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Dimension and Rigidity of the Harmonic Measure on Julia Sets

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Irina Popovici

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Major professor
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Rigidity and Dimension of the Harmonic Measure on Julia Sets

By

Irina Popovici

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ABSTRACT

Rigidity and Dimension of the Harmonic Measure on Julia Sets

By

Irina Popovici

The harmonic measure in a dynamical context appeared for the first time in a paper of Brolin, where it was established that the harmonic measure ω associated with the unbounded component of the complement of Julia set for a polynomial is equal to the measure of maximal entropy.

The comparisson of these measures turns out to be very helpful in understanding the generalized polynomial like systems (GPL). Douady and Hubbard have proved that such GPL are quasi-conformally conjugated to polynomials. The first part of my thesis contains the proof of a necessary and sufficient condition for a polynomial like system to be conformally conjugated to a polynomial and a necessary condition for GPL. It also contains the proof of existence of invariant harmoinc measures for GPL.

The final part of the thesis is related to a problem that goes back to Carleson and to P. Jones, T. Wolf and N. Makarov, of comparing the Hausdorff dimension of the harmonic measure on a compact K and the Hausdorff dimension of the set K

itself. It has been conjectured (A. Volberg) that for all disconnected Julia sets J the harmonic measure has dimension smaller than J.

The second chapter of the thesis contains the proof of a Boundary Harnack Principle for Denjoy domains whose boundaries are uniformly perfect.

This result is used in the final chapter where it is proved that A. Volberg's conjecture is true for the Julia sets of Blaschke products with one parabolic point.

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Introduction

Among the dynamical systems, polynomials and rational systems are the symplest. Starting with the works of L. Böttcher, P. Fatou and G. Julia, they were studied throughout this century, and very intensively recently. The Julia set J_f of a rational map f consists of points that behave chaotically under iteration:

$$J_f = \bar{C} \setminus \{z : \exists U \ni z \text{ such that } F^n|_U \text{ is normal family } \}.$$

There are various ways to investigate the structure and properties of these complicated sets in the plane. One of the possible ways to study the dynamically relevant measures supported on the set. We will be interested in three measures: harmonic measure - expressing complex analytical properties, the Hausdorff measure - reflecting geometrical content, and the measure of maximal entropy - giving information about the dynamics.

Let $\Omega = \bar{C} \setminus K$ be an open set on the Riemann sphere $\bar{C}$ and let $\omega_\Omega(E, z)$ be the harmonic measure of $E \subset K$ with respect to Ω, evaluated at $z \in \Omega$. During last several years there was a considerable interest in the metric properties of such sets. In particular, the estimations (and even calculations) of

$$\text{Hdim}(\omega) \stackrel{def}{=} \inf\{\text{Hdim}(E) : E \text{ is a Borel support of harmonic measure } \omega\}$$

have been done. Here the symbol Hdim stands for the Hausdorff dimension. As a result of this attention and especially due to works of Carleson [Ca1]-[Ca3], Makarov [Ma1], Jones, Wolff [JW1], [Jw2], Wolff [W], and Bourgain [B] the structure of the harmonic measure of general plane sets become much more comprehensible. The deep analogy between the behavior of sums of (almost) independent random variables and the behavior of the Green function of the domain plays a crucial part in this subject ([Ma2]). This analogy becomes still more conspicuous if a domain for which the harmonic measure is investigated has regular self-similar structure. As Carleson showed in [Ca3] the methods of ergodic theory turn out to be relevant in this case.

It is necessary to point out that harmonic measure in dynamical context appeared for the first time in Brolin's paper [Br], where it was established that backward orbits of a polynomial f are equidistributed (or balanced) with respect to harmonic measure ω of the unbounded component of the Julia set $J(f)$.

A measure μ satisfying

$$(\text{degree } f) \cdot \mu(A) = \mu(f(A)) \quad \text{if } f \text{ is injective on } A$$

is called *balanced* measure. The uniqueness of the balanced measure was established later by A. Freire, A. Lopez and R. Mañe in [FLM], [Mañ].

Another way to view the balanced measure is to notice that as n tends to infinity, preimages $f^{-n}(z)$ have uniform distribution with respect to it, hence one can construct the balanced measure as a weak limit of the sums

$$w - \lim_{n \to \infty} \frac{1}{d^n} \sum_{y \in f^{-n} z} \delta_y .$$

Here δ_y denotes the Dirac measure supported at y. This process can be viewed as considering the operator $L^\star$ acting on measures:

$$d(L^\star \mu)(y) = \frac{1}{\deg f} d\nu(f(y))$$

and analyzing its iterates. This is M. Lyubich's construction of balanced measures for rational functions ([Ly]). He also showed that there exists a unique invariant measure of maximal entropy; it coincides with the balanced measure and has entropy equal to $\log \deg f$, which is the topological entropy of this dynamical system. Moreover, the balanced measure has nice ergodic properties: it is Gibbs for totally disconnected Julia sets and for Cantor repellers, it is mixing for polynomials. Later Brolin's result was interpreted as the coincidence of ω and the unique measure of maximal entropy for polynomials.

When we have a dynamical system other than polynomial, the natural question of comparison of these two measures arises. For rational f it was considered by Lopes in [Lo], where it was proved that if $\infty \in \overline{C} \setminus J(f)$ is a fixed point of f, then it follows from $m = \omega$ that f is a polynomial. We will consider the local setting of the problem when f is defined only on a neighborhood of an invariant compact set J_f. The question is to characterize the situation when $\omega \approx m$, where "$\approx$" denotes mutual absolute continuity. It certainly happens when f is conformally equivalent to a polynomial. The first chapter of the thesis covers the converse problem in the case when f is a *generalized polynomial-like map* (GPL).

This is also a question of *rigidity*: the absolute continuity of the two measures implies strong information on dynamics.

This problem has been investigated by Lyubich and Volberg in [LV], under the assumption of the hyperbolicity condition. If (f, U, V) is polynomial like without critical points, $\omega \approx m$ implies that (f, U, V) is conformally conjugated to a polynomial. Recall that without the assumption $\omega \approx m$ the conjugation with a polynomial can be done, but in the class of quasiconformal mappings. The proof in [LyV] relied on the hyperbolicity assumption to derive a Boundary Harnack Principle (BHP) for the Cantor reppeler, to construct invariant harmonic measure ν and then to manipulate the homology equation:

$$\log d - \log J_\nu = u \circ f - u$$

where J_ν denotes the Jacobian of ν. Without a BHP, the proofs in Chapter 1 involve the ergodic properties of the two measures.

The second part of the thesis investigates the relationship between the harmonic measure and the Hausdorff measure on the Julia set of a particular class of maps. It is shown that the Julia set of a rational f can be a really complicated object. For polynomials, Manning's formula [Man] and Brolin's result give the following estimate of the Hausdorff dimension of harmonic measure on $J(f)$:

$$\mathrm{Hdim}(\omega) = \frac{\log d}{\int_{J(f)} \log |f'| d\omega} \leq 1$$

In fact $\int_{J(f)} \log |f'| d\omega = \log d + \sum_i G(c_i)$, where G is the Green function of $\bar{C} \setminus J(f)$ and c_i are critical points of f escaping to infinity (that is lying in an unbounded component of the complement of $J(f)$). This solves the conjecture of Oksendal [O] for compacts which are Julia sets of polynomials. The general conjecture was solved in [JW1], where the estimate

$$Hdim(\omega) \leq 1$$

has been proved for *any* compact set K. Note that for $J(f)$ with polynomial f such that there exists at least one escaping critical point one can see exactly as above that

$$Hdim(\omega) < 1$$

The existence of an escaping critical point means precisely that $J(f)$ is disconnected. Certainly the situations with connected sets are covered by the famous result of Makarov which deals with arbitrary continuum K:

$$\text{Hdim}(\omega) = 1.$$

Coming back to the case when K is a limit set of a holomorphic dynamical system f one can suggest two conjectures:

$$\text{Hdim}(\omega) < 1 \tag{0.0.1}$$

for the harmonic measure on $J(f)$ unless $J(f)$ is connected. This is not true for rational functions. However Zdunik proved this conjecture for the so-called generalized polynomial-like dynamics (GPL) f. See [LV] for the definition of GPL.

Looking at (0.0.1) and having in mind Makarov's result or/and Jones and Wolff solution of Oksendal's conjecture one may conclude that the harmonic measure always find some "thin" set of exposed points to concentrate on. This makes plausible the second conjecture that

$$Hdim(\omega) < Hdim J(f) \tag{0.0.2}$$

for the harmonic measure on $J(f)$ unless $J(f)$ is connected. We certainly cannot expect this to happen for an arbitrary compact set K. This is clear from the example of Ch. Bishop [Bi] : for any $\delta < 1$ Bishop constructed a set K such that $Hdim(\omega) = Hdim(K) = \delta$.

However there are many indications that for $K = J(f)$ the conjecture is correct. First of all Zdunik proved that the Hausdorff dimension of the maximal measure is strictly less than the Hausdorff dimension of $J(f)$ unless $J(f)$ is connected for polynomials f. Now Brolin's result shows that (0.0.2) is true for polynomial dynamics

f. For various types of GPL f (0.0.2) was shown in [MV], [Vo1], [Vo2]. Finally let us mention an interesting result of [Ba] which is in the same vein.

In the last chapters (0.0.2) is proved when f is a parabolic Blaschke product. It is worthwhile to mention that then the assumption

$$J(f) \text{ is disconnected}$$

has an ergodic theory interpretation. The fact that $J(f)$ is disconnected means exactly that f acts non-ergodically on the unit circle T with respect to Lebesgue measure on the circle. In local terms this means that f has only one petal at the parabolic points. The reader can find the discussion of these relationships in Aaranson's papers [A1] and [A2].

The proof has an analytic part (Chapter 2), dynamical part (Chapters 3, 4) and a part that mixes analysis and dynamics (Chapter 4). I think that the analytic part is interesting in its own right. The essence of it is a Boundary Harnack Principle (BHP) for the Fatou set of f. There is an extensive literature on BHP and the reader may consult [An], [Wu] or [JK]. It has been recognized in [Vo2] and [LVo] that BHP may play an important part in metric estimates of harmonic measure on discontinuous fractals. But in all these works mentioned above, the existence of BHP relied upon the fact that the domains under consideration have good geometric localization. They are NTA, John or Lipschitz domains. The Fatou set of a parabolic Blaschke product with one petal does not have any nice localization. The complementary intervals of the Julia set are too small with respect to the distance to the parabolic point (see Appendix). However, Chapter 2 contains a certain BHP which is one of the key points in proving (0.0.2). Another key place is Chapter 3 where we use [DU3] extensively. We couple here our BHP and the technique of [DU3]. After this we prove that the harmonic measure and the δ-packing measure on $J(f)$, ($\delta \stackrel{def}{=} Hdim(J(f))$) are singular. Here one comes to an amusing contradiction: if they are not singular then f can be linearized simultaneously in a common neighborhood of different repelling periodic points and the parabolic fixed point. So these measures are in fact singular. But this is a much weaker statement then (0.0.2). To finish the proof we need a third key consideration which amounts to thermodynamical formalism for certain countable state systems with potentials that can be unbounded. Our potentials are of very special kind and this enables us to adapt certain results of [Bo] to our case. This is done in Chapter 3. After that (0.0.2) follows easily.

We are in the position to state (Chapters' 2, 4) main results. Let D denote the unit disc. The holomorphic coverings $D \longrightarrow D$ of finite degree d are called Blaschke products of degree d. By simple conjugacy we may consider them as coverings $C_+ \longrightarrow C_+$ that fix ∞. We will freely use this two representations. We consider only Blaschke products with parabolic fixed point. As a function with positive imaginary part in C_+, a Blaschke product of degree d which fixes infinity can be written as

$$f(z) = z + c_0 - \sum_{k=1}^{d-1} \frac{c_k}{z - x_k}$$

where $c_0 \in R$, $c_k > 0$ for $k = 1, \dots d-1$. Then f has a petal or two at the parabolic point $p = \infty$ depending on whether $c_0 \neq 0$ or $c_0 = 0$. The former case happens if and only if $J(f) \overset{\subset}{\neq} R$.

The Main Result *Let f be a Blaschke product such that its Julia set $J(f)$ has a parabolic point with just one petal (then the Julia set $J(f)$ is a disconnected subset of the unit circle). Let ω be the harmonic measure in $\bar{C} \setminus J(f)$. The following holds:*

$$Hdim(\omega) < Hdim(J(f)) \quad \square$$

The main analytic tools used to construct and compare the harmonic and the δ-conformal measures are the following two results:

.

Theorem (Boundary Harnack Principle) *If* u *,*v *are positive harmonic functions in* Ω *, vanishing continuously on some uniformly perfect set* K *contained in* R*, and satisfying*

$$u(z) = u(\overline{z}) \text{ and } v(z) = v(\overline{z})$$

then the function $\log \frac{u}{v}$ *is Hölder continuous of order* α *on* $\Omega \bigcup K$*.* □

.

Lemma (On harmonic rigidity) *Let* u, v *be two non-negative subharmonic functions in a disc* B *with diameter* I*. Let* $J \subset I$ *be a closed, uniformly perfect set with infinitly many components.*

If u, v *vanish on* J *and are positive and harmonic in* $B \setminus J$ *, and if*

$$\frac{u(x)}{v(x)} = \lim_{z \to x} \frac{u(z)}{v(z)} = |A(x)|^2 \ , \ \forall x \in J \tag{0.0.3}$$

for some holomorphic function A *in the ball* B *, then* $|A| \equiv$ *constant.* □

CHAPTER 1

Generalized Polynomial Like Systems

If f is a rational function, it was proved by Lopes in [Lo] that if $\infty \in C \setminus J(f)$ is a fixed point of f, then it follows from $m = \omega$ that f is a polynomial. We will consider the local setting of the problem when f is defined only on a neighborhood of an invariant compact set J_f. The question is to characterize the situation when $\omega \approx m$, where "$\approx$" denotes mutual absolute continuity. It certainly happens when f is conformally equivalent to a polynomial. This chapter covers the converse problem in the case when f is a *generalized polynomial-like map* (GPL). As the first section shows, this problem is related to the Straightening Theorem of Douady and Hubbard [DH].

1.1 Necessary and Sufficient Conditions for Conformal Conjugation

Definition: *Let $U, U_1, U_2, \dots U_k$ be $k+1$ topological discs with real analytic boundaries such that $\bar{U}_i \subset U$, $i = 1, 2, \dots k$; $\bar{U}_i \bigcap \bar{U}_j = \emptyset, i \neq j$. A map $f : \bigcup_{i=1}^{k} U_i \to U$ which is a branched covering of degree $d_i < \infty$ on each U_i is called a generalized* *polynomial like system.*

Then $K_f = \bigcap_{n\geq 0} U^n$, *where* $U^n = f^{-n}(U)$. *We call* $J_f = \partial K_f$ *the Julia set of* f. *It is also the boundary of* $A_\infty(f) = \bar{C} \setminus K_f$. *The degree of* f *is* $d = d_1 + d_2 + \ldots + d_k$. *If* $k = 1, d = d_1 \geq 2$. *we say that* f *is polynomial-like (in the sense of Douady and Hubbard [DH]) .*

Saying that two maps f, g are (conformally) conjugate means that there is a (conformal) conjugation in some neighborhoods of the Julia sets.

Theorem 1.1 (Straightening Theorem) *Every polynomial like system* (f, U, V) *is quasyconformally conjugated to a polynomial of the same degree as* f. *Moreover, if* K_f *is connected then the conjugating map is unique up to an afine transformation.*

Julia sets of polynomials are uniformly perfect, a property that is preserved by quasiconformal maps, so the Julia sets of GPL are also uniformly perfect, in particular regular for Dirichlet's problem.

If (f, U, V) and $(g, \tilde{U}, \tilde{V})$ are two polynomial like systems with connected Julia sets, we say that they satisfy the (BiHolo) condition if

$$\exists \phi : U_1 \setminus K_f \to \tilde{U}_1 \setminus K_g \text{ biholomorphic, such that } \phi \circ f = g \circ \phi$$

on the neighborhood U_1 of K_f In [DH] the *external map* of a polynomial was constructed. For a polynomial like (f, U, V) with a connected Julia set, the external map h_f can be obtained as follows: let α map conformally $V \setminus K_f$ onto some standard annulus $\{z, 1 < |z| < R\}$ such that ∂K_f is mapped to the unit circle. Let $W_+ = \alpha(U \setminus K_f)$, let W_- be the image of W_+ under the reflection $z \to 1/\bar{z}$ and let $h_+ = \alpha \circ f \circ \alpha^{-1} : W_+ \to \{z, 1 < |z| < R\}$. By Schwartz' relection principle, h_+ extends analytically to $W_+ \bigcup W_-$. The restriction $h_{|_{S^1}}$ is an expanding real analytic map. We will denote it by h_f. This construction can be generalized to any polynomial like system (see [DH]).

Definition *Two polynomial like systems* $(f, U, V), (g, \tilde{U}, \tilde{V})$ *are externally conjugated if the following conditon, refered to as (ExtMap) is true:*

$$\exists \phi : S^1 \to S^1 \text{ real analytic, such that } \phi \circ h_f = h_g \circ \phi$$

If (f, U, V) and $(g, \tilde{U}, \tilde{V})$ are two polynomial like systems with connected Julia sets then the conditions (BiHolo) and (ExtMap) are equivalent.

Theorem 1.2 *Let* (f, U, V) *be polynomial like of degree d. Then f is holomorphically conjugated to a ploynomial if and only if f is externally equivalent to* $z \to z^d$.

We are going to use the following criterion for conjugation (compare to Shab and Sullivan [SS]): Let $h : S^1 \to S^1$ be analytic and expanding. If the measure of maximal entropy of h is nonsingular with respect to the Lebesque measure on S^1 the h is analytically conjugated to $z \to z^d$.

