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Wei Wang

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S E Newhouse
Major professor

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Entropy Zero Systems and Morse–Smale Systems

By

Wei Wang

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ABSTRACT

Entropy Zero Systems and Morse–Smale Systems

By

Wei Wang

In this thesis, we prove that if f is a real analytic diffeomorphism on a two dimensional compact Riemannian manifold, the non–wandering set of f is finite and f satisfies locally normalized condition, then, f can be C^r $(r > 0)$ approximated by a Morse–Smale diffeomorphism.

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CHAPTER 0

Introduction and Main Result

Let M be a compact Riemannian manifold. f be a C^r $(r > 0)$ diffeomorphism on M. One of the important quantities to describe the complexity of the structure of a system is topological entropy, which tells roughly how many different orbits f has. A formal definition of entropy is as follows.

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n}$$

where $h(f)$ be the topological entropy of f, $s(n, \epsilon) = \max_{E \subset M}\{\text{cardinality of } E\}$ and E is a set such that if $x, y \in E$ then $d(f^k x, f^k y) > \epsilon$ for some $k \in [0, n)$. In other words, $h(f)$ is the asymptotic growth rate of the number of finite length orbits known with precision ϵ as the length goes to infinity.

Topological entropy is a topological invariant and has very nice properties:

1. (Katok–Newhouse–Yomdin Theorem)[5] [8] [13] If the dimension of M is two, then $h(f)$ is continuous with respect to f in the C^∞ case.

2. (Bowen)[2] $h(f) = h(f|\Omega)$, where Ω is the non–wandering set of f, $\Omega = \{x |$ for any neighborhood $U, \bigcup_{|m|>0} f^m(U) \cap U \neq \emptyset\}$.

3. If the dimension of M is two, then the set of systems with zero entropy is a

closed set.

4. The systems with finite non-wandering set have zero entropy.

5. The system whose non-wandering set contains a horse shoe has positive entropy.

A natural question is what we can say about the systems that have zero entropy.

A Morse-Smale system is a system with following properties: (1). its non–wandering set is finite and hyperbolic; (2). all the intersections between stable manifolds and unstable manifolds are transversal.

Morse-Smale systems are very important systems. Their structures are well understood. A interesting question is that whether we can use Morse-Smale systems to approximate systems with zero entropies?

If the dimension of the manifold M is greater than three, the answer to that question is no. Dankner [3] constructed a counterexample example in the early 1980s. If the dimension of M is two, Newhouse conjectures that it is true.

Conjecture (Newhouse) *On the two dimensional compact manifold, the diffeomorphisms with zero entropies are on the boundary of the set of Morse–Smale systems.*

In this direction, the earliest important work was done by Newhouse and Palis [10] in early 1970s. They developed the breaking cycle technique and proved that any diffeomorphism with hyperbolic non-wandering set can be approximated by Ω-stable diffeomorphism. About fifteen years later, Malta and Pacifico [6] generalized this result by relaxing the condition from hyperbolic non-wandering to hyperbolic limit set.

As a corollary of their results, we see that, in dimension two, a diffeomorphism with finite hyperbolic non-wandering set (or limit set) is on the boundary of Morse-Smale system set. Therefore, the next question is that if we drop the hyperbolic condition, which is a very strong condition, will the same result hold?

Problem (Newhouse) *Can any diffeomorphism on the two dimensional compact manifold with finite non-wandering set be approximated by a Morse–Smale diffeomorphism?*

Definition (locally normalized condition) A diffeomorphism f is called to have locally normalized condition at a fixed point o, if it satisfies the following conditions: (1). in a small neighborhood of o, f can be embedded in an analytic vector field with no elliptic sectors; (2). there is an invariant analytic curve through o.

In this paper, we prove the following theorem

Theorem *Let $\mathcal{M}$ be a two dimensional compact manifold, f be a real analytic diffeomorphism on $\mathcal{M}$, the no-wandering set of f $\Omega(f)$ be finite and f satisfy the locally normalized condition. Then, f can be approximated in C^r $(r > 0)$ by a Morse-Smale diffeomorphism.*

In chapter 1, we review some basic definitions and facts. Chapter 2–5 is the proof of above theorem. In Chapter 2, we work with elementary cycles. We will break all the elementary cycles without causing Ω-explosion. Chapter 3 is about advanced cycles. We will study the local structures around fixed points and remove the advanced cycles without causing Ω-explosion. Systems that have no cycles are discussed in chapter 4. Chapter 5 concludes this paper.

CHAPTER 1

Preliminary

In this Chapter, we review some basic definitions and facts. Throughout this paper, we let $\mathcal{M}$ be a two dimensional compact manifold, f be an analytic diffeomorphism on $\mathcal{M}$. $\Omega(f)$ be the non-wandering set of f, *i.e.* the set of points with the property that for every neighborhood U such that $\bigcup_{|m|>0} f^m(U) \bigcap U \neq \emptyset$. When $\Omega(f)$ is finite, $\Omega(f) = \text{per}(f)$, $\text{per}(f)$ is the set of periodic points of f.

Let $p \in \Omega(f)$, we denote the set of points x such that $d(f^n x, f^n p) \to 0 (n \to \infty)$, where $n \in \mathcal{Z}^+$, by $W^s(p)$. $W^s(p)$ is called the stable set of f at p. The stable set of f^{-1} at p is called the unstable set of f at p, denoted by $W^u(p)$. If $T_p(\mathcal{M})$ can be split as a direct sum $E_p^u \bigoplus E_p^s$ so that $T_p f(E_p^s) = E_{fp}^s$ and $T_p f(E_p^u) = E_{fp}^u$, where $E_p^u = \{v \text{ such that } |T_p f v| \geq \lambda |v|\}$ and $E_p^s = \{v \text{ such that } |T_p f v| \leq \lambda^{-1} |v|\}$ for some constant $\lambda > 1$, then, p is called a hyperbolic point. Clearly, if $p \in \text{per}(f)$ and the two eigenvalues of f at p are not on the unit circle, then p is hyperbolic. If one of its two eigenvalues lies inside of the unit circle, while the other one lies outside of the unit circle, then, p is called a saddle. If both eigenvalues lie inside of the unit circle or outside of the unit circle, then, p is called a node. If one of its two eigenvalues lies on the unit circle, the other one is not on the unit circle, then p is called a semi-hyperbolic periodic point.

Let $p \in \text{per}(f)$, $\mathcal{N}(p)$ be a neighborhood of p. Let $\tilde{p}$ be a saddle of a diffeomorphism

$\tilde{f}$, $\mathcal{N}(\tilde{p})$ be a neighborhood of $\tilde{p}$. If, in $\mathcal{N}(p)$, f is topologically equivalent to $\tilde{f}$ in $\mathcal{N}(\tilde{p})$, then, we call p a topological saddle. Similarly, we can define topological node and topological saddle-node. A semi-hyperbolic point can be a saddle-node or a topological saddle-node or a topological saddle or a topological node.

By the Invariant Manifold Theorem, when p is a saddle, the stable set $W^s(p)$ and unstable set $W^u(p)$ are both one dimensional manifolds.

A diffeomorphism f is called a Morse-Smale diffeomorphism, if the following properties are satisfied:

1. $\Omega(f)$ is finite and hyperbolic,

2. the intersections between $W^u(p)$ and $W^s(q)$ for any points p q in $\Omega(f)$ are transversal.

We say that a curve $\Gamma = \{(x(t), y(t)), t \in [0, \epsilon]\}$ enters p under f, if

1. $(x(0), y(0)) = (0, 0)$,

2. $\lim_{t\to 0} y(t)/x(t)$ or $\lim_{t\to 0} x(t)/y(t)$ exists,

3. Γ is invariant under f and $\lim_{n\to\infty} f^n(x(t), y(t)) = p$ (or $\lim_{n\to\infty} f^{-n}(x(t), y(t)) = p$) for $t \in (0, \epsilon)$.

CHAPTER 2

Elementary Cycles

In this chapter, we consider the diffeomorphisms whose nonwandering sets consist of finite fixed points and that has only elementary cycles. We will break the elementary cycles without causing Ω–explosion.

In our following discussion, we only consider orientation preserving diffeomorphism.

Let $\Theta(f)$ be the set of all non-wandering points of f in $\Omega(f)$ which have at least one non-empty hyperbolic sector, one stable separatrix and one unstable separatrix, such that one of which must be a separatrix of this hyperbolic sector. We denote a hyperbolic sector of p by $HS(p)$ and a non-hyperbolic sector by $NHS(p)$.

Definition Let $p,\ q \in \Theta(f)$, $S(p)$ and $S(q)$ be a sector of p and q respectively, and an unstable separatrix $S^u_\varepsilon(p) \subset \overline{S(p)}$ and a stable separatrix $S^s_\varepsilon(q) \subset \overline{S(q)}$. Then, we call that sector $S(q)$ follows sector $S(p)$, if following two properties are satisfied:

1. $S^u(p) \bigcap \widehat{S^s(q)} \neq \emptyset$, where $\widehat{S^s(q)} = S^s(q) - \{q\}$, $S^u(p) = \bigcup_{n\geq 0} f^n(S^u_\varepsilon(p))$, $S^s(q) = \bigcup_{n\geq 0} f^{-n}(S^s_\varepsilon(q))$

2. Let $z \in S^u(p) \bigcap \widehat{S^s(q)}$, $x \in S^u_\varepsilon(p)$, $y \in S^s_\varepsilon(q)$, let D be an arbitrary small neighborhood of curve segment $\gamma = [x\ z] \cup [z\ y]$. Then, $S(p) \bigcap \widetilde{D} \neq \emptyset$ and $S(q) \bigcap \widetilde{D} \neq \emptyset$, where $\widetilde{D}$ is a connected component of $D \setminus \gamma$.

We denote that sector of q, $S(q)$, follows sector of p, $HS(p)$, by $S(p) \succ S(q)$ or $S(q) \prec S(p)$. We also have the corresponding notations when the sectors are hyperbolic and non-hyperbolic sectors.

For any p, $q \in \Omega(f)$, if there is a series of points $p = p_0, p_1, p_2, \ldots, p_k = q$ in $\Omega(f)$ such that $W^u(p_i) \bigcap W^s(p_{i+1}) \neq \emptyset$, $i = 0, 1, \ldots, k-1$, we call that $p \succ q$. If there is a series points $p_1, p_2, \cdots, p_l$ in $\Omega(f)$ such that $p_1 \succ p_2 \succ \cdots \succ p_l$, we call it a chain, denoted by $\mathcal{C}[p_1, p_2, \ldots, p_l]$.

Definition A chain $[p_0, p_1, \ldots, p_k]$, $p_i \in \Theta(f)$, is called an elementary chain, if

$$S^u(p_{i-1}) \bigcap \widehat{S^s(p_i)} \neq \emptyset \text{ for } i = 1, 2, \ldots, k \tag{2.1}$$

and

$$S^\sigma(p_i) \subset \overline{HS(p_i)} \text{ for } i = 1, 2, \ldots, k-1 \text{ and } \sigma = u \text{ or } s \tag{2.2}$$

In this case, we call that $S^\sigma(p_i)$ are in this chain, for $\sigma = u$ or s.

We denote the elementary chain by $\mathcal{C} = [HS(p_0), HS(p_1), \ldots, HS(p_k)]$.

Definition If (2.1) and (2.2) are both true for all $i \in \mathcal{Z}^+$, where $p_i = p_{i \; mod(k+1)}$, then the chain is called an elementary cycle. All the other cycles are called advanced cycles, and we call that $S^\sigma(p_i)$ is in this cycle, for $\sigma = u$ or s.

We denote the elementary cycle by

$$\Delta = [HS(p_0), HS(p_1), \ldots, HS(p_k), HS(p_0)]$$

In the following, we list some simple lemmas which will be used later in this section.

Lemma 2.0.1 *Let $p \in \Theta(f)$ and be a fixed point of f, $HS(p)$ be a hyperbolic sector determined by $S^s(p)$ and $S^u(p)$, V be a neighborhood of p, let $q \in V \bigcap \widehat{S^s(p)}$, $y \in$*

$V \bigcap HS(p)$ and $J = [q, y]$ be a C^r $(r > 0)$ curve segment. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that if $n > n_0$, $f^n J$ is ε-close to $S^u(p)$. □

Definition Let $p \in \Theta(f)$, $q \in \Omega(f)$, $HS(p)$ be determined by $S^u(p)$ and $S^s(p)$, we say that $S^u(p)$ is directly accumulated by $S^u(q)$ through $HS(p)$, if there exists a arc $[x, y] \subset S^u(q) \bigcap \overline{HS(p)}$ and $x \in S^s(p)$.

We say that an open set V meeting $S^\sigma(p)$ ($\sigma = u$ or s) at x, $x \in S^\sigma(p)$, if $V = V' \cap HS(p)$, where V' is a neighborhood of x in $\mathcal{M}$.

