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THE SEIBERG-WITTEN THEORY OF HOMOLOGY 3-SPHERES

By

Weimin Chen

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ABSTRACT

THE SEIBERG-WITTEN THEORY OF HOMOLOGY 3-SPHERES

By

Weimin Chen

In this thesis we study the Seiberg-Witten theory of an oriented homology 3-sphere. The goal is to extract topological invariants — the Seiberg-Witten invariants — by counting the solutions to the Seiberg-Witten equations on the manifold. The first question we consider is whether the Seiberg-Witten invariants depend on the geometric or analytic data involved in their definition. In the first main result of this thesis, we completely determine the dependence of the Seiberg-Witten invariants on the data involved in their definition. In particular, we show that even for the simplest manifold, the 3-sphere S^3, the Seiberg-Witten invariants take infinitely many different values.

The rest of this thesis is devoted to understanding the Seiberg-Witten invariants in a specific geometric setting — the surgery setting. In that context we prove a gluing formula, which identifies the Seiberg-Witten invariants as certain "homological intersection numbers".

To Rong, Xiu and Peter

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This thesis grew out of an unsuccessful endeavor searching for a homology bordism invariant lifting of the Rohlin invariant of an oriented homology 3-sphere via the Seiberg-Witten theory. The existence of such an invariant would imply that there are no Z_2 torsion elements with non-zero Rohlin invariant in the 3-dimensional homology bordism group, which in turn would imply that not every higher dimensional topological manifold is simplicially triangulable. I am very grateful to my thesis advisor Professor Selman Akbulut for suggesting that I work on this problem and for sharing with me his ideas of using gauge theory. I have benefited greatly from numerous discussions with him in the past three years. I would not have been able to survive the hard work in these years without his patience, encouragement, and support.

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TABLE OF CONTENTS

Introduction

Let Y be an oriented homology 3-sphere, i.e. $H_*(Y) = H_*(S^3)$. Equip Y with a Riemannian metric g_0. The unique spin structure on Y gives rise to a (unique) $SU(2)$ vector bundle W on Y such that the oriented volume form of Y acts on W as identity by Clifford multiplication. Consider pairs (A, ψ) where A is an imaginary valued 1-form on Y and ψ is a smooth section of W. The 3-dimensional Seiberg-Witten equations for (A, ψ) read as

$$\begin{cases} D_{g_0}\psi + A\psi = 0 \\ *dA + \tau(\psi, \psi) = 0. \end{cases}$$

Here D_{g_0} is the Dirac operator on Y associated to the metric g_0 and $\tau(\cdot, \cdot)$ is a certain bilinear form on $\Gamma(W)$ with values in the space of imaginary valued 1-forms on Y. The group of gauge transformations $\mathcal{G}(Y) = Map(Y, S^1)$ acts on the pairs (A, ψ) by the following rule:

$$s \cdot (A, \psi) = (A - s^{-1}ds, s\psi) \text{ for } s \in \mathcal{G}(Y).$$

The Seiberg-Witten moduli space $\mathcal{M}(Y)$ is the space of gauge equivalence classes of solutions to the Seiberg-Witten equations (these solutions are called monopoles). It is compact and has virtual dimension zero.

The algebraic count of the elements in $\mathcal{M}(Y)$ is called the Seiberg-Witten invariant of Y and is denoted by $\chi(Y)$ throughout. $\mathcal{M}(Y)$ can be regarded as the set of critical points of the Chern-Simons-Dirac functional and $\chi(Y)$ its Euler characteristic.

The first question we consider is whether the Seiberg-Witten invariant $\chi(Y)$ is independent of the data involved in its definition, such as the Riemannian metric on Y and the perturbations of the Seiberg-Witten equations. Unfortunately, the answer to this question turns out to be negative. To be more precise, suppose that the oriented homology 3-sphere Y bounds a smooth spin 4-manifold X endowed with a Riemannian metric which is a product near Y. We set

$$\alpha(Y) = \chi(Y) - (\mathrm{index} D_X + \frac{1}{8}\mathrm{Sign}(X)),$$

where D_X is the Dirac operator on X defined with the APS global boundary condition ([2]) and $\mathrm{Sign}(X)$ is the signature of X.

In Chapter 1, we give a rigorous definition of $\chi(Y)$ and $\alpha(Y)$ and prove the following theorem.

Theorem A

Let Y be an oriented homology 3-sphere. Then

1. *$\alpha(Y)$ is a topological invariant of Y, and $\alpha(Y) + \alpha(-Y) = 0$.*

2. *$\alpha(Y) \equiv \mu(Y) \pmod 2$, where $\mu(Y)$ is the Rohlin invariant of Y.*

The Casson's invariant satisfies both these properties. Thus this result strongly supports the recent conjecture of Kronheimer and Mrowka ([18]) that $\alpha(Y)$ equals Casson's invariant of Y.

In order to define $\chi(Y)$, we need to consider the following perturbations of the Seiberg-Witten equations:

$$\begin{cases} D_g\psi + A\psi + f\psi = 0 \\ *dA + \tau(\psi,\psi) + \mu = 0, \end{cases}$$

where g is a perturbation of the metric g_0, f is a real valued smooth function on Y and μ is a small, co-closed, imaginary valued 1-form on Y. The topological invariance of $\alpha(Y)$ is roughly saying that the space of pairs (g, f) has a chamber structure and the Seiberg-Witten invariant $\chi(Y)$ depends only on the chamber of the perturbed Dirac operator $D_g + f$ (assuming the perturbation μ is small). In [13], Hitchin studied a family of metrics on S^3 which shows that the Dirac operator associated to this family of metrics has infinitely many different chambers. Using Hitchin's observation, we show that even for the simplest 3-manifold, S^3, the Seiberg-Witten invariant $\chi(S^3)$ takes infinitely many different values.

The rest of this thesis is devoted to understanding the Seiberg-Witten invariant $\chi(Y)$ in the following geometric setting. Assume that Y is decomposed into a union of two submanifolds Y_1 and Y_2 by an embedded torus T^2 where Y_2 is diffeomorphic to $D^2 \times S^1$. We put a Riemannian metric on Y such that a collar neighborhood of T^2 is isometric to $(-1, 1) \times \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ and Y_2 carries a metric whose scalar curvature is non-negative and somewhere positive. By inserting cylinders $[0, 2L+1] \times T^2$, we obtain a family of stretched versions Y_L of Y. Our goal is to express the Seiberg-Witten invariant $\chi(Y_L)$ in terms of Y_1 and Y_2 when the neck is sufficiently long. We regard Y_L as a result of cutting and pasting of two cylindrical end manifolds obtained by attaching infinite cylinders to Y_1 and Y_2 (still denoted by Y_1 and Y_2 for simplicity). It turns out that the (finite energy) Seiberg-Witten moduli spaces of the cylindrical end manifolds Y_1 and Y_2 are generically 1-dimensional manifolds which are immersed into the space of equivalence classes of flat $U(1)$ connections on T^2 via a map which sends a finite energy monopole to its limiting value at the infinity of the cylindrical end. After fixing orientations, these moduli spaces define an "intersection" number $\#\mathcal{S}(Y_1, Y_2)$, which we prove equals to the Seiberg-Witten invariant $\chi(Y_L)$ when the length of the neck is large enough. This result is refered to as the gluing formula of χ.

Theorem B

For large enough L, $\chi(Y_L) = \#\mathcal{S}(Y_1, Y_2)$.

In Chapter 2, we set up the Fredholm theory for Seiberg-Witten equations on cylindrical end 3-manifolds. The issue of perturbation and transversality, and analytic properties of the finite energy monopoles such as exponential decay estimates and "compactness" are discussed. The gluing formula is proved in Chapter 3.

Two technical results needed in Chapters 2 and 3 are included as Appendice A and B.

Part of this thesis has appeared in the Proceedings of 5th Gökova Geometry-Topology Conference (1996) ([6],[7]).

CHAPTER 1

Topological Invariance

1.1 Seiberg-Witten theory in dimension 3

Let Y be an oriented homology 3-sphere equipped with a Riemannian metric g (many facts stated in this section hold for general 3-manifolds). There exists a unique $SU(2)$ vector bundle W_0 over Y as a Clifford module of the Clifford algebra bundle $Cl(TY) \otimes_{\mathbf{R}} \mathbf{C}$ such that the oriented volume form on Y acts as identity on W_0. Let $W = W_0 \otimes L$, where L is the trivial complex line bundle over Y. W is a $U(2)$ vector bundle.

Let (e^1, e^2, e^3) be an oriented local orthonormal basis of T^*Y. This gives rise to a local unitary basis of W_0 and W, within which the Clifford multiplication is given by the following matrices:

$$c(e^1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, c(e^2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c(e^3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\psi = (z, w)$, $\phi = (u, v)$, $\psi, \phi \in W$, we define

$$\tau(\psi, \phi) = \frac{1}{2} \begin{pmatrix} Re(z\bar{u} - w\bar{v}) & z\bar{v} + \bar{w}u \\ \bar{z}v + w\bar{u} & -Re(z\bar{u} - w\bar{v}) \end{pmatrix}.$$

It is straightforward to show

Lemma 1.1.1 *$i\tau(\psi, \phi) = \frac{1}{2}(Re(z\bar{u} - w\bar{v})(e^1) + Im(z\bar{v} + \bar{w}u)(e^2) + Re(z\bar{v} + \bar{w}u)(e^3))$, so $\tau(\psi, \phi) \in \Lambda^1(Y) \otimes i\mathbf{R}$. Moreover, we have*

$$\langle ie \cdot \psi, \phi \rangle_{Re} = -2\langle e, i\tau(\psi, \phi) \rangle$$

for any $e \in \Lambda^1(Y)$, and $|\tau(\psi, \psi)|^2 = \frac{1}{4}|\psi|^4$.

The Levi-Civita connection of the Riemannian metric g lifts to a connection on W_0. Coupled with a $U(1)$ connection A on the complex line bundle L, the Dirac operator D_A: $\Gamma(W) \longrightarrow \Gamma(W)$ is given in a local frame by

$$D_A = \sum_{j=1}^{3} e^j \cdot (\nabla_{e_j} + iA_j).$$

Let $\mathcal{A} = \mathcal{C} \times \Gamma(W)$ where $\mathcal{C}$ is the space of smooth $U(1)$ connections on L. The gauge group $\mathcal{G} = Map(Y, S^1)$ acts on $\mathcal{A}$ by $s \cdot (A, \psi) = (A - s^{-1}ds, s\psi)$, $s \in \mathcal{G}$, $(A, \psi) \in \mathcal{A}$. Note that $\pi_0(\mathcal{G}) = H^1(Y, \mathbf{Z}) = 0$. Each element in $\mathcal{G}$ can be written as e^f with $f \in \Gamma(\Lambda^0(Y) \otimes i\mathbf{R})$ determined up to a constant $2\pi ik$, $k \in \mathbf{Z}$. So $\mathcal{G} = K(\mathbf{Z}, 1)$. Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$. The action of $\mathcal{G}$ is free on the subset $\mathcal{A}^* = \mathcal{A} \setminus \{\psi \equiv 0\}$, and with stabilizer S^1 on the rest. Hence $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ is homotopic to CP^∞.

We shall work within the context of Sobolev spaces and Banach manifolds. By fixing a trivialization of L, $\mathcal{C}$ can be identified with $\Omega^1(Y)\otimes i\mathbf{R}$, the space of imaginary valued 1-forms on Y. Define $\mathcal{A}_1^2 = L_1^2(\Lambda^1(Y)\otimes i\mathbf{R}) \times L_1^2(W_0)$, $\mathcal{G}_1^2 = \{L_2^2$ maps from Y to $S^1\}$. For simplicity, we still use the old symbols to denote the Sobolev objects.

Lemma 1.1.2 *$\mathcal{B}^*$ is a Banach manifold whose tangent space at (A,ψ) is*

$$T\mathcal{B}^*_{(A,\psi)} = \{(a,\phi)\in\mathcal{A}| - d^*a + i\langle i\psi,\phi\rangle_{Re} = 0\}.$$

Proof: Standard arguments. The key point is that the operator $d^*d+|\psi|^2$ is invertible if ψ is not identically zero. See [12]. □

Remark: A neighborhood of $[(A,0)]$ in $\mathcal{B}$ is diffeomorphic to U/S^1, where $U = \{(a,\phi)\in\mathcal{A}|d^*a = 0, \|(a,\phi)\| < \delta\}$.

There is a natural $\mathbf{Z}_4$ action σ on $\mathcal{A}$ given by $\sigma(A,\psi) = (-A, J\psi)$, where J is the quaternion structure on W_0. The action σ descends to an involution on $\mathcal{B}$ and acts freely on $\mathcal{B}^*$.

The Chern-Simons-Dirac functional on $\mathcal{A}$ is defined by

$$\mathcal{CSD}(A,\psi) = -\frac{1}{2}\int_Y A\wedge dA + \frac{1}{2}\int_Y \langle\psi, D_A\psi\rangle_{gRe} Vol_g,$$

which is gauge invariant and descends to $\mathcal{B}$. It is also σ-invariant. The gradient of $\mathcal{CSD}$ at (A,ψ) is given by

$$s(A,\psi) = (*dA + \tau(\psi,\psi), D_A\psi).$$

It can be regarded as a 'weak' tangent vector field on $\mathcal{B}^*$ in the sense that it is not in $T\mathcal{B}^*$ but in its L^2 completion $\mathcal{L}$, i.e., $\mathcal{L}_{(A,\psi)} = \{(a,\phi) \in L^2 | -d^*a + i\langle i\psi, \phi\rangle_{Re} = 0\}$.

The covariant derivative ∇s is given by

$$\nabla s_{(A,\psi)}(a,\phi) = (*da + 2\tau(\psi,\phi) - df(\phi), D_A\phi + a\psi + f(\phi)\psi)$$

where $f(\phi)$ is the unique solution to the equation $(d^*d + |\psi|^2)f = i\langle iD_A\psi, \phi\rangle_{Re}$. As in [29], we have

Lemma 1.1.3 *$\nabla s_{(A,\psi)}$ defines a closed, essentially selfadjoint, Fredholm operator on $\mathcal{L}_{(A,\psi)}$, and its eigenvectors form an L^2-complete orthonormal basis for $\mathcal{L}_{(A,\psi)}$. The domain of $\nabla s_{(A,\psi)}$ is the L^2_1-Sobolev space completion of $\mathcal{L}_{(A,\psi)}$. The eigenvalues form a discrete subset of the real line which has no accumulation points, and which is unbounded in both directions. Each eigenvalue has finite multiplicity.*

The 3-dimensional Seiberg-Witten moduli space $\mathcal{M}$ is the set of critical points of $\mathcal{CSD}$ on $\mathcal{B}$, i.e. the equivalence classes of solutions to the Seiberg-Witten equations

$$\begin{cases} *dA + \tau(\psi,\psi) = 0 \\ D_A\psi = 0. \end{cases}$$

Let $[\theta]$ denote the unique reducible solution $[(0,0)]$. Then the moduli space of irreducible solutions is $\mathcal{M}^* = \mathcal{M} \setminus [\theta]$. As in [17], we have

Lemma 1.1.4 *The moduli space $\mathcal{M}$ can be represented by smooth sections and it is compact.*

In order to define the Seiberg-Witten invariant, i.e. the Euler characteristic of $\mathcal{CSD}$, we need suitable perturbations of $\mathcal{CSD}$.

Definition 1.1.5 *A perturbation $\mathcal{CSD}'$ of $\mathcal{CSD}$ is admissible if:*

1. *The critical points of $\mathcal{CSD}'$ in $\mathcal{B}^*$ are non-degenerate, i.e. $\nabla s'_{[(A,\psi)]}$ is invertible at $[(A,\psi)] \in \mathcal{B}^* \bigcap s'^{-1}(0)$.*

2. *The Dirac operator at the reducible $[\theta]$ is invertible so that $[\theta]$ is isolated.*

Here s' is the gradient of $\mathcal{CSD}'$ and $\nabla s'$ is the covariant derivative of s'. The Dirac operator at $[\theta]$ will be clear when we specify the perturbation.

An admissible perturbation has only finitely many isolated critical points in $\mathcal{B}^*$. This is because the reducible $[\theta]$ is isolated so that $\mathcal{M}^*$ is compact. Each irreducible critical point is assigned a sign by the mod 2 spectral flow of $\nabla s'$. Since $\pi_1(\mathcal{B}^*) = 0$, the spectral flow does not depend on the path chosen. See [29].

We will consider two classes of admissible perturbations. The first class is σ-invariant. First we need to perturb the Dirac operator so that it is invertible and still quaternionic. These perturbations take the form of $D_g + f$ where g stands for the metric and f is a smooth real valued function on Y. The perturbed Chern-Simons-Dirac functional takes the form of

$$\mathcal{CSD}'(A,\psi) = \mathcal{CSD}(A,\psi) + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g + u,$$

where u is some functional on $\mathcal{B}$ which will be constructed in Section 1.5. The corresponding Dirac operator at the reducible $[\theta]$ is $D_g + f$. For convenience, we set

$$\mathcal{CSD}_f(A,\psi) = \mathcal{CSD}(A,\psi) + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g.$$

The following proposition is proved in Section 1.4, in which Met stands for the space of metrics.

Proposition 1.1.6 *Let Y be a closed oriented 3-manifold. For a generic pair $(g,f) \in Met \times C^k(Y)$, the perturbed Dirac operator $D_g + f$ is invertible. Moreover, any two such regular pairs (g_0, f_0) and (g_1, f_1) can be connected by a generic path (g_t, f_t) such that the perturbed Dirac operators $D_{g_t} + f_t$ are invertible except for $t_i \in (0,1)$ with $Ker(D_{g_{t_i}} + f_{t_i}) = \mathbf{H}$, $i = 1, 2, \ldots, n$. Let λ_t, ψ_t be the eigenvalue and eigenvector near t_i, i.e. $(D_{g_t} + f_t)\psi_t = \lambda_t \psi_t$ with $\lambda_{t_i} = 0$ and $\|\psi_t\|_{L^2} = 1$, we have*

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{g_t} + f_t)(t_i)(\psi_{t_i}), \psi_{t_i}\rangle_{Re} \neq 0.$$

As a corollary, the spectral flow of $D_{g_t} + f_t$ at t_i is ± 4 for $i = 1, 2, \ldots, n$.

The next proposition concerning the existence of σ-invariant admissible perturbations is proved in Section 1.5.

Proposition 1.1.7 *Fix a regular pair (g,f) so that the reducible $[\theta]$ is isolated. There exist σ-invariant admissible perturbations of $\mathcal{CSD}_f$ which are supported in the complement of $[\theta]$ and the non-degenerate critical points of $\mathcal{CSD}_f$. Any two such admissible perturbations can be connected by a path supported in the complement of $[\theta]$.*

The second class of admissible perturbations of $\mathcal{CSD}$ has the form of

$$\mathcal{CSD}'_{\mu}(A,\psi) = \mathcal{CSD}(A,\psi) - \int_Y A \wedge *\mu$$

where μ is a generic imaginary valued co-closed 1-form. The gradient of $\mathcal{CSD}'_{\mu}$ at (A,ψ) is

$$s'_{\mu}(A,\psi) = (*dA + \tau(\psi,\psi) + \mu, D_A\psi).$$

$\mathcal{CSD}'_{\mu}$ has a unique reducible critical point $[\theta_\mu] = [(a_\mu, 0)]$ where a_μ is the unique solution to the equations $*da_\mu + \mu = 0$ and $d^* a_\mu = 0$. The covariant derivative $\nabla s'_{\mu}$ is given by

$$\nabla s'_{\mu,(A,\psi)}(a,\phi) = (*da + 2\tau(\psi,\phi) - df(\phi), D_A\phi + a\psi + f(\phi)\psi)$$

where $f(\phi)$ is the unique solution to the equation

$$(d^*d + |\psi|^2)f = i\langle iD_A\psi, \phi\rangle_{Re}.$$

The corresponding Dirac operator at $[\theta_\mu]$ is $D_\mu = D + a_\mu$.

Proposition 1.1.8 *For a generic μ, $\mathcal{CSD}'_{\mu}$ is admissible. Moreover, any two such regular μ_0 and μ_1 can be connected by a path μ_t, $t \in [0,1]$, such that*

1. *s'_{μ_t} is transversal to the zero section of the Hilbert bundle $\mathcal{L}$ over $\mathcal{B}^* \times [0,1]$.*
2. *D_{μ_t} is invertible for all but finitely many points $t_i \in (0,1)$ with $Ker D_{\mu_{t_i}} = \mathbf{C}$. Moreover, if λ_t and ψ_t are the eigenvalue and eigenvector of D_{μ_t} near t_i, i.e.*

$D_{\mu_t}\psi_t = \lambda_t\psi_t$ with $\|\psi_t\|_{L^2} = 1$ and $\lambda_{t_i} = 0$, then

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{\mu_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re} \neq 0.$$

In particular, the spectral flow of D_{μ_t} (as complex linear operators) at t_i is equal to ± 1.

Proof: The "universal" gradient $s(\mu, A, \psi) = (*dA + \tau(\psi,\psi) + \mu, D_A\psi)$ is a section of the Hilbert bundle $\mathcal{L}$ over Ker $d^* \times \mathcal{A}^*$ which is transversal to the zero section. So $s^{-1}(0)$ is a Banach manifold, and so is $s^{-1}(0)/\mathcal{G}$. The projection P: $s^{-1}(0)/\mathcal{G} \to$ Ker d^* is a Fredholm map of index 0. So for a generic μ, $\nabla s'_\mu$ is invertible at ${s'}_\mu^{-1}(0)$, and any two such regular μ_0 and μ_1 can be connected by a path μ_t, $t \in [0,1]$, such that s'_{μ_t} is transversal to the zero section of the Hilbert bundle $\mathcal{L}$ over $\mathcal{B}^* \times [0,1]$.

Consider the real Hilbert bundle $\mathcal{E}$ over Ker $d^* \times (L^2_1(W_0) \setminus \{0\})$ given by $\mathcal{E}_{(a,\psi)} = \{\phi \in L^2(W_0) | \phi$ is orthogonal to $i\psi\}$. Then $L(a,\psi) = D\psi + a\psi$ is a section of $\mathcal{E}$ which is transversal to the zero section. Therefore $L^{-1}(0)$ is a Banach manifold. The projection Π: $L^{-1}(0) \to$ Ker d^* is a Fredholm map of index 1. Since $D_a = D + a$ is complex linear, by Sard-Smale theorem, for a generic $a \in$ Ker d^*, $\Pi^{-1}(a)$ is empty, i.e. D_a is invertible. Two such regular a_0 and a_1 can be connected by a path a_t which is transversal to Π. We can take an analytic path a_t so that for all but finitely many points t_i, D_{a_t} is invertible and Ker$D_{a_{t_i}} = \mathbf{C}$ by index counting. If $D_{a_t}\psi_t = \lambda_t\psi_t$ with $\|\psi_t\|_{L^2} = 1$ and $\lambda_{t_i} = 0$, then

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{a_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re}.$$

Since a_t is transversal to the projection Π, $\int_Y \langle \frac{d}{dt}(D_{a_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re} \neq 0$. □

Remark: The same conclusions hold if we also allow the metrics to change.

1.2 The definition of χ and α

Fix an admissible perturbation $\mathcal{CSD}'$ of $\mathcal{CSD}$ with gradient s'. Denote the Dirac operator at the reducible critical point $[\theta]$ by D'. Let $\mathcal{M}^* = \{[(A,\psi)] \in \mathcal{B}^* | s'(A,\psi) = 0\}$. We define for $\beta_j \in \mathcal{M}^*$,

$$\chi^j = \sum_{\beta_i \in \mathcal{M}^*} (-1)^{SF(\beta_j,\beta_i)}$$

where $SF(\beta_j,\beta_i)$ is the spectral flow between $\nabla s'_{\beta_j}$ and $\nabla s'_{\beta_i}$. As in [29], it is easy to show that $|\chi^j|$ is independent of the choice of β_j. In order to give a sign to $|\chi^j|$, we need to fix a sign near the reducible critical point $[\theta]$.

At $(A,\psi) \in \mathcal{A}^*$, we have a short exact sequence

$$0 \longrightarrow T\mathcal{G}_{id} \xrightarrow{d_{(A,\psi)}} T\mathcal{A}^* \xrightarrow{\pi_*} T\mathcal{B}^* \longrightarrow 0$$

where $d_{(A,\psi)}(f) = (-df, f\psi)$ and $\pi : \mathcal{A}^* \to \mathcal{B}^*$. This enables us to extend any endomorphism of $T\mathcal{B}^*$ to a $\mathcal{G}$-equivariant one of $T\mathcal{A} \oplus T\mathcal{G}_{id}$. An endomorphism L of $T\mathcal{B}^*$ is extended to

$$\mathcal{K}'_L = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & d_{(A,\psi)} \\ 0 & d^*_{(A,\psi)} & 0 \end{pmatrix},$$

an endomorphism of $T\mathcal{A} \oplus T\mathcal{G}_{id} = T\mathcal{B}^* \oplus Im(d_{(A,\psi)}) \oplus T\mathcal{G}_{id}$. $\mathcal{K}'_L$ is self-adjoint if and only if L is. For $L = \nabla {s'}_\mu$, we use $\mathcal{K}'$ for $\mathcal{K}'_L$.

At $(A, \psi) \in \mathcal{A}$, we define a self-adjoint endomorphism of $T\mathcal{A} \oplus T\mathcal{G}_{id}$:

$$\mathcal{K}_{(A,\psi)}(a, \phi, f) = (*da + 2\tau(\psi, \phi) - df, D_A\phi + a\psi + f\psi, -d^*a + i\langle i\psi, \phi\rangle_{Re})$$

or

$$\mathcal{K}_{(A,\psi)} = \begin{pmatrix} D_A & \psi\cdot & \psi\cdot \\ 2\tau(\psi, \cdot) & *d & -d \\ i\langle i\psi, \cdot\rangle_{Re} & -d^* & 0 \end{pmatrix}.$$

As in [29], we have

Lemma 1.2.1 *For smooth $(A, \psi) \in \mathcal{A}$, $\mathcal{K}_{(A,\psi)}$ extends to $L^2(\Lambda^1(Y) \otimes i\mathbf{R} \oplus W_0 \oplus \Lambda^0(Y) \otimes i\mathbf{R})$ as a closed, essentially selfadjoint, Fredholm operator. It has discrete spectrum with no accumulation points, and each eigenvalue has finite multiplicity. The spectrum is unbounded from above and below. The same holds for $\mathcal{K}'_{(A,\psi)}$ if $(A, \psi) \in \mathcal{A}^*$. Moreover, one can replace $\nabla s'$ by $\mathcal{K}$ for the purpose of computing the spectral flow.*

For any $(a, \phi) \in \mathcal{A}^*$, we need to study the small eigenvalues of $\mathcal{K}_t(a, \phi) = \mathcal{K}_0 + tC(a, \phi)$ as $t \to 0$ where

$$\mathcal{K}_0 = \begin{pmatrix} D' & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix}, \text{ and } C(a, \phi) = \begin{pmatrix} a & \phi\cdot & \phi\cdot \\ 2\tau(\phi, \cdot) & 0 & 0 \\ i\langle i\phi, \cdot\rangle_{Re} & 0 & 0 \end{pmatrix}.$$

Here D' is the Dirac operator at $[\theta]$ which is invertible. $\mathcal{K}_0$ has only one zero eigenvector which is the constant function i. $\mathcal{K}_t(a,\phi)$ is expected to have exactly one small eigenvalue λ_t which is analytic in t as $t \to 0$. See [14].

Lemma 1.2.2 $\dot{\lambda}_t(0) = 0$, $\ddot{\lambda}_t(0) = -2\int_Y \langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re}$ *where* $\tilde{\phi} = (D')^{-1}(i\phi)$.

Proof: For simplicity let $K_t = \mathcal{K}_t(a,\phi)$, $C = C(a,\phi)$. Suppose $(K_t - \lambda_t)f_t = 0$ where $\|f_t\| = 1$, $f_0 = i$. By differentiating the equation, we have

$$(C - \dot{\lambda}_t)f_t + (K_t - \lambda_t)\dot{f}_t = 0.$$

So $\dot{\lambda}_t = (C(f_t), f_t)$, and $\dot{\lambda}_t(0) = (C(i), i) = (i\phi, i) = 0$. $K_0(\dot{f}_t(0)) = -C(f_0) = -i\phi$. Let $\tilde{\phi} = (D')^{-1}(i\phi)$, then $\ddot{\lambda}_t(0) = (C(\dot{f}_t(0)), f_0) + (C(f_0), \dot{f}_t(0)) = -2\int_Y \langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re}$.

□

Corollary 1.2.3 *For a generic* ϕ, $\ddot{\lambda}_t(0) \neq 0$. $\lambda_t \sim \lambda t^2$ *where* $\lambda = -\int_Y \langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re}$ *and* $\tilde{\phi} = (D')^{-1}(i\phi)$.

For $\beta_j \in \mathcal{M}^*$, we define

$$\operatorname{sign}(\beta_j) = -\operatorname{sign}\left(\int_Y \langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re}\right) \cdot (-1)^{SF(\beta_j,\phi)}$$

for a generic ϕ, where $SF(\beta_j,\phi)$ is the spectral flow between $\mathcal{K}_{\beta_j}$ and $\mathcal{K}_t(a,\phi)$ for small t.