Since Lebesque measure on S^1 is sent by α^{-1} to the class of the harmonic measure on J_f, the problem of conformal conjugation to a polynomial is reduced to comparing the maximal measure m_f and the harmonic measure ω_f on J_f .

Theorem 1.3 *Let* (f, U, V) *be a polynomial like system of degree d. Then the following are equivalent:*

1. $\exists H$ *a conformal isomorphism in a neighborhood of* K_f *and a polynomial P of degree d such that* $f = H^{-1} \circ P \circ H$.

2. $\exists \phi$ *a conformal isomorphism in a neighborhood of* K_f *and a polynomial system* $(g, \tilde{U}, \tilde{V})$ *of degree d such that* $f = \phi^{-1} \circ g \circ \phi$ *and* $\omega_g = m_g$.

A GPL system (g, U, V) satisfying $\omega_g = m_g$ will be called maximal. Recall that by [Br], all polynomials P are maximal; this can also be derived from the following property of Green's function in $C \setminus K_P$ with pole at infinity: $G(P(z)) = G(z)$. The following result was proved in [LV] (also see [BPV]):

Theorem 1.4 *Let (f, U, V) be GPL. then the following are equivalent:*

1. *$\exists \phi$ a conformal isomorphism in a neighborhood of K_f and a GPL system $(g, \tilde{U}, \tilde{V})$ of degree d such that $f = \phi^{-1} \circ g \circ \phi$ and $\omega_g = m_g$.*

2. *there exists a function τ satisfying :*
   1) *τ is subharmonic in U, $\tau \geq 0$*
   2) *τ vanishes on K_f; $\tau > 0$ on $U \setminus K_f$*
   3) *τ is harmonic in $U \setminus K_f$*
   4) *$\tau(fz) = d\tau(z)$.*

Such a function τ will be called an automorphic function.

Proof : The complete proof of this theorem can be found in [BPV]. Since some of the arguments in the implication $1 \rightarrow 2$ help in understanding the construction of the automorphic function (next section) I decided to include it.

Assume that the system (g, U, V) is maximal; we need to construct an automorphic function for this system (clearly the existence of automorphic function is conformally invariant). Let $\{x_i(\xi)\}_{i=1}^d$ denote all g-preimages of $\xi \in J_g$ counting with multiplicity. Let $u \in C(J_g)$ and $g : E \rightarrow g(E)$ be injective. Then the fact that the Jacobian J_m equals d implies

$$d^{-1} \int_{g(E)} [u \circ (g \mid E)^{-1}] dm = \int_E u\, dm$$

Since the maximal measure has no atoms, m-almost all J_g can be covered by d disjoint sets $E_1 \ldots E_d$ such that, on each E_i, g is univalent. Thus

$$\int_{J_g} \sum_{i=1}^{d} u(x_i(\xi)) dm(\xi) = d \int_{J_g} u\, dm$$

The maximality assumption serves to claim that

$$\int_{J_g} \sum_{i=1}^{d} u(x_i(\xi))d\omega(\xi) = d \int_{J_g} u\, d\omega \tag{1.1.1}$$

Define $\Phi(z) = \int_{J_g} \frac{d\omega(\xi)}{\xi - z}$, and $F(z) = \int \frac{g'(z)d\omega(\xi)}{\xi - g(z)} - d\int_{J_g} \frac{d\omega(\xi)}{\xi - z} \in \mathrm{Hol}(U \setminus J_g)$. Let us prove that $F \in \mathrm{Hol}(U_g)$. To do this choose a contour C in $V \setminus U$. Then for every $n \geq 0$

$$\begin{aligned} \int_C z^n F(z)dz &= \int_{J_g} d\omega(\xi) \int_C \frac{g'(z)z^n dz}{\xi - g(z)} - d \int_{J_g} d\omega(\xi) \int_C \frac{z^n dz}{\xi - z} \\ &= \int_{J_g} \sum_{i=1}^{d} (x_i(\xi))^n d\omega(\xi) - d \int_{J_g} \xi^n d\omega(\xi) = 0 \end{aligned}$$

according to (1.1.1). This proves that the singularities of F are removable. In other words

$$\Phi(g(z))g'(z) - d\Phi(z) = A(z) \in \mathrm{Hol}(U).$$

As Green's function G satisfies $G(z) = \int \log|z - \xi| d\omega(\xi) + \text{const}$, we rewrite this line using the notation G' for $\frac{\partial}{\partial z} G = \Phi$, and H' for $\frac{\partial}{\partial z} H$.

$$G'(g(z))g'(z) - dG'(z) = H'(z),$$

for a certain real harmonic function H in U. As G and H are real valued we also get $\nabla(G \circ g) - d \cdot \nabla G = \nabla H$ and so

$$G \circ g - d \cdot G = H + \text{const} \stackrel{\text{def}}{=} H_0 \tag{1.1.2}$$

Two cases may occur: a) $H_0 \equiv 0$ in U, b) $\{z \in U : H_0(z) = 0\}$ is locally a finite union of real analytic curves. If the first case occurs we got G as our harmonic automorphic function. So let us consider b).

Let N be a neighborhood of J_g, $\bar{N} \subset U$, put $\Gamma = \{z \in N : H_0(z) = 0\}$. Then

$\Gamma = \bigcup_{i=1}^{n} \Gamma_i$, where each Γ_i is a real analytic arc. Clearly Γ covers J_g (as H_0 restricted to J_g equals $(G \circ g - G)$, so it is zero). As J_g has no isolated points we can throw away those Γ_ℓ for which $\#(J_g \cap \Gamma_\ell) < \infty$. After this operation the rest of Γ_k will cover J_g. So let $J_g \subset \Gamma_0 = \bigcup_{i=1}^{m} \Gamma_i$ and $\#(J_g \cap \Gamma_i) = \infty$, $i = 1, \ldots, m$. Now it is clear that $g^{-1}(\Gamma_0) \subset \Gamma_0$.

We call a cross-point any point of J_g which is an intersection of two different arcs Γ_i, $i = 1, \ldots, m$. If p_0 is a cross-point then the set $g^{-n}(p_0)$ consists of cross-points (as $g^{-1}\Gamma_0 \subset \Gamma_0$). But the number of cross-points is obviously finite. So there is no cross points at all.

Let $\mathcal{O}_0$ be a thin neighborhood of Γ_0 in which a holomorphic symmetry $z \to z^*$ with respect to Γ_0 is defined. In $\mathcal{O}_1 = g^{-1}\mathcal{O}_0$ we then get

$$gz^* = (gz)^*. \tag{1.1.3}$$

Let us put $\hat{G}(z) = G(z) + G(z^*)$, $\hat{H}_0(z) = H_0(z) + H_0(z^*)$. Then (1.1.2), (1.1.3) give us

$$(\hat{G} \circ g - d \cdot \hat{G})(z) = \hat{H}_0(z), \quad z \in \mathcal{O}_1.$$

By definition $\hat{H}_0 \equiv 0$ on Γ_0. But also this function is symmetric with respect to Γ_0 and so $\frac{\partial \hat{H}_0}{\partial n} \equiv 0$ on Γ_0. Thus $\hat{H}_0 \equiv 0$ on $\mathcal{O}_1$ and we have a neighborhood $\mathcal{O}_1$ of J_g in which

$$\hat{G} \circ g = d \cdot \hat{G}. \tag{1.1.4}$$

Then a standard extension "by means of equation" gives us $\hat{G}$ on the whole U with the same automorphic property (1.1.4) and the implication $1 \to 2$ is proved.

1.2 Construction of Invariant Harmonic Measure and of the Automorphic Function

In order to construct the automorphic harmonic function we need an invariant version of the harmonic measure.

Theorem 1.5 *Let (f, U, V) be a GPL. Then there exists a finite measure ν on J_f such that ν is f-invariant and $\nu \approx \omega_f$*

Proof: We will use the following result of Y. N. Dowker and A. Calderon which can be found in [Fo] (reformulated in a convenient form):

Theorem 1.6 *Let μ be a probability measure on a compact set X. Let $T : X \to X$ be a continuous endomorphism such that μ is completely non-singular with respect to T. Then there exists a T-invariant probability measure λ absolutely continuous with respect to μ if and only if the following holds*

$$\mu(E) < 1 \Rightarrow \sup_n \mu(f^{-n}E) < 1$$

If μ is ergodic then λ is ergodic.

Proof: Fix an arbirary Borel set $E \subset J$, $\omega(E) = 1 - \epsilon < 1$. Let Γ be a smooth curve encircling J and separating it from ∂U and let $\Gamma^n = f^{-n}(\Gamma)$. The main things now are six notations.

Let ω, v denote the harmonic measures of E with respect to $A_\infty(f)$, $U \setminus K_f$ respectively. Let V, W denote the harmonic measures of $J \setminus E$ with respect to $A_\infty, U \setminus K_f$ respectively. For any function ϕ let ϕ^n denote $\phi \circ f^n$ where defined. As usual $\omega_\Omega(S, z)$ denotes the harmonic measure of S evaluated at z with respect to Ω.

First we need a simple lemma. Fix a compact set K in Ω and consider two harmonic measures on $\partial\Omega$ - with respect to Ω and with respect to $\Omega \setminus K$ evaluated at the same point $a \in \Omega \setminus K$.

Lemma 1.7 *The two harmonic measures on $\partial\Omega$ are boundedly equivalent.*

Proof: We present the proof in the case when all points of $\partial\Omega$ are regular. Only this case is used in what follows. Let O be a neighborhood of $\partial\Omega$ with smooth boundary and satisfying $O \bigcap (K \bigcup \{a\}) = \emptyset$. Let G, g be Green's functions of $\Omega, \Omega \setminus K$ respectively, with pole at a. Then by Harnack's principle $cG \leq g \leq G$ on the boundary of O. By maximum principle this inequality extends to O (both functions vanish on $\partial\Omega$). Then clearly the functions $g - cG$ and $G - g$ are subharmonic in O and so their Riesz measures are nonnegative. The Riesz measures of G and g being equal to our harmonic measures, we are done.

Coming back to the proof of Theorem 1.5: we must show that

$$\omega_{A_\infty}(E, \infty) \leq 1 - \epsilon \Rightarrow \omega_{A_\infty}(f^{-n}E, \infty) \leq 1 - \delta.$$

As $\omega(\infty) \leq 1 - \epsilon$, we get $W(\infty) \geq \epsilon$.

Then $W(\xi) \geq c\epsilon$ on Γ and by previous Lemma, $V(\xi) \geq \delta$ on Γ. Then $V^n(\xi) \geq \delta$ on Γ_n. But these functions vanish on ∂U^n and so $\omega_{A_\infty}(f^{-n}(J \setminus E), \xi) \geq \delta$ on Γ_n. So $\omega_{A_\infty}(f^{-n}(E), \xi) \leq 1 - \delta$ on Γ_n. As Γ_n separates J from ∞ we obtain $\omega(f^{-n}E) = \omega_{A_\infty}(f^{-n}(E), \infty) \leq 1 - \delta$.

We are going to prove that the just constructed *invariant harmonic measure* ν is boundedly equivalent to ω.

Theorem 1.8 *There exist constants $0 < c_1, c_2 < \infty$ such that*

$$c_1 \leq \frac{d\nu}{d\omega} \leq c_2$$

Proof: Let us prove first the right inequality. We wish to repeat the above considerations but it seems hard to get rid of the influence of ∂U. However we are going to prove that $\omega(E) \leq \epsilon \Rightarrow \omega(f^{-n}E) \leq c\epsilon$ for a certain finite c. The construction of invariant measure in [Fo] then gives the right inequality. Let us fix Γ as above and such that

$$\max_{\xi \in \Gamma} \omega_{U \setminus K_f}(\partial U, \xi) \leq \frac{1}{2}.$$

Then if Ω_n is any component of U^n and $\gamma_n = \Gamma^n \bigcap \Omega_n$, we can write $\omega_{U\setminus K_f}(\partial U, f^n(\xi)) = \omega_{\Omega_n \setminus K_f}(\partial \Omega_n, \xi)$ and thus

$$\omega_{\Omega_n \setminus K_f}(\partial \Omega_n, \xi) \leq \frac{1}{2}, \quad \xi \in \gamma_n \tag{1.2.1}$$

We start with the chain of implications:

$$\omega(\infty) \leq \epsilon \Rightarrow \omega(\xi) \leq C_\Gamma \epsilon \quad \text{for } \xi \in \Gamma \Rightarrow$$

$$\Rightarrow v(\xi) \leq C_\Gamma \epsilon \ \text{ for } \xi \in \Gamma \Rightarrow v^n(\xi) \leq C_\Gamma \epsilon \ \text{ for } \xi \in \gamma_n.$$

Let us compare $u_1(\xi) = v^n(\xi)$, with $u_2(\xi) = \omega_{A_\infty}(f^{-n}E, \xi)$ on γ_n for each component Ω_n of U^n. By Poisson formula in $\Omega_n \setminus K_f$ we have:

$$u_2(\xi) - \int_{\partial \Omega_n} u_2(\eta) d\omega_{\Omega_n \setminus K_f}(\eta, \xi) = u_1(\xi)$$

Let $u_2(\xi_0) = \max_{\gamma_n} \max_{\xi \in \gamma_n} u_2(\xi) = \max_{\xi \in \Gamma^n} u_2(\xi)$.

Then By (1.2.1) we have $u_2(\xi_0)(1 - \frac{1}{2}) \leq u_1(\xi_0) \leq C_\Gamma \epsilon$. But Γ^n separates J_f from ∞ and so

$$\omega_{A_\infty}(f^{-n}E, \infty) = u_2(\infty) \leq 2C_\Gamma \epsilon$$

The left inequality can be proved exactly the same way.

Theorem 1.9 *Let (f, U, V) be a GPL. Then the following assertions are equivalent:*

1. *f is conformally maximal*

2. *$\omega_f \approx m_f$.*

Proof: We will be using the following notations: if u is a subharmonic function then $\mu_u = \Delta u$ is its Riesz measure. Let G be Green's function of $A_\infty(f)$ with pole at ∞. We know that $\mu_G = \omega$. Let $\Phi = \frac{d\mu_{G\circ f}}{d\mu_G}$, $\phi = \log \Phi$. Clearly Φ is bounded away from zero and infinity (this is just Harnack's inequality essentially). It will be important for us that Φ is the Jacobian of ω with respect to f. Let $\rho = \frac{d\nu}{d\omega}$, $\gamma = \log \rho$. The measures ν, m are finite, invariant and ergodic. So $\nu \approx m$ implies $\nu = m$. We start with the homology equation:

$$\phi - \log d = \gamma \circ f - \gamma, \quad \omega - a.e \text{ on } J \tag{1.2.2}$$

This is obvious from the computation of the Jacobians of the measures $\nu = m$.

To prove our result it is sufficient (and necessary) to construct the automorphic harmonic function τ.

The first step is to find a disc $B = B(x, r)$ centered at the Julia set and to construct a nonnegative subharmonic function u in B such that

$$\frac{d\omega}{d\mu_u} = e^{\gamma} \quad \text{on } B \tag{1.2.3}$$

The function τ will be an extension of u if the Julia set J_f does not lie on an analytic curve. Otherwise take τ to be an extension of the average between u and the symmetrization of u over the analytic curve.

Let F be be a set with $\omega(F) > 0$ on which γ is continuous. Let $F_0 \subset F, \omega(F_0) > 0$

be such that

$$\lim_{\epsilon\to 0}\frac{\omega(B(y,\epsilon)\bigcap F)}{\omega(B(y,\epsilon))}=1 \tag{1.2.4}$$

Let $(\tilde{J},\tilde{f},\tilde{m})$ be the natural extension of (J,f,m) to the space of inverse orbits of f, with $\tilde{f}$ being the left shift. We denote by $\pi:\tilde{J}\to J$ the projection onto the "0" coordinate. Then $\tilde{m}(\pi^{-1}(F_0))>0$ and by the ergodicity of $\tilde{m}$ one can choose $\tilde{x}$ such that $\tilde{f}^{-n}(\tilde{x})\in\pi^{-1}(F_0)$ with positive frequency. In particular we have chosen $x\in J_f$ and a sequence of compatible inverse images x_n of x such that $x_n\in F_0$ with positive frequency. But one can do more (see [FLM], [Z1]): we can choose $B=B(x,r)$ such that on $3B$ there are univalent compatible inverse branches F_n such that

$$\text{diam}F_n(2B)\le e^{-n\delta}$$

$$x_n=F_n(x)\ \text{ meets } F_0 \text{ with positive frequency} \tag{1.2.5}$$

Let us consider the family $u_n=d^nG\circ F_n$ in $3B$. Then by (1.2.2)

$$\frac{d\omega}{d\mu_{u_n}}(y)=e^{\gamma(y)-\gamma(F_n(y))},\ y\in J_f\bigcap 3B.$$

In particular $\|\mu_{u_n}\|\le C<\infty$ and moreover $c_1\omega\le\mu_{u_n}\le c_2\omega$ (see Theorem 4 which gives the boundedness of γ).