Lemma 2.0.2 *Let p, q be fixed points of f in $\Theta(f)$, $HS(p)$ and $HS(q)$ be determined by $S^\sigma(p)$ and $S^\sigma(q)$ respectively, ($\sigma = s, u$), $HS(p) \succ HS(q)$, let V be an open set meeting $S^s(p)$ at x, $x \in S^s(p) \bigcap \overline{HS(p)}$, U be an open set meeting $S^u(q)$ at y, $y \in S^u(q) \bigcap \overline{HS(q)}$. Then, there exists an integer $m > 0$ such that $f^m(V) \cap U \neq \emptyset$.*

Proof:

Let J be a segment in V that is transversal with $S^s(p)$ at x. By Lemma 2.0.1, there exists n_1 such that $f^{n_1}(J)$ is ε -close to $S^u(p)$. Let $z \in S^u(p) \bigcap S^s(q)$ and N' be a neighborhood of z in $\mathcal{M}$, such that $N = N' \bigcap HS(p) \neq \emptyset$. Then, there exists an integer m_1 such that $f^{m_1} J \bigcap N \neq \emptyset$, let $f^{m_1} J \bigcap N = J'$. We know that there exists n_2 such that $f^{n_2} z \in \overline{HS(q)}$, since $HS(p) \succ HS(q)$, we have an open disk D, as in the definition at page 6, such that $z \in D$, $f^{n_2} z \in D$ and $HS(p) \cap D$, $HS(q) \cap D$ both are contained in the same connected component $\widetilde{D}$ of $D \setminus \gamma$.

We claim that

$$f^{n_2}(J') \bigcap \widetilde{D} \bigcap HS(q) \neq \emptyset$$

To prove the claim, If $S^u(p) = S^s(q)$, the claim is clear; if $S^u(p) \neq S^u(q)$ and $S^u(p)$ directly accumulates on $S_u(q)$, then $\exists\, G' \subset G$ such that $f^{n_2}(G') \subset HS(q) \cap \widetilde{D}$, since $f^{n_2}(z) \in \widetilde{D} \cap HS(q)$. Note that $HS(q) \cap \widetilde{D}$ is open, so $f^{n_2}(J) \cap \widetilde{D} \cap HS(q) \neq \emptyset$; if

$S^u(p) \neq S^s(q)$ and $S^s(q)$ is directly accumulates on $S^s(p)$, we use the similar argument to get the claim.

For the situation when f is orientation reversing, we can follow the similar process as above to prove the claim.

Now, in $HS(q)$, from Lemma 2.0.1. we have

$$S^s_\epsilon(q) \subset \overline{\bigcup_{n\geq 0} f^{-n}(U)}$$

therefore

$$z \in \overline{\bigcup_{n\geq 0} f^{-n}(U)}$$

so

$$f^{n_2}(J')\bigcap(\bigcup_{n\geq 0} f^{-n}(U)) \neq \emptyset$$

this means that there exists $m > 0$ such that $f^m(J')\bigcap U \neq \emptyset$, thus $f^m(V) \cap U \neq \emptyset$, the lemma is proved. □

Definition Let $\{q_j\}$ be fixed points of f in $\Omega(f)$, $HS(q_j)$ be determined by $S^\sigma(q_j)$, $(\sigma = s, u)$ and $j = 0, 1, 2, \ldots, k$. Then, an elementary cycle

$$\Delta = [HS(q_0), HS(q_1), \ldots, HS(q_k), HS(q_0)]$$

$(k \geq 0)$ is called a simple cycle if $S^u(q_i) = S^s(q_{i+1})$ and $HS(q_i) \succ HS(q_{i+1})$, where $i = 0, 1, 2, \cdots; q_i = q_{i \ mod(k+1)}$.

Corollary 2.0.3 *Let* $\Delta = [HS(q_0), HS(q_1), \ldots, HS(q_k), HS(q_0)]$, $k \geq 0$, *be a simple cycle, then,* $S^\sigma(q_i) \subset \Omega(f)$, *where* $S^\sigma(q_i) \subset \overline{HS(q_i)}$, $0 \leq i \leq k$, $\sigma = s, u$.

Proof:

It is enough to prove that $S^s(q_0) \subset \Omega(f)$. For any $x \in S^s(q_0)$, let V be an open set meeting $S^s(q_0)$ at x. To prove $x \in \Omega(f)$, we only have to prove that there is a

$m > 0$ such that $f^m(V) \cap V \neq \emptyset$. Since, $S^u(q_k) = S^s(q_0)$, there exists n_x such that $f^{-n_x}x \in HS(q_k)$ and $f^{-n_x}x \in \overline{\bigcup_{n>0} f^{-n}V}$. Let U be an open set such that it meets $S^u(q_k)$ at $f^{-n_x}x$ and $U \subset \bigcup_{n\geq 0} f^{-n}(V)$. We claim that there is an integer $l > 0$ such that

$$f^l(V) \bigcap U \neq \emptyset$$

therefore,

$$f^m(V) \bigcap (\bigcup_{n\geq 0} f^{-n}(V)) \neq \emptyset$$

it follows that there is a $m > 0$ such that $f^m(V) \bigcap V \neq \emptyset$.

Now, we prove the claim. Let $x_1 \in S^u(q_1) \bigcap \overline{HS(q_1)}$, V_1 be an open set meeting $S^u(q_1)$ at x_1. Since, $HS(q_2) \succ HS(q_1)$, by Lemma 2.0.2, there is a $m_1 > 0$ such that $f^{m_1}(V) \bigcap V_1 \neq \emptyset$. Since, $S^u(q_1) = S^s(q_2)$, there is a $n_{x_1} > 0$ such that $f^{n_{x_1}}x_1 \in S^s(q_2) \bigcap HS(q_2)$. Let U_1 be an open set contained in $f^{n_{x_1}}V_1$ and meeting $S^s(q_2)$ at $f^{n_{x_1}}x_1$, then, $f^{m_1+n_{x_1}}(V) \bigcap U_1 \neq \emptyset$. We repeat above process by starting with U_1 instead of V_1, we find an integer $l > 0$ such that $f^l(V) \bigcap U \neq \emptyset$. This proves the corollary. □

Definition An elementary chain $\mathcal{C} = [HS(q_0), HS(q_1), \cdots, HS(q_k)]$ is called an improper chain if

$$HS(q_0) \succ HS(q_1) \succ \cdots \succ HS(q_k)$$

If in above definition $q_0 = q_k$, then it is called an improper cycle.

Corollary 2.0.4 *Let q_i be fixed points of f for $i = 0, 1, 2, \ldots, k$ and*

$$\Delta = [HS(q_0), HS(q_1), \cdots, HS(q_k), HS(q_0)]$$

be an improper cycle. If there are two points q_i, q_j on this cycle such that $S^u(q_i) \cap S^s(q_j) \neq \emptyset$ and $S^u(q_j)$ is directly accumulated by $S^u(q_i)$ or $S^s(q_i)$ is directly *accumulated by* $S^s(q_j)$, *then* $S^u(q_i) \cap S^s(q_j) \subset \Omega(f)$.

Proof:

It is enough to prove this corollary when $S^u(q_j)$ is directly accumulated by $S^u(q_i)$. Let $x \in S^s(q_j) \bigcap S^u(q_i) \bigcap \overline{HS(q_j)}$, $y \in S^s(q_i) \bigcap \overline{S^s(q_j)}$, then there exists $n_j > 0$ such that $f^{-n_j}(x) \in S^u(q_i) \cap \overline{HS(q_i)}$. Let V be an open set meeting $S^s(q_j)$ at x and $V \subset \overline{HS(q_j)}$, U be an open set meeting $S^u(q_i)$ at $f^{-n_i}(x)$ and $U \subset \overline{HS(q_j)} \bigcap f^{-n_j}(V)$. By using the similar argument as in the proof of Corollary 2.0.3 and Lemma 2.0.2, we find a $m > 0$ such that $f^m(V) \cap U \neq \emptyset$, it follows $f^{m+n_j}(V) \cap V \neq \emptyset$, this proves the corollary. □.

Proposition 2.0.5 *If* $\dim(\mathcal{M}) = 2$, $f \in \mathrm{Diff}^r(\mathcal{M})$, $\Omega(f)$ *is finite, then there are no improper cycles.*

Proof:

Let us suppose, by the way of contradiction, that there are improper cycles and $\Delta = [HS(q_0), HS(q_1), \ldots, HS(q_k), HS(q_0)]$ be one of them. That means that $S^u(q_i) \bigcap S^s(q_{i+1}) \neq \emptyset$ $(0 \leq i < k)$ and $S^u(q_k) \cap S^s(q_0) \neq \emptyset$ and $HS(q_{i+1}) \prec HS(q_i)$, $HS(q_0) \prec HS(q_k)$.

If there exists a q_i such that $S^u(q_i)$ does not coincides with $S^s(q_{i+1})$. Since $HS(q_{i+1})$ follows $HS(q_i)$, then, $S^u(q_{i+1})$is either directly accumulated by $S^u(q_i)$ or $S^s(q_i)$ is accumulated by $S^s(q_{i+1})$. By Corollary 2.0.4, we know that $S^s(q_i) \cap S^u(q_{i+1}) \subset \Omega(f)$. However, the set $\{S^s(q_i) \cap S^u(q_i)\}$ is infinite. It contradicts with the assumption that $\Omega(f)$ is finite. So we only have to consider the case when $S^u(q_i) = S^s(q_{i+1})$, $0 \leq i < k$ and $S^u(q_k) = S^s(q_0)$, it means that Δ is just a simple cycle. By Corollary 2.0.3, we know that in this case $S^s(q_i) \subset \Omega(f)$. This makes $\Omega(f)$ infinite. So there are no improper cycles. □

Definition Let $q, p \in \Omega(f)$, $S(p)$ and $S(q)$ be two sectors, we say that $S^u(q)$ visits $S(p)$ if there exists a sequence $\{x_n \mid n = 1, 2, \ldots\} \subset S^u(q) \bigcap \overline{S(p)}$, such that

$\lim_{n \to \infty} x_n = p$.

Definition Let $q, p \in \Omega(f)$, $S^u(q)$ is said to die at p through $NHS(p)$, if for any $x \in \widehat{S^u(q)}$, the ω-limit set of x is $\{p\}$.

Let p, q be any two points in $\Omega(f)$, we call that p is equivalent to q, if there exist a series of points in $\Omega(f)$

$$p = p_0, p_1, p_2, \ldots, p_{k-1}, p_k = q = q_0, q_1, \ldots, q_{l-1}, q_l = p$$

such that for any p_i and q_j, the following are satisfied:

$$W^u(p_i) \bigcap W^s(p_{i+1}) \neq \emptyset, \quad i = 0, 1, \ldots, k-1$$

$$W^u(q_j) \bigcap W^s(q_{j+1}) \neq \emptyset, \quad j = 0, 1, \ldots, l-1$$

Notice that this relation between points in $\Omega(f)$ is a equivalence relation, therefore, there is a classification of $\Omega(f)$ according to this equivalence relation. Let the equivalence classes are $\{\gamma_i\}$, $i = 1, 2, \ldots, m$, then, $\Omega(f) = \bigcup_1^m \gamma_i$.

Remark:

1. The set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ of equivalence classes is naturally partially ordered by $\gamma_i \leq \gamma_j$, if there exist $p_i \in \gamma_i$ and $q_j \in \gamma_j$ such that $W^u(q_j) \bigcap W^s(p_i) \neq \emptyset$.
2. All the cycles (elementary cycles and advanced cycles) in $\Omega(f)$ are contained in γ_i for $i = 1, 2, \ldots, m$.
3. We say that an equivalence class γ_i is trivial, if γ_i only contains a fixed point.

Let Ξ be the set of all fixed points in γ whose unstable separatrix does not cross any stable separatrix of fixed points in γ.

Definition For any $p \in \gamma_i \cap \Xi(f)$, we call that $S^u(p)$ is free if

$$S^u(p) \subset \bigcup_{\gamma_j < \gamma_i} W^s(\gamma_j)$$

or $S^u(p)$ dies at $\tilde{q}$ through a $NHS(\tilde{q})$, where $W^s(\gamma_j) = \bigcup_{q \in \gamma_j} S^s(q)$.

We also have a corresponding definition by interchanging u and s.

Lemma 2.0.6 *Let q_2, q be two fixed points of f, q_1 be a fixed point in $\Theta(f)$, $S^\sigma_\varepsilon(q_1) \subset \overline{HS(q_1)}$, $\sigma = s, u$, $S^u(q_1)$ is directly accumulated by $S^u(q)$ through $HS(q_1)$ and $S^u(q_1) = S^s(q_2)$, then, $S^u(q)$ will visit $S(q_2)$, where $S(q_2)$ is a sector of q_2 such that $S(q_2) \prec HS(q_1)$.*

Proof:

Since $S^u(q_1)$ is directly accumulated by $S^u(q)$ through $HS(q_1)$. By the definition, there exists a point $x \in \overline{HS(q_1)} \bigcap S^u(q_1)$ and $y \in HS(q_1) \bigcap S^s(q_1)$ such that $[x, y] \subset HS(q_1)$. Let $J = [x, y]$. By Lemma 2.0.1, we know that there exists $n_0 \in \mathbf{N}$ such that for any $n > n_0$ $f^n J$ ε-close $S^u(q_1)$. Since $S^u(q_1) = S^s(q_2)$, hence $f^n J$ ε-close $S^s(q_2)$.

