in general in the case that the boundary terms can not drop out. One possible solution is to approximate such boundary terms by the finite element method in the y variable and the Gauss-Galerkin method in the x variable. For singular diffusion equations with singularities in both x and y variables, a possible solution may need the Gauss-

Galerkin method in both directions. For example, given time t, we may alternately use one dimensional Gauss-Galerkin method in the x direction and then the y direction repeatedly. Obviously much more work needs to be done in the future.

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