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POLYHEDRAL HOMOTOPY AND ITS APPLICATIONS TO POLYNOMIAL SYSTEM SOLVING

By

Hwee Hoon Chung

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ABSTRACT

POLYHEDRAL HOMOTOPY AND ITS APPLICATIONS TO POLYNOMIAL SYSTEM SOLVING

By

Hwee Hoon Chung

In this dissertation, we use the polyhedral homotopy to solve polynomial systems. Polyhedral homotopy is based on Bernshtein's Theory, which states that for a polynomial system with generic coefficients, the number of isolated zeros in the algebraic torus $(\mathbb{C}^*)^n$, counting multiplicities, is equal to the mixed volume of its supports. In the past, the construction of the start system of the homotopy continuation method is always based on the total degree of the original system. For polyhedral homotopy, cells of the right type provide a binomial start system which can be used to solve a polynomial system with the same monomials as the given polynomial system but with randomly chosen coefficients. This system is then used as the start system to solve the original polynomial system. This homotopy is particularly efficient for solving systems with no special structure.

An important merit of this work is its practical application in solving algebraic systems in robot kinematics, chemical reactions, engineering, economic modelling,

neural network, computer vision and many others.

Our solver finds all isolated solutions to satisfactory accuracy and speed; furthermore, its generality should establish itself as the method of choice for systems of moderate size.

To my family

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CHAPTER 1

Introduction

Let $P(x) = (p_1(x), \ldots, p_n(x)) = 0$ with $x = (x_1, ..., x_n)$ be a system of n polynomial equations in n unknowns. We want to find all *isolated* solutions of the system

$$\begin{aligned} p_1(x_1, ..., x_n) &= 0 \\ &\vdots \\ p_n(x_1, ..., x_n) &= 0. \end{aligned}$$

In order to approximate all isolated solutions of $P(x) = 0$, we look for a *homotopy* $H : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ that starts with a trivial (i.e., easily solved) set of polynomial equations

$$\begin{aligned} q_1(x_1, ..., x_n) &= 0 \\ &\vdots \\ q_n(x_1, ..., x_n) &= 0. \end{aligned}$$

Let $Q(x)$ denote $(q_1(x), \ldots, q_n(x))$. We then follow the curves in the real variable t which make up the solution set of

$$H(x,t) = (1-t)Q(x) + tP(x) = 0. \tag{1.1}$$

Note that $H(x,0) = Q(x)$ and $H(x,1) = P(x)$.

If $Q(x) = 0$ is chosen appropriately, then the following three properties should hold:

(1) **Triviality**: The solutions of $Q(x) = 0$ are known.

(2) **Smoothness**: The solution set of $H(x,t) = 0$ for $0 \leq t < 1$ consists of a finite number of smooth paths, each parameterized by t in [0,1).

(3) **Accessibility**: Every isolated solution of $H(x,1) = P(x) = 0$ can be reached by some path originating from $t = 0$. It follows that this path starts at a solution of $H(x,0) = Q(x) = 0$.

If the three properties hold, the solution paths can be followed from the initial points at $t = 0$ to all solutions of the original problem $P(x) = 0$ at $t = 1$ using standard numerical techniques [1, 2].

Drexler [9], Garcia and Zangwill [15] independently and almost simultaneously started to solve polynomial systems by the Homotopy Continuation Method. A few years later, their homotopies were superseded by that constructed by T. Y. Li [25]. It was shown (via the use of generalized Sard's Theorem and implicit function theorem) that with the following choice of $Q(x)$:

$$q_1(x) \quad = \quad a_1 x_1^{d_1} - b_1$$

$$(1.2) \qquad \vdots$$

$$q_n(x) = a_n x_n^{d_n} - b_n$$

where $d_1, ..., d_n$ are the degrees of $p_1(x), ..., p_n(x)$ respectively and a_i, b_i are random complex numbers, that the three properties hold. Rewrite the homotopy (1.1) as

$$(1.3) \qquad H(x(t), t) = 0.$$

Differentiating (1.3) with respect to t, we have

$$H_x \frac{dx}{dt} + H_t = 0$$

and

$$(1.4) \qquad \frac{dx}{dt} = -H_x^{-1} H_t$$

where H_x, H_t are partial derivatives of H with respect to x and t respectively. The curves $x(t)$ are the integral solutions of the initial value problems of the ordinary differential equation (1.4) with $x(0) = x_0$. The total number of curves of $H^{-1}(0)$ produced by the choice of $Q(x)$ in (1.2) is equal to $d = d_1 \cdots d_n$, the *total degree* of the system $P(x)$. The curves emanating from $t = 1$ must merge with one of those that start from $t = 0$. Thus, by following all the curves of $H^{-1}(0)$, we obtain all the isolated zeros of $P(x)$, the polynomial system that we want to solve.

Polynomial systems arise in diverse areas such as robot kinematics, chemical reactions, engineering, economic modelling, neural network, computer vision and many others. Very often, we encounter deficient polynomial systems with the actual root count being less than the total degree $d = d_1 \cdots d_n$ of the system. In fact, a great majority of the systems in application have only a small fraction of the total degree number of solutions. The following examples illustrate the point.

Example 1.0.1 Consider the generalized eigenvalue problem: $Ax = \lambda x$, where $A \in \mathbb{C}^{n \times n}$. We wish to compute the eigenvalues λ and eigenvectors $x = (x_1, \ldots, x_n)$. This reduces to solving a system of $n+1$ equations in $n+1$ variables $(\lambda, x_1, \ldots, x_n)$: $\sum_j a_{ij}x_j - \lambda x_i = 0$; $\sum_i x_i = 1$ for $i = 1, \ldots, n$. There are at most n isolated solutions to this system, with total degree 2^n. Homotopy method, using $Q(x)$ in (1.2) as the start system, requires the tracing of 2^n curves but at most n of them will converge to the target solutions when $t \to 1$.

Example 1.0.2 The Cassou-Nogues System

$$
\begin{aligned}
15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e - \\
28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120 = 0 \\
30b^4c^3d - 32cde^2 - 720b^2cd - 24b^2c^3e - 432b^2c^2 + 576ce - \\
576de + 16b^2cd^2e + 16d^2e^2 + 16c^2e^2 + 9b^4c^4 + 39b^4c^2d^2 + \\
18b^4cd^3 - 432b^2d^2 + 24b^2d^3e - 16b^2c^2de - 240c + 5184 = 0 \\
216b^2cd - 162b^2d^2 - 81b^2c^2 + 1008ce - 1008de + \\
15b^2c^2de - 15b^2c^3e - 80cde^2 + 40d^2e^2 + 40c^2e^2 + 5184 = 0 \\
4b^2cd - 3b^2d^2 - 4b^2c^2 + 22ce - 22de + 261 = 0
\end{aligned}
$$

The total degree of this system is $7 \times 8 \times 6 \times 4 = 1344$. This means that if we were to use the system $Q(x)$ in (1.2) as the start system, we need to trace 1344 curves. However, this system only has 16 isolated solutions. Thus sending 1344 curves out in search for the 16 solutions represents wasted computations.

The existence of extraneous paths, i.e., those paths that diverge to infinity as $t \to 1$, represents wasted computations and substantially limits the power of the method in most occasions. The choice of $Q(x)$ in (1.2) to solve the system $P(x) = 0$ requires an amount of computational effort proportional to $d = d_1 \cdots d_n$ and roughly, proportional to the size of the system. We would like to derive methods for solving deficient systems for which the computational effort is instead proportional to the actual number of solutions.

In this dissertation, we propose to use the polyhedral homotopy, where the choice of the start system $Q(x)$ is based on the Bernshtein's theory [3] and the number of homotopy curves that needed to be traced is greatly reduced in the case of deficient systems. This homotopy is particularly efficient for solving polynomial systems with no special structures.

CHAPTER 2

Polyhedral Homotopy

2.1 Notations

For a system of polynomials $P(x) = (p_1(x), \ldots, p_n(x))$ with $x = (x_1, \ldots, x_n)$, write

$$p_i(x) = \sum_{a \in \mathcal{A}_i} c_{i,a} x^a, \quad i = 1, \ldots, n,$$

where $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $c_{i,a} \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and $x^a = x_1^{a_1} \cdots x_n^{a_n}$. Here $\mathcal{A}_i$, a finite subset of $\mathbb{Z}^n$, is called the *support* of $p_i(x)$ and its convex hull, denoted by $\mathcal{Q}_i = \text{conv}(\mathcal{A}_i)$, is called the *Newton polytope* of $p_i(x)$. We call $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the *support* of $P(x)$.

The *Minkowski sum* of polytopes $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ is defined by

$$\mathcal{Q}_1 + \cdots + \mathcal{Q}_n = \{a_1 + \cdots + a_n \mid a_1 \in \mathcal{Q}_1, \ldots, a_n \in \mathcal{Q}_n\}.$$

The n-dimensional Euclidean volume of the polytope $\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n$ with nonnegative variables $\lambda_1, \ldots, \lambda_n$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ of degree n [48]. The *mixed volume* of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, denoted by $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, is defined to be the coefficient of $\lambda_1 \cdots \lambda_n$ in this polynomial. Equivalently, using the principle of inclusion and exclusion, the mixed volume can be formulated as

$$(2.1)\quad \mathcal{M}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (-1)^{n-1} \sum \mathrm{vol}(\mathcal{Q}_i) + (-1)^{n-2} \sum_{i<j} \mathrm{vol}(\mathcal{Q}_i + \mathcal{Q}_j) + \cdots + \mathrm{vol}(\mathcal{Q}_1 + \cdots + \mathcal{Q}_n).$$

The polyhedral homotopy is based on the following theorem:

Theorem 2.1.1 (Bernshtein) *The number of isolated zeros in $(\mathbb{C}^*)^n$, counting multiplicities, of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ is bounded above by the mixed volume $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. For a generic choice of coefficients, the system $P(x) = 0$ has exactly $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ isolated zeros in $(\mathbb{C}^*)^n$.*

The root count in the above theorem was discovered by Bernshtein [3], Khovanskii [20] and Kushnirenko [21] and is sometimes referred to as the BKK bound. While this bound is, in general, significantly sharper than the variant Bézout numbers, its apparent limitation is that it only counts the zeros of $P(x)$ in the algebraic torus $(\mathbb{C}^*)^n$. Root count in $\mathbb{C}^n$ via mixed volumes was first attempted in [47] and an upper bound was derived by introducing the notion of a *shadowed set*. Later, a significantly much tighter bound was given in the following theorem:

Theorem 2.1.2 [29] *The number of isolated zeros in $\mathbb{C}^n$, counting multiplicities, of a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is bounded above by the mixed volume $\mathcal{M}(\mathcal{A}_1 \cup \{0\}, \ldots, \mathcal{A}_n \cup \{0\})$.*

Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a sequence of finite subsets of $\mathbb{Z}^n$ whose union affinely spans $\mathbb{R}^n$. By a *cell* of $\mathcal{A}$ we mean a tuple $C = (C^{(1)}, \ldots, C^{(n)})$ of subsets $C^{(i)} \subset \mathcal{A}_i$, for $i = 1, \ldots, n$. For a cell $C = (C^{(1)}, \ldots, C^{(n)})$, define

$$\begin{aligned} \mathrm{type}(C) &= (\dim(\mathrm{conv}(C^{(1)})), \ldots, \dim(\mathrm{conv}(C^{(n)}))), \\ \mathrm{conv}(C) &= \mathrm{conv}(C^{(1)}) + \cdots + \mathrm{conv}(C^{(n)}), \\ \text{and} \quad \mathrm{vol}(C) &= \mathrm{vol}(\mathrm{conv}(C)). \end{aligned}$$

A *face* of C is a subcell $F = (F^{(1)}, \ldots, F^{(n)})$ of C where $F^{(i)} \subset C^{(i)}$ and some linear functional $\alpha \in (\mathbb{R}^n)^*$ attains its minimum over $C^{(i)}$ at $F^{(i)}$, for $i = 1, \ldots, n$. We call such an α an *inner normal* of F. If F is a face of C, then $\mathrm{conv}(F^{(i)})$ is a face of the polytope $\mathrm{conv}(C^{(i)})$ for $i = 1, \ldots, n$.

Definition 2.1.3 *A fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a collection $S = \{C_1, ..., C_m\}$ of cells such that*

(a) $\dim(\mathrm{conv}(C_i)) = n$ for all $i = 1, \ldots, m$,

(b) $\mathrm{conv}(C_i) \bigcap \mathrm{conv}(C_j)$ is a proper common face of both $\mathrm{conv}(C_i)$ and $\mathrm{conv}(C_j)$, whenever the intersection is nonempty for $i \neq j$,

(c) $\bigcup_{i=1}^m \mathrm{conv}(C_i) = \mathrm{conv}(\mathcal{A})$.

For $i = 1, \ldots, m$, we write $C_i = (C_i^{(1)}, ..., C_i^{(n)})$, and

(d) $\dim(\mathrm{conv}(C_i^{(1)})) + \cdots + \dim(\mathrm{conv}(C_i^{(n)})) = n$,

(e) $(\#(C_i^{(1)}) - 1) + \cdots + (\#(C_i^{(n)}) - 1) = n$.

A fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_n)$ can be found by the following process: Choose a real-valued function $\omega_i : \mathcal{A}_i \to \mathbb{R}$ for each $i = 1, \ldots, n$. We call the $n-$tuple $\omega = (\omega_1, ..., \omega_n)$ a lifting function on $\mathcal{A}$ and ω_i lifts $\mathcal{A}_i$ to its graph $\hat{\mathcal{A}}_i = \{(a, \omega_i(a)) : a \in \mathcal{A}_i\} \subset \mathbb{R}^{n+1}$. This notation is extended in the obvious way: $\hat{\mathcal{A}} = (\hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_n)$, $\hat{\mathcal{Q}}_i = \text{conv}(\hat{\mathcal{A}}_i)$, $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}_1 + \ldots + \hat{\mathcal{Q}}_n$, etc.

Let $S_\omega = \{C_1, \ldots, C_m\}$ be the set of cells of $\mathcal{A}$ which satisfy for $1 \leq j \leq m$,

(a) $\dim(\text{conv}(\hat{C}_j)) = n$,

(b) $\text{conv}(\hat{C}_j)$ is a facet, i.e., an n-dimensional face, of $\hat{\mathcal{Q}} = \text{conv}(\hat{\mathcal{A}})$ whose inner normal $\alpha \in (\mathbb{R}^{n+1})^*$ has positive last coordinate. In other words, $\text{conv}(\hat{C}_j)$ is a facet in the lower hull of $\hat{Q}$.

If the lifting function ω is chosen generically, then S_ω is a fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ induced by ω [13, 22].

Let $S = \{C_1, \ldots, C_m\}$ be a fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, then the cell C_j of type $(1, \ldots, 1)$ is a sequence of subsets $(C_j^{(1)}, \ldots, C_j^{(n)})$ where each $C_j^{(i)} = \{a_{i0}, a_{i1}\}$ is a 2-element subset of $\mathcal{A}_i$ for $1 \leq i \leq n$. Let $V(C_j)$ be the $n \times n$-matrix whose rows are $a_{i1} - a_{i0}$, $1 \leq i \leq n$. It can be shown that

$$\text{vol}(C_j) = |\det(V(C_j))|.$$

The following is a special case of Theorem 2.4 of [16].

Theorem 2.1.4 *Let* $S = \{C_1, ..., C_m\}$ *be a fine mixed subdivision of* $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. *Then*

$$\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \sum \text{vol}(C_i) = \sum |\det(V(C_i))|,$$

where the summation is taken over all cells C_i of type $(1, \ldots, 1)$ in S.

2.2 Algebraic deformations

In order to find all isolated zeros of a given polynomial system $P(x) = (p_1(x), ..., p_n(x))$ in $\mathbb{C}^n$, we apply the following procedure: According to Theorem 2.1.2, if all p_i's have constant terms, then the mixed volume is in fact, an upper bound for the number of isolated zeros of $P(x)$ in $\mathbb{C}^n$; otherwise we augment the monomial x^0 to those p_i's that do not have constant terms. We assign generic coefficients to all the monomials in $P(x)$. Denote the new system by $Q(x) = (q_1(x), \ldots, q_n(x))$ and let $\mathcal{A}' = (\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$ be its support, so

$$q_i(x) = \sum_{a' \in \mathcal{A}'_i} \bar{c}_{i,a'} x^{a'}, \quad i = 1, ..., n,$$

where $a' = (a'_1, \ldots, a'_n)$ and $x^{a'} = x_1^{a'_1} \cdots x_n^{a'_n}$. Consider the linear homotopy

$$H(x, t) = (1 - t)Q(x) + tP(x) = 0.$$

We wish to obtain zeros of $P(x)$ in $\mathbb{C}^n$ at $t = 1$ by following the solution curves of $H(x, t) = 0$ emanating from the zeros of $Q(x)$ in $\mathbb{C}^n$ at $t = 0$. By the following lemma, the zeros of $Q(x)$ in $\mathbb{C}^n$ are isolated, nonsingular and are contained in $(\mathbb{C}^*)^n$.

Lemma 2.2.1 [29] *Given a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ in the variables $x = (x_1, \ldots, x_n)$, there exist an open dense subset V of $\mathbb{C}^n$ such that if*

$\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in V$, *then* $t\epsilon \in V$ *for all* $t \neq 0$ *in* $\mathbb{R}$, *and if* $y \in \mathbb{C}^n$ *is a solution of*

$$\begin{aligned} q_1 = p_1(x_1, \ldots, x_n) + \epsilon_1 &= 0 \\ &\vdots \\ q_n = p_n(x_1, \ldots, x_n) + \epsilon_n &= 0, \end{aligned}$$

then

$$\text{rank}\frac{\partial(q_1, \ldots, q_n)}{\partial(x_1, \ldots, x_n)}(y) = n \quad \textit{and} \quad y \in (\mathbb{C}^*)^n,$$

where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

In Section 2.2.1, we will employ the polydedral homotopy to find these isolated zeros of $Q(x)$ in $(\mathbb{C}^*)^n$. We will see in Section 2.2.2 that every isolated zero of $P(x)$ in $\mathbb{C}^n$, at $t = 1$, can be reached by a solution curve of $H(x, t) = 0$ emanating from an isolated zero of $Q(x)$ in $(\mathbb{C}^*)^n$, at $t = 0$, obtained in Section 2.2.1.

2.2.1 The nonlinear homotopy

We solve $Q(x) = (q_1(x), \ldots, q_n(x)) = 0$ by lifting its support $\mathcal{A}' = (\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$ by a generically chosen lifting function $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i : \mathcal{A}'_i \to \mathbb{R}$ for $i = 1, \ldots, n$.

Consider the polynomial system $\hat{Q}(x, t) = (\hat{q}_1(x, t), \ldots, \hat{q}_n(x, t))$ in the $n + 1$ variables $x_1, \ldots, x_n, t$ where

$$\hat{q}_i(x,t) = \sum_{a' \in \mathcal{A}'_i} \bar{c}_{i,a'} x^{a'} t^{\omega_i(a')}, \quad i = 1, ..., n. \tag{2.2}$$

For $i = 1, \ldots, n$, we write $\mathcal{A}'_i = \{a_1^{(i)}, \ldots, a_{k(i)-1}^{(i)}, 0\}$ and $\sum_{i=1}^n k_i = N,\ N \in \mathbb{Z}$. Let $b = (b_1, \ldots, b_n)$ where b_i represents the constant term of $q_i(x)$ for $i = 1, \ldots, n$, and $c = (c_1, \ldots, c_N) = (\bar{c}_{1,a_1^{(1)}}, \ldots, \bar{c}_{1,a_{k(1)-1}^{(1)}}, b_1, \ldots, \bar{c}_{n,a_1^{(n)}}, \ldots, \bar{c}_{n,a_{k(n)-1}^{(n)}}, b_n) \in \mathbb{C}^N$ be all the corresponding coefficients of $\hat{Q}(x,t)$.

$\hat{Q}(x,t)$ provides a homotopy with t as the parameter and $\hat{Q}(x,1) = Q(x)$. The lifting function $\omega = (\omega_1, \ldots, \omega_n)$ induces a fine mixed subdivision S_ω of $\mathcal{A}' = (\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$. By projecting the facets in the lower hull of $\hat{\mathcal{Q}}$ down, we obtain cells of type $(1, \ldots, 1)$ in the fine mixed subdivision S_ω. By Theorem 2.1.4, the mixed volume $\mathcal{M}(\mathcal{A}'_1, ..., \mathcal{A}'_n)$ is equal to the sum of the volumes of these cells.