Let N_δ be a δ-neighborhood of q_2, then $\exists\ m > 0$ such that $f^m J \bigcap N_\delta \neq \emptyset$. Choose $x_\delta \in f^m J \bigcap N_\delta$, let $\delta \longrightarrow 0$, then we get a sequence $\{x_\delta\}$ such that $x_\delta \in S^u(q)$ and $x_\delta \longrightarrow q_2$. The lemma is proved. □

The 2-dimensional curve Γ_1 is called to cross 2-dimensional curve Γ_2, if $\Gamma_1 \bigcap \Gamma_2 \neq \emptyset$ and $\exists$ a neighborhood V of $x \in \Gamma_1 \bigcap \Gamma_2$ such that $N_1 \bigcap \Gamma_2 \neq \emptyset$, $N_2 \bigcap \Gamma_2 \neq \emptyset$, where N_1 and N_2 are two different connected components of $N \setminus \Gamma_1$.

Lemma 2.0.7 *Let q, q_2 be fixed points of f, q_1 be a point in $\Theta(f)$ and $S^\sigma(q_1) \subset \overline{HS(q_1)}$, $\sigma = s, u$, if $S^u(q)$ crosses $S^s(q_1)$ and $S^u(q_1)$ crosses $S^s(q_2)$, then $S^u(q)$ crosses $S^s(q_2)$.* □

In the following discussion, we denote by γ a non-trivial equivalence class in $\{\gamma_i\}$, $i = 1, 2, \ldots, m$, that contains an elementary cycle.

Corollary 2.0.8 *There exist points $q_0, q_1 \in \gamma \cap \Xi(f)$ such that*

$$S^u(q_0) \bigcap S^s(q_1) \neq \emptyset$$

and $S^u(q_0)$ does not cross any $S^s(p)$ for all $p \in \gamma$.

Proof:

Since γ is non-trivial and contains an elementary cycle, we have q, $p \in \gamma$ such that

$$S^u(q) \bigcap S^s(p) \neq \emptyset$$

Suppose, by the way of contradiction, that for any $q_0 \in \gamma$, if $S^u(q_0) \cap S^s(q_1) \neq \emptyset$ for some $q_1 \in \gamma$, then, $S^u(q_0)$ must cross $S^s(q_1)$. Then, by the definition of γ and Lemma 2.0.7, we have that $S^u(q_0)$ crosses $S^s(q_0)$. By Corollary 2.0.4, this implies $S^u(q_0) \cap S^s(q_0) \subset \Omega(f)$. This contradicts the assumption that $\Omega(f)$ is finite. This proves the corollary. □

By above Corollary 2.0.8, we know that $\Xi \neq \emptyset$, if $\Omega(f)$ is finite.

For any $p \in \gamma$, we denote $\Lambda(p) = \{q \in \gamma$ such that $S^u(p) \cap S^s(q) \neq \emptyset\}$.

Lemma 2.0.9 *Suppose $\Omega(f)$ is finite, then there exist points $q \in \Xi \cap \Theta(f)$ and $p \in \gamma$ such that one of the following properties is satisfied:*

1. *$S^u(q) \neq S^s(p)$ and $S^u(q) \cap S^s(p) \neq \emptyset$*

2. *$S^u(q) = S^s(p)$ and there exists a hyperbolic sector of p, $HS_2(p)$, such that a unstable separtrix $S^{u'}(p)$ is free, where $S^{u'}_\varepsilon(p) \subset \overline{HS_2(p)}$ and $S^s_\varepsilon(p) \subset \overline{HS_2(p)}$.*

3. $S^u(q) = S^s(p)$ and there exists a non-hyperbolic sector of p, NHS(p), such that $NHS(p) \prec HS(q)$, where $S^u_\varepsilon(q) \subset \overline{HS(q)}$ and $S^s_\varepsilon(p) \subset \overline{NHS(p)}$.

Proof:

Suppose there are no $q \in \Xi \cap \Theta(f)$, $p \in \gamma$ such that

$$S^u(q) \bigcap S^s(p) \neq \emptyset \text{ and } S^u(q) \neq S^s(p)$$

that means that, for any two fixed points p and q in γ, if $S^u(q) \cap S^s(p) \neq \emptyset$ and $S^u(q)$ does not cross $S^s(p)$ then $S^u(q) = S^s(p)$.

Let $p_0 \in \gamma \cap \Xi$. If $\exists\ p_1 \in \Lambda(p_0)$ such that $S^u(p_0) = S^s(p_1)$ and either (ii) or (iii) is satisfied by changing p_0 to q and p_1 to p, then we are done. If there is no such p_1 in $\Lambda(p_0)$, then we can choose a p_1 in $\Lambda(p_0)$ such that

$$S^u(p_0) = S^s(p_1) \text{ and } HS(p_1) \prec HS(p_0)$$

where $S^u_\varepsilon(p_0) \subset \overline{HS(p_0)}$ and $S^s_\varepsilon(p_1) \subset \overline{HS(p_1)}$.

We claim that $p_1 \in \Xi$. In fact, if $p_1 \notin \Xi$ then $\exists\ p \in \gamma$, such that $S^u(p_1)$ crosses $S^s(p)$, where $S^u_\varepsilon(p_1) \subset \overline{HS(p_1)}$. By Lemma 2.0.7, we get that $S^u(p_0)$ crosses $S^s(p)$, which contradicts the fact that $p_0 \in \Xi$.

Based on point p_1, we do the same thing as we just did based on p_0 and repeat this process n times (or we may have already got (ii) or (iii) and hence, finished the proof), then we will get an elementary chain

$$\mathcal{C} = [HS(p_0), HS(p_1), \cdots, HS(p_n)]$$

where $HS(p_k) \prec HS(p_{k-1})$ and $S^u(p_{k-1}) = S^s(p_k)$, $k = 1, 2, \ldots, n$. Because γ is finite, n can not go to infinite. So, either we stop at some step by getting a point p_j such that (ii) or (iii) is satisfied if changing p_{j-1} to q and p_j to p, or we get a elementary cycle

$$\Delta = [HS(p_0), HS(p_1), \cdots, HS(p_0)]$$

Note that Δ actually is a simple cycle. By Corollary 2.0.3, we know that in this case, $S^u(p_k) \subset \Omega(f)$, for $k = 0, 1, \ldots, n$. This contradicts the assumption that $\Omega(f)$ is finite. The lemma is proved. □

Proposition 2.0.10 *Suppose that $\Omega(f)$ be finite, then there exist $q_1 \in \gamma$ and $q_0 \in \Xi$ such that*

$$S^u(q_0) \bigcap S^s(q_1) \neq \emptyset$$

Moreover,

1. *If $S^u(q_0) = S^s(q_1)$. Then, either $S^u(q_1)$ is free or there exists a point $q_2 \in \gamma$ such that $S^u(q_1) = S^s(q_2)$, $HS(q_2) \not\prec HS(q_1)$ and $S^s(q_2) \subset \overline{NHS(q_2)}$, where $S^\sigma(q_1) \subset \overline{NHS(q_1)}$, for $\sigma = u$ or s.*

2. *If $S^u(q_0) \neq S^s(q_1)$. Then, either $HS(q_1) \prec HS(q_0)$, $S^u(q_1)$ is accumulated by $S^u(q_0)$ and $S^u(q_1)$ is free or $S^u(q_0)$ dies at q_1 through the $NHS(q_1)$ or there exists q_2 such that $S^u(q_1) \bigcap S^s(q_2) \neq \emptyset$ and $S^u(q_0)$ dies at q_2 through $NHS(q_2)$.*

Proof:

By Lemma 2.0.9, we only have consider the case when $\exists\ q \in \Xi$ and q_1 in γ such that

$$S^u(q) \bigcup S^s(q_1) \neq \emptyset$$

and

$$S^u(q) \neq S^s(q_1)$$

Now, there are only two possible situations between $S^u(q)$ and $S^u(q_1)$. The first case is that there is a hyperbolic sector $HS(q_1)$ determined by $S^u(q_1)$ and $S^s(q_1)$ through which $S^u(q_1)$ is directly accumulated by $S^u(q)$; The second case is that $S^u(q_1)$ dies at q_1 through a non-hyperbolic sector $NHS(q_1)$, where $S^s_\varepsilon(q_1) \subset \overline{NHS(q_1)}$.

If second case occurs, it is just a part of (ii) of the proposition, then we are done.

Now, let us consider the first situation. If in this case, $S^u(q_1)$ is free, then it is a part of (ii) of proposition, we finish. So, we only have to consider the case when $S^u(q_1)$ is not free. We claim that in this case, $q_1 \in \Xi$. As a matter of fact, if $q_1 \notin \Xi$, that is to say that there exists a $p \in \gamma$ such that $S^u(q_1)$ crosses $S^s(p)$. By Lemma 2.0.7, $S^u(q)$ will cross $S^s(p)$, this contradicts the fact that $q \in \Xi$.

Let $q_2 \in \gamma$ such that $S^u(q_1) \bigcap S^s(q_2) \neq \emptyset$. We have to consider following six possible situations.

Case 1. $S^u(q_1) \neq S^s(q_2)$, $HS(q_2) \prec HS(q_1)$ and $S^u(q_1)$ dies at q_2 through $NHS(q_2)$, where $S^s_\varepsilon(q_2) \subset \overline{NHS(q_2)}$ and $S^\sigma_\varepsilon(q_i) \subset \overline{HS(q_i)}$, where $\sigma = u,\ s$ and $i = 1,\ 2$.

Since $S^u(q_1)$ does not cross $S^s(q_2)$, $S^s(q_1)$ is accumulated by $S^s(q_2)$ through $HS(q_1)$; However, by the assumption, we know that $S^u(q_1)$ is accumulated by $S^u(q)$ through $HS(q_1)$ and $HS(q_1)$ is determined by $S^u(q_1)$ and $S^s(q_1)$. So, $S^u(q)$ crosses $S^s(q_2)$ in the hyperbolic sector $HS(q_1)$, this contradicts the fact that $q \in \Xi$. Hence, this case can not occur.

Case 2. $S^u(q_1) \neq S^s(q_2)$, $HS(q_2) \not\prec HS(q_1)$, $S^u(q_1)$ directly accumulates on $S^u(q_2)$ through $HS(q_2)$, where $S^\sigma_\varepsilon(q_i) \subset \overline{HS(q_i)}$, where $\sigma = s,\ u$ and $i = 1.\ 2$.

Since $S^u(q_1)$ does not cross $S^s(q_2)$, $S^s(q_2)$ will directly accumulate on $S^s(q_1)$. By using the same argument as in (1), we get that $S^u(q)$ crosses $S^u(q_2)$, this contradicts with the fact that $q \in \Xi$. This means this case can not occur.

Case 3. $S^u(q_1) \neq S^s(q_2)$, $HS(q_2) \not\prec HS(q_1)$, $S^u(q_1)$ dies at q_2 through a non-hyperbolic sector $NHS(q_2)$ of q_2, where $S^u_\varepsilon(q_2) \subset \overline{NHS(q_2)}$ and $S^\sigma_\varepsilon(q_i) \subset \overline{HS(q_i)}$, where $\sigma = s,\ u$ and $i = 1,\ 2$.

Since $S^u(q_1)$ is directly accumulated by $S^u(q)$ through $HS(q_1)$, $S^u(q)$ will visit $NHS(q_2)$, therefore, $S^u(q)$ will die at q_2. This proves the proposition in this case.

Case 4. $S^u(q_1) = S^s(q_2)$, $HS(q_2) \not\prec HS(q_1)$, where $S^u_\varepsilon(q_1) \subset \overline{HS(q_1)}$, $S^s_\varepsilon(q_2) \subset \overline{HS(q_2)}$.

Let $S^u(q_2)$ be the unstable sepratrix of q_2 such that

$$S^u_\varepsilon(q_2) \subset \overline{HS(q_2)}$$

If there exists a hyperbolic sector $HS_2(q_2)$ of q_2 such that

$$S^s_\varepsilon(q_2) \subset \overline{HS_2(q_2)} \text{ and } S^u_\varepsilon(q_2) \bigcap \overline{HS_2(q_2)} = \emptyset \tag{2.3}$$

Let $S^{u'}_\varepsilon(q_2)$ be the unstable sepratrix of q_2 such that

$$S^{u'}_\varepsilon(q_2) \subset \overline{HS_2(q_2)}$$

Then, since, $HS(q_2) \not\prec HS(q_1)$, $S^{u'}_\varepsilon(q_2)$ is free.

If there is no hyperbolic sector of q_2 satisfying (3), then, there exists a non-hyperbolic sector $NHS(q_2)$ of q_2 such that $NHS(q_2) \prec HS(q_1)$. Since $S^u(q_1)$ is directly accumulated by $S^u(q)$ and $S^u(q_1) = S^s(q_2)$, $S^u(q)$ will visit $NHS(q_2)$. So $S^u(q)$ will die at q_2 through $NHS(q_2)$ This proves the proposition in Case 4.

Case 5. $S^u(q_1) \neq S^s(q_2)$, $HS(q_2) \prec HS(q_1)$ and $S^u(q_1)$ accumulates directly on $S^u(q_2)$, where $S^\sigma(q_2) \subset \overline{HS(q_2)}$, $\sigma = s,\ u$.

We know that if $S^u(q_2)$ is free then we are done. Now, suppose $S^u(q_2)$ is not free, Then, $S^s(q_2)$ must not cross any unstable separatrix of fixed point in γ. In fact, if there exists a p_0 such that $S^u(q_2)$ cross $S^s(p_0)$, then, $S^s(q_2)$ will be directly accumulated by $S^s(p_0)$ through $HS(q_2)$. So, $S^u(q_1)$ crosses $S^s(p_0)$. This contradicts the assumption that $S^u(q_1)$ does not cross any stable separatrix in γ. Now, we start with q_2 as we did with q_1, repeating our analysis from Case 1 to Case 6. This procedure must be stopped at one of two situations (a) and (b) listed below or else we get contradiction.