Definition 1.2.4 $\chi = sign(\beta_j) \cdot \chi^j$.

It is easy to see that $\operatorname{sign}(\beta_j)$ is independent of (a,ϕ), and χ is independent of β_j as in [29].

Lemma 1.2.5 $\chi(Y) = -\chi(-Y)$, *and* $\chi \equiv 0 \pmod 2$ *if* $\mathcal{CSD}'$ *is a* σ*-invariant admissible perturbation.*

Proof: W_0 still can serve for $-Y$ if we change the Clifford multiplication by a factor of -1. Under this change, $\mathcal{CSD}'(Y) = -\mathcal{CSD}'(-Y)$, $\nabla s'(Y) = -\nabla s'(-Y)$, $\mathcal{M}(Y) = \mathcal{M}(-Y)$, and $\int_Y \langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re} = -\int_{-Y}\langle D'\tilde{\phi}, \tilde{\phi}\rangle_{Re}$. So $\chi(Y) = -\chi(-Y)$. The other statement is obvious. □

Let X be a smooth compact spin 4-manifold with $\partial X = Y$. Equip X with a Riemannian metric such that a neighborhood of Y is isometric to $(-1,0] \times Y$. Suppose D_X is a perturbed Dirac operator on X which takes the form

$$c(dt)(\frac{d}{dt} + D')$$

near the boundary Y. Here D' is the Dirac operator at $[\theta]$ for an admissible perturbation of the Chern-Simons-Dirac functional and takes the form of $D_g + f + a$ where a is a co-closed imaginary valued 1-form, g stands for the metric and f is a smooth real valued function on Y. D' is invertible. IndexD_X is the L^2 index if an infinite cylinder is attached to X, or the index of D_X satisfying the APS global boundary condition.

Lemma 1.2.6 *([2]) $IndexD_X + \frac{1}{8}Sign(X)$ is independent of X, and*

$$(IndexD_X^1 + \frac{1}{8}Sign(X)) - (IndexD_X^2 + \frac{1}{8}Sign(X)) = -SF(D_1', D_2'),$$

where $D_X^i = c(dt)(\frac{d}{dt} + D_i')$ *near* Y. *In the case that* $a = 0$ *and* (g, f) *is a regular pair,* $IndexD_X + \frac{1}{8}Sign(X) \equiv \mu(Y) \pmod 2$ *where* $\mu(Y)$ *is the Rohlin invariant.*

$IndexD_X + \frac{1}{8}Sign(X)$ changes by a factor of -1 if the orientation of Y is reversed.

Definition 1.2.7 *For any admissible perturbation, define*

$$\alpha = \chi - (IndexD_X + \frac{1}{8}Sign(X)).$$

Here D_X takes the form of $c(dt)(\frac{d}{dt} + D')$ near Y where D' is the Dirac operator at the reducible critical point $[\theta]$ associated to the admissible perturbation.

1.3 Topological invariance of α

In this section, we shall prove that α is independent of the choice of the Riemannian metric and admissible perturbation.

Given any two metrics and admissible perturbations $\mathcal{CSD}'_{\mu_i}$, $i = -1, 1$, we can connect them by a path $\mathcal{CSD}'_{\mu_t}$ $t \in [-1, 1]$ for which Proposition 1.1.8 holds. We only need to consider the following two situations:

1. D_{μ_t} is invertible for all t.

2. D_{μ_t} is invertible for all t but $t = 0$.

Here D_{μ_t} is the Dirac operator at the reducible point $[\theta_{\mu_t}]$. In the first case, Index$D_X +$ $\frac{1}{8}$Sign(X) does not change, neither does χ. In fact, we have

Lemma 1.3.1 *Suppose two admissible perturbations μ_0 and μ_1 are connected by a path μ_t which provides a partial cobordism Z between part of $\mathcal{M}_0^*$ and part of $\mathcal{M}_1^*$. If $\beta_0 \in \mathcal{M}_0^*$ is cobordant to $\beta_1 \in \mathcal{M}_1^*$ via Z, then $SF(\beta_0, \beta_1)$ is even. If $\beta_0 \in \mathcal{M}_0^*$ is* *cobordant to* $\beta_1 \in \mathcal{M}_0^*$ *via* Z*, then* $SF(\beta_0, \beta_1)$ *is odd. Here* $SF(\beta_0, \beta_1)$ *stands for the spectral flow between* $\nabla s'_{\beta_0}$ *and* $\nabla s'_{\beta_1}$.

Proof: The lemma follows from the fact that the cobordism Z can be arranged so that the projection from Z to $[0,1]$ is a Morse function. See [9], p.143. □

In the second case, Index$D_X + \frac{1}{8}$Sign(X) changes by ± 1. We shall prove that χ also changes by ± 1 which is compatible to the change of Index$D_X + \frac{1}{8}$Sign(X) so that α remains unchanged. This is done by analyzing the Kuranishi model near the reducible point at $t = 0$.

Nonlinear Fredholm maps between Hilbert spaces admit local reductions to finite dimensional maps. Suppose Ψ: $X \to Y$ is a nonlinear Fredholm map satisfying $\Psi(0) = 0$. Let $T = (d\Psi)_0$. Then there are splittings $X = Ker\ T \oplus (Ker\ T)^{\perp}$, $Y = ImT \oplus CokerT$ and a map $\psi : X \to CokerT$ so that Ψ is equivalent to $T + \psi$ near 0 by a diffeomorphism of X, and $\psi(0) = 0$, $(d\psi)_0 = 0$. Moreover, $\Psi^{-1}(0)$ is diffeomorphic to $\{\psi|_{KerT} = 0\}$ near 0. If there is a group action, the above can be made equivariant.

The detailed construction goes as follows. Let $\pi_k : X \to KerT$, $\pi_c : Y \to CokerT$ be the orthogonal projections. Then $\chi : X \to X$ given by $\chi : x \to \pi_k(x) + T^{-1}(1 - \pi_c)(\Psi(x))$ is a local diffeomorphism at 0. Define $\psi(y) = \pi_c(\Psi(\chi^{-1}(y)))$. Then $\Psi \circ \chi^{-1} = T + \psi$, and $\Psi^{-1}(0) = \{\psi|_{KerT} = 0\}$. See [12].

Suppose two admissible perturbations μ_{-1} and μ_1 are connected by a path μ_t, $t \in [-1, 1]$, in the sense of Proposition 1.1.8 and D_{μ_t} is invertible except for $t = 0$. We will study the Kuranishi model near the reducible point at $t = 0$ of the following family of Seiberg-Witten equations

$$\begin{cases} *_t dA + \tau_t(\psi, \psi) = 0 \\ (D_{\mu_t} + A)\psi = 0 \end{cases}$$

where $A \in Ker\ d^*$. Here d^* stands for d^{*_t} at $t = 0$.

Consider map $\Psi : \mathbf{R} \oplus L^2_1(Ker\ d^* \oplus W_0) \to L^2(Ker\ d^* \oplus W_0)$ given by

$$\Psi(t, A, \psi) = (\pi(*_t dA + \tau_t(\psi, \psi)), (D_{\mu_t} + A)\psi)$$

where $\pi : \Omega^1(Y) \otimes i\mathbf{R} \to Ker\ d^*$ is the L^2 orthogonal projection. Then $Ker\ (d\Psi)_0 = \mathbf{R} \oplus Ker\ D_0$, $Coker(d\Psi)_0 = Ker\ D_0$. Here D_0 stands for D_{μ_0}. Write $\psi = \psi_0 + \psi_1$ where $\psi_0 \in Ker\ D_0$ and $\psi_1 \in (Ker\ D_0)^{\perp}$, then we have a local diffeomorphism $\chi : \mathbf{R} \oplus L^2_1(Ker\ d^* \oplus W_0) \to \mathbf{R} \oplus L^2_1(Ker\ d^* \oplus W_0)$,

$$\begin{aligned} \chi : (t, A, \psi_0 + \psi_1) \quad \to \quad & (t, (*d)^{-1}(\pi(*_t dA + \tau_t(\psi_0 + \psi_1, \psi_0 + \psi_1))), \\ & \psi_0 + D_0^{-1}(1 - \pi_k)((D_{\mu_t} + A)(\psi_0 + \psi_1))), \end{aligned}$$

and $\chi^{-1}(t, 0, \psi_0) = (t, A, \psi_0 + \psi_1)$ where $A = A(t, \psi_0)$, $\psi_1 = \psi_1(t, \psi_0)$ satisfy

$$\begin{cases} A + (\pi *_t d)^{-1}(\pi \tau_t(\psi_0 + \psi_1)) = 0 \\ \psi_1 + D_0^{-1}(1 - \pi_k)(D_{\mu_t} - D_0 + A)(\psi_0 + \psi_1) = 0. \end{cases}$$

Lemma 1.3.2 *$(D_{\mu_t} + A(t, \psi_0))(\psi_0 + \psi_1(t, \psi_0)) \in Ker\ D_0$. If we write*

$$(D_{\mu_t} + A(t, \psi_0))(\psi_0 + \psi_1(t, \psi_0)) = a\psi_0 + ib\psi_0$$

where a, b are real numbers, then $b = 0$.

Proof: For simplicity, denote $D_{\mu_t} + A(t, \psi_0)$ by D. Then $b\|\psi_0\|^2 = \int_Y \langle ib\psi_0, i\psi_0\rangle_{Re} = \int_Y \langle D(\psi_0 + \psi_1) - a\psi_0, i\psi_0\rangle_{Re} = \int_Y \langle D\psi_1, i\psi_0\rangle_{Re} = -\int_Y \langle i\psi_1, D\psi_0\rangle_{Re} = -\int_Y \langle i\psi_1, a\psi_0 + ib\psi_0 - D\psi_1\rangle_{Re} = 0$. □

Lemma 1.3.3 *There exists a constant* C *so that for small* s, *if* $\|\psi_0\|_{L_1^2} \le s$, $t \le s$, *then*

$$\|\psi_1(t, \psi_0)\|_{L_1^2} \le Cs^2, \text{ and } \|A(t, \psi_0)\|_{L_1^2} \le Cs^2.$$

Proof: We have continuous maps $L_1^2 \times L_1^2 \to L^2$ and $(*d)^{-1}, D_0^{-1} : L^2 \to L_1^2$. Apply Banach lemma to the map

$$B(A, \psi_1) = ((\pi *_t d)^{-1}(\pi\tau_t(\psi_0 + \psi_1)), D_0^{-1}(1 - \pi_k)(D_{\mu_t} - D_0 + A)(\psi_0 + \psi_1)),$$

which maps $\{\|A\|_{L_1^2} \le Cs^2, \|\psi_1\|_{L_1^2} \le Cs^2\}$ into itself when $t \le s$ and $\|\psi_0\|_{L_1^2} \le s$ for small s. The lemma follows easily. □

Next we examine the finite dimensional reduction $\phi|_{Ker\,(d\Psi)_0} : \mathbf{R} \oplus Ker\ D_0 \to Ker\ D_0$. Let $\psi_0 \in Ker\ D_0$, $\|\psi_0\|_{L^2} = 1$. We have

$$\phi|_{Ker\,(d\Psi)_0}(t, s\psi_0) = \pi_k(D_{\mu_t} + A(t, s\psi_0))(s\psi_0 + \psi_1(t, s\psi_0)).$$

Without loss of generality, we assume that s is real and positive. By Lemma 1.3.2, $\phi|_{Ker\,(d\Psi)_0}(t, s\psi_0) = 0$ if and only if

$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} + \int_Y \langle A(t, s\psi_0)(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} = 0.$$

Lemma 1.3.4 *Let* $D_{\mu_t}\psi_t = \lambda_t\psi_t$*,* $\lambda_t(0) = 0$*,* $\psi_t(0) = \psi_0$ *as in Proposition 1.1.8. Then*

1.
$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} = s^2(\lambda_t + O(st + t^2))$$
as $t, s \to 0$.

2.
$$\begin{aligned}\int_Y \langle A(t, s\psi_0)(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} &= 2s^4(-\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \\ &\quad +O(s+t))\end{aligned}$$
as $t, s \to 0$.

Proof: Let $D_{\mu_t}\psi_t = \lambda_t\psi_t$, and $\psi_t = a_t\psi_0 + b_t\psi_t^{\perp}$ where $\psi_t^{\perp} \in (KerD_0)^{\perp}$, $\|\psi_t^{\perp}\|_{L^2} = 1$, $a_t \to 1$, $b_t = O(t)$. Then

$$\lambda_t = |a_t|^2(D_{\mu_t}\psi_0, \psi_0) + 2|b_t|^2\lambda_t - |b_t|^2(D_{\mu_t}\psi_t^{\perp}, \psi_t^{\perp}).$$

Since $a_t \to 1$, $b_t = O(t)$, we have $(D_{\mu_t}\psi_0, \psi_0) = \lambda_t + O(t^2)$.

On the other hand, for any $\psi_2 \in (KerD_0)^{\perp}$, we have

$$(D_{\mu_t}\psi_2, \psi_0) = a_t^{-1}b_t(\lambda_t(\psi_t^{\perp}, \psi_2) - (D_{\mu_t}\psi_t^{\perp}, \psi_2)) = O(\|\psi_2\| \cdot t).$$

So

$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} = s^2(\lambda_t + O(st + t^2))$$

as $t, s \to 0$.

For the second assertion, we have

$$\begin{aligned} A(t, s\psi_0) &= -(\pi *_t d)^{-1}(\pi\tau_t(s\psi_0 + \psi_1(t, s\psi_0))) \\ &= -(*d)^{-1}(\tau(\psi_0))s^2 + O(ts^2 + s^3). \end{aligned}$$

So

$$\begin{aligned} \int_Y \langle A(t, s\psi_0)(s\psi_0 + \psi_1(t, s\psi_0)), s\psi_0\rangle_{Re} &= 2s^4(-\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \\ &\quad + O(s+t)) \end{aligned}$$

as $t, s \to 0$. □

Corollary 1.3.5 *The equation $\phi|_{Ker\,(d\Psi)_0}(t, s\psi_0) = 0$ has exactly one solution s for and only for those t such that λ_t and $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle$ have the same sign, if $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \neq 0$. Moreover, we have $t \sim cs^2$ as $t, s \to 0$.*

Remark: $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \neq 0$ is generically true by slightly perturbing μ_t near $t = 0$, observing that $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0, \phi)\rangle = 0$ for any ϕ implies that $\psi_0 = 0$, and also observing that μ_t is transversal to the projection Π (see Proposition 1.1.8).

Lemma 1.3.6 *Let (A, ψ) be the solution to*

$$\begin{cases} *_t dA + \tau_t(\psi, \psi) = 0 \\ (D_{\mu_t} + A)\psi = 0 \end{cases}$$

near the reducible and $t = 0$, then $SF(\mathcal{K}_{(A,\psi)}, \mathcal{K}_{\mu_t,s}(0,\psi_0))$ is odd as $t, s \to 0$.

Proof: $\mathcal{K}_{(A,\psi)}$ is an analytic perturbation in $s = (\psi, \psi_0)$ of

$$\mathcal{K}_0 = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix}.$$

$\mathcal{K}_0$ has three zero eigenvectors $E^1 = \psi_0$, $E^2 = \frac{1}{\sqrt{2}}(i\psi_0 + i)$, $E^3 = \frac{1}{\sqrt{2}}(i\psi_0 - i)$. Let $\mathcal{K}_{(A,\psi)}E^i_s = \lambda^i_s E^i_s$ where $E^i_s(0) = E^i$, $\lambda^i_s(0) = 0$. Then

$$\dot{\lambda}^1_s(0) = 0,\ \ddot{\lambda}^1_s(0) = -8\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle,\ \dot{\lambda}^2_s(0) = 1,\ \dot{\lambda}^3_s(0) = -1.$$

So $\lambda^1_s \sim \lambda s^2$, $\lambda^2_s \sim s$ and $\lambda^3_s \sim -s$ where λ has the same sign with $-\lambda_t$ (see Corollary 1.3.5).

On the other hand, by Lemma 1.2.2, $\mathcal{K}_{\mu_t,s}(0,\psi_0)$ has three small eigenvalues $\lambda_t, \lambda_t, \lambda_1 s^2$ as $t \to 0$ and $s = o(t)$ where $\lambda_1 = -(D_{\mu_t}\tilde{\psi_0}, \tilde{\psi_0})$ and $\tilde{\psi_0} = D^{-1}_{\mu_t}(i\psi_0)$. It is easy to see that λ_1 has the same sign with $-(D_{\mu_t}\psi_0, \psi_0) \sim -\lambda_t$ as $t \to 0$. So $SF(\mathcal{K}_{(A,\psi)}, \mathcal{K}_{\mu_t,s}(0,\psi_0))$ is odd as $t, s \to 0$. □

Theorem 1.3.7 *Let Y be an oriented homology 3-sphere. Then*

1. *$\alpha(Y)$ is a topological invariant of Y, and $\alpha(Y) + \alpha(-Y) = 0$.*

2. *$\alpha(Y) \equiv \mu(Y) \pmod 2$, where $\mu(Y)$ is the Rohlin invariant of Y.*

Proof: There is a family of irreducible critical points disappearing or being created when t passes 0. Call it β_t. It is easy to see from Lemma 1.3.6 that $\mathrm{sign}(\beta_t) = \mathrm{sign}\lambda_t$.

The rest of $\mathcal{M}^*_{\mu_t}$ provides a cobordism between the rest of $\mathcal{M}^*_{\mu_{-1}}$ and $\mathcal{M}^*_{\mu_1}$. The sign convention fixed near the reducibles does not change since $\mathcal{K}_{\mu_t,s}(0,\psi_0)$ has a spectral flow equal to ± 1 when t passes 0 (the point is that D_{μ_t} is complex linear). So we have $\chi_{\mu_{-1}} - \chi_{\mu_1} = -SF(D_{\mu_{-1}}, D_{\mu_1})$ and α remains unchanged. As for $\alpha(Y) + \alpha(-Y) = 0$, it follows from Lemmas 1.2.5 and 1.2.6.

The second assertion is an easy consequence of the existence of σ-invariant admissible perturbations. We will construct them in the next two sections. □

Remark: In [13], Hitchin studied a family of Riemannian metrics on S^3 which shows that the second term in the definition of α may take infinitely many different values. Therefore we prove that even for the simplest manifold, S^3, the Seiberg-Witten invariant $\chi(S^3)$ takes infinitely many different values.

1.4 Perturbations of Dirac operator

In this section, we show that the perturbed Dirac operators $D_g + f$ are invertible for generic pairs of (g, f) and they admit a chamber structure.

Throughout this section, we assume that Y is a closed oriented 3-manifold. Given a metric g on Y, let P_{SO} be the orthonormal tangent frame bundle of Y. Let $H \subset GL(3, \mathbf{R})$ be the subset of symmetric matrices with positive eigenvalues, then $C^k(P_{SO} \times_{Ad} H)$ which is the set of C^k sections of the associated fiber bundle $P_{SO} \times_{Ad} H$ parameterizes the C^k–smooth Riemannian metrics on Y. We use the C^k–norm of $C^k(P_{SO} \times_{Ad} H)$ to topologize it. Let h be a section of $P_{SO} \times_{Ad} H$, g^h be the corresponding metric, and P^h_{SO} be the orthonormal tangent frame bundle associated to g^h. Let ξ be a given spin structure on Y, $\pi : P_{Spin(\xi)} \to P_{SO}$, $\pi : P^h_{Spin(\xi)} \to P^h_{SO}$ be the

$Spin(3)$ bundles correspondent to the metrics g and g^h, then we have a lifting $\tilde{h}$

$$\begin{array}{ccc} P_{Spin(\xi)} & \xrightarrow{\tilde{h}} & P^h_{Spin(\xi)} \\ \downarrow \pi & & \downarrow \pi \\ P_{SO} & \xrightarrow{h} & P^h_{SO}. \end{array}$$

Note that if h is not symmetric, we may not remain in the same spin structure. Let $V = P_{Spin(\xi)} \times_\rho \mathbf{C}^2$, $V^h = P^h_{Spin(\xi)} \times_\rho \mathbf{C}^2$ be the spinor bundles where $\rho : Spin(3) \longrightarrow SU(2)$ is the standard representation. We have an isometry $\tilde{h} : V \longrightarrow V^h$ given by $\tilde{h}(\sigma, \theta) = (\tilde{h}(\sigma), \theta)$.

Let $\mathcal{D} : \Gamma(V) \times C^k(P_{SO} \times_{Ad} H) \longrightarrow \Gamma(V)$ be the map defined by $\mathcal{D}(\psi, h) = \tilde{h}^{-1} \cdot D_{g^h} \cdot \tilde{h}(\psi)$ where $\psi \in \Gamma(V)$ and $h \in C^k(P_{SO} \times_{Ad} H)$. Let σ be a local frame of $P_{Spin(\xi)}$, $\pi(\sigma) = (e_1, e_2, e_3)$, and $(f_1, f_2, f_3) = (e_1, e_2, e_3)h$ which is the local orthonormal frame with respect to the metric g^h. Write $\psi = (\sigma, \theta)$,$h = (\pi(\sigma), (h_{ij}))$, then

$$\begin{aligned} \mathcal{D}(\psi, h) &= \tilde{h}^{-1} \cdot D_{g^h} \cdot (\tilde{h}(\sigma), \theta) \\ &= \tilde{h}^{-1} \cdot (\tilde{h}(\sigma), \sum_{i=1}^{3} (c_i f_i(\theta) - \frac{1}{2} \sum_{k<j} \omega^i_{kj}(h) c_i c_k c_j \theta)) \\ &= (\sigma, \sum_{i=1}^{3} (c_i h_{si} e_s(\theta) - \frac{1}{2} \sum_{k<j} \omega^i_{kj}(h) c_i c_k c_j \theta)) \end{aligned}$$

where $\omega^i_{kj}(h)$ is the Levi-Civita connection 1-forms of the metric g^h with respect to (f_1, f_2, f_3), i.e., $\nabla^h_{f_i} f_j = f_k \omega^i_{kj}(h)$, and

$$c_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, c_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Direct calculation shows that

$$\begin{aligned}
\omega^i_{kj}(h) &= \frac{1}{2}(h^{-1}_{kr}h_{li}h_{sj} + h^{-1}_{jr}h_{lk}h_{si} - h^{-1}_{ir}h_{lj}h_{sk})(\omega^l_{rs} - \omega^s_{rl}) \\
&\quad + \frac{1}{2}h^{-1}_{ks}h_{li}e_l(h_{sj}) - \frac{1}{2}h^{-1}_{kl}h_{sj}e_s(h_{li}) + \frac{1}{2}h^{-1}_{js}h_{lk}e_l(h_{si}) \\
&\quad - \frac{1}{2}h^{-1}_{jl}h_{si}e_s(h_{lk}) - \frac{1}{2}h^{-1}_{is}h_{lj}e_l(h_{sk}) + \frac{1}{2}h^{-1}_{il}h_{sk}e_s(h_{lj})
\end{aligned}$$

where $\nabla_{e_i}e_j = e_k\omega^i_{kj}$, $h^{-1}_{ij}h_{jk} = \delta_{ik}$. See [19] and [16].

Lemma 1.4.1 *$\mathcal{D}(\cdot, h)$: $\Gamma(V) \longrightarrow \Gamma(V)$ is smooth in h. Moreover, $\mathcal{D}(\cdot, h)$ is self-adjoint if* $\det(h) = 1$ *pointwise on Y.*

Proof: That $\mathcal{D}(\cdot, h)$ is smooth in h follows from the local expressions of $\mathcal{D}(\cdot, h)$ and $\omega^i_{kj}(h)$. For the self-adjointness of $\mathcal{D}(\cdot, h)$, we have

$$\begin{aligned}
\int_Y \langle \mathcal{D}(\psi, h), \phi\rangle_g Vol_g &= \int_Y \langle \tilde{h}^{-1}\cdot D_{g^h}\cdot \tilde{h}(\psi), \phi\rangle_g Vol_g \\
&= \int_Y \langle D_{g^h}\cdot \tilde{h}(\psi), \tilde{h}(\phi)\rangle_{g^h} Vol_{g^h} \\
&= \int_Y \langle \tilde{h}(\psi), D_{g^h}(\tilde{h}(\phi))\rangle_{g^h} Vol_{g^h} \\
&= \int_Y \langle \psi, \tilde{h}^{-1}\cdot D_{g^h}\cdot \tilde{h}(\phi)\rangle_g Vol_g \\
&= \int_Y \langle \psi, \mathcal{D}(\phi, h)\rangle_g Vol_g
\end{aligned}$$

where $Vol_g = Vol_{g^h}$ since $\det(h) = 1$ pointwise on Y. □

Lemma 1.4.2 *Given any metric g on Y, let (e_1, e_2, e_3) be an oriented local orthonormal frame in an open subset A of Y. Let f be a smooth real valued function on Y.*

Suppose $\psi, \phi \in Ker\ (D_g + f)$. *If*

$$\frac{d}{dt}\left(\int_Y \langle \mathcal{D}(\psi, e^{tX}), \phi\rangle_g Vol_g\right) = 0$$

at $t = 0$ *for any symmetric matrix function* X *compactly supported in* A *satisfying* $tr(X) = 0$, *then in* A *we have*

$$\langle e_j \nabla_{e_j} \psi, \phi\rangle_g + \langle \psi, e_j \nabla_{e_j} \phi\rangle_g = -\frac{2}{3}\langle f\psi, \phi\rangle_g$$

for $j = 1, 2, 3$, *and*

$$\langle e_j \nabla_{e_i} \psi, \phi\rangle_g + \langle \psi, e_j \nabla_{e_i} \phi\rangle_g = -\frac{1}{2} e_k(\langle \psi, \phi\rangle_g)$$

for any i, j, k *such that* $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$.

Remark: The same conclusions hold with the hermitian product $\langle \cdot, \cdot\rangle_g$ replaced by its real part, if $\langle \cdot, \cdot\rangle_g$ is replaced by its real part in the condition $\frac{d}{dt}(\int_Y \langle \mathcal{D}(\psi, e^{tX}), \phi\rangle_g Vol_g) = 0$.

The proof of this lemma is a lengthy calculation which is given at the end of this section.

Let Met_0 be the subspace of $Met = C^k(P_{SO} \times_{Ad} H)$ given by

$$Met_0 = \{h \in Met \mid \det(h) = 1\}.$$

Every metric in Met is conformal to a metric in Met_0.