Proposition 2.2.2 *For almost every choice of constant terms of $Q(x)$, 0 is a regular value of $\hat{Q}(x,t)$ on $(\mathbb{C}^*)^n \times (0,1]$.*

PROOF: Consider $\hat{Q}(b,x,t) : \mathbb{C}^n \times (\mathbb{C}^*)^n \times (0,1] \to (\mathbb{C}^*)^n$ where b here is also regarded as a variable of $\hat{Q}$. The Jacobian matrix of $\hat{Q}$ with respect to b (the constant terms of $Q(x)$), denoted by $D_b\hat{Q}$, is of rank n in $(\mathbb{C}^*)^n$ for $t \in (0,1]$. This implies that 0 is a regular value of $\hat{Q}$ on $\mathbb{C}^n \times (\mathbb{C}^*)^n \times (0,1]$. It follows from generalized Sard's Theorem that for almost every choice of constant terms b, 0 is a regular value of $\hat{Q}(b,\cdot,\cdot)$ on $(\mathbb{C}^*)^n \times (0,1]$. $\square$

As a consequence of Proposition 2.2.2, for $t \in (0,1]$, all isolated zeros of $\hat{Q}(x,t)$ are nonsingular.

The system $Q(x)$ is said to be in *general position* if its coefficients c satisfy $G(c) \neq 0$, for $G(y) = (g_1(y), \ldots, g_m(y))$, where $\{g_1(y), \ldots, g_m(y)\}$ is a set of polynomals determined by the monomials of $Q(x)$.

Proposition 2.2.3 *For all $t \in (0,1]$, the system $\hat{Q}(x,t)$ is in general position.*

PROOF: Let $G(y) = (g_1(y), \ldots, g_m(y))$ be the polynomial system determined by the monomials of $Q(x)$ in the definition of $Q(x)$ being in general position, and $Z = \{y \in \mathbb{C}^N \mid G(y) = 0\}$. The Lebesgue measure of Z is zero and $\mathbb{C}^N \backslash Z$ is open and dense. Since the coefficients of $Q(x)$ are randomly chosen, $c \notin Z$ and thus the system $\hat{Q}(x,1) = Q(x)$ is in general position. For $j = 1, \ldots, N$, let γ_j be the power of t associated with the term in $\hat{Q}(x,t) = (\hat{q}_1(x,t), \ldots, \hat{q}_n(x,t))$ with c_j as coefficient. Write $\gamma = (\gamma_1, \ldots, \gamma_N)$.

For each fixed $t \neq 0$, the support of $\hat{Q}(x,t)$ is the same as that of $Q(x)$. Since $G(c) \neq 0$, there exists i_0 such that $g_{i_0}(c) \neq 0$, which implies $g_{i_0}(ct^\gamma) \not\equiv 0$ where ct^γ represents $(c_1 t^{\gamma_1}, \ldots, c_N t^{\gamma_N})$. If $\gamma \in \mathbb{Z}^N$, then $g_{i_0}(ct^\gamma)$ is a nontrivial polynomial in t. Thus, there are only finitely many t such that $g_{i_0}(ct^\gamma) = 0$. This implies that $G(ct^\gamma)$ has only finitely many zeros. If $\gamma \in \mathbb{Q}^N$, then factor out the reciprocal of the least common multiple of the denominator of the γ_i's from the polynomials in $G(ct^\gamma)$. This results in a set of nontrivial polynomials in t and we again conclude that $G(ct^\gamma)$, considered as a system of polynomials in t, can only have finitely many zeros.

Let $\{t_1, \ldots, t_\delta\}$ be the set of values of t in $\mathbb{C}$ such that $G(ct^\gamma) = 0$. Write $t_s = r_s e^{i\theta_s}$ $(1 \leq s \leq \delta)$. For $t = t_s$, $G(ct^\gamma)$ becomes

$$
\begin{aligned}
g_1(c_1 r_s^{\gamma_1} e^{i\gamma_1\theta_s}, \ldots, c_N r_s^{\gamma_N} e^{i\gamma_N\theta_s}) &= 0 \\
&\vdots
\end{aligned}
$$

$$g_m(c_1 r_s^{\gamma_1} e^{i\gamma_1\theta_s}, \ldots, c_N r_s^{\gamma_N} e^{i\gamma_N\theta_s}) \;=\; 0.$$

Thus as long as $\theta \neq \theta_s$, $\forall\ 1 \leq s \leq \delta$, then $G(c(e^{i\theta}t)^\gamma) = G((c(e^{i\theta})^\gamma)t^\gamma) \neq 0$, $\forall\ t \in (0,1]$. Hence, with the coefficients of $Q(x)$ being randomly chosen, the system $\hat{Q}(x,t)$ is in general position for all $t \in (0,1]$. $\Box$

By Proposition 2.2.3 and Theorem 2.1.1, for each fixed $t \in (0,1]$, $\hat{Q}(x,t)$ has $\mathcal{M}(\mathcal{A}_1', \ldots, \mathcal{A}_n')$ number of isolated zeros in $(\mathbb{C}^*)^n$. Thus there are $\mathcal{M}(\mathcal{A}_1', \ldots, \mathcal{A}_n')$ number of solution curves of $\hat{Q}(x,t) = 0$ in $(\mathbb{C}^*)^n \times (0,1]$. These curves can only diverge to infinity when $t \to 0$.

Let $S_\omega = \{C_1, \ldots, C_m\}$ and $C_j = (C_j^{(1)}, \ldots, C_j^{(n)})$ be a cell of type $(1, \ldots, 1)$ in S_ω. For $i = 1, \ldots, n$, let $C_j^{(i)} = \{a_{i0}', a_{i1}'\} \subset \mathcal{A}_i'$. Since S_ω is a fine mixed subdivision, the vectors $a_{11}' - a_{10}', \ldots, a_{n1}' - a_{n0}'$ are linearly independent. Thus

$$\mathrm{vol}(C_j)) = \left|\det \begin{bmatrix} a_{11}' - a_{10}' \\ \vdots \\ a_{n1}' - a_{n0}' \end{bmatrix}\right|$$

and $\mathrm{conv}(\hat{C}_j)$ is a facet of $\hat{\mathcal{Q}}' = (\hat{\mathcal{Q}}_1', \ldots, \hat{\mathcal{Q}}_n')$ whose inner normal $\hat{\alpha} \in (\mathbb{R}^{n+1})^*$ has positive last coordinate. Let $\hat{\alpha} = (\alpha, 1) = (\alpha_1, \ldots, \alpha_n, 1)$ be the inner normal of $\mathrm{conv}(\hat{C}_j) = \mathrm{conv}(\{\hat{a}_{10}', \hat{a}_{11}'\}, \ldots, \{\hat{a}_{n0}', \hat{a}_{n1}'\})$ where $\hat{a}_{il}' = (a_{il}', \omega_i(a_{il}'))$ for $i = 1, \ldots, n$ and $l = 0, 1$. Let $x(t)$ represent the general solution curves of $\hat{Q}(x,t) = 0$. With $x(t) = (x_1(t), \ldots, x_n(t))$, let

$$\begin{aligned} x_1(t) &= t^{\alpha_1} y_1(t) \\ &\vdots \end{aligned}$$

$$x_n(t) \;=\; t^{\alpha_n} y_n(t),$$

or, simply, $x(t) = t^{\alpha} y(t)$. Substituting this into (2.2) yields, for $i = 1, ..., n$,

$$\begin{aligned}
\hat{q}_i(y,t) &= \sum_{a' \in \mathcal{A}'_i} \bar{c}_{i,a'} y^{a'} t^{\langle \alpha, a' \rangle} t^{\omega_i(a')} \\
&= \sum_{a' \in \mathcal{A}'_i} \bar{c}_{i,a'} y^{a'} t^{\langle (\alpha,1),(a',\omega_i(a')) \rangle} \\
&= \sum_{a' \in \mathcal{A}'_i} \bar{c}_{i,a'} y^{a'} t^{\langle \hat{\alpha}, \hat{a'} \rangle}.
\end{aligned} \tag{2.3}$$

Let $\beta_i = \min_{a' \in \mathcal{A}'_i} \langle \hat{\alpha}, \hat{a}' \rangle$. Since $\text{conv}(\hat{C}_j)$ is a facet of $\hat{\mathcal{Q}}' = (\hat{\mathcal{Q}}'_1, \ldots, \hat{\mathcal{Q}}'_n)$ with inner normal $\hat{\alpha}$, $\text{conv}(\hat{C}_j^{(i)}) = \text{conv}(\{\hat{a}'_{i0}, \hat{a}'_{i1}\})$ is a face of $\hat{\mathcal{Q}}'_i$ and $\hat{\alpha} = (\alpha, 1)$ also serves as an inner normal of $\text{conv}(\hat{C}_j^{(i)})$ for each $i = 1, ..., n$. It follows that $\langle \hat{\alpha}, \hat{a}'_{i0} \rangle = \langle \hat{\alpha}, \hat{a}'_{i1} \rangle = \beta_i$ and $\langle \hat{\alpha}, \hat{a}' \rangle > \beta_i$ for $\hat{a}' \in \hat{\mathcal{A}}'_i \setminus \hat{C}_j^{(i)}$. Hence, factoring out t^{β_i} in (2.3), we have

$$\hat{q}_i(y,t) = t^{\beta_i} (\bar{c}_{i,a'_{i0}} y^{a'_{i0}} + \bar{c}_{i,a'_{i1}} y^{a'_{i1}} + \sum_{a' \in \mathcal{A}'_i \setminus C_j^{(i)}} \bar{c}_{i,a'} y^{a'} t^{\langle \hat{\alpha}, \hat{a'} \rangle - \beta_i}), \quad i = 1, \ldots, n.$$

Consider the homotopy $R(y,t) = (r_1(y,t), \ldots, r_n(y,t)) = 0$ where

$$r_i(y,t) = \bar{c}_{i,a'_{i0}} y^{a'_{i0}} + \bar{c}_{i,a'_{i1}} y^{a'_{i1}} + \sum_{a' \in \mathcal{A}'_i \setminus C_j^{(i)}} \bar{c}_{i,a'} y^{a'} t^{\langle \hat{\alpha}, \hat{a'} \rangle - \beta_i}, \; i = 1, \ldots, n. \tag{2.4}$$

We have,

$$\hat{q}_i(y,t) = t^{\beta_i} r_i(y,t), \; i = 1, \ldots, n. \tag{2.5}$$

Since $\langle \hat{\alpha}, \hat{a'} \rangle > \beta_i$ for $a' \in \mathcal{A}'_i \setminus C_j^{(i)}$, $i = 1, \ldots, n$, $R(y,0) = 0$ is the binomial system with generic coefficients:

$$
\begin{aligned}
\bar{c}_{1,a'_{10}} y^{a'_{10}} + \bar{c}_{1,a'_{11}} y^{a'_{11}} &= 0 \\
&\vdots \\
\bar{c}_{n,a'_{n0}} y^{a'_{n0}} + \bar{c}_{n,a'_{n1}} y^{a'_{n1}} &= 0.
\end{aligned} \tag{2.6}
$$

From (2.5), the zeros of $R(y,t)$ coincides with those of $\hat{Q}(y,t)$ for $t \neq 0$ and since $x(t) = t^{\alpha} y(t)$, $R(y,1) = \hat{Q}(x,1) = Q(x)$, and zeros of $R(y,t)$ at $t = 1$ are precisely the zeros of $Q(x)$. The system (2.6) has

$$
\left| \det \begin{bmatrix} a'_{11} - a'_{10} \\ \vdots \\ a'_{n1} - a'_{n0} \end{bmatrix} \right| (= \mathrm{vol}(C_j))
$$

solutions in $(\mathbb{C}^*)^n$. Thus by following the solution curves $(y(t),t)$ of $R(y,t) = 0$ starting from the solutions of $R(y,0) = 0$ in (2.6), we can obtain the vol(C_j) number of solutions of $Q(x) = 0$ in $(\mathbb{C}^*)^n$ at $t = 1$. According to Theorem 2.1.4, we can obtain all $\mathcal{M}(\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$ number of isolated zeros of $Q(x)$ in $(\mathbb{C}^*)^n$ if we repeat the same procedure for each cell of type $(1,\ldots,1)$ in S_ω.

2.2.2 The linear homotopy

Consider the linear homotopy

$$
H(x,t) = (1-t)Q(x) + tP(x) = 0. \tag{2.7}
$$

We wish to obtain zeros of $P(x) = (p_1(x), \ldots, p_n(x))$ in $\mathbb{C}^n$ at $t = 1$ by following the solution curves of $H(x,t) = 0$ emanating from the zeros of $Q(x) = (q_1(x), \ldots, q_n(x))$ in $\mathbb{C}^n$, which by Lemma 2.2.1, are contained in $(\mathbb{C}^*)^n$, at $t = 0$. Recall that the system $Q(x)$ is constructed from $P(x)$, with generic coefficients assigned to all monomials in $P(x)$, and have the monomial x^0 augmented to those p_i's that do not have constant terms. Thus, the linear homotopy (2.7) is essentially a special case of the Cheater's homotopy [27].

Theorem 2.2.4 [27] (The Cheater's homotopy) *Let $c = (c_1, \ldots, c_m) \in \mathbf{C}^{\mathbf{m}}$ and d be the total degree of $P(x) = (p_1(x), \ldots, p_n(x))$ with $x = (x_1, \ldots, x_n)$. There exists an open, dense, full-measure subset U of $\mathbf{C}^{\mathbf{n+m}}$ such that for $(b_1^*, \ldots, b_n^*, c_1^*, \ldots, c_m^*) \in U$, the following holds:*

(a) The set X^* of solutions $x = (x_1, \ldots, x_n)$ of

$$\begin{aligned} q_1(x_1, \ldots, x_n) = p_1(c_1^*, \ldots, c_m^*, x_1, \ldots, x_n) + b_1^* &= 0 \\ &\vdots \\ q_n(x_1, \ldots, x_n) = p_n(c_1^*, \ldots, c_m^*, x_1, \ldots, x_n) + b_n^* &= 0 \end{aligned}$$

consists of d_0 isolated points, for some $d_0 \leq d$.

(b) The smoothness and accessibility properties hold for the homotopy

$$H(x,t) = P((1-t)c_1^* + tc_1, \ldots, (1-t)c_m^* + tc_m, x_1, \ldots, x_n) + (1-t)b^*$$

where $b^* = (b_1^*, \ldots, b_n^*)$. It follows that every solution of $P(x) = 0$ is reached by a path beginning at a point of X^*.

To apply Theorem 2.2.4 to our situation, let $m = \sum_{i=1}^{n} k(i) - n$, $b_i^* = b_i$, $1 \leq i \leq n$ and $(c_1^*, \ldots, c_m^*)$ be the coefficients of $Q(x)$ excluding the constant terms. By construction, the coefficients $(b_1^*, \ldots, b_n^*, c_1^*, \ldots, c_m^*)$ of $Q(x)$ are in U. By part (b) of Theorem 2.2.4, every zero of $P(x)$ can be reached at $t = 1$, by a solution curve of $H(x,t) = 0$ emanating from a zero of $Q(x)$ at $t = 0$. Thus, by following all the solution curves of $H(x,t) = 0$ starting from the isolated zeros of $Q(x)$ in $(\mathbb{C}^*)^n$ at $t = 0$, we can obtain all the isolated zeros of $P(x)$ in $\mathbb{C}^n$ at $t = 1$.

CHAPTER 3

Algorithms

In chapter 2, we have seen that by following the homotopy paths of

$$H(x,t) = (1-t)Q(x) + tP(x) = 0,$$

starting from the zeros of $Q(x) = (q_1(x), \ldots, q_n(x))$ in $\mathbb{C}^n$ at $t = 0$, which by Lemma 2.2.1 are contained in $(\mathbb{C}^*)^n$, we may obtain the zeros of $P(x) = (p_1(x), \ldots, p_n(x))$, the polynomial system that we want to solve, in $\mathbb{C}^n$, at $t = 1$. To do so, we must first find the zeros of $Q(x)$ in $(\mathbb{C}^*)^n$. The method outlined in chapter 2 uses a generic lifting $\omega = (\omega_1, \ldots, \omega_n)$, and then projects the lower hull(i.e. union of facets whose inner normal α has positive last coordinate) of the lifted Newton polytope $\hat{Q}' = (\hat{Q}'_1, \ldots, \hat{Q}'_n)$ of the polynomial system $\hat{Q}(x,t)$, defined by (2.2), down to obtain cells of type $(1, \ldots, 1)$ in the induced fine mixed subdivision S_ω. In this chapter, we first of all discuss the details of the implementation of the procedure to obtain cells of type $(1, \ldots, 1)$ in the induced fine mixed subdivision S_ω, and secondly, solve the binomial system that corresponds to each such cell, and thirdly, give an outline of the homotopy curve tracing procedure.

3.1 Vertex-set algorithm

From the formula given by (2.1), the non-vertex points of the Newton polytope $\mathcal{Q}'_i = \text{conv}(\mathcal{A}'_i)$, $1 \leq i \leq n$, do not contribute to the mixed volume $\mathcal{M}(\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$ of the system $Q(x)$. To make our method more efficient, we intend to exclude those non-vertex points in the supports from further considerations in the first place. Deciding whether a point is a vertex of $\mathcal{Q}'_i$ reduces to a linear programming problem:

Let $\mathcal{A}'_i = \{a_1^{(i)}, \ldots, a_{k(i)}^{(i)}\}$, $1 \leq i \leq n$ and $\sum_{i=1}^{n} k(i) = N$. To test if $a_l^{(i)}$ $(1 \leq l \leq k(i))$ is a vertex of $\mathcal{Q}'_i$, we solve the following problem:

$$\text{minimize} \quad \mu$$

$$\text{subject to} \begin{cases} \sum_{j=1,\ j\neq l}^{k(i)} \lambda_j a_j^{(i)} + \mu a_l^{(i)} = a_l^{(i)} \\ \sum_{j=1,\ j\neq l}^{k(i)} \lambda_j + \mu = 1 \\ \forall \lambda_j \geq 0 \\ \mu \geq 0. \end{cases} \tag{3.1}$$

If (3.1) has an optimal solution with $\mu = 0$, then $a_l^{(i)}$ is not a vertex of $\mathcal{Q}'_i$, or $a_l^{(i)} \notin \text{vert}(\mathcal{Q}'_i)$, where $\text{vert}(\mathcal{Q}'_i)$ denotes the vertex set of $\mathcal{Q}'_i$. We can obtain $\text{vert}(\mathcal{Q}'_i)$ by repetitive applications of (3.1) as described in the following algorithm.

Algorithm VERTEX_SET

Input: $\mathcal{A}'_i = \{a_1^{(i)}, \ldots, a_{k(i)}^{(i)}\}$, $1 \leq i \leq n$

S1 Set $i = 1$.

S2 Set $\mathrm{vert}(\mathcal{Q}_i') = \{a_1^{(i)}, \ldots, a_{k(i)}^{(i)}\}$.

S3 Let j range from 1 to $k(i)$. If $a_j^{(i)} \notin \mathrm{vert}(\mathcal{Q}_i')$, then $\mathrm{vert}\mathcal{Q}_i' = \mathrm{vert}\mathcal{Q}_i' \backslash \{a_j^{(i)}\}$.

S4 Set $i = i + 1$. If $i \leq n$, goto **S2**.

3.2 Edge-set algorithm

To find cells of type $(1, \ldots, 1)$ in the fine mixed subdivision S_ω of $\mathcal{A}' = (\mathcal{A}_1', \ldots, \mathcal{A}_n')$ induced by the lifting ω, let $C_j = (C_j^{(1)}, \ldots, C_j^{(n)})$ be such a cell. Then for each i, $\mathrm{conv}(C_j^{(i)})$ is an edge (one-dimensional face) in the lower hull of the lifted polytope $\hat{\mathcal{Q}}_i'$ of $\hat{q}_i(x, t)$ $(1 \leq i \leq n)$. For a random lifting $\omega = (\omega_1, \ldots, \omega_n)$, we begin by constructing edge sets to contain edges that lie in the lower hull of $\hat{\mathcal{Q}}_i'$ $(1 \leq i \leq n)$.