Let $\{n_k\}$ be a subsequence such that $\gamma(x_{n_k})\to c,\ k\to\infty,\ x_{n_k}\in F_0$. Without loss of generality we can think that

$$|\gamma(x_{n_k})-c|\le 2^{-k},$$

$$\frac{\omega(F_{n_k}(2B)\setminus F)}{\omega(F_{n_k}(2B))}\le 2^{-k}$$

The last assertion follows from (1.2.4), (1.2.5). Let $E_k = F_{n_k}^{-1}(F \bigcap F_{n_k}(2B))$. Then

$$|\frac{d\omega}{d\mu_{n_k}} - e^{\gamma - c}| \leq C_1 2^{-k} \text{ on } E_k \tag{1.2.6}$$

$$\mu_{n_k}(2B \setminus E_k) \leq C_2 2^{-k} \tag{1.2.7}$$

Let K be a relatively closed subset of the disc B. We denote by $S_+(B, K)$ the set of bounded subharmonic functions in B vanishing on K and positive and harmonic in $B \setminus K$. We use two results from potential theory:

Lemma 1.10 *Let $\{v_j\}$ be a sequence of uniformly bounded functions from $S_+(B, K)$. Let K be regular for the Dirichlet problem in $C \setminus K$. Then there exists a subsequence which converges pointwisely to a function from $S_+(B, K)$.*

Lemma 1.11 *Let u belong to $S_+(2B, K)$, for some ball $2B$ of diameter less than 1 and having $cap(B \bigcap K) > 0$. Then*

$$\sup_B u \leq C_{B,K} \parallel \mu_u \parallel .$$

So as $\parallel \mu_{u_{n_k}} \parallel \leq C$ we conclude that u_{n_k} are uniformly bounded. We may think that the subsequence in Lemma 1.10 is $\{u_{n_k}\}$ itself; put $u_0 = \lim_{k \to \infty} u_{n_k}$ in B. The convergence is pointwise bounded and so $\mu_{u_{n_k}} \to \mu_{u_0}$ weakly. But (1.2.6) and (1.2.7) show that $\mu_{u_{n_k}} \to e^{-\gamma + c} d\omega$ weakly. Thus $d\mu_{u_0} = e^{-\gamma + c} d\omega$ and $u_0 e^c$ satisfies (1.2.3).

Now the construction of τ follows word by word the construction in [LV] and [BV]. For the sake of completeness, here is a sketch: Consider B_θ a component of $f^{-n}B$ and define τ on it as follows:

$$\tau(z) \stackrel{def}{=} \frac{1}{d^n} u(f^n z)$$

In [LV] or [BV] it is shown in detail that τ (or its symmetrization over J_f, if J_f lies on an analytic curve) does not depend on θ or n, that is if $B_{\theta_1} \bigcap B_{\theta_2} \neq \emptyset$ then τ (or its symmetrization) is the same on this intersection. This follows quite easily from the homology equation. Now τ is defined on the set $O = \bigcup f^{-n} B$.

To define τ on U use the homology equation to push forward the extension: if $\tau(x)$ has been defined, $\tau(f^n(x)) \stackrel{def}{=} d^n \tau(x)$. Since the backward orbit of any point in U is dense, eventually one of its preimages lands in B, where τ has been defined. This extension is well defined in U, harmonic and satisfies the homology equation on an open subset of each component of U, therefore in the whole U.

This concludes our construction and proves the theorem.

CHAPTER 2

Boundary Harnack Principle

2.1 The Main Lemma.

Definition *A compact set K is uniformly perfect if*

$$cap(K_f \bigcap B(x,r)) \geq c \; cap(B(x,r)) \; \forall x \in K_f \text{ and } \forall r \leq r_0$$

Through this section we are going to work with domains Ω of the form $\Omega \overset{def}{=} B(0,R) \setminus K$ for some uniformly perfect compact set $K \subset R$ and for some ball $B(0,R)$ with $\text{dist}(K, \partial B(0,R)) \geq \text{diam}K$.
We will denote by $\Omega_+ \overset{def}{=} \Omega \bigcap \{z : \Im z \geq 0\}$ and by $\Omega_- \overset{def}{=} \Omega \bigcap \{z : \Im z \leq 0\}$. Given a point $\xi \in K$ we will denote by $\xi^{r+} \overset{def}{=} \xi + ir/2$ and by $\xi^{r-} \overset{def}{=} \xi - ir/2$.

We are going to deduce a couple of Harnack-type results in Ω_+. Some of them are inspired from [JK] . Most of the constants M that appear depend only on the constant c from the definition of uniformly perfect sets ; when this is the case we will write $M = M(\text{unif.perf})$

Proposition 2.1 *There exists a universal constant M_1 and some positive ϵ_0 ,* *such that for any* $\epsilon \leq \epsilon_0$ *and any two points* $z_1, z_2 \in \Omega_+$ *satisfying* $dist(z_1, z_2) < 2^k\epsilon$ *and* $dist(z_i, \partial\Omega_+) > \epsilon$ *can be joined by a chain of at most* $M_1 k$ *balls* $B(c_1, r_1), \ldots, B(c_{k'}, r_{k'})$ *contained in* Ω *and having*

$$r_p \geq \frac{1}{M_1} dist(z_i, B(c_p, r_p)) \text{ and } r_p \geq dist(B(c_p, r_p), \partial\Omega) \quad \square$$

Proposition 2.2 *There exists some universal constant* M_2 *such that for any function* u *positive and harmonic in* Ω *and for any* $z_1, z_2 \in \Omega_+$ *satisfying* $dist(z_1, z_2) < 2^k\epsilon$ *and* $dist(z_i, \partial\Omega) > \epsilon$ *, the following holds:*

$$\frac{u(z_1)}{M_2^k} \leq u(z_2) \leq M_2^k\, u(z_1) \quad \square$$

Proof. Use Harnack's principle k times.

Remark For any M_1 there exists M_2 such that if

$$\frac{u(z_1)}{u(z_2)} \geq M_2^k \text{ and } \mathrm{dist}(z_1, z_2) < M_1^k\epsilon \tag{2.1.1}$$

for some $z_1, z_2 \in \Omega_+$ then $\min_{i=1,2} \mathrm{dist}(z_i, \partial\Omega) < \epsilon$

Lemma 2.3 *There exists* $M_3 = M_3(unif.perf)$ *such that for all* $q_0 \in K$ *and for any* u *positive harmonic function in* Ω *such that* u *vanishes continuously on* $B(q, r) \bigcap K$ *the following holds:*

$$\sup\{u(x);\ x \in B(q_0, s/M_3)\} \leq \frac{1}{2} \sup\{u(x)\ ; x \in B(q_0, s)\} \text{ for all } s < r \quad \square$$

Proof: Denote by $a = \sup\{u(x),\ x \in B(q_0, s)\}$. Then

$$u(x) \le a\ \omega_{B(q_0,s)\setminus K}(\partial B(q_0, s), x)$$

By the uniformly perfectness of K there exists some $\gamma = \gamma(\text{unif.perf})$ such that

$$\omega_{B(q_0,s)\setminus K}(\partial B(q_0, s), x) \le \gamma < 1 \text{ for all } x \in B(q_0, s/2).$$

Let $M_3 = 2^n$ for some n large such that $\gamma^n \le \frac{1}{2}$. Let $z \in B(q, s/M_3)$. Then

$$u(z) \le \gamma^n a \le \frac{1}{2} \sup\{u(x),\ x \in B(q_0, s)\} \quad \square$$

Lemma 2.4 *There exists $M_4 = M_4(\text{unif.perf})$ such that if $q_0 \in K$ and if u is a positive harmonic function in Ω that vanishes continuously on $K \bigcap B(q_0, 2r)$ then the following holds:*

$$u(x) \le M_4\ [\ u(q_0^{r+}) + u(q_0^{r-})\ ]\ \text{ for all } x \in B(q_0, r) \quad \square$$

Proof. Let $x \in \Omega_+$. Let M_3 as in Lemma 2.4. Let M_2 satisfy:

$$\frac{u(z_1)}{u(z_2)} > M_2^k \text{ and } \operatorname{dist}(z_1, z_2) < \epsilon \Rightarrow \min \operatorname{dist}(z_i, K) < \frac{\epsilon}{M_3^k}$$

Let N be large such that $2^N > M_2$. Claim that $u(x) \le M_2^{N+4}\ u(q_0^{r+})$ for all $x \in B(q_0, r) \bigcap \Omega_+$.

If not, $\exists x_1 \in B(q_0, r)$ such that $u(x_1) \ge M_2^{N+4}\ u(q_0^{r+})$. By

$$\frac{u(x_1)}{u(q_0^{r+})} \ge M_2^{N+4} \text{ and } \operatorname{dist}(x_1, q_0^{r+}) < 2r$$

we have $\text{dist}(x_1, K) \le \frac{2r}{M_3^{N+4}}$. Let q_1 be a point in K closest to x_1. Then $\text{dist}(q_1, q_0) < r + \frac{4r}{M_3^{N+4}}$. By Lemma 2.4

$$\begin{aligned} \sup\{u(x); x \in B(q_1, r/M_3^3)\} &\ge 2^N \sup\{u(x); x \in B(q_1, r/M_3^{N+3})\} \\ &\ge 2^N u(x_1) \ge M_2 u(x_1) \ge M_2^{N+5}\, u(q_0^{r+}) \qquad (2.1.2) \end{aligned}$$

In this manner we can find $x_2 \in B(q_1, r/M_3^3)$ such that $u(x_2) \ge M_2^{N+5} u(q_0^{r+})$

By (2.1.1), $\text{dist}(x_2, K) \le \frac{4r}{M_3^{N+5}}$. Let q_2 be a point in K closest to x_2. Then

$$\text{dist}(q_2, q_0) \le r + 4r/M_3^{N+4} + r/M_3^3 + 4r/M_3^{N+5} \le 2r$$

and

$$\begin{aligned} \sup\{u(x),\ x \in B(q_2, r/M_3^4\} &\ge 2^N \sup\{u(x),\ x \in B(q_2, r/M_3^{N+4})\} \\ &\ge 2^N u(x_2) \ge M_2^{N+6} u(q_0^{r+}) \qquad (2.1.3) \end{aligned}$$

Finally we can find a sequence of points $x_1, x_2, \dots$ such that $x_n \to K$ with $\text{dist}(x_n, q_0) < 2r$ and $u(x_n) \ge M_2^{N+n} u(q_0^{r+}) \to \infty$. This is impossible because u vanishes continuously on $B(q_0, 2r)$. Therefore the claim is true.

Theorem 2.5 *If u ,v are positive harmonic functions in Ω , vanishing continuously on K, and satisfying*

$$u(z) = u(\overline{z}) \text{ and } v(z) = v(\overline{z})$$

then the function $\log \frac{u}{v}$ can be extended to a Hölder continuous function of order $\alpha = \alpha(unif.perf)$ on $\Omega \bigcup K$. □

Proof. Let $q \in K$ and $r > 0$ be fixed. Denote by $\Gamma_n \stackrel{def}{=} \partial B(q, r/4^n)$ and by

$\Gamma_{n+1/2} \stackrel{def}{=} \partial B(q, r/4^{n+1/2})$. Following [LV] it is enough to show that

$$1 = \min_{\Gamma_n} \frac{u}{v} \leq \max_{\Gamma_n} \frac{u}{v} \leq 1 + \epsilon_n \Rightarrow \frac{\max_{\Gamma_{n+1}} \frac{u}{v}}{\min_{\Gamma_{n+1}} \frac{u}{v}} \leq 1 + \beta\epsilon_n \tag{2.1.4}$$

for some $\beta = \beta(\text{unif.perf}) < 1$.

Decompose Γ_n into

$$\Gamma_n^L = \{1 + \frac{\epsilon_n}{2} < \frac{u}{v} \leq 1 + \epsilon_n\} \tag{2.1.5}$$

and

$$\Gamma_n^S = \{1 \leq \frac{u}{v} \leq 1 + \frac{\epsilon_n}{2}\} \tag{2.1.6}$$

We are going to use the following harmonic measures:

$$\omega(\cdot) = \omega_\Omega(\cdot, \infty)$$

$$\omega_{\Gamma_n}(\cdot, \xi) = \omega_{B(q,r/4^n)\setminus K}(\cdot, \xi) \quad \text{for } \xi \in B(q, r/4^n)$$

$$\omega_{n+\frac{1}{2}}(\cdot, \xi) = \omega_{B(q,r/4^{n+1/2})\setminus K}(\cdot, \xi) \quad \text{for } \xi \in B(q, r/4^{n+1/2})$$

By Poisson's formula we get:

$$u(\xi) = \int_{\Gamma_n^L} u(\eta) d\omega_{\Gamma_n}(\eta, \xi) + \int_{\Gamma_n^S} u(\eta) d\omega_{\Gamma_n}(\eta, \xi)$$

Denote by $v^L(\xi) = \int_{\Gamma_n^L} v(\eta) d\omega_{\Gamma_n}(\eta, \xi)$ and by $v^S(\xi) = \int_{\Gamma_n^S} v(\eta) d\omega_{\Gamma_n}(\eta, \xi)$

Then $v^S + (1 + \epsilon_n/2)\, v^L \leq u(\xi) \leq (1 + \epsilon_n/2)\, v^S + (1 + \epsilon_n)\, v^L$

Let $\xi_+ \in \Gamma_{n+1/2}$ be $\xi_+ = q + ir/4^{n+1/2}$ and $\xi_- \in \Gamma_{n+1/2}$ be $\xi_- = q - ir/4^{n+1/2}$

Suppose that

$$v^S(\xi_+) \geq v^L(\xi_+) \tag{2.1.7}$$

Then the same is true for ξ_-.

Claim: There exists $c = c(\text{unif.perf})$ such that

$$v^S(\xi) \geq c\, v^L(\xi) \text{ for all } \xi \in \Gamma_{n+1}. \tag{2.1.8}$$

Assume the claim is true. Then by (2.1.5) and by $v^L + v^S = v$ we get

$$u(\xi) \leq (1 + \epsilon_n/2)\, v^S(\xi) \;+ (1 + \epsilon_n)\, v^L(\xi) \;\leq (1 + \epsilon_n/2)\, v(\xi) \;+\; \epsilon_n/2\, v^L(\xi)$$

therefore

$$u(\xi) \leq (1 + \epsilon_n/2)\, v(\xi) \;+ \epsilon_n/2\, \frac{1}{1+c}\, v(\xi) \;\leq\; (1 + \beta\epsilon_n)\, v(\xi)$$

and

$$\frac{\max_{\Gamma_{n+1}} \frac{u}{v}}{\min_{\Gamma_{n+1}} \frac{u}{v}} \leq \frac{1 + \beta\epsilon_n}{\min_{\Gamma_n} \frac{u}{v}} = (1 + \beta\epsilon_n)$$

If the opposite of (2.1.7) is true, one can prove that the opposite of the inequality (2.1.8) holds and the minimum of $\frac{u}{v}$ can be estimated from below in order to deduce (2.1.4).

Proof of the claim: Let $\xi \in \Gamma_{n+1}$. For convenience suppose that $q = 0$. Decompose $\Gamma_{n+1/2}$ into four arcs

$$I_+ = (e^{i\pi/20}, e^{i19\pi/20})\frac{r}{4^{n+1/2}},\, J- = (e^{i19\pi/20}, e^{i21\pi/20})\frac{r}{4^{n+1/2}},$$

$$I- = (e^{i21\pi/20}, e^{i39\pi/20})\frac{r}{4^{n+1/2}},\, J_+ = (e^{i39\pi/20}, e^{i\pi/20})\frac{r}{4^{n+1/2}}.$$

The arcs I_+ and I_- are far from the boundary of Ω. By Harnack's principle $v(\eta)$ and $v(\xi_+)$ are comparable for $\eta \in I_+$. Therefore

$$v^S(\xi) \geq \int_{I_+ \bigcup I_-} v^S(\eta) d\omega_{n+\frac{1}{2}}(\eta, \xi) \;\geq c_1\, v^S(\xi_+)\, \omega_{n+\frac{1}{2}}(I_+ \textstyle\bigcup I_-, \xi) \tag{2.1.9}$$

and

$$v^L(\xi) \le c_2\, v^L(\xi_+)\, \omega_{n+\frac{1}{2}}(I_+ \bigcup I_-, \xi) \;+\; \int_{J_+ \bigcup J_-} v^L(\eta) d\omega_{n+\frac{1}{2}}(\eta, \xi) \qquad (2.1.10)$$

The estimate for v^L will be done in two steps:

1. Prove that $v^L(\xi) \le c'\, v^S(\xi_+)$ for $\xi \in J_+ \cup J_-$.

2. Prove that $\omega_{n+\frac{1}{2}}(J_+ \cup J_-, \xi) \le c'\, \omega_{n+\frac{1}{2}}(I_+ \cup I_-, \xi_+)$ for $\xi \in \Gamma_{n+1}$.

Once the two steps are concluded the proof of the claim becomes trivial.

Step 1. Consider the smallest disc that contains J_+. If it doesn't meet K there is nothing to prove. If it contains a point q_0 of K, apply Lemma 2.6 for the function v^L and the disc $B(q_0, 2\mathrm{diam}(J_+))$. We get that

$$v^L(\xi) \le M_4[\, v^L(q_0^{r+}) + v^L(q_0^{r-})] \;\le M_4'[\, v^S(q_0^{r+}) + v^S(q_0^{r-})\,],\;\; \forall \xi \in J_+$$

By the Harnack principle $v^S(q_0^{r+})$ and $v^S(\xi_+)$ are comparable.

