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SEQUENTIAL PREDICTOR–CORRECTOR METHODS FOR VARIABLE REGULARIZATION OF ILL-POSED VOLTERRA PROBLEMS

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ABSTRACT

SEQUENTIAL PREDICTOR–CORRECTOR METHODS FOR VARIABLE REGULARIZATION OF ILL-POSED VOLTERRA PROBLEMS

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Inverse problems based on first-kind Volterra integral equations appear naturally in the study of many applications, from geophysical problems to the inverse heat conduction problem. The ill-posedness of such problems means that a regularization technique is required, but classical regularization schemes like Tikhonov regularization destroy the causal nature of the underlying problem and, in general, produce oversmoothed results. In this paper we investigate a new class of predictor–corrector methods for regularization in which the original (unstable) problem is approximated by a parameterized family of well-posed, second–kind Volterra equations. Being Volterra, these approximating second–kind equations retain the causality of the original problem.

Lamm (1995) was the first to place these methods in a generalized framework and

to provide a mathematical analysis of their convergence using approximating equations which were parameterized by a single numeric parameter to regularize problems with convolution kernels. Here, we extend the analysis to nonconvolution kernels. Moreover, we use approximating equations whose regularizing parameter is a *function* (rather than a single constant), allowing for more or less smoothing at different points in the (one–dimensional) domain.

We also introduce another class of predictor–corrector methods, one that employs a penalty term. Here again our regularization parameter is a function, and the addition of the penalty term does not significantly alter the regularizing equation that we solved in the above-mentioned class of predictor–corrector methods. Nevertheless, its presence provides for significantly stronger convergence theorems in comparison to those we are able to prove for the first class.

To my patient, loving wife Pam

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Introduction

We consider the scalar Volterra first-kind integral problem: given a suitable function $f(\cdot)$ defined on $[0,1]$, find $u(\cdot)$ satisfying for a.e. $t \in [0,1]$

$$\mathcal{A}u(t) = f(t), \tag{1}$$

where

$$\mathcal{A}u(t) := \int_0^t k(t,s)u(s)\,ds, \qquad \text{a.e. } t \in [0,1].$$

Problem (1) is an important one, having many applications. One is the inverse heat conduction problem (IHCP), which in the one-dimensional case can be stated as follows:

Given $f(t)$ in some appropriate space, find u so that the problem

$$\begin{aligned} w_t &= w_{xx}, \\ w(0,t) &= u(t), \\ w(1,t) &= f(t), \\ w(x,0) &= 0, \end{aligned}$$

has a bounded solution $w = w(x,t)$ for $x \in (0,\infty), t > 0$.

When written in the form of (1), this form of the IHCP has a convolution kernel $k(t,s) = \kappa(t-s)$, where

$$\kappa(t) = \frac{1}{2\sqrt{\pi}t^{3/2}} e^{-1/4t}.$$

Another application arises from capillary viscometry (see [12]). The integral equation takes the form of (1) with $k(t,s) = 4(s/t)^3$, a nonconvolution kernel. Here f (the known function) is the apparent wall shear rate (itself a measured quantity rather than the true wall shear rate) and the desired quantity, the reciprocal of $u(\cdot)$, is the viscosity.

It is well-known that in the typical case for (1) where k a non-degenerate square-integrable function, one has that $\mathcal{R}(\mathcal{A})$ is not closed, and hence problem (1) is ill-posed, lacking stability. When using measured data, as is the case in the example from capillary viscometry (and as is typically the case in all applications), direct recovery of the actual solution is hopeless. One attempts to overcome the problem's instability through some "regularization" method.

Indeed, there are a number of regularization techniques. The best-known is Tikhonov regularization, which amounts to a constrained minimization performed upon a suitably–restricted set $\mathcal{D}$ of admissable functions $u(\cdot) \in \mathcal{U}$ ($\mathcal{U}$ a Hilbert space). The method is usually transformed into the global minimization problem

$$\min_{u \in \mathcal{D}} ||\mathcal{A}u - f||^2_{\mathcal{F}} + \alpha ||\mathcal{L}u||^2_{\mathcal{G}} \tag{2}$$

through a Lagrange–multiplier type approach, introducing a penalty term, the effect of which upon the computed solution is relegated by the single positive "regularization parameter" α. Here $\mathcal{L} : \mathcal{D} \to \mathcal{G}$ ($\mathcal{G}$ a Hilbert space) is a closed linear operator and $||\cdot||_{\mathcal{F}}, ||\cdot||_{\mathcal{G}}$ are norms in the spaces $\mathcal{F}$ (the range space of $\mathcal{A}$, another Hilbert space) and $\mathcal{G}$ respectively. If $\mathcal{L}$ is, say, a derivative operator, then increases (decreases) in α generally cause greater smoothing (roughness) in the computed solution.

The theory of Tikhonov regularization is a highly-developed one (see, for example, [5]). To give the flavor of the types of convergence results to be proved later for another regularization method, we (loosely) summarize results for Tikhonov in the following theorem:

Theorem 0.1 (Tikhonov Regularization) *Under classical assumptions, there is for each* $\alpha > 0$ *a unique minimizer* $\hat{u}_\alpha(f)$ *of (2). This* $\hat{u}_\alpha(f)$ *depends continuously upon* f*. Moveover, if* f *(the "true data") is replaced by* f^δ *("noisy" data) where* $\|f^\delta - f\| \leq \delta$*, then* $\alpha = \alpha(\delta)$ *can be chosen in such a way that* $\hat{u}_{\alpha(\delta)}(f^\delta) \to \bar{u}$ *as* $\delta \to 0$.

It is in the sense suggested by this theorem that we say problem (2) is a "regularized approximation" of (1).

There are undesirable side effects inherent in the application of Tikhonov regularization to our problem. One is that it replaces the original "causal" problem with a full-domain one. Problem (1) is considered causal in that for any $t \in (0, 1]$, the solution u on the interval $[0, t]$ is determined only from values of f on that same interval. Differentiating problem (2) leads to the necessary condition that the solution satisfies

$$(\mathcal{A}^*\mathcal{A} + \alpha\mathcal{L}^*\mathcal{L})u = A^*f.$$

This problem is, in fact, equivalent to (2) when $\mathcal{D} = \mathcal{D}(\mathcal{L})$. Yet $\mathcal{A}^*f$ is determined, in general, using data values from the interval $[t, 1]$ ("future values"), thus destroying the causal nature of the original problem.

Another side effect arises in the usage of a single regularization parameter when *a priori* information calls for a solution that is rough in some areas of the domain and smooth in others (see, for example, Figure 5.2). While newer regularization methods based on (2) can provide this type of variability in the solution, they do so outside of the Hilbert space framework, using a norm in $\mathcal{G}$ that is L^1-type, resulting in a non-differentiable problem. Such formulations are, therefore, difficult and costly to implement.

A different class of regularization methods, "predictor–corrector" techniques, has been the focus of study in recent years. One of the earliest of these was developed by J. V. Beck in the 1960's for application to the IHCP. In this method, Beck held solutions rigid for a short time into the future (regularized "prediction") and then truncated the prediction to improve accuracy ("correction"). More on this method can be found in [1]. The method is easy to implement numerically and provides fast results in almost real-time — that is, the causal nature of the original problem is not badly compromised.

The method of Beck is a discretization of a special case of the more general class of predictor–corrector methods considered in [9], [8] and [10]. In what follows, we lay the groundwork for this class of methods.

We will make the following assumption throughout this work.

Hypothesis 0.1 *The kernel $k(t,s)$ from (1) is continuous, and can be extended so that it is defined on $0 \le s \le t \le T$ for some $T > 1$. Along with this extension of our kernel, we assume that $f \in L^2(0,T)$ is such that there exists a function (necessarily unique) $\bar{u} \in L^2(0,T)$ satisfying (1) for a.e. $t \in [0,T]$.*

Thus, for a.e. $t \in [0,1]$ and a.e. $\rho \in [0, T-1]$, $\bar{u}$ satisfies

$$\int_0^{t+\rho} k(t+\rho, s)u(s)\,ds = f(t+\rho).$$

Splitting the left-hand side of this equation up into the sum of two integrals, we have (after a change in variables)

$$\int_0^t k(t+\rho,s)u(s)\,ds + \int_0^\rho k(t+\rho, s+t)u(s+t)\,ds \quad = \quad f(t+\rho). \qquad (3)$$

Now let us assume that $r : [0,1] \to (0, T-1]$ is a given continuous function and that η_r is an associated finite positive Borel measure on $[0, \|r\|_\infty]$. Let us also assume that for each $t \in [0,1]$, $f(t+\rho)$ is η_r–integrable (in the variable ρ). We note that, should f be just piecewise continuous, then this last assumption is obtained.

On the set of η_r–integrable functions whose domains include $[0, r(t)]$ we may define the continuous linear functional $\Omega_{r(t)}$ by

$$\Omega_{r(t)}\phi := \int_0^{r(t)} \phi(\rho)\, d\eta_r(\rho).$$

Applying $\Omega_{r(t)}$ to both sides of (3) we get that $\bar{u}$ satisfies

$$\begin{aligned} \int_0^{r(t)} \int_0^t k(t+\rho, s)u(s)\, ds\, d\eta_r(\rho) \\ + \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)u(s+t)\, ds\, d\eta_r(\rho) &= \int_0^{r(t)} f(t+\rho)\, d\eta_r(\rho), \end{aligned} \tag{4}$$

for a. e. $t \in [0,1]$. After a change in order of integration in the first term of (4) (valid by Fubini's Theorem), we have a term that defines a new integral operator on $L^2(0,T)$, namely

$$\mathcal{A}_r u(t) := \int_0^t \tilde{k}(t,s;r)u(s)\, ds, \qquad t \in [0,1],$$

where

$$\tilde{k}(t,s;r) := \int_0^{r(t)} k(t+\rho, s)\, d\eta_r(\rho), \tag{5}$$

for $0 \le s \le t \le 1, r(t) \in (0, T-1]$. So now (4) becomes

$$\begin{aligned} \int_0^t \tilde{k}(t,s;r)u(s)\, ds + \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)u(s+t)\, ds\, d\eta_r(\rho) \\ = \int_0^{r(t)} f(t+\rho)\, d\eta_r(\rho), \end{aligned} \tag{6}$$

an equation that is satisfied by $\bar{u}(\cdot)$ for a.e. $t \in [0,1]$.

The instability in the original problem manifests itself (as we shall see later in numerical examples) even when the problem is discretized making it a matrix problem (hence well-posed). We shall see that solutions of such a discretized system tend to be highly oscillatory, becoming more so as the stepsize in the discretization shrinks. Our hope is that some change to equation (6) will yield a new equation whose solution is not so oscillatory, even though this solution will no longer be $\bar{u}$. We further hope that our new equation will serve as a regularizing approximation of the old one — that is, its solutions approach that of (1) as the parameter controlling the amount of perturbation shrinks. In [9], Lamm proposed the following equation in place of (6):

$$\int_0^t \tilde{k}(t,s;r)u(s)\,ds + \alpha(t;r)u(t) \quad = \quad \tilde{f}(t;r), \tag{7}$$

where

$$\alpha(t,r) := \int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\,ds\,d\eta_r(\rho) \tag{8}$$

and

$$\tilde{f}(t;r) := \int_0^{r(t)} f(t+\rho)\,d\eta_r(\rho). \tag{9}$$

Here r serves as a functional regularization parameter, with $r(t)$ indicating the length of a future interval for given t. In [9] the author considered only convolution kernels with a constant $r(t) \equiv r$, making (7) a generalization of the equation put forth there. We can view (7) as a perturbation of (1) in the following way. At a fixed $t \in [0,1]$, if we divide (7) through by $r(t)$ and consider integral expressions of the form

$$\frac{1}{r(t)} \int_0^{r(t)} \cdot \quad d\eta_r(\rho)$$

to be average values in some sense, then (heuristically) the resulting equation collapses to (1) as $r(t) \to 0$. As motivation for the jump from (6) to (7) we may think of (7) as holding $u(\cdot)$ constant (temporarily) on a small interval — i.e., $u(s+t) = u(t)$ for $s \in [0, \rho], \rho \in [0, r(t)]$.

As (7) is a new equation, we must direct our attention to questions of existence and uniqueness of solutions. If we have assumptions that are sufficient to give that $\alpha(t;r) \neq 0$ for $t \in [0,1]$ (for instance, $k(t,s) > 0$ for $0 \leq s \leq t \leq T$ and a positive measure that is nonzero on intervals of the form $[0,r]$, for $r > 0$), then (7) can be written as the second-kind equation

$$u(t) + \alpha^{-1}(t;r) \int_0^t \tilde{k}(t,s;r)u(s)\,ds \;=\; \alpha^{-1}(t;r)\tilde{f}(t;r). \tag{10}$$

Now if the function

$$|\alpha^{-1}(t;r)| \left(\int_0^{r(t)} d\eta_r \right)$$

is square-integrable (in t) on $[0,1]$, then using the terminology found in Chapter 9 of [6],

$$K(t,s) := \alpha^{-1}(t;r)\tilde{k}(t,s;r)$$

is a type L^2 kernel on $[0,1]$ (by Proposition 9.2.7) which has a type L^2 resolvent (by Corollary 9.3.16). So, by Theorem 9.3.6, (7) has a unique solution $u(\cdot;r) \in L^2(0,1)$ if $\alpha^{-1}(t;r)\tilde{f}(t;r) \in L^2(0,1)$. (Here the references to numbered results all come from [6]). Incidentally, it is also true that $u(\cdot;r)$ depends continuously in the norm of $L^2(0,1)$ on $\tilde{f}(\cdot;r)$, and hence on f in, say, $L^\infty(0,1)$ for example — that is, (7) is a well-posed problem. Henceforth we shall assume this well-posedness.

Numerical examples (seen later) seem to indicate that our hope that solutions of (7) are more well-behaved than solutions of (6) is indeed realized. There is still the question of whether solutions of (7) are good approximations to those of (6), and the need for a theoretical justification of the continuous dependence of these solutions upon data. In [9], Lamm proved the following theorem.

Theorem 0.2 ([9]) *Let us assume that the kernel k in (1) is a convolution kernel with $k \in C^1([0,T])$ and $k(0) > 0$. We assume also a constant function $r(t) \equiv r$ and that f and the measures η_r are such that (7) has a solution $u = u(\cdot; r)$ for all $r \in (0, T-1]$. If the true solution $\bar{u}$ of (1) on $[0,T]$ is in C^1, then $u(\cdot; r)$ converges uniformly on $[0,1]$ to $\bar{u}$ as $r \to 0$. In the presence of noisy data, $r = r(\delta)$ can be chosen appropriately for the level δ of noise so that convergence occurs as $\delta \to 0$.*

In [8], Lamm went on to consider a discretized version of (1), looking at a step function $u_{\Delta t}(t; r; f^\delta)$ that satisfies (7) with (possibly) noisy data f^δ at $N = 1/\Delta t$ collocation points $t_i = i\Delta t, i = 1, \ldots, N$. In this case, both r and Δt are regularization parameters. The following theorem was proved.

Theorem 0.3 ([8]) *Let k be a convolution kernel in C^1 with $k(t) > 0$ for $t \in [0,1]$. Assume that $\bar{u} \in W^{1,\infty}$, that $r(t) \equiv r$, that $f^\delta \in L^\infty(0,T)$ and that r is directly proportional to Δt. Given a class of measures η_r for which*

$$f^\delta(t) := \int_0^r f^\delta(t+\rho)\, d\eta_r(\rho)$$

(the new right-hand side of (7) corresponding to the function f^δ) is well-defined for all $t \in [0,1]$, $r \in (0, T-1]$, the solution $u_{\Delta t}(t; r; f^\delta)$ converges to $\bar{u}(t)$ at each of the collocation points $t = t_i, i = 1, \ldots, N$ as the noise in the data goes to zero, provided Δt is proportional to the square root of that noise.

We seek here to extend these results.

In each special case with "noise-free" data f that we consider, we will look at the difference

$$y(t; r) := u(t; r) - \bar{u}(t), \qquad t \in [0,1],$$

of the solutions to (1) and (7) (i.e., the error), trying to show that this difference goes to zero in some sense as $\|r\|_\infty \to 0$. Subtracting (6) from (7) we have that $y(\cdot; r)$

satisfies

$$\int_0^t \tilde{k}(t,s;r)y(s)\,ds + \alpha(t;r)y(t)$$
$$= \int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\,[\bar{u}(s+t) - \bar{u}(t)]\,ds\,d\eta_r(\rho),$$

for a.e. $t \in [0,1]$; or, still assuming that $\alpha^{-1}(t;r)$ exists for each t, $y(\cdot;r)$ satisfies

$$y(t) = -\alpha^{-1}(t;r)\int_0^t \tilde{k}(t,s;r)y(s)\,ds + F(t;r), \tag{11}$$

where

$$F(t;r) := \alpha^{-1}(t;r)\int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\,[\bar{u}(s+t) - \bar{u}(t)]\,ds\,d\eta_r(\rho).$$

We will use equation (11) — the equation solved by $y(\cdot;r)$ — throughout in showing that y goes to zero with $\|r\|_\infty$.

We have assumed throughout the discussion thus far that (1) is a scalar equation. Nevertheless, the entire analysis to this point can be generalized to vector equations with very little change. Our arguments in the chapters that follow will not be so easily generalizable, and we will continue to assume a scalar equation.