After applying Algorithm VERTEX_SET, each polytope $\mathcal{Q}_i'$ $(1 \leq i \leq n)$ can be represented as $\mathrm{conv}(\{v_1^{(i)}, \ldots, v_{m_i}^{(i)}\})$, where for $1 \leq l \leq m_i$, $v_l^{(i)}$ are the vertices of $\mathcal{Q}_i'$ and m_i is the cardinality of the vertex set of $\mathcal{Q}_i'$. As an illustration, for the polytope $\mathcal{Q}_1'$, to test if the edge connecting $\hat{v}_1^{(1)}$ and $\hat{v}_2^{(1)}$ lies in the lower hull of $\hat{\mathcal{Q}}_1'$, we have the following constraints:

$$
\begin{cases} \hat{v}_1^{(1)}\hat{\alpha} = \hat{v}_2^{(1)}\hat{\alpha} \\ \hat{v}_1^{(1)}\hat{\alpha} \leq \hat{v}_{i_1}^{(1)}\hat{\alpha}, \; i_1 = 3, \ldots, m_1 \end{cases} \tag{3.2}
$$

where $\hat{\alpha} = (\alpha_1, \ldots, \alpha_n, 1)$ and $\alpha_1, \ldots, \alpha_n$ are the unknowns.

We use the equation in (3.2) to solve for one of the α_i's. Upon substituting the value of that α_i into the rest of the inequalities in (3.2), we obtain inequalities of the form

$$A\alpha \;\leq\; b, \tag{3.3}$$

where the number of variables(i.e., the number of components of α) is now one less than before. The solvability of the problem (3.2) is then equivalent to the feasibility of (3.3) which can be tested by solving the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & \epsilon \\ \text{subject to} & A\alpha - \epsilon e \leq b, \quad \epsilon > 0, \;\; \text{where } e = (1, \ldots, 1). \end{array} \tag{3.4}$$

If $\epsilon_0 = \min \epsilon$ is equal to zero, then (3.3) is feasible. Thus the edge connecting $\hat{v}_1^{(1)}$ and $\hat{v}_2^{(1)}$ lies in the lower hull of lifted polytope $\hat{Q}_1'$. We repeat the above procedure for all other possible forming of edges among vertices in Q_1' and subsequently those in Newton polytopes Q_k' $(2 \leq k \leq n)$.

Algorithm EDGE_SET

Input: $\text{vert}(Q_i') = \{v_1^{(i)}, \ldots, v_{m_i}^{(i)}\}, \; 1 \leq i \leq n$

S1 Set $i = 1$.

S2 Let s_1 range from 1 to $m_i - 1$ and s_2 from $s_1 + 1$ to m_i. If $s_1 < s_2$, then for each edge connecting $\hat{v}_{s_1}^{(i)}$ and $\hat{v}_{s_2}^{(i)}$ (vertices from Q_i'), conduct the feasibility test on the corresponding problem of the form (3.3). If the problem is feasible, then store the pair $(v_{s_1}^{(i)}, v_{s_2}^{(i)})$ in edge set E_i.

S3 Set $i = i + 1$. If $i \leq n$, then go to **S2**.

3.3 Algorithm for finding cells of type $(1, \ldots, 1)$

We construct cells of type $(1, \ldots, 1)$ by starting with an edge e_1 from $\mathcal{Q}'_1$ and adding one edge e_k from $\mathcal{Q}'_k$ $(2 \leq k \leq n)$ at a time. As each edge is added, the k-tuples $(e_1, \ldots, e_k)$ is tested on whether $\hat{e}_1 + \cdots + \hat{e}_k$ lies in the lower hull of $\hat{\mathcal{Q}}'_1 + \cdots + \hat{\mathcal{Q}}'_k$. (3.2) is modified to have k equations and more inequalities:

$$
(3.5) \qquad \begin{cases}
\hat{v}^{(1)}_{j_{11}} \hat{\alpha} = \hat{v}^{(1)}_{j_{12}} \hat{\alpha} \\
\hat{v}^{(2)}_{j_{21}} \hat{\alpha} = \hat{v}^{(2)}_{j_{22}} \hat{\alpha} \\
\vdots \\
\hat{v}^{(k)}_{j_{k1}} \hat{\alpha} = \hat{v}^{(k)}_{j_{k2}} \hat{\alpha} \\
\hat{v}^{(1)}_{j_{11}} \hat{\alpha} \leq \hat{v}^{(1)}_{j_{r_1}} \hat{\alpha}, \ 1 \leq r_1 \leq m_1, \ r_1 \neq j_{11}, j_{12} \\
\hat{v}^{(2)}_{j_{21}} \hat{\alpha} \leq \hat{v}^{(2)}_{j_{r_2}} \hat{\alpha}, \ 1 \leq r_2 \leq m_2, \ r_2 \neq j_{21}, j_{22} \\
\vdots \\
\hat{v}^{(k)}_{j_{k1}} \hat{\alpha} \leq \hat{v}^{(k)}_{j_{r_k}} \hat{\alpha}, \ 1 \leq r_k \leq m_k, \ r_k \neq j_{k1}, j_{k2}
\end{cases}
$$

where $\hat{\alpha} = (\alpha_1, \ldots, \alpha_n, 1)$ with unknowns $\alpha_1, \ldots, \alpha_n$, $\hat{e}_i$ is the edge connecting the vertices $\hat{v}^{(i)}_{j_{i1}}$ and $\hat{v}^{(i)}_{j_{i2}}$ of $\hat{\mathcal{Q}}'_i$, and m_i is the cardinality of the vertex set of $\mathcal{Q}'_i$. We use the k equations in (3.5) to solve for k of the α_i's. Upon substituting those k values of the α_i's into the rest of the inequalities in (3.5), we obtain inequalities of the form

$$
(3.6) \qquad A\alpha \ \leq \ b,
$$

where the number of variables(i.e., the number of components of α) is now k less than before. We conduct the feasibility test on (3.6) by solving the following linear programming problem:

$$\begin{array}{lll} & \text{minimize} & \epsilon \\ (3.7) & \text{subject to} & A\alpha - \epsilon e \leq b, \quad \epsilon > 0, \ \text{where } e = (1, \ldots, 1). \end{array}$$

If $\epsilon_0 =$ min ϵ is equal to zero, then (3.6) is feasible. Thus $\hat{e}_1 + \cdots + \hat{e}_k$ lies in the lower hull of $\hat{Q}'_1 + \cdots + \hat{Q}'_k$. We repeat the above procedure for all other possible edge combinations from $Q'_1 + \cdots + Q'_k$. Only those edges that pass the feasibility test are eligible to be augmented.

Algorithm LOWER_HULL

Input: edge sets $E_1, \ldots, E_n$ from Algorithm EDGE_SET

S1 Set $k = 1$ and set the current k-tuples to contain edges in E_1. Set $k = k + 1$.

S2 $\forall\ e_k \in E_k$ and each $(k-1)$-tuple $(e_1, \ldots, e_{k-1})$ of edges stored, if e_k is found such that $\sum_{j=1}^{k-1} \hat{e}_j + \hat{e}_k$ lies in the lower hull of $\sum_{j=1}^{k-1} \hat{Q}'_j + \hat{Q}'_k$, then add that edge e_k to the current $(k-1)$-tuple and store as a k-tuple.

S3 If $k = n$, then each n-tuple of edges stored corresponds to a type $(1, \ldots, 1)$ cell.

S4 Set $k = k + 1$. If $k \leq n$, then go to **S2**.

3.4 Finding zeros of $R(y, 0)$

From Algorithm LOWER_HULL, we obtain a set of n-tuples of edges, each member corresponding to a type $(1, \ldots, 1)$ cell, along with its inner normal α. For each such cell C_j and its inner normal α, recall that by making the change of variables $x(t) = t^{\alpha}y(t)$ and factoring out the minimal power of t, we obtain the homotopy $R(y,t)$ given by (2.4). When $t=0$, $R(y,0)=0$ is a binomial system given by (2.6):

$$\begin{aligned} \bar{c}_{1,a'_{10}} y^{a'_{10}} + \bar{c}_{1,a'_{11}} y^{a'_{11}} &= 0 \\ &\vdots \\ \bar{c}_{n,a'_{n0}} y^{a'_{n0}} + \bar{c}_{n,a'_{n1}} y^{a'_{n1}} &= 0. \end{aligned}$$

To solve this system in $(\mathbb{C}^*)^n$, we rewrite it as

$$\begin{aligned} y^{v_1} &= b_1 \\ &\vdots \\ y^{v_n} &= b_n \end{aligned} \tag{3.8}$$

where $v_i = a'_{i1} - a'_{i0}$, $b_i = -\frac{\bar{c}_{i,a'_{i0}}}{\bar{c}_{i,a'_{i1}}} \neq 0$, $i = 1, \ldots, n$. Let

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

For abbreviation, we write $y^V = (y^{v_1}, \ldots, y^{v_n})$ and $b = (b_1, \ldots, b_n)$, then (3.8) becomes

$$y^V = b. \tag{3.9}$$

With this notation, it can be verified that for an $n \times n$ integer matrix U, the following holds:

$$(y^U)^V = y^{(VU)}.$$

If V is a lower triangular matrix, namely,

$$V = \begin{bmatrix} v_{11} & 0 & & 0 \\ v_{21} & v_{22} & & 0 \\ \vdots & \vdots & \ddots & \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$$

where v_{ij}'s are all integers and $v_{ii} \neq 0$ for all $i = 1, ..., n$, then since $\det V \neq 0$, (3.9) becomes

$$\begin{aligned} y_1^{v_{11}} &= b_1 \\ y_1^{v_{21}} y_2^{v_{22}} &= b_2 \\ &\vdots \\ y_1^{v_{n1}} y_2^{v_{n2}} \cdots y_n^{v_{nn}} &= b_n. \end{aligned} \tag{3.10}$$

By forward substitution, (3.10) has $|v_{11}| \times \cdots \times |v_{nn}| = |\det V| = \mathrm{vol}(C_j)$ solutions. In general, we may lower triangularize V by multiplying on the right by an integer matrix U with $|\det U| = 1$. This matrix U can be found by the following procedure:

The greatest common divisor d of two integers a and b is given by

$$d = \gcd(a, b) = ka + lb, \qquad k, l \in \mathbb{Z}$$

where the integers k and l are obtained by the Euclidean algorithm. In matrix notation, this can be formulated as

$$U \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

where

$$U = \begin{bmatrix} k & l \\ -\frac{b}{d} & \frac{a}{d} \end{bmatrix}$$

and $\det(U) = 1$.

In the multivariate case, let $U(a,b)$ be the identity matrix except for the entries

$$(U(a,b))_{ii} = k, \quad (U(a,b))_{ij} = l,$$
$$(U(a,b))_{ji} = -\frac{b}{d}, \quad (U(a,b))_{jj} = \frac{a}{d}.$$

A product of a series of matrices of the form $U(a,b)$ can be chosen to upper triangularize a matrix from the left. To lower triangularize V, let U be an integer matrix with $|\det U| = 1$ such that $U^T V^T$ is upper triangular. Thus VU is lower triangular.

Now, let $z^U = y$ and substitute it into (3.9), we have

$$y^V = (z^U)^V = z^{VU} = b. \tag{3.11}$$

Since VU is lower triangular, $z = (z_1, ..., z_n)$ in (3.11) can be solved and the number of solutions equals $|\det(VU)| = |\det(V)| \cdot |\det(U)| = |\det(V)|$. Consequently, we have as many solutions of $y = (y_1, ..., y_n)$ as in (3.9).

3.5 Numerical curve tracing

The central part of the solver is the tracing of the solution curves of the nonlinear homotopy $R(y,t) = 0$ given by (2.4), and the linear homotopy $H(x,t) = 0$ given by (2.7). The curve tracing process of both the nonlinear and linear homotopy involves the following predictor-corrector method.

Algorithm TRACE_PATH

input: starting point (x_*, t_*)

S1 (**Evaluate**) From (x_*, t_*) on the solution curve $(x(t), t)$ of $H(x,t) = 0$, evaluate the tangent vector $(\frac{dx}{dt}(t_*), 1)$, where $H_x(x_*, t_*)\frac{dx}{dt}(t_*) + H_t(x_*, t_*) = 0$.

S2 (**Predictor**) Along the tangent vector $(\frac{dx}{dt}(t_*), 1)$ with stepsize δ, predict $(x_0, t_0) = (x_*, t_*) + \delta(\frac{dx}{dt}(t_*), 1)$.

S3 (**Corrector**) From the predicted point (x_0, t_0), use Newton's method to find a sequence (x_i, t_0), $i = 0, 1, \ldots$ that converge to the point (x^*, t_0) on the solution curve, as a step forward from (x_*, t_*). If the correction is unsucccessful (i.e., the sequence does not converge), then cut the stepsize δ by half and return to **S2** for a finer prediction.

S4 (**Update**) If $t_0 = 1$, stop and output x^* as the target solution; otherwise, replace (x_*, t_*) with (x^*, t_0), adjust the stepsize if necessary, and go to **S1** for further forward tracing.

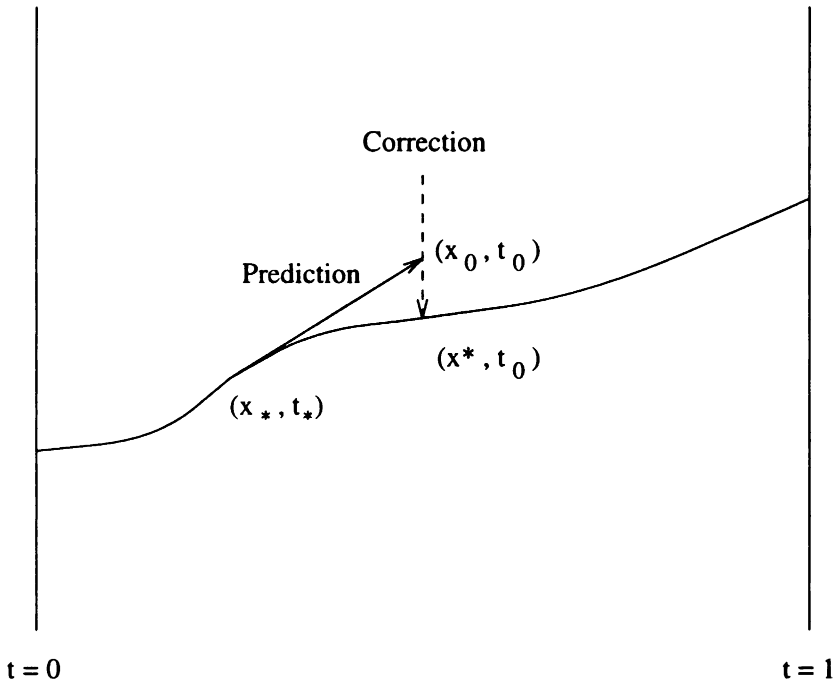

Figure 3.1. The prediction-correction process

Remark 3.5.1 *From $H_x(x,t)\frac{dx}{dt} + H_t = 0$, we have $H_x(x,t)\frac{dx}{dt} = -H_t$. Here, the matrix $H_x(x,t)$ is of full rank. We solve for the tangent vector $\frac{dx}{dt}$ by the use of Gaussian elimination on the matrix $H_x(x,t)$.*

Remark 3.5.2 *The algorithm in fixed-t correction is Newton's iterations:*

$$x_j = x_{j-1} - H'(x_{j-1}, t_0)^{-1} H(x_{j-1}, t_0), \ j = 1, 2, 3, \ldots$$

Upon rearranging, we have $H'(x_{j-1}, t_0)(x_{j-1} - x_j) = H(x_{j-1}, t_0)$, $j = 1, 2, 3, \ldots$. The Gaussian elimination method is used to solve for $x_{j-1} - x_j$.

For the first part of the curve tracing process with the homotopy $R(y,t)$, discussed in Section 2.2.1, the solutions of the binomial system (2.6) serve as starting points at

$t = 0$. The total number of curves to trace is equal to the sum of volumes of cells of type $(1, \ldots, 1)$ which equals the mixed volume $\mathcal{M}(\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$. By Theorem 2.1.1 and Proposition 2.2.3, we obtain all the $\mathcal{M}(\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$ number of isolated zeros of $Q(x)$ in $(\mathbb{C}^*)^n$ at $t = 1$, at the end of the execution of Algorithm TRACE_PATH for the homotopy $R(y, t) = 0$ given by (2.4).

The second part of the curve tracing process with the homotopy $H(x, t)$ discussed in Section 2.2.2 takes all the isolated zeros of $Q(x)$ in $(\mathbb{C}^*)^n$ as starting points at $t = 0$. The total number of curves to trace is equal to $\mathcal{M}(\mathcal{A}'_1, \ldots, \mathcal{A}'_n)$. We obtain all the isolated zeros of $P(x)$ in $\mathbb{C}^n$ at $t = 1$, upon execution of Algorithm TRACE_PATH for the homotopy $H(x, t) = 0$ given by (2.7).

CHAPTER 4

Minimal Bézout number

Polynomial systems that arise in applications are very often deficient in the number of roots, i.e., the actual root count is less than the total degree of the system. The choice of the start system is important for solving polynomial systems by the homotopy continuation method because it ultimately determines the number of homotopy paths to trace in the process. In the past, the classical way of constructing the start system is based on variant Bézout numbers of the given system. The natural approach is to find among all possible homogenizations of the given system by grouping the variables in different ways the one that corresponds to the minimal Bézout number. A smaller Bézout number translates into less homotopy paths that need to be traced. Many papers on the topic have addressed reductions in the number of solution curves to trace in the homotopy continuation method [8, 24, 38, 39].

This chapter describes the procedure for finding the minimal Bézout number over all possible homogenizations of a given polynomial system. We wish to make a comparision of this traditional approach with the use of the polyhedral homotopy where the total number of curves that need to be traced in the continuation process is equal to the mixed volume of the support of the given system. The computational results of which are presented in the next chapter.

4.1 Notations and definitions

The complex n-space $\mathbb{C}^n$ can be naturally embedded in the complex projective space $\mathbb{P}^n = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \backslash (0, \ldots, 0)\}/\sim$ where the equivalence relation $\sim$ is given by $x \sim y$ if $x = cy$ for nonzero $c \in \mathbb{C}$. Similarly, the space $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_m}$ can be naturally embedded in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$. A point $(y_1, \ldots, y_m)$ in $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_m}$ with $y_j = (y_1^{(j)}, \ldots, y_{k_j}^{(j)})$, $j = 1, \ldots, m$, corresponds to a point $(z_1, \ldots, z_m)$ in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ with $z_j = (z_0^{(j)}, \ldots, z_{k_j}^{(j)})$ and $z_0^{(j)} = 1$, $j = 1, \ldots, m$.

For a polynomial system $P(x) = (p_1(x), \ldots, p_n(x))$ with $x = (x_1, \ldots, x_n)$, if we partition the variables $x_1, \ldots, x_n$ into m groups $y_1 = (x_1^{(1)}, \ldots, x_{k_1}^{(1)}), y_2 = (x_1^{(2)}, \ldots, x_{k_2}^{(2)}), \ldots, y_m = (x_1^{(m)}, \ldots, x_{k_m}^{(m)})$ with $k_1 + \cdots + k_m = n$ and let d_{ij} be the degree of p_i with respect to y_j, $j = 1, \ldots, m$, then the m-homogenization of p_i $(1 \le i \le n)$ is defined as

$$\tilde{p}_i(z_1, \ldots, z_m) = (z_0^{(1)})^{d_{i1}} \times \cdots \times (z_0^{(m)})^{d_{im}} p_i(y_1/z_0^{(1)}, \ldots, y_m/z_0^{(m)}).$$

$\tilde{p}_i$ is homogeneous with respect to each $z_j = (z_0^{(j)}, \ldots, z_{k_j}^{(j)})$, $j = 1, \ldots, m$. Here $z_j^{(i)} = x_j^{(i)}$, for $j \neq 0$. The polynomial $\tilde{p}_i$ is said to be *m-homogeneous*, and $(d_{i1}, \ldots, d_{im})$ is the m-homogeneous degree of $\tilde{p}_i$.