The Proof of Proposition 1.1.6:

Consider the real Hilbert bundle E over the Banach manifold $B = Met_0 \times C^k(Y) \times (L^2_1(V) \setminus \{0\})$. At $(h, f, \psi) \in B$, $E_{(h,f,\psi)} = \{\phi \in L^2(V) | \phi$ is orthogonal to $i\psi, j\psi, k\psi\}$. Here $i, j, k \in \mathbf{H}$ satisfying

$$ij = k, \;\; jk = i, \;\; ki = j, \;\; \text{and} \;\; i^2 = j^2 = k^2 = -1.$$

The map $L : (h, f, \psi) \longrightarrow \mathcal{D}(\psi, h) + f\psi$ defines a section of the bundle E over the Banach manifold B. Suppose that $(h, f, \psi) \in L^{-1}(0)$, then the differential of L at (h, f, ψ) is

$$\delta L_{(h,f,\psi)}(H, F, \Psi) = \mathcal{D}(\Psi, h) + f\Psi + \delta\mathcal{D}(\psi, \cdot)(h)(H) + F\psi,$$

from which it is easy to see that if $\phi \in (Im\delta L)^{\perp}$, then $\phi \in Ker\ (\mathcal{D}(\cdot, h) + f)$ and $\phi = a_1(i\psi) + a_2(j\psi) + a_3(k\psi)$ for some real functions a_1, a_2, a_3. Moreover, by Lemma 1.4.2,

$$\int_Y \langle \delta\mathcal{D}(\psi, \cdot)(h)(H), \phi \rangle_{Re} Vol = 0$$

for any H implies that

$$\langle e_i \nabla_{e_i} \psi, \phi \rangle_{Re} + \langle \psi, e_i \nabla_{e_i} \phi \rangle_{Re} = -\frac{2}{3} \langle f\psi, \phi \rangle_{Re}$$

for $i = 1, 2, 3$, and

$$\langle e_j \nabla_{e_i} \psi, \phi \rangle_{Re} + \langle \psi, e_j \nabla_{e_i} \phi \rangle_{Re} = -\frac{1}{2} e_k (\langle \psi, \phi \rangle_{Re})$$

for i, j, k such that $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$. From this we obtain that

$$\langle \psi, e_s \cdot (e_l(a_1)(i\psi) + e_l(a_2)(j\psi) + e_l(a_3)(k\psi)) \rangle_{Re} = 0$$

for any $s, l = 1, 2, 3$. Since ψ is not identically zero, we have $e_l(a_i) = 0$ for any $l, i = 1, 2, 3$. Hence a_1, a_2, a_3 are constant. So L is transversal to the zero section of E and $L^{-1}(0)$ is a Banach submanifold in B. The projection

$$P : L^{-1}(0) \longrightarrow Met_0 \times C^k(Y)$$

is a Fredholm map of index 3. Note that $L(h, f, \cdot) = \mathcal{D}(\cdot, h) + f$ is quaternionic, so by Sard-Smale theorem, for a generic pair $(h, f) \in Met_0 \times C^k(Y)$, $P^{-1}(h, f)$ is empty, i.e., $\mathcal{D}(\cdot, h) + f$ is invertible. Any two such regular pairs (h_0, f_0) and (h_1, f_1) can be connected by an analytic path (h_t, f_t) which is transversal to the projection P. The operators $\mathcal{D}(\cdot, h_t) + f_t$ are invertible except for finitely many points $t_i \in (0, 1)$, $i = 1, 2, \ldots, n$. The fact that $Ker\ (\mathcal{D}(\cdot, h_{t_i}) + f_{t_i}) = \mathbf{H}$ follows from index counting. Suppose that $\mathcal{D}(\psi_t, h_t) + f_t\psi_t = \lambda_t\psi_t$ near t_i with $\lambda_{t_i} = 0$ and $\|\psi_t\|_{L^2} = 1$, then

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(\mathcal{D}(\psi_{t_i}, h_t) + f_t\psi_{t_i})(t_i), \psi_{t_i} \rangle_{Re}.$$

Since the path (h_t, f_t) is transversal to the projection P, we have

$$\int_Y \langle \frac{d}{dt}(\mathcal{D}(\psi_{t_i}, h_t) + f_t\psi_{t_i})(t_i), \psi_{t_i} \rangle_{Re} \neq 0.$$

Suppose $h_1 \in Met$ is conformal to $h \in Met_0$ and $g^{h_1} = e^{2u} g^h$. Let $m : V^{h_1} \to V^h$

be the isometry. The Dirac operators are related in the following way (see [13] or [19]):

$$D_{g^h} = e^{2u} m D_{g^{h_1}} m^{-1} e^{-u}.$$

It is easy to see from this that $D_{g^{h_1}} + f$ is invertible if and only if $D_{g^h} + e^u f$ is. Similar arguments justify the chamber structure. □

The Proof of Lemma 1.4.2:

Let $\psi = (\sigma, \theta)$, $\pi(\sigma) = (e_1, e_2, e_3)$, then

$$\begin{aligned}
\mathcal{D}(\psi, h) &= (\sigma, c_1 e_1(\theta) + c_2 e_2(\theta) + c_3 e_3(\theta) - \frac{1}{2}((\omega_{12}^2(h) + \omega_{13}^3(h))c_1\theta \\
&\quad + (\omega_{23}^3(h) - \omega_{12}^1(h))c_2\theta - (\omega_{13}^1(h) + \omega_{23}^2(h))c_3\theta \\
&\quad + (\omega_{12}^3(h) - \omega_{13}^2(h) + \omega_{23}^1(h))c_1c_2c_3\theta)).
\end{aligned}$$

For $h = e^{tX}$, where $X = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$\begin{aligned}
\omega_{12}^2(h) + \omega_{13}^3(h) &= (\omega_{12}^2 + \omega_{13}^3)(1 + tx) + (1 + tx)^2 e_1(1 - tx) + O(t^2), \\
\omega_{23}^3(h) - \omega_{12}^1(h) &= (\omega_{23}^3 - \omega_{12}^1)(1 - tx) + (1 - tx)^2 e_2(1 + tx) + O(t^2), \\
\omega_{13}^1(h) + \omega_{23}^2(h) &= -(1 - tx)e_3(1 + tx) - (1 + tx)e_3(1 - tx) + O(t^2), \\
\omega_{12}^3(h) - \omega_{13}^2(h) + \omega_{23}^1(h) &= \frac{1}{2}((1 + tx)^2(\omega_{23}^1 + \omega_{12}^3) + (1 - tx)^2(\omega_{12}^3 - \omega_{13}^2)) \\
&\quad + O(t^2).
\end{aligned}$$

So we have

$$\begin{aligned}\frac{d}{dt}(\mathcal{D}(\psi,h))(0) &= (\sigma, xc_1e_1(\theta) - xc_2e_2(\theta) - \frac{1}{2}((x(\omega_{12}^2 + \omega_{13}^3) - e_1(x))c_1\theta \\ &\quad - (x(\omega_{23}^3 - \omega_{12}^1) - e_2(x))c_2\theta + x(\omega_{23}^1 + \omega_{13}^2)c_1c_2c_3\theta)).\end{aligned}$$

If we write $\psi = (\sigma, \theta)$, $\phi = (\sigma, \xi)$, then

$$\begin{aligned}\int_Y \langle \frac{d}{dt}(\mathcal{D}(\psi,h))(0), \phi\rangle Vol &= \int_Y (\langle xc_1e_1(\theta), \xi\rangle - \langle xc_2e_2(\theta), \xi\rangle \\ &\quad - \frac{1}{2}(x\omega_{12}^2 + x\omega_{13}^3 - e_1(x))\langle c_1\theta, \xi\rangle \\ &\quad - \frac{1}{2}(x\omega_{23}^3 - x\omega_{12}^1 - e_2(x))\langle c_2\theta, \xi\rangle \\ &\quad + \frac{1}{2}x(\omega_{23}^1 + \omega_{13}^2)\langle\theta, \xi\rangle)Vol.\end{aligned}$$

Let (e^1, e^2, e^3) be the dual to (e_1, e_2, e_3), then

$$\begin{aligned}d(x\langle c_1\theta, \xi\rangle * e^1) &= e_1(x)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3 \\ &\quad + x(\langle c_1e_1(\theta), \xi\rangle + \langle c_1\theta, e_1(\xi)\rangle)e^1 \wedge e^2 \wedge e^3 \\ &\quad - x(\omega_{12}^2 + \omega_{13}^3)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

Integration by parts, we have

$$\begin{aligned}\int_Y e_1(x)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3 &= -\int_Y x(\langle c_1e_1(\theta), \xi\rangle + \langle c_1\theta, e_1(\xi)\rangle)e^1 \wedge e^2 \wedge e^3 \\ &\quad + \int_Y x(\omega_{12}^2 + \omega_{13}^3)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_Y e_2(x)\langle c_2\theta,\xi\rangle e^1\wedge e^2\wedge e^3 &= -\int_Y x(\langle c_2e_2(\theta),\xi\rangle + \langle c_2\theta, e_2(\xi)\rangle)e^1\wedge e^2\wedge e^3\\ &\quad+\int_Y x(\omega_{23}^3-\omega_{12}^1)\langle c_2\theta,\xi\rangle e^1\wedge e^2\wedge e^3.\end{aligned}$$

These give us

$$\begin{aligned}\int_Y\langle\frac{d}{dt}(\mathcal{D}(\psi,h))(0),\phi\rangle Vol &= \frac{1}{2}\int_Y x(\langle e_1\nabla_{e_1}\psi,\phi\rangle + \langle\psi, e_1\nabla_{e_1}\phi\rangle)e^1\wedge e^2\wedge e^3\\ &\quad-\frac{1}{2}\int_Y x(\langle e_2\nabla_{e_2}\psi,\phi\rangle + \langle\psi, e_2\nabla_{e_2}\phi\rangle)e^1\wedge e^2\wedge e^3.\end{aligned}$$

Therefore, if

$$\int_Y\langle\frac{d}{dt}(\mathcal{D}(\psi,h))(0),\phi\rangle Vol = 0$$

for all $h=e^{tX}$ where $X=\begin{pmatrix} x & 0 & 0\\ 0 & -x & 0\\ 0 & 0 & 0\end{pmatrix}$, we have

$$\langle e_1\nabla_{e_1}\psi,\phi\rangle + \langle\psi, e_1\nabla_{e_1}\phi\rangle = \langle e_2\nabla_{e_2}\psi,\phi\rangle + \langle\psi, e_2\nabla_{e_2}\phi\rangle.$$

Similarly, we have

$$\langle e_1\nabla_{e_1}\psi,\phi\rangle + \langle\psi, e_1\nabla_{e_1}\phi\rangle = \langle e_3\nabla_{e_3}\psi,\phi\rangle + \langle\psi, e_3\nabla_{e_3}\phi\rangle.$$

But $\psi, \phi \in Ker\ (D_g + f)$, we have

$$\begin{aligned}\sum_{i=1}^{3}(\langle e_i \nabla_{e_i}\psi, \phi\rangle + \langle \psi, e_i\nabla_{e_i}\phi\rangle) &= \langle D_g\psi, \phi\rangle + \langle \psi, D_g\phi\rangle \\ &= -2\langle f\psi, \phi\rangle.\end{aligned}$$

So we have

$$\langle e_i \nabla_{e_i}\psi, \phi\rangle + \langle \psi, e_i\nabla_{e_i}\phi\rangle = -\frac{2}{3}\langle f\psi, \phi\rangle$$

for $i = 1, 2, 3$. Similar computation with $X = \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ yields

$$\langle e_2 \nabla_{e_1}\psi, \phi\rangle + \langle \psi, e_2\nabla_{e_1}\phi\rangle + \langle e_1 \nabla_{e_2}\psi, \phi\rangle + \langle \psi, e_1\nabla_{e_2}\phi\rangle = 0.$$

Combined with

$$(\langle e_2 \nabla_{e_1}\psi, \phi\rangle + \langle \psi, e_2\nabla_{e_1}\phi\rangle) - (\langle e_1 \nabla_{e_2}\psi, \phi\rangle + \langle \psi, e_1\nabla_{e_2}\phi\rangle) = -e_3(\langle \psi, \phi\rangle),$$

we have

$$\langle e_2 \nabla_{e_1}\psi, \phi\rangle + \langle \psi, e_2\nabla_{e_1}\phi\rangle = -\frac{1}{2}e_3(\langle \psi, \phi\rangle).$$

In general, we have

$$\langle e_j \nabla_{e_i}\psi, \phi\rangle + \langle \psi, e_j\nabla_{e_i}\phi\rangle = -\frac{1}{2}e_k(\langle \psi, \phi\rangle)$$

for i, j, k such that $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$. □

1.5 The σ-invariant perturbations

In this section, we give the construction of the σ-invariant admissible perturbations using holonomy along embedded loops. Assume that (g, f) is regular. Let s^f denote the gradient of $\mathcal{CSD}_f$ and $\mathcal{M}_f$ denote the set of critical points where

$$\mathcal{CSD}_f = \mathcal{CSD} + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g, \quad \text{and} \quad s^f(A, \psi) = (*dA + \tau(\psi, \psi), D_A\psi + f\psi).$$

The moduli space $\mathcal{M}_f$ is compact and can be represented by smooth sections.

Definition 1.5.1 *A thickened loop is an embedding $\gamma : S^1 \times D^2 \to Y$, together with a bump function $\eta(y)$ on D^2 centered at $0 \in D^2$, with $\int_{D^2} \eta(y)dy = 1$.*

Given a thickened loop $\lambda = (\gamma, \eta)$, one can define a pair of σ-invariant functions $(p, q)_\lambda : \mathcal{B} \to [-1, 1] \times \mathbf{R}^+$ by

$$p_\lambda(A, \psi) = \int_{D^2} \cos(\theta_y)\eta(y)dy,$$

where $e^{i\theta_y}$ is the holonomy of A along the loop $\gamma_y = S^1 \times \{y\}$, and

$$q_\lambda(A, \psi) = \int_{D^2 \times S^1} |\psi|^2 \eta(y)dydt.$$

Lemma 1.5.2 *The function (p, q) is smooth on $\mathcal{A}$.*

Proof: The same arguments as in [29]. It is useful to know that

$$dp_\lambda|_{(A,\psi)}(a, \phi) = \int_{D^2} i\sin(\theta_y)\eta(y)(\int_{S^1 \times \{y\}} \gamma_y^* a)dy$$

and

$$dq_\lambda|_{(A,\psi)}(a,\phi) = 2\int_{D^2\times S^1} \langle \psi, \phi\rangle \eta(y)dydt.$$

□

For any set Λ of finitely many thickened loops, we have a smooth map $\Phi_\Lambda : \mathcal{B}^* \to \prod_{\lambda\in\Lambda}([-1,1]\times \mathbf{R}^+)_\lambda$ given by

$$\Phi_\Lambda([(A,\psi)]) = ((p,q)_\lambda(A,\psi), \lambda\in\Lambda).$$

The map Φ_Λ is σ-invariant and continuous on $\mathcal{B}$.

Lemma 1.5.3 *There is a set Λ of finitely many thickened loops such that*

1. *$Ker\ \nabla s^f \cap \bigcap_{\lambda\in\Lambda} Ker\ (d(p,q)_\lambda) = \{0\}$ at any $[(A,\psi)] \in \mathcal{M}_f{}^*$.*
2. *Φ_Λ is injective up to the σ action on $\mathcal{M}_f$. Therefore we can identify $\mathcal{M}_f/\sigma$ with a compact subset of $\prod_{\lambda\in\Lambda}([-1,1]\times\mathbf{R}^+)_\lambda$.*

Proof: Suppose $[(A,\psi)] \in \mathcal{M}_f{}^*$, and $(a,\phi) \in Ker\ \nabla s^f_{(A,\psi)}$, i.e., (a,ϕ) satisfies

$$\begin{cases} D_A\phi + f\phi + a\psi = 0 \\ *da + 2\tau(\psi,\phi) = 0 \\ -d^*a + i\langle i\psi, \phi\rangle_{Re} = 0. \end{cases}$$

Since A is not flat, if $(a,\phi)\in Ker\ (d(p,q)_\lambda)$ for all thickened loops, then $\int_{S^1\times\{y\}} \gamma_y^* a = 0$ for all γ. So $da = 0$. $da = 0$ implies $\tau(\psi,\phi) = 0$. So $\phi = v\psi$ for some function $v \in \Omega^0(Y)\otimes i\mathbf{R}$ wherever $\psi \neq 0$. This implies $dv + a = 0$ and $\int_Y(|dv|^2 + |v|^2|\psi|^2) = 0$ by plugging into the equations. Since ψ is not identically zero, we have $(a,\phi) = 0$.

So for each $[(A,\psi)] \in \mathcal{M}_f{}^*$, there is a set of finitely many thickened loops such that the first assertion holds for $[(A,\psi)]$. Then the first assertion follows by the compactness of $\mathcal{M}_f^*$ and the smoothness of the function (p,q).

For the second assertion, suppose $[(A_1,\psi_1)],[(A_2,\psi_2)] \in \mathcal{M}_f{}^*$ such that $(p,q)_\lambda(A_1,\psi_1) = (p,q)_\lambda(A_2,\psi_2)$ for all loops. Then $dA_1 = \pm dA_2$, and $|\psi_1|^2 = |\psi_2|^2$. Assume $dA_1 = dA_2$, then $\tau(\psi_1) = \tau(\psi_2)$. By writing in a local frame, it is easy to see that $\psi_1 = s\psi_2$ for some $s \in Map(Y,S^1)$. Then it is easy to see that $[(A_1,\psi_1)] = [(A_2,\psi_2)]$. In the case of $dA_1 = -dA_2$, apply σ.

Now for any $[(A_1,\psi_1)] \neq [(A_2,\psi_2)]$ in $\mathcal{M}_f{}^*/\sigma$, there is a thickened loop λ separating them. By the compactness of $\mathcal{M}_f{}^*/\sigma$ and the smoothness of (p,q), there exists a set of finitely many loops separating any two points in $\mathcal{M}_f{}^*/\sigma$ with distance greater then a fixed number. Combining with the first assertion, since each point in $\mathcal{M}_f{}^*/\sigma$ has a neighborhood described by a Kuranishi model, the second assertion follows.

□

For any smooth function h on $\prod_{\lambda\in\Lambda}([-1,1]\times \mathbf{R}^+)_\lambda$, the composition $u = h\circ\Phi_\Lambda$ is a smooth function on $\mathcal{A}$. We will perturb $\mathcal{CSD}_f$ by adding u, i.e., $\mathcal{CSD}' = \mathcal{CSD}_f + u$. Denote the gradient of $\mathcal{CSD}'$ by s'. The following lemma is standard (see [29]).

Lemma 1.5.4 *1. ∇s^f and $\nabla s'$ are continuous families of Fredholm operators from bundle $T\mathcal{B}^*$ to $\mathcal{L}$ over $\mathcal{B}^*$, and $\nabla s^f - \nabla s'$ is compact.*

2. $\mathcal{M} = {s'}^{-1}(0)$ can be represented by smooth sections.

3. There exists a constant $\epsilon > 0$ such that when $|dh| < \epsilon$, $\mathcal{M}$ is compact.

4. When $|dh| \to 0$, the distance between $\mathcal{M}_f$ and $\mathcal{M}$ goes to zero.

Next we define a section G of the bundle $\mathcal{L}$ over $\mathcal{B}^* \times V$ where V is the dual of the vector space $\prod_{\lambda\in\Lambda}(\mathbf{R}\times\mathbf{R})_\lambda$:

$$G((A,\psi),(v,w)_\lambda) = s^f(A,\psi) + grad(\rho(\Phi_\Lambda)(\sum_{\lambda\in\Lambda}(v_\lambda p_\lambda + w_\lambda q_\lambda)))(A,\psi).$$

Here the set Λ of thickened loops satisfies the conditions in Lemma 1.5.3, and ρ is a cutoff function on $\prod_{\lambda\in\Lambda}(\mathbf{R}\times\mathbf{R})_\lambda$ satisfying that $\rho\equiv 0$ in a neighborhood of $\prod_{\lambda\in\Lambda}([-1,1]\times\{0\})_\lambda$ and $\Phi_\Lambda([(A,\psi)])$ where $[(A,\psi)]\in\mathcal{B}^*$ is a non-degenerate critical point of $\mathcal{CSD}_f$, and $\rho\equiv 1$ in a neighborhood of the rest of $\Phi_\Lambda(\mathcal{M}_f{}^*)$.

Lemma 1.5.5 *There exists $\epsilon > 0$ (depending on ρ) such that G is transversal to the zero section of $\mathcal{L}$ when restricted to $\mathcal{B}^*\times B(\epsilon)$, where $B(\epsilon)$ is a ball of radius ϵ centered at the origin in $V = (\prod_{\lambda\in\Lambda}(\mathbf{R}\times\mathbf{R})_\lambda)^*$.*

Proof: G is transversal to the zero section of $\mathcal{L}$ over $\mathcal{M}_f{}^*\times\{0\}$ by the choice of the set Λ. By continuity and Lemma 1.5.4 (4), this lemma is proved. □

The Proof of Proposition 1.1.7:

Apply Sard-Smale theorem to the projection $\Pi : G^{-1}(0)\to B(\epsilon)$. For a generic $(v,w)_\lambda\in B(\epsilon)$, the perturbation $\mathcal{CSD}' = \mathcal{CSD}_f + u$ is admissible where

$$u = \rho(\Phi_\Lambda)(\sum_{\lambda\in\Lambda}(v_\lambda p_\lambda + w_\lambda q_\lambda)).$$

□

CHAPTER 2

Seiberg-Witten Equations on Cylindrical End Manifolds

Throughout this chapter, we assume that Y is an oriented 3-manifold with boundary which is the complement of a tubular neighborhood of a knot in an integral homology 3-sphere (many results proved in this chapter hold for general 3-manifolds with toroidal boundary). Equip Y with a Riemannian metric g_0 such that a neighborhood of $\partial Y = T^2$ is orientedly isometric to $(-1, 0] \times \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$. We attach $[0, \infty) \times T^2$ to Y and still denote it by Y.

Given a spin structure of Y, there is a unique $SU(2)$ vector bundle W over Y such that the oriented volume form acts on W as identity by the Clifford multiplication. The spinor bundle W is cylindrical, i.e. on $[0, \infty) \times T^2$, W is isometric to the pull back $\pi^* W_0$ where $\pi : [0, \infty) \times T^2 \to T^2$ is the projection and W_0 is the total spinor bundle on T^2 associated to the spin structure induced from Y.

2.1 The Fredholm theory

In this section, we set up the Fredholm theory for Seiberg-Witten equations on Y. Throughout $\mathcal{H}^1(T^2)$ stands for the space of harmonic 1-forms on T^2. We fix a cut-off function ρ on Y which equals to 0 on $Y \setminus [0,\infty) \times T^2$ and 1 on $[1,\infty) \times T^2$.

Definition 2.1.1 *For $\delta > 0$, let*

$$\begin{aligned} \mathcal{A}_\delta &= \{(A,\psi) | A = B + \rho\pi^* a, \\ &\quad B \in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}), \psi \in L^2_{2,\delta}(W), a \in \mathcal{H}^1(T^2) \otimes i\mathbf{R}\}, \end{aligned}$$

where $\pi : [0,\infty) \times T^2 \to T^2$ is the projection.

Here $L^2_{k,\delta}$ denotes the weighted Sobolev spaces with weight δ ([21]). $\mathcal{A}_\delta$ is a Hilbert space (over real numbers) with the norm

$$\|(A,\psi)\|_{\mathcal{A}_\delta} = \|(B,\psi)\|_{L^2_{2,\delta}} + \|a\|_{L^2}.$$

Note that the decomposition of A as $B + \rho\pi^* a$ is unique. We define a map $R : \mathcal{A}_\delta \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ by $R(A,\psi) = a$.

Definition 2.1.2 *The group of gauge transformations is*

$$\begin{aligned} \mathcal{G}_\delta &= \{s \in L^2_{3,loc}(Y, S^1) | s^{-1}ds = g + \rho\pi^* h, \\ &\quad g \in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}), h \in \mathcal{H}^1(T^2) \otimes i\mathbf{R}\}. \end{aligned}$$

$\mathcal{G}_\delta$ acts on $\mathcal{A}_\delta$ by the formula $s \cdot (A,\psi) = (A - s^{-1}ds, s\psi)$ for $s \in \mathcal{G}_\delta$ and $(A,\psi) \in \mathcal{A}_\delta$.

Lemma 2.1.3 *$\mathcal{G}_\delta$ is an Abelian Hilbert Lie group acting smoothly on $\mathcal{A}_\delta$ with the Lie algebra $T\mathcal{G}_{\delta,id} = L^2_{3,\delta}(\Lambda^0(Y)\otimes i\mathbf{R})\oplus i\mathbf{R}$. Moreover, for $s\in\mathcal{G}_\delta$, if $s^{-1}ds$ is decomposed as $g+\rho\pi^*h$, then h has zero period along the longitude and periods in $2\pi i\mathbf{Z}$ along the meridian.*

Proof: Suppose that $s\in\mathcal{G}_\delta$ is in the component of identity, then $s=e^f$ for some $f\in L^2_{3,loc}(\Lambda^0(Y)\otimes i\mathbf{R})$. By Definition 2.1.2, $df = s^{-1}ds$ can be decomposed as $g+\rho\pi^*h$, from which it follows that $h=0$ and $df\in L^2_{2,\delta}(\Lambda^1(Y)\otimes i\mathbf{R})$. By Taubes inequality (Lemma 5.2 in [28]), there exists an imaginary valued constant f_0 on Y such that

$$\int_Y |f-f_0|^2 e^{2\delta t} \le C(\delta)\int_Y |df|^2 e^{2\delta t},$$

which proves that the Lie algebra $T\mathcal{G}_{\delta,id}$ is $L^2_{3,\delta}(\Lambda^0(Y)\otimes i\mathbf{R})\oplus i\mathbf{R}$.

Let γ_1,γ_2 be the longitude and meridian, and F be the Seifert surface that γ_1 bounds in Y. For $s\in\mathcal{G}_\delta$, if $s^{-1}ds$ is decomposed as $g+\rho\pi^*h$, then we have

$$\int_{\gamma_1} h = \int_F d(s^{-1}ds) = 0 \text{ and } \int_{\gamma_2} h = \int_{\gamma_2} s^{-1}ds \in 2\pi i\mathbf{Z}.$$

The rest follows easily from the Sobolev theorems for weighted spaces. □

Lemma 2.1.4 *The de Rham cohomology group $H^1_{DR}(Y)$ can be represented by the space of "bounded" harmonic forms*

$$\mathcal{H}^1(Y) = \{a\in\Omega^1(Y) | da = d^*a = 0, \|a\|_{C^0(Y)} < \infty, \lim_{t\to\infty} a(\frac{\partial}{\partial t}) = 0\}.$$

Moreover, each element $a\in\mathcal{H}^1(Y)$ can be decomposed as $b+\rho\pi^*a_\infty$ with $a_\infty\in\mathcal{H}^1(T^2)$

and $b \in L^2_{k,\delta}$ *for some* $\delta > 0$. *The map* $R : \mathcal{H}^1(Y) \to \mathcal{H}^1(T^2)$ *defined by* $R(a) = a_\infty$ *represents the embedding* $H^1_{DR}(Y) \to H^1_{DR}(T^2)$. *As a corollary, for any* $\kappa \in H^1(Y, \mathbf{Z})$, *there is an* $s_\kappa \in C^\infty(Y, S^1)$ *such that* $s_\kappa^{-1} ds_\kappa \in \mathcal{H}^1(Y) \otimes i\mathbf{R}$ *and* $[s_\kappa^{-1} ds_\kappa] = 2\pi i\kappa$. *So* $\pi_0(\mathcal{G}_\delta) = H^1(Y, \mathbf{Z}) = \mathbf{Z}$.

Proof: The Laplacian $d^*_{T^2} d_{T^2} : L^2_2(\Lambda^0(T^2)) \to L^2(\Lambda^0(T^2))$ restricted to $(Ker d^*_{T^2} d_{T^2})^\perp$ is invertible. Let G be the inverse. Suppose a closed form $A \in \Omega^1(Y)$ is written as $A_0 dt + A_1$ on $[0, \infty) \times T^2$ with $A_1 \in \Omega^1(T^2)$. Then $f = G(d^*_{T^2} A_1)$ is a smooth function on $[0, \infty) \times T^2$. We extend f to the rest of Y and still call it f. Let $B = A - df$. We can further modify f by a function of t so that $\int_{T^2} B_0 = 0$, where $B = B_0 dt + B_1$ on $[0, \infty) \times T^2$. (B_0, B_1) satisfies the following equations:

$$\frac{\partial B_1}{\partial t} = d_{T^2} B_0, \;\; d_{T^2} B_1 = 0 \text{ and } d^*_{T^2} B_1 = 0,$$

which shows that B_1 is in $\mathcal{H}^1(T^2)$ and constant in t and $B_0 = 0$. Since $d^* B$ is compactly supported, there is a unique solution $g \in L^2_{k,\delta}(\Lambda^0(Y))$ to the equation $d^* B = d^* dg$ (see Lemma 2.1.7 below). Let $C = B - dg$, then $C \in \mathcal{H}^1(Y)$ and the cohomology classes $[A]$ and $[C]$ are equal in $H^1_{DR}(Y)$.

Suppose $a \in \mathcal{H}^1(Y)$ and $a = a_0 dt + a_1$ on $[0, \infty) \times T^2$. Then the pair (a_0, a_1) satisfies the following system of equations

$$\begin{cases} \frac{\partial a_1}{\partial t} - d_{T^2} a_0 = 0 \\ \frac{\partial a_0}{\partial t} - d^*_{T^2} a_1 = 0 \\ d_{T^2} a_1 = 0. \end{cases}$$

The operator $L = \begin{pmatrix} 0 & d_{T^2} \\ d^*_{T^2} & 0 \end{pmatrix}$ is formally self-adjoint and elliptic on $Kerd \oplus \Omega^0(T^2)$. By expanding (a_1, a_0) in terms of an orthonormal basis of eigenvectors of L, we see that $a = a_0 dt + a_1$ can be decomposed as $b + \rho\pi^* a_\infty$ where $a_\infty \in \mathcal{H}^1(T^2)$ and $b \in L^2_{k,\delta}$ for some $\delta > 0$.

Assume $a_1, a_2 \in \mathcal{H}^1(Y)$, if $a_1 - a_2 = df$ for a smooth function f on Y, then $df \in L^2_{k,\delta}$, and by Taubes inequality and integration by parts, $df = 0$. Hence the map $\mathcal{H}^1(Y) \to H^1_{DR}(Y)$ is also injective. The rest of the lemma follows easily. □

Definition 2.1.5 *Let $\mathcal{B}_\delta = \mathcal{A}_\delta / \mathcal{G}_\delta$ and $\mathcal{B}^*_\delta = \mathcal{A}^*_\delta / \mathcal{G}_\delta$ where $\mathcal{A}^*_\delta = \mathcal{A}_\delta \setminus \{\psi \equiv 0\}$.*

Lemma 2.1.6 *1. $\mathcal{B}^*_\delta$ is a Hilbert manifold with the slice at $(A, \psi) \in \mathcal{A}^*_\delta$ given by $T_{(A,\psi),\epsilon} = U \times V$ where*

$$\begin{aligned} U = (B, \psi) + \{(a, \phi) &\in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{2,\delta}(W) | - d^* a \\ &+ i \langle i\psi, \phi \rangle_{Re} = 0, \|(a, \phi)\|_{L^2_{2,\delta}} < \epsilon\}, \end{aligned}$$

$$V = R(A, \psi) + \{a_\infty \in \mathcal{H}^1(T^2) \otimes i\mathbf{R} | \|a_\infty\|_{L^2} < \epsilon\},$$

where A is decomposed into $B + \rho\pi^* R(A, \psi)$. The tangent space of $\mathcal{B}^*_\delta$ at (A, ψ) is

$$\begin{aligned} T\mathcal{B}^*_{\delta,(A,\psi)} = &\{(a, \phi) \in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{2,\delta}(W) | - d^* a \\ &+ i \langle i\psi, \phi \rangle_{Re} = 0\} \oplus \mathcal{H}^1(T^2) \otimes i\mathbf{R}. \end{aligned}$$

2. *A neighborhood of* $[(A,0)]$ *in* $\mathcal{B}_\delta$ *is diffeomorphic to* $T_{(A,0),\epsilon}/S^1$. $T_{(A,0),\epsilon} = U \times V$ *and*

$$U = (B,0) + \{(a,\phi) \in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{2,\delta}(W) | d^*a = 0, \|(a,\phi)\|_{L^2_{2,\delta}} < \epsilon\},$$
$$V = R(A,\psi) + \{a_\infty \in \mathcal{H}^1(T^2) \otimes i\mathbf{R} | \|a_\infty\|_{L^2} < \epsilon\},$$

where A *is decomposed into* $B + \rho\pi^* R(A,\psi)$. *The action of* S^1 *on* $T_{(A,0),\epsilon}$ *is given by the complex multiplication on the factor* ϕ.