Finally, to motivate the need for a specialized argument like the one involving an approximate identity to be found in Chapter 1, we note that if our ("true") data f is such that the "true solution" $\bar{u} \in C^1[0,T]$, and if $k(t,s) > 0$ for $0 \le s \le t \le T$, then

$$\begin{aligned} |F(t;r)| &\le \frac{1}{\alpha(t;r)}\int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\,|\bar{u}'(\zeta_t(s))|\,s\,ds\,d\eta_r(\rho) \\ &\le \frac{r(t)\|\bar{u}'\|_\infty}{\alpha(t;r)}\int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\,ds\,d\eta_r(\rho) \\ &= \mathcal{O}(r(t)), \end{aligned}$$

and thus

$$\|F(\cdot; r(\cdot))\| \leq \mathcal{O}(\|r\|_\infty).$$

It might appear, then, that an immediate application of the Gronwall Inequality to (11) will show that our error goes to zero with $\|r\|_\infty$. This would indeed be the case if we had a uniform bound on

$$\left|\frac{\tilde{k}(t,s;r)}{\alpha(t;r)}\right|$$

as $\|r\|_\infty \to 0$. Nevertheless, a close inspection of this expression shows that it is not reasonable to expect such a bound. For instance, when $d\eta_r(\rho) \equiv d\rho$ (Lebesgue measure) and $k(t,s) \equiv 1$, we have

$$\frac{\tilde{k}(t,s;r)}{\alpha(t;r)} = \frac{2}{r(t)}.$$

Therefore, it will be necessary to perform a more in-depth analysis upon the error to show that it goes to zero.

CHAPTER 1

The Constant-r Case

First we work to extend the results of [9] to nonconvolution kernels. To obtain results throughout this paper, we will require assumptions on η_r, k, $\bar{u}$, f and f^δ. The following will suffice for this chapter, and will be standing assumptions even when not explicitly mentioned. Often we will require similar, or even identical hypotheses in other chapters to the ones we state here. When identical, we will simply refer back to these by number.

Hypothesis 1.1 *We assume that Hypothesis 0.1 holds, that* $k \in C^1([0,T] \times [0,T])$ *and that* $k(t,s) > 0$ *for* $0 \le s \le t \le T$. *We also assume that* $\bar{u} \in C^1([0,T])$. *Having assumed these, it is without loss of generality that we further assume* $k(t,t) = 1$ *for* $t \in [0,T]$, *for problem (1) can always be divided by* $k(t,t)$ *yielding a new right-hand side* $f(t)/k(t,t)$ *for which the above assumptions hold.*

Hypothesis 1.2 *The measure* η_r, *parameterized by* $r > 0$ *is a positive finite Borel measure on* $[0,r]$ *satisfying the following condition: if we have a positive integrand* $g(t) > 0$ *for* $t \in (0,r]$, *then we require*

$$\int_0^r g(\rho)\, d\eta_r(\rho) > 0.$$

When r *is a continuous function on* $[0,1]$ *into* $(0, T-1]$, *we assume* η_r *to be a (single)* *associated finite positive Borel measure on* $[0, \|r\|_\infty]$ *for which*

$$\int_0^x g(\rho)\, d\eta(\rho) > 0,$$

for all $x \in [r_{min}, \|r\|_\infty]$, *where* $r_{min} := \min\{r(t) : t \in [0,1]\}$.

We further assume that, for a measure η_r *of the variety described above, we have that the quantities*

$$\frac{\int_0^{r(t)} d\eta_r}{\alpha(t;r)} \quad \text{and} \quad \frac{\tilde{f}(t;r)}{\alpha(t;r)}$$

are square-integrable on $[0,1]$, *where* α *and* $\tilde{f}$ *are defined in (8) and (9) respectively.*

Hypothesis 1.3 *When noisy data* f^δ *is used in place of* f, *we assume that* $f^\delta(t) = f(t) + d(t)$, *where* $d(\cdot) \in L^\infty(0,T)$ *satisfies* $\|d\|_\infty \leq \delta$ *for some fixed* $\delta > 0$ *(here* $\|\cdot\|_\infty$ *denotes the* $L^\infty(0,T)$ *norm). For* η_r *a family of measures parameterized by* $r \in (0, T-1]$ *and satisfying Hypothesis 1.2, we assume that* $\int_0^r d(t+\rho)\, d\eta_r(\rho)$ *is well-defined for all* $t \in [0,1]$, *and that*

$$\frac{\int_0^r f^\delta(t+\rho)\, d\eta_r(\rho)}{\alpha(t;r)}$$

is square-integrable (in t*) on* $[0,1]$. *If the measures are parameterized by continuous functions* $r : [0,1] \to (0, T-1]$, *we assume that* $\int_0^{r(t)} d(t+\rho)\, d\eta_r(\rho)$ *is well-defined for all* $t \in [0,1]$ *and that*

$$\frac{\int_0^{r(t)} f^\delta(t+\rho)\, d\eta_r(\rho)}{\alpha(t;r)}$$

is square integrable on $[0,1]$.

As in [9], we will assume in this chapter that $r(t) \equiv r$ (constant) for $t \in [0,1]$. Notice that Hypotheses 1.2 and 1.1 together imply that (7) has a unique solution $u(\cdot;r) \in L^2(0,T)$. The goal in the remainder of this chapter is to demonstrate the convergence of $u(\cdot)$ solving equation (7) to the solution $\bar{u}$ of (1) in the case of $r(t) = r$

a constant. We will do this in the absence of noise in the data, and also for the case of noisy data.

1.1 Definitions and Preliminary Equations

For $t \in [0,1]$, $r \in (0, T-1]$ let us define

$$\beta_{t,r} := \frac{\int_0^r \int_0^\rho [\rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta)]\, ds\, d\eta_r(\rho)}{\int_0^r \int_0^\rho ds\, d\eta_r(\rho)}, \tag{1.1}$$

where, for each fixed $t \in [0,1]$, $(\rho, s) \in [0, T-1] \times [0, T-1]$, $\xi = \xi_t(\rho, s)$ and $\zeta = \zeta_t(\rho, s)$ are chosen by Taylor's Theorem so that

$$k(t+\rho, s+t) = k(t,t) + \rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta). \tag{1.2}$$

Because we assume that k satisfies Hypothesis 1.1, we have that

$$\begin{aligned} |\beta_{t,r}| &\le \|k\|_{1,\infty} \frac{\int_0^r \int_0^\rho (\rho + s)\, ds\, d\eta_r(\rho)}{\int_0^r \int_0^\rho ds\, d\eta_r(\rho)} \\ &\le 2r\|k\|_{1,\infty} \\ &= \mathcal{O}(r), \end{aligned}$$

uniformly in $t \in [0,1]$.

Now for r small enough that $|\beta_{t,r}| < 1$, we have the expansion

$$\begin{aligned} \frac{1}{\alpha(t;r)} &= \frac{1}{\int_0^r \int_0^\rho k(t+\rho, s+t)\, ds\, d\eta_r(\rho)} \\ &= \frac{1}{\int_0^r \int_0^\rho [k(t,t) + \rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta)]\, ds\, d\eta_r(\rho)} \\ &= \frac{1}{\int_0^r \int_0^\rho ds\, d\eta_r(\rho)} \cdot \frac{1}{1+\beta_{t,r}} \\ &= \frac{1+\gamma_{t,r}}{\int_0^r \rho\, d\eta_r(\rho)}, \end{aligned} \tag{1.3}$$

where $\gamma_{t,r} = \mathcal{O}(r)$ uniformly in $t \in [0,1]$. Employing another Taylor expansion, we have from (5) that

$$\begin{aligned}\tilde{k}(t,s;r) &= \int_0^r [k(t,s) + \rho D_1 k(t+\xi_{t,\rho,s}, s)]\, d\eta_r(\rho) \\ &= k(t,s)\left(\int_0^r d\eta_r\right) + g(t,s;r),\end{aligned}$$

where

$$|g(t,s;r)| \leq \|k\|_{1,\infty} \int_0^r \rho\, d\eta_r(\rho)$$

for all $0 \leq t \leq 1$.

With these definitions, we write equation (11) (the equation in the error $y(t;r) = u(t;r) - \bar{u}(t)$) as

$$\begin{aligned}y(t) &= -\frac{1+\gamma_{t,r}}{\int_0^r \rho\, d\eta_r(\rho)} \int_0^t \tilde{k}(t,s;r) y(s)\, ds + F(t;r) \\ &= -\frac{1+\gamma_{t,r}}{\int_0^r \rho\, d\eta_r(\rho)} \left[\left(\int_0^r d\eta_r\right) \int_0^t k(t,s) y(s)\, ds + \int_0^t g(t,s;r) y(s)\, ds\right] \\ &\quad + F(t;r). \end{aligned} \tag{1.4}$$

Employing a technique used in [9] we define

$$\psi(t,\varepsilon) := \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{\varepsilon} e^{-t/\varepsilon}, & \text{if } t \geq 0, \end{cases} \tag{1.5}$$

$$\varepsilon(r) := \frac{\int_0^r \rho\, d\eta_r(\rho)}{\int_0^r d\eta_r},$$

and convolving both sides of equation (1.4) with $\psi(t, \varepsilon(r))$, we get

$$\begin{aligned}&\int_0^t \psi(t-s, \varepsilon(r)) y(s)\, ds \\ &= \frac{-1}{\int_0^r \rho\, d\eta_r(\rho)} \int_0^t \psi(t-\tau, \varepsilon(r)) \int_0^\tau (1+\gamma_{\tau,r}) g(\tau,s;r) y(s)\, ds\, d\tau\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\varepsilon(r)}\int_0^t \psi(t-\tau,\varepsilon(r))\int_0^\tau (1+\gamma_{\tau,r})k(\tau,s)y(s)\,ds\,d\tau + \psi(t,\varepsilon(r)) * F(t;r)\\
= \;& \frac{-1}{\int_0^r \rho\, d\eta_r(\rho)}\int_0^t\int_s^t \psi(t-\tau,\varepsilon(r))(1+\gamma_{\tau,r})g(\tau,s;r)\,d\tau\, y(s)\,ds\\
&-\frac{1}{\varepsilon(r)}\int_0^t\int_s^t \psi(t-\tau,\varepsilon(r))(1+\gamma_{\tau,r})k(\tau,s)\,d\tau\, y(s)\,ds + \psi(t,\varepsilon(r)) * F(t;r)\\
= \;& \frac{-1}{\int_0^r \rho\, d\eta_r(\rho)}\int_0^t\int_s^t \psi(t-\tau,\varepsilon(r))(1+\gamma_{\tau,r})g(\tau,s;r)\,d\tau\, y(s)\,ds\\
&-\frac{1}{\varepsilon(r)}\int_0^t \left[k(t,s) - e^{-(t-s)/\varepsilon(r)}k(s,s) - \int_s^t e^{-(t-\tau)/\varepsilon(r)}D_1 k(\tau,s)\,d\tau\right] y(s)\,ds\\
&-\frac{1}{\varepsilon(r)}\int_0^t\int_s^t \psi(t-\tau,\varepsilon(r))\gamma_{\tau,r}\, k(\tau,s)\,d\tau\, y(s)\,ds + \psi(t,\varepsilon(r)) * F(t;r). \qquad (1.6)
\end{aligned}
$$

Subtracting equation (1.6) from equation (1.4) we get

$$
y(t) = \int_0^t G(t,s;r)y(s)\,ds + H_1(t;r), \qquad t \in [0,1], \tag{1.7}
$$

where we define the quantities

$$
G(t,s;r) := G_1(t,s;r) + G_2(t,s;r), \quad 0 \le s \le t \le 1 \tag{1.8}
$$

with

$$
\begin{aligned}
G_1(t,s;r) \;&:=\; \psi(t-s,\varepsilon(r))\,[1-k(s,s)] - \int_s^t \psi(t-\tau,\varepsilon(r))\, D_1 k(\tau,s)\,d\tau,\\
&=\; -\int_s^t \psi(t-\tau,\varepsilon(r))\, D_1 k(\tau,s)\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
&G_2(t,s;r)\\
&:= \frac{1}{\int_0^r \rho\, d\eta_r(\rho)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))(1+\gamma_{\tau,r})g(\tau,s;r)\,d\tau - (1+\gamma_{t,r})g(t,s;r)\right]\\
&\quad+\frac{1}{\varepsilon(r)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))\gamma_{\tau,r}k(\tau,s)\,d\tau - \gamma_{t,r}\, k(t,s)\right],
\end{aligned}
$$

for $0 \le s \le t \le 1$ and

$$H_1(t;r) := F(t;r) - \psi(t,\varepsilon(r)) * F(t;r), \quad t \in [0,1].$$

1.2 Convergence Using Noise-Free Data

We wish to show that solutions of (7) do well in approximating solutions of (1) as the equations themselves become more alike — that is, as $r \to 0$. This, along with the results in section 1.3 will justify the solving of (7) as a regularization scheme for (1).

We begin by assuming that the data itself is free of error. Though this is not likely to be the case in numerical representations on a computer, this is a good first test-case, as we could never hope to have convergence in the noisy-data case if we did not have it in the noise-free one. Furthermore, even in those rare instances where the true data can be represented exactly numerically, the numerical solution process itself will introduce errors that call for some stabilizing scheme.

Theorem 1.1 *Let η_r, k, f and $\bar{u}$ satisfy Hypotheses 1.2 and 1.1. If there exists a constant $C \ge 1$ for which*

$$\int_0^r \rho\, d\eta_r(\rho) \ge \frac{r}{C} \int_0^r d\eta_r,$$

for each $r > 0$ sufficiently small, then the solution $u(\cdot;r)$ of (7) converges uniformly on $[0,1]$ to $\bar{u}$ as $r \to 0$.

To show (ultimately) convergence of y to zero as $r \to 0$, we will use the Gronwall Inequality on equation (1.7). We will show that G_1 and G_2 have uniform bounds (independent of r, t and s), and that $H_1 \to 0$ uniformly.

Proof of Theorem 1.1: First, we have that

$$
\begin{aligned}
|G_1(t,s;r)| &\leq \int_s^t \psi(t-\tau,\varepsilon(r))|D_1k(\tau,s)|\,d\tau \\
&\leq \frac{\|k\|_{1,\infty}}{\varepsilon(r)}\int_s^t e^{-(t-\tau)/\varepsilon(r)}\,d\tau \\
&= \|k\|_{1,\infty}\left(1-e^{-(t-s)/\varepsilon(r)}\right) \\
&\leq \|k\|_{1,\infty}
\end{aligned}
$$

for $0 \leq s \leq t \leq 1$.

Next, we see that

$$
\begin{aligned}
&|G_2(t,s;r)| \\
&\leq \frac{1}{\int_0^\tau \rho\,d\eta_r(\rho)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))(1+|\gamma_{\tau,r}|)|g(\tau,s;r)|\,d\tau + (1+|\gamma_{t,r}|)|g(t,s;r)|\right] \\
&\quad + \frac{1}{\varepsilon(r)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))|\gamma_{\tau,r}|k(\tau,s)\,d\tau + |\gamma_{t,r}|k(t,s)\right] \\
&\leq \|k\|_{1,\infty}\left[\int_s^t \psi(t-\tau,\varepsilon(r))(1+|\gamma_{\tau,r}|)\,d\tau + (1+|\gamma_{t,r}|)\right] \\
&\quad + \frac{\|k\|_\infty}{\varepsilon(r)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))|\gamma_{\tau,r}|\,d\tau + |\gamma_{t,r}|\right] \\
&\leq 2\|k\|_{1,\infty}\left[1+\int_s^t \psi(t-\tau,\varepsilon(r))\,d\tau\right] + \frac{\|k\|_\infty}{\varepsilon(r)}\left[\int_s^t \psi(t-\tau,\varepsilon(r))|\gamma_{\tau,r}|\,d\tau + |\gamma_{t,r}|\right]
\end{aligned}
$$

for small enough r, since $\gamma_{t,r} = \mathcal{O}(r)$. Now by assumption we have

$$\frac{1}{\varepsilon(r)} \leq Cr^{-1},$$

so that $|\gamma_{\tau,r}|/\varepsilon(r) \leq C_1$ for some $C_1 > 0$ independent of r. Thus,

$$
\begin{aligned}
|G_2(t,s;r)| &\leq [2\|k\|_{1,\infty} + C_1\|k\|_\infty]\left[1+\int_s^t \psi(t-\tau,\varepsilon(r))\,d\tau\right] \\
&\leq 2(2+C_1)\|k\|_{1,\infty},
\end{aligned}
$$

for $0 \leq s \leq t \leq 1$. Thus G_1 and G_2 (and hence G) are bounded uniformly in $0 \leq s \leq t \leq 1$, $r > 0$ as $r \to 0$.

Since $\bar{u} \in C^1[0,T]$ we argue similarly to the argument found near the end of the introduction that, for each $t \in [0,1]$,

$$\begin{aligned}
|F(t;r)| &\leq \frac{1}{\alpha(t;r)} \int_0^r \int_0^\rho k(t+\rho, s+t)|\bar{u}(s+t) - \bar{u}(t)|\, ds\, d\eta_r(\rho) \\
&= \frac{1}{\alpha(t;r)} \int_0^r \int_0^\rho k(t+\rho, s+t)|\bar{u}'(\zeta_t(s))| s\, ds\, d\eta_r(\rho) \\
&\leq r\frac{\|\bar{u}'\|_\infty}{\alpha(t;r)} \int_0^r \int_0^\rho k(t+\rho, s+t)\, ds\, d\eta_r(\rho) \\
&= r\frac{\|\bar{u}'\|_\infty}{\alpha(t;r)} \alpha(t;r) \\
&= \mathcal{O}(r),
\end{aligned}$$

which shows that $\|F(\cdot,r)\|_\infty = \mathcal{O}(r)$. Thus,

$$\begin{aligned}
|H_1(t;r)| &\leq \|F(\cdot,r)\|_\infty \left[1 + \int_0^t \psi(t-\tau, \varepsilon(r))\, d\tau\right] \\
&\leq 2\|F(\cdot,r)\|_\infty \\
&= \mathcal{O}(r).
\end{aligned}$$

The theorem is now a trivial consequence of the Gronwall Inequality. 2

1.3 Noisy Data

If our data has noise in it then recovery of the "true" $\bar{u}$ (corresponding to the true f) becomes hopeless because of the instability of problem (1). In fact, it is likely that (1) has no solution when f is replaced by a function f^δ satisfying Hypothesis 1.3, as

f^δ need not be in $\mathcal{R}(\mathcal{A})$. In contrast, for this function f^δ we can define

$$\tilde{f}^\delta(t;r) := \int_0^r f^\delta(t+\rho)\,d\eta_r(\rho),$$

and have that

$$\alpha(t;r)u(t) + \int_0^t \tilde{k}(t,s;r)u(s)\,ds \;=\; \tilde{f}^\delta(t;r), \tag{1.9}$$

— equation (7) with $\tilde{f}^\delta$ in place of $\tilde{f}$ — has a unique solution $u^\delta(t;r) \in L^2(0,1)$.