For the m-homogeneous system $\tilde{P}(z) = (\tilde{p}_1(z), \ldots, \tilde{p}_n(z))$ with $z = (z_1, \ldots, z_m)$, the m-homogeneous Bézout number $\mathcal{B}$ [38] of the system with respect to z is given by the coefficient of $\alpha_1^{k_1} \cdots \alpha_m^{k_m}$ in the product

$$(d_{11}\alpha_1 + \cdots + d_{1m}\alpha_m)(d_{21}\alpha_1 + \cdots + d_{2m}\alpha_m) \cdots (d_{n1}\alpha_1 + \cdots + d_{nm}\alpha_m).$$

The system $\tilde{P}(z)$ has no more than $\mathcal{B}$ isolated solutions, counting multiplicities, in $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ [49]. However, direct application of this definition (i.e., expanding the product and finding the appropriate coefficient) does not lead to an efficient computer algorithm. In Section 4.2, we describe an efficient algorithm for computing the Bézout number by forming degree products, and in Section 4.4, we discuss how to find the minimal Bézout number among all the possible homogenizations by exhaustive search.

4.2 Bézout number calculations

Let D denote the $n \times m$ degree matrix of the polynomial system $P(x)$, for a given partition of its n variables into m groups, each group with cardinality k_j, $1 \leq j \leq m$, where $\sum_{j=1}^{m} k_j = n$. The elements of D are d_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$. Let K be the vector $[k_1, \ldots, k_m]$. The Bézout number is the sum of *degree products* of the form $\prod_{i=1}^{n} d_{il_i}$, where among integers $l_1, \ldots, l_n$, integer j $(1 \leq j \leq m)$ appears exactly k_j times, i.e., we sum degree products over all possible ways to choose each row once while choosing k_j entries from each column j. The row expansion algorithm in Section 4.2.1 forms degree products using a method resembling the evaluation of a determinant via expansion by minors.

4.2.1 Row expansion algorithm

Form the degree products starting with the element d_{1j} in row 1 of D. Choose one element from each of the remaining rows while including only $k_j - 1$ elements from the jth column. We form a minor D' corresponding to d_{1j} by deleting row 1 of D. The row expansion algorithm computes the Bézout number as the sum along the first row of each $d_{1j}(k_j > 0)$ times the Bézout number of the corresponding minor. The Bézout number of each minor is then computed recursively by the same row expansion procedure. Let K' be the vector $K = [k_1, \ldots, k_m]$ with k_j replaced by $k_j - 1$. With the following recurrence relation:

$$b(D, K, i) = \sum_{j=1,\ k_j \neq 0}^{m} d_{ij}\ b(D', K', i+1),$$

the Bézout number is given by

$$\mathcal{B} = b(D, K, 1).$$

If the degree matrix D is sparse, we may skip over computations where $d_{ij} = 0$ and avoid expanding the recursion below that branch. We expand along the row with the most zero elements, by exchanging rows if necessary.

4.3 Multi-homogenizations

The number of possible multi-homogenizations of an n-variable system is essentially the same as the number of ways to partition n distinct objects into m identical boxes, $m = 1, \ldots, n$. Denote these numbers as $p(n, m), m = 1, \ldots, n$. Then $p(n, 1) = 1$, since there is only one way to place all n objects into one box. Also, $p(n, n) = 1$ corresponds to putting 1 object in each of the n boxes. We have the following relation:

$$p(n, m) = m * p(n-1, m) + p(n-1, m-1)$$

since for each of the $p(n-1, m)$ partitionings of $n-1$ items, we may add the nth item into any one of the m boxes; also, for each of the $p(n-1, m-1)$ partitionings of $n-1$ things into $m-1$ boxes, we can put the nth item in the mth box by itself.

Values of $p(n, m)$ are listed in the following table, along with the total number of partitions $\mathcal{P}(n)$ for n objects: $\mathcal{P}(n) = \sum_{m=1}^{n} p(n, m)$. The numbers $p(n, m)$ are known as *Stirling numbers of the second kind* [50]. Each of the $\mathcal{P}(n)$ partitions yields a distinct multi-homogenization of the n-variable system.

n	$p(n,1)$	$p(n,2)$	$p(n,3)$	$p(n,4)$	$p(n,5)$	$p(n,6)$	$p(n,7)$	$p(n,8)$	$p(n,9)$	$p(n,10)$	$\mathcal{P}(n)$
1	1										1
2	1	1									2
3	1	3	1								5
4	1	7	6	1							15
5	1	15	25	10	1						52
6	1	31	90	65	15	1					203
7	1	63	301	350	140	21	1				877
8	1	127	966	1701	1050	266	28	1			4140
9	1	255	3025	7770	6951	2646	462	36	1		21147
10	1	511	9330	34105	42525	22827	5880	750	45	1	115975
11											678570
12											4213597
13											27644437

Example 4.3.1 Consider the following system which arises from a test for numerical bifurcation:

$$\begin{aligned} 5x_1^9 - 6x_1^5x_2^2 + x_1x_2^4 + 2x_1x_3 &= 0 \\ -2x_1^6x_2 + 2x_1^2x_2^3 + 2x_2x_3 &= 0 \\ x_1^2 + x_2^2 - 0.265625 &= 0 \end{aligned}$$

There are $\mathcal{P}(3) = 5$ ways to multi-homogenize this system. For each partition of the variables, we form the corresponding degree matrix D and apply the row expansion algorithm:

$\{x_1, x_2, x_3\}$	$\{x_1, x_2\}, \{x_3\}$	$\{x_1, x_3\}, \{x_2\}$	$\{x_1\}, \{x_2, x_3\}$	$\{x_1\}, \{x_2\}, \{x_3\}$
$K = [3]$	$K = [2, 1]$	$K = [2, 1]$	$K = [1, 2]$	$K = [1, 1, 1]$
$D = \begin{bmatrix} 9 \\ 7 \\ 2 \end{bmatrix}$	$D = \begin{bmatrix} 9 & 1 \\ 7 & 1 \\ 2 & 0 \end{bmatrix}$	$D = \begin{bmatrix} 9 & 4 \\ 6 & 3 \\ 2 & 2 \end{bmatrix}$	$D = \begin{bmatrix} 9 & 4 \\ 6 & 3 \\ 2 & 2 \end{bmatrix}$	$D = \begin{bmatrix} 9 & 4 & 1 \\ 6 & 3 & 1 \\ 2 & 2 & 0 \end{bmatrix}$
$\mathcal{B} = 126$	$\mathcal{B} = 32$	$\mathcal{B} = 210$	$\mathcal{B} = 126$	$\mathcal{B} = 44$

Thus, the minimal Bézout number is 32, which corresponds to the partition $\{x_1, x_2\}, \{x_3\}$.

4.4 Efficiency measures

For each partitioning of the variables, we form the corresponding degree matrix. We do not actually form the homogenized polynomials, but rather, we scan through the terms for each group of variables, find the term with the largest degree with respect to that group. Since the degrees are all nonnegative, the Bézout number is the sum of nonnegative degree products. As we test homogenizations in search of minimal Bézout numbers, we may abort the calculation if the running subtotal exceeds the current minimal Bézout number. When applying the row expansion algorithm, it is helpful to skip over any degree that is zero. This not only avoids unnecessary computation, but it also assures that any subterm we compute at any level of the tree of partitionings has a string of strictly positive degrees above it. When any of the subterms at the leaves of the tree exceeds the current minimum, we are safe in aborting the Bézout number calculations for the corresponding partitions.

CHAPTER 5

Numerical experiments

The techniques discussed so far find their natural application in polynomial systems arising in a variety of fields and modelling geometric and kinematic constraints. As mentioned in the previous chapter, for those systems, the traditional approach is to find among all possible homogenizations of the given system the one that corresponds to the minimal Bézout number. A smaller Bézout number translates into proportionately less computer time when we intend to find all isolated solutions of the given polynomial system using the homotopy continuation method. In this chapter, we make a comparision of this traditional approach with the use of the polyhedral homotopy where the total number of curves that need to be traced in the continuation process is equal to the mixed volume of the support of the given system. We present the numerical results of applying our algorithm to an extensive list of polynomial systems that arise in applications.

5.1 Problem statements

5.1.1 Robot manipulator PUMA

This is the inverse position problem of a PUMA robot. To find the relative joint

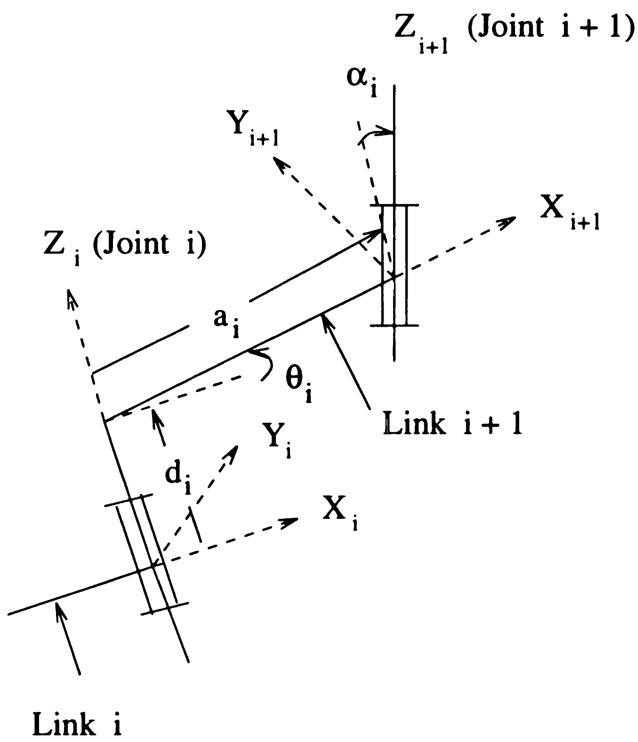

Figure 5.1. The basic notation

displacements given the hand position and orientation of the arm of the robot [37] (a_i, d_i and α_i are constants while θ_i is a variable), we have the following system:

$$x_1^2 + x_2^2 - 1 = 0$$

$$x_3^2 + x_4^2 - 1 = 0$$

$$
\begin{aligned}
x_5^2 + x_6^2 - 1 &= 0 \\
x_7^2 + x_8^2 - 1 &= 0 \\
0.004731x_1x_3 - 0.3578x_2x_3 - 0.1238x_1 - 0.001637x_2 - 0.9338x_4 + x_7 - 0.3571 &= 0 \\
0.2238x_1x_3 + 0.7623x_2x_3 + 0.2638x_1 - 0.07745x_2 - 0.6734x_4 - 0.6022 &= 0 \\
x_6x_8 + 0.3578x_1 + 0.004731x_2 &= 0 \\
-0.7623x_1 + 0.2238x_2 + 0.3461 &= 0.
\end{aligned}
$$

Here x_i ($i = 1, \ldots, 8$) represent the sine and cosine functions of the robot's joint angles θ_i. The number of variables of this system is 8 with total degree $2^7 = 128$. The optimal Bézout number is 16 with the partition $\{x_1, x_2\}, \{x_3, x_4, x_7, x_8\}, \{x_5, x_6\}$. The mixed volume is 16 and there are 16 isolated zeros.

5.1.2 Robot manipulator ROMIN

This is the inverse position problem of the robot manipulator Romin [14]. Let l_2, l_3 represent the length of the two arms of the robot, and $\theta_1, \theta_2, \theta_3$ represent the joint angles placing the robot at a given position $P = (a, b, c)$.

Denote $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$, $i = 1, 2, 3$. We obtain the following system (once l_2 and l_3 are fixed):

$$
\begin{aligned}
-s_1(l_2c_2 + l_3c_3) &= a \\
c_1(l_2c_2 + l_3c_3) &= b \\
l_2s_2 + l_3s_3 &= c \\
s_i^2 + c_i^2 &= 1, \ i = 1, 2, 3.
\end{aligned}
$$

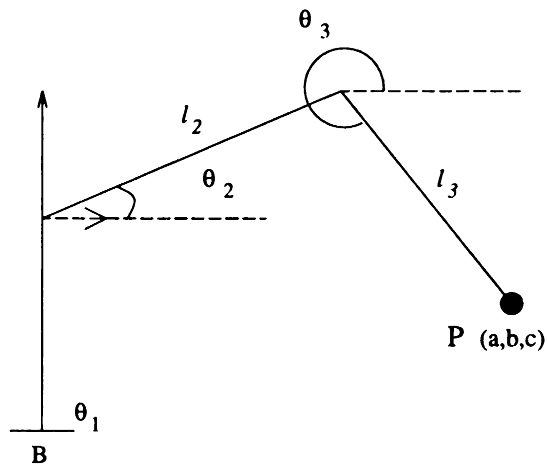

Figure 5.2. The ROMIN manipulator

The number of variables of this system is 6 with total degree $2^5 = 32$. The optimal Bézout number is 16, with the partition $\{s_1, c_1\}, \{c_2, c_3, s_2, s_3\}$. The mixed volume is 4. For $l_2 = 718$mm, $l_3 = 600$mm and the point $(500, 500, 0)$, the number of isolated zeros is 4.

5.1.3 Neural network – adaptive Lotka-Volterra System

A neural network can be thought of as a network of interconnectred cells in which activity levels at each cell are inhibited or excited by the activity of the other cells. The Lotka-Volterra model(the oldest predator-prey model), with interaction coefficients dependent on time, (allowing the weights of the connections between cells to vary) is one such network. The model to be analyzed consists of n interconnected cells. The rate of change of activity level at the ith cell is the nonconstant Lotka-Volterra equations [46]:

$$X_i'(t) = X_i(t)[1 - cX_i(t) + \sum_{i=1}^{n} \delta_{ij} A_{ij}(t) X_j(t)],\ 1 \le i \le n.$$

An *interior critical point* of the Lotka-Volterra system is any positive vector $(x_1, ..., x_n)$ satisfying

$$1 - cx_i + \sum_{j=1}^{n} \delta_{ij} x_i x_j^2 = 0,\ 1 \le i \le n$$

where the connection matrix $\Delta_p^* = (\delta_{ij})$ is given by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } j \le p, i \ne j \\ -1 & \text{if } j > p, i \ne j \end{cases} \qquad 1 \le i, j \le n$$

where p is a given integer between 0 and n.

We solve the following system in order to find the interior critical points of the Lotka-Volterra system for $n = p = 4$:

$$\begin{aligned} x_1x_2^2 + x_1x_3^2 + x_1x_4^2 - cx_1 + 1 &= 0 \\ x_2x_1^2 + x_2x_3^2 + x_2x_4^2 - cx_2 + 1 &= 0 \\ x_3x_1^2 + x_3x_2^2 + x_3x_4^2 - cx_3 + 1 &= 0 \\ x_4x_1^2 + x_4x_2^2 + x_4x_3^2 - cx_4 + 1 &= 0. \end{aligned}$$

The number of variables of this system is 4 with total degree $3^4 = 81$. The optimal Bézout number is 81 with the partition $\{x_1, x_2, x_3, x_4\}$. The mixed volume is 73 and for a generic choice of c, there are 73 isolated zeros.

5.1.4 Symmetrized four-bar mechanism

This is the problem of synthesizing a four-bar with two given fixed pivots $(0,0)$ and $a = (1,0)$, to generate a path through five precision points $d_j = (d_{j1}, d_{j2})$, $j = 0, \ldots, 4$ [41]. The following system is set up to determine all sets of lengths $r_1, \ldots, r_5$ that allow the coupler point to pass through all the precision points:

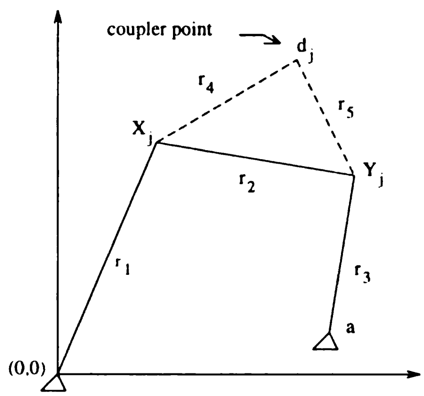

Figure 5.3. Four-bar linkage at the jth position

$$
\begin{aligned}
& a_{l1}x_1^2x_3^2 + a_{l2}x_1^2x_3x_4 + a_{l3}x_1^2x_3 + a_{l4}x_1^2x_4^2 + a_{l5}x_1^2x_4 \\
& +a_{l6}x_1^2 + a_{l7}x_1x_2x_3^2 + a_{l8}x_1x_2x_3x_4 + a_{l9}x_1x_2x_3 + a_{l10}x_1x_2x_4^2 \\
& +a_{l11}x_1x_2x_4 + a_{l12}x_1x_3^2 + a_{l13}x_1x_3x_4 + a_{l14}x_1x_3 + a_{l15}x_1x_4^2 \\
& +a_{l16}x_1x_4 + a_{l17}x_2^2x_3^2 + a_{l18}x_2^2x_3x_4 + a_{l19}x_2^2x_3 + a_{l20}x_2^2x_4^2 \\
& +a_{l21}x_2^2x_4 + a_{l22}x_2^2 + a_{l23}x_2x_3^2 + a_{l24}x_2x_3x_4 + a_{l25}x_2x_3
\end{aligned}
$$

$$+a_{l26}x_2x_4^2 + a_{l28}x_2x_4 + a_{l28}x_3^2 + a_{l29}x_4^2 \quad = \quad 0, \quad l = 1, 4.$$

The number of variables of this system is 4 with total degree $4^4 = 256$. The optimal Bézout number is 96 with the partition $\{x_1, x_2\}, \{x_3, x_4\}$. The mixed volume is 80. For generic choice of coefficients, there are 36 isolated zeros.

5.1.5 The nine-point problem

This is the problem of finding all four-bar linkages whose coupler curve passes through nine precision points [58].

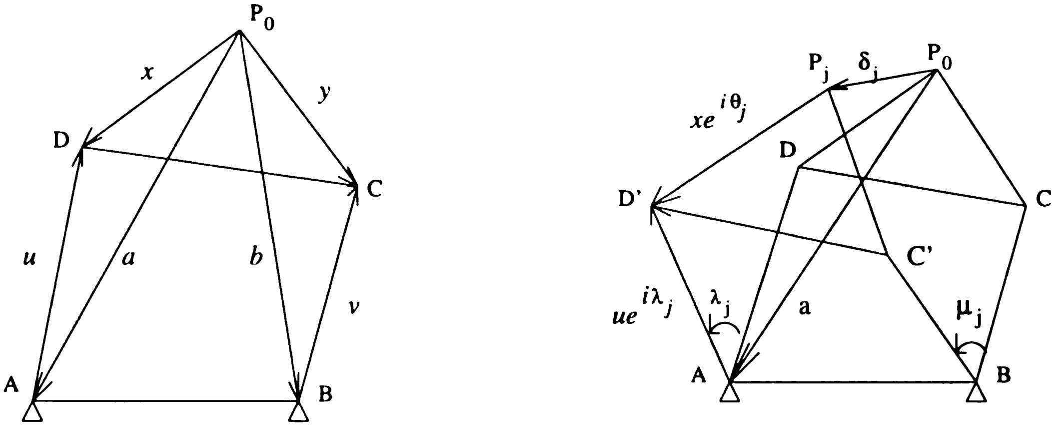

Figure 5.4. Four-bar $ABCD$ with coupler triangle CP_0D at the initial precision point P_0 (left); Displacement of the four-bar to a new precision point P_i (right)

The 8 variables are $a, \hat{a}, x, \hat{x}, b, \hat{b}, y, \hat{y}$ and there are a set of constants $\delta_j, \hat{\delta}_j$ ($j = 1, \ldots, 8$). After expunging the lower order terms, we have for $j = 1, \ldots, 8$ the following systems of equations:

$$(\hat{a}xb\hat{y})[(\hat{b}y - b\hat{y})(\delta_j(\hat{a} - \hat{x}) + \hat{\delta}_j(a - x)) + (a\hat{x} - \hat{a}x)(\delta_j(\hat{b} - \hat{y}) + \hat{\delta}_j(b - y))] = 0.$$

The number of variables of this system is 8 with total degree $7^8 = 5764801$. The optimal Bézout number is 645120 with the partition $\{a, \hat{a}\}, \{b, \hat{b}\}, \{x, \hat{x}\}, \{y, \hat{y}\}$. The mixed volume is 83977. There are 4326 isolated zeros [58].

5.1.6 Chemical equilibrium

This is the problem of combustion of propane(C_3H_8) in air(O_2 and N_2) [33] to form the ten products listed in table 5.1 :

$$\begin{aligned}
y_1y_2 + y_1 - 3y_5 &= 0 \\
2y_1y_2 + y_2y_3^2 + y_1 + R_7y_2y_3 + R_8y_2 + R_9y_2y_4 + 2R_{10}y_2^2 - Ry_5 &= 0 \\
2y_2y_3^2 + 2R_5y_3^2 + R_6y_3 + R_7y_2y_3 - 8y_5 &= 0 \\
2y_4^2 + R_9y_2y_4 - 4Ry_5 &= 0 \\
y_1y_2 + y_2y_3^2 + y_4^2 + y_1 + R_5y_3^2 + R_6y_3 + R_7y_2y_3 + R_8y_2 + R_9y_2y_4 + R_{10}y_2^2 - 1 &= 0
\end{aligned}$$

where the variables are y_1, y_2, y_3, y_4, y_5.