(a). we get a fixed point q_i and q_{i+1} in γ such that $S^u(q_i)$ does not cross any unstable separatrix in γ. $S^u(q_i) = S^s(q_{i+1})$ and either $S^s(q_{i+1})$ is free or there exists q_{i+2} such that $S^u(q_{i+1}) = S^s(q_{i+2})$ and $HS(q_{i+2})$ does not follow $HS(q_{i+1})$, q_{i+2} has a non-hyperbolic sector $NHS(q_{i+2})$ such that $NHS(q_{i+2}) \prec HS(q_{i+1})$ and $S^u_\varepsilon(q_{i+2}) \subset \overline{NHS(q_{i+2})}$.

(b). we get two fixed points q_i and q_{i+1} in γ such that either $HS(q_{i+1})$ follows $HS(q_i)$ and $S^u(q_{i+1})$ is accumulated by $S^u(q_i)$ and $S^u(q_{i+1})$ is free or $S^u(q_i)$ dies at q_{i+1} through a non-hyperbolic sector $NHS(q_{i+1})$ or there exist q_{i+2} such that $S^u(q_i) \cap S^s(q_{i+1}) \neq \emptyset$ and $S^u(q_i)$ dies at q_{i+2} through the non-hyperbolic sector $NHS(q_{i+2})$.

If (a) or (b) happens, then, let $q_0 = q_i, q_1 = q_{i+1}, q_2 = q_{i+2}$, we prove the proposition. If our above procedure is not stopped at (a) or (b), then, we get a improper chain. Since the number of fixed points in γ is finite, we actually get a improper cycle $\Delta = [HS(q), HS(q_1), \cdots, HS(q)]$. By Proposition 2.0.5, we know that it is impossible.

Case 6. $S^u(q_1) = S^s(q_2)$ and $HS(q_2) \prec HS(q_1)$.

If $S^u(q_2)$ is free, then we are done. If $S^u(q_2)$ is not free, it must not cross any stable separatrix in γ. As a matter of fact, if there exists a $p \in \gamma$ such that $S^u(q_2)$ crosses $S^s(p)$, then $S^s(q_2)$ is directly accumulated at $S^s(p)$ through $HS(q_2)$. Since $S^u(q_2) = S^s(q_1)$ and $S^u(q)$ accumulated on $S^u(q_1)$, $S^u(q)$ will visit $HS(q_2)$, hence $S^u(q)$ must intersect with $S^s(p)$. It contradicts with the assumption that $S^u(q)$ does not cross any stable separatrix. Now, we base on q_2 to repeat our process as we did from Case 1 to Case 6. By using the same argument as in Case 5, we then finish the proof of this proposition. □

Theorem 2.0.11 *Suppose that $\Omega(f)$ is finite, then, f can be approximated in* $\mathrm{Diff}^r(\mathcal{M})$ *by a diffeomorphism g which has no elementary cycles and $\Omega(g)$ is finite.*

Proof:

We take on $\{\gamma_i\}$, for $i = 1, 2, \ldots, m$, a simple ordering compatible with $\leq$, so that $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_m$. Suppose that for $1 \le j < l$ γ_j either is trivial or does not contain elementary cycles. Let $p, q \in \gamma_l$ given by Proposition 2.0.10. We can perform an arbitrary small C^r perturbation of f such that γ_l remains same, while, $S^u(p) \cap S^s(q) = \emptyset$. Moreover, there is no new intersections with $S^u(p)$ and such that each free unstable separatrix $S^u(p')$ of $p' \in \gamma_j$, for $1 \le j \le l$ remains free. (see Theorem B in [10]). Continue this process, we can achieve an arbitrary small C^r perturbation of f which has the same non-wandering set as f and has one more free unstable separatrix. Finally, we will obtain a diffeomorphism g, C^r close to f, such that $\Omega(g) = \Omega(f)$ and g has no elementary cycles. This proves the theorem. □

CHAPTER 3

Advanced Cycles

In this chapter, we consider the diffeomorphism that has advanced cycles, we will remove all the advanced cycles without causing Ω-explosion.

Let p be a fixed point of f, N be a neighborhood of p, f be analytic in N. We denote by $J_p^1(f)$ the first jet of f at p. In the following sections, we will discuss the local structures of f in a small neighborhood of p, when $J_p^1(f)$ has different Jordan forms.

Suppose $\Omega(f)$ is finite. Let

$$\begin{aligned}
F_1^{\pm} &= \left\{ p \in \Omega(f) \text{ such that } J_p^1(f) \text{ has Jordan form } \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \right\} \\
F_2^{\pm} &= \left\{ p \in \Omega(f) \text{ such that } J_p^1(f) \text{ has Jordan form } \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \\
F_3 &= \left\{ p \in \Omega(f) \text{ such that } J_p^1(f) \text{ has Jordan form } \begin{pmatrix} e^{2\pi\alpha i} & 0 \\ 0 & e^{-2\pi\alpha i} \end{pmatrix} \right\}
\end{aligned}$$

where $\alpha \notin \mathcal{Z}$

Let $\widetilde{\Delta}_1 = [p_0, p_1, \ldots, p_k]$ be an advanced cycle in $\Omega(f)$, $S^{\sigma}(p_i)$, $(\sigma = u, s)$, be separatrices that are in $\tilde{\Delta}_1$, that is $S^u(p_i) \bigcap S^s(p_{i+1}) \neq \emptyset$, for $i = 0, 1, \ldots, k-1$, and $S^u(p_k) \bigcap S^s(p_0) \neq \emptyset$. We denote by Ω_1 the set

$$\Omega_1 = \{p \in \tilde{\Delta} \text{ such that } S^u(p) \text{ and } S^s(p) \text{ are not in a same sector } \}$$

In the following discussion, we always assume that the diffeomorphism is orientation preserving and satisfies locally normalized condition unless we explicitly specify others.

3.1 When $p \in F_1^{\pm}$

Without lose generality, we only consider the case when $p \in F_1^+$.

Theorem 3.1.1 (1) *Let o be a singular point of a real analytic vector field*

$$(y + X(x,y))\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y}$$

where $X(x,y) = o(\sqrt{x^2+y^2})$ *and* $Y(x,y) = o(\sqrt{x^2+y^2})$. *Then, o can only be either saddle or node or center or focus or saddle-node or* $S_\delta(o)$ *consists of a hyperbolic sector and an elliptic sector, where* $S_\delta(o)$ *is a* δ*-neighborhood of o.* □

Combine the locally normalized condition, Corollary 3.0.13 and Theorem 3.1.1, we have

Proposition 3.1.2 *Let* $p \in F_1^{\pm}$. *Then,* $p \in \Omega_1$ *if and only if p is a topological saddle-node and* $S^u(p)$ *and* $S^s(p)$ *are not in a same sector, where* $S^u(p)$ *and* $S^s(p)$ *are in the advanced cycle.* □

3.2 When $p \in F_3$

Lemma 3.2.1 *Let f be an analytic diffeomorphism, $o = (0,0)$ be its fixed point. Suppose the*

$$J_o^1(f) = \begin{pmatrix} e^{2\pi\alpha i} & 0 \\ 0 & e^{-2\pi\alpha i} \end{pmatrix}$$

where $\alpha \notin \mathcal{Z}$, then, there is an analytic map h, and an analytic diffeomorphism g such that $hf = gh$, g is

$$(x,y) \longrightarrow (x\cos 2\pi\alpha + y\sin 2\pi\alpha, -x\sin 2\pi\alpha + y\cos 2\pi\alpha + \phi(x,y))$$

where ϕ is analytic and $\phi(0,0) = 0$, $\phi(x,y) = o(\sqrt{x^2+y^2})$.

Proof:

Since

$$J_o^1(f) = \begin{pmatrix} e^{2\pi\alpha i} & 0 \\ 0 & e^{-2\pi\alpha i} \end{pmatrix}$$

By Taylor Theorem, f can be written as:

$$(x,y) \longrightarrow (x\cos 2\pi\alpha + y\sin 2\pi\alpha + \tilde{\psi}(x,y), -x\sin 2\pi\alpha + y\cos 2\pi\alpha + \tilde{\phi}(x,y))$$

with $\tilde{\psi}(x,y)$ and $\tilde{\phi}(x,y)$ are the series with degree greater than 1.

Define h as a transformation

$$(x,y) \longrightarrow (x, y + \frac{\tilde{\phi}(x,y)}{\sin 2\pi\alpha})$$

Since $\alpha \notin \mathcal{Z}$, h is well defined and is analytic. Moreover, it is invertible, its inverse is

$$(x,y) \longrightarrow (x, y + \chi(x,y))$$

where

$$\chi(x,y)\sin 2\pi\alpha + \tilde{\psi}(x, y+\chi(x,y)) = 0$$

Consider hfh^{-1}, we have

$$\begin{aligned} hfh^{-1} &= hf(x, y+\chi(x,y)) \\ &= h(x\cos 2\pi\alpha + y\sin 2\pi\alpha + \tau(x,y), \\ &\quad -x\sin 2\pi\alpha + y\cos 2\pi\alpha + \phi^*(x,y)) \\ &= (x\cos 2\pi\alpha + y\sin 2\pi\alpha, -x\sin 2\pi\alpha + y\cos 2\pi\alpha + \phi(x,y)) \end{aligned}$$

where

$$\begin{aligned} \tau(x,y) &= \chi(x,y)\sin 2\pi\alpha + \tilde{\psi}(x, y+\chi(x,y)) = 0 \\ \phi^*(x,y) &= \chi(x,y)\cos 2\pi\alpha + \tilde{\phi}(x, y+\chi(x,y)) = 0 \end{aligned}$$

and

$$\begin{aligned} \phi(x,y) &= \chi(x,y) \\ &+ \frac{\tilde{\phi}(x\cos 2\pi\alpha + \sin(2\pi\alpha)\chi^*(x,y), -x\sin 2\pi\alpha + \cos(2\pi\alpha)\chi^*(x,y)}{\sin 2\pi\alpha} \end{aligned}$$

where $\chi^*(x,y) = y + \chi(x,y)$.

Let $g = hfh_{-1}$. We can directly check that $\phi(x,y)$ is as desired. This proves this lemma. □

Proposition 3.2.2 *Let f be analytic in a neighborhood of a fixed point p, and*

$$J_p^1(f) = \begin{pmatrix} e^{2\pi\alpha i} & 0 \\ 0 & e^{-2\pi\alpha i} \end{pmatrix}$$

where $\alpha \notin \mathcal{Z}$. Then, there is no separatrix in N.

Proof:

Without lose generality, we let $p = o(0,0)$ be a fixed point of f. Then, by Lemma 3.2.1, f is analytically conjugate with $\tilde{f}$:

$$(x, y) \longrightarrow (x \cos 2\pi\alpha + y \sin 2\pi\alpha, -x \sin 2\pi\alpha + y \cos 2\pi\alpha + \phi(x, y))$$

where $\phi(x, y)$ is analytic and $\phi(x, y) = o(\sqrt{x^2 + y^2})$.

Suppose, by the way of contradiction, that there is a separatrix $\Gamma = \{(x, y)$ such that $\lambda(x, y) = 0\}$. By the definition of separatrix, we know that either $\frac{\partial\lambda}{\partial x}\Big|_{(0,0)} \neq 0$ or $\frac{\partial\lambda}{\partial y}\Big|_{(0,0)} \neq 0$.

Let us first consider the case when $\frac{\partial\lambda}{\partial y}\Big|_{(0,0)} \neq 0$. Then, Γ can be written as $y = \mu(x)$. By Corollary 3.0.14, $\mu(x)$ is analytic. Suppose its Taylor expansion is

$$\mu(x) = \sum_{i=1}^{\infty} a_i x^i$$

Since $f\Gamma = \Gamma$, we have that

$$-\sin(2\pi\alpha)x + \cos(2\pi\alpha)\mu(x) + \phi(x, \mu(x)) = \mu(\cos(2\pi\alpha)x + \sin(2\pi\alpha)\mu(x))$$

or

$$\phi(x, \mu(x)) = \mu(\cos(2\pi\alpha)x + \sin(2\pi\alpha)\mu(x)) + \sin(2\pi\alpha)x - \cos(2\pi\alpha)\mu(x) \tag{3.1}$$

Let $\phi(x, \mu(x)) = \sum_{i=o}^{\infty} b_i x^i$. Since $\phi(x, y) = o(\sqrt{x^2 + y^2})$, we have that $b_0 = 0$ and $b_1 = 0$. Therefore, equation (5) becomes

$$\sum_{i=2}^{\infty} b_i x^i = \sum_{i=1}^{\infty} a_i (\cos(2\pi\alpha)x + \sin(2\pi\alpha)\mu(x))^i + \sin(2\pi\alpha)x - \cos(2\pi\alpha)\sum_{i=1}^{\infty} a_i x^i$$

by comparing the coefficients of first order term in both sides of above equation, we have

$$a_1 x \cos 2\pi\alpha + a_1^2 x \sin 2\pi\alpha + x \sin 2\pi\alpha - x a_1 \cos 2\pi\alpha = 0$$

that is

$$(a^2 + 1) \sin 2\pi\alpha = 0$$

since $\alpha \notin \mathcal{Z}$, $\sin 2\pi\alpha \neq 0$, therefore, we get a contradiction, This means that Γ does not exist.