If we denote by y^δ the error $y^\delta(t) = y^\delta(t;r) := u^\delta(t;r) - \bar{u}(t)$, then we find that y^δ solves an equation analogous to (11), namely

$$y(t) \;=\; -\frac{1}{\alpha(t;r)}\int_0^t \tilde{k}(t,s;r)y(s)\,ds + F(t;r) + E(t;r), \tag{1.10}$$

a.e. $t \in [0,1]$, where

$$E(t;r) := \frac{1}{\alpha(t;r)}\int_0^r d(t+\rho)\,d\eta_r(\rho), \quad t \in [0,1].$$

Let us define

$$k_{min} := \min\{k(t,s) : 0 \le s \le t \le T\}.$$

Then

$$\begin{aligned} |E(t;r)| &\le \frac{\int_0^r |d(t+\rho)|\,d\eta_r(\rho)}{\int_0^r\int_0^\rho k(t+\rho, s+t)\,ds\,d\eta_r(\rho)} \\ &\le \frac{\|d\|_\infty}{k_{min}}\cdot\frac{\int_0^r d\eta_r}{\int_0^r \rho\,d\eta_r(\rho)} \\ &\le \frac{\delta}{k_{min}\,\varepsilon(r)}, \qquad t \in [0,1]. \end{aligned}$$

Convolving (1.9) with $\psi(t, \varepsilon(r))$ and subtracting as before we get that

$$y^\delta(t) = \int_0^t G(t, s; r) y^\delta(s)\, ds + H(t; r),$$

where $H(t; r) := H_1(t; r) + H_2(t; r)$ with

$$H_2(t; r) := E(t; r) - \psi(t, \varepsilon(r)) * E(t; r), \quad t \in [0, 1].$$

Here G and H_1 are as in the last section. Now

$$\begin{aligned} |H_2(t; r)| &= \left| E(t; r) - \int_0^t \psi(t - s, \varepsilon(r)) E(s; r)\, ds \right| \\ &\leq \|E(\cdot; r)\|_\infty \left[1 + \int_0^t \psi(t - s, \varepsilon(r))\, ds \right] \\ &\leq \frac{2\delta}{k_{min}\, \varepsilon(r)}, \qquad t \in [0, 1]. \end{aligned}$$

Thus, if $r = r(\delta)$ is chosen so that the quantities

$$\frac{\delta}{\varepsilon(r)} = \frac{\delta \int_0^r d\eta_r}{\int_0^r \rho\, d\eta_r(\rho)}$$

and $r(\delta)$ both go to zero as $\delta \to 0$, then, since we have $H(\cdot)$ (in place of $H_1(\cdot)$) converging uniformly to zero on $[0, 1]$, the argument used to prove Theorem 1.1 still goes through; that is, we have the following result.

Theorem 1.2 *Let η_r, k, f, f^δ and $\bar{u}$ satisfy Hypotheses 1.2 – 1.3. Suppose also that, for some $C \geq 1$, the η_r satisfy the condition*

$$\int_0^r \rho\, d\eta_r(\rho) \geq \frac{r}{C} \int_0^r d\eta_r$$

for all $r > 0$ sufficiently small. If $r = r(\delta)$ is chosen so that both $r(\delta) \to 0$ and

$$\delta \frac{\int_0^{r(\delta)} d\eta_{r(\delta)}}{\int_0^{r(\delta)} \rho \, d\eta_{r(\delta)}(\rho)} \to 0$$

as $\delta \to 0$, then the solution $u^\delta(t;r)$ of (1.9) converges uniformly to $\bar{u}$ on $[0,1]$ as $\delta \to 0$.

We have the following simple corollary.

Corollary 1.1 *Let us assume the conditions of Theorem 1.1. If for fixed $p \in (0,1)$, $C_1 > 0$ we take $r(\delta) = C_1\delta^p$, then $u^\delta(t;r) \to \bar{u}$ uniformly on $[0,1]$ as $\delta \to 0$.*

Proof of Corollary 1.1: Since $p \in (0,1)$ it is clear that $r(\delta) \to 0$ as $\delta \to 0$. And by assumption we have

$$\frac{\delta}{\varepsilon(r)} \le \frac{C\delta}{r} = \frac{C}{C_1}\delta^{1-p} \to 0$$

as $\delta \to 0$. 2

What this theorem and corollary demonstrate is that, in the presence of noisy data, r cannot be allowed to go to zero too quickly. It must be tied to the level of noise in the data, providing an adequate level of regularization appropriate for the amount of noise present. Apparently, the instability of the original problem (1) begins to manifest itself in the perturbed problem (1.9) as the regularization parameter $r \to 0$, and it is necessary to relegate the extent to which this happens in relation to the exactness of the data. This is the same kind of phenomenon that is suggested in Theorem 0.1 concerning the regularization parameter α for Tikhonov regularization.

CHAPTER 2

The Variable $r(\cdot)$ Case

In Chapter 1 we extended the results of [9] to cover nonconvolution kernels. While this is a significant step, we have our sights set upon more. In particular, our goal is to extend the method to accommodate a function $r(\cdot)$ that varies throughout $[0, 1]$, allowing for variable amounts of regularization.

2.1 Preliminaries

We will use the symbols r_{min} and r_{max} (or $\|r\|_\infty$) to denote the minimum and maximum values of $r(\cdot)$ on the interval $[0, 1]$. When r_n is one in a sequence of functions, we denote these values by $r_{n,min}$ and $r_{n,max}$ respectively.

For a fixed function $r(\cdot)$ we define

$$a_\nu(t; r) := \int_0^{r(t)} \rho^\nu \, d\eta_r(\rho), \qquad \nu = 0, 1. \tag{2.1}$$

Using this notation, we can write

$$\begin{aligned} \tilde{k}(t, s; r) &= \int_0^{r(t)} [k(t, s) + \rho D_1 k(t + \xi_{t,s}(\rho), s)] \, d\eta_r(\rho) \\ &= a_0(t; r) k(t, s) + \int_0^{r(t)} \rho D_1 k(t + \xi_{t,s}(\rho), s) \, d\eta_r(\rho). \end{aligned} \tag{2.2}$$

It can also be shown exactly like in (1.3) that

$$\frac{1}{\alpha(t;r)} = \frac{1+\gamma(t;r)}{a_1(t;r)}, \tag{2.3}$$

where $\gamma(t;r) = \mathcal{O}(r(t)) \leq \mathcal{O}(r_{max})$ for $t \in [0,1]$.

Using these estimates, we write

$$\begin{aligned}
\frac{\tilde{k}(t,s;r)}{\alpha(t;r)} &= \frac{a_0(t;r)}{\alpha(t;r)}k(t,s) + D(t,s;r) \\
&= \left(\frac{a_0(t;r)}{\alpha(t;r)} - \frac{a_0(t;r_{min})}{\alpha(t;r_{min})} + \frac{a_0(t;r_{min})[1+\gamma(t;r_{min})]}{a_1(t;r_{min})}\right) k(t,s) + D(t,s;r) \\
&= A(t,s;r) + B(t,s;r) + C(t,s;r) + D(t,s;r),
\end{aligned}$$

where

$$\begin{aligned}
A(t,s;r) &:= \frac{a_0(t;r_{min})}{a_1(t;r_{min})}k(t,s), && (2.4)\\
B(t,s;r) &:= \left(\frac{a_0(t;r)}{\alpha(t;r)} - \frac{a_0(t;r_{min})}{\alpha(t;r_{min})}\right) k(t,s), && (2.5)\\
C(t,s;r) &:= \frac{a_0(t;r_{min})\gamma(t;r_{min})}{a_1(t;r_{min})}k(t,s), && (2.6)\\
D(t,s;r) &:= \frac{1}{\alpha(t;r)}\int_0^{r(t)} \rho D_1 k(t+\xi_{t,s}(\rho), s)\, d\eta_r(\rho), && (2.7)
\end{aligned}$$

each defined for $0 \leq s \leq t \leq 1$. Thus, we write the error equation (1.10) in $y^\delta(t) = u^\delta(t;r) - \bar{u}(t)$ as

$$\begin{aligned}
y(t) &= -\int_0^t [A(t,s;r) + B(t,s;r) + C(t,s;r) + D(t,s;r)]\, y(s)\, ds \\
&\quad + F(t;r) + E(t;r) && (2.8)
\end{aligned}$$

for $t \in [0,1]$.

We choose to break down B further, writing it as

$$\begin{aligned} B(t,s;r) &= \left(\frac{a_0(t;r)[1+\gamma(t;r)]}{a_1(t;r)} - \frac{a_0(t;r_{min})[1+\gamma(t;r_{min})]}{a_1(t;r_{min})}\right) k(t,s) \\ &= [B_1(t,s;r) + B_2(t,s;r) + B_3(t,s;r)]\, k(t,s), \end{aligned}$$

where, for $0 \le s \le t \le 1$,

$$\begin{aligned} B_1(t,s;r) &:= \frac{\int_{r_{min}}^{r(t)} d\eta_r}{a_1(t;r)}, \\ B_2(t,s;r) &:= a_0(t;r_{min}) \left(\frac{1}{a_1(t;r)} - \frac{1}{a_1(t;r_{min})}\right), \\ B_3(t,s;r) &:= \frac{a_0(t;r)}{a_1(t;r)}\gamma(t;r) - \frac{a_0(t;r_{min})}{a_1(t;r_{min})}\gamma(t;r_{min}). \end{aligned}$$

We intend to convolve the equation (2.8) with $\psi(t,\varepsilon)$ defined in (1.5), as in the proof of Theorem 1.1. In that proof it became clear that, when $g(\cdot)$ is a bounded function on $[0,1]$, then so is

$$\int_s^t \psi(t-\tau,\varepsilon) g(\tau)\, d\tau,$$

for $0 \le s \le t \le 1$. The first issue, then, will be to get conditions sufficient for a uniform bound (in t, s and $r(\cdot)$) on as many of the elements of the kernel in (2.8) as possible — namely, on B, C and D.

2.2 Convergence Using Abstract Measures

In Chapter 1 we had a numeric parameter r which served as regularization parameter, with $r \to 0$, and to each such r we associated a measure η_r. In this chapter we replace the constant parameter r with a function $r(\cdot)$, and we will talk about sequences of such functions going to zero. As before, we associate a single measure η_r to each (fixed) function $r : [0,1] \to (0, T-1]$.

To show the boundedness of the various quantities in the last section, we will need to make several assumptions about how the sequence of functions $(r_n(\cdot))$ and the associated measures η_{r_n} are chosen.

Hypothesis 2.1 *We assume that $(r_n(\cdot))$ is a sequence of continuous functions on $[0,1]$ into $(0, T-1]$ converging uniformly to 0 as $n \to \infty$, and that the associated sequence of measures (η_{r_n}) is such that η_{r_n} satisfies Hypothesis 1.2 for each $n = 1, 2, \ldots$. We assume further that these sequences are chosen to satisfy the following conditions:*

(i) The sequence

$$\frac{\int_{r_{n,min}}^{\|r_n\|_\infty} d\eta_{r_n}}{\int_0^{r_{n,min}} \rho \, d\eta_{r_n}(\rho)}$$

is bounded.

(ii) The sequence

$$\frac{\int_0^{\|r_n\|_\infty} d\eta_{r_n}}{\int_0^{r_{n,min}} \rho \, d\eta_{r_n}(\rho)} \|r_n\|_\infty$$

is bounded.

We can now prove the following theorem.

Theorem 2.1 *Assume that k, f and $\bar{u}$ satisfy Hypothesis 1.1. Suppose we have a sequence of continuous functions $r_n : [0,1] \to (0, T-1]$ (so $r_{n,min} > 0$ for each n) converging uniformly to zero and (η_{r_n}) is a corresponding sequence of measures satisfying Hypothesis 2.1. Then $u(t; r_n)$, the solution of*

$$\int_0^t \tilde{k}(t, s; r_n) u(s) \, ds + \alpha(t; r_n) u(t) = \tilde{f}(t; r_n),$$

converges uniformly to $\bar{u}$ on $[0,1]$ as $n \to \infty$. Moreover, if f^{δ_n} satisfies Hypothesis 1.3 for (δ_n), a sequence of positive numbers converging to zero, then the solution $u^{\delta_n}(t; r_n)$

of

$$\int_0^t \tilde{k}(t,s;r_n)u(s)\,ds + \alpha(t;r_n)u(t) = \int_0^{r_n(t)} f^{\delta_n}(t+\rho)\,d\eta_{r_n}(\rho)$$

converges uniformly to $\bar{u}$ *on* $[0,1]$, *provided that* $r_n(\cdot) = r_n(\cdot;\delta_n)$ *is chosen so that the quantities*

$$\delta_n \frac{\int_0^{\|r_n\|_\infty} d\eta_{r_n}}{\int_0^{r_{n,min}} \rho\, d\eta_{r_n}(\rho)} \to 0$$

and $\|r_n\|_\infty \to 0$ *as* $n \to \infty$.

As in chapter 1, the first assertion is really a corollary of the second, arising from the special case where $\delta_n = 0$ and $E(t;r_n) \equiv 0$ for all n. Note that, in that instance, the condition about $r_n(\cdot,\delta_n)$ being chosen to have the expression above converge to zero is really no condition at all. Thus, we will prove only the second assertion. Throughout the proof we will suppress the n, writing simply r or $r(\cdot)$ for a function that comes from the sequence. Whenever a bound is asserted, it will be emphasized that this bound is independent of n.

Proof of Theorem 2.1: We know that the difference $y(t;r) = u(t;r) - \bar{u}(t)$ satisfies (2.8). Taking

$$\varepsilon = \varepsilon(r_{min}) = \frac{a_1(t;r_{min})}{a_0(t;r_{min})},$$

(note that $\varepsilon(r_{min})$ is independent of t) we convolve (2.8) with $\psi(t,\varepsilon(r_{min}))$ to get

$$\begin{aligned}
&\int_0^t \psi(t-s,\varepsilon)y(s)\,ds \\
&= -\int_0^t \psi(t-\tau,\varepsilon)\int_0^\tau \frac{\tilde{k}(\tau,s;r)}{\alpha(\tau;r)}\,y(s)\,ds\,d\tau + \psi(t,\varepsilon) * [E(t;r)+F(t;r)] \\
&= -\int_0^t\int_s^t \psi(t-\tau,\varepsilon)\frac{\tilde{k}(\tau,s;r)}{\alpha(\tau;r)}\,d\tau\,y(s)\,ds + \psi(t,\varepsilon) * [E(t;r)+F(t;r)].
\end{aligned}$$

Subtracting this equation from equation (2.8) we get

$$y(t) = \int_0^t G(t,s;r)y(s)\,ds + H_1(t;r) + H_2(t;r), \tag{2.9}$$

where

$$
\begin{aligned}
&G(t,s;r)\\
&\quad := \int_s^t \psi(t-\tau,\varepsilon)[A(\tau,s;r)+B(\tau,s;r)+C(\tau,s;r)+D(\tau,s;r)]\,d\tau\\
&\qquad +\psi(t-s,\varepsilon)-A(t,s;r)-B(t,s;r)-C(t,s;r)-D(t,s;r) \qquad (2.10)
\end{aligned}
$$

and

$$H_1(t;r) := F(t;r)-\psi(t,\varepsilon)*F(t;r), \qquad (2.11)$$

$$H_2(t;r) := E(t;r)-\psi(t,\varepsilon)*E(t;r), \qquad (2.12)$$

for $t\in[0,1]$.

As mentioned earlier, the boundedness of

$$B(t,s;r)-\int_s^t \psi(t-\tau,\varepsilon)B(\tau,s;r)\,d\tau$$

hinges upon the boundedness of B itself, which in turn rests upon the boundedness of B_1, B_2 and B_3. We have

$$B_1(t,s;r) \le \frac{\int_{r_{min}}^{r_{max}} d\eta_r}{\int_0^{r_{min}} \rho\, d\eta_r(\rho)},$$

which is bounded (uniformly in n) by Hypothesis 2.1(i). Next we have

$$
\begin{aligned}
|B_2(t,s;r)| &= \left(\int_0^{r_{min}} d\eta_r\right)\frac{\int_{r_{min}}^{r(t)} \rho\, d\eta_r(\rho)}{\left(\int_0^{r(t)} \rho\, d\eta_r(\rho)\right)\left(\int_0^{r_{min}} \rho\, d\eta_r(\rho)\right)}\\
&\le \left(\frac{\int_0^{r_{max}} d\eta_r}{\int_0^{r_{min}} \rho\, d\eta_r(\rho)} r_{max}\right)\left(\frac{\int_{r_{min}}^{r_{max}} d\eta_r}{\int_0^{r_{min}} \rho\, d\eta_r(\rho)}\right),
\end{aligned}
$$

the first factor of which is bounded by Hypothesis 2.1(ii), and the second factor by

(i) (both uniformly in n). To get, finally, that $B(t, s; r)$ is bounded, we have

$$|B_3(t, s; r)| \;\leq\; \frac{\int_0^{r_{max}} d\eta_r}{\int_0^{r_{min}} \rho\, d\eta_r(\rho)} \left[\gamma(t; r) + \gamma(t; r_{min})\right],$$

which is bounded (uniformly in n) by Hypothesis 2.1(ii) and the fact that there exists a $C_1 > 0$ for which $|\gamma(t; r)| \leq C_1 r_{max}$ uniformly in $t \in [0, 1], n = 1, 2, \ldots$.