The number of variables of this system is 5 with total degree $2 \times 3 \times 3 \times 2 \times 3 = 108$. The optimal Bézout number is 56 with the partition $\{y_1\}, \{y_2, y_4, y_5\}, \{y_3\}$. The mixed volume is 16. With $p = 40$ and $R = 10$, there are 16 isolated zeros.

Table 5.1. Products of Propane Combustion

Product	Subscript	Description
CO_2	1	carbon dioxide
H_2O	2	water
N_2	3	nitrogen
CO	4	carbon monoxide
H_2	5	hydrogen
H	6	hydrogen atom
OH	7	hydroxyl radical
O	8	oxygen atom
NO	9	nitric oxide
O_2	10	oxygen

Table 5.2. Definitons of constants

Constant	Definiton
R_5	K_5
R_6	$K_6p^{-1/2}$
R_7	$K_7p^{-1/2}$
R_8	K_8p^{-1}
R_9	$K_9p^{-1/2}$
R_{10}	$K_{10}p^{-1}$

Table 5.3. Equilibrium Constants

Constant	Value
K_5	1.930×10^{-1}
K_6	2.597×10^{-3}
K_7	3.448×10^{-3}
K_8	1.799×10^{-5}
K_9	2.155×10^{-4}
K_{10}	3.846×10^{-5}

5.1.7 Lumped-parameter chemically reacting system

The following system arises as a result of the isothermal catalytic reaction sequence between two adsorbed species [5, 28]:

$$\begin{aligned}
-a_1x_1(1-x_3-x_4)+a_2x_3-(x_1-a_6) &= 0\\
-a_3x_2(1-x_3-x_4)+a_4x_4-(x_2-a_7) &= 0\\
a_1x_1(1-x_3-x_4)-a_2x_3-a_5x_3x_4 &= 0\\
a_3x_2(1-x_3-x_4)-a_4x_4-a_5x_3x_4 &= 0.
\end{aligned}$$

The number of variables of this system is 4 with total degree $2^4 = 16$. The optimal Bézout number is 8 with the partition $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$. The mixed volume is 7 and there are 4 isolated zeros.

5.1.8 Heart-dipole

A system of eight nonlinear equations in eight unknowns is derived for the determination of the magnitudes, directions, and locations of two independent dipoles in a two-dimensional conducting region from boundary potential measurements [45]. It has biomedical significance in electrocardiographic applications:

$$\begin{aligned}
x_1+x_2-0.6325 &= 0\\
x_3+x_4+1.34534 &= 0\\
x_1x_5+x_2x_6-x_3x_7-x_4x_8+0.8365348 &= 0\\
x_1x_7+x_2x_8+x_3x_5+x_4x_6-1.7345334 &= 0
\end{aligned}$$

$$x_1x_5^2 - x_1x_7^2 - 2x_3x_5x_7 + x_2x_6^2 - x_2x_8^2 - 2x_4x_6x_8 - 1.352352 = 0$$

$$x_3x_5^2 - x_3x_7^2 + 2x_1x_5x_7 + x_4x_6^2 - x_4x_8^2 + 2x_2x_6x_8 + 0.843453 = 0$$

$$x_1x_5^3 - 3x_1x_5x_7^2 + x_3x_7^3 - 3x_3x_7x_5^2 + x_2x_6^3 - 3x_2x_6x_8^2 + x_4x_8^3 - 3x_4x_8x_6^2 + 0.9563453 = 0$$

$$x_3x_5^3 - 3x_3x_5x_7^2 - x_1x_7^3 + 3x_1x_7x_5^2 + x_4x_6^3 - 3x_4x_6x_8^2 - x_2x_8^3 + 3x_2x_8x_6^2 - 1.2342523 = 0.$$

The number of variables of this system is 8 with total degree $2^2 \times 3^2 \times 4^2 = 576$. The optimal Bézout number is 193 with the partition $\{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8\}$. The mixed volume is 121 and there are 4 isolated zeros.

5.1.9 Conformal analysis of cyclic molecules

This problem arises in computational biology. The molecule has a cyclic backbone of

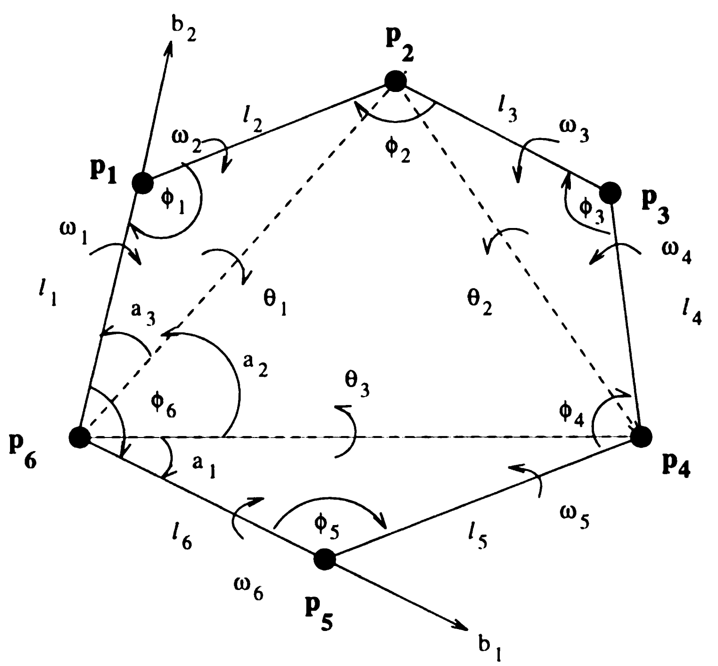

Figure 5.5. The cyclic molecule

6 atoms, typically of carbon. The bond lengths and angles provide the constraints while the six dihedral angles are allowed to vary. In kinematic terms, atoms and bonds are analogous to links and joints of a serial mechanism in which each pair of consecutive axes intersects at a link. In Figure 5.5, backbone atoms are regarded as points $p_1, ..., p_6 \in \mathbb{R}^3$; the unknown dihedrals are the angles $\omega_1, ..., \omega_6$ about axes (p_6, p_1) and (p_{i-1}, p_i) for $i = 2, ..., 6$.

The following system is set up to compute all conformations of the cyclic molecule [10]:

$$
\begin{aligned}
f_1 = \beta_{11} + \beta_{12}t_2^2 + \beta_{13}t_3^2 + \beta_{14}t_2^2t_3^2 + \beta_{15}t_2t_3 &= 0 \\
f_2 = \beta_{21} + \beta_{22}t_3^2 + \beta_{23}t_1^2 + \beta_{24}t_3^2t_1^2 + \beta_{25}t_3t_1 &= 0 \\
f_3 = \beta_{31} + \beta_{32}t_1^2 + \beta_{33}t_2^2 + \beta_{34}t_1^2t_2^2 + \beta_{35}t_1t_2 &= 0
\end{aligned}
$$

where β_{ij} is the (i, j)-th entry of the matrix

$$
\begin{bmatrix}
-9 & -1 & -1 & 3 & 8 \\
-9 & -1 & -1 & 3 & 8 \\
-9 & -1 & -1 & 3 & 8
\end{bmatrix}
$$

The number of variables of this system is 3 with total degree $4^3 = 64$. The optimal Bézout number is 16 with the partition $\{x_1\}, \{x_2\}, \{x_3\}$. The mixed volume is 16 and there are 16 isolated zeros.

5.1.10 Camera motion from point matches

This is the problem of computing the displacement of a camera between two positions in a static environment [10], given the coordinates of certain points in the two views, under perspective projection on calibrated cameras. Using quaternions formulation

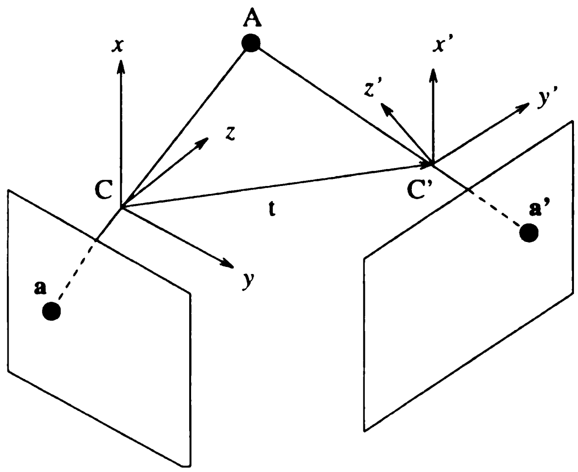

Figure 5.6. Single point seen by two camera positions

and scaling the coordinates of the given frames(dividing all the components by 1000), the following system is obtained:

$$
\begin{aligned}
-d_1q_1 - d_2q_2 - d_3q_3 + 1 &= 0 \\
-3.6d_1q_1 + 4.1d_1q_2 + 2.0d - 1q_3 + 0.1d_1 + 4.1d_2q_1 + 1.8d_2q_2 + 3.7d - 2q - 3 - 0.2d_2 & \\
+2.0d_3q_1 + 3.7d_3q_2 - 4.0d_3q_3 + 0.3d_3 + 0.1q_1 - 0.2q_2 + 0.3q_3 + 5.8 &= 0 \\
0.3464d - 1q_1 + 0.1732d_1q_2 - 5.999648d_1q_3 - 0.1732d_1 + 0.1732d_2q_1 - 5.999648d_2q_2 & \\
-0.1732d_2q_3 + 0.3464d_2 - 5.999648d_3q_1 - 0.1732d_3q_2 - 0.3464d_3q_3 - 0.1732d_3 & \\
-0.1732q_1 + 0.3464q_2 - 0.1732q_3 + 5.999648 &= 0
\end{aligned}
$$

Table 5.4. Coordinates of the two frames

point	first frame	second frame
1	(1000,2000,1000)	(1100,1900,900)
2	(1414,-1414,1414)	(1314,-1514,1314)
3	(-1732,0,1732)	(-1832,100,1632)
4	(2000,1000,3000)	(-1100,-900,1900)
5	(-1000,-1000,2000)	(2100,1100,2900)

$$\begin{aligned}
-5701.3d_1q_1 - 2.9d_1q_2 + 3796.7d_1q_3 - 1902.7d_1 - 2.9d_2q_1 - 5698.7d_2q_2 & \\
+1897.3d_2q_3 + 3803.3d_2 + 3796.7d_3q_1 + 1897.3d_3q_2 + 5703.1d_3q_3 + 0.7d_3 & \\
-1902.7q_1 + 3803.3q_2 + 0.7q_3 + 5696.6 &= 0 \\
-6.8d_1q_1 - 3.2d_1q_2 + 1.3d_1q_3 + 5.1d_1 - 3.2d_2q_1 - 4.8d_2q_2 - 0.7d_2q_3 - 7.1d_2 & \\
+1.3d_3q_1 - 0.7d_3q_2 + 9.0d_3q_3 - d_3 + 5.1q_1 - 7.1q_2 - q_3 + 2.6 &= 0 \\
-2.140796d_1q_1 - 3.998792d_1q_2 + 3.715992d_1q_3 - 3.998792d_3q_2 - 2.140796d_3q_3 & \\
+0.2828d_3 - 0.2828q_1 + 0.2828q_3 + 5.856788 &= 0.
\end{aligned}$$

The number of variables of this system is 6 with total degree $2^6 = 64$. The optimal Bézout number is 20 with the partition $\{d_1, d_2, d_3\}, \{q_1, q_2, q_3\}$. The mixed volume is 20 and there are 20 isolated zeros.

5.1.11 Electrical network

The following are the steady-state equations for the load flow in an electrical network [26]:

$$x_3(c_{11}x_1 + c_{12}x_2 + c_{13}) - c_{14} = 0$$

$$
\begin{aligned}
x_4(c_{21}x_1 + c_{22}x_2 + c_{23}) - c_{24} &= 0 \\
x_1(c_{31}x_1 + x_{32}x_2 + c_{33}) - c_{34} &= 0 \\
x_2(c_{41}x_1 + x_{42}x_2 + c_{43}) - c_{44} &= 0
\end{aligned}
$$

with

$$
C = (c_{ij}) = \begin{pmatrix} 1-3i & -4-2i & -3+8i & 1+i \\ 4+i & 2-3i & -4-3i & -3+5i \\ 1+3i & -4+2i & -3-8i & 1-i \\ 4-i & 2+3i & -4+3i & -3-5i \end{pmatrix}.
$$

The number of variables of this system is 4 with total degree $2^4 = 16$. The optimal Bézout number is 6 with the partition $\{x_1, x_2\}, \{x_3, x_4\}$. The mixed volume is 6 and there are 6 isolated zeros.

5.1.12 Vibrating systems

The following represent equations of motion for the vibrating system formulated by means of the Lagrangian method [26, 23]:

$$
\begin{aligned}
P(\lambda)x &= 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + 1 &= 0
\end{aligned}
$$

where

$$
P(\lambda) = A_2\lambda^2 + A_1\lambda + A_0
$$

and

$$A_2 = \begin{bmatrix} -10 & 2 & -1 & 1 & 3 \\ 2 & -11 & 2 & -2 & 1 \\ -1 & 2 & -12 & -1 & 1 \\ 1 & -2 & -1 & -10 & 2 \\ 3 & 1 & 1 & 2 & -11 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 3 \\ 1 & 2 & 0 & -2 & -2 \\ 2 & 1 & -2 & 2 & 3 \\ 1 & 3 & -2 & 3 & 3 \end{bmatrix}, A_0 = \begin{bmatrix} 10 & 2 & -1 & 2 & -2 \\ 2 & 9 & 3 & -1 & -2 \\ -1 & 3 & 10 & 2 & -1 \\ 2 & -1 & 2 & 12 & 1 \\ -2 & -2 & 1 & 1 & 10 \end{bmatrix}.$$

The number of variables of this system is 6 with total degree $3^5 = 243$. The optimal Bézout number is 10 with the partition $\{x_1, x_2, x_3, x_4, x_5\}, \{\lambda\}$. The mixed volume is 10 and 10 real eigenvalues along with its corresponding eigenvectors are found.

5.1.13 6R inverse position problem

This is the inverse kinematics of the 6R manipulator problem in 8 equations in 8 unknowns [40, 53]:

$$x_{2l-1}^2 + x_{2l}^2 - 1 = 0, \quad l = 1, ..., 4$$

$$a_{l1}x_1x_3 + a_{l2}x_1x_4 + a_{l3}x_2x_3 + a_{l4}x_2x_4 +$$

$$a_{l5}x_5x_7 + a_{l6}x_5x_8 + a_{l7}x_6x_7 + a_{l8}x_6x_8 +$$

$$a_{l9}x_1 + a_{l10}x_2 + a_{l11}x_3 + a_{l12}x_4 + a_{l13}x_5 +$$

$$a_{l14}x_6 + a_{l15}x_7 + a_{l16}x_8 + a_{l17} = 0, \quad l = 1, ..., 4.$$

In the equations, $x_i (i = 1, ..., 8)$ represents the sine and cosine functions of the robot's joint angles. The number of variables of this system is 8 with total degree $2^8 = 256$. The optimal Bézout number is 96 with the partition $\{x_1, x_2, x_5, x_6\}, \{x_3, x_4, x_7, x_8\}$. The mixed volume is 64. With the coefficients in [53], there are 32 isolated zeros.

5.1.14 6R2 inverse position problem

The following is the inverse kinematics of the 6R manipulator problem in 11 equations in 11 unknowns [57]:

$$
\begin{aligned}
c_1^2 + z_{21}^2 + z_{22}^2 - 1 &= 0 \\
z_{31}^2 + z_{32}^2 + z_{33}^2 - 1 &= 0 \\
z_{41}^2 + z_{42}^2 + z_{43}^2 - 1 &= 0 \\
z_{51}^2 + z_{52}^2 + z_{53}^2 - 1 &= 0 \\
c_1 z_{33} - c_2 + z_{21} z_{31} + z_{22} z_{32} &= 0 \\
-c_3 + z_{31} z_{41} + z_{32} z_{42} + z_{33} z_{43} &= 0 \\
-c_4 + z_{41} z_{51} + z_{42} z_{52} + z_{43} z_{53} &= 0 \\
-c_1 + z_{51} z_{61} + z_{52} z_{62} + z_{53} z_{63} &= 0 \\
-c_1 e_2 z_{32} + d_2 z_{21} + d_3 z_{31} + d_4 z_{41} + d_5 z_{51} - e_1 z_{22} + e_2 z_{22} z_{33} + e_3 z_{32} z_{43} & \\
-e_3 z_{33} z_{42} + e_4 z_{42} z_{53} - e_4 z_{43} z_{52} + e_5 z_{52} z_{63} - e_5 z_{53} z_{62} - p_{61} &= 0 \\
c_1 e_2 z_{31} + d_2 z_{22} + d_3 z_{32} + d_4 z_{42} + d_5 z_{52} + e_1 z_{21} - e_2 z_{21} z_{33} - e_3 z_{31} z_{43} & \\
+e_3 z_{33} z_{41} - e_4 z_{41} z_{53} + e_4 z_{43} z_{51} - e_5 z_{51} z_{63} + e_5 z_{53} z_{61} - p_{62} &= 0 \\
c_1 d_2 + d_3 z_{33} + d_4 z_{43} + d_5 z_{53} + e_2 z_{21} z_{32} - e_2 z_{22} z_{31} + e_3 z_{31} z_{42} & \\
-e_3 z_{32} z_{41} + e_4 z_{41} z_{52} - e_4 z_{42} z_{51} + e_5 z_{51} z_{62} - e_5 z_{52} z_{61} - p_{63} &= 0.
\end{aligned}
$$

The number of variables of this system is 11 with total degree $2^{11} = 1024$. The optimal Bézout number is 320 with the partition $\{z_{21}, z_{23}, z_{41}, z_{42}, z_{43}\}$, $\{z_{31}, z_{32}, z_{33}, z_{51}, z_{52}, z_{53}\}$. The mixed volume is 288 and there are 16 isolated zeros.

5.1.15 An interpolating quadrature formula over a grid

The following system occurred in the context of wavelet functions [52]. To introduce wavelets, we introduce the notion of multi-resolution analysis: A multi-resolution analysis of $L_2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_j \subset L_2(\mathbb{R})$, $j \in \mathbb{N}$, with the following properties:

1. $V_j \subset V_{j+1}$,
2. $v(x) \in V_j \Leftrightarrow v(2x) \in V_{j+1}$,
3. $v(x) \in V_0 \Leftrightarrow v(x+1) \in V_0$,
4. $\cup_{j=-\infty}^{+\infty} V_j$ is dense in $L_2(\mathbb{R})$ and $\cap_{j=-\infty}^{+\infty} V_j = \{0\}$,
5. A *scaling function* $\varphi \in V_0$, with a non-vanishing integral, exists so that the collection $\{\varphi(x-l) | l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

We assume that the scaling function φ has compact support $[0,L]$ and satisfies a refinement equation

$$\varphi(x) = 2 \sum_k h_k \varphi(2x-k)$$

with $L+1$ non-zero coefficients h_k. To construct an n-point interpolating quadrature formula (with degree of precision n) with points $x_k = \frac{k}{2} + \tau$ for integrating a function defined on a grid, the polynomial system $F(w) = 0$, with the jth equation given as

$$f(w) = \sum_{k=0}^{n-1} \omega_k \left(\frac{k}{2} + \tau \right)^j - M_j = 0, \quad j = 0, 1, ..., n$$

has to be solved. Here the constants M_j are the pth moments of the scaling function φ. The unknowns are $\omega = (\omega_0, ..., \omega_{n-1})$ where ω_i, $0 \le i \le n-1$, are the weights of the quadrature formula and τ is the range of the shift. For $n = 4$, the system becomes

$$
\begin{aligned}
\omega_0 + \omega_1 + \omega_2 + \omega_3 - 1 &= 0 \\
\omega_0\tau + \omega_1\tau + \omega_2\tau + \omega_3\tau + \frac{1}{2}\omega_1 + \omega_2 + \frac{3}{2}\omega_3 - 0.63397459621556 &= 0 \\
\omega_0\tau^2 + \omega_1\tau^2 + \omega_2\tau^2 + \omega_3\tau^2 + \omega_1\tau + 2\omega_2\tau + 3\omega_3\tau + \frac{1}{4}\omega_1 + \omega_2 + & \\
\frac{9}{2}\omega_3 - 0.40192378864668 &= 0 \\
\omega_0\tau^3 + \omega_1\tau^3 + \omega_2\tau^3 + \omega_3\tau^3 + \frac{3}{2}\omega_1\tau^2 + 3\omega_2\tau^2 + \frac{9}{2}\omega_3\tau^2 + \frac{3}{4}\omega_1\tau + & \\
3\omega_2\tau + \frac{27}{4}\omega_3\tau + \frac{1}{8}\omega_1 + \omega_2 + \frac{27}{8}\omega_3 - 0.13109155679036 &= 0 \\
\omega_0\tau^4 + \omega_1\tau^4 + \omega_2\tau^4 + \omega_3\tau^4 + 2\omega_1\tau^3 + 4\omega_2\tau^3 + 6\omega_3\tau^3 + \frac{3}{2}\omega_1\tau^2 + & \\
6\omega_2\tau^2 + \frac{27}{2}\omega_3\tau^2 + \frac{1}{2}\omega_1\tau + 4\omega_2\tau + \frac{27}{2}\omega_3\tau + \frac{1}{16}\omega_1 + & \\
\omega_2 + \frac{81}{16}\omega_3 + 0.30219332850656 &= 0.
\end{aligned}
$$

The number of variables of this system is 5 with total degree $5 \times 4 \times 3 \times 2 \times 1 = 120$. The optimal Bézout number is 10 with the partition $\{\tau\}, \{\omega_0, \omega_1, \omega_2, \omega_3\}$. The mixed volume is 5 and there are 5 isolated zeros.