Step 2. Let $G_{n+\frac{1}{2}}$ be the Green function in $B(q, r/4^{n+1/2}) \setminus K$. Denote by c_J the middle point of J_+; let ϕ be a smooth function such that

$$\phi = 1 \text{ on } B(c_J, |J_+|) \text{ and } \;\phi = 0 \;\text{ outside } B(c_J, 2|J_+|)$$

We get:

$$\begin{aligned} \omega_{n+\frac{1}{2}}(J_+, \xi) &\le \int_{B(c_J, |J_+|)} \Delta G_{n+\frac{1}{2}}(z, \xi) dz \le \\ &\le \int_{B(c_J, 2|J_+|)} \phi(z) \Delta G_{n+\frac{1}{2}}(z, \xi) dz = \\ &= \int_{B(c_J, 2|J_+|)} \Delta\phi(z) G_{n+\frac{1}{2}}(z, \xi) dz() \end{aligned} \qquad (2.1.11)$$

Recall that ξ is a point in Γ_{n+1} . Apply Lemma 2.6 to the harmonic function

$G_{n+\frac{1}{2}}(\cdot,\xi)$ and the ball $B(c_J, 2|J_+|)$ in the domain $B(c_J, 10|J_+|)$. To simplify notations let $p_+ = c_J + i\frac{|J_+|}{2}$ and $p_- = c_J - i\frac{|J_+|}{2}$. We get:

$$G_{n+\frac{1}{2}}(z,\xi) \leq M_5\,[G_{n+\frac{1}{2}}(p_+,\xi) + G_{n+\frac{1}{2}}(p_-,\xi)] \text{ for all } z \in B(c_{+J}, 2|J_+|) \qquad (2.1.12)$$

Use (2.1.13) and the fact that the function ϕ has $|\Delta\phi| \leq \frac{\text{const}}{|J_+|^2}$ in the inequality (2.1.11). We get

$$\omega_{n+\frac{1}{2}}(J_+,\xi) \leq M_5\,[G_{n+\frac{1}{2}}(p_+,\xi) + G_{n+\frac{1}{2}}(p_-,\xi)] \qquad (2.1.13)$$

Comparing $G_{n+\frac{1}{2}}(\cdot,p_+)$ and $\omega_{n+\frac{1}{2}}(I_+ \bigcup I_-,\cdot)$ on the domain $B(q_0, \frac{r}{4^{n+1/2}}) \setminus B(p_+, \frac{r}{100\,4^{n+1/2}}) \setminus K$ we get that

$$G_{n+\frac{1}{2}}(\xi,p_+) \leq \text{const}\;\omega_{n+\frac{1}{2}}(I_+ \bigcup I_-,\xi) \text{ for all } \xi \in \Gamma_{n+1}$$

The last inequality and (2.1.13) conclude the proof of Step 2. □

Corollary 2.6 *If $J_f \subset R$ is the Julia set of a rational map and if G is the Green function in $C \setminus J_f$, then the function $\log\frac{G(f(z))}{G(z)}$ can be extended to J_f as a Hölder continuous function.*

2.2 Harmonic Rigidity.

Consider u, v two positive, harmonic functions in a domain Ω, vanishing on $\partial\Omega$. that extend as nonnegative subharmonic functions in the plane. If the boundary of the domain is "nice", one can prove that the ratio u/v has Hölder extension to the boundary. This limit relates to the Riesz measures of the subharmonic functions u, v.

Lemma 2.7 (Grishin's Lemma) *Let Ω be a neighborhood of a compact K and let*

u, v be two subharmonic functions in Ω, vanishing on K and positive and harmonic in $\Omega \setminus K$. Suppose that for every $x \in \partial K$ there exists the limit

$$\rho(x) \;=\; \lim_{\xi \to x, \xi \in V \setminus K} \frac{u(x)}{v(x)}$$

and this limit is continuous. Then $d\mu_u = \rho\, d\mu_v$, where μ_u, μ_v are the Riesz measures of u and v.

See [Gr] and [F] for proof.

Lemma 2.8 (Volberg's Lemma on Harmonic Rigidity) *Let u, v be two non-negative subharmonic functions in the disc B with diameter I. Let $J \subset I$ be a closed, uniformly perfect set with infinitely many components and with linear measure $H_1(J) = 0$.*

If u, v vanish on J and are positive and harmonic in $B \setminus J$, and if

$$\frac{u(x)}{v(x)} = \lim_{z \to x} \frac{u(z)}{v(z)} = |A(x)|^2 \,,\ \forall x \in J \qquad (2.2.1)$$

for some holomorphic function A in the ball B, then $|A| \equiv$ constant.

Proof: Let us consider $\partial u(z)$ and $\partial v(z)$. These are holomorphic functions in $B \setminus J$ and their $\bar{\partial}$ derivatives are the Riesz measures μ_u, μ_v, respectively. Consider the following function:

$$w_1(z) = \partial u(z) - A(z)\overline{A(\bar{z})}\partial v(z).$$

It is holomorphic in $B \setminus J$ and its distributional $\bar{\partial}$ derivative is equal to

$$\bar{\partial}\, w_1 \;=\; \mu_u - |A(x)|^2 \mu_v \;\equiv\; 0$$

by the assumption (2.2.1) . So $w_1(z)$ is a holomorphic function in B. In a similar way one can show that $w_2 \stackrel{def}{=} \frac{\partial u}{\partial \bar{z}} - \overline{A(z)}A(\bar{z})\frac{\partial v}{\partial \bar{z}}$ is anti-holomorphic in the ball B.

In particular $\partial u(x) - |A(x)|^2 \partial v(x)$ is continuous and locally bounded on I, the diameter of the ball B.

Without loss of generality we can assume that $u(\bar{z}) = u(z)$, $v(\bar{z}) = v(z)$; otherwise we can consider $u^\star(z) = u(z) + u(\bar{z})$, $v^\star(z) = v(z) + v(\bar{z})$ and (2.2.1) still holds with symmetrized functions.

But if u, v are symmetric, then on I we have $\partial u = \frac{\partial u}{\partial x}$, $\partial v = \frac{\partial v}{\partial x}$ and we conclude that

$$\frac{\partial u}{\partial x} - |A(x)|^2 \frac{\partial v}{\partial x} \quad \text{is continuous on } I \tag{2.2.2}$$

Next we will prove that $W(x) \stackrel{def}{=} u(x) - |A(x)|^2 v(x)$ is locally C^1 on I.

By $\frac{\partial}{\partial x} = (\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})$ we get that for $a, b \in R$ and $\epsilon > 0$

$$W(b + i\epsilon) - W(a + i\epsilon) = \int_a^b \frac{\partial W}{\partial z}(t + i\epsilon) + \frac{\partial W}{\partial \bar{z}}(t + i\epsilon) dt \tag{2.2.3}$$

By $W(z) = u(z) - A(z)\overline{A(\bar{z})}v(z)$ we get that

$$\frac{\partial W}{\partial z} = \frac{\partial u}{\partial z} - A'(z)\overline{A(\bar{z})}v(z) - A(z)\overline{A'(\bar{z})}v(z) - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial z}$$

and

$$\frac{\partial W}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial \bar{z}}$$

This implies that for $z \in B \setminus J$

$$\frac{\partial W}{\partial x}(z) = [\ \frac{\partial u}{\partial z} - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial z}\ ] - v(z)[\ A'(z)\overline{A(\bar{z})} + A(z)\overline{A'(\bar{z})}\ ] +$$

$$+ [\ \frac{\partial u}{\partial \bar{z}} - A(z)\overline{A(z)}\frac{\partial v}{\partial \bar{z}}\ ] + \frac{\partial v}{\partial \bar{z}}[\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)] = w_1 - w_3 + w_2 + w_4$$

The functions w_1, w_2, w_3 are continuous on a neighborhood of I, because they are either analytic or anti-analytic or the product between two continuous functions.

The function $w_4 = \frac{\partial v}{\partial \bar{z}}[\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)]$ is bounded uniformly in ϵ because

$$|\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)| \leq c\epsilon \quad \text{for Im } z = \epsilon$$

and

$$|\frac{\partial v}{\partial \bar{z}}| \leq \int_J \frac{d\mu_v(\zeta)}{|\zeta - z|} \leq \frac{c}{\epsilon} \quad \text{for Im } z = \epsilon$$

Equation (2.2.3) becomes:

$$|W(b+i\epsilon) - W(a+i\epsilon)| \leq \int_a^b |w_1(t+i\epsilon) - w_3(t+i\epsilon) + w_2(t+i\epsilon) + w_4(t+i\epsilon)|dt$$

Sending ϵ to zero one gets that $|W(b) - W(a)| \leq M|b-a|$, therefore W is absolutely continuous on R and

$$\frac{\partial W}{\partial x} = w_1(x) - w_3 + w_2 + 0$$

is correct for $x \in I \setminus J$, and therefore on I since $H_1(J) = 0$. Therefore W is in $C^1_{loc}(I)$.

Its derivative is equal to

$$[\frac{\partial u}{\partial x} - |A(x)|^2 \frac{\partial v}{\partial x}] - v(x)(|A(x)|^2)'$$

Consider the function W on one interval L of the complement of J in I.

It vanishes at the endpoints of L, so its derivative has a zero at a certain $x_l \in L$:

$$(\ u - |A|^2 v\ )'(x_l) \ = \ 0 \qquad (2.2.4)$$

Because the function $u - |A|^2 v$ vanishes on J which is a perfect set we can conclude that its derivative vanishes on J also.

On the other hand on the real line

$$(u - |A|^2 v)' = \frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x} - v(|A|^2)' = w_1 - (|A|^2)'v$$

and both $(u - |A|^2 v)'$ and $(|A|^2)'v$ vanish on J , therefore $w_1(x) = \frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x}$ has to be zero on J . But this function is holomorphic in a ball containing J, so it has to vanish everywhere : $w_1(x) \equiv 0$ on I and

$$\frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x} \equiv 0 \quad \text{on } I. \tag{2.2.5}$$

Computing the the x -derivative in (2.2.4) we get that

$$\frac{\partial u}{\partial x}(x_l) - |A|^2(x_l) \frac{\partial v}{\partial x}(x_l) - v(x_l)(|A|^2)'(x_l) = 0$$

which according to (2.2.5) gives that $v(x_l)(|A|^2)'(x_l) = 0$. The function v being harmonic outside of J , can not vanish at x_l, which means that $(|A|^2)'(x_l) = 0$.

This is enough to conclude that $|A|^2 =$ constant .

Corollary 2.9 *If one adds* u, v *are symmetric to the assumptions above, then* $u =$ *const* v.

Proof: By the previous lemma, $\Delta(u - cv) = 0$. So the function $u - cv$ is harmonic in the ball B. Its zero set, $Z(u - cv)$, is locally a real analytic curve, otherwise $u \equiv cv$.

Thus if $u \neq c\,v$ then at least $u = c\,v$ on the union of some intervals of the real line, which means that $\frac{\partial u}{\partial x} - c\,\frac{\partial v}{\partial x} = 0$ on some intervals on the real line. Due to the symmetry of u, v the y-derivative of both functions vanishes on R, so the following holds:

$$\partial u - c\,\partial v = 0 \quad \text{on some intervals of the real line}$$

therefore $\partial u - c\,\partial v = 0$ everywhere in B, because it is holomorphic.

Similarly $\bar{\partial} u - c\,\bar{\partial} v = 0$ everywhere in B. This implies that $u - c\,v$ is constant in B , and being zero on the real line, it has to be identically zero in B. Therefore $u \equiv c\,v$

CHAPTER 3

Construction of Invariant Measures

3.1 The Jump Transformation.

In order to compare the harmonic measure and the Hausdorff measure we would like to construct some f-invariant measures equivalent to each of them. If we attempt to construct a finite f-invariant measure equivalent to ω, a necessary condition is :

$$\text{if a set } A \subset J \text{ has } \omega(A) < 1 \text{ then } \omega(f^{-n}A) < 1 - \epsilon < 1.$$

By $\omega(f^{-n}A, x) = \omega(A, f^{n}(x)) \to 1$ as $n \to \infty$, there is no finite f-invariant measure equivalent to ω.

Given a parabolic Blaschke product f with one petal and the harmonic measure ω on J we will construct a transformation T on an open, dense subset $J_{\star}$ of the Julia set, called the jump transformation, such that the triplet $(\omega, T, J_{\star})$ satisfies Walters' conditions [Wa].

First of all ω is going to be totally non-singular:

$$\omega(E) = 0 \Longleftrightarrow \omega(TE) = 0 \qquad (3.1.1)$$

and the transformation will satisfy:

$$\exists \epsilon_0 > 0 \text{ such that } \forall x \in J_\star,\ T^{-1}(B_{2\epsilon_0}(x) \bigcap J_\star) = \sqcup_i A_i(x), \tag{3.1.2}$$

that is the disjoint union of open sets such that $T : A_i \to B_{2\epsilon_0}(x) \cap J_\star$ is a homeomorphism and $\text{dist}(Ty, Ty') \geq \text{dist}(y, y')$ $\forall y, y' \in A_i(x)$.

Also

$$\forall \epsilon > 0\ \exists M \text{ such that } \forall x \in J_\star,\ T^{-M} \text{ is } \epsilon\text{-dense in } J_\star \tag{3.1.3}$$

Notice that (3.1.1) implies the fact that the push-forward measure $T^\star\omega = \omega \circ T^{-1}$ is absolutely continuous with respect to ω. Also the Jacobian of ω with respect to T is well defined and we will introduce the notation

$$\psi = -\log \frac{d\omega \circ T}{d\omega}(x) = -\log J_\omega^T(x)$$

To write down the rest of Walters' conditions we need more notations: let $S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(T^i x)$ for any function $\varphi : J_\star \to R$. Given b an element of $T^{-n}b_0$ for some $b_0 \in J_\star$, let T_b^{-n} denote the inverse branch that sends $b_0 \to b$. Set $c_{\varphi,b}(x, x') = |S_n\varphi(y) - S_n\varphi(y')|$ for $y = T_b^{-n}x, y' = T_b^{-n}x'$.

Let $\mathcal{F}_T$ be the class of functions φ such that

(i) $$\sum_{y \in T^{-1}x} e^{\varphi(y)} \leq K_\varphi < \infty \tag{3.1.4}$$

(ii) $$c_\varphi(x, x') = \sup_{n \geq 1} \sup c_{\varphi,b}(x, x') \leq C_\varphi < \infty \text{ for x,x' close enough} \tag{3.1.5}$$

(iii) $$c_\varphi(x, x') \to 0 \text{ as } d(x, x') \to 0 \tag{3.1.6}$$

Remark If φ is the Jacobian of the harmonic measure with respect to the jump transformation, then

(3.1.4) $\Leftrightarrow$ the density of $\omega \circ T^{-1}$ with respect to ω is bounded

(3.1.5) $\Leftrightarrow$ $\sup_n \sup\{\frac{J_\omega^{T^n}(y)}{J_\omega^{T^n}(y')}; y \in T^{-n}x, y' \in T^{-n}x'\} \leq A(x,x') \leq K < \infty$

(3.1.6) $\Leftrightarrow$ $A(x,x') \to 1$ when $d(x,x') \to 0$

Let $UC(X)$ denote the space of uniformly continuous functions on some set X. For $\varphi \in \mathcal{F}_T$ define the operator $\mathcal{L}_\varphi$ on $UC(X)$:

$$\mathcal{L}_\varphi g(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} g(y)$$

Recall that we have denoted by δ the Hausdorff dimension of J. We are going to show that both potentials $\psi = -\log J_\omega^T$ and $\phi = -\delta \log |T'|$ satisfy Walters' conditions (3.1.4) to (3.1.6) so that the following theorem from [Wa] applies:

Theorem 3.1 *Suppose* $T : X_0 \to X_0$ *satisfies (3.1.2) and (3.1.3) and that* $\varphi \in \mathcal{F}_T$ *. Then* $\mathcal{L}_\varphi$ *extends to a bounded linear operator in* $C(X_0)$ *.*

Furthermore, there exists a unique probability measure μ *satisfying (3.1.1) and some* $\lambda > 0$ *such that*

$$\mathcal{L}_\varphi^\star \mu = \lambda\, \mu. \tag{3.1.7}$$

There exists a unique positive function h, $h \in C(\bar{X}_0)$ *such that*

$$\mathcal{L}_\varphi h = \lambda h \quad \textit{and} \quad \mu(h) = 1. \tag{3.1.8}$$

The measure $h\mu$ *is T-invariant.*

See [Wa] for details.

This could be enough in order to construct invariant measures equivalent to the δ-conformal measure and equivalent to the harmonic measure. However, we need better properties of the densities h in equation (3.1.8). For this reason the actual construction of the invariant measures will be done in the next section, using a theorem of Ionescu Tulcea.

Following M.Denker and M.Urbanski, [DU1], [DU2], [DU3], we will construct the Jump transformation T for conformal parabolic systems. For the sake of completeness we are obliged to repeat parts of these works.

Let J be a compact set consisting of at least two points, let $J \subset U \subset C$ be an open neighborhood of J and let $f : U \to C$ be a holomorphic map. We assume that there exists a point p such that $f(p) = p$ and $f'(p) = 1$.

Definition A system (f, U, J) is called conformally parabolic if there exists a unique parabolic point p in U and if

(i) $f(J) = J$

(ii) there are no critical points in J

(iii) there exists a neighborhood W of p such that if $x \in U$ satisfies $f^n(x) \in W$ for $\forall n > 0$ then $x = p$

(iv) for $x \in U \setminus J$, $\exists r_x > 0$ such that $f^n(B(x, r_x)) \to p$ uniformly

(v) if $f^n(x) \in U, \forall n > 0$ then $x \in J$ or $f^n(x) \to p$

(vi) the mapping $f : J \to J$ is topologically exact.

Lemma 3.2 *Let V be an open neighborhood of the parabolic point p. There exists $\delta_0 = \delta_0(V) > 0$ such that for every $z \in J \setminus V$ all inverse branches of f^n are well defined on $B(z, 3\delta_0)$.*

Proof: See [DU3].

Let now f be a parabolic Blaschke product with Cantor like Julia set, having the parabolic point at $p = 1$. Denote by Γ^α the cone domain with vertex at the parabolic point, that is $\Gamma^\alpha = \text{convh}(1, B(0, \sin \alpha))$.

Lemma 3.3 *There exists α such for every $z \in J$ all inverse branches of f^n are well defined in $B(z, r)$ if $B(z, r) \bigcap \Gamma^\alpha = \emptyset$*

Lemma 3.4 *Such a transformation f is positively expansive, i.e.*

$$\exists\, \delta' > 0 \text{ such that if } \sup_{n>0} d(f^n(x), f^n(y)) \leq \delta' \text{ then } x = y.$$

Proof: See [DU1].

Recall that a cover $\mathcal{R} = \{R_1, R_2, \ldots R_s\},\ s \leq \infty$ of X is said to be a Markov partition if it satisfies:

(i) $R_i = \overline{\text{int} R_i}$

(ii) $\text{int} R_i$ are disjoint

(iii) $f(R_i)$ is the union of some R_j.