For the case when $\frac{\partial \lambda}{\partial x}\Big|_{(0,0)} \neq 0$, we use the similar argument as above to prove this proposition. □

Corollary 3.2.3 $F_3 \bigcap \Theta(f) = \emptyset$, *i.e. points in* F_3 *can not appear in the advanced cycles.* □

3.3 When $p \in F_2^{\pm}$

In this section, we only consider the case when $p \in F_2^+$, For the case when $p \in F_2^-$, the argument is similar.

Proposition 3.3.1 *Suppose* f *be an analytic diffeomorphism with* $O = (0\ 0)$ *as its fixed point in an open neighborhood* $N(O)$ *of* O *and* $\Gamma = \{(x\ y) | f(x\ y) = 0\}$ *be a* C^r *curve entering* O. *If the first jet* $J_O^1(f)$ *of* f *at* O *is* E, *the unit 2-dimensional matrix and* Γ *is invariant under* f, *then there exits a* C^r *diffeomorphism* h *and a* C^r *diffeomorphism*

$$g : (x, y) \longrightarrow (x + \tilde{\varphi}(x, y), \quad y(1 + \tilde{\psi}(x, y))) \tag{3.2}$$

such that $f \circ h = h \circ g$ *and* $h(O) = O$, *where* $\tilde{\varphi}(0,0) = \tilde{\psi}(0,0) = 0$ *and* $\frac{\partial \tilde{\varphi}}{\partial x}\Big|_{(0,0)} = \frac{\partial \tilde{\varphi}}{\partial y}\Big|_{(0,0)} = 0.$

Proof: By the definition of invariant curve entering a fixed point, we know that in equation $f(x,y) = 0$ either $\frac{\partial f}{\partial x}\Big|_{(0,0)} \neq 0$ or $\frac{\partial f}{\partial y}\Big|_{(0,0)} \neq 0$. Without loss generality, suppose that $\frac{\partial f}{\partial x}\Big|_{(0,0)} \neq 0$. By Implicit Function Theorem, $f(x,y) = 0$ has an unique solution $y = \mu(x)$ and $\mu(0) = 0$, $\mu(x) \in C^r$. So $\Gamma = \{(x, \mu(x)) \mid x \in I\}$, I is a appropriate interval. Since the first jet $J_O^1(f)$ of f is E, f has the following form:

$$(x,y) \longrightarrow (x + \varphi(x,y), \quad y + \psi(x,y))$$

where $\varphi(0,0) = \psi(0,0) = 0$ and $\frac{\partial(\varphi\ \psi)}{\partial(x\ y)}\Big|_{(0,0)} = 0_{2\times 2}$. Both $\varphi(x,y)$, $\psi(x,y)$ are analytic. Note that under f, the image of curve Γ is the set

$$f\Gamma = \{\, (\varphi(x, \mu(x) + x), \quad \psi(x, \mu(x) + y)) \mid x \in I\}$$

Since Γ is invariant under f, we have that

$$\psi(x, \mu(x)) + \mu(x) = \mu(\varphi(x, \mu(x)) + x) \tag{3.3}$$

for any $x \in I$.

Consider function $F_x(y) = y + \psi(x,y) - \mu(x + \varphi(x,y))$. It is a C^r function for $x \in I$.

$$\begin{aligned} F_x(y) &= F_x(\mu(x)) + \frac{\partial F_x}{\partial y}\Big|_{y=\mu(x)} (y - \mu(x)) \\ &\quad + \frac{1}{2}\frac{\partial^2 F_x}{\partial y^2}\Big|_{y=\mu(x)} (y - \mu(x))^2 + o(|y - \mu(x)|^2) \end{aligned}$$

From (3.2), we know $F_x(\mu(x)) = 0$ for any $x \in I$. So

$$F_x(y) = \frac{\partial F_x}{\partial y}\Big|_{y=\mu(x)} (y - \mu(x)) + \frac{1}{2}\frac{\partial^2 F_x}{\partial y^2}\Big|_{y=\mu(x)} (y - \mu(x))^2 + o(|y - \mu(x)|^2)$$

Let

$$G_x(y) = \left.\frac{\partial F_x}{\partial y}\right|_{y=\mu(x)} + \frac{1}{2}\left.\frac{\partial^2 F_x}{\partial y^2}\right|_{y=\mu(x)} (y-\mu(x)) + o(|y-\mu(x)|)$$

then

$$F_x(y) = (y-\mu(x))G_x(y)$$

Define h as $(x,y) \longrightarrow (x, y-\mu(x))$, then h^{-1} is $(x,y) \longrightarrow (x, y+\mu(x))$ and h is a C^r diffeomorphism. So

$$\begin{aligned} h \circ f \circ h^{-1}(x,y) &= h \circ f(x, y+\mu(x)) \\ &= h(x+\varphi(x,y+\mu(x)), \quad y+\mu(x)+\psi(x,y+\mu(x))) \\ &= (x+\varphi(x,y+\mu(x)), \quad \Phi(x,y)) \end{aligned}$$

where $\Phi(x,y) = y+\mu(x)+\psi(x,y+\mu(x)) - \mu(x+\varphi(x,y+\mu(x)))$.

Note that $F_x(y) = y+\psi(x,y) - \mu(x+\varphi(x,y))$. We have

$$\begin{aligned} \Phi(x,y) &= F_x(y+\mu(x)) \\ &= (y+\mu(x)-\mu(x))(G_x(y+\mu(x))) \\ &= yG_x(y+\mu(x)) \end{aligned}$$

So,

$$h \circ f \circ h^{-1}(x,y) = (x+\varphi(x,y+\mu(x)), \quad yG_x(y+\mu(x)))$$

Let $\tilde{\varphi}(x,y) = \varphi(x,y+\mu(x))$, $\tilde{\psi}(x,y) = G_x(y+\mu(x)) - 1$ and $g = h \circ f \circ h^{-1}$. We have

$$g : (x,y) \longrightarrow (x+\tilde{\varphi}(x,y), \quad y(1+\tilde{\psi}(x,y))$$

and $\tilde{\varphi}(0,0) = \psi(0,0) = 0$.

Since, $G_0(0) = \left.\frac{\partial F_0}{\partial y}\right|_{y=0} = 1 + \left.\frac{\partial \psi(0,y)}{\partial y}\right|_{y=0} - \mu'(0) \left.\frac{\partial \varphi(0,y)}{\partial y}\right|_{y=0} = 1$, thus

$$\tilde{\psi}(0,0) = G_0(0) - 1 = 0$$

and also, we have

$$\left.\frac{\partial \tilde{\varphi}}{\partial x}\right|_{(0,0)} = \left.\frac{\partial \varphi}{\partial x}\right|_{(0,0)} + \left.\frac{\partial \varphi}{\partial y}\right|_{(0,0)} \mu'(0) = 0$$

$$\left.\frac{\partial \tilde{\varphi}}{\partial y}\right|_{(0,0)} = \left.\frac{\partial \varphi}{\partial y}\right|_{(0,0)} = 0$$

The proposition is proved. □

Definition Let f be a C^r $(r > m)$ function defined on a neighborhood $N = (x_0 - \lambda\ x_0 + \lambda)$ of x_0. If $f^{(i)}(x_0) = 0$ for $i = 0, \cdots, m-1$ and $f^{(m)}(x_0) \neq 0$, then x_0 is called a zero point of f with multiplicity m. If $m = 1$, we call x_0 a simple zero point of f.

Corollary 3.3.2 *Let f be a C^r $(r > m)$ function on $N = (x_0 - \lambda,\ x_0 + \lambda)$, x_0 be its zero point with multiplicity m, then*

$$f(x) = (x - x_0)^m \Phi(x) \tag{3.4}$$

where $\Phi(x)$ is a continues function in N and $\Phi(x_0) \neq 0$. □

Lemma 3.3.3 *Let f be a C^r $(r > m)$ function on $N = (x_0 - \lambda,\ x_0 + \lambda)$, x_0 be its zero point with multiplicity m. Let $\tilde{f}$ be a C^r function such that*

$$d(f, \tilde{f}) < \varepsilon$$

where $d(\ ,\ )$ is the C^m topology in $C^m(N, R)$, then $\tilde{f}$ has at most m zero points in a neighborhood of x_0.

Proof:

Since $f^{(m)}(x_0) \neq 0$, without lose generality, we suppose that $f^{(m)}(x_0) > k > 0$. By the continuity of f, there exist a $\delta > 0$ such that $f^m(x) > k > 0$ for all $x \in I = (x_0 - \delta\ x_0 + \delta)$. Because $d(f, \tilde{f}) < \varepsilon$, we have

$$\tilde{f}^{(m)}(x) > f^{(m)} - \varepsilon > k - \varepsilon > 0 \tag{3.5}$$

for all $x \in I$.

Now suppose $\tilde{f}(x)$ has $m + j$ zero points in $I (j > 1)$. By Roll Theorem, we know that $\tilde{f}^{(1)}(x)$ has $m + j - 1$ zero points in I, $\tilde{f}^{(2)}(x)$ has $m + j - 2$ zero points in I, and so on. After m steps, we get that $\tilde{f}^{(m)}(x)$ has j zero points in I. This is a contradiction. The lemma is proved. □

Lemma 3.3.4 *Let f be a C^r $(r > m)$ function on $\mathcal{N} = (-\lambda\ \lambda)$ and $x = 0$ be its zero point with multiplicity m, then for any $\varepsilon \in (0\ \lambda)$, $\lambda \in (0\ \lambda)$, there exits a C^r function $\tilde{f}$ such that*

1. $d(f, \tilde{f}) < \varepsilon$
2. *$\tilde{f}$ has exactly m different zero points in the interval $\mathcal{I} = [0\ \delta)$*

Proof:

By Corollary 3.3.2, $f(x) = x^m \Phi(x)$, where $\Phi(x)$ is a continuous function and $\Phi(0) \neq 0$. Without lose generality, we suppose that $\Phi(0) > 0$, then there exist a positive number λ_{m-1} such that $0 < \lambda_{m-1} < \delta < \lambda$ and $\Phi(x) > 0$ for all $x \in (0\ \lambda_{m-1})$. Choose a number η_{m-1} in the interval $(0\ \lambda_{m-1})$, we have

$$f(\eta_{m-1}) = \eta_{m-1}^m \Phi(\eta_{m-1}) > 0$$

So we can choose a number β_{m-1} satisfying $0 < \beta_{m-1} < \eta_{m-1}\Phi(\eta_{m-1})$ such that for any $|\alpha_{m-1}| < \beta_{m-1}$ we have

$$\begin{aligned} \alpha_{m-1}\eta_{m-1}^{m-1} + \eta_{m-1}^{m}\Phi(\eta_{m-1}) &= \eta_{m-1}^{m-1}(\alpha_{m-1} + \eta_{m-1}\Phi(\eta_{m-1})) \\ &> \eta_{m-1}^{m-1}(-\beta_{m-1} + \eta_{m-1}\Phi(\eta_{m-1})) \\ &> 0 \end{aligned}$$

Define a function

$$f_1(x) = \alpha_{m-1}x^{m-1} + x^m\Phi(x) = x^{m-1}(\alpha_{m-1} + x\Phi(x))$$

when x is small enough, the sign of function $f_1(x)$ is determined by the sign of α_{m-1}. Choose $\alpha_{m-1} < 0$, we have that there exists a positive number λ_{m-2} such that $0 < \lambda_{m-2} < \eta_{m-1}$ and $f_1(x) < 0$ for all $x \in (0\ \lambda_{m-2})$.

We select $\eta_{m-2} \in (0,\ \lambda_{m-2})$, then

$$f_1(\eta_{m-2}) = \alpha_{m-1}\eta_{m-2}^{m-1} + \eta_{m-2}^{m}\Phi(\eta_{m-2}) < 0$$

Choose β_{m-2} such that

$$0 < \beta_{m-2} < \left|\frac{f_1(\eta_{m-1})}{\eta_{m-2}^{m-2}}\right|$$

then, for any number α_{m-2} satisfying $|\alpha_{m-2}| < \beta_{m-2}$, we have

$$\begin{aligned} \alpha_{m-2}\eta_{m-2}^{m-2} + f_1(\eta_1) &= \eta_{m-2}^{m-2}\left(\alpha_{m-2} + \frac{f_1(\eta_{m-1})}{\eta_{m-2}^{m-2}}\right) \\ &< \eta_{m-2}^{m-2}\left(\beta_{m-2} + \frac{f_1(\eta_{m-1})}{\eta_{m-2}^{m-2}}\right) \\ &< 0 \end{aligned}$$

Define function

$$f_2(x) = \alpha_{m-2}x^{m-2} + \alpha_{m-1}x^{m-1} + x^m\Phi(X)$$

Choose $\alpha_{m-2} > 0$, when x is small enough, say, $|x| < \lambda_{m-3}$, we have

$$f_2(x) = x^{m-2}(\alpha_{m-2} + \alpha_{m-1}x + x^2\Phi(x) > 0$$

Choose $\eta_{m-3} \in (0,\ \lambda_{m-3})$ then $f_2(\eta_{m-3}) > 0$.