One shows that $C(t, s; r)$ has a uniform bound independent of t, s and r (one bound for all n) just as we showed it for B_3. For D, we have

$$\begin{aligned} |D(t, s; r)| \;&\leq\; \frac{1 + |\gamma(t; r)|}{a_1(t; r)} \int_0^{r(t)} \rho |D_1 k(t + \xi_{t,s}(\rho), s)|\, d\eta_r(\rho) \\ &\leq\; \|k\|_{1,\infty}[1 + |\gamma(t; r)|], \end{aligned}$$

which shows that D also has a uniform bound in t, s and r independent of n.

With $A(t, s; r) = \frac{1}{\varepsilon} k(t, s)$, we perform an integration by parts (details like those in (1.6)) to get

$$\begin{aligned} &\left|\psi(t - s, \varepsilon) - A(t, s; r) + \int_s^t \psi(t - \tau, \varepsilon) A(\tau, s; r)\, d\tau\right| \\ &\qquad = \left|\int_s^t \psi(t - \tau, \varepsilon) D_1 k(\tau, s)\, d\tau\right| \\ &\qquad \leq \|k\|_{1,\infty}. \end{aligned}$$

Thus, our kernel G is bounded (uniformly in t, s and r, independent of n).

One shows that $H_1(t; r_n)$ goes to zero uniformly as $\|r_n\|_\infty \to 0$ just as it was shown in the constant-r case for Theorem 1.1. Also, as in the constant-r case, the expression

$$\delta \frac{\int_0^{r_{n,max}} d\eta_{r_n}}{\int_0^{r_{n,min}} \rho\, d\eta_{r_n}(\rho)},$$

comes out of looking at the bound on $H_2(t; r_n)$, and the convergence of this expression to zero is a sufficient condition to get that $\|H_2(\cdot; r_n(\cdot))\|_\infty \to 0$ uniformly as $\delta \to 0$.

Now the result follows from the Gronwall inequality. 2

Theorem 2.1 is, in fact, a generalization of Theorems 1.1 and 1.2. For, if each $r_n(\cdot)$ is a constant function, then the terms of the sequence in Hypothesis 2.1(i) are all zero. Also, (ii) reduces to the condition

$$\int_0^r \rho \, d\eta_r(\rho) \geq \frac{r}{C} \int_0^r d\eta_r,$$

stated in Theorem 1.1.

2.3 Application to Specific Measures

As an application of theorem 2.1, suppose that, for each continuously differentiable $r : [0,1] \to (0, T-1]$ we take η_r to be a weighted Lebesgue measure of the form $d\eta_r(\rho) = \omega(\rho)d\rho$, where we assume that $\omega : [0, T-1] \to (0, \infty)$ is in L^∞ with $0 < \omega_{min} \leq \omega(\rho) \leq \|\omega\|_\infty$, for $\rho \in [0, T-1]$.

Suppose now that $r(\cdot) = r_n(\cdot)$ is one in a sequence of continuous functions satisfying Hypothesis 2.1 with associated measure as described above. Looking at the expression from condition (i) in that hypothesis, we see that

$$\frac{\int_{r_{min}}^{r_{max}} \omega(\rho)\, d\rho}{\int_0^{r_{min}} \rho\omega(\rho)\, d\rho} \leq \frac{2\|\omega\|_\infty (r_{max} - r_{min})}{\omega_{min} r_{min}^2}.$$

Thus, condition (i) under these types of measures comes down to the existence of some $M_1 > 0$ (independent of n) for which

$$r_{n,max} - r_{n,min} \leq M_1 r_{n,min}^2, \tag{2.13}$$

for all n.

From Hypothesis 2.1(ii) we take the expression

$$\frac{\int_0^{r_{max}} \omega(\rho)\,d\rho}{\int_0^{r_{min}} \rho\omega(\rho)\,d\rho} r_{max} \quad \leq \quad \frac{2\|\omega\|_\infty r_{max}^2}{\omega_{min} r_{min}^2}.$$

Hence condition (ii) requires that there be an $M_2 > 0$ such that

$$\frac{r_{n,max}}{r_{n,min}} \leq M_2 \qquad (2.14)$$

for all $n = 1, 2, \ldots$.

We thus have the following corollary.

Corollary 2.1 *Assume that $k, f, \bar{u}$ and r_n satisfy the conditions of Theorem 2.1, and that $d\eta_{r_n} = \omega(\rho)\,d\rho$ for each n as described above. If there exist constants $M_1, M_2 >$ 0 so that the conditions in (2.13) and (2.14) are satisfied for all n, then $u(t; r_n)$ converges uniformly to $\bar{u}$ on $[0,1]$ as $n \to \infty$. Moreover, if f^{δ_n} satisfies Hypothesis 1.3 for (δ_n), a sequence of positive numbers converging to zero, then the solution $u^{\delta_n}(t; r_n)$ of*

$$\int_0^t \tilde{k}(t,s;r_n)u(s)\,ds + \alpha(t;r_n)u(t) = \int_0^{r_n(t)} f^{\delta_n}(t+\rho)\,d\eta_{r_n}(\rho)$$

converges uniformly to $\bar{u}$ on $[0,1]$, provided that $r_n(\cdot) = r_n(\cdot;\delta_n)$ is chosen so that the quantities

$$\frac{\delta_n \|r_n(\cdot;\delta_n)\|_\infty}{r_{n,min}^2} \to 0$$

and $\|r_n(\cdot;\delta_n)\|_\infty \to 0$ as $n \to \infty$.

The condition in (2.13) deserves some scrutiny. It says, basically, that the functions r_n are becoming constant at the square of the rate at which they are converging to zero. The convergence proved in Corollary 2.1 is asymptotically the same type of convergence as proved in Theorem 1.1. Nevertheless, for $\delta > 0$ (which is generally the case), $r(\cdot)$ need not be constant.

CHAPTER 3

Penalty Predictor–Corrector Methods

In Chapter 2, we proved a convergence result for the class of predictor–corrector methods that has been the main focus of this work so far. While some conditions under which this convergence was shown are quite general — the main one being that $r(t)$ is allowed to vary, providing varying amounts of regularization throughout the domain — others were somewhat limiting. While not ruling out the use of truly variable $r(\cdot)$ in the (usual) case of noisy data, Hypothesis 2.1(i) seems to limit the variation allowed in the (functional) regularization parameter $r(\cdot)$, at least in the limit as the noise level converges to zero. In fact, as seen in Corollary 2.1 for a particular choice of measure, the limiting behavior of $r(\cdot)$ must be, in some sense, like a constant.

The goal of obtaining convergence results under less stringent conditions on the sequence $r_n(\cdot)$ of functions remains an open problem. Motivated by this, we explore a new class of regularizing methods that we dub *Penalty Predictor–Corrector* methods for their similarity to the earlier class. In contrast to the results in Chapter 2, we find for this new class of methods that we do not need to constrain the variation on the regularization parameter $r(\cdot)$ as the noise level goes to zero.

3.1 Motivation for the Method

Let us assume for the moment that, at some fixed $t \in [0,1]$, $r(t) > 0$ and $u(\cdot)$ is a known function on $[0,t)$. We assume also that $\mu(t;r) > 0$. We may then seek a constant c that solves

$$\min_{c \in \mathbf{R}} \int_0^{r(t)} \left| \int_0^t k(t+\rho, s) u(s)\, ds + c \int_0^\rho k(t+\rho, s+t)\, ds - f(t+\rho) \right|^2 d\rho$$
$$+ c^2 \mu(t;r). \tag{3.1}$$

This minimization problem harkens back to equation (2), where the set $\mathcal{D}$ consists of constant extensions of $u(\cdot)$, and the norm in $\mathcal{F}$ is an L^2-norm on $[t, t+r(t)]$. Taking $\frac{\partial}{\partial c}$ of this expression, we get the necessary condition on a solution c that

$$\left[\int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)\, ds\ \omega(t,\rho)\, d\rho + \mu(t;r) \right] c$$
$$+ \int_0^{r(t)} \int_0^t k(t+\rho, s) u(s)\, ds\ \omega(t,\rho)\, d\rho \ = \ \int_0^{r(t)} f(t+\rho)\omega(t,\rho)\, d\rho, \tag{3.2}$$

where

$$\omega(t,\rho) := \int_0^\rho k(t+\rho, s+t)\, ds.$$

For the moment, let us say that this process has been carried out at $t = t_i$. This solution c is, in the sense of Tikhonov regularization, the best constant to represent $u(\cdot)$ on the interval $[t_i, t_i + r(t_i)]$. In [11], the authors deal with a discretization of the problem and, under supposition that $r(t_i) = r\Delta t$ for some integer $r > 0$, they find the best step function (represented as a vector in $\mathbf{R}^r$) to approximate $u(\cdot)$ on $[t_i, t_i + r(t_i)]$. They follow this *prediction* of the solution into the future with a *correction*, retaining only the first component of this vector as the value of the approximate solution $\hat{u}(s)$ on $[t_i, t_i + \Delta t]$ and discarding the rest of the vector. Because theirs is a discretization of the problem, they can move on to the next 't'-value, namely $t_{i+1} = t_i + \Delta t$ and repeat the process on the interval $[t_{i+1}, t_{i+1} + r(t_{i+1})]$. As Δt shrinks, the retained value from the vector is used in the definition of $\hat{u}$ on correspondingly shorter intervals at each step.

We are considering the full continuous problem. Following after the discussion in the last paragraph, we might (in the limit) take our approximation $\hat{u}(t) = c$ (just at that particular t value). Identifying $u(t)$ with c, then, (3.2) becomes (after a change in order of integration)

$$\begin{aligned}\left[\int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\, ds\, \omega(t,\rho)\, d\rho + \mu(t;r)\right] u(t) & \\ + \int_0^t \int_0^{r(t)} k(t+\rho, s)\omega(t,\rho)\, d\rho\, u(s)\, ds &= \int_0^{r(t)} f(t+\rho)\omega(t,\rho)\, d\rho,\end{aligned}$$

which is a special case of the more general equation

$$\begin{aligned}\left[\int_0^{r(t)} \int_0^{\rho} k(t+\rho, s+t)\, ds\, d\eta_{t,r}(\rho) + \mu(t;r)\right] u(t) & \\ + \int_0^t \int_0^{r(t)} k(t+\rho, s)\, d\eta_{t,r}(\rho)\, u(s)\, ds &= \int_0^{r(t)} f(t+\rho)\, d\eta_{t,r}(\rho).\end{aligned}$$

If we define $\alpha(t;r), \tilde{k}(t,s;r)$ and $\tilde{f}(t;r)$ using (8), (5) and (9), respectively, corresponding to the measure $\eta_{t,r}$ (for each $t \in [0,1]$), then this equation becomes

$$\int_0^t \tilde{k}(t,s;r)u(s)\, ds + [\alpha(t;r) + \mu(t;r)]\, u(t) = \tilde{f}(t;r), \tag{3.3}$$

an equation that closely resembles (7). The appearance of the new function $\mu(t;r)$ has to do with the difficulties we encountered in the variable-$r(t)$ case. If we make a wise choice for this μ — say, one that when added to $\alpha(t;r)$ makes for something like a constant (in t) coefficient $c = c(r)$ of $u(t)$ (of course, not so much of a constant that we lose the benefits of our function $r(\cdot)$ over the constant parameter r used in

Chapter 1) — our hope is the solution of (3.3), or of

$$\int_0^t \tilde{k}(t,s;r)u(s)\,ds + c(r)u(t) \quad = \quad \tilde{f}(t;r), \tag{3.4}$$

(assuming that one exists) will converge to the solution of (1) as $r_n(\cdot) \to 0$, and that it will do so under less stringent conditions than those we required in Chapter 2.

3.2 Convergence of Solutions

In the last section we motivated a new penalty predictor–corrector type method. We proposed the introduction of a function $\mu(t;r)$ into the perturbed equation of the predictor–corrector methods we studied earlier, hinting that we could do so in such a way as to give ourselves a new perturbed equation in the classic second-kind form — that is, where the coefficient of the $u(t)$-term was constant. As we shall see, the function μ that we will use in the results of this chapter is not actually intended to make this coefficient into a constant. Here, rather, we have chosen μ so that the problem reduces to one quite similar to the one we analyzed in Chapter 1. The comparison of equation (3.1) to (2) together with results from earlier chapters should make us expect convergence in the noise-free case only as $r(\cdot)$, and simultaneously μ, go to zero. We may also expect that, in the presence of error in the data, both functions must somehow be tied to the amount of error in order to get convergence as the error goes to zero.

For this reason, it is reasonable to tie the two parameters together. Let $p \in (0, 1/2]$. Given a continuous function $r : [0,1] \to (0, T-1]$ and a finite Borel measure η_r on $[0, \|r\|_\infty]$, we will set

$$\mu(t;r) \quad := \quad \|r\|_\infty^p \int_0^{r(t)} d\eta_r. \tag{3.5}$$

With this definition for μ, it follows again from the results in [6] to which we referred in the introduction that equation (3.3) has a unique solution $u(t;r)$ if Hypotheses 1.1 – 1.2 are met with $\alpha(t;r)+\mu(t;r)$ in place of $\alpha(t;r)$.

Theorem 3.1 *Suppose that k satisfies the assumptions made in Hypothesis 1.1. Assume that f is such that a continuous solution $\bar{u}$ of (1) exists on $[0,T]$ with $\bar{u}(0)=0$. Let $(r_n(\cdot))$ be a sequence of continuous functions with $r_n:[0,1]\to(0,T-1]$, $r_n\to 0$ uniformly, and assume that the measures η_{r_n} satisfy Hypothesis 1.2. If $\mu(t;r_n)$ is given as in (3.5) (where r_n replaces r), then the solution $u(t;r_n)$ of (3.3) converges uniformly to $\bar{u}(t)$ as $n\to\infty$.*

To prove this theorem, we will need a lemma, the truth of which is asserted in [3] but not proved there. It asserts that our function $\psi(t,\varepsilon)$ defined in (1.5) is an approximate identity in the space $\mathcal{C}([0,\Upsilon])$, at least in a certain sense.

Lemma 3.2 *Suppose that $g:[0,\Upsilon]\to\mathbf{R}$ is a continuous function satisfying $g(0)=0$. Then*

$$\psi(t,\varepsilon)*g(t):=\int_0^t\psi(t-s,\varepsilon)g(s)\,ds$$

(ψ as defined in (1.5)) converges uniformly to $g(t)$ on $[0,\Upsilon]$ as $\varepsilon\to 0^+$.

The proof is fashioned after that for a similar assertion in [4]. Because we want to use the variable ε in another, more traditional sense during this proof, we elect to use the symbol η (in no way related to the measures called by the same name in other parts of this work) as the second argument for the function ψ.

Proof of Lemma 3.2: . Let $\varepsilon>0$. By the uniform continuity of g on $[0,\Upsilon]$ there exists a number $\delta>0$ such that, for each $t\in[0,\Upsilon]$, $0\le s\le\min\{\delta,t\}$,

$$|g(t-s)-g(t)|\le\frac{\varepsilon}{2}.$$

Taking $s = t$, we note that this last condition along with the assumption that $g(0) = 0$ implies that $|g(t)| \leq \varepsilon/2$ for $t \in [0, \delta]$. If we define

$$\beta_\eta(t) := \int_0^t \psi(s,\eta)\, ds = 1 - e^{-t/\eta},$$

then

$$\begin{aligned} |\psi(t,\eta) * g(t) - g(t)| &\leq |\psi(t,\eta) * g(t) - \beta_\eta(t)g(t)| + |g(t)||\beta_\eta(t) - 1| \\ &\leq \int_0^t \psi(s,\eta)|g(t-s) - g(t)|\, ds + |g(t)|e^{-t/\eta}. \end{aligned}$$

Thus, for $t \in [0, \delta]$,

$$\begin{aligned} |\psi(t,\eta) * g(t) - g(t)| &< \frac{\varepsilon}{2} \int_0^\infty \psi(s,\eta)\, ds + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

and, for $t \in [\delta, \Upsilon]$,

$$\begin{aligned} |\psi(t,\eta) * g(t) - g(t)| &\leq \int_0^\delta \psi(s,\eta)|g(t-s) - g(t)|\, ds \\ &\quad + \int_\delta^t \psi(s,\eta)|g(t-s) - g(t)|\, ds + |g(t)| \left(\sup_{\delta \leq \tau \leq T} e^{-\tau/\eta} \right) \\ &\leq \frac{\varepsilon}{2} + 3\|g\|_\infty e^{-\delta/\eta} \\ &< \varepsilon \end{aligned}$$

for all $\eta = \eta(\delta)$ sufficiently small. 2

In proving Theorem 3.1, we will take $r(\cdot)$ to be one element in the sequence $(r_n(\cdot))$ of functions.