Assume that the map $F : X \to X$ is open, surjective and positively expansive. From the proofs in Ruelle's book [Ru] the following result can be deduced:

Theorem 3.5 *If $F : X \to X$ is an open, surjective, positively expansive continuous mapping of a compact metric space X, and if μ is an atomless measure,*
then there exists a finite Markov partition of arbitrary small diameters satisfying: $\mu(\partial R_1 \bigcup \partial R_2 \bigcup \ldots \partial R_s) = 0$.

The transition matrix $A = (A_{i,j})_{i,j \leq s}$ associated with the Markov partition $\mathcal{R}_0$ is defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } F(R_i) \supset R_j \\ 0 & \text{if } F(R_i) \bigcap R_j = \emptyset \end{cases}$$

A sequence $k_1, \ldots, k_n$ is said to be A-admissible if $A_{k_i k_{i+1}} = 1$ for every $i = 1, \ldots n-1$. Coming back to our parabolic system, given a Markov partition and some A-admissible sequence $k_1, \ldots, k_n$ define $A(k_1, \ldots, k_n) = \bigcap_{j=1}^{n} f^{-j+1} R_{k_j}$. We will call such a set an n-cylinder. The family of all n-cylinders will be denoted $\mathcal{R}_0^n$.

Among all cylinders of $\mathcal{R}_0^n$ we distinguish "good" and "bad" cylinders. A cylinder will be called good if $f^{n-1}A(k_1,\dots k_n)\bigcap W = \emptyset$, and will be called bad cylinder if $f^{n-1}A(k_1,\dots k_n)\bigcap W \neq \emptyset$. The set W is the neighborhood of the parabolic point from the condition (iii) in the definition of "conformally parabolic systems". Let $\mathcal{G}_n$ be the collection of the good cylinders of order n, let $\mathcal{B}_n$ be the collection of the bad cylinders of order n, and $\mathcal{R}_\mathcal{G}$ be the collection of all good cylinders, $\mathcal{R}_\mathcal{B}$ be the collection of all bad cylinders.

Remark Since the map f is expansive and so expanding with respect to a metric compatible with the topology on J, (see theorem 2.2 of [DU2]), the diameters of elements of $\mathcal{R}_0^n$ tend to zero and so the family $\mathcal{R}_0$ generates the Borel σ-algebra mod ν for any non-atomic measure ν on J.

Lemma 3.6 *Let ν be a non-atomic measure on J. Then $X_0 \stackrel{def}{=} \bigcup_{A\in\mathcal{R}_G} A$ satisfies $\nu(X_0) = 1$ and*

$$\lim_{n\to\infty} \Sigma_{A\in\mathcal{B}_n}\nu(A) = 0 \tag{3.1.9}$$

See [DU3] for proof.

Now we are ready to introduce the jump transformation of f.

For every $x \in X_0$ take the smallest rank of a good n-cylinder that contains x, i.e.

$$N(x) = \inf\{n\,; x \in A(k_1,\dots k_n) \text{ and } f^{n-1}A(k_1,\dots k_n)\bigcap W = \emptyset\}.$$

In view of previous Lemma, $N(x)$ is almost everywhere finite with respect to any non-atomic measure. Schweiger's jump transformation is defined by

$$T(x) = f^{N(x)}(x).$$

The set $N^{-1}(n)$ consists exactly of good cylinders $A(k_1,\dots k_n) \in \mathcal{G}_n$ that have a "bad" parent: $A(k_1, \dots k_{n-1}) \in \mathcal{B}_{n-1}$.

Let us consider a new family, $\mathcal{R}_\star$, to be the union for all n of families of cylinders forming $N^{-1}(n)$. Denote the elements of $\mathcal{R}_\star$ by C_i. Then (3.1.9) becomes

$$\bigcup_{C \in \mathcal{R}_\star} C = X_0 \text{ and } T(C) = \bigcup_{C' \in \mathcal{R}_\star, C' \bigcap T(C) \neq \emptyset} C' \tag{3.1.10}$$

In this sense $\mathcal{R}_\star$ constitutes a countable Markov partition for T. Also the transition matrix for $(T, \mathcal{R}_\star)$ is aperiodic.

For $x \in X_0$ define the positive integers $N_n(x)$ by:

$$N_1(x) = N(x), \quad N_{n+1}(x) = N_n(x) + N(T^n(x))$$

which gives $T^n(x) = T^{N_n(x)}(x)$. If we denote by $C^n(x)$ the cylinder of $\mathcal{R}_\star^n$ containing x , then also

$$T^n_{|C^n(x)} = f^{N_n(x)}_{|C^n(x)}$$

The following result is proved in [DU3]:

Lemma 3.7 *The jump transformation is expanding, i.e. there exist* $c, \beta > 0$ *such that*

$$|(T^n)'(x)| \geq c\, e^{\beta n} \textit{ for all } x \in X_0, n > 0. \tag{3.1.11}$$

Lemma 3.8 *There exist constants* B, η *depending only on* f*, such that for all pairs of points* x, x' *sufficiently close, if* $y = T_b^{-n}x$*,* $y' = T_b^{-n}x'$*, then* $\forall n$

$$\frac{|(T^n)'(y')|}{|(T^n)'(y)|} \leq e^{\frac{B}{\eta} d(x,x')}. \tag{3.1.12}$$

Let $\Sigma_\star \subset \mathcal{R}_\star^{\mathcal{N}}$ be the space of all admissible sequences of cylinders according to the transition matrix of $(T, \mathcal{R}_\star)$.

Let $\Sigma_\star^0 = \{(C_n)_{n\geq 0} \in \Sigma_\star$ such that $\bigcap_n T^{-n}(C_n) \neq \emptyset\}$. The spaces $\Sigma_\star$ and $\Sigma_\star^0$ may not be compact. Since by lemma 3.8 the diameters of of the partition $\mathcal{R}_\star^n$ converge to zero, one can define the projection $\pi : \Sigma_\star^0 \to X_0$ setting

$$\pi(\ (C_n)_{n\geq 0}\ ) = \bigcap_{n\geq 0} T^{-n}(C_n)$$

Consider the metric ρ on $\Sigma_\star^0$ to be

$$\rho((C_n)_{n\geq 0}, (C'_n)_{n'\geq 0}) \ = \ e^{-\min\{n;\, C_n \neq C'_n\}}$$

Proposition 3.9 *The map* $\pi : \Sigma_\star^0 \to X_0$ *is a Hölder continuous surjection, with exponent* β *from (3.1.11)*

Remark If we identify $x \in X_0$ with $\pi^{-1}x$ then one can transport the metric ρ to X_0. The previous lemma can be formulated as

$$\forall x, x' \in X_0 \quad |x - x'| = d(x, x') \leq c\rho(x, x')^\beta$$

Using the fact that for any cylinder $C \in \mathcal{R}_\star$ there exists a cylinder $A \in \mathcal{R}_0$ such that

$$T(C) = f(A), \tag{3.1.13}$$

one can prove the following lemma:

Lemma 3.10 *1. For any* $\epsilon > 0$ *there exists some* M *such that* $\forall x \in X^0$, $T^{-M}x$ *is* ϵ*-dense in* (X_0, ρ)

2. For any $\epsilon > 0$ *there exists some* M *such that* $\forall x \in X_0$, $T^{-M}x$ *is* ϵ*-dense in* (X_0, d).

Using the fact that f is expanding we can conclude that

Lemma 3.11 *For every $C^{n+1} \in \mathcal{R}_\star^{n+1}$ there exists a unique holomorphic inverse branch T^{-n} of T^n defined on $B(T^n(C^{n+1}), 2\delta_0)$ and sending $T^n(C^{n+1})$ to C^{n+1}.*

Using Köebe distortion properties for the functions T^{-n} and (3.1.12), one can show that for points y, y' lying in the same n-cylinder of $\mathcal{R}_\star^n$, the following holds:

$$d(y,y') \leq \frac{\text{const}}{\text{diam}R_p} e^{-\beta n} d(T^n y, T^n y') \tag{3.1.14}$$

Let C_i denote the family of $\mathcal{R}_\star$ such that $T(C) = f(R_i \cap X_0)$, $i = 1, \ldots, s$ (see equation 3.1.13). We will refer to these cylinders as standard cylinders.

If we select the initial Markov partition to consist of $R_i = J \cap I_i$ where each I_i is an arc of the unit circle that is mapped one-to-one and onto the full circle and the first of them, I_1, is the arc containing the parabolic point, then the standard cylinders turn out to be exactly the sets $f_1^{-n} I_j$, $j = 2, \ldots, s$, where f_1^{-1} denotes the inverse branch that sends the unit circle onto I_1.

The set X_0 that we get for this particular choice of cylinders is $J \setminus \bigcup_n f^{-n}(p)$. We will denote it by $J_\star$.

A similar convention will be made if we work with J_f on the real line instead of the circle.

3.2 Construction of Invariant Measures.

Let $\delta = \text{Hdim}(J)$. We will first construct an invariant measure whose Hausdorff dimension is equal to δ. To do that we need to introduce conformal measures. Recall that a probability m is said to be δ-conformal with respect to some transformation F if

$$m(F(A)) = \int_A |F'|^\delta dm \quad \text{for all sets A such that } F_{|A} \text{ is one-to-one.}$$

It was proved in [DU2] that for parabolic systems there exists exactly one δ-conformal measure m with respect to the jump transformation. Moreover m is non-atomic.

Lemma 3.12 *There exists some* $K < \infty$ *such that*

$$\sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta} \le K \text{ for all } x \in X_0.$$

Proof: Let us denote $L(x) = \sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta}$. Lemma 3.8 implies that if $y, y' \in C_i$, for some standard cylinder C_i, then

$$B^{-1} \le \frac{|T'(y)|}{|T'(y')|} \le B$$

So if $x, x' \in f(R_i \bigcap J_\star)$, then $\frac{L(x)}{L(x')} \le B$. Use the fact $|T'|^\delta$ is the Jacobian of the δ-conformal measure with respect to T. So

$$\int_{f(R_i \bigcap J_\star)} L(x) dm(x) = \sum_{C \in \mathcal{C}} \int_C |T'(y)|^{-\delta} dm(Ty) = \sum_{C \in \mathcal{C}} m(C) = 1$$

But the integral majorizes the quantity $B \sup_{x \in f(R_i \bigcap J_\star)} L(x) m(f(R_i \bigcap J_\star)$. Thus

$$L(x) \le B \frac{1}{\min_{1 \le i \le s} m(f(R_i \bigcap J_\star))} = K < \infty$$

Note that we can actually show that the convergence in the series $\sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta}$ is uniform in $x \in J_\star$, in the sense that

$$\sup_{x \in J_\star} \sum_{y \in T^{-1}x \bigcap (\bigcup_{i=k+1}^{\infty} C_i)} \frac{1}{|T'(y)|^\delta} = \epsilon^k \to 0 \text{ as } k \to \infty \qquad (3.2.1)$$

Recall that if two points y, y' are contained in the same cylinder then

$c^{-1} \leq \frac{|T'(y)|}{|T'(y')|} \leq c$. That means that as x varies in $J_\star$, the sums in (3.2.1) differ at most by a factor of c, therefore the convergence is uniform.

Let $\phi(x) = -\delta \log |T'|(x)$, for $x \in J_\star$. The measure m, the transformation T, and the potential ϕ satisfy all Walters' conditions (3.1.1)-(3.1.6). We conclude that

Theorem 3.13 *There exists a unique T-invariant probability measure μ absolutely continuous with respect to m. Moreover μ is ergodic.*

For the harmonic measure, fix a reference point P_0 on the real line outside the Julia set and let $\omega = \omega(\cdot, P_0)$, $G = G(\cdot, P_0)$. We introduce $\mathcal{J}(z) = \frac{G(f(z))}{G(z)}$, $z \in C$. By Harnack inequality, for z in a certain neighborhood U of J,

$$0 < c \leq \mathcal{J}(z) \leq C < \infty$$

It will be more convenient for us to change coordinates and consider J_f a subset of the real line rather than of the circle, and the parabolic point to be $p = 0$, rather than $p = 1$.

Lemma 3.14 *1. For all $x \in J_f$ there exists the limit*

$$\lim_{z \to x, z \in U \setminus J_f} \mathcal{J}(z) \overset{def}{=} \mathcal{J}(x)$$

2. For $\omega - a.e.$ $x \in J_f$, $\mathcal{J}(x) = Jac_\omega^f(x)$.

Proof: The first part of this lemma follows from Theorem 2.5. For the second part apply Grishin's Lemma (2.7) for the functions $u = G \circ f$, $v = G$ whose Riesz measures are $\mu_u = \Delta(G \circ f)$ and $\mu_v = \Delta G = \omega$. We get that for sets $E \subset J_f$ on which $f : E \to f(E)$ is injective we have

$$\omega(f(E)) = \mu_u(E) = \int_E \mathcal{J}(x) d\mu_v(x) = \int_E \mathcal{J}(x) d\omega(x)$$

therefore $\mathcal{J}$ is the Jacobian of the harmonic measure.

In what follows we mean by $Jac^f_\omega(x)$ this continuous representative $\mathcal{J}(x)$. Let $Jac^{T^n}_\omega$ be the Jacobian of the harmonic measure with respect to T^n.

Lemma 3.15 *Let $C(k_1, \dots k_n)$ be a n-cylinder of $\mathcal{R}_\star$. The map $T^n : C(k_1, \dots k_n) \to X_0$ is injective and if $y \in T^{-n}x$, $y' \in T^{-n}x'$ are two points of the same n-cylinder, then*

$$\left| \frac{Jac^{T^n}_\omega(x)}{Jac^{T^n}_\omega(x')} - 1 \right| \leq c \; dist(x, x')^\beta$$

Proof: Let g be a holomorphic inverse branch of T^n defined on $B(fR_i, 2\delta_0) = B(C_{k_n}, 2\delta_0)$ such that $g(T(C_{k_n})) = C(k_1, \dots k_n)$. Thus $g(x) = y$, $g(x') = y'$. Such a branch exists by Lemma 3.11. We have to estimate the following ratio:

$$\frac{\mathcal{J}(y)\mathcal{J}(Ty)\dots\mathcal{J}(T^{n-1}y)}{\mathcal{J}(y')\mathcal{J}(Ty')\dots\mathcal{J}(T^{n-1}y')} \tag{3.2.2}$$

Let us choose points ζ, ζ' in $U \setminus J$ extremely close to y, y', and let us estimate

$$\frac{G(T^n\zeta)G(\zeta')}{G(T^n\zeta')G(\zeta)} \tag{3.2.3}$$

If we can estimate the previous ratio uniformly in $\zeta \to y$, $\zeta' \to y'$, we get the estimate for the ratio in (3.2.2).

Now let $z = T^n\zeta$, $z' = T^n\zeta'$. Both points can be assumed to be in $B(C_{k_n}, 2\delta_0)$ as $\zeta \sim y, \zeta' \sim y'$. Rewrite the double ratio in (3.2.3) as

$$\frac{G(gz')}{G(z')} : \frac{G(gz)}{G(z)} \tag{3.2.4}$$

The functions $u = G \circ g,\ v = G$ satisfy the conditions of Theorem 2.7, so the latter double ratio has the following estimate with constants independent of $u, v,$ and $z, z' \in B(f(R_i), \delta_0)$:

$$\left|\frac{u(z')}{v(z')} : \frac{u(z)}{v(z)} - 1\right| \leq c\,[d(z,z')]^{\beta}$$

Just let $\zeta \to z,\ \zeta' \to z', z \to x,\ z' \to x'$ and we get the desired estimate.

Now to imitate the results of Section 3 of [DU3] we need the following notations:

$$\mathcal{H}_0 = \{g;\ g : X_0 \to R \text{ bounded}\} \text{ and } \mathcal{H}_\alpha = \{g \in \mathcal{H}_0;\ v_\alpha(g) < \infty\},$$

where $v_\alpha(g) = \sup\{\left|\frac{g(x)-g(x')}{\rho(x,x')^\alpha}\right|,\ x, x' \in J_\star, \rho(x,x') \leq e^{-1}\ \}$.

These are Banach spaces, $\mathcal{H}_0$ with the supremum norm and $\mathcal{H}_\alpha$ with the norm $\| g \|_\alpha = \| g \|_0 + v_\alpha(g)$.