Lemma 2.1.7 *Let* $L_1 = d^*d$ *and* $L_2 = d^*d + |\psi|^2$ *where* $\psi \in L^2_{2,\delta}(W)$. *Then there is* $\delta_0 > 0$ *such that for* $k \geq 2$ *and any* $\delta \in (0,\delta_0]$, $L_1 : L^2_{k,\delta}(\Lambda^0(Y)) \to L^2_{k-2,\delta}(\Lambda^0(Y))$ *is a Fredholm operator of index* -1. *Ker* $L_1 = 0$, *and the range of* L_1 *is the* L^2*-orthogonal complement of the space of constant functions.* $L_2 : L^2_{k,\delta}(\Lambda^0(Y)) \oplus i\mathbf{R} \to L^2_{k-2,\delta}(\Lambda^0(Y))$ *is isomorphic if* ψ *is not identically zero.*

Proof: The operator $L_1 = d^*d : L^2_{k,\delta}(\Lambda^0(Y)) \to L^2_{k-2,\delta}(\Lambda^0(Y))$ is Fredholm of index -1 by Theorem 7.4 of [21]. *Ker* $L_1 = 0$ follows from integration by parts. From index counting it follows that the range of L_1 is the L^2-orthogonal complement of the space of constant functions. For $\psi \in L^2_{2,\delta}(W)$, $L_2 : L^2_{k,\delta}(\Lambda^0(Y)) \to L^2_{k-2,\delta}(\Lambda^0(Y))$ is a compact perturbation of L_1, so it is also a Fredholm operator of index -1. So $L_2 : L^2_{k,\delta}(\Lambda^0(Y)) \oplus i\mathbf{R} \to L^2_{k-2,\delta}(\Lambda^0(Y))$ is an isomorphism if ψ is not identically zero, since *Ker* $L_2 = 0$ and index $L_2 = 0$. □

The Proof of Lemma 2.1.6:

1. The construction of a local slice is standard by applying the implicit function theorem. The key point is the properties of L_2 stated in Lemma 2.1.7. To prove

that $\mathcal{B}_\delta^*$ is Hausdorff and the local slice is embedded into $\mathcal{B}_\delta^*$, the argument in [12] can be used, combined with Taubes inequality (Lemma 5.2 in [28]).

2. Part 2 of this lemma follows similarly with Lemma 2.1.7 understood.

□

Definition 2.1.8 *For $(A,\psi) \in \mathcal{A}_\delta$, we define*

$$\mathcal{L}_{\delta,(A,\psi)} = \{(a,\phi) \in L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W) | -d^*a + i\langle i\psi, \phi\rangle_{Re} = 0\}.$$

$\mathcal{L}_{\delta,(A,\psi)}$ is a closed subspace of $L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W)$.

Lemma 2.1.9 *$\mathcal{L}_\delta = \{\mathcal{L}_{\delta,(A,\psi)}\}$ is a Hilbert bundle over $\mathcal{A}_\delta^*$ which descends to a Hilbert bundle over $\mathcal{B}_\delta^*$ (we still call it $\mathcal{L}_\delta$).*

Proof: For any $(a,\phi) \in L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W)$, we can project (a,ϕ) into $\mathcal{L}_{\delta,(A,\psi)}$ by solving the following equation

$$-d^*(a - df) + i\langle i\psi, \phi + f\psi\rangle_{Re} = 0$$

for $f \in L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \oplus i\mathbf{R}$. By Lemma 2.1.7, the operator $L_2 = d^*d + |\psi|^2$ is an isomorphism from $L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \oplus i\mathbf{R}$ to $L^2_{0,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$ since $(A,\psi) \in \mathcal{A}_\delta^*$. So the above equation has a unique solution $f(a,\phi)$ for any (a,ϕ). If $(a,\phi) \in \mathcal{L}_{\delta,(A_1,\psi_1)}$ with (A_1,ψ_1) close enough to (A,ψ), one can easily show that the projection $(a,\phi) \to (a - df, \phi + f\psi)$ is one to one and onto, again using the invertibility of L_2. This proves the local triviality of $\mathcal{L}_\delta$. The bundle $\mathcal{L}_\delta$ over $\mathcal{A}_\delta^*$ is $\mathcal{G}_\delta$-equivariant, so it descends to a Hilbert bundle over $\mathcal{B}_\delta^*$. □

Definition 2.1.10 *For* $(A, \psi) \in \mathcal{A}_\delta^*$*, we define*

$$s(A, \psi) = (*dA + \tau(\psi, \psi), D_A\psi).$$

Here $D_A = D_{g_0} + A$ *where* D_{g_0} *is the Dirac operator associated to the metric* g_0*.* s *is a section of* $\mathcal{L}_\delta$ *over* $\mathcal{A}_\delta^*$*, which descends to a section of* $\mathcal{L}_\delta$ *over* $\mathcal{B}_\delta^*$*.*

The covariant derivative of s *is a section of* $End(T\mathcal{B}_\delta^*, \mathcal{L}_\delta)$ *over* $\mathcal{A}_\delta^*$ *which descends to* $\mathcal{B}_\delta^*$*, defined by*

$$\nabla s_{(A,\psi)}(a, \phi) = (*da + 2\tau(\psi, \phi) - df(a, \phi), D_A\phi + a\psi + f(a, \phi)\psi)$$

where $f(a, \phi)$ *is the unique solution to the equation*

$$d^*df + f|\psi|^2 = i\langle D_A\psi, i\phi\rangle_{Re}.$$

The map $(a, \phi) \to (-df(a, \phi), f(a, \phi)\psi)$ *from* $T\mathcal{B}^*_{\delta,(A,\psi)}$ *to* $L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W)$ *is compact by the Sobolev theorems for weighted spaces.*

Definition 2.1.11

1. *For any* $r > 0$*, let* $\mathcal{H}(r) = \mathcal{H}^1(T^2) \otimes i\mathbf{R} \setminus \bigcup_{p \in B} D(p, r)$ *where* B *is the lattice of "bad" points for the induced spin structure on* T^2 *(see Appendix A) and* $D(p, r)$ *is the closed disc of radius* r *centered at* p*.*

2. $\mathcal{A}_\delta(r) = R^{-1}(\mathcal{H}(r))$*,* $\mathcal{A}_\delta^*(r) = \mathcal{A}_\delta(r) \bigcap \mathcal{A}_\delta^*$*,* $\mathcal{B}_\delta(r) = \mathcal{A}_\delta(r)/\mathcal{G}_\delta$ *and* $\mathcal{B}_\delta^*(r) = \mathcal{A}_\delta^*(r)/\mathcal{G}_\delta$*, where* $R : \mathcal{A}_\delta \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ *is given by* $R(A, \psi) = a$ *for* $A = B + \rho\pi^*a$*.*

Note that for any $a \in \mathcal{H}(r)$, the twisted Dirac operator $D_a^{T^2} = D^{T^2} + a$ is invertible, where D^{T^2} is the Dirac operator on T^2 (see Appendix A).

Proposition 2.1.12 *For any $r > 0$, there exists a $\delta(r) > 0$ such that for each $\delta \in (0, \delta(r))$, $\nabla s : T\mathcal{B}_\delta^* \to \mathcal{L}_\delta$ is a continuous family of Fredholm operators of index 1 over $\mathcal{A}_\delta^*(r)$. (So s is a Fredholm section of $\mathcal{L}_\delta$ over $\mathcal{B}_\delta^*(r)$).*

At $(A, \psi) \in \mathcal{A}_\delta^*$, we have a short exact sequence

$$0 \to T\mathcal{G}_{\delta,id} \xrightarrow{d_{(A,\psi)}} T\mathcal{A}_\delta^* \xrightarrow{\pi} T\mathcal{B}_\delta^* \to 0$$

where $d_{(A,\psi)}(f) = (-df, f\psi)$ and $\pi : \mathcal{A}_\delta^* \to \mathcal{B}_\delta^*$ is the natural projection. This enables us to extend $\nabla s_{(A,\psi)} : T\mathcal{B}_{\delta,(A,\psi)}^* \to \mathcal{L}_{\delta,(A,\psi)}$ to a $\mathcal{G}_\delta$-equivariant map $\mathcal{K}'_{(A,\psi)}$ (see [29]), where

$$\mathcal{K}'_{(A,\psi)} = \begin{pmatrix} \nabla s_{(A,\psi)} & 0 & 0 \\ 0 & 0 & d_{(A,\psi)} \\ 0 & d^*_{(A,\psi)} & 0 \end{pmatrix}.$$

$\mathcal{K}'_{(A,\psi)}$ is from $T\mathcal{A}_\delta^* \oplus (L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \oplus i\mathbf{R})$ to $L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W) \oplus L^2_{1,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$. Since the operator $\begin{pmatrix} 0 & d_{(A,\psi)} \\ d^*_{(A,\psi)} & 0 \end{pmatrix}$ is invertible, $\nabla s_{(A,\psi)}$ is Fredholm if and only if $\mathcal{K}'_{(A,\psi)}$ is, and $\mathrm{index}\mathcal{K}' = \mathrm{index}\nabla s$.

For $(A, \psi) \in \mathcal{A}_\delta$, we define a map $\mathcal{K}_{(A,\psi)} : \mathcal{A}_\delta \oplus (L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \oplus i\mathbf{R}) \to L^2_{1,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{1,\delta}(W) \oplus L^2_{1,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$ by

$$\mathcal{K}_{(A,\psi)}(a, \phi, f) = (*da + 2\tau(\psi, \phi) - df, D_A\phi + a\psi + f\psi, -d^*a + i\langle i\psi, \phi\rangle_{Re}).$$

Then $\mathcal{K}'_{(A,\psi)}$ is a compact perturbation of $\mathcal{K}_{(A,\psi)}$ and Proposition 2.1.12 follows from

Lemma 2.1.13 *For any $r > 0$, there exists a $\delta(r) > 0$ such that for each $\delta \in (0, \delta(r))$, $\mathcal{K}$ is a continuous family of Fredholm maps on $\mathcal{A}^*_\delta(r)$ of index 1.*

Proof: Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V\mathcal{A}_{\delta,1} \oplus L^2_{1,\delta} & \to & V\mathcal{A}_{\delta,1} \oplus L^2_{1,\delta} & \to & 0 & & \\ & & \uparrow V\mathcal{K}_{(A,\psi)} & & \uparrow \mathcal{K}_{(A,\psi)} & & \uparrow & & \\ 0 & \to & V\mathcal{A}_\delta \oplus L^2_{2,\delta} & \to & \mathcal{A}_\delta \oplus (L^2_{2,\delta} \oplus i\mathbf{R}) & \to & \mathcal{H}^1(T^2) \otimes i\mathbf{R} \oplus i\mathbf{R} & \to & 0 \end{array}$$

where $L^2_{1,\delta} = L^2_{1,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$ and $L^2_{2,\delta} = L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$. $V\mathcal{A}_\delta$ is the fiber of map $R : \mathcal{A}_\delta \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ and $V\mathcal{A}_{\delta,1}$ is its $L^2_{1,\delta}$-completion. $V\mathcal{K}_{(A,\psi)}$ is the restriction of $\mathcal{K}_{(A,\psi)}$. We have a long exact sequence (see [24])

$$\begin{array}{ccccccccc} & & Coker V\mathcal{K}_{(A,\psi)} & \to & Coker \mathcal{K}_{(A,\psi)} & \to & 0 & & \\ 0 & \to & Ker V\mathcal{K}_{(A,\psi)} & \to & Ker \mathcal{K}_{(A,\psi)} & \to & \mathcal{H}^1(T^2) \otimes i\mathbf{R} \oplus i\mathbf{R} & \to & . \end{array}$$

The lemma follows from the claim that for any $r > 0$, there exists a $\delta(r) > 0$ such that for each $\delta \in (0, \delta(r))$, $V\mathcal{K}_{(A,\psi)} : V\mathcal{A}_\delta \oplus L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \to V\mathcal{A}_{\delta,1} \oplus L^2_{1,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$ is a Fredholm map of index -2 for $(A, \psi) \in \mathcal{A}^*_\delta(r)$.

$V\mathcal{K}_{(A,\psi)}$ is a compact perturbation of an operator of form $I(\frac{\partial}{\partial t}+B_a)$ on $[0,\infty)\times T^2$ where

$$I = \begin{pmatrix} dt & 0 & 0 & 0 \\ 0 & *_{T^2} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B_a = \begin{pmatrix} D_a^{T^2} & 0 & 0 & 0 \\ 0 & 0 & -d_{T^2} & *d_{T^2} \\ 0 & -d_{T^2}^* & 0 & 0 \\ 0 & -*d_{T^2} & 0 & 0 \end{pmatrix}$$

acting on $\Gamma(W_0 \oplus (\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^0(T^2)) \otimes i\mathbf{R})$. Here W_0 is the total spinor bundle over T^2, and $D_a^{T^2}$ is the twisted Dirac operator with $a = R(A,\psi) \in \mathcal{H}^1(T^2) \otimes i\mathbf{R}$. For any $r > 0$, let $\delta(r) = \min\{|u| : u \neq 0$ is an eigenvalue of B_a for some $a \in \mathcal{H}(r)\}$. Then $V\mathcal{K}_{(A,\psi)} : V\mathcal{A}_\delta \oplus L^2_{2,\delta}(\Lambda^0(Y) \otimes i\mathbf{R}) \to V\mathcal{A}_{\delta,1} \oplus L^2_{1,\delta}(\Lambda^0(Y) \otimes i\mathbf{R})$ is a Fredholm map of index -2 for any $\delta \in (0, \delta(r))$ by Theorem 7.4 of [21]. □

2.2 Perturbation and transversality

Fix a small $r > 0$ and a $\delta \in (0, \delta(r))$ for the weight of the Sobolev spaces. For simplicity we omit the subscript δ in the discussion. Consider the following perturbations of s over $\mathcal{A}^*(r)$:

$$s_{\mu,f}(A,\psi) = (*dA + \tau(\psi,\psi) + \mu, D_A\psi + f\psi) \text{ for } (A,\psi) \in \mathcal{A}^*(r),$$

where μ is a co-closed imaginary valued 1-form and f is a smooth real valued function on Y, both supported in $Y \backslash [0,\infty) \times T^2$. The metric g being used here is a perturbation of g_0 supported in $Y \setminus [0,\infty) \times T^2$.

Definition 2.2.1 *Define the Seiberg-Witten moduli spaces*

$$\mathcal{M}_{\mu,f}(r) = \{[(A,\psi)] \in \mathcal{B}(r) | (*dA + \tau(\psi,\psi) + \mu, D_A\psi + f\psi) = 0\},$$
$$\mathcal{M}^*_{\mu,f}(r) = \mathcal{M}_{\mu,f}(r) \bigcap \mathcal{B}^*.$$

Let $[R] : \mathcal{B}(r) \to \mathcal{H}(r)/\mathbf{Z}$ be the map induced by $R : \mathcal{A} \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$.

Proposition 2.2.2 *The moduli spaces $\mathcal{M}_{\mu,f}(r)$ and $\mathcal{M}^*_{\mu,f}(r)$ have the following properties.*

1. *For a generic μ, $\mathcal{M}^*_{\mu,f}(r)$ is a collection of 1-dimensional smooth curves, and the map $[R] : \mathcal{M}^*_{\mu,f}(r) \to \mathcal{H}(r)/\mathbf{Z}$ is an immersion.*

2. *Given a set S of immersed curves in $\mathcal{H}(r)/\mathbf{Z}$, for a generic μ, the map $[R]$: $\mathcal{M}^*_{\mu,f}(r) \to \mathcal{H}(r)/\mathbf{Z}$ is transversal to S.*

3. *For a generic (g,f), the L^2-closed extension of the perturbed Dirac operator $D_g + f$ is invertible. Fix such a (g,f), then for any small enough μ, there exists a neighborhood U_μ of $[(a_\mu, 0)]$ in $\mathcal{B}(r)$ such that $U_\mu \bigcap \mathcal{M}^*_{\mu,f}(r) = \emptyset$, where a_μ is the unique solution to $*da_\mu + \mu = 0$, $d^*a_\mu = 0$ such that $R(a_\mu)$ has zero period along the meridian.*

4. *$\mathcal{M}_{\mu,f}(r) \setminus \mathcal{M}^*_{\mu,f}(r) = \{(a_\mu + iA, 0) | A \in \mathcal{H}^1(Y)/\mathcal{H}^1(Y,\mathbf{Z})\} \simeq S^1$. Note that for any $A \in \mathcal{H}^1(Y)$, $R(A)$ is a multiple of dy where e^{iy} parameterizes the meridian.*

For simplicity we omit the subscript f. Consider the section $\tilde{s}$ of $\mathcal{L}$ over $\mathcal{B}^*(r) \times Ker\ d^*$:

$$\tilde{s}([(A,\psi)],\mu) = [(*dA + \tau(\psi,\psi) + \mu, D_A\psi + f\psi)].$$

For any $([(A,\psi)],\mu) \in \tilde{s}^{-1}(0)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{L} & \to & \mathcal{L} & \to & 0 & & \\ & & \uparrow V\nabla\tilde{s} & & \uparrow \nabla\tilde{s} & & \uparrow 0 & & \\ 0 & \to & VT\mathcal{B}^*(r) \times Ker\ d^* & \to & T\mathcal{B}^*(r) \times Ker\ d^* & \stackrel{d[R]}{\to} & \mathcal{H}^1(T^2) \otimes i\mathbf{R} & \to & 0 \end{array}$$

at $([(A,\psi)],\mu)$. This gives rise to a long exact sequence (see [24])

$$\begin{array}{ccccccc} & & Coker(V\nabla\tilde{s}_{([(A,\psi)],\mu)}) & \to & Coker(\nabla\tilde{s}_{([(A,\psi)],\mu)}) & \to 0 & \\ 0 & \to & Ker(V\nabla\tilde{s}_{([(A,\psi)],\mu)}) & \to & Ker(\nabla\tilde{s}_{([(A,\psi)],\mu)}) & \stackrel{d[R]}{\to} & \mathcal{H}^1(T^2) \otimes i\mathbf{R} \ \to . \end{array}$$

Lemma 2.2.3 *$Coker(V\nabla\tilde{s}_{([(A,\psi)],\mu)}) = 0$ for any $([(A,\psi)],\mu) \in \tilde{s}^{-1}(0)$.*

Proof: First observe that $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$ is Fredholm as a map from the $L^2_{1,\delta}$-completion of $VT\mathcal{B}^*_{[(A,\psi)]}$ to the $L^2_{0,\delta}$-completion of $\mathcal{L}_{[(A,\psi)]}$. So by regularity, it suffices to show that the $L^2_{0,\delta}$-orthogonal complement of the image of $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$ is zero dimensional.

Let $(a',\phi') \in \mathcal{L}_{[(A,\psi)]}$ be $L^2_{0,\delta}$-orthogonal to the range of $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$. Set $(a,\phi) = e^{2\delta t}(a',\phi')$, then (a,ϕ) is L^2-orthogonal to the range of $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$ and $e^{-2\delta t}(a,\phi)$ is in $\mathcal{L}_{[(A,\psi)]}$, i.e. $-d^*(e^{-2\delta t}a) + i\langle i\psi, e^{-2\delta t}\phi\rangle_{Re} = 0$. Note that (a,ϕ) is in $L^2_{1,-\delta}$.

Observe that $L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{2,\delta}(W) = VT\mathcal{B}^*(r) \oplus Im(d_{[(A,\psi)]})$ (recall the map $d_{(A,\psi)}$ is defined by $f \to (-df, f\psi)$), and $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$ vanishes on $Im(d_{[(A,\psi)]})$. So if (a,ϕ) is L^2-orthogonal to the range of $V\nabla\tilde{s}_{([(A,\psi)],\mu)}$, then for any $\mu' \in Ker\ d^*$ and $(b,\theta) \in L^2_{2,\delta}(\Lambda^1(Y) \otimes i\mathbf{R}) \oplus L^2_{2,\delta}(W)$, we have

$$(*db + 2\tau(\psi,\theta) + \mu', a) + (D_A\theta + f\theta + b\psi, \phi) = 0.$$

This implies that $D_A\phi + f\phi + a\psi = 0$ and $(b\psi, \phi) = 0$ for any b. Since ψ is not identically zero, by the unique continuation theorem for Dirac operators, we have $\phi = ih\psi$ for a real valued function h. Then $D_A\phi + f\phi + a\psi = 0$ implies that $idh + a = 0$. Hence

$$d^*(e^{-2\delta t}dh) + he^{-2\delta t}|\psi|^2 = 0.$$

That $(a, \phi) \equiv 0$ follows from $h \equiv 0$, which follows by integration by parts from the claim that $e^{-\delta t}|h|$ is bounded on $[0, \infty) \times T^2$.

Next we prove that $e^{-\delta t}|h|$ is bounded on $[0, \infty) \times T^2$. First of all, $idh + a = 0$ implies that $\frac{\partial h}{\partial t} \in L^2_{1,-\delta}$ and $d_{T^2}h \in L^2_{1,-\delta}$. Let $h_0(t)$ be the L^2-orthogonal projection of h onto $Kerd^*_{T^2}d_{T^2}$, then $\|h - h_0(t)\|_{L^2(T^2)} \leq c\|d_{T^2}h\|_{L^2(T^2)}$. So $h - h_0(t) \in L^2_{0,-\delta}$. On the other hand, $|\frac{dh_0}{dt}| \leq C\|\frac{\partial h}{\partial t}\|_{L^2(T^2)}$ so that $\frac{dh_0}{dt}e^{-\delta t}$ is bounded on $[0, \infty) \times T^2$. So

$$|h_0(t) - h_0(0)| \leq \int_0^t |\frac{dh_0}{dt}|dt \leq \frac{C}{\delta}e^{\delta t}.$$

It follows easily that $e^{-\delta t}|h|$ is bounded on $[0, \infty) \times T^2$. □

Lemma 2.2.4 *Given any spin structure on Y, for a generic (g, f), the L^2-closed extension of the perturbed Dirac operator $D_g + f$ is invertible.*

Proof: For a perturbed metric g of g_0 which is supported in $Y \setminus [0, \infty) \times T^2$, the Dirac operator D_g on Y takes the form of $dt(\frac{\partial}{\partial t} + D^{T^2})$ on the cylindrical end. Note that D^{T^2} is invertible (see Appendix A). So the L^2-closed extension of $D_g + f$ is an essentially self-adjoint Fredholm operator (f is vanishing on the cylindrical end). The argument for the proof of Proposition 1.1.6 can be applied to prove this lemma. □

Lemma 2.2.5 *For small enough $\delta > 0$, the operator $*d : L^2_{k,\delta}(\Lambda^1(Y)) \bigcap Ker\ d^* \to$*

$L^2_{k-1,\delta}(\Lambda^1(Y)) \bigcap Ker\ d^*$ *is Fredholm with* $\dim Ker * d = 0$ *and* $\dim Coker * d = 1$. *Moreover, for any compactly supported co-closed 1-form* μ, *there exists a unique* $a_\mu \in Ker\ d^*$ *such that i)* $*da_\mu + \mu = 0$; *ii)* a_μ *can be decomposed as* $b + \rho\pi^* a_\infty$ *where* $b \in L^2_{k,\delta}$ *and* $a_\infty \in \mathcal{H}^1(T^2)$ *with zero period along the meridian. Furthermore,* a_μ *satisfies the estimate:* $||b||_{L^2_{k,\delta}} + |a_\infty| \leq C||\mu||_{L^2_{k-1,\delta}}$.

Proof: The Fredholm property and the index calculation of $*d$ follows from a similar argument as in Proposition 2.1.12. $Ker\ *d = 0$ follows from $H^1(Y, T^2) = 0$. Given $\mu \in Ker\ d^*$ or equivalently $*\mu \in Ker\ d$, since $H^2(Y, \mathbf{R}) = 0$, there exists an $A \in \Omega^1(Y)$ such that $dA + *\mu = 0$ or equivalently $*dA + \mu = 0$. If μ is compactly supported, the argument for Lemma 2.1.4 can be used to modify A with an exact 1-form and a "bounded" harmonic form, and the resulting 1-form a_μ has the claimed properties. □

The Proof of Proposition 2.2.2:

Since $Coker(\nabla\tilde{s}_{([(A,\psi)],\mu)}) = 0$ for any $([(A,\psi)],\mu) \in \tilde{s}^{-1}(0)$, $\tilde{s}^{-1}(0)$ is a Banach manifold. The projection $\Pi : \tilde{s}^{-1}(0) \to Ker\ d^*$ is a Fredholm map of index 1 (Proposition 2.1.12). So by Sard-Smale theorem, for a generic μ, $\mathcal{M}^*_\mu(r) = \Pi^{-1}(\mu)$ is a collection of 1-dimensional smooth curves. In addition, $Ker\ (V\nabla\tilde{s}) \bigcap Ker\ \Pi = 0$ since $V\nabla\tilde{s}$ is formally self-adjoint on $VT\mathcal{B}^*(r)$. So $d[R] : T\mathcal{M}^*_\mu(r) \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ is injective.

Since $Coker(V\nabla\tilde{s}_{([(A,\psi)],\mu)}) = 0$ for any $([(A,\psi)],\mu) \in \tilde{s}^{-1}(0)$, the map $[R]$: $\tilde{s}^{-1}(0) \to \mathcal{H}(r)/\mathbf{Z}$ is a submersion. For any set S of immersed curves in $\mathcal{H}(r)/\mathbf{Z}$, $[R]^{-1}(S)$ is a set of immersed submanifolds of co-dimension 1 in $\tilde{s}^{-1}(0)$. If μ is a regular value of the projection $\Pi : [R]^{-1}(S) \to Ker\ d^*$, then the map $[R] : \mathcal{M}^*_\mu(r) \to \mathcal{H}(r)/\mathbf{Z}$

is transversal to S.

Properties 3, 4 follow easily from Lemmas 2.2.4, 2.2.5 and 2.1.4.

2.3 The finite energy monopoles

Fix a perturbation (g, f, μ) which is supported in $Y \setminus [0, \infty) \times T^2$.

Definition 2.3.1 $(A, \psi) \in \Omega^1(Y) \otimes i\mathbf{R} \oplus \Gamma(W)$ *is said to be* **a monopole of finite energy** *if* (A, ψ) *satisfies*

- *the Seiberg-Witten equations*

$$\begin{cases} *dA + \tau(\psi, \psi) + \mu = 0 \\ D_g\psi + A\psi + f\psi = 0; \end{cases}$$

- *the finite energy condition*

$$\int_Y (|\nabla_A \psi|^2 + \frac{1}{2}|\psi|^4) < \infty.$$

The exponential decay estimates

Lemma 2.3.2 *(Lemma 4 in [17])*

Let X be a compact 3-manifold with boundary. Assume that $(A, \psi) \in \Omega^1(X) \otimes i\mathbf{R} \oplus \Gamma(W)$ satisfies the Seiberg-Witten equations on X. Then there exists a gauge transformation $s \in C^\infty(X, S^1)$ such that for any sub-domain X' with $\overline{X'} \subset intX$,

$s \cdot (A, \psi)$ satisfies:

$$\|s \cdot (\psi)\|_{C^k(X')} \leq C(k, X, X') h_1(\|\psi\|_{L^4(X)}),$$
$$\|s \cdot (A)\|_{C^k(X')} \leq C(k, X, X') h_2(\|\psi\|_{L^4(X)})$$

for a constant $C(k, X, X')$ and polynomials h_1, h_2 with $h_1(0) = 0$.

Corollary 2.3.3 *For a finite energy monopole (A, ψ),*

$$\|\psi\|_{C^0(T^2)}(t) \leq C\|\psi\|_{L^4([t-1,t+1]\times T^2)}.$$

In particular, $\psi \to 0$ as $t \to \infty$. Moreover, there exists a constant K depending only on the geometry of Y and the norm of (μ, f) such that $\|\psi\|_{C^0(Y)} < K$.

Proof: It follows from Lemma 2.3.2, the Weitzenböck formula and maximum principle. □

Throughout this section, we use $a(t)$ to denote the harmonic component of A_1 where $A = A_0 dt + A_1$ on $[0, \infty) \times T^2$. After a gauge transformation, any (A, ψ) takes the **standard form** on $[0, \infty) \times T^2$, i.e. $d^*_{T^2} A_1 = 0$ and $\int_{T^2} A_0 = 0$ (see Lemma 2.1.4).