Continue this process, we get a function

$$f_{m-1}(x) = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{m-1}x^{m-1} + x^m\Phi(x)$$

where

$$|\alpha_i| < \beta_i,\ \frac{|\alpha_i|}{\alpha_i} = (-1)^{2i+1}, i = 1, \cdots, m-1$$

and a sequence

$$0 < \eta_m < \lambda_m < \cdots < \eta_2 < \lambda_2 < \eta_1 < \lambda_1 < \delta < \lambda$$

with the following properties:

$$f_{m-1}(\eta_m) > 0,\ f_{m-1}(\eta_{m-1}) < 0 \text{ if } m \text{ is odd}$$

$$f_{m-1}(\eta_m) < 0,\ f_{m-1}(\eta_{m-1}) > 0 \text{ if } m \text{ is even}$$

Let

$$\tilde{f}(x) = f_{m-1}(x) = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{m-1}x^{m-1} + x^m\Phi(x)$$

Claim $\tilde{f}(\eta_{2i-1}) > 0,\ \tilde{f}(\eta_{2i}) < 0$ for $i = 0, 1, \cdots, \left[\frac{m-1}{2}\right]$.

Proof of the claim.

Let $\beta = \max\{\beta_{m-1}, \cdots, \beta_1\}$

$$\begin{aligned}\tilde{f}(\eta_{2i-1}) &= \alpha_1\eta_{2i-1} + \cdots + \alpha_{2i-1}\eta_{2i-1}^{2i-1} + \cdots + \alpha_{m-1}\eta_{2i-1}^{m-1} + \eta_{2i-1}^{m}\Phi(\eta_{2i-1}) \\ &= \alpha_1\eta_{2i-1} + \cdots + \alpha_{2i-1}\eta_{2i-1}^{2i-1} + f_{2i-1}(\eta_{2i-1})\end{aligned}$$

Since $f_{2i-1}(\eta_{2i-1}) < 0$ and $|\alpha_1\eta_{2i-1} + \cdots + \alpha_{2i-1}\eta_{2i-1}^{2i-1}| < \beta\lambda_{m-1}$, we can choose β small enough such that $\beta\lambda_{m-1} < |f_{2i-1}(\eta_{2i-1})|$ for all i, then

$$\tilde{f}(\eta_{2i-1}) < \beta\lambda_{m-1} + f_{2i-1}(\eta_{2i-1}) < 0$$

By using the same argument, we can prove that $\tilde{f}(\eta_{2i}) > 0$. The claim is proved.

From the above discussion, we know that $\tilde{f}(x)$ has at least one zero point in each interval $(\eta_{i+1},\ \eta_i)$ $i = 1, \cdots, m-1$. Since 0 is a zero point of $\tilde{f}$, $\tilde{f}(x)$ has at least m distinct zero points in the interval $[0,\ \delta)$.

By Lemma 3.3.3, we know that $\tilde{f}$ has at most m zero points. So $\tilde{f}$ has exactly m distinct zero points in the interval $[0,\ \delta)$.

Now we finish the proof of this lemma by showing $d(f, \tilde{f}) < \varepsilon$.

Since

$$\begin{aligned}\tilde{f}(x) &= \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{m-1}x^{m-1} + x^m\Phi(x) \\ &= \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{m-1}x^{m-1} + f(x)\end{aligned}$$

then

$$\tilde{f}^{(k)}(x) = f^{(k)}(x) + k!\alpha_k + \cdots + (m-1)(m-2)\cdots(m-k)x^{m-k-1}\alpha_{m-1}$$

thus

$$\begin{aligned} \left|\tilde{f}^{(k)}(x) - f^{(k)}(x)\right| &= \left|k!\alpha_k + \cdots + (m-1)(m-2)\cdots(m-k)x^{m-k-1}\alpha_{m-1}\right| \\ &< K\beta \end{aligned}$$

where K is a constant and $k = 1, \cdots, m$.

Let $\beta = \varepsilon/K$, we have that $d(f,\ \tilde{f}) < \varepsilon$. This finishes the proof of lemma. □

Lemma 3.3.5 *Let $\tilde{f}$ be the function defined in Lemma 5.13, then all the zero points $\mathcal{Z}(\tilde{f})$ of $\tilde{f}$ are simple zero points.*

Proof:

Suppose $\tilde{f}$ has a zero point $x_0 \in \mathcal{Z}(\tilde{f})$ which is not simple. That means that it has a multiplicity greater than 1, say, $m > 1$. By Lemma 3.3.4, we know that for any $\varepsilon > 0$ and $\delta > 0$, there exists a C^r function, say, $\overline{f}$, such that in the interval $(x_0 - \delta,\ x_0 + \delta)$, we have that $d(\overline{f}, \tilde{f}) < \varepsilon$. $\overline{f}$ has m distinct zero points in the interval $(x_0 - \delta/2,\ x_0 + \delta/2)$. Since $\mathcal{Z}(\tilde{f})$ is finite, we can choose δ small enough such that

$$\mathcal{Z}(\tilde{f}) \bigcap (x_0 - \frac{\delta}{2},\quad x_0 + \frac{\delta}{2}) = \{x_0\}$$

Define a function

$$g(x) = \begin{cases} \tilde{f}(x) & x \in \mathcal{N} \setminus (x_0 - \delta,\ x_0 + \delta) \\ \overline{f}(x) & x \in (x_0 - \frac{\delta}{2},\ x_0 + \frac{\delta}{2}) \\ h(x) & (x_0 - \delta,\ x_0 + \delta) \setminus (x_0 - \frac{\delta}{2},\ x_0 + \frac{\delta}{2}) \end{cases}$$

where $h(x)$ is a C^r function which makes $g(x)$ to be C^r function on $\mathcal{N}$ and $d(\tilde{f}, g) < \varepsilon$ for $x \in (x_0 - \delta,\ x_0 + \delta) \setminus (x_0 - \frac{\delta}{2},\ x_0 + \frac{\delta}{2})$.

We get that

$$d(g, f) \leq d(g, \tilde{f}) + d(\tilde{f}, f) < \varepsilon$$

but g has $m+k-1>m$ zero points in a small neighborhood of O, it contradicts Lemma 3.3.4. This lemma is proved. □

Proposition 3.3.6 *Let g be the C^r diffeomorphism*

$$(x,y) \longrightarrow (x+\varphi(x,y), \quad y(1+\psi(x,y)))$$

where $\varphi(0,0)=\psi(0,0)=0$, $\frac{\partial\varphi(x,y)}{\partial x}\Big|_{(0,0)} = \frac{\partial\varphi(x,y)}{\partial y}\Big|_{(0,0)} = 0$, $\mathcal{N}$ be a neighborhood of $O(0,0)$. Then, for any $\varepsilon>0$, $\delta>0$, there exists a C^r diffeomorphism $\tilde{g}$ with the following properties:

1. *$d(g,\tilde{g})<\varepsilon$*
2. *The fixed point set $Fix(\tilde{g}) \in \mathcal{N}$ of $\tilde{g}$ is finite,*
3. *$Fix(\tilde{g})$ is semi-hyperbolic or hyperbolic, The hyperbolic manifolds of semi-hyperbolic fixed points are contained in x axis.*

Proof:

Consider function $F(x)=\varphi(x,0)$. By Taylor Theorem, we have

$$F(x) = F(0) + F'(0)x + \frac{1}{2}F''(0)x^2 + \cdots + \frac{1}{m!}F^{(m)}(0)x^m + o(x^m)$$

Since, $F(0)=\varphi(0,0)=0$, $F'(0)=\frac{\partial\varphi(x,0)}{\partial x}\Big|_{(0,0)}=0$, we suppose that $F^{(i)}(0)=0$, $i=1,2,\cdots,m-1$ and $F^{(m)}\neq 0$ $(m>1)$. Let $a_m=\frac{1}{m!}F^{(m)}(0)$, then

$$F(x) = a_m x^m + o(x^m)$$

this means that $x=0$ is a zero point of $F(x)$ with multiplicity m. By Lemma 3.3.4, we have a function

$$G(x) = \sum_{k=1}^{m-1} \alpha_k x^k + F(x)$$

such that $d(F,G) < \varepsilon$ and $G(x)$ has m distinct zero points in $(0,\delta)$. We denote them by $0 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < \delta$.

Now, let

$$\tilde{\varphi}(x,y) = \sum_{k=1}^{m-1} \alpha_k x^k + \varphi(x,y)$$

Define $\tilde{g}$ as a following two dimensional map

$$(x,y) \longrightarrow (x + \tilde{\varphi}(x,y), \quad y(1+\psi(x,y)))$$

then $\tilde{g}$ is a C^r diffeomorphism and $d(g,\tilde{g}) < \varepsilon$. Moreover, $\tilde{g}$ has m different fixed points in $\mathcal{N}$. They are

$$P_0 = (0,0), P_1 = (x_1,0), P_2 = (x_2,0), \cdots, P_{m-1} = (x_{m-1},0)$$

Note that these are all the fixed points $\tilde{g}$ has in $\mathcal{N}$.

We claim that all these fixed points are hyperbolic or semi-hyperbolic. That is to say that the first jets $J^1_{P_i}(\tilde{g})$ of $\tilde{g}$ at each fixed points P_i, for $i = 0,1,\cdots,m-1$, have the Jordan form $\begin{pmatrix} \lambda & 0 \\ \sigma & \beta \end{pmatrix}$, where $|\lambda| \neq 1$ and $\sigma = 0$ or 1.

Proof of the claim:

$$\begin{aligned} J^1_{P_i}(\tilde{g}) &= \begin{pmatrix} 1 + \left.\frac{\partial\tilde{\varphi}}{\partial x}\right|_{(x_i,0)} & \left.\frac{\partial\tilde{\varphi}}{\partial y}\right|_{(x_i,0)} \\ 0 & 1+\psi(x_i,0) \end{pmatrix} \\ &= \begin{pmatrix} 1 + \left.\frac{\partial G}{\partial x}\right|_{(x_i,0)} & \left.\frac{\partial\tilde{\varphi}}{\partial y}\right|_{(x_i,0)} \\ 0 & 1+\psi(x_i,0) \end{pmatrix} \end{aligned}$$

By Lemma 5.13, $\left.\frac{\partial G}{\partial x}\right|_{x=x_0} \neq 0$. Let $\lambda = 1 + \left.\frac{\partial G}{\partial x}\right|_{(x_i,0)}$, $\beta = 1 + \psi(x_i,0)$, thus $J^1_{p_i}(\tilde{g})$ has the Jordan normal form $\begin{pmatrix} \lambda & 0 \\ \sigma & \beta \end{pmatrix}$, where $\sigma = 0$ or 1. This finishes the proof of the claim and the of proof the proposition. □

3.4 Removing advanced cycles

Lemma 3.4.1 *Let f has only advanced cycles, then, f can be approximated by an analytic diffeomorphism $\tilde{f}$ such that the following are satisfied:*

1. *$\tilde{f}$ has only advanced cycles,*
2. *$\Omega(f) = \Omega(\tilde{f})$,*
3. *For any p, q in $\Omega(\tilde{f})$, if $S^u(p) \bigcap S^s(q) \neq \emptyset$, then, $S^u(p)$ and $S^s(q)$ intersect transversely.*

Proof:

Let p, $q \in \Omega(f)$ such that $S^u(p) \bigcap S^s(q) \neq \emptyset$. Let $x \in S^u(p) \bigcap S^s(q)$ and $\mathcal{N}^u(p)$ be a fundamental neighborhood for $S^u(p)$ that contains x. In $\mathcal{N}^u(p)$, we make a small perturbation, as in [10], such that $S^u(p)$ intersects $S^s(q)$ transversely at x. We denote by $\tilde{f}$ the new diffeomorphism. Since f has only advanced cycles, this operation does not change the non-wandering set of f. This proves this lemma. □

Theorem 3.4.2 *Let f be an analytic diffeomorphism and $\Omega(f)$ consist of finite fixed points. If f only has advanced cycles, then, f can be approximated in C^r $(r > 0)$ by a diffeomorphism with no cycle and finite non-wandering set.*

Proof:

This theorem can be proved more clearly, if we work with vector fields instead of directly with map by using the techniques developed by Poincare-Bendixson. That proof will appear else where.

By Lemma 3.4.1, we can suppose that all the intersections between stable separatrices and unstable separatrices are transversal. So, small perturbations do not destroy these connections.

Since f only has advanced cycles, γ_i, for $i = 1, 2, \ldots, m$, only contains advanced cycles. Suppose that γ_i is trivial for $1 \leq i < j$. Let $\tilde{\Delta} = [q_0, q_1, \ldots, p, \ldots, p_k]$ be a advanced cycle in γ_j, where $p \in \tilde{\Delta} \bigcap \Omega_1$ or $p \in \tilde{\Delta} \bigcap \Omega_2$.

If $p \in F_1^{\pm}$. By Proposition 3.1.2, p is a topological saddle-node and the stable separatrix $S^s(p)$ and unstable separatrix $S^u(p)$ which are in cycle $\tilde{\Delta}$ are not in a same sector. Choose a small neighborhood $\mathcal{N}(p)$ of p such that $\mathcal{N}(p) \bigcap \Omega(f) = \emptyset$. In $\mathcal{N}(p)$, we make a small perturbation to split topological saddle-node p into a topological saddle p_1 and a topological node p_2. This perturbation does not break old separatrix connections and no new separatrix connection is created. So, Ω-explosion does not occur. While, cycle $\tilde{\Delta} = [q_0, q_1, \ldots, p, \ldots, p_k]$ becomes $\tilde{\Delta}' = [q_0, q_1, \ldots, p_1, p_2, \ldots, p_k]$ which no longer is a cycle, because p_2 is a topological node. We denote by f_1 the new diffeomorphism. Then, f_1 has less advanced cycles than f and $\Omega(f_1)$ remains finite.