Let $\varphi : J_\star \to R$ be a function bounded from above and satisfying $v_\alpha(\varphi) < \infty$. For $y \in J_\star$ let

$$S_n\varphi(y) = \varphi(y) + \varphi(Ty) + \ldots + \varphi(T^{n-1}y)$$

Let $\mathcal{L}_\varphi(\Psi)(x) \stackrel{def}{=} \sum_{y \in T^{-1}x} e^{\varphi(y)}\ \Psi(y)$. Then $\mathcal{L}_\varphi^n(\Psi)(x) = \sum_{y \in T^{-n}x} e^{S_n(y)}\Psi(y)$

Lemma 3.16 *If for some function φ as above the operator $\mathcal{L}_\varphi$ satisfies:*

$$\| \mathcal{L}_\varphi^n \|_0 \leq M \quad \forall n \geq 1$$

Then:

$$\| \mathcal{L}_\varphi^n g \|_\alpha \leq\ Me^{-\alpha n} \| g \|_\alpha + c \| g \|_0, \quad \forall g \in \mathcal{H}_\alpha$$

Proof: Repeat word by word the proof from [DU3]. Let $x,\ x' \in J_\star$ such that $\rho(x,x') = e^{-k}$, for some $k \geq 1$. By the definition of the metric, they are in the same k-cylinder. Choose $y \in T^{-n}x$ and let $y' \in T^{-n}x'$ be in the same (n+k)-cylinder with

y. Then

$$|g(y) - g(y')| \leq \| g \|_\alpha e^{-\alpha(k+n)} \tag{3.2.5}$$

$$|e^{S_n\varphi(y)} - e^{S_n\varphi(y')}| = |\frac{e^{S_n\varphi(y)}}{e^{S_n\varphi(y')}} - 1| e^{S_n\varphi(y')} \leq$$

$$\leq c \| \varphi \|_\alpha e^{-\alpha k} e^{S_n\varphi(y')} \tag{3.2.6}$$

Thus

$$\mathcal{L}_\varphi^n g(x) - \mathcal{L}_\varphi^n g(x') = \sum_{y \in T^{-n}x} [e^{S_n\varphi(y)} g(y) - e^{S_n\varphi(y')} g(y')] =$$

$$= \sum_{y \in T^{-n}x} e^{S_n\varphi(y)} [g(y) - g(y')] + \sum_{y \in T^{-n}x} g(y') [e^{S_n\varphi(y)} - e^{S_n\varphi(y)}] =$$

The second sum is estimated by (3.2.6) as follows :

$$c \| \varphi \|_\alpha \| \mathcal{L}_\varphi^n g \|_0 \rho(x,x')^\alpha \leq cM \| \varphi \|_\alpha \| g \|_0 \rho(x,x')^\alpha$$

The first sum can be majorated by

$$e^{-\alpha n} \| g \|_\alpha \rho(x,x')^\alpha \| \mathcal{L}_\varphi^n 1 \|_0 \leq M e^{-\alpha n} \| g \|_\alpha \rho(x,x')^\alpha$$

Lemma 3.17 *If the function ψ denotes $-\log Jac_\omega^T$, then the operator $\mathcal{L}_\psi$ satisfies:*

$$\| \mathcal{L}_\psi^n g \|_0 \leq M \| g \|_0$$

Proof: The operator $\mathcal{L}_\psi$ being positive, it is enough to prove that $\| \mathcal{L}_\psi^n 1 \|_0 \leq M$. Using the definition of $\mathcal{L}_\psi$, one gets:

$$\mathcal{L}_\psi^n 1(x) = \sum_{y \in T^{-n}x} \frac{1}{Jac_\omega^{T^n}(y)} \tag{3.2.7}$$

As in the proof of Lemma 3.12 one can majorize the right side of (3.2.7) by

$\frac{B}{\min_{1\leq i\leq s}\omega(f(R_i\bigcap J_\star))} = M < \infty.$

Lemma 3.18 *The image under $\mathcal{L}_\psi$ of any bounded subset of $\mathcal{H}_\alpha$ is relatively compact in $\mathcal{H}_0$.*

Proof: Before actually proving this lemma we would like to point out that bounded sequences in $\mathcal{H}_\alpha$ may not have any convergent subsequence in $\mathcal{H}_0$ (Arzela-Ascoli's theorem does not apply because the space $(J_\star, \rho)$ is not compact). Consider for example the sequence χ_{C_n}; it is bounded in $\mathcal{H}_\alpha$ and has no convergent subsequence in the supremum norm. However, after applying $\mathcal{L}_\psi$ the values of χ_{C_n} are "redistributed" over $J_\star$ and multiplied with weights e^ψ that decrease to zero as the cylinders C_k get closer to the parabolic point.

Let $(g_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_\alpha$, $\| g_n \|_\alpha \leq M$. We want to show that one can find a function $g \in \mathcal{H}_0$ and a subsequence g_{n_k} such that $\| \mathcal{L}_\psi g_{n_k} - g \|_0 \to 0$.

By Lemma 3.17, the functions $\mathcal{L}_\psi g_n$ are uniformly bounded in $\mathcal{H}_0$ by some constant M_1.

For $x, x' \in J_\star$ we have:

$$|\mathcal{L}_\psi g_n(x) - \mathcal{L}_\psi g_n(x')| = | \sum_{y\in T^{-1}x} [e^{\psi(y)} g_n(y) - e^{\psi(y')} g_n(y')] | \leq$$

$$\leq \sum_{y\in T^{-1}x} |e^{\psi(y)} - e^{\psi(y')}||g_n(y)| + \sum_{y\in T^{-1}x} e^{\psi(y')}|g_n(y) - g_n(y')| \leq$$

$$\leq \| \mathcal{L}_\psi g_n \|_0 \, \rho(x,x')^\alpha v_\alpha(\psi) + \rho(x,x')^\alpha v_\alpha(g_n) \| \mathcal{L}_\psi 1 \|_0 \leq C\rho(x,x')^\alpha$$

Therefore the family $\mathcal{L}_\psi g_n$ are equicontinuous on $J_\star$. By the Arzela- Ascoli theorem for the sets $\bigcup_{i=1}^k C_i$, we can extract subsequences $(g_n^k)_{n>0}$ such that for each k and for some continuous function g

$$\sup_{x\in\bigcup_{i=1}^k C_i} |\mathcal{L}_\psi g_n^k - g| \to 0 \text{ as } n \to \infty$$

Consider the diagonal subsequence $(g_n^n)_n$ and relabel it g_n. We get that for each k

$$\epsilon_n^k \stackrel{def}{=} \sup_{x\in\bigcup_{i=1}^k C_i} |\mathcal{L}_\psi g_n - g| \to 0 \text{ as } n\to\infty \tag{3.2.8}$$

Finally, to show that the subsequence $\mathcal{L}_\psi g_n$ is uniformly Cauchy, let $\epsilon > 0$ be fixed.

$$|\mathcal{L}_\psi g_n(x) - \mathcal{L}_\psi g_{n+p}(x)| \le \sum_{y\in T^{-1}x} e^{\psi(y)}|g_n(y) - g_{n+p}(y)| \le$$

$$\sum_{y\in T^{-1}x\bigcap(\bigcup_{i=1}^k C_i)} e^{\psi(y)}|g_n(y) - g_{n+p}(y)| + \sum_{y\in T^{-1}x\bigcap(\bigcup_{i=k+1}^\infty C_i)} e^{\psi(y)}|g_n(y) - g_{n+p}(y)| \le$$

$$\le \epsilon_n^k \parallel \mathcal{L}_\psi 1 \parallel_0 + (\parallel g_n \parallel_0 + \parallel g_{n+p} \parallel_0) \sup_{x\in J_\star} \sum_{y\in T^{-1}x\bigcap(\bigcup_{i=k+1}^\infty C_i)} e^{\psi(y)}$$

If we denote by ϵ^k the supremum in the previous inequality, by an argument similar to the one in (3.2.1), we have that $\epsilon^k \to 0$ as $k\to\infty$, so for some large $k = k(\epsilon)$, $\epsilon^k \le \epsilon/2$. For this fixed k there exists some index n_0 such that $\epsilon_n^k \le \epsilon/2$ $\forall n \ge n_0$. Therefore $\parallel \mathcal{L}_\psi g_n - \mathcal{L}_\psi g_{n+p} \parallel_0 \le \epsilon$ $\forall n \ge n_0 = n_0(\epsilon)$

The last three Lemmas allow us to use a result of Ionescu Tulcea and Marinescu [ITM] which states that $\mathcal{L}_\psi$ on the complexification of $\mathcal{H}_0$ can be decomposed into the sum $\mathcal{L}_\psi = P + S$ of a projection P onto the finite dimensional space of eigenvectors with eigenvalues on the unit circle, and a contraction S of $\mathcal{H}_\alpha$.

On the other hand, Schweiger formalism applies for the system $(T, J_\star, \omega, \mathcal{R}_\star)$. In particular there exists a unique T -invariant probability equivalent to ω, call it $d\mu = \rho d\omega$. This measure is ergodic, $\log\rho \in L^\infty(\omega)$, and the following holds:

$$\int_{J_\star} g\rho d\omega = \int_{J_\star} g\circ T^n \rho d\omega = \int_{J_\star} g\mathcal{L}_\psi^n \rho d\omega$$

This computation shows that ρ is an eigenvector of $\mathcal{L}$ with eigenvalue 1 and that there are no other eigenvalues of the form $\lambda = e^{2\pi\frac{p}{q}}$ except for $\lambda = 1$. (Because if ρ_λ

were an eigenvector for such a λ, then $\rho_\lambda d\omega$ would be a T^q-invariant measure, and by uniqness in Schweiger formalism this means that $\rho_\lambda = \rho$.)

The number of eigenvalues in the theorem of Ionescu Tulcea and Marinescu being finite, we get that it is impossible to have eigenvalues of modulus 1 whose arguments are not rational multiples of 2π.

Theorem 3.19 *Let $\psi = -\log Jac_\omega^T$ for the harmonic measure on parabolic Julia set. Then*

1. *The operator $\mathcal{L}_\psi : L^\infty(\omega) \to L^\infty(\omega)$ has only one eigenvalue of absolute value 1, this eigenvalue is 1, and the eigenvector ρ is continuous. The eigenvector ρ is normalized such that $\int_{J_*} \rho d\omega = 1$.*

2. *$\mathcal{L}_\psi = P + S$ where P is the projection $P(g) = (\int_{J_*} g d\omega)\rho$*

3. *The operator S acts on $\mathcal{H}_\alpha$ and $\| S^n \|_\alpha \leq c_1 \gamma_1^n$ for some $\gamma_1 \in (0,1)$.*

In the sections that follow we will denote by ν the T-invariant measure equivalent to ω and by ρ its density with respect to ω. We will denote by μ the T-invariant measure equivalent to the δ-conformal measure m and r will be the density $\frac{d\mu}{dm}$.

Both invariant measures being ergodic by Schweiger's formalism, there are only two possible situations: either $\nu = \mu$ or $\nu \perp \mu$.

If $\nu = \mu$ then their Jacobians (with respect to T) are equal so

$$\log J_\omega^T + \rho - \rho \circ T = \log J_m^T + r - r \circ T$$

Thus

$$\log J_\omega^T - \delta \log |T'| = \gamma \circ T - \gamma \tag{3.2.9}$$

where γ is a bounded function. We use an observation from [DU3] that the homology equation (3.2.9) for T implies a similar homology equation for f.

Recall that $T(x) = f^{N(x)}(x)$ for some $N(x) \geq 2$ then $N(x)$ is constant on cylinders and

$$\log J_\omega^T = \sum_{i=0}^{N(x)-1} \log J_\omega^f(f^i(x))$$

$$\delta \log |T'(x)| = \delta \sum_{i=0}^{N(x)-1} |f'(f^i x)|$$

Therefore if we denote by $\Theta(x) = \log J_\omega^T(x) - \delta \log |T'(x)|$ we get that

$$\Theta(x) - \Theta(fx) = \log J_\omega^f(x) - \delta \log |f'(x)| \tag{3.2.10}$$

On the other hand (3.2.9) for T implies that

$$\Theta(x) = \gamma(f^{N(x)}(x)) - \gamma(x), \;\; \Theta(fx) = \gamma(f^{N(x)}(x)) - \gamma(fx) \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11) we get

$$\log J_\omega^f(x) - \delta \log |f'(x)| = \gamma(fx) - \gamma(x) \;\; a.e. \; x \in J \tag{3.2.12}$$

Since the left side of the equation (3.3) is continuous and the tree of preimages $\bigcup_{n \geq 0} f^{-n}(x)$ of every point is dense in J , one can follow the proof in [PUZ] to show that the function γ has a continuous representative and that the equality (3.3) holds everywhere in J.

The next section contains a brief explanation on why (3.3) leads to contradiction (therefore only $\omega \perp m$ can happen).

3.3 Solving the Homology Equation.

This section is a compilation of results from [PV] where it was shown that the homology equation leads to a contradiction . Most of them are intermediate steps toward the construction of new coordinates (in a neighborhood of some reppeling periodic point) that simultaneously conjugate the dynamics to a linear map and to a translation.

Throughout this section we use the notation $\frac{u(x)}{v(x)} = \lim_{z\to x} \frac{u(z)}{v(z)}$, $x \in J$, for positive harmonic functions u, v in $N \setminus J$, vanishing on J (here N is a neighborhood of J).

First we are going to introduce some notations: let Γ be a cone with the vertex at the parabolic point p, containing J, such that the forward orbit of the critical points does not intersect Γ. for $x \in J$ Let $D(x)$ denote the largest disc centered at x and contained in the cone Γ ,and let $\Gamma_x = \mathrm{conv}(p, D(x))$. Denote by f_1^{-1} the inverse branch of f that sends the parabolic point to itself. All the functions f_1^{-n} are well defined in Γ .

For a set E we will denote $f_1^{-n}E$ by E_{-n} .

The homology equation can be written as follows:

$$\log \frac{G(x)}{G(x_{-n})} - \delta \log |(f^n)'(x_{-n})| = \gamma(x) - \gamma(x_{-n}) \tag{3.3.1}$$

Lemma 3.20 *If the homology equation (3.3.1) holds for some continuous* γ, *then* $\forall z \in \Gamma \setminus J$

$$\exists \text{ the non-zero and finite limit } \lim_{n\to\infty} |(f^n)'(z_{-n})|^\delta \, G(z_{-n}).$$

Proof: Let us estimate $\log(\frac{\beta_{n+k}(z)}{\beta_k(z)})$ where $\beta_n \stackrel{def}{=} |(f^n)'(z_{-n})|^\delta \, G(z_{-n})$.

Let x be a point of J closest to z that contains z in $D(x)$. Then:

$$\log \frac{\beta_{n+k}(z)}{\beta_k(z)} = \log \left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)} \right] + \log \frac{\beta_{n+k}(x)}{\beta_k(x)} =$$

$$= \log \left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)} \right] + \log \frac{G(x_{-n-k})}{G(x_{-k})} \, |(f^n)'(x_{-n-k})|^\delta =$$

$$= \log\left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)}\right] + \gamma(x_{-n-k}) - \gamma(x_{-k})$$

The last term tends to zero (uniformly in n and z) when $k \to \infty$. The first term can be written as:

$$\log[\frac{G(z_{-n-k})}{G(z_{-k})} : \frac{G(x_{-n-k})}{G(x_{-k})}] - \delta \log|\frac{(f^k)'(z_{-k})}{(f^{n+k})'(z_{-n-k})} : \frac{(f^k)'(x_{-k})}{(f^{n+k})'(x_{-n-k})}|$$

Let us consider the first term of the previous sum; it can be written as :

$$\log[\frac{G(f_1^{-n}z_{-k})}{G(f_1^{-n}x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}]$$

or as

$$\log[\frac{(G \circ f_1^{-n})(z_{-k})}{(G \circ f_1^{-n})(x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}].$$

Use $\frac{\text{dist}(z_{-k},x_{-k})}{r(D(x_{-k}))} \asymp \frac{1}{k} \to 0$ as $k \to \infty$ to apply Theorem 2.5 for $u = G \circ f_1^{-n}$, $v = G$ in the domain $D(x_{-k})$. We get that

$$\log[\frac{(G \circ f_1^{-n})(z_{-k})}{(G \circ f_1^{-n})(x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}] \le c\frac{1}{k^\eta} \to 0$$

uniformly in n and z.

The second term can be written as

$$\delta \log|\frac{(f_1^{-n})'(x_{-k}}{(f_1^{-n})'(z_{-k})}| \le c\frac{1}{k} \to 0$$

uniformly in z and n as f_1^{-n} are univalent functions and $\frac{\text{dist}(z_{-k},x_{-k})}{r(D(x_{-k}))} \asymp \frac{1}{k}$. This is enough to conclude that the limit exists.

Let us denote $v(z) = \lim_{n\to\infty} |(f^n)'(z_{-n})|^\delta\, G(z_{-n})$ for $x \in \Gamma$. As the limit in

Lemma 6.1 exists, we conclude that

$$v(fz) = v(z)|f'(z)|^{\delta} \text{ if } z, f(z) \in \Gamma. \tag{3.3.2}$$

The function v is not harmonic.

Let $\rho\Gamma$ denote a smaller cone with vertex at p.

Lemma 3.21 *In the cone $\rho\Gamma$ the following holds:*

$$c_{\rho}^{-1} \leq \frac{v(z)}{G(z)} \leq c_{\rho}$$

Moreover, for $x \in J \bigcap \Gamma$,

$$\frac{v(x)}{G(x)} = \lim_{\substack{z \to x \\ z \in \Gamma}} \frac{v(z)}{G(z)} = e^{-\gamma(x)} e^{\gamma(p)}.$$

Proof: It is in the spirit of the previous proof. The interested reader can find it in [PV].

Notation We are going to denote $v_p \overset{def}{=} v$ and we will call it "parabolic point automorphic function". We will introduce another function of this type and show that they are proportional.

Choose x_0 to be a repelling periodic point of f and let it be close to p such that

$$\mathrm{orb}(\mathrm{Crit}(f)) \bigcap D(x_0, |x_0 - p|) = \emptyset.$$

Let l be the period of x_0 and let $(f^l)'(x_0) = \lambda > 1$.

Let $F = f^l$ and F^{-n} be the holomorphic inverse branches that send x_0 to x_0. These functions are single-valued holomorphic functions in the whole disc $D(x_0, |x_0 - p|)$.

Lemma 3.22 *For all* $z \in D(x_0, |x_0 - p|) \setminus J$,

$$\exists \text{ the non zero and finite limit } \lim_{n\to\infty} |(F^n)'(F^{-n}z)|^\delta G(F^{-n}z).$$

Proof: Let us estimate $\log[b_{n+k}(z) : b_k(z)]$ where $b_n \stackrel{def}{=} |(F^n)'(F^{-n}z)|^\delta G(F^{-n}z)$.

Let $x \in J$ be the point closest to z. Then $z \in D(x_0, |x_0 - p|)$ and

$$\log \frac{b_{n+k}(z)}{b_k(z)} = \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + \log \frac{b_{n+k}(x)}{b_k(x)} =$$

$$= \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + \log \frac{G(F^{-n-k}x)}{G(F^{-k}x)} |(F^n)'(F^{-n-k}x)|^\delta =$$

$$= \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + (\gamma(F^{-n-k}x) - \gamma(F^{-k}x)).$$

The last term tends to zero when $k \to \infty$ uniformly in n and in $z \in \rho D(x_0, |x_0 - p|)$ for every $\rho < 1$.