Lemma 2.3.4 *Assume that the finite energy monopole (A, ψ) is in the standard form. Then the following holds for a constant c:*

a) $\int_{T^2}(|A_0|^2 + |d_{T^2} A_0|^2) \leq c \int_{T^2} |\psi|^4$;

b) $\int_{T^2}(|A_1 - a(t)|^2 + |\nabla^{T^2}(A_1 - a(t))|^2) \leq c \int_{T^2} |\psi|^4$;

c) $\int_{T^2} |\frac{\partial}{\partial t}(A_1 - a(t))|^2 \leq c \int_{T^2} |\psi|^4$;

d) $\int_{T^2} |\frac{\partial}{\partial t} a(t)|^2 \leq c \int_{T^2} |\psi|^4$;

e) $\|\frac{\partial}{\partial t} A_0\|_{L_1^2(T^2)} \leq c \|\frac{\partial \psi}{\partial t}\|_{L^2(T^2)}$.

Proof: On $[0,\infty) \times T^2$, the equation $*dA + \tau(\psi,\psi) = 0$ reads as

$$d_{T^2}(A_1 - a(t)) + q_1(\psi) = 0,$$
$$\frac{\partial}{\partial t}(A_1 - a(t)) + \frac{\partial}{\partial t} a(t) - d_{T^2} A_0 + q_2(\psi) = 0$$

for some quadratic forms q_1, q_2. Observe that $\frac{\partial}{\partial t}(A_1 - a(t))$, $\frac{\partial}{\partial t} a(t)$ and $d_{T^2} A_0$ are L^2 orthogonal to each other. The estimates $a), b), c), d)$ follow easily.

For $e)$, note that $d^*_{T^2} d_{T^2}(\frac{\partial}{\partial t} A_0) = d^*_{T^2}(\frac{\partial}{\partial t} q_2(\psi))$. So we have

$$\|\frac{\partial}{\partial t} A_0\|_{L_1^2(T^2)} \leq c \|d^*_{T^2} d_{T^2}(\frac{\partial}{\partial t} A_0)\|_{L^2_{-1}(T^2)} \leq c \|\frac{\partial}{\partial t} q_2(\psi))\|_{L^2(T^2)} \leq c \|\frac{\partial \psi}{\partial t}\|_{L^2(T^2)}$$

since $\int_{T^2} A_0 = 0$ and $\|\psi\|_{C^0(Y)} < K$. □

Lemma 2.3.5 *For any $r > 0$, there exists a $c(r) > 0$ such that $c(r) \int_{T^2} |\psi|^2 \leq \int_{T^2} |D_a^{T^2} \psi|^2$ for any a in the closure of $\mathcal{H}(r)$ (see Definition 2.1.11 for $\mathcal{H}(r)$).*

Proof: Observe that both $\int_{T^2} |\psi|^2$ and $\int_{T^2} |D_a^{T^2} \psi|^2$ are gauge invariant, so we can assume that a is in the compact set $\overline{\mathcal{H}(r)}/(\mathbf{Z} \oplus \mathbf{Z})$. The lemma follows by taking $c(r) = \min\{u^2 : u \text{ is an eigenvalue of } D_a^{T^2} \text{ for some } a \text{ in } \overline{\mathcal{H}(r)}/(\mathbf{Z} \oplus \mathbf{Z})\}$. □

The following estimate turns out to be crucial.

Lemma 2.3.6 *There exists a constant c_1 with the following significance. Let (A, ψ)*

be a finite energy monopole. For any $r > 0$, if $a(t)$ is in $\mathcal{H}(r)$ for $T_1 < t < T_2$, then

$$\int_{T_1}^{T_2}\int_{T^2}|\psi|^2 \leq \frac{c_1}{c(r)}\int_{T_1}^{T_2}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4).$$

Proof: Assume that (A,ψ) is in the standard form without loss of generality. Since $a(t)$ is in $\mathcal{H}(r)$ for $T_1 < t < T_2$, by Lemma 2.3.5, for $T_1 < t < T_2$, we have

$$\begin{aligned} c(r)\int_{T^2}|\psi|^2(t) &\leq \int_{T^2}|D^{T^2}_{a(t)}\psi|^2(t) \leq \int_{T^2}|\nabla^{T^2}_{a(t)}\psi|^2(t) \\ &\leq \int_{T^2}(|\nabla_A\psi|^2 + |(A_1 - a(t))\otimes\psi|^2)(t). \end{aligned}$$

But $\int_{T^2}|(A_1 - a(t))\otimes\psi|^2 \leq K^2\int_{T^2}|(A_1 - a(t))|^2 \leq C\int_{T^2}|\psi|^4$ by Corollary 2.3.3 and Lemma 2.3.4 b). So we have

$$\int_{T_1}^{T_2}\int_{T^2}|\psi|^2 \leq \frac{c_1}{c(r)}\int_{T_1}^{T_2}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4)$$

for a constant c_1. □

Lemma 2.3.7 *Let γ be a loop in T^2. Then there exists a constant $c(\gamma)$ such that for any (A,ψ) satisfying the Seiberg-Witten equations on the cylindrical end, the following estimate holds for any $t_1 < t_2$:*

$$\int_{t_1}^{t_2}\int_{\gamma}|\psi|^2 \leq c(\gamma)(\int_{t_1}^{t_2}\int_{T^2}|\psi|^2 + \int_{t_1-1}^{t_2+1}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4)).$$

Proof: Note that it suffices to prove the estimate for $t_2 = t_1 + 1$. Also note that both sides of the estimate are gauge invariant. By the embedding $L_1^2(T^2) \to L^2(\gamma)$, we have

$$\int_\gamma |\psi|^2 \le C\int_{T^2}(|\nabla^{T^2}\psi|^2+|\psi|^2) \le C\int_{T^2}(|\nabla_A\psi|^2+|A\otimes\psi|^2+|\psi|^2).$$

On the other hand, in $U=[t_1-1,t_1+2]\times T^2$, A can be decomposed into $A=B+h$ in a Hodge gauge (Lemma 4 in [17]) such that $\|B\|_{L^2_1(U)}\le C\|dA\|_{L^2(U)}\le C_1\|\psi\|^2_{L^4(U)}$ and h is harmonic with norm bounded by K. Hence

$$\int_{t_1}^{t_1+1}\int_{T^2}|A\otimes\psi|^2 \le K^2\int_{t_1}^{t_1+1}\int_{T^2}|\psi|^2+\|B\|^2_{L^2_1(U)}\cdot\|\psi\|^2_{L^4(U)}.$$

The lemma follows easily from these estimates. □

Lemma 2.3.8 *There exists a constant c_2 such that the following estimate*

$$\begin{aligned}|a(t_1)-a(t_2)| &\le c_2\Big(\int_{t_1}^{t_2}\int_{T^2}|\psi|^2+\int_{t_1-1}^{t_2+1}\int_{T^2}(|\nabla_A\psi|^2+\frac{1}{2}|\psi|^4)\\ &\quad+\Big(\int_{t_1-1}^{t_2+1}\int_{T^2}(|\nabla_A\psi|^2+\frac{1}{2}|\psi|^4)\Big)^{\frac{1}{2}}\Big)\end{aligned}$$

holds for any finite energy monopole (A,ψ).

Proof: Without loss of generality, we assume (A,ψ) is in the standard form. Then

$$\begin{aligned}|a(t_1)-a(t_2)| &\le \sum_{i=1}^{2}\Big|\int_{\gamma_i}a(t_1)-\int_{\gamma_i}a(t_2)\Big|\\ &\le \sum_{i=1}^{2}\Big(\Big|\int_{\gamma_i}A_1(t_1)-\int_{\gamma_i}A_1(t_2)\Big|\\ &\quad+\int_{\gamma_i}(|A_1(t_1)-a(t_1)|+|A_1(t_2)-a(t_2)|)\Big)\\ &\le \sum_{i=1}^{2}\Big(\int_{t_1}^{t_2}\int_{\gamma_i}|dA|+C(\|A_1(t_1)-a(t_1)\|_{L^2_1(T^2)}\end{aligned}$$

$$
\begin{aligned}
& +\|A_1(t_2) - a(t_2)\|_{L_1^2(T^2)})) \\
\leq\ & \sum_{i=1}^{2} C(\int_{t_1}^{t_2}\int_{\gamma_i} |\psi|^2 + (\int_{t_1\times T^2}|\psi|^4)^{\frac{1}{2}} + (\int_{t_2\times T^2}|\psi|^4)^{\frac{1}{2}} \\
\leq\ & c_2(\int_{t_1}^{t_2}\int_{T^2}|\psi|^2 + \int_{t_1-1}^{t_2+1}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4) \\
& +(\int_{t_1-1}^{t_2+1}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4))^{\frac{1}{2}})
\end{aligned}
$$

holds for any $t_1 < t_2$. Here γ_1, γ_2 are the longitude and meridian. □

Definition 2.3.9 *Choose an increasing function $\Gamma(r) > 0$ satisfying*

$$(\frac{c_1}{c(r)} + 1)\Gamma(r) + \Gamma(r)^{\frac{1}{2}} < c_2^{-1} r.$$

A finite energy monopole (A, ψ) is said to be **"r-good"** *if there are t_0 and T with $T \leq t_0$ such that $a(t_0) \in \overline{\mathcal{H}(2r)}$ and $\int_{T-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4) < \Gamma(r)$.*

The "r-good" monopoles of finite energy have the following good property.

Lemma 2.3.10 *Let (A, ψ) be an "r-good" monopole of finite energy with T as in the Definition 1.3.9. Then for all $t \in [T, \infty)$, $a(t)$ is in $\mathcal{H}(r)$. Moreover, $a_\infty = \lim_{t\to\infty} a(t)$ exists in $\mathcal{H}(r)$ and the following estimate holds for any $t \in [T, \infty)$:*

$$
\begin{aligned}
|a(t) - a_\infty| \ \leq\ & c_2((\frac{c_1}{c(r)} + 1)\int_{t-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4) \\
& +(\int_{t-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4))^{\frac{1}{2}}).
\end{aligned}
$$

Proof: It follows easily from the definition of "r-goodness" and Lemmas 2.3.6 and 2.3.8. □

Lemma 2.3.11 *For any $r > 0$, there exists a $\delta_0(r) > 0$ with the following significance. For any $\epsilon > 0$, there exists an $\epsilon_1 > 0$ such that for any "r-good" monopole (A, ψ), when $\int_{t_0-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2+\frac{1}{2}|\psi|^4) < \epsilon_1$ for $t_0 \in [1,\infty)$, we have $\int_{2t_0}^{\infty}\int_{T^2}|\psi|^2 e^{2\delta t} < \epsilon$ for any $\delta \leq \delta_0(r)$.*

Proof: Without loss of generality, assume that (A, ψ) is in the standard form. By Lemma 2.3.10, there is a number $T > 0$ such that for all $t \in [T, \infty)$, $a(t)$ is in $\mathcal{H}(r)$ and $a_\infty = \lim_{t\to\infty} a(t)$ exists in $\mathcal{H}(r)$. Moreover, the estimate

$$\begin{aligned} |a(t) - a_\infty| \;\; &\leq \;\; c_2\left(\left(\frac{c_1}{c(r)} + 1\right)\int_{t-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4)\right. \\ &\quad \left. + \left(\int_{t-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4)\right)^{\frac{1}{2}}\right). \end{aligned}$$

holds for $t \in [T, \infty)$. We can further apply a gauge transformation so that a_∞ lies in the compact set $\overline{\mathcal{H}(r)}/(\mathbf{Z}\oplus\mathbf{Z})$. There exists a $\delta_0(r) > 0$ such that for any $a \in \overline{\mathcal{H}(r)}/(\mathbf{Z}\oplus\mathbf{Z})$,

$$\|D_a^{T^2}\psi\|^2_{L^2(T^2)} \geq 4\delta_0(r)\|\psi\|^2_{L_1^2(T^2)}$$

for any $\psi \in \Gamma(W_0)$. Set $u(t) = \int_{t\times T^2}|\psi|^2$. Then we have $\frac{\partial^2}{\partial t^2}u(t) = 2\int_{t\times T^2}(\langle\frac{\partial^2}{\partial t^2}\psi, \psi\rangle + |\frac{\partial}{\partial t}\psi|^2)$. But

$$\begin{aligned} \int_{t\times T^2}\langle\frac{\partial^2}{\partial t^2}\psi, \psi\rangle \;\; &= \;\; \int_{t\times T^2}(|D_{A_1}^{T^2}\psi|^2 - |A_0\psi|^2 - \langle\frac{\partial A_1}{\partial t}\psi + \frac{\partial A_0}{\partial t}\psi, \psi\rangle \\ &\geq \;\; \int_{t\times T^2}|D_{a_\infty}^{T^2}\psi|^2 - \theta(t)\|\psi\|^2_{L_1^2(T^2)}(t) \end{aligned}$$

where $\theta(t) \to 0$ as $\int_{t-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2 + \frac{1}{2}|\psi|^4) \to 0$ by the estimates in Lemma 2.3.4, Corollary 2.3.3, and Lemma 2.3.10. So when $\int_{t_0-1}^{\infty}\int_{T^2}(|\nabla_A\psi|^2+\frac{1}{2}|\psi|^4)$ is small enough, we have

$$\frac{\partial^2}{\partial t^2}u(t) \geq 4\delta_0(r)u(t)$$

for $t \geq t_0$. By the maximum principle, we have $u(t) \leq e^{4\delta_0(r)(t_0-t)}u(t_0)$ for $t \geq t_0$. Hence

$$\int_{2t_0}^{\infty} u(t)e^{2\delta t}dt \leq C(\delta_0(r))u(t_0)$$

holds for any $\delta \leq \delta_0(r)$. The lemma follows easily. □

Proposition 2.3.12 *Assume that the "r-good" monopole of finite energy (A, ψ) is in the standard form. Then (A, ψ) is in $\mathcal{A}_\delta(r)$ for any weight $\delta \in (0, \min(\delta(r), \delta_0(r)))$. Moreover, the following estimate*

$$\|(A - a_\infty, \psi)\|_{L^2_{2,\delta}([T,\infty)\times T^2)} \leq c_3(\delta)(\int_{T-1}^{\infty}\int_{T^2} |\psi|^2 e^{2\delta t})^{\frac{1}{2}}$$

holds for a constant $c_3(\delta)$ and any $T \in [1, \infty)$. Here $\delta(r)$ is referred to Proposition 2.1.12.

Proof: It follows from Lemma 2.3.11, the estimates in Lemma 2.3.4, Taubes inequality and standard elliptic estimates. □

The convergence of "r-good" monopoles of finite energy

Proposition 2.3.13 *Let (A_n, ψ_n) be a sequence of "r-good" monopoles of finite energy. Then a subsequence of (A_n, ψ_n) converges in $\mathcal{M}_\delta(\frac{r}{2})$ to a "$\frac{1}{2}r$-good" monopole of finite energy (A_0, ψ_0) for any weight $\delta \in (0, \min(\delta(\frac{r}{2}), \delta_0(\frac{r}{2})))$.*

Proof: The Weitzenböck formula and maximum principle yield an upper bound K for the C^0 norm of the spinors (see Corollary 2.3.3). Then the existence of a local

Hodge gauge ([17]) plus elliptic regularity and a patching argument ([30]) imply the existence of a "weak" limit. More precisely, there exist a finite energy monopole (A_0, ψ_0), a subsequence of (A_n, ψ_n) (still denoted by (A_n, ψ_n)) and a sequence of gauge transformations s_n such that $s_n \cdot (A_n, \psi_n)$ converges to (A_0, ψ_0) in C^∞ over any compact subset of Y. We can further assume that $s_n \cdot (A_n, \psi_n)$ are in the standard form and therefore the limit (A_0, ψ_0) is also in the standard form. For simplicity we still use (A_n, ψ_n) to denote $s_n \cdot (A_n, \psi_n)$.

Take T_0 large enough so that $\int_{T_0-1}^{\infty} \int_{T^2} (|\nabla_{A_0}\psi_0|^2 + \frac{1}{2}|\psi_0|^4) < \Gamma(\frac{r}{2})$ (see Definition 2.3.9). Note that for any finite energy monopole (A, ψ) the Weitzenböck formula yields the following equation

$$\int_0^\infty \int_{T^2} (|\nabla_A \psi|^2 + \frac{1}{2}|\psi|^4) = \int_{0\times T^2} \langle D_{A_1}^{T^2}\psi, \psi\rangle.$$

It follows from the "weak" convergence of (A_n, ψ_n) to (A_0, ψ_0) that there is an N such that when $n > N$ we have

$$\int_{T_0-1}^\infty \int_{T^2} (|\nabla_{A_n}\psi_n|^2 + \frac{1}{2}|\psi_n|^4) < \Gamma(\frac{r}{2}) < \Gamma(r).$$

Since (A_n, ψ_n) are "r-good", by Lemma 2.3.10, $a_n(t)$ is in $\mathcal{H}(r)$ for any $n > N$ and $t \in [T_0, \infty)$. From this it follows that (A_0, ψ_0) is a "$\frac{1}{2}r$-good" monopole of finite energy, and therefore is in $\mathcal{A}_\delta(\frac{r}{2})$ for any weight $\delta \in (0, \min(\delta(\frac{r}{2}), \delta_0(\frac{r}{2}))$. It is also easy to see that $a_{n,\infty} \to a_{0,\infty}$.

In order to prove that (A_n, ψ_n) converges to (A_0, ψ_0) in $\mathcal{A}_\delta(\frac{r}{2})$ for any given weight $\delta \in (0, \min(\delta(\frac{r}{2}), \delta_0(\frac{r}{2}))$, it suffices to prove that given any $\epsilon > 0$, there exist $t_0 \in [1, \infty)$

and N such that when $n > N$,

$$\|(A_n - a_{n,\infty}, \psi)\|_{L^2_{2,\delta}([2t_0+1,\infty)\times T^2)} < \epsilon.$$

This goes as follows. By Lemma 2.3.11, there exists an $\epsilon_1 > 0$ such that when

$$\int_{t_0-1}^{\infty}\int_{T^2}(|\nabla_{A_n}\psi_n|^2 + \frac{1}{2}|\psi_n|^4) < \epsilon_1,$$

we have

$$\int_{2t_0}^{\infty}\int_{T^2}|\psi_n|^2 e^{2\delta t} < (c_3^{-1}(\delta)\epsilon)^2.$$

Now take t_0 large enough so that

$$\int_{t_0-1}^{\infty}\int_{T^2}(|\nabla_{A_0}\psi_0|^2 + \frac{1}{2}|\psi_0|^4) < \frac{\epsilon_1}{2}.$$

Then there exists an N such that when $n > N$ we have

$$\int_{t_0-1}^{\infty}\int_{T^2}(|\nabla_{A_n}\psi_n|^2 + \frac{1}{2}|\psi_n|^4) < \epsilon_1.$$

Therefore we have

$$\|(A_n - a_{n,\infty}, \psi)\|_{L^2_{2,\delta}([2t_0+1,\infty)\times T^2)} < \epsilon$$

by Proposition 2.3.12. Hence the proposition is proved. □

CHAPTER 3

The Gluing Formula

3.1 The gluing of moduli spaces

Assume that Y_i $(i = 1, 2)$ are oriented cylindrical end 3-manifolds over T_i^2 where Y_2 is actually diffeomorphic to $D^2 \times S^1$ carrying a metric whose scalar curvature is non-negative and somewhere positive. By the Weitzenböck formula, the moduli space $\mathcal{M}(Y_2)$ actually only consists of reducible solutions. Assume that there exists an orientation reversing isometry $h : T_1^2 \to T_2^2$ which is covered by the corresponding bundle maps. For any $L > 0$, we can form an oriented Riemannian 3-manifold Y_L as follows:

$$Y_L = Y_1 \setminus [L+1, \infty) \times T_1^2 \bigcup_h Y_2 \setminus [L+1, \infty) \times T_2^2,$$

where $h : (L, L+1) \times T_1^2 \to (L+1, L) \times T_2^2$ is given by $h(L+t, x) = (L+1-t, h(x))$ for $t \in (0,1)$ and $x \in T_1^2$. Note that the isometry $h : T_1^2 \to T_2^2$ induces an isometry $h : \mathcal{H}^1(T_1^2) \to \mathcal{H}^1(T_2^2)$.

Throughout this section, we fix a small $r > 0$ and a small weight δ. For simplicity, we omit the dependence of r and δ in the discussion. We also omit the perturbation data since it is vanishing on the neck.

Let $\tilde{\mathcal{G}}(Y_i)$ be the normal subgroup of $\mathcal{G}(Y_i)$ which consists of elements in the component of identity. We define $\tilde{\mathcal{M}}(Y_i)$ to be the space of $\tilde{\mathcal{G}}(Y_i)$-equivalence classes of the solutions to the Seiberg-Witten equations on Y_i ($i = 1, 2$). Then $\tilde{\mathcal{M}}(Y_i)$ is a $\mathbf{Z}$-fold cover of $\mathcal{M}(Y_i)$:

$$0 \to \mathbf{Z} \to \tilde{\mathcal{M}}(Y_i) \to \mathcal{M}(Y_i) \to 0.$$

The irreducible part of $\tilde{\mathcal{M}}(Y_1)$ is denoted by $\tilde{\mathcal{M}}^*(Y_1)$, which is a $\mathbf{Z}$-fold cover of $\mathcal{M}^*(Y_1)$.

Let $\mathcal{S}(Y_1, Y_2)$ be the set of pairs $(\alpha_1, \alpha_2) \in \tilde{\mathcal{M}}^*(Y_1) \times \tilde{\mathcal{M}}(Y_2)$ such that there are smooth representatives (A_1, ψ_1) and (A_2, ψ_2) satisfying $hR_1(A_1, \psi_1) = R_2(A_2, \psi_2)$, where $R_i : \mathcal{A}(Y_i) \to \mathcal{H}^1(T_i^2) \otimes i\mathbf{R}$.

Definition 3.1.1 *$(\alpha_1, \alpha_2) \in \mathcal{S}(Y_1, Y_2)$ is said to be regular if*

1. *the map $[R_1] : \tilde{\mathcal{M}}^*(Y_1) \to \mathcal{H}^1(T_1^2) \otimes i\mathbf{R}$ is injective at α_1.*

2. *$h[R_1](\tilde{\mathcal{M}}^*(Y_1))$ and $[R_2](\tilde{\mathcal{M}}(Y_2))$ intersect transversally at $[R_2](\alpha_2)$.*

Note that for a generic perturbation, $\mathcal{S}(Y_1, Y_2)$ consists of regular pairs (Proposition 2.2.2). We assume that $\mathcal{S}(Y_1, Y_2)$ is regular throughout this chapter.

For any $L > 0$, fix a cut-off function $\rho_L(t)$ which equals to one for $t < L$ and equals to zero for $t > L + 1$ with $|\rho'_L| < 2$. Given a regular pair (α_1, α_2) with smooth representatives (A_1, ψ_1) and (A_2, ψ_2), we construct an "almost" monopole (A_L, ψ_L)

on Y_L as follows:

$$\begin{aligned}
\psi_L &= \rho_L\psi_1 + (1-\rho_L)h^{-1}\psi_2 && \text{on } [L, L+1]\times T_1^2 \\
A_L &= \rho_L(A_1 - R_1(A_1)) + R_1(A_1) \\
&\quad +(1-\rho_L)h^{-1}(A_2 - R_2(A_2)) && \text{on } [L, L+1]\times T_1^2 \\
\psi_L &= \psi_i && \text{on } Y_i \setminus [L,\infty)\times T_i^2 \\
A_L &= A_i && \text{on } Y_i \setminus [L,\infty)\times T_i^2.
\end{aligned}$$

Note that ψ_L is compactly supported in $Y_1 \setminus [L+1,\infty)\times T^2$.

Proposition 3.1.2 *Assume that (α_1, α_2) is regular. Then for large L, the "almost" monopole (A_L, ψ_L) can be deformed to a non-degenerate monopole $T(A_L, \psi_L)$ satisfying*

$$\|T(A_L,\psi_L) - (A_L,\psi_L)\|_{L_1^2(Y_L)} \le CL^2 e^{-\delta L}.$$

Moreover, any monopole (A, ψ) on Y_L which is in the L_1^2-ball of radius $K_1 L^{-6}$ centered at (A_L, ψ_L) is gauge equivalent to $T(A_L, \psi_L)$. In particular, there is a well-defined gluing map $T : \mathcal{S}(Y_1, Y_2) \to \mathcal{M}^*(Y_L)$ by $T(\alpha_1, \alpha_2) = [T(A_L, \psi_L)]$.

The following estimate on (A_L, ψ_L) is straightforward.

Lemma 3.1.3 $\|(*dA_L + \tau(\psi_L,\psi_L), D_{A_L}\psi_L)\|_{L^2(Y_L)} \le Ce^{-\delta L}$.

Next we estimate the lowest eigenvalue of $\Delta_L = d^*d + |\psi_L|^2$. Set

$$\lambda_L = \inf_{f\neq 0} \frac{\int_{Y_L} |\Delta_L f|^2}{\int_{Y_L} |f|^2}.$$

Lemma 3.1.4 *Assume that one of ψ_1 or ψ_2 is not identically zero. For any function $\gamma(L) = o(L^{-4})$, there exists an $L_0 > 0$ such that when $L > L_0$, we have $\lambda_L > \gamma(L)$.*

The basic idea of the proof is the same as in Theorem 4 of Appendix B, but the argument is more difficult. We postpone the proof to the end of this section. From now on, we assume that one of ψ_1 or ψ_2 is not identically zero.

Corollary 3.1.5 *The norm of $\Delta_L^{-1} : L^2(Y_L) \to L_2^2(Y_L)$ is at most L^3 for large L.*

Proof: In Lemma 3.1.4, take $\gamma(L) = K^2 L^{-6}$ with K to be determined later. There exists a constant C independent of L such that for any $f \in L_2^2(Y_L)$, we have

$$
\begin{aligned}
\|f\|_{L_2^2(Y_L)} &\leq C(\|\Delta_L f\|_{L^2(Y_L)} + \|f\|_{L^2(Y_L)}) \\
&\leq C(\|\Delta_L f\|_{L^2(Y_L)} + K^{-1}L^3\|\Delta_L f\|_{L^2(Y_L)}) \\
&\leq L^3\|\Delta_L f\|_{L^2(Y_L)}
\end{aligned}
$$

for large L and a suitable choice of K. This proves the lemma. □

Let $T\mathcal{B}^*_{(A_L,\psi_L)}$ be the tangent space of $\mathcal{B}^*(Y_L)$ at (A_L, ψ_L), $\mathcal{L}_{(A_L,\psi_L)}$ be the L^2 completion of $T\mathcal{B}^*_{(A_L,\psi_L)}$. Then $T\mathcal{B}^*_{(A_L,\psi_L)} = \{(a, \phi) \in \mathcal{A}(Y_L) | -d^*a + i\langle i\psi_L, \phi\rangle_{Re} = 0\}$.

Lemma 3.1.6 *There exist constants K_1, K_2 with the following significance. When L is sufficiently large, for any $(A, \psi) \in \mathcal{A}(Y_L)$ satisfying*

$$
\|(A, \psi) - (A_L, \psi_L)\|_{L_1^2(Y_L)} \leq K_1 L^{-6},
$$

there exists a gauge transformation $s \in \mathcal{G}(Y_L)$ such that $s \cdot (A, \psi) - (A_L, \psi_L) \in$

$T\mathcal{B}^*_{(A_L,\psi_L)}$ *and*

$$\|s\cdot(A,\psi)-(A_L,\psi_L)\|_{L^2_1(Y_L)}\le K_2L^3\|(A,\psi)-(A_L,\psi_L)\|_{L^2_1(Y_L)}.$$

Proof: The point of this lemma is to have an estimate on the size of the local slice at (A_L,ψ_L) if an upper bound for the norm of Δ_L^{-1} is known (Corollary 3.1.5).