5.1.16 Optimizing the Wood function

The following system is derived from optimizing the Wood function [35]:

$$
\begin{aligned}
200x_1^3 - 200x_1x_2 + x_1 - 1 &= 0 \\
-100x_1^2 + 110.1x_2 + 9.9x_4 - 20 &= 0
\end{aligned}
$$

$$\begin{aligned} 180x_3^3 - 180x_3x_4 + x_3 - 1 &= 0 \\ -90x_3^2 + 9.9x_2 + 100.1x_4 - 20 &= 0. \end{aligned}$$

The number of variables of this system is 4 with total degree $3^2 \times 2^2 = 36$. The optimal Bézout number is 25 with the partition $\{x_1\}, \{x_2, x_4\}, \{x_3\}$. The mixed volume is 9 and there are 9 isolated zeros.

5.1.17 An application in electrochemistry

The following system is derived from an application in the field of electrochemistry [60]:

$$\begin{aligned} c_1x_2^6 + c_2x_2^5 + c_3x_2^4 + c_4x_1^2x_3 + c_5x_2^3 + c_6x_2^2 + c_7x_2 + c_8 &= 0 \\ c_9x_2^5 + c_{10}x_2^4 + c_{11}x_1^2x_2 + c_{12}x_1^2x_3 + c_{13}x_2^3 + c_{14}x_1x_2 + c_{15}x_2^2 + c_{16}x_2 + c_{17} &= 0 \\ c_{18}x_1^2 + c_{19}x_1x_3 + c_{20}x_2 + c_{21} &= 0. \end{aligned}$$

The number of variables of this system is 4 with total degree 60. The optimal Bézout number is 52 with the partition $\{x_1, x_2\}, \{x_3\}$. The mixed volume is 18 and there are 15 isolated zeros.

5.1.18 Benchmark i1 from the interval arithmetics benchmarks

The following system is derived from the standard benchmarks in interval arithmetic papers [17, 34]:

$$\begin{aligned}
x_1 - 0.25428722 - 0.18324757x_4x_3x_9 &= 0\\
x_2 - 0.37842197 - 0.16275449x_1x_{10}x_6 &= 0\\
x_3 - 0.27162577 - 0.16955071x_1x_2x_{10} &= 0\\
x_4 - 0.19807914 - 0.15585316x_7x_1x_6 &= 0\\
x_5 - 0.44166728 - 0.19950920x_7x_6x_3 &= 0\\
x_6 - 0.14654113 - 0.18922793x_8x_5x_{10} &= 0\\
x_7 - 0.42937161 - 0.21180484x_2x_5x_8 &= 0\\
x_8 - 0.07056438 - 0.17081208x_1x_7x_6 &= 0\\
x_9 - 0.34504906 - 0.19312740x_{10}x_6x_4 &= 0\\
x_{10} - 0.42651102 - 0.21466544x_4x_8x_1 &= 0.
\end{aligned}$$

The number of variables of this system is 10 with total degree $3^{10} = 59049$. The optimal Bézout number is 452 with the partition $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_7\}, \{x_6\}, \{x_8\}, \{x_9\}, \{x_{10}\}$. The mixed volume is 66 and there are 50 isolated zeros.

5.1.19 Optimal multi-dimensional integration formula

The following system arises in the derivation of optimal multi-dimensional integration formula [36, 18]:

$$
\begin{aligned}
x_1 + x_3 + x_5 + 2x_7 - 1 &= 0 \\
x_1x_2 + x_3x_4 + 2x_5x_6 + 2x_7x_8 + 2x_7x_9 - \frac{2}{3} &= 0 \\
x_1x_2^2 + x_3x_4^2 + 2x_5x_6^2 + 2x_7x_8^2 + 2x_7x_9^2 - \frac{2}{5} &= 0 \\
x_1x_2^3 + x_3x_4^3 + 2x_5x_6^3 + 2x_7x_8^3 + 2x_7x_9^3 - \frac{2}{7} &= 0 \\
x_1x_2^4 + x_3x_4^4 + 2x_5x_6^4 + 2x_7x_8^4 + 2x_7x_9^4 - \frac{2}{9} &= 0 \\
x_5x_6^2 + 2x_7x_8x_9 - \frac{1}{9} &= 0 \\
x_5x_6^4 + 2x_7x_8x_9^2 - \frac{1}{25} &= 0 \\
x_5x_6^3 + x_7x_8x_9^2 + x_7x_8^2x_9 - \frac{1}{15} &= 0 \\
x_5x_6 + x_7x_8x_9^3 + x_7x_8^3x_9 - \frac{1}{21} &= 0.
\end{aligned}
$$

The number of variables of this system is 9 with total degree $1 \times 2 \times 3 \times 4 \times 5 \times 3 \times 5 \times 4 \times 5 = 36000$. The optimal Bézout number is 8852 with the partition $\{x_1, x_3, x_5, x_7\}, \{x_2, x_4, x_6, x_8, x_9\}$. The mixed volume is 136 and there are 16 isolated zeros.

5.1.20 The system of A. H. Wright

The following system is presented in [61]:

$$\begin{aligned}
x_1^2 - x_1 + x_2 + x_3 + x_4 + x_5 - 10 &= 0 \\
x_2^2 + x_1 - x_2 + x_3 + x_4 + x_5 - 10 &= 0 \\
x_3^2 + x_1 + x_2 - x_3 + x_4 + x_5 - 10 &= 0 \\
x_4^2 + x_1 + x_2 + x_3 - x_4 + x_5 - 10 &= 0 \\
x_5^2 + x_1 + x_2 + x_3 + x_4 - x_5 - 10 &= 0.
\end{aligned}$$

The number of variables of this system is 5 with total degree $2^5 = 32$. The optimal Bézout number is 32 with the partition $\{x_1, x_2, x_3, x_4, x_5\}$. The mixed volume is 32 and there are 32 isolated zeros.

5.1.21 The Reimer5 System

The following system is available at the PoSSo test suite:

$$\begin{aligned}
2x^2 - 2y^2 + 2z^2 - 2t^2 + 2u^2 - 1 &= 0 \\
2x^3 - 2y^3 + 2z^3 - 2t^3 + 2u^3 - 1 &= 0 \\
2x^4 - 2y^4 + 2z^4 - 2t^4 + 2u^4 - 1 &= 0 \\
2x^5 - 2y^5 + 2z^5 - 2t^5 + 2u^5 - 1 &= 0 \\
2x^6 - 2y^6 + 2z^6 - 2t^6 + 2u^6 - 1 &= 0.
\end{aligned}$$

The number of variables of this system is 5 with total degree $2 \times 3 \times 4 \times 5 = 720$. The optimal Bézout number is 720 with the partition $\{x, y, z, t, u\}$. The mixed volume is 720 and there are 144 isolated zeros.

5.1.22 Butcher's problem

The following system is available at the PoSSo test suite:

$$
\begin{aligned}
x_3x_5 + x_2x_6 + x_4x_7 - x_7^2 - \frac{1}{2}x_7 - \frac{1}{2} &= 0 \\
x_3x_5^2 + x_2x_6^2 - x_4x_7^2 + x_7^3 + x_7^2 - \frac{1}{3}x_4 + \frac{4}{3}x_7 &= 0 \\
x_1x_3x_6 - x_4x_7^2 + x_7^3 - \frac{1}{2}x_4x_7 + x_7^2 - \frac{1}{6}x_4 + \frac{2}{3}x_7 &= 0 \\
x_3x_5^3 + x_2x_6^3 + x_4x_7^3 - x_7^4 - \frac{3}{2}x_7^3 + x_4x_7 - \frac{5}{2}x_7^2 - \frac{1}{4}x_7 - \frac{1}{4} &= 0 \\
x_1x_3x_5x_6 + x_4x_7^3 - x_7^4 + \frac{1}{2}x_4x_7^2 - \frac{3}{2}x_7^3 + \frac{1}{2}x_4x_7 - \frac{7}{4}x_7^2 - \frac{3}{8}x_7 - \frac{1}{8} &= 0 \\
x_1x_3x_6^2 + x_4x_7^3 - x_7^4 + x_4x_7^2 - \frac{3}{2}x_7^3 + \frac{2}{3}x_4x_7 - \frac{7}{6}x_7^2 - \frac{1}{12}x_7 - \frac{1}{12} &= 0 \\
-x_4x_7^3 + x_7^4 - x_4x_7^2 + \frac{3}{2}x_7^3 - \frac{1}{3}x_4x_7 + \frac{13}{12}x_7^2 + \frac{7}{24}x_7 + \frac{1}{24} &= 0.
\end{aligned}
$$

The number of variables of this system is 7 with total degree $2 \times 3 \times 3 \times 4 \times 4 \times 4 \times 4 = 4608$. The optimal Bézout number is 1361 with the partition $\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7\}$. The mixed volume is 24 and there are 5 isolated zeros.

5.1.23 Construction of Virasoro algebras

The following system can be found in [51]:

$$
\begin{aligned}
8x_1^2 + 8x_1(x_2 + x_3) - 8x_2x_3 + 2x_1(x_4 + x_5 + x_6 + x_7) - 2x_4x_7 - 2x_5x_6 - x_1 &= 0 \\
8x_2^2 + 8x_2(x_1 + x_3) - 8x_1x_3 + 2x_2(x_4 + x_5 + x_6 + x_7) - 2x_4x_6 - 2x_5x_7 - x_2 &= 0 \\
8x_3^2 + 8x_3(x_1 + x_2) - 8x_1x_2 + 2x_3(x_4 + x_5 + x_6 + x_7) - 2x_4x_5 - 2x_6x_7 - x_3 &= 0 \\
8x_4^2 + 2x_1(x_4 - x_7) + 2x_2(x_4 - x_6) + 2(x_3 + 3x_8)(x_4 - x_5) + 2x_4(4x_5 + x_6 + x_7) - x_4 &= 0 \\
8x_5^2 + 2x_1(x_5 - x_6) + 2x_2(x_5 - x_7) + 2(x_3 + 3x_8)(x_5 - x_4) + 2x_5(4x_4 + x_6 + x_7) - x_5 &= 0
\end{aligned}
$$

$$
\begin{aligned}
8x_6^2+2x_1(x_6-x_5)+2x_2(x_6-x_4)+2(x_3+3x_8)(x_6-x_7)+2x_4(4x_7+x_4+x_5)-x_6 &= 0\\
8x_7^2+2x_1(x_7-x_4)+2x_2(x_7-x_5)+2(x_3+3x_8)(x_7-x_6)+2x_4(4x_6+x_4+x_5)-x_7 &= 0\\
8x_8^2+6(x_4+x_5+x_6+x_7)x_8-6x_4x_5-6x_6x_7-x_8 &= 0.
\end{aligned}
$$

The number of variables of this system is 8 with total degree $2^8 = 256$. The optimal Bézout number is 256 with the partition $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. The mixed volume is 200 and there are 200 isolated zeros.

5.1.24 The cyclic n-roots problem

The following system [4, 11] arises from the problem of finding all *bi-equimodular* vectors $x \in C^n$, i.e., all x with coordinates of constant absolute value such that the Fourier transform of x is a vector with coordinates of constant absolute value. This is equivalent to the problem of finding all solutions $(z_0, z_1, ..., z_{n-1})$ of A_n(the alternating group) with all $|z_j|$=1: (The relation between x and z is $z_j = x_{j+1}/x_j$)

$$
\begin{aligned}
z_0+z_1+...+z_{n-1} &= 0\\
z_0z_1+z_1z_2+...+z_{n-1}z_{n-1}+z_{n-1}z_0 &= 0\\
z_0z_1z_2+z_1z_2z_3+...+z_{n-1}z_0z_1 &= 0\\
&\vdots\\
z_0z_1...z_{n-2}+...+z_{n-1}z_0...z_{n-3} &= 0\\
z_0z_1...z_{n-1} &= 1.
\end{aligned}
$$

Table 5.5. Results for the cyclic n-roots problem

n	*totdeg*	*mdeg*	$\mathcal{M}(\mathcal{Q})$	$\mathcal{N}(\mathcal{Q})$
5	120	120	70	70
6	720	720	156	156
7	5040	5040	924	924

For $n = 7$, we have the following system:

$$
\begin{aligned}
z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0 \\
z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_6 + z_6z_0 &= 0 \\
z_0z_1z_2 + z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_6 + z_5z_6z_0 + z_6z_0z_1 &= 0 \\
z_0z_1z_2z_3 + z_1z_2z_3z_4 + z_2z_3z_4z_5 + z_3z_4z_5z_6 + z_4z_5z_6z_0 + z_5z_6z_0z_1 + z_6z_0z_1z_2 &= 0 \\
z_0z_1z_2z_3z_4 + z_1z_2z_3z_4z_5 + z_2z_3z_4z_5z_6 + z_3z_4z_5z_6z_0 + z_4z_5z_6z_0z_1 &+ \\
z_5z_6z_0z_1z_2 + z_6z_0z_1z_2z_3 &= 0 \\
z_0z_1z_2z_3z_4z_5 + z_1z_2z_3z_4z_5z_6 + z_2z_3z_4z_5z_6z_0 + z_3z_4z_5z_6z_0z_1 + z_4z_5z_6z_0z_1z_2 &+ \\
z_5z_6z_0z_1z_2z_3 + z_6z_0z_1z_2z_3z_4 &= 0 \\
z_0z_1z_2z_3z_4z_5z_6 - 1 &= 0.
\end{aligned}
$$

The number of variables of this system is 7 with total degree $7! = 5040$. The optimal Bézout number is 5040 with the partition $\{z_0, z_1, z_2, z_3, z_4, z_5, z_6\}$. The mixed volume is 924 and there are 924 isolated zeros.

For other values of n, the results are summarized in Table 5.5. Here *totdeg* denotes the total degree of the system, *mdeg* the optimal multi-homogeneous Bézout number, $\mathcal{M}(\mathcal{Q})$ its mixed volume and $\mathcal{N}(\mathcal{Q})$ the total number of isolated zeros.

5.1.25 Combustion chemistry

The following is a combustion chemistry problem [42, 43]:

$$
\begin{aligned}
x_2 + 2x_6 + x_9 + 2x_{10} - 10^{-5} &= 0 \\
x_3 + x_8 - 3.10^{-5} &= 0 \\
x_1 + x_3 + 2x_5 + 2x_8 + x_9 + x_{10} - 5.10^{-5} &= 0 \\
x_4 + 2x_7 - 10^{-5} &= 0 \\
0.5140437.10^{-7}x_5 - x_1^2 &= 0 \\
0.1006932.10^{-6}x_6 - x_2^2 &= 0 \\
0.7816278.10^{-15}x_7 - x_4^2 &= 0 \\
0.1496236.10^{-6}x_8 - x_1x_3 &= 0 \\
0.6194411.10^{-7}x_9 - x_1x_2 &= 0 \\
0.2089296.10^{-14}x_{10} - x_1x_2^2 &= 0.
\end{aligned}
$$

Due to the range of the coefficients, the scaling routine as described in [42] is applied. The scaled system:

$$
\begin{aligned}
1.018154833016\mathbf{69}.10^{-1}x_2 + 9.89610100506422.10^{-1}x_6 + 1.34637048100730x_9 + & \\
3.46970317210432x_{10} - 2.12454115933396 &= 0 \\
5.76739795357135.10^{-1}x_3 + 7.89949754301577.10^{-1}x_8 - 2.19492968782850 &= 0 \\
7.19621954936988.10^{-1}x_9 + 9.2726133519085.10^{-1}x_{10} + 8.95128807036246.10^{-1}x_3 + & \\
2.45208250541714x_8 + 8.73159766974099.10^{-2}x_1 + 1.37722259176202x_5 - & \\
5.67773314994310 &= 0
\end{aligned}
$$

$$\begin{aligned}
2.50030218520604.10^{-3}x_4 + 1.99987913687438.10x_7 - 1.99987913687440.10 &= 0 \\
-1.37722259176203x_1^2 + 7.26099038733160.10^{-1}x_5 &= 0 \\
-9.89610100506425.10^{-1}x_2^2 + 1.01049898287039x_6 &= 0 \\
-1.99987913687438.10x_4^2 + 5.00030217607503.10^{-2}x_7 &= 0 \\
-1.93702197268146x_3x_1 + 5.16256404988361.10^{-1}x_8 &= 0 \\
-9.68877757611935.10^{-1}x_2x_1 + 1.03212194948595x_9 &= 0 \\
-3.21732159608141x_2^2x_1 + 3.10817545009477.10^{-1}x_{10} &= 0.
\end{aligned}$$

The number of variables of this system is 10 with total degree $2^5 \times 3 = 96$. The optimal Bézout number is 44 with the partition $\{x_1, x_4, x_7\}, \{x_2, x_3, x_5, x_6\}, \{x_8, x_9, x_{10}\}$. The mixed volume is 16 and there are 16 finite solutions.

5.1.26 Economic modelling

The following system arises in the field of economic modelling [42]:

$$\begin{aligned}
(x_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7)x_8 - 1 &= 0 \\
(x_2 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_6 + x_5x_7)x_8 - 2 &= 0 \\
(x_3 + x_1x_4 + x_2x_5 + x_3x_6 + x_4x_7)x_8 - 3 &= 0 \\
(x_4 + x_1x_5 + x_2x_6 + x_3x_7)x_8 - 4 &= 0 \\
(x_5 + x_1x_6 + x_2x_7)x_8 - 5 &= 0 \\
(x_6 + x_1x_7)x_8 - 6 &= 0 \\
x_7x_8 - 7 &= 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= 0.
\end{aligned}$$

Table 5.6. Results for the Economic modelling problem

n	*totdeg*	*mdeg*	$\mathcal{M}(\mathcal{Q})$	$\mathcal{N}(\mathcal{Q})$
5	54	20	8	8
6	162	48	16	16
7	486	112	32	32
8	1458	256	64	64

The number of variables of this system is 8 with total degree $3^6 \times 2 \times 1 = 1458$. The optimal Bézout number is 256 with the partition $\{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\{x_7, x_8\}$. The mixed volume is 64 and there are 64 isolated zeros.

For other values of n, the results are summarized in Table 5.6. Here *totdeg* denotes the total degree of the system, *mdeg* the optimal multi-homogeneous Bézout number, $\mathcal{M}(\mathcal{Q})$ its mixed volume and $\mathcal{N}(\mathcal{Q})$ the total number of isolated zeros.