The first term can be written as

$$\log[\frac{G(F^{-n-k}z)}{G(F^{-k}z)} : \frac{G(F^{-n-k}x)}{G(F^{-k}x)}] + \delta \log |\frac{(F^{n+k})'(F^{-n-k}z)}{(F^k)'(F^{-k}z)} : \frac{(F^{n+k})'(F^{-n-k}x)}{(F^k)'(F^{-k}x)}| \quad (3.3.3)$$

Let us rewrite it as:

$$\log[\frac{(G \circ F^{-n-k})(z)}{(G \circ F^{-k})(z)} : \frac{(G \circ F^{-n-k})(x)}{(G \circ F^{-k})(x)}] =$$

$$\log[\frac{(G \circ F^{-n})(F^{-k}z)}{G(F^{-k}z)} : \frac{(G \circ F^{-n})(F^{-k}x)}{G(F^{-k}x)}].$$

But if $z \in \rho D(x_0, |x_0 - p|)$ then

$$\frac{\text{dist}(F^{-k}z, F^{-k}x)}{|x_0 - p|} \le c_\rho \lambda^{-k} \quad (3.3.4)$$

Considering $u = G \circ F^{-n}$, $v = G$ in $D(x_0, |x_0 - p|)$, from the Holder continuity of $\log \frac{u}{v}$ we conclude that

$$|\log \frac{u(F^{-k}z)}{v(F^{-k}z)} : \frac{u(F^{-k}x)}{v(F^{-k}x)}| \leq c_1(\rho)\lambda^{-k\eta} \to 0 \text{ as } k \to \infty$$

uniformly in u, v, z when $z \in \rho D(x_0, |x_0 - p|)$. That means that the first term of (3.3.3) tends to zero when $k \to \infty$ uniformly in n and $z \in \rho D(x_0, |x_0 - p|)$.

The second term of (3.3.3) can be written as

$$\delta \log |\frac{(F^n)'(F^{-n-k}z)}{(F^n)'(F^{-n-k}x)}| = \delta \log |\frac{(F^{-n})'(F^{-k}x)}{(F^{-n})'(F^{-k}z)}|$$

Now (3.3.4) and Koebe's principle applied to the univalent function F^{-n} in $D(x_0, |x_0 - p|)$ show that

$$\delta \log |\frac{(F^{-n})'(F^{-k}x)}{(F^{-n})'(F^{-k}z)}| \leq c_2(\rho)\lambda^{-k} \to 0 \text{ as } k \to \infty \tag{3.3.5}$$

where the convergence is uniform in n and $z \in \rho D(x_0, |x_0 - p|)$.

Remark: Let us denote $v_{x_0}(z) = \lim_{n\to\infty} |(F^n)'(F^{-n}z)|^\delta G(F^{-n}z)$, for $z \in D(x_0, |x_0 - p|)$. Then

$$v_{x_0}(Fz) = v_{x_0}(z)|F'(z)|^\delta \text{ for } Fz \in D(x_0, |x_0 - p|). \tag{3.3.6}$$

The function v_{x_0} is NOT harmonic.

Lemma 3.23 *In the disc* $\rho D(x_0, |x_0 - p|)$,

$$c_3(\rho)^{-1} \leq \frac{v_{x_0}(z)}{G(z)} \leq c_3(\rho)$$

Moreover, for $x \in J \bigcap D(x_0, |x_0 - p|)$

$$\frac{v_{x_0}(x)}{G(x)} = \lim_{z\to x, z\in D} \frac{v_{x_0}(z)}{G(z)} = e^{-\gamma(x)} e^{\gamma(x_0)}$$

Proof:

$$\frac{v_{x_0}(z)}{G(z)} = \lim_{n\to\infty} [\frac{G(F^{-n}z)}{G(z)} : \frac{G(F^{-n}x_0)}{G(x_0)}][\frac{G(F^{-n}x_0)}{G(x_0)} |(F^n)'(x_0)|^\delta]|\frac{(F^{-n})'(x_0)}{(F^{-n})'(z)}|^\delta$$

The middle factor equals 1 by homology equation. The last factor is bounded independently of n and $z \in \rho D(x_0, |x_0 - p|)$, by Koebe's principle. The first factor is bounded independedly of n and $z \in \rho D(x_0, |x_0 - p|)$ by Theorem 2.5 for $u = G \circ F^{-n}$, $v = G$ in $D(x_0, |x_0 - p|)$.

Let z approach $x \in \rho D(x_0, |x_0 - p|)$, for some $\rho < 1$.

$$\frac{v_{x_0}(z)}{G(z)} = \lim_{n\to\infty} [\frac{G(F^{-n}z)}{G(z)} : \frac{G(F^{-n}x)}{G(x)}][\frac{G(F^{-n}x)}{G(x)} |(F^n)'(F^{-n}x)|^\delta]|\frac{(F^{-n})'(x)}{(F^{-n})'(z)}|^\delta$$

The middle factor equals $e^{-\gamma(x)} e^{\gamma(F^{-n}x)}$ and thus tends to $e^{-\gamma(x)} e^{\gamma(x_0)}$ uniformly in $z \in \rho D(x_0, |x_0 - p|)$ because $x_{-n} \in \rho D(x_0, |x_0 - p|)$ when $n \to \infty$.

The logarithms of the first and the last factor can be estimated uniformly in n by $c(\rho)|x - z|^\eta$ and by $c(\rho)|x - z|$ respectively, using Theorem 2.5 and Koebe's distortion. $\square$

Let us consider the following functions:

$$\beta_p(z) = \sum_{n\geq 1} [\log |f'(z_{-n})| - \log |f'(x_{-n})|]$$

where $z_{-n} = f_1^{-n} z$ and $x_{-n} = f_1^{-n} x_0$ for $z \in \Gamma$, and

$$\beta_{x_0}(z) = \sum_{n\geq 1} [\log |F'(F^{-n}z)| - \log |F'(x_0)|]$$

for $z \in D(x_0, |x_0 - p|)$. These are symmetric harmonic functions in the domains Γ, $D(x_0, |x_0 - p|)$, respectively.

Consider

$$h_p(z) \stackrel{def}{=} \int_{x_0}^{z} e^{-(\beta_p(z)+i\tilde{\beta}_p(z))}\, dz \tag{3.3.7}$$

and

$$h_{x_0}(z) \stackrel{def}{=} \int_{x_0}^{z} e^{-(\beta_{x_0}(z)+i\tilde{\beta}_{x_0}(z))}\, dz. \tag{3.3.8}$$

Due to the symmetry of $\beta_p(z)$ and $\beta_{x_0}(z)$ one can choose their complex conjugates such that h_p and h_{x_0} are real on the real axis.

Because

$$\beta_p(fz) - \beta_p(z) = \log|f'(z)| \quad \text{for } z, f(z) \in \Gamma$$

$$\beta_{x_o}(Fz) - \beta_{x_0}(z) = \log|F'(z)| - \log\lambda \quad \text{for } z, F(z) \in D(x_0, |x_0 - p|)$$

we have that:

$$\left|\frac{h_p'(fz)}{h_p'(z)}\right| = \frac{e^{-\beta_p(fz)}}{e^{-\beta_p(z)}} = \frac{1}{|f'(z)|} \quad \text{for } z, fz \in \Gamma \tag{3.3.9}$$

$$\left|\frac{h_{x_0}'(Fz)}{h_{x_0}'(z)}\right| = \frac{\lambda}{|F'(z)|} \quad \text{for } z, Fz \in D(x_0, |x_0 - p|) \tag{3.3.10}$$

Let us consider Koenig's function (or Fatou coordinates) Φ_{x_0}:

$$\Phi_{x_0}(z) = \lim_{n\to\infty} \frac{F^{-n}(z) - x_0}{(F^{-n})'(x_0)}$$

It is well defined in $D(x_0, |x_0 - p|)$. See [CG] for details.

We can see that $\Phi_{x_0}'(z) = h_{x_0}'(z)$ and that both Φ_{x_0} and h_{x_0} vanish at x_0. Therefore

$$\Phi_{x_0} \equiv h_{x_0}$$

Similarly, if

$$\Phi_p(z) = \lim_{n\to\infty} \frac{f_1^{-n}(z) - f_1^{-n}(x_0)}{(f_1^{-n})'(x_0)}$$

then $\Phi_p(x_0) = 0$, and $\Phi_p'(z) = h_p'(z)$. Therefore $\Phi_p \equiv h_p$. In particular

$$h_{x_0} \circ F(z) = \lambda\, h_{x_0}(z) \quad \text{for } z, Fz \in D(x_0, |x_0 - p|), \;\; \lambda > 1 \tag{3.3.11}$$

$$h_p \circ f(z) = h_p(z) + a \quad \text{for } z, fz \in \Gamma, \;\; a > 0 \tag{3.3.12}$$

$$h_p(x_0) = h_{x_0}(x_0) = 0, \;\; h_p(p) = \infty \tag{3.3.13}$$

Let us notice also that

$$v_p(z) = e^{\delta\beta_p(z)} \lim_{n\to\infty} G(z_{-n})|(f^n)'(x_{-n})|^\delta \quad \text{for } z \in \Gamma$$

$$v_{x_0}(z) = e^{\delta\beta_{x_0}(z)} \lim_{n\to\infty} G(F^{-n}z)|(F^n)'(x_0)|^\delta \quad \text{for } z \in D(x_0, |x_0 - p|)$$

In particular the limits in the right part exist. Let us denote them by $\tilde{\tau}_p(z)$ for $z \in \Gamma$, $\tilde{\tau}_{x_0}(z)$ for $z \in D(x_0, |x_0 - p|)$. These are positive harmonic functions defined in $\Gamma \setminus J$ and $D(x_0, |x_0 - p|) \setminus J$, respectively, and subharmonic in Γ and $D(x_0, |x_0 - p|)$ respectively, and vanishing on J.

By the first inequalities of Lemma 3.21 and Lemma 3.23, if $\rho < 1$, then:

$$\frac{\tilde{\tau}_p(z)}{G(z)} \asymp e^{-\delta\beta_p(z)} c_\rho \quad \text{for } z \in \rho\Gamma$$

$$\frac{\tilde{\tau}_{x_0}(z)}{G(z)} \asymp e^{-\delta\beta_{x_0}(z)} c_\rho \quad \text{for } z \in \rho D(x_0, |x_0 - p|)$$

Use the equalities in Lemma 3.21 and Lemma 3.23 to derive:

$$\frac{\tilde{\tau}_p(x)}{G(x)} = e^{-\delta\beta_p(x)} e^{-\gamma(x)} e^{\gamma(p)} \quad \text{for } x \in \rho\Gamma \bigcap J$$

$$\frac{\tilde{\tau}_{x_0}(x)}{G(x)} = e^{-\delta\beta_{x_0}(x)}e^{-\gamma(x)}e^{\gamma(x_0)} \quad \text{for } x \in \rho D(x_0, |x_0 - p|) \bigcap J$$

Denoting $\tau_p = \tilde{\tau}_p(z)e^{-\gamma(p)}$, $\tau_{x_0} = \tilde{\tau}_{x_0}(z)e^{-\gamma(x_0)}$ we conclude that

$$\frac{\tau_p(x)}{\tau_{x_0}(x)} = \left|\frac{h_p'(x)}{h_{x_0}'(x)}\right|^{\delta}$$

If we introduce

$$A(z) \stackrel{def}{=} (\frac{h_p'(z)}{h_{x_0}'(z)})^{\delta/2}$$

we see that A is a well defined holomorphic function in $\Gamma \bigcap D(x_0, |x_0 - p|)$ and

$$\frac{\tau_p(x)}{\tau_{x_0}(x)} = |A(x)|^2 \tag{3.3.14}$$

Applying Lemma 2.8 on harmonic rigidity (according to [A1] or [DU3] the Julia set has linear measure zero) and its corollary for the functions τ_p, τ_{x_0} that satisfy (3.3.14) in $\Gamma \bigcap D(x_0, |x_0 - p|)$, we get that the two functions are proportional. On the real axis

$$\frac{h_p'(x)}{h_{x_0}'(x)} = |\frac{h_p'(x)}{h_{x_0}'(x)}| = \text{ const } = |\frac{h_p'(x_0)}{h_{x_0}'(x_0)}| = \frac{e^{-\beta_p(x_0)}}{e^{-\beta_p(x_0)}} = 1$$

so actually $\tau_p(z) = \tau_{x_0}(z) \stackrel{def}{=} \tau(z)$ for $z \in \Gamma \bigcap D(x_0, |x_0 - p|)$.

So $h_p'(z) = h_{x_0}'(z)$ and $h_p(z) \equiv h_{x_0}(z)$ in $\Gamma \bigcap D(x_0, |x_0 - p|)$ because $h_p(x_0) = h_{x_0}(x_0) = 0$.

Let $h(z) \stackrel{def}{=} h_p(z) = h_{x_0}(z)$ for $z \in \Gamma \bigcap D(x_0, |x_0 - p|)$. The functions h_p and h_{x_0} being defined as Fatou coordinates are univalent and the following conjugations (linearizations) hold:

$$h(z) = \lambda h(F^{-1}z) \quad z, F^{-1}z \in D(x_0, |x_0 - p|)$$

$$h(z) = h(f_1^{-1}z) + a \quad a > 0, z, f_1^{-1}z \in \Gamma$$

$$h(x_0) = 0, h(p) = \infty \tag{3.3.15}$$

Consider the domain $\mathcal{O} = h(\Gamma \bigcap D(x_0, |x_0 - p|))$ and let $R(w) = \tau(h^{-1}w)$, $w \in \mathcal{O}$.

The map R is defined on $\mathcal{O}$ and

$$R(\lambda z) = \lambda^{\delta} R(z) \text{ for } z, \lambda z \in \mathcal{O}$$

$$R(z + a) = R(z) \quad \text{for } z, z + a \in \mathcal{O}$$

Extend the map R to the whole plane using the first equation and denote the extension by R_1. Extend the map R using the second equation and denote this extension by R_2. Being holomorphic extensions to C of the same function, R_1, R_2 are equal.

In particular the zero set of R is invariant under the group of transformations generated by $g_1(z) = \lambda z$, $g_2(z) = z + a$, which contains elements $g_n = z \pm \epsilon_n$, for some $\epsilon_n \to 0$. So as $R_{|h(J)} = 0$, then $R_{|h(J) \pm \epsilon_n} = 0$.

But this is impossible. Just take a point y to be one of the endpoints of a complementary interval of $h(J)$. Then $R(y - \epsilon_n)$ is positive starting with certain n. We come to a contradiction.

CHAPTER 4

Thermodynamical Formalism for Countable State Systems

4.1 Entropy

Let μ be a T invariant probability measure on $J_\star$. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_k \ldots\}$ be a finite or countable partition of $J_\star$. Denote by $\underline{Q} = Q_{i_1} \ldots, Q_{i_n}$ the set $Q_{i_1} \bigcap T^{-1} Q_{i_2} \bigcap \ldots \bigcap T^{-n+1} Q_{i_n}$ and by $\mathcal{W}_n(\mathcal{Q})$ or $\mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \ldots T^{-n+1}\mathcal{Q}$ the set of all such words.

Notation: Let $H_\mu(\mathcal{Q}) \stackrel{def}{=} \Sigma_n \mu(Q_k) \log \frac{1}{\mu(Q_k)}$.

Lemma 4.1 *For any partitions $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$,*

$$H_\mu(\mathcal{Q} \vee \tilde{\mathcal{Q}}) \leq H_\mu(\mathcal{Q}) + H_\mu(\tilde{\mathcal{Q}}).$$

Proof: See Lemma 1.17 in [Bo], page 28.

Lemma 4.2 *If $\mathcal{Q}$ is a finite partition of $J_\star$ then there exists the limit*

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \ldots \vee T^{-n+1}\mathcal{Q}) \stackrel{def}{=} H_\mu(T, \mathcal{Q}).$$

Proof: See Lemma 1.18 in [Bo], page 28.

Lemma 4.3 *If $\mathcal{Q}$ is a finite partition, then for any k,*

$$H_\mu(T, \mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \ldots \vee T^{-k+1}\mathcal{Q}) = H_\mu(T, \mathcal{Q}).$$

Proof: See Lemma 2.2 in [Bo], page 46.

Definition The entropy of the measure μ is defined by

$$h_\mu(T) \stackrel{def}{=} \sup H_\mu(T, \mathcal{Q}),$$

where the supremum is taken over all finite partitions $\mathcal{Q}$ of $J_\star$.

Note that the entropy may be equal to zero or infinity.

Lemma 4.4 *Let $\mathcal{B} = \{B_1 \ldots, B_m\}$ be a finite partition of $J_\star$ and let $\mathcal{D}_n$ be a sequence of partitions with diam($\mathcal{D}_n$) $\to 0$ as $n \to \infty$.*

Then there exists a sequence of partitions $\mathcal{E}_n$ of cardinal m, $\mathcal{E}_n = \{E_1^n, \ldots, E_m^n\}$ such that each E_i^n is the union of some elements of $\mathcal{D}_n$ and
$\lim_{n\to\infty} \mu(E_i^n \Delta B_i) = 0$ *for each $i = \overline{1, m}$.*

Proof: As in Lemma 1.23 from [Bo], page 33, except that the partitions $\mathcal{D}_n$ may be countable.

Lemma 4.5 *Let $\epsilon > 0$ and let $\mathcal{B}$ be a finite partition of $J_\star$.*

Then there exists $\delta_0 > 0$ such that for all partitions $\mathcal{D}$ with diam($\mathcal{D}$) $< \delta_0$ the following holds :

$$H_\mu(\mathcal{B} \vee \mathcal{D}) - H_\mu(\mathcal{D}) \leq \epsilon$$

Proof: As in Lemma 2.3 from [Bo], page 47.

Lemma 4.6 *Suppose $\mathcal{D}_n$ is a sequence of finite partitions with diam$(\mathcal{D}_n) \to 0$. Then*

$$h_\mu(T) = \lim_{n\to\infty} H_\mu(T, \mathcal{D}_n)$$

Proof: See Proposition 2.4 in [Bo], page 48.