Proof of Theorem 3.1: From (6) we see that the true solution $\bar{u}$ of (1) satisfies

$$\begin{aligned}&\int_0^t \tilde{k}(t,s;r)\bar{u}(s)\,ds + [\alpha(t;r)+\mu(t;r)]\,u(t)\\ =\;& \tilde{f}(t;r) - \int_0^{r(t)}\int_0^{\rho} k(t+\rho, s+t)\,[\bar{u}(s+t)-\bar{u}(t)]\,ds\,d\eta_r(\rho) + \mu(t;r)\bar{u}(t), \quad (3.6)\end{aligned}$$

for $t \in [0,1]$. Setting $y(t;r) = u(t;r) - \bar{u}(t)$, we get that $y(t;r)$ solves

$$y(t) \;=\; \frac{-1}{\alpha(t;r)+\mu(t;r)}\int_0^t \tilde{k}(t,s;r)y(s)\,ds + \tilde{F}(t;r), \qquad t\in[0,1], \qquad (3.7)$$

where

$$\tilde{F}(t;r) := \frac{\int_0^{r(t)}\int_0^{\rho} k(t+\rho,s+t)\,[\bar{u}(s+t)-\bar{u}(t)]\,ds\,d\eta_r(\rho) - \mu(t;r)\bar{u}(t)}{\alpha(t;r)+\mu(t;r)}.$$

As in (2.2), we write

$$\begin{aligned}\tilde{k}(t,s;r) \;&=\; \int_0^{r(t)} [k(t,s) + \rho D_1 k(t+\xi_{t,s}(\rho), s)]\,d\eta_r(\rho)\\ &=\; \left(\int_0^{r(t)} d\eta_r\right)[k(t,s) + H(t,s;r)], \qquad t\in[0,1], \qquad (3.8)\end{aligned}$$

where, for $0 \le s \le t \le 1$, we have

$$\begin{aligned}|H(t,s;r)| \;&=\; \frac{\left|\int_0^{r(t)} \rho D_1 k(t+\xi_{t,s}(\rho), s)d\eta_r(\rho)\right|}{\int_0^{r(t)} d\eta_r}\\ &\le\; \|k\|_{1,\infty}\frac{\int_0^{r(t)}\rho\,d\eta_r(\rho)}{\int_0^{r(t)} d\eta_r}\\ &\le\; \|k\|_{1,\infty}\|r\|_\infty.\end{aligned}$$

Employing a Taylor expansion as in (1.3), we also write

$$\alpha(t;r) = \int_0^{r(t)} \rho\,d\eta_r(\rho) + h(t;r),$$

for $t \in [0,1]$, where

$$h(t;r) := \int_0^{r(t)} \int_0^{\rho} [\rho D_1 k(t+\xi, t+\zeta) + s D_2 k(t+\xi, t+\zeta)]\, ds\, d\eta_r(\rho).$$

Then

$$\alpha(t;r) = \|r\|_\infty^p \left(\int_0^{r(t)} d\eta_r \right) \tilde{h}(t;r),$$

where

$$\begin{aligned}
\left|\tilde{h}(t;r)\right| &= \frac{1}{\|r\|_\infty^p \left(\int_0^{r(t)} d\eta_r \right)} \left| \int_0^{r(t)} \rho\, d\eta_r(\rho) + h(t;r) \right| \\
&\le \frac{1}{\|r\|_\infty^p \left(\int_0^{r(t)} d\eta_r \right)} \left[\|r\|_\infty \left(\int_0^{r(t)} d\eta_r \right) + \frac{3}{2} \|k\|_{1,\infty} \int_0^{r(t)} \rho^2\, d\eta_r(\rho) \right] \\
&\le \|r\|_\infty^{1-p} \left[1 + \frac{3}{2} \|k\|_{1,\infty} \|r\|_\infty \right] \\
&= \mathcal{O}(\|r\|_\infty^{1-p}).
\end{aligned}$$

Using our definition for μ from (3.5), we then have for $\|r\|_\infty$ sufficiently small that

$$\begin{aligned}
\frac{1}{\alpha(t;r) + \mu(t;r)} &= \frac{1}{\|r\|_\infty^p \int_0^{r(t)} d\eta_r} \cdot \frac{1}{1 + \tilde{h}(t;r)} \\
&= \frac{1 + \gamma_p(t;r)}{\|r\|_\infty^p \int_0^{r(t)} d\eta_r},
\end{aligned} \tag{3.9}$$

where $\gamma_p(t;r) = \mathcal{O}(\|r\|_\infty^{1-p})$ as $\|r\|_\infty \to 0$.

Returning to equation (3.7), we employ (3.8) and (3.9) to get that

$$\begin{aligned}
y(t) &= -\frac{1 + \gamma_p(t;r)}{\|r\|_\infty^p} \int_0^t [k(t,s) + H(t,s;r)]\, y(s)\, ds + \tilde{F}(t;r) \\
&= \frac{-1}{\|r\|_\infty^p} \int_0^t k(t,s) y(s)\, ds - \int_0^t G(t,s;r) y(s)\, ds + \tilde{F}(t;r), \qquad t \in [0,1]
\end{aligned} \tag{3.10}$$

where, for $0 \le s \le t \le 1$,

$$
\begin{aligned}
|G(t,s;r)| &= \frac{1}{\|r\|_\infty^p} \left| k(t,s)\gamma_p(t;r) + H(t,s;r)\left[1+\gamma_p(t;r)\right] \right| \\
&= \mathcal{O}(\|r\|_\infty^{1-2p}).
\end{aligned}
$$

Since $p \in (0, 1/2]$, G is bounded (at least) for all $0 \le s \le t \le 1$.

Now as in previous chapters, we will convolve (3.10) with $\psi(t, \varepsilon(r))$ and subtract the result from (3.10). In this instance, we take $\varepsilon(r) = \|r\|_\infty^p$, and the result is

$$
y(t) = \int_0^t K(t,s;r)y(s)\,ds + \tilde{F}(t;r) - \psi(t,\varepsilon(r)) * \tilde{F}(t;r), \qquad (3.11)
$$

where

$$
K(t,s;r) := \int_s^t \psi(t-\tau,\varepsilon(r))G(\tau,s;r)\,d\tau - G(t,s;r) - \int_s^t \psi(t-\tau,\varepsilon(r))D_1k(\tau,s)\,d\tau.
$$

Here the suppressed details are much like those seen when applying this convolution technique in previous chapters. We see that, for $0 \le s \le t \le 1$,

$$
|K(t,s;r)| \le 2\|G\|_\infty + \|k\|_{1,\infty},
$$

showing that K is bounded.

Next we look at the terms of

$$
\tilde{F}(t;r) = \frac{\int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)\left[\bar{u}(s+t) - \bar{u}(t)\right] ds\, d\eta_r(\rho) + \alpha(t;r)\bar{u}(t)}{\alpha(t;r) + \mu(t;r)} - \bar{u}(t).
$$

First, we have from (3.9) for all $t \in [0,1]$ that

$$
\frac{\left| \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)\left[\bar{u}(s+t) - \bar{u}(t)\right] ds\, d\eta_r(\rho) \right|}{\alpha(t;r) + \mu(t;r)}
$$

$$\begin{aligned}
&\leq \frac{2\|k\|_\infty \|\bar{u}\|_\infty \int_0^{r(t)} \rho \, d\eta_r(\rho)}{\alpha(t;r) + \mu(t;r)} \\
&\leq 2[1 + \gamma_p(t;r)]\|k\|_\infty \|\bar{u}\|_\infty \|r\|_\infty^{1-p} \\
&= \mathcal{O}(\|r\|_\infty^{1-p}).
\end{aligned}$$

Likewise, for all $t \in [0,1]$,

$$\frac{\alpha(t;r)}{\alpha(t;r) + \mu(t;r)} \leq [1 + \gamma_p(t;r)]\|k\|_\infty \|r\|_\infty^{1-p}.$$

Thus,

$$\tilde{F}(t;r) - \psi(t, \varepsilon(r)) * \tilde{F}(t;r) = \psi(t, \varepsilon(r)) * \bar{u}(t) - \bar{u}(t) + \mathcal{O}(\|r\|_\infty^{1-p}).$$

And, by Lemma 3.2

$$\psi(t, \varepsilon(r)) * \bar{u}(t) - \bar{u}(t) = \psi(t, \|r\|_\infty^p) * \bar{u}(t) - \bar{u}(t)$$

converges to zero uniformly as $\|r\|_\infty \to 0$.

With these observations about the boundedness of the kernel K and the uniform convergence of

$$\psi * \tilde{F} - \tilde{F}$$

to zero, we may apply the Gronwall Inequality to (3.11) to get the desired result. 2

We note here that, should we replace our expression for $\mu(t;r)$ in (3.5) with a constant multiple of it (say, multiplied by some $c > 0$), then the proof of Theorem 3.1 still goes through. Evidently, though, the proof fails if $\mu(t;r) \equiv 0$, as the presence of a nonzero constant added to $\tilde{h}(t;r) = \mathcal{O}(\|r\|^{1-p}$ in the denominator in (3.9) after the $\|r\|_\infty^p \int_0^{r(t)} d\eta_r$ has been factored out is crucial.

As in earlier chapters, we need to know that the convergence asserted in Theorem 3.1 will happen in the presence of noisy data. As before, the proof does not change much from the "pure data" case.

Theorem 3.1 *Along with all of the assumptions in Theorem 3.1, we also assume that* f^{δ_n} *satisfies Hypothesis 1.3 with respect to the measures* η_{r_n} *for* (δ_n)*, a sequence of positive numbers converging to zero. If for some* $c > 0, q \in (0,2)$ *we have* $\|r_n\|_\infty = c\delta_n^q$*, then the solution* $u^{\delta_n}(t;r_n)$ *of*

$$\int_0^t \tilde{k}(t,s;r_n)u(s)\,ds + [\alpha(t;r_n) + \mu(t;r_n)]\,u(t) = \int_0^{r_n(t)} f^{\delta n}(t+\rho)\,d\eta_{r_n}(\rho)$$

converges uniformly to $\bar{u}$ *as* $n \to \infty$.

Proof of Theorem 3.1: As in the proof of Theorem 3.1 we form the difference $y(t;r) = u^\delta(t;r) - \bar{u}(t)$ (here we are suppressing the dependence upon n), convolve the equation for y with $\psi(t, \|r\|_\infty^p)$ and subtract to get equation (3.11) with the additional terms

$$E(t;r) - \psi(t, \varepsilon(r)) * E(t;r),$$

where

$$E(t;r) = \frac{1}{\alpha(t;r) + \mu(t;r)} \int_0^{r(t)} d(t+\rho)\,d\eta_r(\rho).$$

But, from (3.9),

$$\begin{aligned} |E(t;r)| &= \frac{|1+\gamma_p(t;r)|}{\|r\|_\infty^p \int_0^{r(t)} d\eta_r} \int_0^{r(t)} d(t+\rho)\,d\eta_r(\rho) \\ &\leq \frac{\delta}{\|r\|_\infty^p}[1 + |\gamma_p(t;r)|]. \end{aligned}$$

Since $p \in (0, 1/2]$, it is clear that $\|E(\cdot;r)\|_\infty \to 0$ (and hence $E - \psi * E$) for any choice of $q \in (0,2)$. 2

CHAPTER 4

The Discretized Problem

Most problems of type (1) are solved numerically on a computer, and nearly always with some error introduced as a result. It is impractical, in general, to represent a function on $[0, 1]$ (or any interval) perfectly. Most likely, the functions involved are known only by samplings at specific times, and represented on a machine as vectors.

One numerical approach to problem (1) without using a special regularization technique is to simply partition the interval $[0, 1]$ into N subintervals $[t_i, t_{i+1}]$, $i = 0, \dots N-1$, each of width $\Delta t = 1/N$ and to seek constants $\alpha_i, i = 0, \dots, N-1$, so that the step function

$$u(t) = \sum_{i=0}^{N-1} \alpha_i \chi_i(t)$$

satisfies (1) at the collocation points $t = t_i$, $i = 1, \dots, N$. In this discretized form, (1) becomes a lower-triangular matrix problem with nonzero diagonal elements (under suitable conditions upon the kernel) and as such is well-posed. Nevertheless, the instability of (1) (the infinite-dimensional problem) manifests itself even in this discretized setting, with the condition number of the matrix growing as Δt shrinks. Often it is necessary to keep the value of N so small that solutions are hardly of any use in order to keep those solutions from becoming highly oscillatory. We will not delve deeply into these issues, but more can be found in [10].

Thus, in considering a discretized version of problem (1), it makes sense for us to also consider a discretization of some perturbed, well-posed problem that we will solve in lieu of it. In this chapter, we will consider a discretized form of the penalty predictor-corrector equation (3.3) (with penalty function $\mu(t;r)$ for $t \in [0,1]$) and show that the solution of this equation converges to the true solution of (1) at collocation points as the grid size shrinks. Further (in contrast to the continuous theory in Chapter 3), our convergence theory will also apply to the case of $\mu(\cdot;r) \equiv 0$. The significance of this result is that we will obtain convergence of solutions of standard discretizations of equation (7) in the case of regularization parameters $r(\cdot)$ that are not limited by a variability constraint such as that given in Hypothesis 2.1(i).

4.1 The Setup

We assume that $[0,1]$ has been partitioned up into N (a positive integer) subintervals $[t_i, t_{i+1}]$ where $t_i = i\Delta t$, $i = 0, \ldots, N-1$, $\Delta t = 1/N$. We further assume that $k, \bar{u}$ and f satisfy Hypothesis 1.1. So we have $\bar{u}$ satisfies (1) at each $t \in [0,1]$; in particular, at $t = t_i$, $i = 1, \ldots, N$.

We take as our perturbation of (1) the equation (3.3) for $\mu(t;r) \geq 0$, namely

$$\int_0^t \tilde{k}(t,s;r)u(s)\,ds + [\alpha(t;r) + \mu(t;r)]u(t) \quad = \quad \tilde{f}(t;r). \tag{4.1}$$

where for now, we take $r(\cdot)$ to be a continuous function on $[0,1]$ into $(0, T-1]$.

In discretizing (4.1) we seek constants $\alpha_0, \ldots, \alpha_{N-1}$ so that the step function

$$u(t;\Delta t) = \sum_{i=0}^{N-1} \alpha_i \phi_i(t) \tag{4.2}$$

satisfies (4.1) at each of the points $t_1, \ldots, t_N$. Here ϕ_i is the indicator function on the half-open interval $(t_i, t_{i+1}]$ for $i = 1, \ldots, N-1$

$$\phi_i(t) := \begin{cases} 1, & \text{if } t \in (t_i, t_{i+1}] \\ 0, & \text{otherwise,} \end{cases}$$

and ϕ_0 is the indicator function on the closed interval $[t_0, t_1]$. Substituting (4.2) into (4.1) and setting $t = t_{j+1}$ gives us

$$\sum_{i=0}^{j} \alpha_i \int_{t_i}^{t_{i+1}} \tilde{k}(t_{j+1}, s; r)\, ds + [\alpha(t_{j+1}; r) + \mu(t_{j+1}; r)]\alpha_j \quad = \quad \tilde{f}(t_{j+1}; r) \tag{4.3}$$

for $j = 0, \ldots, N-1$. (Notice that we have taken to writing r_j for $r(t_j)$.) Equation (4.3) can be written as a lower-triangular matrix system (in the α_i's) where the diagonal elements

$$\int_{t_j}^{t_{j+1}} \int_0^{r_{j+1}} k(t_{j+1} + \rho, s)\, d\eta_r(\rho)\, ds + \alpha(t_{j+1}; r) + \mu(t_{j+1}; r)$$

are all positive if we assume that $k > 0$, $\mu(t; r) \geq 0$ and η_r satisfies Hypothesis 1.2. Under these assumptions, constants $\alpha_0, \ldots, \alpha_{N-1}$ do exist (uniquely) so that (4.2) satisfies (4.1) at t_j, $j = 1, \ldots, N$.

In previous chapters we have talked about convergence of an approximate solution to the true solution as a (function) parameter $r(\cdot) \to 0$. In the discretized setting, we also want to see what happens as $\Delta t \to 0$, and we will link these two parameters together.

Hypothesis 4.1 *Let $\gamma : [0, 1] \to (0, \infty)$ be a piecewise-continuous function. We will assume that $r(\cdot) := \gamma(\cdot)\Delta t$.*

Because of this assumption, it makes sense to write $\mu(t_j; \gamma, \Delta t)$ in place of $\mu(t_j; r)$ for $j = 0, \ldots, N-1$. We will adopt a notation $\mu_j(\Delta t)$, suppressing the dependence upon γ. Likewise, in place of η_r we will write $\eta_{\Delta t}$ (also suppressing the measure's dependence upon γ).

Let us then define constants $c_j(\Delta t) = c_j(\mu, \Delta t)$ by

$$c_j(\Delta t) := \frac{\alpha(t_j; r) + \mu_j(\Delta t)}{a_0(t_j; r)}, \quad j = 0, \ldots, N-1, \tag{4.4}$$

where a_ν is defined as in (2.1) for $\nu = 0, 1$. (Note that the $a_\nu(t; r_j), \alpha(t_j; r)$ depend ultimately upon γ and Δt by Hypothesis 4.1, but they do so through the r_j.) Using these, (4.3) can be written as

$$\sum_{i=0}^{j} \alpha_i \int_{t_i}^{t_{i+1}} \tilde{k}(t_{j+1}, s; r)\, ds + a_0(t_{j+1}; r) c_{j+1}(\Delta t)\alpha_j \quad = \quad \tilde{f}(t_{j+1}; r).$$

Alternatively, if $d\eta_{\Delta t}$ is a positive Borel measure satisfying Hypothesis 1.2 and f^δ satisfies Hypothesis 1.3 with respect to $d\eta_{\Delta t}$, then with f^δ in place of f in (4.1) there is a unique step function $u(\cdot; \Delta t, f^\delta)$ (whose dependence upon γ is suppressed) of the form (4.2) satisfying

$$\sum_{i=0}^{j} \alpha_i \int_{t_i}^{t_{i+1}} \frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)}\, ds + c_{j+1}(\Delta t)\alpha_j \quad = \quad \int_0^{r_{j+1}} \frac{f^\delta(t_{j+1} + \rho)}{a_0(t_{j+1}; r)}\, d\eta_{\Delta t}(\rho), \tag{4.5}$$

for $j = 0, \ldots, N-1$.