5.1.27 An application from neurophysiology

The following system is obtained from the newsgroup sci.math.num-analysis and sci.math.symbolic:

$$
\begin{aligned}
x_1^2 + x_3^2 - 1 &= 0 \\
x_2^2 + x_4^2 - 1 &= 0 \\
x_5 x_3^3 + x_6 x_4^3 - c_1 &= 0 \\
x_5 x_1^3 + x_6 x_2^3 - c_2 &= 0 \\
x_5 x_3^2 x_1 + x_6 x_4^2 x_2 - c_3 &= 0 \\
x_5 x_3 x_1^2 + x_6 x_4 x_2^2 - c_4 &= 0.
\end{aligned}
$$

The number of variables of this system is 6 with total degree $2^2 \times 4^4 = 1024$. The optimal Bézout number $= 216$ with the partition $\{x_1, x_2, x_3, x_4\}, \{x_5, x_6\}$. The mixed volume is 20 and there are 8 isolated zeros.

5.1.28 The system of E. R. Speer

The following system is given in [54]:

$$4\beta(n + 2a_1 - 8x_1)(a_2 - a_3) - x_2x_3x_4 + x_2 + x_4 = 0$$

$$4\beta(n + 2a_1 - 8x_2)(a_2 - a_3) - x_1x_3x_4 + x_1 + x_3 = 0$$

$$4\beta(n + 2a_1 - 8x_3)(a_2 - a_3) - x_1x_2x_4 + x_2 + x_4 = 0$$

$$4\beta(n + 2a_1 - 8x_4)(a_2 - a_3) - x_1x_2x_3 + x_1 + x_3 = 0$$

where $a_1 = x_1 + x_2 + x_3 + x_4$, $a_2 = x_1x_2x_3x_4$, $a_3 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ and β and n are parameters to the system.

The number of variables of this system is 4 with total degree $5^4 = 625$. The optimal Bézout number is 384 with the partition $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$. The mixed volume is 96. With $4\beta = 0.657958984375000 - 0.187652587890625i$, $n = 0.937683105468750 - 0.288299560546875i$, there are 43 isolated zeros.

5.1.29 Solotarev system

The following system is available at the Frisco test suite:

$$\begin{aligned} 3x_1^2 - 2x_1 - x_3 &= 0 \\ x_1^3 - x_1^2 - x_1x_3 + x_3 - 2x_4 - 2 &= 0 \\ 3x_2^2 - 2x_2 - x_3 &= 0 \\ x_2^3 - x_2^2 - x_2x_3 - x_3 + 2 &= 0. \end{aligned}$$

The number of variables of this system is 4 with total degree $2 \times 3 \times 2 \times 3 = 36$. The optimal Bézout number is 10 with the partition $\{x_1, x_3\}, \{x_2\}, \{x_4\}$. The mixed volume is 6 and there are 6 isolated zeros.

5.1.30 Trinks system

This system arises in Number Theory and is available at the Frisco test suite:

$$\begin{aligned} 45x_2 + 35x_5 - 165x_6 - 36 &= 0 \\ 35x_2 + 25x_3 + 40x_4 - 27x_5 &= 0 \\ 25x_2x_5 - 165x_6^2 + 15x_1 - 18x_3 + 30x_4 &= 0 \\ 15x_2x_3 + 20x_4x_5 - 9x_1 &= 0 \\ -11x_6^3 + x_1x_2 + 2x_3x_4 &= 0 \\ -11x_5x_6 + 3x_6^2 + 99x_1 &= 0. \end{aligned}$$

The number of variables of this system is 6 with total degree $2^3 \times 3 = 24$. The optimal Bézout number is 24 with the partition $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. The mixed volume is

10 and there are 10 isolated zeros.

5.1.31 A small system from constructive Galois theory: $s9_1$

This system is obtained from the Frisco test suite:

$$\begin{aligned}
-x_1x_2 - 2x_3x_4 &= 0 \\
9x_1 + 4x_5 &= 0 \\
-4x_4x_6 - 2x_1x_7 - 3x_2x_3 &= 0 \\
-7x_6 + 9x_8 - 8x_7 &= 0 \\
-4x_3x_7 - 5x_2x_6 - 6x_4 - 3x_1 &= 0 \\
-5x_3 - 6x_3x_7 - 7x_2 + 9x_5 &= 0 \\
9x_3 + 6x_8 - 5x_5 &= 0 \\
9x_6 - 7x_8 + 8 &= 0.
\end{aligned}$$

The number of variables of this system is 8 with total degree $2^4 = 16$. The optimal Bézout number is 10 with the partition $\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8\}$. The mixed volume is 10 and there are 10 isolated zeros.

5.1.32 n-dimensional reaction-diffusion problem

For a general n, the following system has 2^n solutions:

$$\begin{aligned}
x_0 = x_{n+1} &= 0 \\
x_{k-1} - 2x_k + x_{k+1} + \alpha x_k(1 - x_k) &= 0, \quad k = 1, 2, ..., n.
\end{aligned}$$

For $n = 3$, the system becomes:

$$\begin{aligned} -2x_1 + x_2 + 0.835634534x_1(1 - x_1) &= 0 \\ x_1 - 2x_2 + x_3 + 0.835634534x_2(1 - x_2) &= 0 \\ x_2 - 2x_3 + 0.835634534x_3(1 - x_3) &= 0. \end{aligned}$$

with total degree $2^3 = 8$. The optimal Bézout number is 8 with the partition $\{x_1, x_2, x_3\}$. The mixed volume is 7 and there are 7 isolated zeros.

5.1.33 4-dimensional Lorentz attractor

The equilibrium points of a chaotic attractor in 4-dimension [31] is given by:

$$\begin{aligned} x_1(x_2 - x_3) - x_4 + c &= 0 \\ x_2(x_3 - x_4) - x_1 + c &= 0 \\ x_3(x_4 - x_1) - x_2 + c &= 0 \\ x_4(x_1 - x_2) - x_3 + c &= 0 \end{aligned}$$

where c is a given constant. The number of variables of this system is 4 with total degree $2^4 = 16$. The optimal Bézout number is 14 with the partition $\{x_1, x_2\}, \{x_3, x_4\}$. The mixed volume is 12 and there are 11 isolated zeros.

5.1.34 A bifurcation problem

The following system arises from a test for numerical bifurcation [19]:

$$\begin{aligned}
5x_1^9 - 6x_1^5x_2^2 + x_1x_2^4 + 2x_1x_3 &= 0\\
-2x_1^6x_2 + 2x_1^2x_2^3 + 2x_2x_3 &= 0\\
x_1^2 + x_2^2 - 0.265625 &= 0.
\end{aligned}$$

The number of variables of this system is 3 with total degree $9 \times 7 \times 2 = 126$. The optimal Bézout number is 32 with the partition $\{x_1, x_2\}, \{x_3\}$. The mixed volume is 16 and there are 16 isolated zeros.

5.1.35 Benchmark D1 from the interval arithmetics benchmarks

This system is derived from the standard benchmarks in interval arithmetics papers [17, 34] and is available at the Frisco Test Suite:

$$\begin{aligned}
x_1^2 + x_2^2 - 1 &= 0\\
x_3^2 + x_4^2 - 1 &= 0\\
x_5^2 + x_6^2 - 1 &= 0\\
x_7^2 + x_8^2 - 1 &= 0\\
x_9^2 + x_{10}^2 - 1 &= 0\\
x_{11}^2 + x_{12}^2 - 1 &= 0\\
3x_3 + 2x_5 + x_7 - 3.9701 &= 0
\end{aligned}$$

$$\begin{aligned}
3x_1x_4 + 2x_1x_6 + x_1x_8 - 1.7172 &= 0 \\
3x_2x_4 + 2x_2x_6 + x_2x_8 - 4.0616 &= 0 \\
x_3x_9 + x_5x_9 + x_7x_9 - 1.9791 &= 0 \\
x_2x_4x_9 + x_2x_6x_9 + x_2x_8x_9 + x_1x_{10} - 1.9115 &= 0 \\
-x_3x_{10}x_{11} - x_5x_{10}x_{11} - x_7x_{10}x_{11} + x_4x_{12} + x_6x_{12} + x_8x_{12} - 0.4077 &= 0.
\end{aligned}$$

The number of variables of this system is 12 with total degree $2^{12} = 4068$. The optimal Bézout number is 320 with the partition $\{x_1, x_2\}, \{x_3, x_4, x_5, x_6, x_7, x_8\}, \{x_9, x_{10}\}, \{x_{11}, x_{12}\}$. The mixed volume is 192 and there are 48 isolated zeros.

5.1.36 Caprasse's system

The following system is available at the Frisco test suite:

$$\begin{aligned}
x_2^2x_3 + 2x_1x_2x_4 - 2x_1 - x_3 &= 0 \\
2x_2x_3x_4 + x_1x_4^2 - x_1 - 2x_3 &= 0 \\
-x_1^3x_3 + 4x_1x_2^2x_3 + 4x_1^2x_2x_4 + 2x_2^3x_4 + 4x_1^2 - 10x_2^2 + 4x_1x_3 - 10x_2x_4 + 2 &= 0 \\
-x_1x_3^3 + 4x_2x_3^2x_4 + 4x_1x_3x_4^2 + 2x_2x_4^3 + 4x_1x_3 + 4x_3^2 - 10x_4^2 + 2 &= 0.
\end{aligned}$$

The number of variables of this system is 4 with total degree $3^2 \times 4^2 = 144$. The optimal Bézout number is 62 with the partition $\{x_1, x_2\}, \{x_3, x_4\}$. The mixed volume is 48 and there are 48 isolated zeros.

5.1.37 Arnborg7 system

The following system is obtained from the Frisco test suite:

$$
\begin{aligned}
x^2yz + xy^2z + xyz^2 + xyz + xy + xz + yz &= 0 \\
x^2y^2z + xy^2z^2 + x^2yz + xyz + yz + x + z &= 0 \\
x^2y^2z^2 + x^2y^2z + xy^2z + xyz + xz + z + 1 &= 0.
\end{aligned}
$$

The number of variables of this system is 3 with total degree $4 \times 5 \times 6 = 120$. The optimal Bézout number is 48 with the partition $\{x, y, z\}$. The mixed volume is 20 and there are 20 isolated zeros.

5.1.38 Rose system

The following represents a general economic equilibrium model and is available at the PoSSo test suite:

$$
\begin{aligned}
y^4 - \frac{20}{7}x^2 &= 0 \\
x^2z^4 + \frac{7}{10}xz^4 + \frac{7}{48}z^4 - \frac{50}{27}x^2 - \frac{35}{27}x - \frac{49}{216} &= 0 \\
\frac{3}{5}x^6y^2z + x^5y^3 + \frac{3}{7}x^5y^2z + \frac{7}{5}x^4y^3 - \frac{7}{20}x^4yz^2 - \frac{3}{20}x^4z^3 & \\
+\frac{609}{1000}x^3y^3 + \frac{63}{200}x^3y^2z - \frac{77}{125}x^3yz^2 - \frac{21}{50}x^3z^3 & \\
+\frac{49}{1250}x^2y^3 + \frac{147}{2000}x^2y^2z - \frac{23863}{60000}x^2yz^2 & \\
-\frac{91}{400}x^2z^3 - \frac{27391}{800000}xy^3 + \frac{4137}{800000}xy^2z - \frac{1078}{9375}xyz^2 - \frac{5887}{200000}xz^3 & \\
-\frac{1029}{160000}y^3 - \frac{24353}{1920000}yz^2 - \frac{343}{128000}z^3 &= 0.
\end{aligned}
$$

The number of variables of this system is 3 with total degree $4 \times 6 \times 9 = 216$. The optimal Bézout number is 144 with the partition $\{x\}, \{y, z\}$. The mixed volume is 136 and there are 136 isolated zeros.

5.1.39 Moeller4 System

The following system is available at the PoSSo test suite:

$$
\begin{aligned}
y + u + v - 1 &= 0 \\
z + t + 2u - 3 &= 0 \\
y + t + 2v - 1 &= 0 \\
x - y - z - t - u - v &= 0 \\
-\frac{1569}{31250} yz^3 + x^2 tu &= 0 \\
-\frac{587}{15625} yt + zv &= 0.
\end{aligned}
$$

The number of variables of this system is 6 with total degree $4 \times 2 = 8$. The optimal Bézout number is 8 with the partition $\{x, y, z, t, u, v\}$. The mixed volume is 7 and there are 7 isolated solutions.

5.1.40 KatsuraN System

This is a problem of magnetism in physics and is available at the PoSSo test suite:

$$
\begin{aligned}
2x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2 + v^2 - v &= 0 \\
xy + yz + 2zt + 2tu + 2uv - u &= 0 \\
2xz + 2yt + 2zu + u^2 + 2tv - t &= 0
\end{aligned}
$$

$$\begin{aligned} 2xt + 2yu + 2tu + 2zv - z &= 0 \\ t^2 + 2xv + 2yv + 2zv - y &= 0 \\ 2x + 2y + 2z + 2t + 2u + v - 1 &= 0. \end{aligned}$$

The number of variables of this system is 6 with total degree $2^5 = 32$. The optimal Bézout number is 32 with the partition $\{x, y, z, t, u\}$. The mixed volume is 32 and there are 32 isolated zeros.

5.1.41 Cohn-2 system

The following system is available at the PoSSo test suite:

$$\begin{aligned} x^3y^2 + 4x^2y^2 - x^2yz^2 + 288x^2y^2 + 207x^2yz + 1152xy^2z + 156xyz^2 + \\ xz^3 - 3456x^2y + 20736xy^2 + 19008xyz + 82944y^2z + 432xz^2 - \\ 497664xy + 62208xz + 2985984x &= 0 \\ y^3t^3 + 4y^3t^2 - y^2zt^2 + 4y^2t^3 - 48y^2t^2 - 5yzt^2 + 108yzt + z^2t + 144zt - 1728z &= 0 \\ -x^2z^2t + 4xz^2t^2 + z^3t^2 + x^3t^2 + x^3z + 156x^2zt + 207xz^2t + \\ 1152xzt^2 + 288z^2t^2 + 432x^2z + 19008xzt - 3456z^2t + 82944xt^2 + \\ 20736zt^2 + 62208xz - 497664zt + 2985984z &= 0 \\ y^3t^3 - xy^2t^2 + 4y^3t^2 + 4y^2t^3 - 5xy^2t - 48y^2t^2 + x^2y + 108xyt + 144xy - 1728x &= 0. \end{aligned}$$

The number of variables of this system 4 with total degree $5 \times 6 \times 5 \times 6 = 900$. The optimal Bézout number is 450 with the partition $\{x, y, z\}, \{t\}$. The mixed volume is 124 and there are 18 isolated zeros.

5.1.42 Cassou-Nogues

The following system is available at the Frisco test suite:

$$\begin{aligned}
5b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e - 28b^2cde - & \\
648b^2d + 36b^2d^2e + 9b^4d^3 - 120 &= 0 \\
30b^4c^3d - 32cde^2 - 720b^2cd - 24b^2c^3e - 432b^2c^2 + 576ce - 576de + & \\
16b^2cd^2e + 16d^2e^2 + 16c^2e^2 + 9b^4c^4 + 39b^4c^2d^2 + 18b^4cd^3 - 432b^2d^2 + & \\
24b^2d^3e - 16b^2c^2de - 240c + 5184 &= 0 \\
216b^2cd - 162b^2d^2 - 81b^2c^2 + 1008ce - 1008de + 15b^2c^2de - & \\
15b^2c^3e - 80cde^2 + 40d^2e^2 + 40c^2e^2 + 5184 &= 0 \\
4b^2cd - 3b^2d^2 - 4b^2c^2 + 22ce - 22de + 261 &= 0.
\end{aligned}$$

The number of variables of this system is 4 with total degree $7 \times 8 \times 6 \times 4 = 1344$. The optimal Bézout number is 368 with the partition $\{b\}, \{c, d, e\}$. The mixed volume is 24 and there are 16 isolated zeros.

5.1.43 A "dessin d'enfant" I system

The following system is available at the Frisco test suite:

$$\begin{aligned}
6a_{33}a_{10}a_{20} + 10a_{22}a_{10}a_{31} + 8a_{32}a_{10}a_{21} - 162a_{10}^2a_{21} + & \\
16a_{21}a_{30} + 14a_{31}a_{20} + 48a_{10}a_{30} &= 0 \\
15a_{33}a_{10}a_{21} - 162a_{10}^2a_{22} - 312a_{10}a_{20} + 24a_{10}a_{30} + & \\
27a_{31}a_{21} + 24a_{32}a_{20} + 18a_{22}a_{10}a_{32} + 30a_{22}a_{30} + 84a_{31}a_{10} &= 0
\end{aligned}$$

$$-240a_{10} + 420a_{33} - 64a_{22} + 112a_{32} = 0$$

$$180a_{33}a_{10} - 284a_{22}a_{10} - 162a_{10}^2 + 60a_{22}a_{32} +$$

$$50a_{32}a_{10} + 70a_{30} + 55a_{33}a_{21} + 260a_{31} - 112a_{20} = 0$$

$$66a_{33}a_{10} + 336a_{32} + 90a_{31} + 78a_{22}a_{33} - 1056a_{10} - 90a_{21} = 0$$

$$136a_{33} - 136 = 0$$

$$4a_{22}a_{10}a_{30} + 2a_{32}a_{10}a_{20} + 6a_{20}a_{30} - 162a_{10}^2a_{20} + 3a_{31}a_{21}a_{10} = 0$$

$$28a_{22}a_{10}a_{33} + 192a_{30} + 128a_{32}a_{10} + 36a_{31}a_{20} -$$

$$300a_{10}a_{21} + 40a_{32}a_{21} - 648a_{10}^2 + 44a_{22}a_{31} = 0$$

where the variables are $a_{10}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{32}, a_{33}$.

The number of variables of this system is 8 with total degree $3^4 \times 2^2 = 324$. The optimal Bézout number is 108 with the partition $\{a_{33}, a_{10}, a_{20}, a_{22}, a_{31}, a_{32}, a_{21}\}, \{a_{30}\}$. The mixed volume is 46 and there are 46 isolated zeros.

5.1.44 A "dessin d'enfant" II system

The following system is available at the Frisco test suite:

$$\begin{aligned}
16a_{20}a_{32} + 18a_{21}a_{31} + 20a_{22}a_{30} &= 0 \\
-80a_{23} + 180a_{34} + 855a_{35} &= 0 \\
7a_{21}a_{31} + 8a_{21}a_{30} &= 0 \\
210a_{35} - 210 &= 0 \\
40a_{20}a_{34} + 44a_{21}a_{33} + 48a_{22}a_{32} + 52a_{23}a_{31} + 280a_{30} &= 0 \\
27a_{20}a_{33} + 30a_{21}a_{32} + 33a_{22}a_{31} + 36a_{23}a_{30} &= 0
\end{aligned}$$

$$
\begin{aligned}
55a_{20}a_{35} + 60a_{21}a_{34} + 65a_{22}a_{33} + 70a_{23}a_{32} + 80a_{30} + 375a_{31} &= 0 \\
78a_{21}a_{35} + 84a_{22}a_{34} + 90a_{23}a_{33} - 170a_{20} + 102a_{31} + 480a_{32} &= 0 \\
136a_{23}a_{35} - 114a_{22} + 152a_{33} + 720a_{34} &= 0 \\
105a_{22}a_{35} + 112a_{23}a_{34} - 144a_{21} + 126a_{32} + 595a_{33} &= 0
\end{aligned}
$$

where the variables are: $a_{20}, a_{21}, a_{22}, a_{23}, a_{30}, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}$.

The number of variables of this system is 10 with total degree $2^8 = 256$. The optimal Bézout number is 126 with the partition $\{a_{20}, a_{32}, a_{21}, a_{31}\}$, $\{a_{22}, a_{30}, a_{23}, a_{34}, a_{35}, a_{33}\}$. The mixed volume is 42 and there are 42 isolated zeros.

5.1.45 Sendra system

The following system is available at the Frisco test suite:

$$
\begin{aligned}
-270x^4y^3 - 314xy^4 - 689xy^3 + 1428 = 0 \\
36x^7 + 417x^6y - 422x^5y^2 - 270x^4y^3 + 1428x^3y^4 - 1475x^2y^5 + 510xy^6 - 200x^4 - 174x^5y \\
-966x^4y^2 + 529x^3y^3 + 269x62y^4 + 49xy^5 - 267y^6 + 529x^4y + 1303x^2y^3 - 314xy^4 + 262y^5 \\
+36x^4 - 788x^2y^2 - 689xy^3 + 177y^4 = 0.
\end{aligned}
$$

The number of variables of this system is 2 with total degree $7^2 = 49$. The optimal Bézout number is 49 with the partition $\{x, y\}$. The mixed volume is 46 and there are 46 isolated zeros.