Assume that (A,ψ) is in $\mathcal{A}(Y_L)$. Set $(a,\phi)=(A,\psi)-(A_L,\psi_L)$ for simplicity. To construct the local slice, we look for $f\in L^2_2$ such that

$$-d^*(A-A_L-df)+i\langle i\psi_L,e^f\psi-\psi_L\rangle_{Re}=0.$$

This can be written in terms of (a,ϕ) as

$$(\Delta_L+\langle i\psi_L,i\phi\rangle_{Re})f+G(\phi,f)=d^*a-i\langle i\psi_L,\phi\rangle_{Re},$$

where $G(\phi,f)=i\langle i\psi_L,(e^f-f-1)(\phi+\psi_L)\rangle_{Re}$ satisfying

$$\|G(\phi,f_1)-G(\phi,f_2)\|_{L^2}\le C\max(\|f_1\|_{L^2_2},\|f_2\|_{L^2_2})\|f_1-f_2\|_{L^2_2}$$

for some constant C and ϕ,f_i satisfying $\|\phi\|_{L^2_1}<1$ and $\|f_i\|_{L^2_2}<1$ for $i=1,2$. The lemma follows by applying Banach lemma to the map

$$B(f)=(\Delta_L+\langle i\psi_L,i\phi\rangle_{Re})^{-1}(d^*a-i\langle i\psi_L,\phi\rangle_{Re}-G(\phi,f))$$

mapping an L^2_2-ball of radius KL^{-3} into itself for some constant K. □

Next we deform the "almost" monopole (A_L, ψ_L) to a monopole. Let Π be the L^2 orthogonal projection onto $\mathcal{L}_{(A_L,\psi_L)}$. For any $(a, \phi) \in T\mathcal{B}^*_{(A_L,\psi_L)}$, we define

$$\begin{aligned} L(a,\phi) &= \Pi(*d(A_L + a) + \tau(\psi_L + \phi), D_{(A_L+a)}(\psi_L + \phi)) \\ &= (*dA_L + \tau(\psi_L), D_{A_L}\psi_L) + \nabla s_{(A_L,\psi_L)}(a,\phi) + \Pi Q(a,\phi). \end{aligned}$$

Here $\nabla s_{(A_L,\psi_L)} : T\mathcal{B}^*_{(A_L,\psi_L)} \to \mathcal{L}_{(A_L,\psi_L)}$ is given by

$$\nabla s_{(A_L,\psi_L)}(a,\phi) = (*da + 2\tau(\psi_L,\phi) - df(\phi), D_{A_L}\phi + a\psi_L + f(\phi)\psi_L)$$

with $f(\phi)$ given by the equation

$$\Delta_L f = i\langle D_{A_L}\psi_L, i\phi\rangle_{Re}.$$

$Q(a,\phi) = (\tau(\phi), a\phi)$ satisfies

$$\|Q(a_1,\phi_1) - Q(a_2,\phi_2)\|_{L^2(Y_L)} \le C(\|(a_1,\phi_1)\|_{L^2_1} + \|(a_2,\phi_2)\|_{L^2_1})\|(a_1,\phi_1) - (a_2,\phi_2)\|_{L^2_1}.$$

Lemma 3.1.7 *There exists a constant K_3 such that when $\|(a,\phi)\|_{L^2_1(Y_L)} \le K_3 L^{-3}$ for large enough L, $L(a,\phi) = 0$ implies that*

$$(*d(A_L + a) + \tau(\psi_L + \phi), D_{(A_L+a)}(\psi_L + \phi)) = 0.$$

Proof:

$$\begin{aligned} L(a,\phi) &= \Pi(*d(A_L+a)+\tau(\psi_L+\phi), D_{(A_L+a)}(\psi_L+\phi)) \\ &= (*d(A_L+a)+\tau(\psi_L+\phi)-dg(a,\phi), D_{(A_L+a)}(\psi_L+\phi)+g(a,\phi)\psi_L) \end{aligned}$$

where $g(a,\phi)$ satisfies the equation

$$\Delta_L g = i\langle D_{(A_L+a)}(\psi_L+\phi), i\phi\rangle_{Re}.$$

If $L(a,\phi)=0$, then $D_{(A_L+a)}(\psi_L+\phi)+g(a,\phi)\psi_L=0$. So for large L, we have

$$\begin{aligned} \|g(a,\phi)\|_{L^2_2(Y_L)} &\leq L^3\|\Delta_L g(a,\phi)\|_{L^2(Y_L)} \\ &\leq L^3\|\langle g(a,\phi)\psi_L, i\phi\rangle_{Re}\|_{L^2(Y_L)} \\ &\leq CL^3\|\psi_L\|_{C^0}\|\phi\|_{L^2_1(Y_L)}\|g(a,\phi)\|_{L^2_2(Y_L)}. \end{aligned}$$

If $\|(a,\phi)\|_{L^2_1(Y_L)} \leq K_3L^{-3}$ for a small enough constant K_3, we have $g(a,\phi)=0$, which proves the lemma. □

Lemma 3.1.8 *Assume that* (α_1,α_2) *is regular. Then* $\nabla s_{(A_L,\psi_L)} : T\mathcal{B}^*_{(A_L,\psi_L)} \to \mathcal{L}_{(A_L,\psi_L)}$ *is invertible for large L. Moreover, the norm of* $(\nabla s_{(A_L,\psi_L)})^{-1}$ *is at most* L^2 *for large L.*

Proof: First of all, $\nabla s_{(A_L,\psi_L)} : T\mathcal{B}^*_{(A_L,\psi_L)} \to \mathcal{L}_{(A_L,\psi_L)}$ can be extended to an operator

$$\mathcal{K}'_{(A_L,\psi_L)} = \begin{pmatrix} \nabla s_{(A_L,\psi_L)} & 0 & 0 \\ 0 & 0 & d_{(A_L,\psi_L)} \\ 0 & d^*_{(A_L,\psi_L)} & \end{pmatrix},$$

where $d_{(A_L,\psi_L)}(f) = (-df, f\psi_L)$. Secondly, by Theorem 4 in Appendix B, the lowest eigenvalue of

$$\mathcal{K}_{(A_L,\psi_L)} = \begin{pmatrix} D_{A_L} & \psi_L\cdot & \psi_L\cdot \\ 2\tau(\psi_L,\cdot) & *d & -d \\ i\langle i\psi_L,\cdot\rangle_{Re} & -d^* & 0 \end{pmatrix}$$

is at least KL^{-2} for any constant K when L is sufficiently large. Here we essentially use the fact that ψ_L is identically zero on the Y_2 side and not identically zero on the Y_1 side and the assumption that (α_1, α_2) is regular so that the regularity and the transversality conditions in Theorem 4 of Appendix B hold. Finally, the difference between $\mathcal{K}'_{(A_L,\psi_L)}$ and $\mathcal{K}_{(A_L,\psi_L)}$ can be ignored since the norm of $\mathcal{K}'_{(A_L,\psi_L)} - \mathcal{K}_{(A_L,\psi_L)}$ is bounded from above by $CL^6\|D_{A_L}\psi_L\|_{C^0} \leq cL^6 e^{-\delta L}$ by Lemma 3.1.3 and Corollary 3.1.5. So the lowest eigenvalue of $\nabla s_{(A_L,\psi_L)}$ is bounded from below by KL^{-2} for any constant K when L is large enough. The lemma follows easily. □

The Proof of Proposition 3.1.2:

In order to deform the "almost" monopole (A_L, ψ_L) to a monopole, we need to solve the equation $L(a,\phi) = 0$ for $(a,\phi) \in T\mathcal{B}^*_{(A_L,\psi_L)}$. This equation can be written as

$$(a,\phi) = -(\nabla s_{(A_L,\psi_L)})^{-1}((*dA_L + \tau(\psi_L), D_{A_L}\psi_L) + \Pi Q(a,\phi)).$$

Assuming that (α_1, α_2) is regular, it then follows from Lemmas 3.1.3 and 3.1.8 that the map

$$B(a,\phi) = -(\nabla s_{(A_L,\psi_L)})^{-1}((*dA_L + \tau(\psi_L), D_{A_L}\psi_L) + \Pi Q(a,\phi))$$

maps an L_1^2-ball of radius KL^{-2} in $T\mathcal{B}^*_{(A_L,\psi_L)}$ into itself and satisfies

$$||B(a_1,\phi_1) - B(a_2,\phi_2)||_{L_1^2(Y_L)} < ||(a_1,\phi_1) - (a_2,\phi_2)||_{L_1^2(Y_L)}$$

for large enough L and some small enough constant K. Therefore the equation $L(a,\phi) = 0$ has a unique solution (a_L, ϕ_L) in the L_1^2-ball of radius KL^{-2}. By Lemma 3.1.7, the resulting monopole is $T(A_L, \psi_L) = (A_L, \psi_L) + (a_L, \phi_L)$, and

$$||T(A_L,\psi_L) - (A_L,\psi_L)||_{L_1^2(Y_L)} \leq CL^2 e^{-\delta L}.$$

Suppose that monopole (A,ψ) is in an L_1^2-ball of radius $K_1 L^{-6}$ centered at (A_L, ψ_L). By Lemma 3.1.6, (A,ψ) is gauge equivalent to a monopole in the local slice with distance from (A_L, ψ_L) less than $K_2 K_1 L^{-3}$, and it must be $T(A_L, \psi_L)$ by the uniqueness of the solution (a_L, ϕ_L) to the equation $L(a,\phi) = 0$. In particular, $[T(A_L, \psi_L)]$ depends only on the homotopy class of (A_L, ψ_L), which implies that there is a well-defined gluing map $T : \mathcal{S}(Y_1, Y_2) \to \mathcal{M}^*(Y_L)$. By Lemma 3.1.8 and the estimate

$$||T(A_L,\psi_L) - (A_L,\psi_L)||_{L_1^2(Y_L)} \leq CL^2 e^{-\delta L},$$

$[T(A_L, \psi_L)]$ is non-degenerate. Therefore the proposition is proved. □

The Proof of Lemma 3.1.4:

Suppose that there is a sequence of $L_n \to \infty$ such that

$$\lambda_{L_n} \leq \gamma(L_n) = o(L_n^{-4}),$$

then there exists a sequence of $c_n > 0$, $f_n \neq 0$ such that $\Delta_{L_n} f_n = c_n^2 f_n$ and $c_n = o(L_n^{-1})$.

Claim: *There exist constants M and L_0 with the following significance. The f_n's can be chosen such that $||f_n||_{C^0(Y_{L_n})} \leq M$, and one of the following is true:*

1. *either $\int_{Y_1(L_0)} |f_n|^2$ or $\int_{Y_2(L_0)} |f_n|^2$ is equal to one;*

2. *either $||f_n||_{L^2(T_1^2)}(L_0)$ or $||f_n||_{L^2(T_2^2)}(L_0)$ is greater than or equal to one.*

Here $Y_i(L_0) = Y_i \setminus (L_0, \infty) \times T_i^2$, $i = 1, 2$.

Assuming the **Claim**, Lemma 3.1.4 is proved as follows. By elliptic estimates, we can select a subsequence of f_n which is convergent in C^∞ to f_i on Y_i $(i = 1, 2)$ on any compact subset. Moreover, f_1, f_2 satisfy the following conditions:

a) $d^* df_i + |\psi_i|^2 f_i = 0$ on Y_i, $i = 1, 2$;

b) $||f_i||_{C^0(Y_i)} \leq M$, $i = 1, 2$;

c) one of f_1 or f_2 is not identically zero.

Lemma 3.1.4 is proved if we show that conditions $a)$ and $b)$ contradict condition $c)$. In fact, by condition $a)$, for any t, we have

$$0 \;=\; \int_{Y_i(t)} \langle d^* df_i + |\psi_i|^2 f_i, f_i \rangle$$

$$= \int_{Y_i(t)} (|df_i|^2 + |\psi_i|^2|f_i|^2) - \frac{1}{2}\frac{\partial}{\partial t}(\int_{T_i^2} |f_i|^2)(t).$$

Since $\xi_i(t) = \int_{T_i^2} |f_i|^2(t)$ is bounded by b), there exists a sequence of t_n such that $\frac{\partial \xi_i}{\partial t}(t_n) \to 0$ as $t_n \to \infty$. Therefore $\int_{Y_i}(|df_i|^2 + |\psi_i|^2|f_i|^2) = 0$, $i = 1, 2$. So f_1 and f_2 are constant functions and one of them is zero, since one of ψ_1 or ψ_2 is not identically zero. But in the proof of the **Claim**, it is easy to see that $f_1 = f_2$. So both of f_1 and f_2 are zero, contradicting c).

The Proof of the Claim:

For simplicity, we omit the subscript L_n or n if no confusion is caused. Write $f = g_1 + g_2$ where $g_1 \in \mathcal{K}er\ d^*_{T^2}d_{T^2}$ and $g_2 \in (\mathcal{K}er\ d^*_{T^2}d_{T^2})^\perp$.

Pick $L_0 > 0$ large enough, there are two possibilities:

- On $[L_0, 2L+1-L_0]$, $\max|g_1| \leq \max\|g_2\|_{L^2(T_1^2)}$. In this case, by the maximum principle, for large L_0, $\|g_2\|_{L^2(T_1^2)}$ can not reach its maximum in the interior of $[L_0, 2L+1-L_0]$. The **Claim** follows easily in this case.

- On $[L_0, 2L+1-L_0]$, $\max\|g_2\|_{L^2(T_1^2)} \leq \max|g_1|$. In this case, we need to show that for large enough L either $|g_1|$ reaches its maximum at the endpoints of $[L_0, 2L+1-L_0]$, from which the **Claim** follows, or either $\max|g_1| \leq K|g_1(L_0)|$ or $\max|g_1| \leq K|g_1(2L+1-L_0)|$ holds for a constant K independent of L.

Assume that we are in the second case. On $[L_0, 2L+1-L_0]$, we have

$$-\frac{\partial^2}{\partial t^2} g_1(t) + h(t) = c^2 g_1(t)$$

where $h(t)$ is the L^2-projection of $|\psi_L|^2 f$ into $\mathcal{K}er\ d^*_{T^2}d_{T^2}$. $c = o(L^{-1})$ and $h(t)$

satisfies

$$|h(t)| \leq Ke^{-2\delta t} \max|g_1| \ \ on \ \ [L_0, L+1)$$

and

$$|h(t)| \leq Ke^{-2\delta(2L+1-t)} \max|g_1| \ \ on \ \ (L, 2L+1-L_0].$$

Set $g_3(t) = c^{-1}\frac{\partial}{\partial t}g_1(t)$, then we have

$$\frac{\partial}{\partial t}g_1(t) = cg_3(t)$$
$$\frac{\partial}{\partial t}g_3(t) = -cg_1(t) + c^{-1}h(t).$$

These equations can be written equivalently as

$$\frac{\partial}{\partial t}\left[e^{Ct}\begin{pmatrix} g_1(t) \\ g_3(t) \end{pmatrix}\right] = e^{Ct}\begin{pmatrix} 0 \\ c^{-1}h(t) \end{pmatrix}$$

where $C = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$. Note that $e^{Ct} = \begin{pmatrix} \cos ct & -\sin ct \\ \sin ct & \cos ct \end{pmatrix}$.

Since $c = o(L^{-1})$, we have $\cos ct > \frac{1}{2}$ and

$$|\int_{L_0}^{t} \sin cs \cdot c^{-1}h(s)ds| \leq K\int_{L_0}^{t} |sh(s)|ds \leq Ke^{-\delta L_0} \max|g_1|$$

for large L and any $t \in [L_0, L+1)$. On the other hand,

$$g_1(t)\cos ct - g_3(t)\sin ct = -\int_{L_0}^{t} \sin cs \cdot c^{-1}h(s)ds + g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0.$$

So if $|g_1(t)|$ reaches its maximum in the interior of $[L_0, 2L+1-L_0]$, without loss of generality, assuming that it is in $(L_0, L+1)$, then

$$\frac{1}{2}\max|g_1| \leq Ke^{-\delta L_0}\max|g_1| + |g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0|.$$

Hence for large L_0, we have

$$\max|g_1| \leq 4|g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0|.$$

On the other hand,

$$g_3(t) = \int_{L_0}^{t} \cos c(s-t)\cdot c^{-1}h(s)ds + g_3(L_0)\cos c(L_0-t) + g_1(L_0)\sin c(L_0-t).$$

Assume that $|g_1(t)|$ has its maximum at $t_0 \in (L_0, L+1)$. Then $g_3(t_0) = 0$ and

$$\begin{aligned} & |g_1(L_0)\sin c(L_0-t_0) + g_3(L_0)\cos c(L_0-t_0)| \\ \leq\; & \int_{L_0}^{t_0} |c^{-1}h(s)|ds \\ \leq\; & Kc^{-1}e^{-2\delta L_0}\max|g_1| \\ \leq\; & 4Kc^{-1}e^{-2\delta L_0}|g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0|. \end{aligned}$$

Since $c = o(L^{-1})$ as $L \to \infty$ and $L_0 e^{-2\delta L_0} = o(1)$ as $L_0 \to \infty$, we have

$$|g_3(L_0)| \leq 10Kc^{-1}e^{-2\delta L_0}|g_1(L_0)| + |g_1(L_0)|.$$

Hence

$$\begin{aligned}\max|g_1| &\le 4|g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0| \\ &\le 4|g_1(L_0)| + (10Kc^{-1}e^{-2\delta L_0}|g_1(L_0)| + |g_1(L_0)|)|\sin cL_0| \\ &\le 5|g_1(L_0)| + 10KL_0e^{-2\delta L_0}|g_1(L_0)| \\ &\le K_1|g_1(L_0)|\end{aligned}$$

with K_1 independent of L when L_0 is sufficiently large. This proves the **Claim**. (Observe that $|(g_1(t_0)\cos ct_0 - g_3(t_0)\sin ct_0) - (g_1(L_0)\cos cL_0 - g_3(L_0)\sin cL_0)| \le Ke^{-\delta L_0}$ and $|(g_1(t_0)\cos ct_0 - g_3(t_0)\sin ct_0) - (g_1(2L+1-L_0)\cos cL_0 - g_3(2L+1-L_0)\sin cL_0)| \le Ke^{-\delta L_0}$ where $t_0 = L + \frac{1}{2}$, from which one sees $f_1 = f_2$).

3.2 Geometric limits

Let Y be an oriented integral homology 3-sphere decomposed into $Y = Y_1 \bigcup_{T^2} Y_2$, where Y_1 is the complement of a tubular neighborhood of a knot and Y_2 is diffeomorphic to $D^2 \times S^1$. Equip Y with a Riemannian metric such that a collar neighborhood of T^2 is orientedly isometric to $(-1,1)\times\mathbf{R}/2\pi\mathbf{Z}\times\mathbf{R}/2\pi\mathbf{Z}$ with $(-1,0)\times\mathbf{R}/2\pi\mathbf{Z}\times\mathbf{R}/2\pi\mathbf{Z}$ in Y_1 and Y_2 carries a non-negative, somewhere positive scalar curvature metric. We insert cylinders of lengths $2L+1$ and obtain a family of stretched versions Y_L of Y. We also use Y_1 and Y_2 to denote the corresponding cylindrical end manifolds. Note that the finite energy monopoles on Y_2 are reducible by Weitzenböck formula, so the moduli space $\tilde{\mathcal{M}}(Y_2)$ is identified with the line $\mathcal{H}^1(Y_2)\otimes i\mathbf{R}$ of imaginary valued "bounded" harmonic 1-forms on Y_2 which is embedded into $\mathcal{H}^1(T^2)\otimes i\mathbf{R}$ by the map

R_2 (Lemma 2.1.4). With a small perturbation, we assume that $\tilde{\mathcal{M}}(Y_2)$ misses the lattice of "bad" points for the spin structure on T^2 where the twisted Dirac operators are not invertible.

Let (A_n, ψ_n) be a sequence of monopoles on Y_{L_n}, $L_n \to \infty$. Weitzenböck formula and the maximum principle yield an upper bound K for the C^0 norm of the spinors, which depends only on the scalar curvature of the manifolds. Then the existence of a local Hodge gauge ([17]) plus elliptic regularity and a patching argument ([30]) imply the existence of geometric limits as we stretch the neck.

Lemma 3.2.1 *There exists an $r > 0$ with the following significance. Let (A_n, ψ_n) be a sequence of monopoles on Y_{L_n}, $L_n \to \infty$. Then there exist a sequence of gauge transformations s_n and a pair of finite energy monopoles (A_i, ψ_i) on Y_i ($i = 1, 2$) such that a subsequence of $s_n \cdot (A_n, \psi_n)$ converges in C^∞ to (A_i, ψ_i) on any compact subset of Y_i ($i = 1, 2$). Moreover, (A_1, ψ_1) and (A_2, ψ_2) are "r-good" monopoles and have the same limiting value, i.e. $R_1(A_1, \psi_1) = R_2(A_2, \psi_2)$.*

Proof: Exhaust Y_i by a sequence of compact subsets $K_{i,n}$ such that $K_{i,n} \subset K_{i,n+1}$, $i = 1, 2$. There exist a subsequence of (A_n, ψ_n) (still labeled by n), a sequence of gauge transformations $s_{i,n}$ defined on $K_{i,n}$, and monopoles (A_i, ψ_i) on Y_i such that $s_{i,n} \cdot (A_n, \psi_n)$ converges to (A_i, ψ_i) in C^∞ on any compact subset of Y_i. Note that (A_i, ψ_i) are of finite energy by Weitzenböck formula.

First we show that there is an $r > 0$ such that the geometric limits (A_i, ψ_i) ($i = 1, 2$) are "r-good". Let d be the distance between the lattice of "bad" points and the line $R_2(\mathcal{M}(Y_2))$ in $\mathcal{H}^1(T^2) \otimes i\mathbf{R}$. We simply take $r = \frac{d}{100}$. Note that $\psi_2 \equiv 0$ and $a_2(t) = a_{2,\infty}$ for all $t \in [0, \infty)$, so (A_2, ψ_2) is "r-good". On the other hand, there are

t_0 and N such that $|(D^{T^2}_{A_{1,1}}\psi_1, \psi_1)(t_0)| < \frac{1}{2}\Gamma(r)$ and $|(D^{T^2}_{A_{n,1}}\psi_n, \psi_n)(t_0)| < \Gamma(r)$ when $n > N$ (see Definition 2.3.9 for $\Gamma(r)$). For large n, $a_{s_{2,n}\cdot(A_n)}(2L_n)$ is in $\mathcal{H}(4r)$, so is $a_n(2L_n)$. By Weitzenböck formula, $\int_{Y_2(t_0)}(|\nabla_{A_n}\psi_n|^2 + \frac{1}{2}|\psi_n|^4) < |(D^{T^2}_{A_{n,1}}\psi_n, \psi_n)(t_0)| < \Gamma(r)$ when $n > N$, where $Y_2(t_0) = Y_2 \setminus (2L_n + 1 - t_0, \infty) \times T^2$. So for large n, $a_n(t_0+1)$ is in $\mathcal{H}(3r)$ (Lemma 2.3.10) and so is $a_{s_{1,n}\cdot(A_n)}(t_0 + 1)$. So $a_1(t_0 + 1)$ is in $\mathcal{H}(2r)$. It follows easily that (A_1, ψ_1) is "r-good".

Next we show that

1. The $s'_{i,n}s$ can be chosen so that as $n \to \infty$, $s^{-1}_{1,n}|_{T^2} \cdot s_{2,n}|_{T^2}$ is in the component of identity of $Map(T^2, S^1)$. As a consequence, $s_{1,n}$ and $s_{2,n}$ can be extended to an $s_n \in \mathcal{G}(Y_{L_n})$.

2. $R_1(A_1, \psi_1) = R_2(A_2, \psi_2)$.

Given any $\epsilon > 0$, pick L_0 large enough so that

$$\|(A_i, \psi_i)(L_0) - R_i(A_i, \psi_i)\|_{C^k(T^2)} < \epsilon, \;\; i = 1, 2.$$

For large enough n, we have

$$\|s_{1,n} \cdot (A_n, \psi_n)(L_0) - (A_1, \psi_1)(L_0)\|_{C^k(T^2)} < \epsilon$$

and

$$\|s_{2,n} \cdot (A_n, \psi_n)(2L_n + 1 - L_0) - (A_2, \psi_2)(L_0)\|_{C^k(T^2)} < \epsilon.$$

Let γ be a generator of $H_1(T^2)$. Then we have

$$
\begin{aligned}
& |\int_\gamma (R_1(A_1,\psi_1) - R_2(A_2,\psi_2))| \\
\leq\ & 3C\epsilon + |\int_\gamma (s_{1,n}\cdot A_n(L_0) - s_{2,n}\cdot A_n(2L_n+1-L_0))| \\
\leq\ & 3C\epsilon + |\int_{L_0}^{2L_n+1-L_0}\int_\gamma dA_n + 2\pi i[s_{1,n}^{-1}|_{T^2}\cdot s_{2,n}|_{T^2}]|,
\end{aligned}
$$

where $[s_{1,n}^{-1}|_{T^2}\cdot s_{2,n}|_{T^2}]$ is the degree of the map $s_{1,n}^{-1}|_{T^2}\cdot s_{2,n}|_{T^2} : \gamma \to S^1$. On the other hand, we have estimates

$$
\begin{aligned}
|\int_{L_0}^{2L_n+1-L_0}\int_\gamma dA_n| \ &\leq\ C\int_{L_0}^{2L_n+1-L_0}\int_\gamma |\psi_n|^2 \\
&\leq\ C_1\int_{L_0-1}^{2L_n+2-L_0}\int_{T^2}(|\nabla_{A_n}\psi_n|^2 + \frac{1}{2}|\psi_n|^4) \\
&\leq\ C_1(|(D_{A_{n,1}}^{T^2}\psi_n,\psi_n)(L_0-1)| \\
&\quad +|(D_{A_{n,1}}^{T^2}\psi_n,\psi_n)(2L_n+2-L_0)|
\end{aligned}
$$

by Lemmas 2.3.6, 2.3.7 and Weitzenböck formula, from which it follows that $R_1(A_1,\psi_1) = R_2(A_2,\psi_2)$ in $\mathcal{H}^1(T^2)\otimes i\mathbf{R}/(\mathbf{Z}\oplus\mathbf{Z})$. It follows that (A_i,ψ_i) can be modified by an element in $\mathcal{G}(Y_i)$ so that $R_1(A_1,\psi_1) = R_2(A_2,\psi_2)$ in $\mathcal{H}^1(T^2)\otimes i\mathbf{R}$ (due to the fact that Y is a homology 3-sphere) and $s_{1,n}^{-1}|_{T^2}\cdot s_{2,n}|_{T^2}$ is in the identity component of $Map(T^2,S^1)$ for large n. □

In the following discussion, we fix the number r in Lemma 3.2.1, a weight δ small and a generic perturbation (g,f,μ) with μ sufficiently small.

Recall the set $\mathcal{S}(Y_1,Y_2)$ of pairs $(\alpha_1,\alpha_2)\in\tilde{\mathcal{M}}^*(Y_1)\times\tilde{\mathcal{M}}(Y_2)$ such that there are smooth representatives (A_1,ψ_1) and (A_2,ψ_2) satisfying $R_1(A_1,\psi_1) = R_2(A_2,\psi_2)$. By

Proposition 3.1.2 and Lemma 3.2.1, each pair (α_1, α_2) in $\mathcal{S}(Y_1, Y_2)$ is "r-good". Therefore by Proposition 2.3.13 (convergence of "r-good" monopoles) and Proposition 2.2.2 (3), $\mathcal{S}(Y_1, Y_2)$ is compact and hence consists of finitely many points.

Proposition 3.2.2 *For large enough L, the gluing map $T : \mathcal{S}(Y_1, Y_2) \to \mathcal{M}^*(Y_L)$ given by $T(\alpha_1, \alpha_2) = [T(A_L, \psi_L)]$ is one to one and onto.*

Assume that a sequence of irreducible monopoles (A_n, ψ_n) on Y_{L_n} converges to geometric limits (A_i, ψ_i) on Y_i $(i = 1, 2)$. Note that $\psi_2 \equiv 0$ and (A_1, ψ_1) is irreducible since the (perturbed) Dirac operator at the reducible point on Y_{L_n} is invertible for large n and the norm is uniformly bounded from below (Theorem B in [3]). Our next goal is to show that for large enough n, (A_n, ψ_n) is in the image of the gluing map T. This is done by showing that up to a gauge transformation the L_1^2 distance between (A_n, ψ_n) and the "almost" monopole (A_{L_n}, ψ_{L_n}) is less than $K_1 L_n^{-6}$ (see Proposition 3.1.2).

For simplicity we omit the subscript n in the notation if no confusion is caused. As in Lemma 2.3.11, there exists an $L_0 > 0$ such that for any $t \in [L_0, 2L + 1 - L_0]$, we have

$$\int_{T^2} |\psi|^2(t) \le e^{4\delta(L_0 - t)} \int_{T^2} |\psi|^2(L_0) + e^{4\delta(t - 2L - 1 + L_0)} \int_{T^2} |\psi|^2(2L + 1 - L_0).$$

For each $L > 0$, fix a cut-off function ρ_L which equals to one for $t \le L$ and equals to zero for $t \ge L + 1$. We construct an "almost" monopole $(\tilde{A}_1, \tilde{\psi}_1)$ on Y_1 as follows:

$$\tilde{A}_1 \quad = \quad \rho_L(A - a(L+1)) + a(L+1) \quad \text{on } Y_1 \setminus [L+1, \infty) \times T^2$$

$$\begin{aligned} \tilde{A}_1 &= a(L+1) && \text{on } [L+1,\infty)\times T^2 \\ \tilde{\psi}_1 &= \rho_L\psi && \text{on } Y_1\setminus[L+1,\infty)\times T^2 \\ \tilde{\psi}_1 &= 0 && \text{on } [L+1,\infty)\times T^2. \end{aligned}$$

(Note that we have omitted the subscript n in the notation; here $A = A_n$ and $L = L_n$). Here $a(t)$ is the harmonic component of $A|_{\{t\}\times T^2}$.

The following estimate is straightforward.

Lemma 3.2.3 $\|(*d\tilde{A}_1 + \tau(\tilde{\psi}_1), D_{\tilde{A}_1}\tilde{\psi}_1)\|_{L^2_{1,\delta}(Y_1)} \leq Ce^{-\delta L}$ *holds for* $(\tilde{A}_1, \tilde{\psi}_1)$ *on* Y_1.