We will use a differencing technique similar to that used in [8] to analyze convergence. To this end, we make a shift in the indices j in (4.5) to get that $u(t; \Delta t, f^\delta)$ satisfies

$$\sum_{i=0}^{j-1} \alpha_i \int_{t_i}^{t_{i+1}} \frac{\tilde{k}(t_j, s; r)}{a_0(t_j; r)}\, ds + c_j(\Delta t)\alpha_{j-1} \quad = \quad \int_0^{r_j} \frac{f^\delta(t_j + \rho)}{a_0(t_j; r)}\, d\eta_{\Delta t}(\rho),$$

for $j = 1, \ldots, N$. Subtracting this equation from (4.5) yields

$$c_{j+1}(\Delta t)\alpha_j + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} \alpha_j \phi_j(s)\, ds$$

$$
\begin{aligned}
= \quad & \frac{1}{a_0(t_{j+1};r)} \int_0^{r_{j+1}} [f(t_{j+1}+\rho) + d(t_{j+1}+\rho)] \, d\eta_{\Delta t}(\rho) \\
& -\frac{1}{a_0(t_j;r)} \int_0^{r_j} [f(t_j+\rho) + d(t_j+\rho)] \, d\eta_{\Delta t}(\rho) + c_j(\Delta t)\alpha_{j-1} \\
& -\sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} - \frac{\tilde{k}(t_j,s;r)}{a_0(t_j;r)} \right] \alpha_i \phi_i(s) \, ds,
\end{aligned} \tag{4.6}
$$

which holds for $j = 1, \ldots, N-1$. We have seen in (3.6) that the true solution $\bar{u}$ satisfies

$$
\begin{aligned}
& \int_0^t \tilde{k}(t,s;r)\bar{u}(s)\,ds + \int_0^{r(t)} \int_0^\rho k(t+\rho, s+t)\,[\bar{u}(s+t) - \bar{u}(t)]\,ds\,d\eta_{\Delta t}(\rho) \\
& \qquad +[\alpha(t;r) + \mu(t;r)]\bar{u}(t) \;=\; \int_0^{r(t)} f(t+\rho)\,d\eta_{\Delta t}(\rho) + \mu(t;r)\bar{u}(t),
\end{aligned} \tag{4.7}
$$

for all $t \in [0,1]$. If, as we did above, we evaluate (4.7) at $t = t_j$ and divide through by $a_0(t_j;r)$, and then subtract the resulting equation from the one arising from evaluation at $t = t_{j+1}$ and division by $a_0(t_{j+1};r)$, we get

$$
\begin{aligned}
& c_{j+1}(\Delta t)\bar{u}(t_{j+1}) + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} \bar{u}(s)\,ds \\
= \quad & -\int_0^{t_j} \left[\frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} - \frac{\tilde{k}(t_j,s;r)}{a_0(t_j;r)} \right] \bar{u}(s)\,ds + c_j(\Delta t)\bar{u}(t_j) \\
& -\int_0^{r_{j+1}} \int_0^\rho k(t_{j+1}+\rho, s+t_{j+1}) \frac{\bar{u}(s+t_{j+1}) - \bar{u}(t_{j+1})}{a_0(t_{j+1};r)}\,ds\,d\eta_{\Delta t}(\rho) \\
& +\int_0^{r_j} \int_0^\rho k(t_j+\rho, s+t_j) \frac{\bar{u}(s+t_j) - \bar{u}(t_j)}{a_0(t_j;r)}\,ds\,d\eta_{\Delta t}(\rho) \\
& +\int_0^{r_{j+1}} \frac{f(t_{j+1}+\rho)}{a_0(t_{j+1};r)}\,d\eta_{\Delta t}(\rho) - \int_0^{r_j} \frac{f(t_j+\rho)}{a_0(t_j;r)}\,d\eta_{\Delta t}(\rho) \\
& +\mu_{j+1}(\Delta t)\frac{\bar{u}(t_{j+1})}{a_0(t_{j+1};r)} - \mu_j(\Delta t)\frac{\bar{u}(t_j)}{a_0(t_j;r)},
\end{aligned} \tag{4.8}
$$

for $j = 1, \ldots, N-1$. Subtracting (4.6) from (4.8) gives us

$$
c_{j+1}(\Delta t)\,[\bar{u}(t_{j+1}) - \alpha_j] + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} [\bar{u}(s) - \alpha_j\phi_j(s)]\,ds
$$

$$
\begin{aligned}
= & -\int_0^{r_{j+1}} \frac{d(t_{j+1}+\rho)}{a_0(t_{j+1};r)}\, d\eta_{\Delta t}(\rho) + \int_0^{r_j} \frac{d(t_j+\rho)}{a_0(t_j;r)}\, d\eta_{\Delta t}(\rho) + c_j(\Delta t)[\bar{u}(t_j) - \alpha_{j-1}] \\
& -\sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} - \frac{\tilde{k}(t_j,s;r)}{a_0(t_j;r)}\right] [\bar{u}(s) - \alpha_i \phi_i(s)]\, ds \\
& -\int_0^{r_{j+1}} \int_0^{\rho} k(t_{j+1}+\rho, s+t_{j+1}) \frac{\bar{u}(s+t_{j+1}) - \bar{u}(t_{j+1})}{a_0(t_{j+1};r)}\, ds\, d\eta_{\Delta t}(\rho) \\
& +\int_0^{r_j} \int_0^{\rho} k(t_j+\rho, s+t_j) \frac{\bar{u}(s+t_j) - \bar{u}(t_j)}{a_0(t_j;r)}\, ds\, d\eta_{\Delta t}(\rho) \\
& +\mu_{j+1}(\Delta t) \frac{\bar{u}(t_{j+1})}{a_0(t_{j+1};r)} - \mu_j(\Delta t) \frac{\bar{u}(t_j)}{a_0(t_j;r)}.
\end{aligned} \tag{4.9}
$$

By a Taylor expansion we can write

$$
\bar{u}(t) = \bar{u}(t_{j+1}) + (t - t_{j+1})\bar{u}'(z_j(t)), \tag{4.10}
$$

for some $z_j(t)$ between t and t_{j+1}. Thus for $t \in (t_j, t_{j+1}]$ we have

$$
\bar{u}(t) - \alpha_j \phi_j(t) = \Delta t \left[\beta_j + \frac{t - t_{j+1}}{\Delta t} \bar{u}'(z_j(t))\right],
$$

for $j = 0, \ldots, N-1$, where

$$
\beta_j := \frac{\bar{u}(t_{j+1}) - \alpha_j}{\Delta t}.
$$

Using this and dividing through by Δt, (4.9) becomes

$$
\begin{aligned}
& \left[c_{j+1}(\Delta t) + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)}\, ds\right] \beta_j + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} \frac{(s - t_{j+1})}{\Delta t} \bar{u}'(z_j(s))\, ds \\
= \; & c_j(\Delta t)\beta_{j-1} - \frac{1}{\Delta t} \left\{ \int_0^{r_{j+1}} \frac{d(t_{j+1}+\rho)}{a_0(t_{j+1};r)}\, d\eta_{\Delta t}(\rho) - \int_0^{r_j} \frac{d(t_j+\rho)}{a_0(t_j;r)}\, d\eta_{\Delta t}(\rho) \right. \\
& \qquad + \int_0^{r_{j+1}} \int_0^{\rho} k(t_{j+1}+\rho, s+t_{j+1}) \frac{\bar{u}(s+t_{j+1}) - \bar{u}(t_{j+1})}{a_0(t_{j+1};r)}\, ds\, d\eta_{\Delta t}(\rho) \\
& \qquad \left. - \int_0^{r_j} \int_0^{\rho} k(t_j+\rho, s+t_j) \frac{\bar{u}(t_j+s) - \bar{u}(t_j)}{a_0(t_j;r)}\, ds\, d\eta_{\Delta t}(\rho) \right\} \\
& -\sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} - \frac{\tilde{k}(t_j,s;r)}{a_0(t_j;r)}\right] \left[\beta_i + \frac{s - t_{i+1}}{\Delta t} \bar{u}'(z_i(s))\right] ds \\
& +\frac{1}{\Delta t} \left[\mu_{j+1}(\Delta t) \frac{\bar{u}(t_{j+1})}{a_0(t_{j+1};r)} - \mu_j(\Delta t) \frac{\bar{u}(t_j)}{a_0(t_j;r)}\right],
\end{aligned} \tag{4.11}
$$

for $j = 1, \ldots, N-1$.

Finally, we define the quantities (for which the dependence upon γ is suppressed)

$$\begin{aligned}
D_j(\Delta t) &:= c_{j+1}(\Delta t) + \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)}\, ds, && (4.12)\\
W_j(\Delta t) &:= \frac{c_j(\Delta t)}{D_j(\Delta t)}, && (4.13)\\
E_j(\delta, \Delta t) &:= \frac{1}{D_j(\Delta t)} \left[\int_0^{r_{j+1}} \frac{d(t_{j+1}+\rho)}{a_0(t_{j+1}; r)}\, d\eta_{\Delta t}(\rho) \right.\\
&\qquad \left. - \int_0^{r_j} \frac{d(t_j+\rho)}{a_0(t_j; r)}\, d\eta_{\Delta t}(\rho) \right], && (4.14)\\
V_{j,i}(\Delta t) &:= \frac{1}{D_j(\Delta t)\Delta t} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} - \frac{\tilde{k}(t_j, s; r)}{a_0(t_j; r)} \right] ds, && (4.15)\\
Z_j(\Delta t) &:= \frac{R_j(\Delta t)}{D_j(\Delta t)\Delta t}, && (4.16)
\end{aligned}$$

where

$$\begin{aligned}
R_j(\Delta t) := &\int_0^{r_{j+1}} \int_0^{\rho} k(t_{j+1}+\rho, s+t_{j+1}) \frac{\bar{u}(s+t_{j+1}) - \bar{u}(t_{j+1})}{a_0(t_{j+1}; r)}\, ds\, d\eta_{\Delta t}(\rho)\\
&- \int_0^{r_j} \int_0^{\rho} k(t_j+\rho, s+t_j) \frac{\bar{u}(t_j+s) - \bar{u}(t_j)}{a_0(t_j; r)}\, ds\, d\eta_{\Delta t}(\rho)\\
&+ \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} - \frac{\tilde{k}(t_j, s; r)}{a_0(t_j; r)} \right] (s - t_{i+1}) \bar{u}'(z_i(s))\, ds\\
&+ \int_{t_j}^{t_{j+1}} \frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} (s - t_{j+1}) \bar{u}'(z_j(s))\, ds\\
&- \mu_{j+1}(\Delta t) \frac{\bar{u}(t_{j+1})}{a_0(t_{j+1}; r)} + \mu_j(\Delta t) \frac{\bar{u}(t_j)}{a_0(t_j; r)}, \qquad (4.17)
\end{aligned}$$

for $i = 0, \ldots, j-1$, $j = 1, \ldots, N-1$. With these expressions, (4.11) can be written as

$$\beta_j = W_j(\Delta t)\beta_{j-1} - \Delta t \sum_{i=0}^{j-1} V_{j,i}(\Delta t)\beta_i - \frac{1}{\Delta t} E_j(\delta, \Delta t) - Z_j(\Delta t), \qquad (4.18)$$

for $j = 1, \ldots, N-1$.

Now if we evaluate (4.5) at $j = 0$ and (4.7) at $t = t_1$ (dividing through the resulting equation by $a_0(t_1; r)$) and then subtract the two equations, we get

$$\beta_0 = -\frac{1}{\Delta t} E_0(\delta, \Delta t) - Z_0(\Delta t), \tag{4.19}$$

where

$$\begin{aligned} D_0(\Delta t) &:= c_1(\Delta t) + \int_0^{t_1} \frac{\tilde{k}(t_1, s; r)}{a_0(t_1; r)}\, ds, & (4.20)\\ E_0(\delta, \Delta t) &:= \frac{1}{D_0(\Delta t)} \int_0^{r_1} \frac{d(t_1 + \rho)}{a_0(t_1; r)}\, d\eta_{\Delta t}(\rho), & (4.21)\\ Z_0(\Delta t) &:= \frac{R_0(\Delta t)}{D_0(\Delta t)\Delta t}, & (4.22) \end{aligned}$$

with

$$\begin{aligned} R_0(\Delta t) := \frac{1}{a_0(t_1; r)} \Big[&\int_0^{r_1} \int_0^{\rho} k(t_1 + \rho, s + t_1) \left[\bar{u}(s + t_1) - \bar{u}(t_1)\right] ds\, d\eta_{\Delta t}(\rho) \\ &+ \int_0^{t_1} \tilde{k}(t_1, s; r)(s - t_1)\bar{u}'(z_0(s))\, ds - \mu_1(\Delta t)\bar{u}(t_1) \Big]. \end{aligned} \tag{4.23}$$

4.2 Conditions for Convergence

In the last section we defined expressions that arose out of a "differencing" approach to the solutions $\bar{u}$ and $u(t; \Delta t, f^\delta)$. While the quantities themselves are somewhat different, the purpose they serve is the same as similarly-defined expressions in [8]. The next theorem and proof also come from [8]. We provide the proof here for convenience.

Theorem 4.1 *Assume that point evaluations of the solution $\bar{u}$ to (1) make sense, and that $d\eta_{\Delta t}$, f^δ, k and μ satisfy the conditions discussed above. Suppose positive* *numbers w, M, v and z exist such that*

$$\begin{aligned}
W_j(\Delta t) &\le w, \qquad j = 1, \dots, N-1, \\
V_{j,i}(\Delta t) &\le v, \qquad i = 0, \dots, j-1, \; j = 1, \dots, N-1, \\
E_j(\delta, \Delta t) &\le M \frac{\delta}{\Delta t}, \qquad j = 0, \dots, N-1, \\
Z_j(\Delta t) &\le z, \qquad j = 0, \dots, N-1,
\end{aligned}$$

uniformly in $\Delta t > 0$, with $w \in (0,1)$. If $\Delta t = \Delta t(\delta) = c\sqrt{\delta}$ for a fixed $c > 0$, then the solution $u(t; \Delta t, f^\delta)$ of (4.5) converges to $\bar{u}$ at the collocation points t_j, $j = 0, \dots, N(\delta)$ ($N(\delta) = 1/\Delta t(\delta)$) as $\delta \to 0$. This convergence is at the best possible rate with respect to δ; that is,

$$|u(t_j; \Delta t, f^\delta) - \bar{u}(t_j)| \le K\delta^{1/2} + \mathcal{O}(\delta),$$

for $j = 1, \dots, N(\delta)$, as $\delta \to 0$, where K is a positive constant independent of δ and Δt.

Proof of Theorem 4.2: Arguing as in [8] we define constants $B_j = B_j(\Delta t, \delta)$, $j = 0, \dots, N-1$ satisfying

$$\begin{aligned}
B_0 &= \frac{1}{\Delta t^2} M\delta + z, \\
B_j &= w B_{j-1} + \Delta t \, v \sum_{i=0}^{j-1} B_i + \frac{1}{\Delta t^2} M\delta + z, \quad j = 1, \dots, N-1.
\end{aligned}$$

If we assume that $\Delta t = \Delta t(\delta)$ is chosen so that $\delta/\Delta t^2(\delta)$ remains bounded as $\delta \to 0$, then the coefficients in the relations above are bounded. It is easily verified that these B_j satisfy the second-order difference equation

$$\begin{aligned}
B_0 &= \frac{1}{\Delta t^2} M\delta + z, \\
B_1 &= (w + \Delta t \, v) B_0 + \frac{1}{\Delta t^2} M\delta + z,
\end{aligned}$$

$$B_j \;=\; (1+w+\Delta t\, v)B_{j-1} - wB_{j-2}, \quad j=2,\ldots,N-1.$$

By the theory of difference equations we find that

$$B_j = C_1\,\tau_1^j + C_2\,\tau_2^j,$$

for $j=0,\ldots,N-1$. Here

$$\begin{aligned}
\tau_1 &= 1+\frac{v}{1-w}\Delta t + \mathcal{O}(\Delta t^2),\\
\tau_2 &= w\left[1-\frac{v}{1-w}\Delta t\right] + \mathcal{O}(\Delta t^2),\\
C_1(\delta,\Delta t) &= \frac{z+M\frac{\delta}{\Delta t^2}}{1-w} + \mathcal{O}(\Delta t),\\
C_2(\delta,\Delta t) &= -w\,\frac{z+M\frac{\delta}{\Delta t^2}}{1-w} + \mathcal{O}(\Delta t).
\end{aligned}$$

From the definitions of the B_j it is clear that each $B_j > 0$, and that $\tau_1 > \tau_2 > 0$. Since $C_2 < 0$ for Δt sufficiently small, we have

$$\begin{aligned}
B_j &= C_1(\delta,\Delta t)\tau_1^j + C_2(\delta,\Delta t)\tau_2^j\\
&\le C_1(\delta,\Delta t)\tau_1^j\\
&= C_1(\delta,\Delta t)\left[1+\frac{v}{1-w}\Delta t + \mathcal{O}(\Delta t^2)\right]^j\\
&\le 2C_1(\delta,\Delta t)exp\left(\frac{2v}{1-w}\right),
\end{aligned}$$

and thus taking $\Delta t(\delta) = c\sqrt{\delta}$ gives us that the B_j have a uniform bound for $j = 0,\ldots,N-1$, independent of N. A simple induction argument gives that

$$|\beta_j| \le B_j, \quad j=0,\ldots,N-1,$$

showing that

$$|\alpha_j - \bar{u}(t_{j+1})| \le 2C_1(\delta, \Delta t)\Delta t\, exp\left(\frac{v}{1-w}\right) \to 0$$

as δ (and hence $\Delta t(\delta)$) goes to 0. 2

Theorem 4.2 *Suppose that $k, \bar{u}$ and f satisfy Hypothesis 1.1, that we have measures $\eta_{\Delta t} = \eta_{\gamma\Delta t}$ for $\Delta t > 0$ (sufficiently small) that satisfy Hypothesis 1.2, and that f^δ satisfies Hypothesis 1.3 with respect to these measures. Assume also that Hypothesis 4.1 holds, and that $\|\gamma\|_\infty < k_{min}/\|k\|_\infty$. If we take*

$$\mu_j(\Delta t) := c a_0(t_j; r)\Delta t^q,$$

for $c \ge 0, q \ge 1$, then the conclusions of Theorem 4.1 hold concerning the convergence of $u(t; \Delta t, f^\delta)$ to $\bar{u}$ as $\delta \to 0$ at the best possible rate with respect to δ, provided $\Delta t = \Delta t(\delta) = c\sqrt{\delta}$.