If $p \in F_2$. Let $\mathcal{N}(p)$ be the neighborhood of p as above, $S^u(p)$ be an unstable separatrix in $\tilde{\Delta}$. Since $\tilde{\Delta}$ is an advanced cycle, the separatrix of p which is in the same sector as $S^u(p)$ must be free, we denote it by $S^{s'}(p)$. By Proposition 3.3.1 and Proposition 3.3.6, we can choose a coordinate system in $\mathcal{N}(p)$ such that $S^u(p)$ is a part of x-axis. We then make a small perturbation in $\mathcal{N}(p)$ so that p is split into several hyperbolic or semi-hyperbolic points $\{p, p_1, p_2 \ldots, p_l\}$, all these points are in x-axis and the two of the hyperbolic separatrices of each point are part of x-axis. We denote by $\widetilde{S^s}(p)$ the image of $S^{s'}(p)$ under this perturbation and denote by $S^{u'}(p)(\neq$ the image of $S^u(p)$ under this perturbation) the separatrix of p which is also in a same sector of p as $\widetilde{S^s}(p)$. Then, $S^{u'}(p)$ is also a part of x-axis. Notice that $\widetilde{S^s}(p)$ remains free and no seapratrix connection is broken. No new cycle is created.

If $\{p, p_1, p_2, \ldots, p_l\}$ contains a node, then we are done. We find a diffeomorphism f_1 which is close to f and has less advanced cycles than f and its non-wandering set remains finite.

If $\{p, p_1, p_2, \ldots, p_l\}$ does not contains node. We choose a point $y \in S^{u'}(p)$ and let $\mathcal{N}^u(p)$ be a fundamental neighborhood for $S^{u'}(p)$ that contains y. Since $\widetilde{S^s}(p)$ is free, we can make a perturbation in $\mathcal{N}^u(p)$ to break x-axis at y, as in [10]. Therefore, the cycle $\tilde{\Delta}$ is broken. This operation does not make Ω-explosion. We denote by f_1 the new diffeomorphism. Then, we have that f_1 has less one advanced cycle than f and $\Omega(f_1)$ remains finite. This proves this theorem. □

As a corollary of the proof of above theorem, we have

Corollary 3.4.3 *Let f be an real analytic diffeomorphism, $\Omega(f)$ be finite and f satisfy locally normalized condition, let P be a fixed point of f. Then, there is a neighborhood of P, $N(p)$, and a diffeomorphism g such that g is ϵ close to f, $\Omega(g)$ is finite, g only has hyperbolic or semi-hyperbolic fixed points in $N(p)$ and g has no cycles in $N(P)$.*

CHAPTER 4

No Cycles

In this chapter, we discuss the diffeomorphism that has no cycles. We will approximate it by a Morse–Smale diffeomorphism.

Let f be a diffeomorphism on $\mathcal{M}$, its non-wandering set $\Omega(f)$ be finite. If for any chain $C[p_1, p_2, \cdots, p_l]$ in $\Omega(f)$, $p_i \neq p_j$ for $i \neq j$, we call that f satisfies no cycle condition or f has no cycles.

Corollary 4.0.4 *If f has no advanced cycles and elementary cycles, then f satisfies no cycle condition.* □

Lemma 4.0.5 *Suppose that $f \in \mathrm{Diff}^r(\mathcal{M})$, $(r > 0)$, and $\Omega(f)$ is finite, then*

$$\bigcup_{p \in \Omega(f)} W^u(p) = \bigcup_{p \in \Omega(f)} W^s(p) = \mathcal{M}$$

Proof:

We first prove that $\bigcup_{p \in \Omega(f)} W^u(p) = \mathcal{M}$.

It suffices to prove that $\mathcal{M} \subset \bigcup_{p \in \Omega(f)} W^u(p)$.

Since $\Omega(f)$ is finite, we can let $\Omega(f) = \{\mathcal{O}(p_1), \mathcal{O}(p_2), \cdots, \mathcal{O}(p_n)\}$, where $\mathcal{O}(p_i)$ denotes the periodic orbit of periodic point p_i and $\mathcal{O}(p_i) \bigcap \mathcal{O}(p_j) = \emptyset$ for $i \neq j$. Let N_i be a neighborhood of $\mathcal{O}(p_i)$ such that $N_i \bigcap N_j = \emptyset$ for $i \neq j$. $N = \bigcup_{i=1}^n N_i$. Let

x be any point in $\mathcal{M} - \Omega(f)$, then x is not a periodic point, it follows that $\{f^{-m}x\}$ must be an infinite sequence. Since $\mathcal{M} - N$ is closed, hence it is compact, and $(\mathcal{M} - N) \bigcap \Omega(f) = \emptyset$, $\{f^{-m}x\}$ can not stay in $\mathcal{M} - N$ for all $m > 0$, thus there exists an integer $K > 0$ such that $f^{-m}x \in N$ for $m > K$. Note that $N_i \bigcap N_j = \emptyset$ for $i \neq j$, there must be a large integer $L \geq K > 0$ such that when $m > L$, $f^{-m}x$ are contained in one particular N_k, it follows that α-limit set $\alpha(x)$ of x is contained in $N_k \bigcap \Omega(f)$, that is $\alpha(x) \subset N_k \bigcap \Omega(f)$, hence, $x \in W^u(\mathcal{O}(p_k))$. So, $\mathcal{M} \subset \bigcup_{p \in \Omega(f)} W^u(p)$.

To prove $\bigcup_{p \in \Omega(f)} W^s(p) = \mathcal{M}$, we substitute f for f^{-1} and repeat the above argument. □

Let

$$\Omega_1 = \{p \in \Omega(f) \text{ such that } W^u(p) \bigcap \{p\} = \{p\}\}$$

Ω_1 consists of all the sinks of f on $\mathcal{M}$

$$\begin{aligned} \Omega_2 &= \{p \in \Omega(f) - \Omega_1 \text{ such that } \overline{W^u(p)} \bigcap \Omega_1 \neq \emptyset \\ &\quad \text{and } \overline{W^u(p)} \bigcap (\Omega(f) - \Omega_1) = \emptyset\} \\ &\dots \\ \Omega_k &= \{p \in \Omega(f) - \bigcup_{i<k} \Omega_i \text{ such that } \overline{W^u(p)} \bigcap \bigcup_{i<k} \Omega_i \neq \emptyset \\ &\quad \text{and } \overline{W^u(p)} \bigcap (\Omega(f) - \bigcup_{i<k} \Omega_i) = \emptyset\} \end{aligned}$$

Lemma 4.0.6 *Suppose $\Omega(f)$ be finite, then for any $p,\ q \in \Omega(f)$,*

$$\overline{W^u(p)} \bigcap \mathcal{O}(q) \neq \emptyset \textit{ if and only if } W^u(p) \bigcap W^s(\mathcal{O}(q)) \neq \emptyset$$

Proof:

Suppose $W^u(p) \bigcap W^s(\mathcal{O}(q)) \neq \emptyset$. Let $x \in W^u(p) \bigcap W^s(\mathcal{O}(q))$, then $\{f^n x\} \in W^u(p)$ for $n > 0$. Since $x \in W^s(\mathcal{O}(q))$, the ω-limit set of x $\omega(x) \in \mathcal{O}(q)$, it follows that $f^n x \longrightarrow \tilde{q} \in \mathcal{O}(q)$, $(n \to \infty)$. So $\tilde{q} \in \overline{W^u(p)}$, thus, $\overline{W^u(p)} \cap \mathcal{O}(q) \neq \emptyset$.

Suppose $\overline{W^u(p)} \cap \mathcal{O}(q) \neq \emptyset$. Let U be a small neighborhood of $\mathcal{O}(q)$ such that $U \cap \Omega(f) = \{\mathcal{O}(q)\}$. Suppose, by the way of contradiction, that $W^u(p) \bigcap W^s(\mathcal{O}(q)) = \emptyset$, then, there is no x in $W^u(p) \bigcap U$ such that $f^n x \in U$ for all $n > 0$. Since $\overline{W^u(p)} \bigcap \mathcal{O}(q) \neq \emptyset$, $W^u(p) \bigcap U$ must be an infinite set.

Consider set $D = f(W^u(p) \bigcap U) \bigcap U - W^u(p) \bigcap U$. Since $\overline{D} \cap \Omega(f) = \emptyset$, $W^u(p) \cap D$ is a finite set. Let $W^u(p) \cap D = \{x_1, x_2, \ldots, x_k\}$ and n_i be the largest number such that $f^{n_i} x_i \in U$, then there are at most total $\sum_{i=1}^{k} n_i$ points in $W^u(p) \cap U$, it contradicts the fact that $W^u(p) \cap U$ is infinite. this proves the lemma. □

Lemma 4.0.7 *Suppose $f \in \mathrm{Diff}^r(\mathcal{M})$, $(r > 0)$. Let $\Omega(f)$ be finite and satisfy the no cycle condition. Then, there exists an integer $N > 0$ such that $\Omega_k \neq \emptyset$ for $k \leq N$ and $\Omega_k = \emptyset$ for $k > N$.*

Proof:

By the definition of Ω_k, we know that $\Omega_1 \neq \emptyset$.

Now we prove $\Omega_2 \neq \emptyset$.

Let $S = \{p \in \Omega(f)$ such that $W^u(p) \bigcap W^s(\Omega_1) \neq \emptyset\} = \{p_1, p_2, \ldots, p_s\}$. Suppose, by the way of contradiction, that $\Omega_2 = \emptyset$, then, for each $i = 1, 2, \ldots, s$, there exists a periodic points $q_i \in \Omega(f) - \Omega_1$ such that

$$W^u(p_i) \bigcap W^s(q_i) \neq \emptyset$$

Consider $p_1 \in S$. $W^u(p_1) \bigcap W^s(q_1) \neq \emptyset$, $q_1 \in \Omega(f) - \Omega_1$. If $q_1 \in S$, by relabeling the points in S, we can let q_1 be p_2.

Now we suppose that $q_1 \notin S$. Since $q_1 \notin \Omega_1$, $W^u(q_1) \neq \emptyset$. By Lemma 4.0.5, there is a periodic point $q_2 \in \Omega(f)$ such that $W^u(q_1) \bigcap W^s(q_2) \neq \emptyset$. Since $q_1 \notin S$, $q_2 \notin \Omega_1$. If $q_2 \in S$, by relabeling the points in S, we can let $p_2 = q_2$. If $q_2 \notin$

S, we continue above process. Note that $\Omega(f)$ is finite, we must have a series of periodic points $p_1, q_1, q_2, \ldots, q_n \in \Omega(f)$ such that $q_n = p_2$ and $W^u(p_1) \bigcap W^s(q_1) \neq \emptyset$, $W^u(q_i) \bigcap W^s(q_{i+1}) \neq \emptyset$ for $i = 1, 2, \ldots, n-1$, it means that $p_1 \succ p_2$.

We substitute p_1 by p_2 and repeat above argument, then, we have either $p_2 \succ p_1$ or $p_2 \succ p_3$. If $p_2 \succ p_1$ occurs, a cycle appears, this contradicts the fact that f satisfies the no cycle condition. So, we must have $p_1 \succ p_2 \succ p_3$. Continue this process, because S is finite, we must encounter a cycle, which is a contradiction. This proves $\Omega_2 \neq \emptyset$.

By repeating above argument, we finally can find a $N > 0$ such that Ω_N consists of all the sources of f on $\mathcal{M}$. This proves the lemma. □

Corollary 4.0.8 *Suppose the $f \in \text{Diff}^r(\mathcal{M})$, $r > 0$, $\Omega(f)$ be finite and f satisfy no cycle condition. Let $\Omega_i = \{\mathcal{O}(p_1), \mathcal{O}(p_2), \ldots, \mathcal{O}(p_{k_i})\}$, where $\mathcal{O}(p_j)$ is the periodic orbit of periodic point p_j and $\mathcal{O}(p_j) \cap \mathcal{O}(p_l) = \emptyset$ for $j \neq l$. Then, $W^u(\mathcal{O}(p_j)) \bigcap W^s(\mathcal{O}(p_l)) = \emptyset$ for $j \neq l$.* □

Corollary 4.0.9 *Suppose the $f \in \text{Diff}^r(\mathcal{M})$, $r > 0$, $\Omega(f)$ be finite and f satisfy no cycle condition. Then*

$$\Omega(f) = \bigcup_{k=1}^{N} \Omega_k$$

where $\Omega_i \neq \Omega_j$ for $i \neq j$. □

Corollary 4.0.10 *Suppose the $f \in \text{Diff}^r(\mathcal{M})$, $r > 0$, $\Omega(f)$ be finite and f satisfy no cycle condition. Then $\overline{W^u(\Omega_i)} \bigcap \Omega_j \neq \emptyset$ if and only if $i \geq j$* □

Lemma 4.0.11 (Smale) *[7] Suppose F be a compact $f-$invariant set and Q be a compact neighborhood of F such that $\bigcap_{m>0} f^m(Q) = F$. Then there is a compact neighborhood V of F such that $V \subset \text{int}(Q)$ and $f(V) \subset \text{int}(V)$.* □

Definition A series of compact subset $M_k, M_{k-1}, \ldots, M_1, M_0$ of $\mathcal{M}$ is called a filtration of $\mathcal{M}$ associated with f if

$$\mathcal{M} = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$$

and

$$f(M_i) \subset \text{int}(M_i)$$

We denote it by $[M_k, M_{k-1}, \ldots, M_1, M_0]$.