5.1.46 Parallel robot (left-arm robot)

The following system is available at the Frisco test suite. The problem is to find all the possible positions of the upper platform, given the length of the arms(between the ground and the platform) of the parallel robot. Using quaternions, with the given matrix X of base points and the matrix Y of points of the platform

$$X = \begin{bmatrix} 0 & 1/2 & 3/2 & 3/2 & 1/3 & 2 \\ 0 & -1/2 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and the length $lg = [1, 1, 0.8, 2, 2, 2]$, the following system is obtained:

$$u^2 + v^2 + w^2 - 1 = 0$$

$$-12b - 4a + 4c + 4ab + 4ac + 4bc + a^2 + 5b62 + 13c^2 + 8w +$$
$$2u + 9 + 8acu + 8acw + 8bav + 8bcv + 2ua^2 - 6ub^2 - 6uc^2 +$$
$$2va^2 + 2vb^2 + 2vc^2 + 8wc^2 + 8bu + 8cv - 8av - 8bw + 2v = 0$$

$$8.86 - 6b + 2a + 4abu + 6 - 6ab - 6ac + 6bc + 2.86a^2 + 4.86b^2 +$$
$$6.86c^2 + 4w - 3u + 4bcw + 4aw - 4cu + 4acu + 4bcv - 3ua^2 -$$
$$3ub^2 - 3uc^2 - va^2 + 3vb^2 - vc^2 + 4wc^2 + 4bu - 4av + 3v = 0$$

$$-2b + 2a + 2c + 2ab - 10ac - 2bc + 3.5a^2 - 2.5b^2 + 1.5c^2 + 4w -$$
$$5u + 7.5 + 4acu - 4acw + 4bcv - 5ua^2 - ub^2 - uc^2 -$$
$$va^2 - vb^2 - vc^2 + 4wc^2 + 4bu - 4cv - 4av + 4bw - v = 0$$

$$-4a - 4abu - 1.333333333c + 1.333333333ab - 4bc - 1.638888889a^2 +$$
$$0.361111111b^2 - 1.638888889c^2 + 2w - 0.666666667u - 4bcw - 4aw +$$

$$0.361111111 + 2wb^2 + 4cu - 0.666666667ua^2 - 0.666666667ub^2 -$$
$$0.666666667uc^2 + va^2 - 3vb^2 + vc^2 + 2wc^2 - 3v = 0$$
$$-8b + 4abu + 8c - 8ab - 8ac + 8b^2 + 8c^2 + 2w - 2u + 4bcw +$$
$$4aw - 2wb^2 - 2wa^2 - 4cu + 4acu + 4acw + 4bav + 4bcv -$$
$$2ua^2 - 6ub^2 - 6uc^2 - 2va^2 + 2vb^2 - 2vc^2 + 2wc^2 + 4bu +$$
$$4cv - 4av - 4bw + 2v = 0$$

where the variables are u, v, w, a, b, c.

The number of variables of this system is 6 with total degree $2 \times 3^5 = 486$. The optimal Bézout number is 160 with the partition $\{u, v, w\}, \{a, b, c\}$. The mixed volume is 160 and there are 40 isolated zeros.

5.1.47 Parallel robot with 24 real roots

The following system can be found in [44] and is obtained from the Frisco test suite:

$$62500x_1^2 + 62500y_1^2 + 62500z_1^2 - 74529 = 0$$
$$625x_2^2 + 625y_2^2 + 625z_2^2 - 1250x_2 - 2624 = 0$$
$$12500x_3^2 + 12500y_3^2 + 12500z_3^2 + 2500x_3 - 44975y_3 - 10982 = 0$$
$$400000x_1x_2 + 400000y_1y_2 + 400000z_1z_2 - 400000x_2 + 178837 = 0$$
$$1000000x_{1x}x_3 + 100000y_1y_3 + 100000z_1z_3 + 100000x_3 - 1799000y_3 - 805427 = 0$$
$$2000000x_2x_3 + 2000000y_2y_3 + 2000000z_2z_3 - 2000000x_2 +$$
$$200000x_3 - 3598000y_3 - 1403 = 0$$
$$113800000000000x_3y_2z_1 - 113800000000000x_2y_3z_1 - 113800000000000 *$$
$$x_3y_1z_2 + 113800000000000x_1y_3z_2 + 113800000000000x_2 *$$

$$y_1y_3 - 113800000000000x_1y_2z_3 - 206888400000000x_2y_1 +$$
$$206888400000000x_3y_1 + 206888400000000x_1y_2 - 206888400000000 *$$
$$x_3y_2 - 206888400000000x_1y_3 + 206888400000000x_2y_3 - 20142600000000 *$$
$$x_2z_1 + 2014260000000x_3z_1 - 61907200000000y_2z_1 + 61907200000000 *$$
$$y_3z_1 + 2014000000000x_1z_2 - 2014260000000x_3z_2 + 61907200000000 *$$
$$y_1z_2 - 61907200000000y_3z_2 - 2014260000000x_1z_3 + 2014260000000 *$$
$$x_2z_3 - 61907200000000y_1z_3 + 61907200000000y_2z_3 - 36290716800000 *$$
$$x_1 + 38025201600000x_2 + 292548849600000x_3 + 11809567440000y_1 +$$
$$1475978220000y_2 - 825269402280000y_3 - 1212982689600000z_1 -$$
$$151600474800000z_2 + 825859951200000z_3 - 19295432410527 = 0$$

$$-777600000000x_3y_2z_1 + 777600000000x_2y_3z_1 + 777600000000 *$$
$$x_3y_1z_2 - 777600000000x_1y_3z_2 - 777600000000x_2y_1z_3 +$$
$$777600000000x_1y_2z_3 - 1409011200000x_2y_1 + 1409011200000x_3 *$$
$$y_1 + 1409011200000x_1y_2 - 1409011200000x_3y_2 - 1409011200000 *$$
$$x_1y_3 + 1409011200000x_2y_3 - 106512000000x_2z_1 + 1065312000000 *$$
$$x_3z_1 - 805593600000y_2z_1 + 805593600000y_3z_1 + 1065312000000 *$$
$$x_1z_2 - 1065312000000x_3z_2 + 805593600000y_1z_2 - 805593600000 *$$
$$y_3z_2 - 1065312000000x_1z_3 + 1065312000000x_2z_3 - 805593600000 *$$
$$y_1z_3 + 805593600000y_2z_3 + 23585027200x_1 + 398417510400x_2 +$$
$$158626915200x_3 - 311668424000y_1 - 268090368000y_2 + 72704002800 *$$
$$y_3 + 412221302400z_1 + 354583756800z_2 + 307085438400z_3 + 282499646407 = 0$$

$$3200x_2 + 1271 = 0.$$

Note that the coefficients need to be scaled. The number of variables of this system is 9 with total degree $2^6 \times 3^2 = 576$. The optimal Bézout number is 80 with the partition $\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \{x_3, y_3, z_3\}$. The mixed volume is 80 and there are 40 isolated zeros.

5.1.48 3 Torus system

The following system is available at the Frisco test suite:

$$\begin{aligned}
x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4 - 2x^3 - 2xy^2 - & \\
2xz^2 + x^2 + 1.3333333333y^2 + 1.3333333333z^2 &= 0 \\
x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4 - 2x^3 - 2x^2z - 2xy^2 - 2xz^2 - 2y^2z - & \\
2z^3 + 2.960784314x^2 + 5.921568627xz + 3.921568627y^2 + 2.960784314z^2 - & \\
3.921568627x - 3.921568627x + 1.960784314 &= 0 \\
x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4 - 2x^2z - 2y^2z - 2z^3 + & \\
5.263157895x^2 + 5.263157895y^2 + z^2 &= 0.
\end{aligned}$$

The number of variables of this system is 3 with total degree $4^3 = 64$. The optimal Bézout number is 64 with the partition $\{x, y, z\}$. The mixed volume is 64 and there are 16 isolated zeros.

5.1.49 Emulsion chemistry

The variables of the following system are r_1, r_2, r_3, r_4:

$$r_1(r_2 + r_3 + r_4) = A$$

$$
\begin{aligned}
r_2(r_1+r_3+r_4) &= B \\
r_3(r_1+r_2+r_4) &= C \\
r_4(r_1+r_2+r_3) &= D.
\end{aligned}
$$

The number of variables of this system is 4 with total degree $2^4 = 16$. The optimal Bézout number is 16 with the partition $\{r_1, r_2, r_3, r_4\}$. The mixed volume is 8 and there are 8 isolated zeros.

5.1.50 Commodities market

The variables of the following system are s, t, u, v, w, x, y, z:

$$
\begin{aligned}
u+s-2A &= 0 \\
v+t-2B &= 0 \\
v-u-t+s &= 0 \\
vz-ty-ux+sw &= 0 \\
vz+ty+ux+sw-2ce &= 0 \\
-vz-ty+ux+sw-2cd &= 0 \\
-su(z+y)+st(z+x)+uv(y+w)-tv(x+w)-2p_0(u-t)(v-s) &= 0 \\
u(z-w)+t(w-z)+s(y-x)+v(x-y)-r_0(u-t)(v-s) &= 0.
\end{aligned}
$$

The number of variables of this system is 8 with total degree $2^4 \times 3 = 48$. The optimal Bézout number is 7 with the partition $\{s, t, u, v\}, \{w, x, y, z\}$. The mixed volume s 7. There are 6 isolated zeros.

5.1.51 Raksanyi system

The following system arises from Systems theory with rational function coefficients and is available at the Frisco test suite:

$$\begin{aligned} t + (v - a) &= 0 \\ x + y + z + t - (u + w + a) &= 0 \\ xz + xt + yz + zt - (uw - ua - wa) &= 0 \\ xzt - uwa &= 0 \end{aligned}$$

where the variables are x, y, z, t.

The number of variables of this system is 4 with total degree $3 \times 2 = 6$. The optimal Bézout number is 3 with the partition $\{x, y\}, \{z\}, \{t\}$. The mixed volume is 3 and there are 3 isolated zeros.

5.1.52 Runge-Kunga

This system arises in an application of the Runge-Kutta space. The problem is to construct a modified version of an explicit 3 stage Runge-Kutta method with order 4 [6]:

$$\begin{aligned} b_1 + b_2 + b_3 - (\alpha - \beta) = 0 \\ b_2c_2 + b_3c_3 - (\frac{1}{2} + \frac{1}{2}\beta + \beta^2 - \alpha\beta) = 0 \\ b_2c_2^2 + b_3c_3^2 - (\alpha(\frac{1}{3} + \beta^2) - \frac{4}{3}\beta - \beta^2 - \beta^3) = 0 \\ b_3a_{32}c_2 - (\alpha(\frac{1}{6} + \frac{1}{2}\beta + \beta^2) - \frac{2}{3}\beta - \beta^2 - \beta^3) = 0 \\ b_2c_2^3 + b_3c_3^3 - (\frac{1}{4} + \frac{1}{4}\beta + \frac{5}{2}\beta^2 + \frac{3}{2}\beta^3 + \beta^4 - \alpha(\beta + \beta^3)) = 0 \end{aligned}$$

$$b_3c_3a_{32}c_2 - (\frac{1}{8} + \frac{3}{8}\beta + \frac{7}{4}\beta^2 + \frac{3}{2}\beta^3 + \beta^4 - \alpha(\frac{1}{2}\beta + \frac{1}{2}\beta^2 + \beta^3)) = 0$$
$$b_3a_{32}c_2^2 - (\frac{1}{12} + \frac{1}{12}\beta + \frac{7}{6}\beta^2 + \frac{3}{2}\beta^3 + \beta^4 - \alpha(\frac{2}{3}\beta + \beta^2 + \beta^3)) = 0$$
$$\frac{1}{24} + \frac{7}{24}\beta + \frac{13}{12}\beta^2 + \frac{3}{2}\beta^3 + \beta^4 - \alpha(\frac{1}{3}\beta + \beta^2 + \beta^3) = 0$$

where the variables are $b_1, b_2, b_3, c_2, c_3, a_{32}, \alpha, \beta$.

The number of variables of this system is 8 with total degree $4^4 \times 3^2 \times 2 = 4608$. The optimal Bézout number is 1361 with the partition $\{b_1\}, \{b_2, b_3, \alpha\}, \{\beta\}, \{c_2, c_3\}, \{a_{32}\}$. The mixed volume is 24 and there are 5 isolated zeros.

5.1.53 Solubility of silver chloride in water

The problem is to find the concentrations of all species in a saturated silver chloride solution [32]. After reduction, the following system is generated:

$$\begin{aligned} \alpha_1 x_1^4 + \alpha_2 x_1^3 x_3 + \alpha_3 x_1^3 + \alpha_4 x_1 + \alpha_5 &= 0 \\ \beta_1 x_1 x_3^2 + \beta_2 x_3^2 + \beta_3 &= 0. \end{aligned}$$

The number of variables of this system is 2 with total degree $4 \times 3 = 12$. The optimal Bézout number is 9 with the partition $\{x_1\}, \{x_3\}$. The mixed volume is 9. With the given coefficients, there are 9 isolated zeros.

Table 5.7. Coefficients for the butler's problem

coefficients	
α_1	1.069D-04
α_2	2D04
α_3	1.000
α_4	-1.8D-10
α_5	-1.283D-24
β_1	2D16
β_2	1D14
β_3	-1.000

5.1.54 Enumerative geometry(Hypersurface schubert conditions)

This system is available at the Frisco test suite. The problem is to find those p-plane which intersect $m * p$ given m-planes in $\mathbb{C}^{m+p}$ which osculate the rational normal curve. For $m = 2, p = 3$,

$$F(s_1) = \ldots = F(s_6) = 0$$

where $s_1, \ldots, s_6$ are six distinct real numbers, and

$$F(s) = \det \begin{bmatrix} 1 & 0 & a & b & c \\ s & 1 & e & f & g \\ s^2 & 2s & 1 & 0 & 0 \\ s^3 & 3s^2 & 0 & 1 & 0 \\ s^4 & 4s^3 & 0 & 0 & 1 \end{bmatrix}.$$

Table 5.8. Characteristics of systems that arise in the Schubert Calculus

m	p	*totdeg*	*mdeg*	$\mathcal{M}(\mathcal{Q})$	$\mathcal{N}(\mathcal{Q})$
2	2	16	6	4	2
2	3	64	20	17	5
2	4	256	70	66	14
2	5	1024	252	247	42

The first two columns of the matrix give the 2-planes in $\mathbb{R}^5$ which osculate the rational normal curve. The last three columns are local coordinates on the Grassmanian of 3-planes in $\mathbb{R}^5$. The system $F(s) = 0$ is given by:

$$1 - 2s_ie - 3s_i^2 f - 4s_i^3 g + s_i^2 a + 2s_i^3 b + 3s_i^4 c + s_i^4 af - s_i^4 eb +$$
$$2s_i^5 ag - 2s_i^5 ec + s_i^6 bg - s_i^6 cf = 0, \ i = 1, \ldots, 6$$

where the variables are a, b, c, d, e, f.

The number of variables of this system is 6 with total degree $2^6 = 64$. The optimal Bézout number is 20 with the partition $\{a, b, c\}, \{d, e, f\}$. The mixed volume is 17 and there are 5 isolated zeros.

Table 5.8 summarizes the results for different values of m and p. Here *totdeg* denotes the total degree of the system, *mdeg* its optimal Bézout number $\mathcal{M}(\mathcal{Q})$ its mixed volume and $\mathcal{N}(\mathcal{Q})$ the total number of isolated zeros.

5.2 Summary

Table 5.9 and Table 5.10 summarize the various characteristics of the 54 polynomial systems presented: For each system, n denotes the number of variables, *totdeg* the total degree, *mdeg* the optimal multi-homogeneous Bézout number, $M(\mathcal{Q})$ the mixed volume and $N(\mathcal{Q})$, the total number of isolated zeros.

5.3 Conclusions

Our solver finds all isolated zeros of the above polynomial systems to satisfactory accuracy and speed. As an observation, these systems are characterized by substantially low mixed volume $\mathcal{M}(\mathcal{Q})$, compared to *totdeg*, their total degree. By Theorem 2.1.2, the mixed volume $\mathcal{M}(\mathcal{Q})$ of a given polynomial system is an upper bound on $\mathcal{N}(\mathcal{Q})$, the total number of isolated zeros in $\mathbb{C}^n$. A great majority of the polynomial systems presented here actually have $\mathcal{M}(\mathcal{Q})$ number of isolated zeros. Of both theoretical as well as practical interest is the fact that the numbers $\mathcal{M}(\mathcal{Q})$ are either less than or equal to the optimal multi-homogeneous Bézout number, *mdeg*. We would like to work towards a possible theoretical explanation in the future. For those systems with a discrepancy between $\mathcal{M}(\mathcal{Q})$ and $\mathcal{N}(\mathcal{Q})$, we wish to study and explore the structure of the systems. We hope to derive a method that respect these structures and requires computational efforts proportional to the actual number of isolated zeros of these systems. Overall, the generality of the current solver should establish itself as the method of choice for systems of moderate size.

Table 5.9. Characteristics of the polynomial systems I

no.	name	description	n	*totdeg*	*mdeg*	$M(Q)$	$N(Q)$
1	puma	robot manipulator PUMA	8	128	16	16	16
2	romin	robot manipulator ROMIN	6	32	16	4	4
3	LV	Lotka-Volterra System	4	81	81	73	73
4	sym4	symmetrized four-bar	4	256	96	80	36
5	ninept	nine-point problem	8	7^8	645120	83977	4326
6	chemeqm	chemical equilibrium	5	108	56	16	16
7	lumped	lumped-parameter	4	16	8	7	4
8	heart	heart-dipole	8	576	193	121	4
9	cycmol	cyclic molecules	3	64	16	16	16
10	camera	camera motion	6	64	20	20	20
11	elect	electrical network	4	16	6	6	6
12	vib	vibrating systems	6	243	10	10	10
13	6R	6R inverse position	8	256	96	64	32
14	6R2	6R inverse position	11	1024	320	288	16
15	quad	quadrature formula	5	120	10	10	5
16	wood	the wood function	4	36	25	9	9
17	ec	electrochemistry problem	4	60	52	18	15
18	i1	benchmark i1	10	59049	452	66	50
19	mdi	integration formula	9	36000	8852	136	16
20	AHW	The system of A. H. Wright	5	32	32	32	32
21	R5	The system called Reimer5	5	720	720	720	144
22	butcher	Butcher's problem	7	4608	1361	24	5
23	viralg	Virasoro algebras	8	256	256	200	200
24	cycn	the cyclic n-roots problem	7	5040	5040	924	924
25	comb	combustion chemistry	10	96	44	16	16
26	econ	economic modelling	8	1458	256	64	64
27	neu	neurophysiology	6	1024	216	20	8

Table 5.10. Characteristics of the polynomial systems II

no.	name	description	n	*totdeg*	*mdeg*	$M(\mathcal{Q})$	$N(\mathcal{Q})$
28	speer	the system of E. R. Speer	4	625	384	96	43
29	sol	solotarev	4	36	10	6	6
30	trinks	number theory	6	24	24	10	10
31	galois	$s9_1$ from Galois theory	8	16	10	10	10
32	rediff3	reaction-diffusion problem	3	8	8	7	7
33	lorentz	lorentz attractor	4	16	14	12	11
34	bif	a bifurcation problem	3	126	32	16	16
35	D1	benchmark D1	12	4068	320	192	48
36	capr	caprasse's system	4	144	62	48	48
37	arn	arnborg7	3	120	48	20	20
38	rose	economic equilibrium model	3	216	144	136	136
39	moe	Moeller4	6	8	8	7	7
40	kat	Katsura	6	32	32	32	32
41	cohn2	cohn2	4	900	450	124	18
42	CN	Cassou-Noggues	4	1344	368	24	16
43	dessinI	dessin d'enfant I	8	324	108	46	46
44	dessinII	dessin d'enfant II	10	256	126	42	42
45	sendra	sendra	2	49	49	46	46
46	rbpl	parallel robot(left-arm)	6	486	160	160	40
47	rbpl24	parallel robot	9	576	80	80	40
48	3torus	torus	3	64	64	64	16
49	emulchem	emulsion chemistry	4	16	16	8	8
50	commod	commodities market	8	48	7	7	6
51	raksanyi	systems theory	4	6	3	3	3
52	RK	runge-kutta	8	4608	1361	24	5
53	butler	solubility	2	12	9	9	9
54	schubert	Schubert conditions	10	1024	252	247	42

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