Proof of Theorem 4.2: From our assumptions, we have that $D_j(\Delta t) > 0$ for each $j = 0, \ldots, N-1$. Furthermore, we have for each of these j that

$$\begin{aligned}\frac{1}{D_j(\Delta t)} &= \frac{a_0(t_{j+1}; r)}{\alpha(t_{j+1}; r) + \mu_{j+1}(\Delta t) + \int_{t_j}^{t_{j+1}} \int_0^{r_{j+1}} k(t_{j+1} + \rho, s)\, d\eta_{\Delta t}(\rho)\, ds} \\ &\le \frac{a_0(t_{j+1}; r)}{\int_{t_j}^{t_{j+1}} \int_0^{r_{j+1}} k(t_{j+1} + \rho, s)\, d\eta_{\Delta t}(\rho)\, ds} \\ &\le \frac{1}{k_{min}\Delta t}. \end{aligned} \tag{4.24}$$

From this we see that

$$\begin{aligned}|E_j(\delta, \Delta t)| &\le \frac{\delta}{k_{min}\Delta t}\left(\frac{1}{a_0(t_{j+1}; r)}\int_0^{r_{j+1}} d\eta_{\Delta t}(\rho) + \frac{1}{a_0(t_j; r)}\int_0^{r_j} d\eta_{\Delta t}(\rho)\right) \\ &= \frac{2\delta}{k_{min}\Delta t},\end{aligned}$$

for $j = 1, \ldots, N-1$, and likewise we have the same bound for $|E_0(\delta, \Delta t)|$. We can thus take M (in the statement of Theorem 4.1) to be $M = 2/k_{min}$.

In (2.2) we showed that

$$\tilde{k}(t,s;r) = a_0(t;r)k(t,s) + \int_0^{r(t)} \rho D_1 k(t + \xi_{s,t}(\rho), s)\, d\eta_{\Delta t}(\rho). \tag{4.25}$$

Thus,

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \left| \frac{\tilde{k}(t_{j+1},s;r)}{a_0(t_{j+1};r)} - \frac{\tilde{k}(t_j,s;r)}{a_0(t_j;r)} \right| ds \\
&\leq \int_{t_i}^{t_{i+1}} \left(|k(t_{j+1},s) - k(t_j,s)| + \frac{\int_0^{r_j} \rho |D_1 k(t_j + \xi_{s,t_j}(\rho), s)|\, d\eta_{\Delta t}(\rho)}{a_0(t_j;r)} \right. \\
&\qquad \left. + \frac{\int_0^{r_{j+1}} \rho |D_1 k(t_{j+1} + \xi_{s,t_{j+1}}(\rho), s)|\, d\eta_{\Delta t}(\rho)}{a_0(t_{j+1};r)} \right) ds \\
&\leq \int_{t_i}^{t_{i+1}} \left[|D_1 k(t_j + \xi_{s,t_j}(\Delta t), s)| \Delta t + \|k\|_{1,\infty}(r_j + r_{j+1}) \right] ds \\
&\leq \|k\|_{1,\infty}(1 + 2\|\gamma\|_\infty)\Delta t^2.
\end{aligned}$$

This along with (4.24) shows that we may bound the $V_{j,i}, i = 0, \ldots, j-1, j = 1, \ldots, N-1$ by

$$v = \frac{\|k\|_{1,\infty}}{k_{min}}(1 + 2\|\gamma\|_\infty).$$

Turning to the Z_j, we see that by (4.24) we need only show that each of the terms of the R_j is $\mathcal{O}(\Delta t^2)$. We have that the first term

$$\begin{aligned}
&\left| \int_0^{r_{j+1}} \int_0^\rho k(t_{j+1} + \rho, s + t_{j+1}) \frac{\bar{u}(s + t_{j+1}) - \bar{u}(t_{j+1})}{a_0(t_{j+1};r)}\, ds\, d\eta_{\Delta t}(\rho) \right| \\
&\leq \frac{\|k\|_\infty}{a_0(t_{j+1};r)} \int_0^{r_{j+1}} \int_0^\rho s|\bar{u}'(z_j(t_{j+1} + s))|\, ds\, d\eta_{\Delta t}(\rho) \\
&\leq \frac{\|k\|_\infty \|\bar{u}'\|_\infty}{a_0(t_{j+1};r)} r_{j+1} \int_0^{r_{j+1}} \rho\, d\eta_{\Delta t}(\rho) \\
&\leq \|k\|_\infty \|\bar{u}'\|_\infty \|\gamma\|_\infty^2 \Delta t^2,
\end{aligned}$$

for $j = 1, \ldots, N-1$. The first term of R_0 is handled similarly, as is the second term in the expression for R_j, $j = 1, \ldots, N-1$.

Our work in bounding the $V_{j,i}$ also shows that

$$\begin{aligned}
&\left| \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left[\frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} - \frac{\tilde{k}(t_j, s; r)}{a_0(t_j; r)} \right] (s - t_{i+1}) \bar{u}'(z_i(s))\, ds \right| \\
&\leq \|\bar{u}'\|_\infty \Delta t \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left| \frac{\tilde{k}(t_{j+1}, s; r)}{a_0(t_{j+1}; r)} - \frac{\tilde{k}(t_j, s; r)}{a_0(t_j; r)} \right| ds \\
&\leq \|k\|_{1,\infty} \|\bar{u}'\|_\infty (1 + 2\|\gamma\|_\infty) N \Delta t^3 \\
&= \|k\|_{1,\infty} \|\bar{u}'\|_\infty (1 + 2\|\gamma\|_\infty) \Delta t^2,
\end{aligned}$$

which takes care of the summation term of the R_j, $j = 1, \ldots, N-1$. The fourth term for these same values of j is handled similarly, as is the second term in the expression for R_0.

Using our definition for the μ_j, we handle the remaining terms from the expression for the R_j, $j = 1, \ldots, N-1$ as follows:

$$\begin{aligned}
\left| \mu_{j+1}(\Delta t) \frac{\bar{u}(t_{j+1})}{a_0(t_{j+1}; r)} - \mu_j(\Delta t) \frac{\bar{u}(t_j)}{a_0(t_j; r)} \right| &= c \left| \bar{u}(t_{j+1}) - \bar{u}(t_j) \right| \Delta t^q \\
&\leq c \|\bar{u}'\|_\infty \Delta t^{q+1}.
\end{aligned}$$

Since $q \geq 1$, this term is (at least) $\mathcal{O}(\Delta t^2)$. The final term in the expression for R_0 can be written as

$$c \left[\bar{u}(t_1) - \bar{u}(0) \right] \Delta t^q,$$

since we assume $\bar{u}(0) = 0$, and is then handled the same way.

Next, we have that

$$W_j(\Delta t) = \frac{c_j(\Delta t)}{c_{j+1}(\Delta t) + \frac{1}{a_0(t_{j+1}; r)} \int_{t_j}^{t_{j+1}} \tilde{k}(t_{j+1}, s; r)\, ds}$$

$$
\begin{aligned}
&= \frac{c_j(\Delta t)}{c_j(\Delta t) + [c_{j+1}(\Delta t) - c_j(\Delta t)] + \frac{1}{a_0(t_{j+1};r)} \int_{t_j}^{t_{j+1}} \tilde{k}(t_{j+1}, s; r)\, ds} \\
&= \frac{1}{1 + K_j},
\end{aligned}
$$

for $j = 1, \ldots, N-1$, where

$$
\begin{aligned}
K_j &:= \frac{1}{c_j(\Delta t)} \left(c_{j+1}(\Delta t) - c_j(\Delta t) + \frac{1}{a_0(t_{j+1};r)} \int_{t_j}^{t_{j+1}} \tilde{k}(t_{j+1}, s; r)\, ds \right) \\
&= \frac{1}{c_j(\Delta t)} \left(\frac{\alpha(t_{j+1};r) + \mu_{j+1}(\Delta t)}{a_0(t_{j+1};r)} - \frac{\alpha(t_j;r) + \mu_j(\Delta t)}{a_0(t_j;r)} \right. \\
&\qquad \left. + \frac{1}{a_0(t_{j+1};r)} \int_{t_j}^{t_{j+1}} \int_0^{r_{j+1}} k(t_{j+1} + \rho, s)\, d\eta_{\Delta t}(\rho)\, ds \right) \\
&= \frac{1}{c_j(\Delta t)} \left(\frac{\alpha(t_{j+1};r)}{a_0(t_{j+1};r)} + c\Delta t^q - \frac{\alpha(t_j;r)}{a_0(t_j;r)} - c\Delta t^q \right. \\
&\qquad \left. + \frac{1}{a_0(t_{j+1};r)} \int_{t_j}^{t_{j+1}} \int_0^{r_{j+1}} k(t_{j+1} + \rho, s)\, d\eta_{\Delta t}(\rho)\, ds \right) \\
&\geq \frac{1}{c_j(\Delta t)} \left(\frac{\alpha(t_{j+1};r)}{a_0(t_{j+1};r)} - \frac{\alpha(t_j;r)}{a_0(t_j;r)} + k_{min}\Delta t \right) \\
&\geq \frac{1}{c_j(\Delta t)} \left(0 - \frac{\|k\|_\infty \int_0^{r_j} \rho\, d\eta_{\Delta t}(\rho)}{a_0(t_j;r)} + k_{min}\Delta t \right) \\
&\geq \frac{\Delta t}{c_j(\Delta t)} \left(-\|k\|_\infty \|\gamma\|_\infty + k_{min} \right).
\end{aligned}
$$

Now we have assumed that

$$
\|\gamma\|_\infty < \frac{k_{min}}{\|k\|_\infty},
$$

which means that there exists an $L > 0$ such that

$$
\begin{aligned}
K_j &\geq \frac{L\Delta t}{c_j(\Delta t)} \\
&= \frac{L a_0(t_j;r)\Delta t}{\int_0^{r_j} \int_0^{\rho} k(t_j + \rho, s + t_j)\, d\eta_{\Delta t}(\rho)\, ds + c a_0(t_j;r)\Delta t^q} \\
&\geq \frac{L a_0(t_j;r)\Delta t}{\|k\|_\infty \int_0^{r_j} \rho\, d\eta_{\Delta t}(\rho) + c a_0(t_j;r)\Delta t^q} \\
&\geq \frac{L}{\|k\|_\infty \|\gamma\|_\infty + c\Delta t^{q-1}}
\end{aligned}
$$

$$\geq \frac{L}{\|k\|_\infty \|\gamma\|_\infty + c},$$

since $\Delta t \leq 1$ and $q \geq 1$. Thus, if we define $w = [1 + L/(\|k\|_\infty \|\gamma\|_\infty + c)]^{-1}$, then we have $W_j(\Delta t) \leq w < 1$ for all $j = 1, \ldots, N-1$. 2

One surprising thing about this theorem is that the proof goes through with the constant $c = 0$, and thus with $\mu(t;r) \equiv 0$. Recall that in the continuous case (see Chapter 3) we could not prove convergence with $\mu(t;r) \equiv 0$ without restricting the variability of $r(t)$ in the limit as the noise level goes to zero (as in Chapter 2). This shows that, at least in the case of the discretized problem, the penalty predictor-corrector class of methods we have analyzed here is a generalization of the predictor-corrector class of methods we discussed in Chapters 1 and 2.

The condition assumed in Theorem 4.2 that

$$\|\gamma\|_\infty < \frac{k_{min}}{\|k\|_\infty}$$

is used at the end of the proof in order to get the existence of an $L > 0$ for which

$$k_{min} - \|k\|_\infty \|\gamma\|_\infty > L,$$

and, ultimately, an upper bound $w \in [0, 1)$ upon the W_j. A closer inspection shows that what is really necessary is an $L > 0$ for which

$$\frac{\alpha(t_{j+1};r)}{a_0(t_{j+1};r)\Delta t} - \frac{\alpha(t_j;r)}{a_0(t_j;r)\Delta t} + k_{min} > L.$$

We exploit this observation to arrive at several corollaries.

Corollary 4.1 *If we assume all of the stated conditions of Theorem 4.2, replacing only the condition* $\|\gamma\|_\infty < k_{min}/\|k\|_\infty$ *with the assumption that the function* γ *in*

Hypothesis 4.1 is constant, then the conclusions of Theorem 4.2 hold.

Proof of Corollary 4.1: If we take $L = k_{min}$, then

$$W_j(\Delta t) \leq \frac{1}{1 + L/(||k||_\infty ||\gamma||_\infty + c)} < 1$$

for $j = 1, \ldots, N-1$. 2

Corollary 4.1 gives us a chance to compare the generality of Theorem 4.2 with the one found in [8] (summed up in Theorem 0.3). This result is more general both in that it allows for nonconvolution kernels and in the few requirements placed upon the measures, as Theorem 0.3 is proved only for special types of measures.

Corollary 4.2 *Let us assume all of the stated conditions of Theorem 4.2, replacing only the condition* $||\gamma||_\infty < k_{min}/||k||_\infty$ *with the assumptions that* $\gamma \in C^1$ *and the following condition (like in Theorem (1.1)): there exists a constant* $C \geq 1$ *such that, for each* $\Delta t > 0$ *(sufficiently small) and each* $x \in [r_{min}, r_{max}]$ *(that is,* $x/\Delta t \in [\gamma_{min}, \gamma_{max}]$*),*

$$\int_0^x \rho \, d\eta_{\Delta t}(\rho) \geq \frac{x}{C} \int_0^x d\eta_{\Delta t}. \tag{4.26}$$

If

$$||\gamma||_\infty < \frac{k_{min}}{||k||_\infty - \frac{k_{min}}{C}},$$

Then the conclusions of Theorem 4.2 hold.

Note that, should a positive C exist so that (4.26) is satisfied, it necessarily the case that $C \geq 1$. This is because

$$\begin{aligned} C &\geq \frac{x \int_0^x d\eta_{\Delta t}}{\int_0^x \rho \, d\eta_{\Delta t}(\rho)} \\ &\geq \frac{x \int_0^x d\eta_{\Delta t}}{x \int_0^x d\eta_{\Delta t}}. \end{aligned}$$

Proof of Corollary 4.2: For $j = 1, \ldots, N-1$ we have

$$\begin{aligned}\frac{\alpha(t_j;r)}{a_0(t_j;r)} &\geq k_{min}\frac{\int_0^{r_j}\rho\, d\eta_{\Delta t}(\rho)}{\int_0^{r_j} d\eta_{\Delta t}} \\ &\geq \frac{k_{min}\gamma_j\Delta t}{C}.\end{aligned}$$

Also,

$$\frac{\alpha(t_j;r)}{a_0(t_j;r)} \leq \|k\|_\infty \gamma_j \Delta t.$$

Thus

$$\begin{aligned}\frac{\alpha(t_{j+1};r)}{a_0(t_{j+1};r)\Delta t} - \frac{\alpha(t_j;r)}{a_0(t_j;r)\Delta t} + k_{min} &\geq \frac{k_{min}\gamma_{j+1}}{C} - \|k\|_\infty\gamma_j + k_{min} \\ &= \frac{k_{min}}{C}(\gamma_{j+1}-\gamma_j) + \gamma_j(\frac{k_{min}}{C} - \|k\|_\infty) + k_{min} \\ &= \frac{k_{min}}{C}\gamma'(\xi_j)\Delta t + \gamma_j(\frac{k_{min}}{C} - \|k\|_\infty) + k_{min}.\end{aligned}$$

Now $k_{min}\|\gamma'\|_\infty/C$ is bounded, and so the first term above can be made as small as needed as $\Delta t \to 0$. Thus, if

$$\gamma_j(\frac{k_{min}}{C} - \|k\|_\infty) + k_{min} > 0,$$

for each $j = 1, \ldots, N-1$, the conclusion holds. Our assumption above upon the size of $\|\gamma\|_\infty$ is sufficient to imply this. 2

CHAPTER 5

Numerical Results

In Chapter 4 we described a collocation scheme which uses no *special* regularization method in its attempt to find a step-function solution that matches the data at N discrete points. We emphasize the word "special", because the act of discretization so as to consider a finite-dimensional problem in place of the original infinite-dimensional one is itself a regularizing process. Nevertheless, we claimed that without a relatively large stepsize, such solutions become highly oscillatory. Figure 5.1, supports this assertion.