Recall from Definition 2.1.10 that for $(A,\psi) \in \mathcal{A}^*$, $\nabla s_{(A,\psi)} : T\mathcal{B}^*_{(A,\psi)} \to \mathcal{L}_{(A,\psi)}$ is given by

$$\nabla s_{(A,\psi)}(a,\phi) = (*da + 2\tau(\psi,\phi) - df(a,\phi), D_A\phi + a\psi + f(a,\phi)\psi)$$

where $f(a,\phi)$ is the unique solution to the equation

$$d^*df + f|\psi|^2 = i\langle D_A\psi, i\phi\rangle_{Re}.$$

Lemma 3.2.4 *For all sufficiently large* L, $\nabla s_{(\tilde{A}_1,\tilde{\psi}_1)} : T\mathcal{B}^*_{(\tilde{A}_1,\tilde{\psi}_1)} \to \mathcal{L}_{(\tilde{A}_1,\tilde{\psi}_1)}$ *is surjective. So there exists a bounded right inverse* $P : \mathcal{L}_{(\tilde{A}_1,\tilde{\psi}_1)} \to T\mathcal{B}^*_{(\tilde{A}_1,\tilde{\psi}_1)}$ *satisfying*

$$\|P(a,\phi)\|_{\mathcal{A}} \leq K\|(a,\phi)\|_{L^2_{1,\delta}(Y_1)}$$

for a constant K *independent of* L *(see Definition 2.1.1 for the norm* $\|\ \|_{\mathcal{A}}$*).*

Proof: Let Π be the L^2-orthogonal projection onto $\mathcal{L}_{(A_1,\psi_1)}$, π be the L^2-orthogonal projection onto $T\mathcal{B}^*_{(\tilde{A}_1,\tilde{\psi}_1)}$ and I be the right inverse of $\nabla s_{(A_1,\psi_1)}$ (I exists by the assumption that $\mathcal{S}(Y_1, Y_2)$ is regular). For $(a,\phi) \in \mathcal{L}_{(\tilde{A}_1,\tilde{\psi}_1)}$, we have

$$\nabla s_{(\tilde{A}_1,\tilde{\psi}_1)} \pi I \Pi(a,\phi) = (a,\phi) + o(1)(a,\phi)$$

as $L \to \infty$. Here the key point is that $d^*d + |\tilde{\psi}_1|^2$ is invertible and the norm of the inverse is bounded uniformly in L (Lemma 2.1.7). □

Next we deform the "almost" monopoles $(\tilde{A}_1, \tilde{\psi}_1)$ to monopoles. Let Π_1 be the L^2 orthogonal projection onto $\mathcal{L}_{(\tilde{A}_1,\tilde{\psi}_1)}$. For any $(a,\phi) \in T\mathcal{B}^*_{(\tilde{A}_1,\tilde{\psi}_1)}$, we define

$$\begin{aligned} L(a,\phi) &= \Pi_1(*d(\tilde{A}_1 + a) + \tau(\tilde{\psi}_1 + \phi), D_{(\tilde{A}_1+a)}(\tilde{\psi}_1 + \phi)) \\ &= (*d\tilde{A}_1 + \tau(\tilde{\psi}_1), D_{\tilde{A}_1}\tilde{\psi}_1) + \nabla s_{(\tilde{A}_1,\tilde{\psi}_1)}(a,\phi) + \Pi_1 Q(a,\phi) \end{aligned}$$

where $Q(a,\phi) = (\tau(\phi), a\phi)$ satisfying

$$\|Q(a_1,\phi_1) - Q(a_2,\phi_2)\|_{L^2_{1,\delta}} \le C(\|(a_1,\phi_1)\|_{\mathcal{A}} + \|(a_2,\phi_2)\|_{\mathcal{A}})\|(a_1,\phi_1) - (a_2,\phi_2)\|_{\mathcal{A}}.$$

Lemma 3.2.5 $L(a,\phi) = 0$ *implies that*

$$(*d(\tilde{A}_1 + a) + \tau(\tilde{\psi}_1 + \phi), D_{(\tilde{A}_1+a)}(\tilde{\psi}_1 + \phi)) = 0$$

when $\|(a,\phi)\|_{\mathcal{A}}$ *is sufficiently small.*

Proof: A similar argument as in the proof of Lemma 3.1.7. The key point is that $d^*d + |\tilde{\psi}_1|^2$ is invertible and the norm of the inverse is bounded uniformly in L (Lemma

2.1.7). □

Lemma 3.2.6 *The "almost" monopole $(\tilde{A}_1, \tilde{\psi}_1)$ can be deformed to a monopole $(\tilde{A}_1{}', \tilde{\psi}_1{}')$ such that $(\tilde{A}_1{}', \tilde{\psi}_1{}') - (\tilde{A}_1, \tilde{\psi}_1) \in T\mathcal{B}^*_{(\tilde{A}_1, \tilde{\psi}_1)}$ and $\|(\tilde{A}_1{}', \tilde{\psi}_1{}') - (\tilde{A}_1, \tilde{\psi}_1)\|_{\mathcal{A}} \leq Ce^{-\delta L}$.*

Proof: A similar argument as in the proof of Proposition 3.1.2. The fact that $d^*d + |\tilde{\psi}_1|^2$ is invertible and the norm of the inverse is bounded uniformly in L (Lemma 2.1.7) is also used here to get an estimate $\|\Pi_1 Q(a,\phi)\|_{L^2_{1,\delta}} \leq c\|Q(a,\phi)\|_{L^2_{1,\delta}}$. □

The Proof of Proposition 3.2.2:

We need an estimate on the restriction of (A, ψ) on the Y_2 side. Similarly we construct "almost" monopoles $(\tilde{A}_2, \tilde{\psi}_2)$ on Y_2:

$$
\begin{aligned}
\tilde{A}_2 &= (1-\rho_L)(A - a(L)) + a(L) \quad \text{on } Y_2 \setminus [L+1, \infty) \times T^2 \\
\tilde{A}_2 &= a(L) \quad \text{on } [L+1, \infty) \times T^2 \\
\tilde{\psi}_2 &= (1-\rho_L)\psi \quad \text{on } Y_2 \setminus [L+1, \infty) \times T^2 \\
\tilde{\psi}_2 &= 0 \quad \text{on } [L+1, \infty) \times T^2.
\end{aligned}
$$

By Weitzenböck formula and the exponential decay estimate for the spinor, we have

$$\int_{Y_2(L+1)} (|\nabla_A \psi|^2 + \frac{1}{2}|\psi|^4) \leq |(D^{T^2}_{A,1}\psi, \psi)(L)| \leq Ce^{-6\delta L}$$

for a small $\delta > 0$ where $Y_2(L+1) = Y_2 \setminus (L+1, \infty) \times T^2$. It then follows that

$$\| * d\tilde{A}_2\|_{L^2_\delta} \leq Ce^{-\delta L} \text{ and } \|\tilde{\psi}_2\|_{L^2_{1,\delta}} \leq Ce^{-\delta L}.$$

Therefore the distance between $R_2(\tilde{A}_2, 0)$ and $[R_2](\tilde{\mathcal{M}}(Y_2))$ is controlled by $Ce^{-\delta L}$ (Lemmas 2.1.4, 2.2.5). On the other hand, the distance between $R_2(\tilde{A}_2, 0)$ and $R_1(\tilde{A}_1, \tilde{\psi}_1)$ which is given by $|a(L+1)-a(L)|$ is also controlled by $Ce^{-\delta L}$ (Lemma 2.3.4 (d) and the exponential decay estimate for the spinor ψ). So is the distance between $R_1(\tilde{A}_1{}', \tilde{\psi}_1{}')$ and $[R_2](\tilde{\mathcal{M}}(Y_2))$ by Lemma 3.2.6. By the assumption of transversality (Definition 3.1.1 (2)), we have

$$R_1(T\tilde{\mathcal{M}}^*(Y_1)_{(A_1,\psi_1)}) \bigcap R_2(T\tilde{\mathcal{M}}(Y_2)_{(A_2,\psi_2)}) = \{0\}.$$

Then it follows that the distance between $R_1(\tilde{A}_1{}', \tilde{\psi}_1{}')$ and $R_1(A_1, \psi_1)$ is controlled by $Ce^{-\delta L}$. Since $[R_1] : \tilde{\mathcal{M}}^*(Y_1) \to \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ is an immersion at $[(A_1, \psi_1)]$ (Definition 3.1.1 (1)), the distance between $[(\tilde{A}_1{}', \tilde{\psi}_1{}')]$ and $[(A_1, \psi_1)]$ is controlled by $Ce^{-\delta L}$. So is the distance between $[(\tilde{A}_1, \tilde{\psi}_1)]$ and $[(A_1, \psi_1)]$ by Lemma 3.2.6. The distance between $[(\tilde{A}_2, \tilde{\psi}_2)]$ and $[(A_2, \psi_2)]$ is also controlled by $Ce^{-\delta L}$. Now it is easy to see that up to a gauge transformation (A_n, ψ_n) is within an L_1^2 ball of radius $Ce^{-\delta L_n}$ centered at the "almost" monopole (A_{L_n}, ψ_{L_n}) for large enough n. By Proposition 3.1.2, (A_n, ψ_n) is in the image of the gluing map T. On the other hand, it follows from the "weak"convergence of the gauge transformations that the gluing map $T : \mathcal{S}(Y_1, Y_2) \to \mathcal{M}^*(Y_L)$ is also one to one. Hence the proposition is proved.

3.3 Spectral flow, Maslov index and the gluing formula

First we recall the basic relation between Maslov index and the spectral flow of a one-parameter family of first-order, self-adjoint, elliptic differential operators of APS type on a stretched manifold. The basic references are [3] and [4].

Let M be a closed, oriented, smooth manifold that is decomposed into the union of two submanifolds M_1, M_2 by a co-dimension 1, compact oriented submanifold Σ,

$$M = M_1 \bigcup M_2, \quad \Sigma = M_1 \bigcap M_2 = \partial M_1 = \partial M_2.$$

Equip M with a Riemannian metric such that the hypersurface Σ has a collar neighborhood isometric to $(-1,1) \times \Sigma$, and $\Sigma = 0 \times \Sigma, (-1,0) \times \Sigma \subset M_1$. We stretch M by inserting cylinders $[0, 2L] \times \Sigma$ and obtain a family of manifolds $M(L)$. Let $M_1(\infty), M_2(\infty)$ be the cylindrical end manifolds obtained by attaching $[0,\infty) \times \Sigma$ to M_1, and $(-\infty, 0] \times \Sigma$ to M_2.

Let $D : \Gamma(E) \to \Gamma(E)$ be a first-order, self-adjoint, elliptic differential operator acting on the space of smooth sections of a real Riemannian vector bundle $E \to M$ which is of the APS type near Σ. More precisely, on $(-1,1) \times \Sigma$, E is isometric to the pull-back bundle $\pi^* E_0$ and D can be written as

$$D = \sigma(\frac{\partial}{\partial t} + D_0),$$

where $\pi : (-1,1) \times \Sigma \to \Sigma$ is the projection, $E_0 \to \Sigma$ is a Riemannian vector bundle on Σ, $\sigma : E_0 \to E_0$ is a bundle isometry, and D_0 is a first-order, self-adjoint, elliptic operator acting on $\Gamma(E_0)$. Then E and D naturally extend to a vector bundle $E(L)$ and an operator $D(L)$ on the stretched manifold $M(L)$, and to $E_j(\infty)$ and $D_j(\infty)$ on the cylindrical end manifold $M_j(\infty)$, $j = 1, 2$.

Let l_j be the space of limiting values of the extended L^2-solutions of $D_j(\infty) = 0$ over $M_j(\infty)$. Denote $\mathcal{K}er\ D_0$ by $\mathcal{H}$. Then we have (see [3])

Lemma 3.3.1 *1. $\mathcal{H}$ is a symplectic vector space with the preferred symplectic form*

$$\{x, y\} = \int_\Sigma \langle x, \sigma y\rangle.$$

2. l_1, l_2 are Lagrangian subspaces in $\mathcal{H}$.

We call l_j the Lagrangian subspace associated to $D_j(\infty)$.

Let E_1, E_2 be the restriction of the vector bundle E and D_1, D_2 be the restriction of the operator D on the submanifolds M_1 and M_2 of M. For any pair of Lagrangian subspaces l_1, l_2 of the symplectic vector space $\mathcal{H} = \mathcal{K}er\ D_0$, we have a pair of self-adjoint Fredholm operators $D_1(l_1), D_2(l_2)$ defined with global boundary conditions:

$$D_1(l_1) : L_1^2(E_1, P_+ \oplus l_1) \to L^2(E_1)$$
$$D_2(l_2) : L_1^2(E_2, P_- \oplus l_2) \to L^2(E_2)$$

where $P_\pm$ are the subspaces of $L^2(E_0)$ spanned by the eigenvectors of positive/negative eigenvalues of D_0, and the space $L_1^2(E_1, P_+ \oplus l_1)$ is the L_1^2-Sobolev completion of smooth sections of bundle E_1 whose restrictions on Σ lie in the space $P_+ \oplus l_1$ and similarly is the other space $L_1^2(E_2, P_- \oplus l_2)$ understood.

Each homotopy class (with fixed ends) of one-parameter families of pairs of Lagrangian subspaces $(l_1(s), l_2(s)) : a \leq s \leq b$ is associated with an integer which is called the Maslov index of $(l_1(s), l_2(s))$ and denoted by $Mas\{(l_1(s), l_2(s)) : a \leq s \leq b\}$ (see [4],[5]).

The (ϵ_1, ϵ_2)-spectral flow is defined as follows. Let $D(s) : a \leq s \leq b$ be a family of real self-adjoint operators such that for some fixed $\delta > 0$ the total spectrum of $D(s)$ in the range of eigenvalues λ with $|\lambda| < \delta$ is finite-dimensional and has no essential spectrum. Furthermore, after taking into consideration of multiplicities, these eigenvalues λ with $|\lambda| < \delta$ vary continuously with respect to s. Let ϵ_1, ϵ_2 be real numbers with $|\epsilon_1| < \delta, |\epsilon_2| < \delta$, such that ϵ_1 is not an eigenvalue of $D(a)$ and ϵ_2 is not an eigenvalue of $D(b)$. Then the (ϵ_1, ϵ_2)-spectral flow of $D(s) : a \leq s \leq b$ is equal to the number of times the eigenvalues λ of $D(s)$ in the range $|\lambda| < \delta$ cross the line joining (a, ϵ_1) and (b, ϵ_2) from below, minus the number of times crossing from above (see [4] for details). The (ϵ, ϵ)-spectral flow will be called briefly as ϵ-spectral flow.

Let $D(s) : a \leq s \leq b$ be a one-parameter family of first-order, self-adjoint, elliptic differential operators on M which are of the APS type, i.e. in the collar neighborhood $(-1, 1) \times \Sigma$, $D(s) = \sigma(\frac{\partial}{\partial t} + D_0(s))$. Furthermore, there exists a $\delta > 0$ such that there are no eigenvalues of $D_0(s)$ in the range $(-\delta, 0)$ and $(0, \delta)$, and $\mathcal{H} = \mathcal{K}er\ D_0(s)$ is a fixed symplectic vector space for $a \leq s \leq b$. A one-parameter family of pairs of Lagrangian subspaces $(l_1(s), l_2(s)) : a \leq s \leq b$ in $\mathcal{H}$ is said to satisfy the endpoint condition if $(l_1(s), l_2(s))$ is the pair of Lagrangian subspaces associated to $(D_1(\infty)(s), D_2(\infty)(s))$ at the endpoints $s = a, b$.

The basic relation between Maslov index and spectral flow is given by the following

Theorem 3.3.2 *(Theorem C in [4])*

There exists an $L_0 > 0$ such that for any choice of smoothly varying pairs of Lagrangian subspaces $(l_1(s), l_2(s)) : a \leq s \leq b$ satisfying the endpoint condition, for all $L > L_0$, the (L^{-2})-spectral flow of $D(s)(L)$ on $M(L)$ for $a \leq s \leq b$ equals to

$$\sum_{j=1}^{2} SF_\epsilon\{D_j(s)(l_j(s)) : a \leq s \leq b\} + Mas\{(l_1(s), l_2(s)) : a \leq s \leq b\}$$

where $SF_\epsilon\{D_j(s)(l_j(s)) : a \leq s \leq b\}$ is the ϵ-spectral flow of $D_j(s)(l_j(s)) : a \leq s \leq b$. Here $\epsilon > 0$ is chosen so that the eigenvalues of $D_j(s)(l_j(s))$ in the range $[-\epsilon, \epsilon]$ consist of at most zero eigenvalues for the endpoints $s = a, b$.

Now let's go back to our own problem. Suppose that Y is an oriented integral homology 3-sphere that is decomposed as $Y = Y_1 \bigcup_{T^2} Y_2$ with Y_1 being the complement of a tubular neighborhood of a knot and $Y_2 = D^2 \times S^1$. Y carries a Riemannian metric such that a collar neighborhood of T^2 is orientedly isometric to $(-1,1) \times \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ with $(-1,0) \times \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z} \subset Y_1$, where we assume that the first and second factors in $\mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ represent the longitude and meridian respectively ([1]), and the metric on Y_2 has non-negative, somewhere positive scalar curvature. By inserting cylinders $[0, 2L+1] \times T^2$, we obtain a family of stretched versions Y_L of Y. We also use Y_1 and Y_2 to denote the corresponding cylindrical end manifolds if no confusion occurs.

The basic result we've obtained so far (Proposition 3.2.2) is that for a large enough L, the irreducible Seiberg-Witten moduli space $\mathcal{M}^*(Y_L)$ of Y_L is identified via the gluing map T with the set of "intersection points" $\mathcal{S}(Y_1, Y_2)$. Here $\mathcal{S}(Y_1, Y_2)$ consists of the pairs $(\alpha_1, \alpha_2) \in \tilde{\mathcal{M}}^*(Y_1) \times \tilde{\mathcal{M}}(Y_2)$ such that there are smooth representatives

(A_1, ψ_1) and (A_2, ψ_2) having the same limiting value, i.e. $R_1(A_1, \psi_1) = R_2(A_2, \psi_2)$. Our next goal is to orient $\tilde{\mathcal{M}}^*(Y_1)$ and $\tilde{\mathcal{M}}(Y_2)$ appropriately so that their "intersection number" $\#\mathcal{S}(Y_1, Y_2)$ equals to the Seiberg-Witten invariant $\chi(Y_L)$ as the oriented sum of the points in the moduli space $\mathcal{M}^*(Y_L)$. This is referred to as the gluing formula of χ.

Fix a generic perturbation (g, f, μ) compactly supported on the Y_1 side according to Proposition 2.2.2 and thereafter omit it in the discussion for simplicity. Assume that L_0 is large enough so that Proposition 3.2.2 holds for Y_{L_0}. Pick a smooth section ϕ of the spinor bundle $W \to Y_{L_0}$ which is compactly supported in $Y_1 \setminus [0, \infty) \times T^2$ and satisfies $((D_g + f)^{-1}(i\phi), (i\phi)) < 0$. Then by Lemma 1.2.2, for small enough $t > 0$, the self-adjoint operator (on Y_{L_0})

$$\mathcal{K}_{(t,\phi)} = \begin{pmatrix} D_g + f & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix} + t \begin{pmatrix} 0 & \phi\cdot & \phi\cdot \\ 2\tau(\phi, \cdot) & 0 & 0 \\ i\langle i\phi, \cdot\rangle_{Re} & 0 & 0 \end{pmatrix}$$

acting on $\Gamma(W \oplus (\Lambda^1 \oplus \Lambda^0) \otimes i\mathbf{R})$ is invertible and has one small eigenvalue

$$\lambda_t \sim -((D_g + f)^{-1}(i\phi), (i\phi))t^2 > 0.$$

According to Definition 1.2.4, the Euler characteristic $\chi(Y_{L_0})$ is defined by

$$\chi(Y_{L_0}) = \sum_{\beta \in \mathcal{M}^*(Y_{L_0})} \mathrm{sign}\beta, \quad \text{where } \mathrm{sign}\beta = (-1)^{SF(\mathcal{K}_\beta, \mathcal{K}_{(t,\phi)})}$$

for small $t > 0$ (SF denotes the spectral flow). Here if β is represented by (A, ψ), then

$$\mathcal{K}_\beta = \mathcal{K}_{(A,\psi)} = \begin{pmatrix} D_A & \psi\cdot & \psi\cdot \\ 2\tau(\psi,\cdot) & *d & -d \\ i\langle i\psi,\cdot\rangle_{Re} & -d^* & 0 \end{pmatrix}.$$

Let (A_{L_0}, ψ_{L_0}) be the "almost" monopole being deformed to (A, ψ) under the gluing map T (note that ψ_{L_0} is compactly supported in $Y_1 \setminus (L_0+1, \infty) \times T^2$). It is obvious that $\mathcal{K}_\beta$ can be replaced by $\mathcal{K}_{(A_{L_0},\psi_{L_0})}$ for the purpose of spectral flow calculation. For any $L > 0$, we insert cylinders of lengths $2L$ into Y_{L_0} and obtain a family of manifolds $Y_{L_0,L}$ and operators $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ on $Y_{L_0,L}$ from $\mathcal{K}_{(A_{L_0},\psi_{L_0})}$ in the obvious way.

Lemma 3.3.3 *For large enough L_0, $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ are invertible for any $L > 0$. In particular, the spectral flow between $\mathcal{K}_{(A_{L_0},\psi_{L_0})}$ and $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ is zero for any $L > 0$.*

Proof: For large enough $L_0 > 0$, the operators $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ are invertible for all $0 < L < L_0$ by Theorem 4 in Appendix B. Suppose that $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ has a non-zero kernel for some $L \geq L_0$, i.e. there is an $x \neq 0$ such that $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)x = 0$. On the inserted cylinder, $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ has the form $I(\frac{\partial}{\partial t} + B)$ where

$$I = \begin{pmatrix} dt & 0 & 0 & 0 \\ 0 & *_{T^2} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} D_a^{T^2} & 0 & 0 & 0 \\ 0 & 0 & -d_{T^2} & *d_{T^2} \\ 0 & -d_{T^2}^* & 0 & 0 \\ 0 & -*d_{T^2} & 0 & 0 \end{pmatrix}$$

acting on $\Gamma(W_0 \oplus (\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^0(T^2)) \otimes i\mathbf{R})$. Here W_0 is the total spinor bundle over T^2, and $D_a^{T^2}$ is an invertible twisted Dirac operator. It follows that x can be decomposed as $x = x_0 + x_+ + x_-$ with $x_0 \in Ker\ B$ constant in t and $x_\pm$ have exponential decay to the right/left. Take a cut-off function γ in the middle of the inserted cylinder, define $y_\pm$ on the cylindrical end manifolds Y_1/Y_2 by:

$$y_+ = \gamma(x - x_0) + x_0, \ \ y_- = (1-\gamma)(x - x_0) + x_0.$$

Then it follows that

$$\|\mathcal{K}_{(A_{L_0},\psi_{L_0}),1}(\infty)y_+\|_{L^2_\delta} \leq ce^{-\delta L}(\|y_+ - x_0\|_{L^2_\delta} + \|y_- - x_0\|_{L^2_\delta})$$

and

$$\|\mathcal{K}_{(A_{L_0},\psi_{L_0}),2}(\infty)y_-\|_{L^2_\delta} \leq ce^{-\delta L}(\|y_+ - x_0\|_{L^2_\delta} + \|y_- - x_0\|_{L^2_\delta})$$

for some small $\delta > 0$ and a constant c. Here $\mathcal{K}_{(A_{L_0},\psi_{L_0}),j}(\infty)$ is the corresponding operator on the cylindrical end manifold Y_j, $j = 1, 2$. On the other hand, observe that y_+ and y_- have the same limiting value x_0 and for all large enough L_0, the Lagrangian subspaces associated to $\mathcal{K}_{(A_{L_0},\psi_{L_0}),j}(\infty)$ $(j = 1, 2)$ are transversal to each other with angles larger than a fixed number (due to the fact that $\mathcal{S}(Y_1, Y_2)$ is regular). Then the above estimates yield

$$\|x_0\| \leq c_1e^{-\delta L}(\|y_+ - x_0\|_{L^2_\delta} + \|y_- - x_0\|_{L^2_\delta}).$$

Since both of $\mathcal{K}_{(A_{L_0},\psi_{L_0}),1}(\infty)$ and $\mathcal{K}_{(A_{L_0},\psi_{L_0}),2}(\infty)$ have no L^2 kernels, we have estimates

$$\|y_\pm - x_0\|_{L^2_\delta} \leq c_2e^{-\delta L}(\|y_+ - x_0\|_{L^2_\delta} + \|y_- - x_0\|_{L^2_\delta})$$

which imply that for large L_0 (therefore $L \geq L_0$ large) $y_\pm$ vanish identically, contradicting the assumption that $x \neq 0$. Therefore the lemma is proved. □

The operators considered here have the APS form $I(\frac{\partial}{\partial t} + B)$ on the inserted cylinder where

$$I = \begin{pmatrix} dt & 0 & 0 & 0 \\ 0 & *_{T^2} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} D_a^{T^2} & 0 & 0 & 0 \\ 0 & 0 & -d_{T^2} & *d_{T^2} \\ 0 & -d^*_{T^2} & 0 & 0 \\ 0 & - * d_{T^2} & 0 & 0 \end{pmatrix}.$$

The symplectic vector space $\mathcal{H} = Ker\ B$ is $\mathcal{H}^1(T^2) \otimes i\mathbf{R} \oplus i\mathbf{R} \oplus i\mathbf{R}$. Let's fix the notation about $\mathcal{H}$ first. Recall that T^2 is orientedly isometric to $\mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ (longitude, meridian). Let (x, y) be the oriented coordinates, then we orient $\mathcal{H}^1(T^2) \otimes i\mathbf{R}$ by $idx \wedge idy$. Furthermore, the 3rd component of $\mathcal{H}$ corresponds to the dt-component of the 1-forms and the 4th one is from the Lie algebra of the gauge group.

Let $\mathcal{K}_0$ (acting on $\Gamma(W \oplus (\Lambda^1 \oplus \Lambda^0) \otimes i\mathbf{R})$) be the operator at the reducible point $(0, 0)$:

$$\mathcal{K}_0 = \begin{pmatrix} D_g + f & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix}.$$

The corresponding operators $\mathcal{K}_{0,j}(\infty)$ on the cylindrical end manifolds Y_j have no L^2 kernels and the associated Lagrangian subspaces of $\mathcal{K}_{0,1}(\infty)$ and $\mathcal{K}_{0,2}(\infty)$ are spanned by $(idy, (0, 0, 0, 1))$ and $([R_2](T\tilde{\mathcal{M}}(Y_2)), (0, 0, 0, 1))$ respectively. Note that $[R_2](T\tilde{\mathcal{M}}(Y_2))$ is transversal to idy since Y is a homology 3-sphere.

Now we are ready to orient the moduli spaces $\tilde{\mathcal{M}}^*(Y_1)$ and $\tilde{\mathcal{M}}(Y_2)$. Assume that

$\alpha_1 \in \tilde{\mathcal{M}}^*(Y_1)$ is represented by (A,ψ). For any vector $V \in \mathcal{H}^1(T^2)\otimes i\mathbf{R}$ with positive idx-component which is not in $R_1(T\tilde{\mathcal{M}}(Y_1)_{(A,\psi)})$, let $v \in T\tilde{\mathcal{M}}(Y_1)_{(A,\psi)}$ such that $V\wedge R_1(v) = idx\wedge idy$. Pick an $L_0 > 0$ and cut down (A,ψ) at L_0+1. Denote the result by (A_{L_0},ψ_{L_0}). We assume that L_0 is large enough so that the Lagrangian subspace associated to $\mathcal{K}_{(A_{L_0},\psi_{L_0}),1}(\infty)$ is transversal to the Lagrangian subspace spanned by $(V,(0,0,0,1))$. Connect (A_{L_0},ψ_{L_0}) to the reducible point $(0,0)$ by a path $(A,\psi)_s$ which is constant in t on $[L_0+1,\infty)\times T^2$. Choose a smooth path of Lagrangian subspaces $l_1(s)$ which equals to the Lagrangian subspace $(idy,(0,0,0,1))$ associated to $\mathcal{K}_{0,1}(\infty)$ or the Lagrangian subspace associated to $\mathcal{K}_{(A_{L_0},\psi_{L_0}),1}(\infty)$ at the endpoints of the path.

Definition 3.3.4 *1. The orientation of $\tilde{\mathcal{M}}^*(Y_1)$ at $\alpha_1 = [(A,\psi)]$ is determined by the tangent vector $(-1)^m v$. Here m is the sum of the (ϵ)-spectral flow of operators $\mathcal{K}_{(A,\psi)_s,1}(L_0+1)(l_1(s))$ (for a small $\epsilon > 0$) and the Maslov index $Mas\{(l_1(s), l_V)\}$, where l_V is the Lagrangian subspace spanned by $(V,(0,0,0,1))$.*

2. The orientation of $\tilde{\mathcal{M}}(Y_2)$ is determined so that the positive direction of $[R_2](T\tilde{\mathcal{M}}(Y_2))$ has positive idx-component. Note that $[R_2](T\tilde{\mathcal{M}}(Y_2))$ is transversal to idy-axis since Y is a homology 3-sphere.