Lemma 4.0.12 *Suppose $f \in \text{Diff}^r(\mathcal{M})$, $(r > 0)$, $\Omega(f)$ be finite and f satisfy no cycle condition. Then, there exists a filtration $[M_N, M_{N-1}, \ldots, M_1, M_0]$ of $\mathcal{M}$ associated with f such that*

$$\begin{aligned} \Omega_i &\subset \text{int}(M_i - M_{i-1}) \\ \Omega_i &= \bigcap_{-\infty<j<\infty} f^j(M_i - M_{i-1}) \end{aligned}$$

where $i = 1, 2, \ldots, N$.

Proof:

From Corollary 4.0.9 and Corollary 4.0.10, we have that $\Omega(f) = \bigcup_{i=1}^{N} \Omega_k$ and $\overline{W^u(\Omega_i)} \bigcap \Omega_j = \emptyset$ for $i < j$. So, $\overline{W^u(\Omega_1)} \bigcap \Omega_1 = \Omega_1$. Let Q_1 be a compact neighborhood of Ω_1 such that $Q_1 \bigcap \bigcup_{j>1} \Omega_j = \Omega(f)$. Then, if $x \in \bigcap_{n\geq 0} f^n Q_1$, then, $f^m x \in Q_1$ for $m > 0$, it follows that the α-limit set $\alpha(x)$ of x is in Q_1, i.e. $\alpha(x) \in Q_1 \bigcap \Omega(f) = \Omega_1$, thus $x \in W^u(\Omega_1) = \Omega_1$. So, we have that $\Omega_1 = \bigcap_{n\geq 0} f^n Q_1$. By Lemma 4.0.11, there is a compact neighborhood M_1 of Ω_1 such that $\Omega_1 \subset M_1 \subset \text{int}(Q_1)$, $\Omega_1 \subset \bigcap_{n\geq 0} f^n M_1 \subset \bigcap_{n\geq 0} f^n Q_1 = \Omega_1$, and $f(M_1) \subset \text{int}(M_1)$.

Let Q_2 be a compact neighborhood of $\overline{W^u(\Omega_2)}$. Since

$$\overline{W^u(\Omega_2)} \bigcap \bigcup_{j>2} \Omega_j = \emptyset$$

we can let Q_2 be such that $Q_2 \bigcap \bigcup_{j>2} \Omega_j = \emptyset$. We claim that

$$\bigcap_{n\geq 0} f^n(Q_2 \bigcup M_1) = \bigcup_{i\leq 2} W^u(\Omega_i)$$

In fact, let $x \in \bigcap_{n\geq 0} f^n(Q_2 \bigcup M_1)$, then, $f^m x \in Q_2 \bigcup M_2$ for all $m > 0$, hence, $\alpha(x) \in (Q_2 \bigcup M_2) \bigcap \Omega(f) = \Omega_1 \bigcup \Omega_2$, it follows that $x \in \bigcup_{i\leq 2} W^u(\Omega_i)$. So,

$$\bigcap_{n\geq 0} f^n(Q_2 \bigcap M_1) = \bigcup_{i\leq 2} W^u(\Omega_i) \subset \text{int}(Q_1 \bigcup M_1)$$

By Lemma 4.0.11, there is a compact neighborhood M_2 of $\bigcup_{i\leq 2} W^u(\Omega_i)$ such that $f(M_2) \subset \text{int}(M_2)$ and $\bigcup_{i\leq 2} W^(\Omega_i) \subset M_2 \subset \text{int}(Q_1 \bigcup M_1)$. We can suppose that $M_1 \subset M_2$. As a matter of fact, if $M_1 \not\subset M_2$, we can substitute M_2 by $M_2 \bigcup (M_1 - M_2)$.

Now we check M_2 has the required properties. First, we note that since $\Omega_2 \subset \text{int}(M_2)$ and $M_1 \bigcap \Omega_2 \subset Q_1 \bigcap \Omega_2 = \emptyset$, we have $\Omega_2 \subset \text{int}(M_2 - M_1)$. Because Ω_2 is invariant under f, $\Omega_2 \subset \bigcap_{-\infty<j<\infty} f^j(M_2 - M_1)$, on the other hand, if $x \in \bigcap_j f^j(M_2 - M_1)$, then $\alpha(x) \subset (M_2 - M_1) \bigcap \Omega(f) = \Omega_2$ and $\omega(x) \subset \Omega_2$, $x \in W^u(\Omega_2) \bigcap W^s(\Omega_2) = \Omega_2$. So, $\Omega_2 = \bigcap_{-\infty<j<\infty} f^j(M_2 - M_1)$.

By repeating above process, we can find a series of compact set $\emptyset = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_N = \mathcal{M}$ which have the required properties. This proves the lemma. □

Let $\Omega_i = \{\mathcal{O}(p_1), \mathcal{O}(p_2), \ldots, \mathcal{O}(p_{k_i})\}$ and $[M_N, M_{N-1}, \ldots, M_1, M_0]$ be the filtration in above lemma. Then, we have

Lemma 4.0.13 *There exist a series compact subsets of* M_i

$$M_{i-1} = M_{i_0} \subset M_{i_1} \subset M_{i_2} \subset \cdots \subset M_{i_{k_i}} = M_i$$

such that

$$\begin{aligned} f(M_{i_j}) &\subset \operatorname{int}(M_{i_j}) \\ \mathcal{O}(p_j) &\subset \operatorname{int}(M_{i_j} - M_{i_{j-1}}) \\ \mathcal{O}(p_j) &= \bigcap_{-\infty<n<\infty} f^n(M_{i_j} - M_{i_{j-1}}) \end{aligned}$$

where $j = 1, 2, \ldots, k_i$.

Proof:

From Corollary 4.0.8, we know that for $i \neq j$, $W^u(\mathcal{O}(p_i)) \bigcap W^s(\mathcal{O}(p_j)) = \emptyset$ and for $j = 1, 2, \ldots, k_i$, $\overline{W^u(\mathcal{O}(p_j))} \bigcap \Omega_i = \mathcal{O}(p_j)$. Let Q be a compact neighborhood of $\overline{W^u(\mathcal{O}(p_1))}$ such that $Q \bigcap (\bigcup_{j>1} \mathcal{O}(p_j)) = \emptyset$ and $Q \cup M_{i-1} \subset M_i$.

We claim that $\bigcap_{n\geq 0} f^n(Q \bigcup M_{i-1}) = \bigcup_{j\leq i-1} W^u(\Omega_j) \bigcup W^u(\mathcal{O}(p_1))$. In fact, if $x \in \bigcap_{n\leq 0} f^n(Q \cup M_{i-1})$, then $f^{-m}x \in Q \cup M_{i-1}$, $\alpha(x) \subset (Q \cup M_{i-1}) \cap \Omega(f)$, it follows that $x \in \bigcup_{j\leq i-1} W^u(\Omega_j) \bigcup W^u(\mathcal{O}(p_1))$. So

$$\bigcap_{n\geq 0} f^n(Q \bigcup M_{i-1}) = \bigcup_{j\leq i-1} W^u(\Omega_j) \bigcup W^u(\mathcal{O}(p_1))$$

By Lemma 4.0.11, there is a compact neighborhood $M_{i_1} \subset \operatorname{int}(Q \cup M_{i-1})$ such that $M_{i-1} \subset M_{i_1} \subset M_i$ and $f(M_{i_1}) \subset \operatorname{int}(M_{i_1})$. By using the similar argument as in the proof of Lemma 4.0.12, we have that $\mathcal{O}(p_1) \subset \operatorname{int}(M_{i_1} - M_{i-1})$ and $\mathcal{O}(p_1) = \bigcap_{-\infty<n<\infty} f^n(M_{i_1} - M_{i-1})$.

Repeating above process, we can find $M_{i_2}, M_{i_3}, \ldots, M_{i_{k_i}}$ that have the required properties listed in Lemma 4.0.13. This proves the lemma. □

Let $\Omega(f) = \{\mathcal{O}(q_1), \mathcal{O}(q_2), \ldots, \mathcal{O}(q_n)\}$, where $\mathcal{O}(q_i)$ is the periodic orbit of periodic point q_i, $\mathcal{O}(q_i) \cap \mathcal{O}(q_j) = \emptyset$. Suppose f have no cycle condition. Then, by combining Lemma 4.0.14 and Lemma 4.0.13, we have

Proposition 4.0.14 *There is a filtration of $\mathcal{M}$ associated with f,*

$$[M_n, M_{n-1}, \ldots, M_1, M_0]$$

such that

$$\begin{aligned} \mathcal{O}(q_i) &\subset \operatorname{int}(M_i - M_{i-1}) \\ \mathcal{O}(q_i) &= \bigcap_{-\infty<n<\infty} f^n(M_i - M_{i-1}) \end{aligned}$$

where $i = 1, 2, \ldots, n$. □

Theorem 4.0.15 *Let f be a diffeomorphism on $\mathcal{M}$, $\Omega(f)$ be finite and f satisfies no cycle condition. Then, for any $\epsilon > 0$, there exists a Morse-Smale diffeomorphism g on $\mathcal{M}$ such that $d(f, g) < \epsilon$.*

Proof:

Let $\Omega(f) = \{\mathcal{O}(q_1), \mathcal{O}(q_2), \ldots, \mathcal{O}(q_n)\}$. By relabeling the periodic points in $\Omega(f)$, we can suppose that q_i be a hyperbolic periodic point for $i < k_0$ and q_i be a degenerate or non-hyperbolic periodic point for $i \geq k_0$. Let $[M_n, M_{n-1}, \ldots, M_1, M_0]$ be the filtration defined in Proposition 4.0.14, then $\mathcal{O}(q_{k_0}) \subset \operatorname{int}(M_{k_0} - M_{k_0-1})$. Let V be neighborhood of $\mathcal{O}(q_{k_0})$ such that $\mathcal{O}(q_{k_0}) \subset V \subset \operatorname{int}(M_{k_0} - M_{k_0-1})$. By Corollary 3.4.7, we know that for any $\epsilon > 0$, there is a diffeomorphism h defined on V such that h has no non-hyperbolic periodic points in V and $d(f|_V, h) < \epsilon$.

We define

$$g_1(x) = \begin{cases} h(x) & x \in V \\ \overline{h}(x) & x \in (M_{k_0} - M_{k_0-1}) - V \\ f(x) & x \in M - (M_{k_0} - M_{k_0-1}) \end{cases} \tag{4.1}$$

where $\overline{h}(x)$ is a diffeomorphism which makes $g(x)$ as smooth as required. It follows that $d(f, g_1) < \epsilon$ on $\mathcal{M}$ and $\Omega(g_1)$ contains less degenerate periodic points than $\Omega(f)$. Continue this process, we can find a diffeomorphism $\tilde{g}$ such that $\tilde{g}$ has no non-hyperbolic periodic points on whole $\mathcal{M}$ and $d(f, \tilde{g}) < \epsilon$. Moreover, $\Omega(\tilde{g})$ remains finite and satisfy the no cycle condition.

Now, we prove that $\tilde{g}$ can be approximated by a Morse-Smale diffeomorphism. Let $p, q \in \Omega(\tilde{g})$ such that $W^u(p) \cap W^s(q) \neq \emptyset$ and the intersection is not transversal. Let $x \in W^u(p) \cap W^s(q)$ and $\mathcal{N}^u(p)$ be a fundamental neighborhood for $W^u(p)$ at x. In $\mathcal{N}(p)$, we make a small perturbation such that the intersection becomes transversal. Because $\tilde{g}$ has no cycles, this operation does not cause Ω-explosion. We denote by $\tilde{g_1}$ the new diffeomorphism, then, $\Omega(\tilde{g_1})$ is finite and hyperbolic and there are less non-transversal connections between stable manifolds and unstable manifolds than in the case of $\tilde{g}$. Continue above process, since $\Omega(\tilde{g_1})$ is finite, we finally can find a diffeomorphism g with finite, hyperbolic $\Omega(g)$ which is close to f and all the connections between its stable manifolds and unstable manifolds are transversal. This proves the theorem. □

CHAPTER 5

Conclusion and Remarks

Combine Theorem 2.0.11, Theorem 3.4.2 and Theorem 4.0.15, we have

Theorem 5.0.16 *Let $\mathcal{M}$ be a two dimensional compact manifold, f be an analytic diffeomorphism on $\mathcal{M}$, its no-wandering set $\Omega(f)$ be finite and f satisfy locally normalized condition. Then, f can be approximated in C^r $(r > 0)$ by a Morse-Smale diffeomorphism.* □

There is a long way to go to study the structures of systems that have zero entropies and to prove or disprove Newhouse conjecture. Following the path presented in this paper, we first have to remove the real analytic and locally normalized conditions, and then consider the diffeomorphisms that have finite Birkhoff centers instead of non–wandering sets, and then do further study. Also, we may study this problem from other angles. For instance, we may try to connect the structures of entropy zero systems with the non-exponential correlation decay under an invariant measure. In one word, this is a fascinating but very hard problem to work with.

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