To produce the results in Figure 5.1, a convolution kernel of $k(t) \equiv 1$ was used along with a *known* true $\bar{u}(t) = t$, so that the true data is $f(t) = t^2/2$. A random amount of noise not exceeding 10^{-2} (making for a relative error of 2%) was then added to f to produce noisy data, which then was used (in place of f) in the collocation process described above. The resulting approximate solution from four different stepsizes corresponding to $N = 8, 15, 25$ and 45 subintervals of $[0, 1]$ is plotted against the known true solution, with the approximate solution plotted in dashing. The approximate solution, which we said was a step function, has not actually been plotted as such. Instead, the constant interval in the solution has been condensed down to

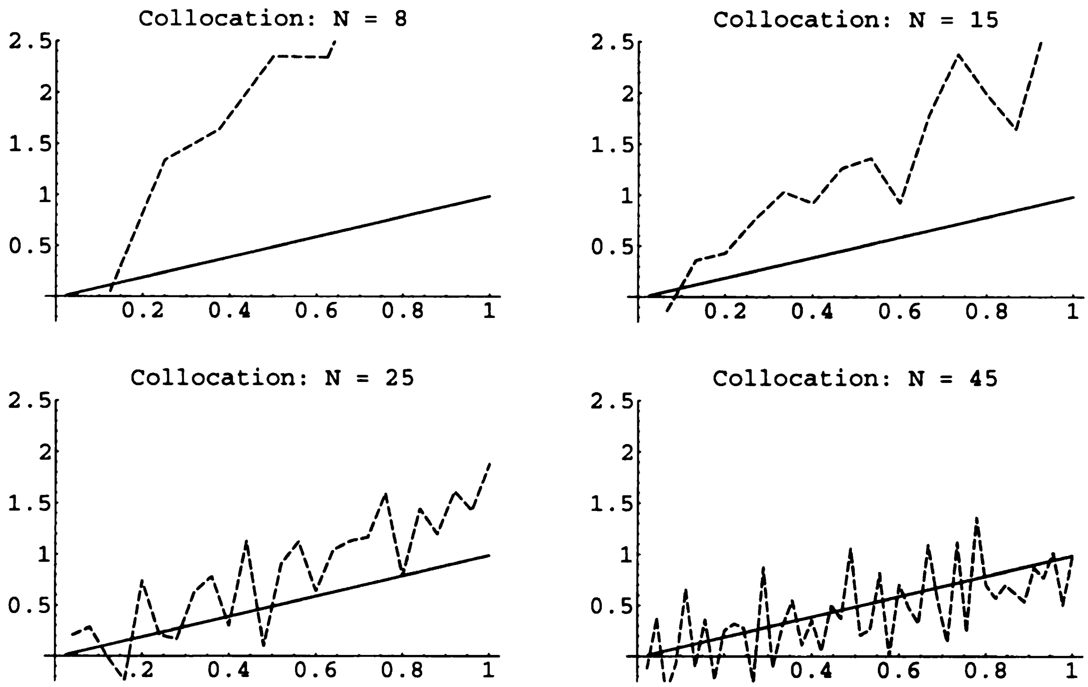

Figure 5.1. Simple Collocation at Various Stepsizes

a point, with these points then being joined by straight lines. The other numerical examples we include in this work are obtained and displayed in similar fashion.

In the Introduction we claimed that, using standard Tikhonov regularization, it is difficult to find a value for the parameter α that provides for a smooth solution in certain sections of the domain and a rough one in others. We demonstrate this with an example using a discretized Tikhonov scheme. In Figure 5.2 we have discretized $[0,1]$ into 40 subintervals and used this scheme for four different choices of α, $\alpha = 0$ (equivalent to the discretized scheme of Figure 5.1), $10^{-11}, 5\times10^{-10}$, and 5×10^{-9}. In this example, the "spike" function that is the solid graph in each of the four plots was numerically integrated against the kernel $k(t) = t^2$ to obtain the quasi-true f. Random noise not exceeding 10^{-6} was then added to the data, with Tikhonov regularization then being applied to the resulting perturbed data to obtain the (dashed) approximate solutions.

We note that, at $\alpha = 10^{-11}$, the approximate solution recovers the spike quite well, but the smooth sections of the true solution are correspondingly rough in the approximate one. As α is increased, this roughness is smoothed out nicely, but at the expense of recovering the spike. This is a tendency that is found in all regularization methods, that as the regularization parameter is increased solutions tend to become oversmoothed.

In Figure 5.3 we have applied our discretized method (with $\mu \equiv 0$) to the same problem as that in Figure 5.2 to allow for comparison of its success against that of Tikhonov. Three of the plots are for constant $\gamma(t)$, with these values set at $1, 2$ and 3. In the fourth plot, we have set γ to 5 through most of the domain, but it equals 1 in the region of the spike.

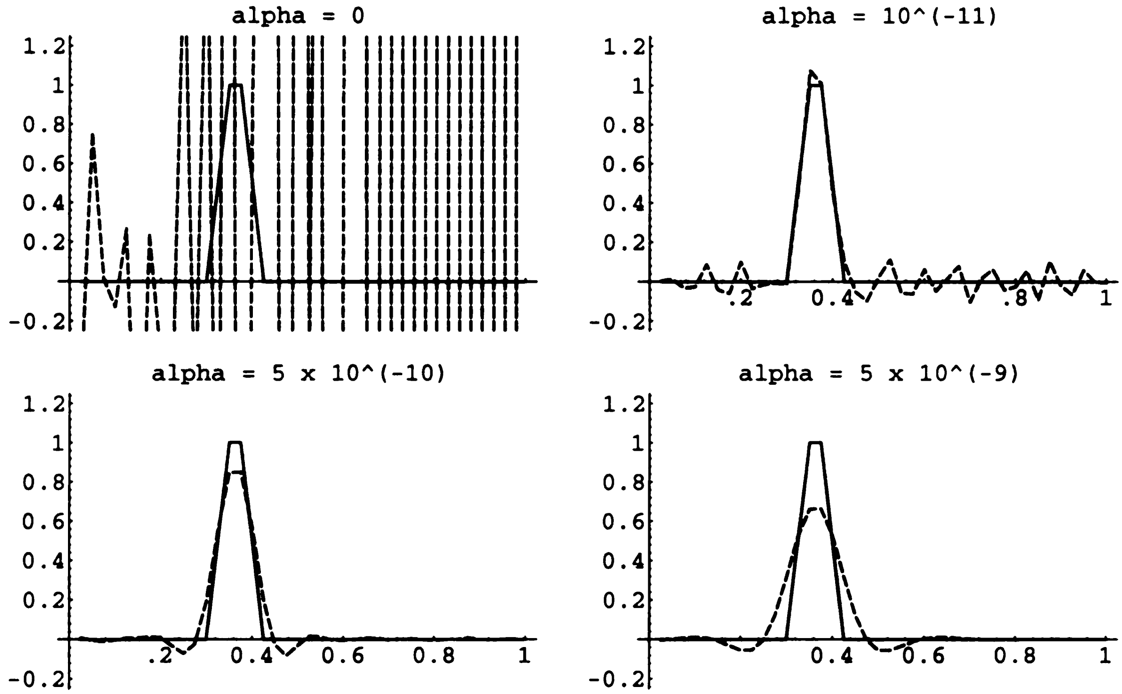

Figure 5.2. Tikhonov Regularization for Several α Values

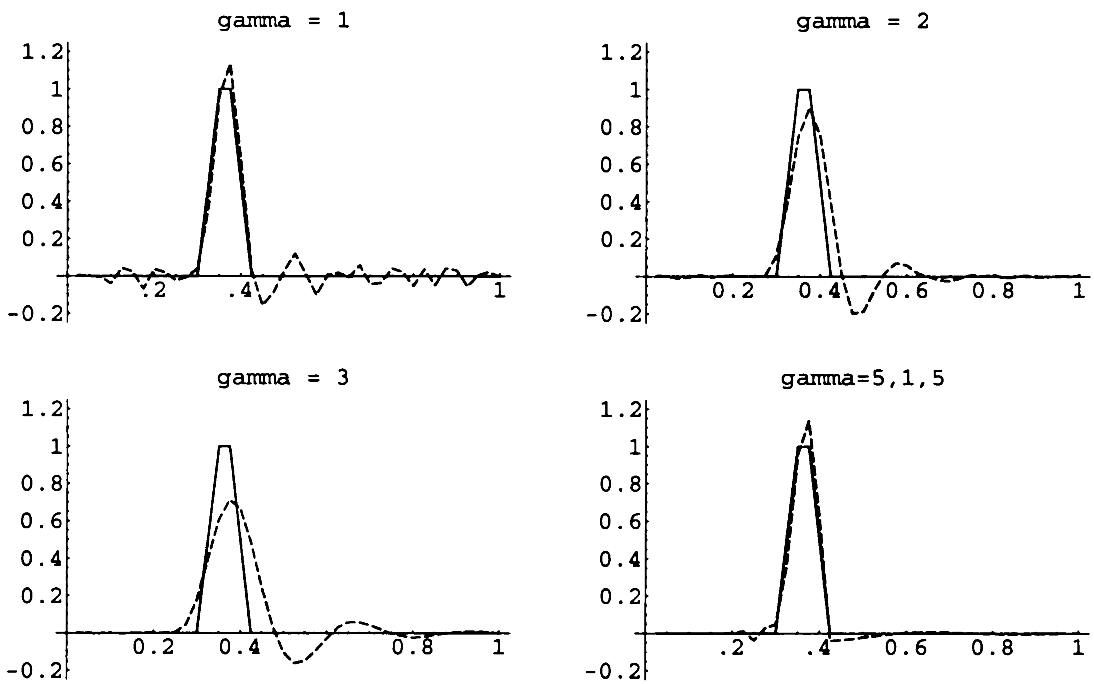

Figure 5.3. Tikhonov Regularization for Several α Values

The relationship between Figures 5.4 and 5.5 is analogous to that between Figures 5.2 and 5.3. The true solution is a step function (though our plotter renders it as looking continuous). We have again discretized to 40 subintervals, numerically integrated the true solution against the kernel $k(t) = t$, added relative error in the amount of around 0.05%, and applied a regularization process to this perturbed data, using Tikhonov at several values of α in Figure 5.4 and our method in Figure 5.5.

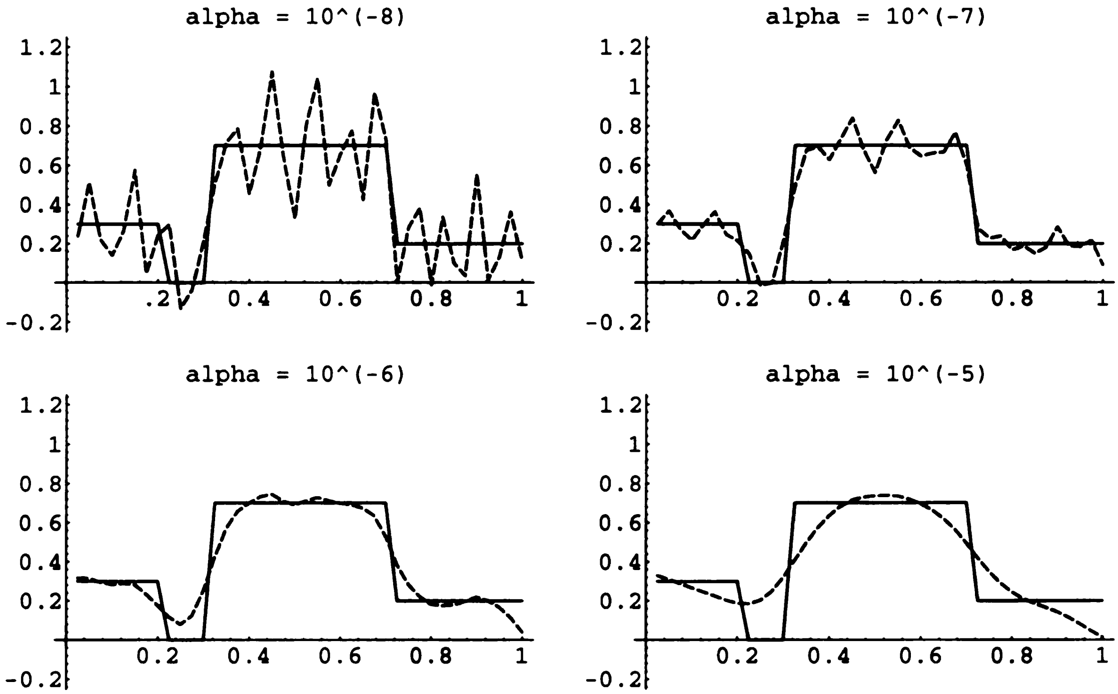

Figure 5.4. More Tikhonov Solutions

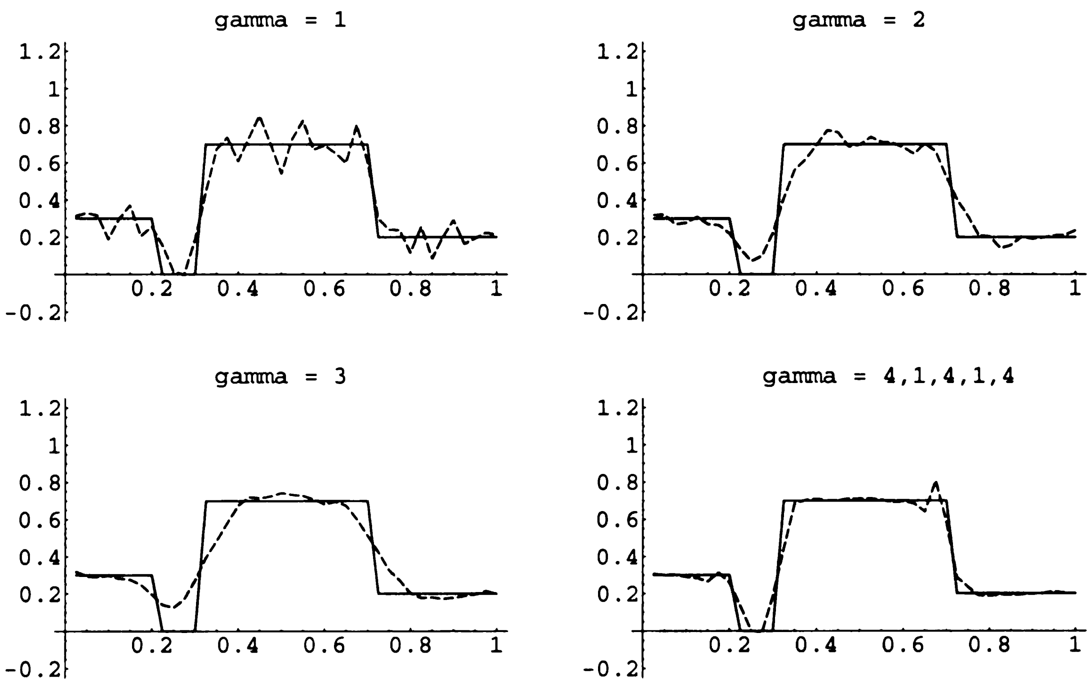

Figure 5.5. More Solutions using Predictor-Corrector Scheme

this new class of methods to be a generalization of the earlier class. It is our hope that we may be able to show this in the case of the infinite-dimensional (continuous) problem as well.

In order to demonstrate the practical value of these methods, we provided numerical examples of computed solutions plotted against known true solutions. Of course, the purpose of any regularization technique is to find approximate solutions for problems the solutions of which are *not* known. One might reasonably ask (regardless of the particular regularization technique of which we are speaking) how we are to choose the correct value for the regularization parameter. Why, after all, would we know that 5×10^{-10} is a better value for α than 5×10^{-9} in the example of Figure 5.2 if we did not have the true solution with which to compare it?

This is a difficult question to address mathematically. In the case of Tikhonov regularization, at least, we have discrepancy principles that provide somewhat satisfactory answers. To summarize one known as the Morozov Discrepancy Principle, we assume that we have perturbed data f^δ for the true f in (1) with an estimate on the noise $\|f^\delta - f\|_{\mathcal{F}} \leq \delta$. It is an established fact that the minimization problem (2) (using f^δ for f) has a unique solution $\hat{u}_\alpha^\delta$ for each $\alpha > 0$, and that the "discrepancy"

$$\|A\hat{u}_\alpha^\delta - f^\delta\|_{\mathcal{F}}$$

is monotone in α. The Morozov Discrepancy Principle says that the correct value of α is the one for which the discrepancy equals δ (see, for example, [5] or [7] for further information on this topic). It is an open problem how to select the regularization parameter $r(\cdot)$ for the predictor-corrector method described in Chapters 1 and 2, or how to select the regularization parameters for the penalty predictor-corrector methods of Chapters 3 and 4. A sequential type of discrepancy principle seems the natural choice for these types of regularization methods, so that the proper value of

r_i is chosen at the i^{th} step in the discretized process. We hope that, after further work, we may be able to get a satisfactory answer to this question.

Finally, the convergence proofs in this work have consistently been under the hypothesis that $k(t,s) > 0$ for all t, s. In the case of a convolution kernel, even just the weaker assumption that $k(0) \neq 0$ is sufficient to quantify the degree of ill-posedness in problem (1) as being of first-order in that a single differentiation of (1) would lead to a well-posed second-kind equation. We call such a kernel 1-smoothing. While problems with 1-smoothing kernels are certainly ill-posed, there are many problems which do not fall under this classification, having ν-smoothing kernels (i.e., kernels for which

$$k(0) = k'(0) = \ldots = k^{(\nu-1)}(0) = 0, \qquad k^{(\nu)}(0) \neq 0,$$

so that the problem (1) becomes a well-posed second-kind equation after ν differentiations) for integer $\nu > 1$. In the case of the IHCP, the kernel is infinitely-smoothing in that no amount of differentiation ever leads to a well-posed equation. We seek a more general theory that applies to problems in some (or all) of these types of problems as well.

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