Lemma 3.3.5 *The orientation on $\tilde{\mathcal{M}}^*(Y_1)$ is well-defined, which induces an orientation on $\mathcal{M}^*(Y_1)$ via the $\mathbf{Z}$-fold covering map $\tilde{\mathcal{M}}^*(Y_1)\to\mathcal{M}^*(Y_1)$.*

Proof: We need to prove that the orientation of $\tilde{\mathcal{M}}^*(Y_1)$ is independent of the choice of α_1 (and its representatives (A,ψ)), the vector $V \in \mathcal{H}^1(T^2)\otimes i\mathbf{R}$, the cut-off point L_0, the path $(A,\psi)_s$ and the path of Lagrangian subspaces $l_1(s)$.

First of all, the independence on the choice of cut-off point L_0, the path $(A,\psi)_s$ and the path of Lagrangian subspaces $l_1(s)$ follows easily from Theorem 3.3.2. Secondly, suppose that two monopoles (A_1,ψ_1) and (A_2,ψ_2) are in the same component. Join them by a path of monopoles (A_s,ψ_s) and then cut down the path at L_0+1 for sufficiently large L_0 (still denote the path by (A_s,ψ_s)). Let $l(s)$ be the Lagrangian subspace associated to $\mathcal{K}_{(A_s,\psi_s),1}(\infty)$. Then the (ϵ)-spectral flow of $\mathcal{K}_{(A_s,\psi_s),1}(L_0+1)(l(s))$ is zero because $\tilde{\mathcal{M}}^*(Y_1)$ is immersed into $\mathcal{H}^1(T^2)\otimes i\mathbf{R}$ so that $\mathcal{K}_{(A_s,\psi_s),1}(\infty)$ have no L^2-kernels for large enough L_0. On the other hand, since (A_s,ψ_s) is irreducible so that the 3rd component of $l(s)$ is non-zero, $\text{Mas}\{(l(s),(V,(0,0,0,1))\}$ (mod 2) equals to the sign change of $V\wedge R_1(v_s)$ where v_s is a smooth tangent vector field in $T\tilde{\mathcal{M}}^*(Y_1)$ along the path (A_s,ψ_s). So the orientation at (A_1,ψ_1) and the orientation at (A_2,ψ_2) are compatible. Finally, suppose that $V_1,V_2\in\mathcal{H}^1(T^2)\otimes i\mathbf{R}$ are two different vectors used in the definition. Then $\text{Mas}\{(l_1(s),(V_1,(0,0,0,1))\}-\text{Mas}\{(l_1(s),(V_2,(0,0,0,1))\}$ (mod 2) equals to the sign change from $V_1\wedge R_1(v)$ to $V_2\wedge R_1(v)$ for any $v\in T\tilde{\mathcal{M}}(Y_1)_{(A,\psi)}$, which implies that the orientation of $\tilde{\mathcal{M}}^*(Y_i)$ at $[(A,\psi)]$ is independent of the choice of the vector V. Therefore we have proved that the orientation of $\tilde{\mathcal{M}}^*(Y_1)$ is well-defined.

Next we prove that the $\mathbf{Z}$-fold covering map $\tilde{\mathcal{M}}^*(Y_1)\to\mathcal{M}^*(Y_1)$ induces an orientation on $\mathcal{M}^*(Y_1)$. Suppose that (A_1,ψ_1) and (A_2,ψ_2) are gauge equivalent by a gauge transformation s_1 not in the identity component of $\mathcal{G}(Y_1)$. Pick an L_0 large enough and cut down (A_1,ψ_1) at L_0+1 and still denote it by (A_1,ψ_1) (we can assume that s_1 is constant in t on $[L_0+1,\infty)\times T^2$). Connect (A_1,ψ_1) with the reducible point $(0,0)$ by a path (A_s,ψ_s) (so $s_1\cdot(A_s,\psi_s)$ is a path joining $s_1\cdot(A_1,\psi_1)=(A_2,\psi_2)$

with $(-s_1^{-1}ds_1, 0))$. Then the (ϵ)-spectral flow of $\mathcal{K}_{(A_s,\psi_s),1}(L_0+1)(l_1(s))$ equals to that of $\mathcal{K}_{s_1\cdot(A_s,\psi_s),1}(L_0+1)(l_1(s))$ where $l_1(s)$ is a path of Lagrangian subspaces which equals to the associated Lagrangian subspace of $\mathcal{K}_{(A_s,\psi_s),1}(\infty)$ at the endpoints. On the other hand, the (ϵ)-spectral flow of $\mathcal{K}_{(-us_1^{-1}ds_1,0),1}(L_0+1)(l_3) : 0 \leq u \leq 1$ is even (Dirac operators are complex linear) where the Lagrangian subspace l_3 is spanned by $(idy, (0,0,0,1))$. Now it is easy to see that the orientation at (A_1, ψ_1) and (A_2, ψ_2) are compatible. So the lemma is proved. □

Now we are ready to define the "intersection number" $\#\mathcal{S}(Y_1, Y_2)$ and prove the gluing formula.

Definition 3.3.6 *1. For any $(\alpha_1, \alpha_2) \in \mathcal{S}(Y_1, Y_2)$, let e_j be the positively oriented tangent vector of $\tilde{\mathcal{M}}(Y_j)$ at α_j ($j = 1, 2$). Then the sign of (α_1, α_2) is the sign of $[R_1]e_1 \wedge [R_2]e_2$ with respect to $idx \wedge idy$.*

2. $\#\mathcal{S}(Y_1, Y_2) = \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(Y_1,Y_2)} sign(\alpha_1, \alpha_2)$.

Theorem 3.3.7 *(Gluing Formula)*

$\chi(Y_L) = \#\mathcal{S}(Y_1, Y_2)$ for sufficiently large $L > 0$.

Proof: Let (A_{L_0}, ψ_{L_0}) be the "almost" monopole being deformed to $\beta \in \mathcal{M}^*(Y_{L_0})$. By Lemma 3.3.3, sign$\beta = (-1)^{m_1+1}$ where m_1 is the L^{-2}-spectral flow between $\mathcal{K}_{(A_{L_0},\psi_{L_0})}(L)$ and $\mathcal{K}_0$ for sufficiently large L. By Theorem 3.3.2, m_1 is equal to

$$\sum_{j=1}^{2} SF_\epsilon\{\mathcal{K}_{(A,\psi)_s,j}(L_0+1)(l_j(s))\} + Mas\{(l_1(s), l_2(s))\}$$

for any choice of $(A, \psi)_s$ joining (A_{L_0}, ψ_{L_0}) with the reducible point $(0,0)$ and any choice of a path of Lagrangian subspaces $(l_1(s), l_2(s))$ satisfying the endpoint condition. Here $SF_\epsilon\{\mathcal{K}_{(A,\psi)_s,j}(L_0+1)(l_j(s))\}$ is the ϵ-spectral flow of $\mathcal{K}_{(A,\psi)_s,j}(L_0+1)(l_j(s))$ for some small $\epsilon > 0$. We choose $(A,\psi)_s$ such that ψ_s is identically zero on the Y_2 side, and choose $l_2(s) = l_2$ to be the Lagrangian subspace spanned by $([R_2](T\tilde{\mathcal{M}}(Y_2)),(0,0,0,1))$. Then the ϵ-spectral flow of $\mathcal{K}_{(A,\psi)_s,2}(L_0+1)(l_2)$ is even. On the other hand, suppose $\beta = T(\alpha_1,\alpha_2)$. Let e_j be the positively oriented tangent vector of $\tilde{\mathcal{M}}(Y_j)$ at α_j ($j = 1,2$). Then by taking $V = [R_2]e_2$ in Definition 3.3.4, we have $\text{sign}(\alpha_1,\alpha_2)$ equals to the sign of $[R_1]e_1 \wedge [R_2]e_2 = (-1)^m[R_1]v \wedge [R_2]e_2 = (-1)^{m+1}idx \wedge idy = (-1)^{m+1}$. Here m and v are referred to Definition 3.3.4. The theorem follows from the relation $m \equiv m_1 \bmod 2$.

APPENDICES

APPENDIX A

The purpose of this appendix is to find out for what $a \in \mathcal{H}^1(T^2) \otimes i\mathbf{R}$ the twisted Dirac operator $D_a = D + a$ is not invertible. Here D is the Dirac operator on T^2 associated to a given spin structure and the flat metric.

First of all, let's recall some basic facts about the spin structures on the torus T^2. There are two equivalent descriptions of spin structures. Topologically, a spin structure on T^2 is a framing of its stabilized tangent bundle $TT^2 \oplus \epsilon$ (a homotopy equivalence class of trivializations). There are four different spin structures on T^2 which are parameterized by $H^1(T^2, \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. It is well-known that among these four different spin structures, three of them are spin boundaries, i.e. spin structures induced from a spin 3-manifold bounded by the torus. The only one left which is not a spin boundary is usually called the Lie group framing. Assume that $T^2 = \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ and let $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ be the tangent vectors of the circles. For $(k,l) = (0,0), (0,1), (1,0), (1,1)$, the following formula defines four different framings $\xi_{(k,l)}$ of the tangent bundle TT^2 which induce all the spin structures on T^2 (framings of $TT^2 \oplus \epsilon$):

$$\xi_{(k,l)}(x,y) = \begin{pmatrix} \cos(kx+ly) & -\sin(kx+ly) \\ \sin(kx+ly) & \cos(kx+ly) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}, \quad (x,y) \in T^2.$$

The Lie group framing is $\xi_{(0,0)}$. See [15] for details.

The geometric aspect of spin structures is related to the groups $Spin(n)$. The groups $Spin(n)$ sit inside the n-dimensional Clifford algebras $Cl(n)$ and double cover the groups $SO(n)$. Let $\pi : Spin(n) \to SO(n)$ be the double covering map. Equip the torus T^2 with a Riemannian metric, assuming that it is the product metric for simplicity. Let $P_{SO(2)}$ be the $SO(2)$ principal bundle to which the tangent frame bundle of T^2 is reduced. A spin structure on T^2 is then defined to be an equivalence class of liftings of the principal bundle $P_{SO(2)}$ to a $Spin(2)$ principal bundle $P_{Spin(2)}$, i.e. $P_{Spin(2)} \xrightarrow{\pi} P_{SO(2)}$ such that π restricts to the double covering map on each fiber. Two liftings $P^{(1)}_{Spin(2)} \xrightarrow{\pi_1} P_{SO(2)}$ and $P^{(2)}_{Spin(2)} \xrightarrow{\pi_2} P_{SO(2)}$ are said to be equivalent if and only if there is a bundle isomorphism i such that the following diagram commutes:

$$\begin{array}{ccc} P^{(1)}_{Spin(2)} & \xrightarrow{i} & P^{(2)}_{Spin(2)} \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ P_{SO(2)} & \xrightarrow{identity} & P_{SO(2)} \end{array}$$

For each spin structure $P_{Spin(2)} \xrightarrow{\pi} P_{SO(2)}$, there is a canonically associated spinor bundle $W = W^+ \oplus W^-$ on T^2, where $W^\pm = P_{Spin(2)} \times_{\varrho_\pm} \mathbf{C}$. The representations $\varrho_\pm : Spin(2) \to U(1)$ are distinguished by the conditions $\varrho_\pm(e_1 e_2) = \mp i$ for any orthonormal basis (e_1, e_2) of $\mathbf{R}^2$.

The topological and geometrical descriptions of spin structures on T^2 are related in the following way. The spin structure induced by the trivialization $\xi_{(k,l)}$ $(k, l = 0, 1)$ corresponds to the unique equivalence class of liftings $P^{(k,l)}_{Spin(2)} \to P_{SO(2)}$ for which the trivialization $\xi_{(k,l)}$ of $P_{SO(2)}$ can be lifted to a trivialization $\tilde{\xi}_{(k,l)}$ of $P^{(k,l)}_{Spin(2)}$, which further induces trivializations for the spinor bundles $W^{\pm}$ and W.

Theorem:

Assume that $T^2 = \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ *carries the product metric and* $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ *is the oriented orthonormal basis. For* $(k, l) = (0,0), (0,1), (1,0), (1,1)$, *define trivializations* $\xi_{(k,l)}$ *of* TT^2 *by the following formula:*

$$\xi_{(k,l)}(x,y) = \begin{pmatrix} \cos(kx+ly) & -\sin(kx+ly) \\ \sin(kx+ly) & \cos(kx+ly) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}, \qquad (x,y) \in T^2.$$

Then within the induced trivialization $\tilde{\xi}_{(k,l)}$ *of the spinor bundles associated to the spin structure given by* $\xi_{(k,l)}$, *the Dirac operator* $D^{(k,l)}$ *is given by the following formula*

$$D^{(k,l)} \begin{pmatrix} u \\ v \end{pmatrix} = dx \left(\frac{\partial}{\partial x} + \frac{i}{2} \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} + dy \left(\frac{\partial}{\partial y} + \frac{i}{2} \begin{pmatrix} l & 0 \\ 0 & -l \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix},$$

where u, v *are complex valued functions on* T^2. *As a consequence, for* $a \in \mathcal{H}^1(T^2) \otimes i\mathbf{R}$, *the twisted Dirac operator* $D^{(k,l)}_a = D^{(k,l)} + a$ *is invertible unless* $a = \frac{i}{2}(kdx + ldy) + sidx + tidy$ *for some integers* s *and* t, *and* $\dim_{\mathbf{C}} Ker\ D^{(k,l)}_a = 2$ *if* $a = \frac{i}{2}(kdx + ldy) + sidx + tidy$ *for some integers* s *and* t.

Remark: The lattice

$$B_{(k,l)} = \{a|a = \frac{i}{2}(kdx + ldy) + sidx + tidy, s, t \in \mathbf{Z}\}$$

is called the lattice of "bad" points for the spin structure $\xi_{(k,l)}$.

Proof: In general, if $(e_1, e_2, ..., e_n)$ is an oriented local orthonormal frame, then within the induced trivialization of the spinor bundles, the induced connection is given by $-\frac{1}{2}\sum_{i<j}\omega_{ij}e_ie_j$, where ω_{ij} is given by the formula $\nabla e_j = e_i\omega_{ij}$ (see [19] for details). Back to our case of the torus, let $\xi_{(k,l)}(x, y) = (e_1, e_2)$, and $\nabla e_2 = e_1\omega_{12}$, then $\omega_{12} = kdx + ldy$ by direct calculation. The theorem follows easily from this.

APPENDIX B

The purpose of this appendix is to give an estimate on the lowest eigenvalue of certain self-adjoint elliptic operators on a manifold containing long necks, a technical result needed in Chapter 3. See [7].

Let X be an oriented Riemannian manifold with a cylindrical end modeled on Y, i.e. there exists a compact subset K such that $X \setminus K$ is isometric to $(-1, \infty) \times Y$. Let E be a cylindrical Riemannian vector bundle over X. By definition, there is a Riemannian vector bundle E_0 over Y such that E is isometric to $\pi^* E_0$ on the cylindrical end $(-1, \infty) \times Y$, where $\pi : (-1, \infty) \times Y \to Y$ is the natural projection. Assume that $D : \Gamma(E) \to \Gamma(E)$ is a first order formally self-adjoint elliptic operator on X, which takes the following form on the cylindrical end $(-1, \infty) \times Y$

$$D = I(\frac{\partial}{\partial t} + A)$$

where I is a bundle automorphism of E_0 which preserves its inner product, and $A : \Gamma(E_0) \to \Gamma(E_0)$ is an elliptic operator on Y independent of t. The self-adjointness of D implies that I and A satisfy the following conditions:

$$I^2 = -1, \;\; I^* = -I, \;\; A^* = A, \;\; IA + AI = 0.$$

Note that the spectrum of A is symmetric about the origin and the automorphism I maps E_λ to $E_{-\lambda}$ where E_λ is the eigenspace correspondent to eigenvalue λ. See [26]. We assume that $Ker\ A \neq 0$. Then the automorphism I defines a complex structure on $Ker\ A$ which induces a symplectic structure on it. In particular, the dimension of $Ker\ A$ is even. The operator D as described will be said cylindrical compatible.

Definition 1

An exponentially small perturbation of a cylindrical compatible operator D is a first order formally self-adjoint elliptic operator D' satisfying the following conditions:

a) D' is a zero order perturbation of D,

b) on the cylindrical end $(-1,\infty)\times Y$, $D' = D + P(t)$ where $P(t) : \Gamma(E_0) \to \Gamma(E_0)$ is a smooth family of zero order self-adjoint operators satisfying the following exponential decay conditions: there exist a small $\delta > 0$, some $T_0 > 0$ and a constant C such that when $t > T_0$,

$$||P(t)\psi||_{L^2(Y)} \le Ce^{-\delta(t-T_0)}||\psi||_{L^2(Y)} \;\; \textit{and} \;\; ||\frac{\partial P}{\partial t}\psi||_{L^2(Y)} \le Ce^{-\delta(t-T_0)}||\psi||_{L^2(Y)}$$

for $\psi \in L^2(E_0)$.

Let D' be an exponentially small perturbation of a cylindrical compatible operator.

The space of "bounded" harmonic sections of D' is denoted by $H_B(D')$, i.e.

$$H_B(D') = \{\psi \in \Gamma(E) | D'\psi = 0, ||\psi||_{C^0(X)} < \infty\}.$$

The space of L^2 harmonic sections of D' is denoted by $H_{L^2}(D')$, i.e.

$$H_{L^2}(D') = \{\psi \in L^2(E) | D'\psi = 0\}.$$

Let β be a fixed cut-off function which is equal to one at ∞, and $\pi : (-1, \infty) \times Y \to Y$ be the natural projection.

Lemma 2

There exists a small $\delta_1 > 0$ such that for any $\psi \in H_B(D')$, there exists a unique limiting value $r(\psi) \in Ker\ A$ such that

$$||\psi - \beta\pi^* r(\psi)||_{L^2_{\delta_1}(E)} < \infty.$$

In particular, $\psi \in H_{L^2}(D')$ if and only if $r(\psi) = 0$. Moreover,

$$dim H_B(D') - dim H_{L^2}(D') = \frac{1}{2} dim Ker\ A.$$

Now consider a pair of triples (X_i, E_i, D_i') for $i = 1, 2$. Suppose that there is an orientation reversing isometry $h : Y_1 \to Y_2$ which is covered by corresponding bundle maps which identify A_1 with A_2 in a suitable way so that for any $L > 0$, we can form a triple (X_L, E_L, D_L') where $X_L = X_1 \setminus [L+1, \infty) \times Y_1 \cup_h X_2 \setminus [L+1, \infty) \times Y_2$

with $h : (L, L+1) \times Y_1 \to (L+1, L) \times Y_2$ given by $h(L+t, y) = (L+1-t, h(y))$, $E_L = E_1 \bigcup_h E_2$, $D_L = D_1 \bigcup_h D_2$ and $P_L = \beta_L P_1 + (1-\beta_L) h^* P_2$ for some cut-off function β_L supported in $(L, L+1) \times Y_1$ with $|\nabla \beta| \leq 2$, and $D'_L = D_L + P_L$. Set

$$\lambda_L = \inf_{\psi \neq 0} \frac{\int_{X_L} |D'_L \psi|^2}{\int_{X_L} |\psi|^2}.$$

The purpose of this appendix is to investigate the behavior of λ_L as $L \to \infty$.

Definition 3

Suppose D', D'_1 and D'_2 are exponentially small perturbations of cylindrical compatible operators.

a) D' is said to be regular if $H_{L^2}(D') = 0$.

b) (D'_1, D'_2) is said to be a transversal pair if

$$r(H_B(D'_1)) \bigcap h^*(r(H_B(D'_2))) = \{0\}.$$

Here is the main result.

Theorem 4

1) $\lambda_L = O(\frac{1}{L^2})$ as $L \to \infty$,

2) if (D'_1, D'_2) is a regular transversal pair, then for any function $\gamma(L) = o(\frac{1}{L^2})$ as $L \to \infty$, there exists $L_0 > 0$ such that when $L > L_0$, we have

$$\lambda_L > \gamma(L).$$

In particular, D'_L is invertible for large L.

We first introduce some notation. Let λ_i, $i \in \mathbf{Z}$ denote the eigenvalues of the operator A, and u_i denote the corresponding eigensections. Set $\mu = \inf_{\lambda_i \neq 0} |\lambda_i|$. For simplicity, we omit the subscript L if no confusion is caused.

Lemma 5

There exist $L_0 > 0$ and $M > 1$ with the following significance. Assume that ψ and c satisfy $D'\psi = c\psi$ with $\psi \neq 0$ and $|c| < \delta(\mu)$ for some small $\delta(\mu)$, then ψ can be rescaled so that $||\psi||_{C^0(X_L)} < M$ and one of the following conditions holds:

- *either $\int_{X_1(L_0)} |\psi|^2$ or $\int_{X_2(L_0)} |\psi|^2$ is equal to one,*
- *either $||\psi||_{L^2(Y_1)}(L_0)$ or $||\psi||_{L^2(Y_2)}(L_0)$ is greater than or equal to one.*

Here $X_i(L_0) = X_i \setminus (L_0, \infty) \times Y_i$, $i = 1, 2$.

Proof: Let Π_1, Π_2 be the L^2-orthogonal projection onto $Ker\ A$ and $(Ker\ A)^\perp$. On the cylindrical neck of X_L, write $\psi = f_1 + f_2$ where $f_1 \in Ker\ A$ and $f_2 \in (Ker\ A)^\perp$. Set $\xi(t) = \int_Y |f_2|^2$.

Direct computation shows that

$$\begin{aligned}
\frac{\partial f_1}{\partial t} &= I\Pi_1 P\psi - cI(f_1) \\
\frac{\partial f_2}{\partial t} &= -Af_2 + I\Pi_2 P\psi - cI(f_2) \\
\frac{\partial^2 f_2}{\partial t^2} &= (A^2 - c^2)f_2 + IA\Pi_2 P\psi + I\Pi_2 \frac{\partial P}{\partial t}\psi + I\Pi_2 P\frac{\partial \psi}{\partial t} + c\Pi_2 P\psi.
\end{aligned}$$

For any $\epsilon > 0$, there exists $L_0 > 0$ such that on the neck $[L_0, 2L + 1 - L_0] \times Y_1$ we have

$$\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} &\geq 2\int_Y (\frac{\partial^2 f_2}{\partial t^2}, f_2) \\ &\geq K(\mu^2 \|f_2\|^2_{L^2_1(Y)} - \epsilon \|f_2\|_{L^2_1(Y)}(\|f_1\|_{L^2(Y)} + \|f_2\|_{L^2(Y)})) \end{aligned}$$

for some constant K. Here $|c| < \delta(\mu)$ for some small $\delta(\mu)$. If $\xi(t)$ reaches its maximum in an interior point $t_0 \in (L_0, 2L+1-L_0)$, then on the neck, we have

$$\max \|f_1\|_{L^2(Y)} \geq \|f_1\|_{L^2(Y)}(t_0) \geq \frac{\mu^2 - \epsilon}{\epsilon} \max \|f_2\|_{L^2(Y)}.$$

Otherwise, $\xi(t) = \|f_2\|^2_{L^2(Y)}$ reaches its maximum at the end points.

On the other hand, we have on the neck that

$$\begin{aligned} \tfrac{\partial f_1}{\partial t} + cI(f_1) &= I\Pi_1 P\psi \\ \tfrac{\partial (If_1)}{\partial t} - c(f_1) &= -\Pi_1 P\psi. \end{aligned}$$

Set $C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, then

$$\begin{pmatrix} f_1 \\ If_1 \end{pmatrix}(t) = e^{-Ct}\int_{L_0}^t e^{Cs} \begin{pmatrix} I\Pi_1 P\psi \\ -\Pi_1 P\psi \end{pmatrix} ds + e^{-C(t-L_0)} \begin{pmatrix} f_1(L_0) \\ If_1(L_0) \end{pmatrix}.$$

This implies that on the interval $[L_0, 2L+1-L_0]$

$$\|f_1\|_{L^2(Y)}(t) \leq c_1 e^{-\delta(L_0 - T_0)}(\max \|f_1\|_{L^2(Y)} + \max \|f_2\|_{L^2(Y)}) + \|f_1(L_0)\|_{L^2(Y)}.$$

If $||f_2||_{L^2(Y)}$ reaches its maximum in the interior, then

$$\max ||f_1||_{L^2(Y)} \leq 2||f_1(L_0)||_{L^2(Y)}$$

for large enough L_0. If $||f_2||_{L^2(Y)}$ reaches its maximum at the end points, assuming that it is the left end point without loss of generality, we have

$$\max(||f_1||_{L^2(Y)} + ||f_2||_{L^2(Y)}) \leq 2(||f_1(L_0)||_{L^2(Y)} + ||f_2(L_0)||_{L^2(Y)})$$

for large enough L_0. Lemma 5 follows easily from these estimates. □

The Proof of Theorem 4:

1). Pick $\phi \in Ker\ A$ with $||\phi||_{L^2(Y)} = 1$. Let ρ_L be a cut-off function which equals to one on $[\frac{3L}{4}, \frac{3L}{4} + \frac{L}{2} + 1]$ and equals to zero outside $[\frac{L}{2}, \frac{L}{2} + L + 1]$ with $|\nabla \rho_L| = O(\frac{1}{L})$. Then

$$\int_{X_L} |D'_L(\rho_L \phi)|^2 \leq \int_{X_L} |\nabla \rho_L|^2 |\phi|^2 + \int_{X_L} |P_L(\rho_L \phi)|^2 = O(\frac{1}{L}), \text{and} \int_{X_L} |\rho_L \phi|^2 \geq \frac{L}{10}.$$

So $\lambda_L = O(\frac{1}{L^2})$ as $L \to \infty$.

2). Suppose that there exists a sequence of $L_n \to \infty$ such that $\lambda_{L_n} \leq \gamma(L_n)$. Then there exist ψ_n, c_n such that $D'_{L_n}\psi_n = c_n \psi_n$ with $c_n^2 = \lambda_{L_n}$. By Lemma 5, there exist $\psi_1 \in H_B(D'_1)$, $\psi_2 \in H_B(D'_2)$ such that a subsequence of ψ_n converges to ψ_1 over X_1 and ψ_2 over X_2 in C^k norm on any compact subset. Note that one of ψ_1 and ψ_2 is nonzero. Part 2 of Theorem 4 follows if we show that $r(\psi_1) = h^* r(\psi_2)$. But this follows from the fact that if we write $\psi = f_1 + f_2$ as in Lemma 5,

$$\begin{aligned}\|f_1(t) - f_1(2L+1-t)\|_{L^2(Y)} \;\leq\; & C(e^{-\delta t} + |\cos(c(2L+1-2t)) - 1| \\ & +|\sin(c(2L+1-2t))|),\end{aligned}$$

for large enough t and L. C is some constant independent of t and L.

The Proof of Lemma 2:

Suppose $\psi \in \Gamma(E)$ and $D'\psi = 0$. On the cylindrical end $(T_0, \infty) \times Y$, write $\psi = \sum_i f_i u_i$ where u_i are the eigensections of the operator A corresponding to eigenvalues λ_i, and f_i are smooth functions in t. Then we have

$$\frac{\partial f_i}{\partial t} + \lambda_i f_i = (IP(t)\psi, u_i).$$

Set $g_i = (IP(t)\psi, u_i)$, then $\sum_i g_i^2 = \|P\psi\|^2_{L^2(Y)}$ and

$$f_i(t) = \int_{T_0}^{t} e^{-\lambda_i(t-s)} g_i(s)ds + f_i(T_0)e^{-\lambda_i(t-T_0)}.$$

Now assume that $\psi \in L^2_{-\gamma}$ for any small enough $\gamma > 0$. Assume that $\delta_1 < \min(\frac{\delta}{2}, \frac{\mu}{4})$ where $\mu = \inf_{\lambda_i \neq 0} |\lambda_i|$.

- For $\lambda_i = 0$, we have for any $t' > t$,

$$e^{\delta_1 t}|f_i(t') - f_i(t)| \leq C(\int_t^{t'} \int_Y e^{-\frac{\delta}{10}s}|\psi|^2 Vol_Y ds)^{\frac{1}{2}},$$

  so $f_i(\infty) = \lim_{t\to\infty} f_i(t)$ exists and $f_i - f_i(\infty) \in L^2_{\delta_1}$.

- For $\lambda_i > 0$, we have for some constant $C(\mu)$ that

$$e^{2\delta_1 t}(\sum_i f_i^2(t)) \leq C(\mu) \int_{T_0}^{\infty} e^{2\delta_1 s}(\sum_i g_i^2(s))ds + (\sum_i f_i^2(T_0))e^{2\delta_1 T_0}.$$

- For $\lambda_i < 0$. First of all, we have

$$f_i(t) = -e^{-\lambda_i t} \int_t^{\infty} e^{\lambda_i s} g_i(s) ds$$

since $\psi \in L^2_{-\gamma}$ for any small enough $\gamma > 0$. On the other hand, for some constant $C(\mu)$, we have

$$e^{2\delta_1 t}(\sum_i f_i^2(t)) \leq C(\mu) \int_t^{\infty} e^{2\delta_1 s}(\sum_i g_i^2(s))ds.$$

Take $r(\psi) = \sum_i f_i(\infty) u_i$ where $u_i \in Ker\ A$, then

$$||\psi - \beta\pi^* r(\psi)||_{L^2_{\delta_1}(E)} < \infty$$

where β is a fixed cut-off function which is equal to one at ∞, and $\pi : (-1,\infty)\times Y \to Y$ is the natural projection. As for $\dim H_B(D') - \dim H_{L^2}(D') = \frac{1}{2}\dim Ker\ A$, it follows from Theorem 7.4 in [21].

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