Theorem 4.7 *Let μ be a T-invariant probability measure on $J_\star$ such that*

$$\sum_i \mu(C_i) \log \frac{1}{\mu(C_i)} < \infty,$$

where C_i are the standard cylinders from the end of section 3.1. Then the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-n+1}\mathcal{C})$$

and it is equal to $h_\mu(T)$.

Proof: By the T-invariance of μ , the sums $\Sigma_i \mu(T^{-k}C_i) \log \frac{1}{\mu(T^{-k}C_i)}$ are finite for each k, and the sequence $\frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-n+1}\mathcal{C})$ has a finite limit which we will denote by L.

To show that $h_\mu(T) \le L$, let $\mathcal{D}_n^0 = \{C_1, C_2, \ldots, C_n, \bigcup_{k\ge n+1} C_k\}$, and $\mathcal{D}_n = \mathcal{D}_n^0 \vee \ldots \vee T^{-n+1}\mathcal{D}_n^0$.

Then diam$\mathcal{D}_n \to 0$ and

$$\lim_{n\to\infty} h_\mu(T, \mathcal{D}_n^0) = \lim_{n\to\infty} h_\mu(T, \mathcal{D}_n) = h_\mu(T). \qquad (4.1.1)$$

Together with $H_\mu(T, \mathcal{D}_n^0 \vee \ldots \vee T^{-k+1}\mathcal{D}_n^0) \le H_\mu(T, \mathcal{C} \vee \ldots \vee T^{-k+1}\mathcal{C})$, it implies that $h_\mu(T, \mathcal{D}_n^0) \le L$ and therefore that $h_\mu(T) \le L$.

For the reversed inequality we will show that for any $\epsilon > 0$, $h_\mu(T) \ge L - \epsilon$.

Claim that if two partitions of $J_\star$, $\mathcal{B} = \{B_1, \ldots B_m, \ldots\}$ and

$\mathcal{F} = \{B_1, \dots B_m, J_\star \setminus \bigcup_{k=1}^{m} B_k\}$ satisfy

$$H_\mu(\mathcal{B}) \leq H_\mu(\mathcal{F}) + \epsilon \tag{4.1.2}$$

then for an arbitrary partition $\mathcal{A}$ of $J_\star$ the following holds:

$$H_\mu(\mathcal{B} \vee \mathcal{A}) \leq H_\mu(\mathcal{F} \vee \mathcal{A}) + \epsilon.$$

To prove the claim let $B = J_\star \setminus \bigcup_{k=1}^{m} B_k$. By (4.1.2) we get

$$\sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{1}{\mu(B_i)} - \mu(B) \log \frac{1}{\mu(B)} \leq \epsilon$$

and

$$\sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{\mu(B)}{\mu(B_i)} \leq \epsilon.$$

$$H_\mu(\mathcal{B} \vee \mathcal{A}) - H_\mu(\mathcal{F} \vee \mathcal{A}) = \sum_{i=m+1}^{\infty} \sum_{a} \mu(B_i \bigcap A_a) \log \frac{\mu(B \bigcap A_a)}{\mu(B_i \bigcap A_a)} =$$

$$= \sum_{i=m+1}^{\infty} \mu(B_i) \Sigma_a \frac{\mu(B_i \bigcap A_a)}{\mu(B_i)} \log \frac{\mu(B \bigcap A_a)}{\mu(B_i \bigcap A_a)}$$

By the concavity of the function $x \mapsto \log x$ we get :

$$H_\mu(\mathcal{B} \vee \mathcal{A}) - H_\mu(\mathcal{F} \vee \mathcal{A}) \leq \sum_{i=m+1}^{\infty} \mu(B_i) \log(\sum_{a} \frac{\mu(B \bigcap A_a)}{\mu(B_i)}) =$$

$$= \sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{\mu(B)}{\mu(B_i)} \leq \epsilon.$$

Therefore the claim is true.

Fix some $\epsilon > 0$. Then for all large enough n, the partitions $\mathcal{D}_n^0$ are fine enough to ensure that

$$H_\mu(\mathcal{C}) \leq H_\mu(\mathcal{D}_n^0) + \epsilon \quad \text{for all } n \geq n_\epsilon.$$

Therefore $H_\mu(T^{-k}\mathcal{C}) \leq H_\mu(T^{-k}\mathcal{D}_n^0) + \epsilon$. Applying the claim k times we get :

$$H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-k}\mathcal{C}) \leq H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-k}\mathcal{C}) + \epsilon \leq$$

$$H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \ldots \vee T^{-k}\mathcal{C}) + 2\epsilon \leq$$

$$\leq H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \ldots \vee T^{-k}\mathcal{D}_n^0) + k\epsilon$$

$$\Rightarrow \frac{1}{k} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-k}\mathcal{C}) \leq \frac{1}{k} H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \ldots \vee T^{-k}\mathcal{D}_n^0) + \epsilon$$

$$\Rightarrow L \leq h_\mu(\mathcal{D}_n^0) + \epsilon = h_\mu(\mathcal{D}_n) + \epsilon$$

$$\Rightarrow L \leq h_\mu(T) + \epsilon.$$ □

4.2 Pressure

We will define the pressure for potentials ϕ in the space of functions

$$\mathcal{H}'_\alpha \stackrel{def}{=} \{\phi : J_\star \to R\,;\phi \text{ is bounded from above, and } v_\alpha(\phi) < \infty\}.$$

For such functions ϕ define :

$$S_n\phi(\underline{A}) = \sup_{x \in \underline{A}} \sum_{k=0}^{n-1} \phi(T^k x) \text{ for a } n-\text{word } \underline{A} \in \mathcal{W}_n(\mathcal{C}) \tag{4.2.1}$$

and

$$Z_n(\phi) = \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \exp S_n\phi(\underline{A}) \tag{4.2.2}$$

Remark For functions $\phi \in \mathcal{H}'_\alpha$ the sequence $\frac{1}{n}\log Z_n(\phi)$ is decreasing . Define the pressure to be

$$P(\phi, \mathcal{C}) \stackrel{def}{=} P(\phi) \stackrel{def}{=} \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi).$$

It can be a real number or it can equal $-\infty$.

Theorem 4.8 *Let $\phi \in \mathcal{H}_\alpha'$ and let μ be a T-invariant probability measure on $J_\star$ satisfying $H_\mu(\mathcal{C}) < \infty$. Then*

$$h_\mu(T) + \int_{J_\star} \phi \, d\mu \;\leq\; P(\phi) \tag{4.2.3}$$

Proof: If $\int \phi d\mu = -\infty$ there is nothing to prove. Assume that $\int \phi d\mu$ is finite. We will show that

$$\frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} (\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \;+\; \int_{\underline{A}} S_n \phi d\mu) \;=$$

$$= \frac{1}{n} (\sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \;+\; \int_{J_\star} S_n \phi d\mu \,) \;=$$

$$= \frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-n+1}\mathcal{C}) + \int_{J_\star} \phi d\mu \;\leq P(\phi) \tag{4.2.4}$$

Let $z_{\underline{A}} \in \underline{A}$ satisfy : $\int_{\underline{A}} S_n \phi d\mu) \;\leq \mu(\underline{A}) S_n \phi(z_A)$. Then

$$\frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} (\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \;+\; \int_{\underline{A}} S_n \phi d\mu) \;\leq$$

$$\leq \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \mu(\underline{A}) \log \frac{\exp S_n \phi(z_A)}{\mu(\underline{A})} \;\leq$$

$$\leq \frac{1}{n} \log \Big(\sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \exp S_n \phi(z_A) \,\Big) \;\leq$$

$$\leq \frac{1}{n} \log \Big(\sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \sup_{\underline{A}} \exp S_n \phi(z) \,\Big) \;=\; \frac{1}{n} Z_n(\phi) \tag{4.2.5}$$

$$\Rightarrow\; h_\mu(T) + \int \phi \, d\mu \;\leq\; P(\phi)$$

Let $\delta = \mathrm{Hdim}(J)$.

Theorem 4.9 *The function $\phi = -\delta \log |T'|$ has $P(\phi) = 0$.*

Proof: Let $\underline{A} \in \mathcal{W}_m(\mathcal{C})$. The map $T^m : \underline{A} \to J_\star$ being one-to-one and onto, there is a natural identification between $\mathcal{W}_m(\mathcal{C})$ and the set $T^{-m}x$, given by $\underline{A} \in \mathcal{W}_m(\mathcal{C}) \longleftrightarrow x_{\underline{A}} = \underline{A} \bigcap T^{-m} J_\star$ for arbitrary points x in $J_\star$.

Fix such an x and let $x_{\underline{A}} = \underline{A} \bigcap T^{-m} J_\star$.

$$S_m\phi(\underline{A}) = \sup_{y \in \underline{A}} \sum_{i=0}^{m-1} \phi(T^i y)$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \sum_{i=0}^{m-1} \sup_{x,y \in \underline{A}} |\phi(T^i x) - \phi(T^i y)| \leq$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \parallel \phi \parallel_\alpha \sum_{i=0}^{m-1} e^{-(m-i)\beta} \leq$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \text{const} \tag{4.2.6}$$

Let Ψ be the fixed point of the Perron Frobenius operator, $\mathcal{L}_\phi \Psi = \Psi$ defined in Section 3.2. According to [DU3], it is bounded away from zero and infinity. By the inequality (4.2.6) we get :

$$S_m\phi(\underline{A}) \leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \text{const} + \log \Psi(x_{\underline{A}}) \quad \Rightarrow$$

$$\exp S_m\phi(\underline{A}) \leq c \exp(\sum_{x_{\underline{A}}} \phi(T^i x_{\underline{A}})) \Psi(x_{\underline{A}}) \quad \Rightarrow$$

$$Z_m(\phi) = \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \exp S_m\phi(\underline{A}) \leq$$

$$\leq c \sum_{x_{\underline{A}} \in T^{-m}x} \exp S_m\phi(x_{\underline{A}}) \Psi(x_{\underline{A}}) = c(\mathcal{L}^m \Psi)(x) = c\Psi(x).$$

The function Ψ being bounded (see Theorem 3.19), we get : $Z_m(\phi) \leq M_2$ and $P(\phi) = \lim_{m\to\infty} \frac{1}{m} Z_m(\phi) \leq 0$. The opposite inequality follows identically, using

$$\sup_{y\in\underline{A}}\sum_{i=0}^{m-1}\phi(T^i y) \;\geq\; \sum_{i=0}^{m-1}\phi(T^i x_{\underline{A}}).$$

Theorem 4.10 (Variational Principle) *Let $\phi = -\delta\log|T'|$. The probability measure μ equivalent to the δ-conformal measure is the unique T-invariant measure that satisfies:*

$$H_\lambda(T) + \int_{J_\star}\phi\, d\lambda \;=\; 0 \quad and \quad H_\lambda(\mathcal{C}) < \infty. \tag{4.2.7}$$

Proof: Let us prove uniqness first. Let λ be a T-invariant measure that satisfies (4.2.7) First let us assume that $\mu\perp\lambda$. Then there exists a Borel subset of $J_\star$, call it B, satisfying

$$\mu(B)=0,\ \lambda(B)=1, \quad \text{and } \ T^{-1}B=B \tag{4.2.8}$$

By Lemma 4.4 for the partitions $\mathcal{B} = \{B, J_\star\setminus B\}$ and $\mathcal{D}_m = \mathcal{D}^0_m \vee T^{-1}\mathcal{D}^0_m\vee\ldots\vee T^{-m+1}\mathcal{D}^0_m$ where $\mathcal{D}^0_m = \{C_1,\ldots C_m, \bigcup_{k=m+1}^{\infty} C_k\}$, we can find sets F_m that are union of standard cylinders and that satisfy

$$(\mu+\lambda)(B\Delta F_m)\to 0 \ \text{ as } \ m\to\infty \tag{4.2.9}$$

By Theorems 4.8 and 4.9 we get:

$$0 = P(\phi) = \inf\frac{1}{n}H_\lambda(\mathcal{C}\vee\ldots\vee T^{-n+1}\mathcal{C}) \;+\; \int_B \phi d\,\lambda \;\leq \tag{4.2.10}$$

$$\leq \frac{1}{n}(H_\lambda(\mathcal{C}\vee\ldots\vee T^{-n+1}\mathcal{C}) \;+\; \int_B S_n\phi d\,\lambda) \;\Rightarrow$$

$$0\leq \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})}(\lambda(\underline{A})\log\frac{1}{\lambda(\underline{A})} \;+\; \int_{\underline{A}\bigcap B} S_n\phi\, d\lambda)$$

By an argument similar to the one in Theorem 4.8, if $x\in J_\star$ is fixed and $x_{\underline{A}} =$

$\underline{A} \cap T^{-n}x$, then

$$\int_B S_n\phi d\lambda \leq \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} \lambda(\underline{A})[\, S_n\phi(x_{\underline{A}}) + \, \| \phi \|_\alpha] \;\leq$$

$$\leq \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} \lambda(\underline{A})\; S_n\phi(x_{\underline{A}}) \; + c_1 \;\Rightarrow$$

$$-c_1 + 1 \leq \sum_{\underline{A}\subset F_n} \lambda(\underline{A}) \; [\, \log \frac{1}{\lambda(\underline{A})} \; \exp S_n\phi(x_{\underline{A}}) \,] + \sum_{\underline{A}\cap F_n = \emptyset} \lambda(\underline{A}) \; [\, \log \frac{1}{\lambda(\underline{A})} + S_n\phi(x_{\underline{A}}) \,] \;\leq$$

$$\leq \lambda(F_n) \log \frac{\sum_{\underline{A}\subset F_n} \exp S_n\phi(x_{\underline{A}})}{\lambda(F_n)} \; + \; \lambda(J_\star \setminus F_n) \log \frac{\sum_{\underline{A}\cap F_n = \emptyset} \exp S_n\phi(x_{\underline{A}})}{\lambda(J_\star \setminus F_n)} \; =$$

$$\leq \lambda(F_n) \log [\, \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} e^{S_n\phi(x_{\underline{A}})} \; \chi_{F_n}(x_{\underline{A}}) \,] \, +$$

$$+ \, \lambda(J_\star \setminus F_n) \log [\, \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} e^{S_n\phi(x_{\underline{A}})} \; \chi_{J_\star \setminus F_n}(x_{\underline{A}}) \,] \; + \; 2 \; \sup_{[0,1]} x \log \frac{1}{x} \tag{4.2.11}$$

By $\phi \in \mathcal{H}_\alpha^{'}$ and by the boundedness of $\log \Psi$ we get that

$$m(\underline{A}) \asymp e^{S_n(x_{\underline{A}})} \int_{\underline{A}} e^{-S_n\phi(z)} \, dm(z) \; = \; e^{S_n(x_{\underline{A}})} \int_{\underline{A}} |(T^n)'|^h \, dm$$

Therefore

$$e^{S_n\phi(x_{\underline{A}})} \; \asymp \; m(\underline{A}) \; \asymp \; \mu(\underline{A}) \tag{4.2.12}$$

and (4.2.11) becomes:

$$-c_2 \; \leq \; \lambda(F_n) \log [\, \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} e^{\mu(\underline{A})} \; \chi_{F_n}(x_{\underline{A}}) \,] \, +$$

$$+ \, \lambda(J_\star \setminus F_n) \log [\, \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} e^{\mu(\underline{A})} \; \chi_{J_\star \setminus F_n}(x_{\underline{A}}) \,]$$

$$= \lambda(F_n) \log(\mu(F_n)) \; + \; \lambda(J_\star \setminus F_n) \log(\mu(J_\star \setminus F_n)) \tag{4.2.13}$$

By $\mu(F_n) \to \mu(B) = 0$ and $\lambda(F_n) \to \lambda(B) = 1$ as $n \to \infty$ the inequality (4.2.13) becomes $-c_2 \leq -\infty$.

Therefore λ and μ can not be orthogonal. In particular this gives that in the class of T-invariant ergodic probabilities there is at most one that satisfies (4.2.7). This will be enough for our purposes.

To prove uniqness for the general case, decompose the measure λ into $\lambda = \tilde{\lambda} + \tilde{\mu}$ where $\tilde{\lambda} \perp \mu$ and $\tilde{\mu} \ll \mu$ and follow the proof of Theorem 1.22, page 31 in [Bo].

In order to prove that the equality (4.2.7) actually holds for the measure μ it is enough to show that $h_\lambda(T) + \int \phi \, d\lambda \geq 0$.

Fix $n > 0$. We have:

$$\frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-n+1}\mathcal{C}) + \int_{J_*} \phi d\mu =$$

$$= \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n} [\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} + \int_{\underline{A}} S_n\phi \, d\mu] \geq$$

$$\geq \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n} [\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} + \mu(\underline{A}) \inf_{x \in \underline{A}} S_n\phi(x)] \tag{4.2.14}$$

Using one more time $\mu(\underline{A}) \asymp e^{S_n\phi(x_{\underline{A}})} = |(T^n)'(x_{\underline{A}})|^{-\delta}$ and the distortion properties of T it follows that

$$\log \frac{1}{\mu(\underline{A})} \geq \delta \sup_{x \in \underline{A}} (\sum_{i=0}^{n-1} \log |T'|(T^i x)) + c = -\inf S_n\phi(x) + c$$

and

$$h_\mu(T) + \int_{J_*} \phi \, d\mu \geq 0$$

which conclude the proof.

Theorem 4.11 *Let λ be a T-invariant measure with positive entropy satisfying*

$H_\lambda(\mathcal{C}) < \infty$. *Then*

$$Hdim(\lambda) \leq \frac{h_\lambda(T)}{\int_{J_\star} \log|T'|\, d\lambda} \tag{4.2.15}$$

Proof: Let $\alpha > \frac{h_\lambda(T)}{\int_{J_\star} \log|T'|\, d\lambda}$. We want to show that $Hdim(\lambda) \leq \alpha$. Let $\delta_0 > 0$ satisfy

$$\delta_o < \frac{\alpha}{\alpha_0} - 1 \quad \text{where } \alpha_0 \stackrel{def}{=} \frac{h_\lambda(T)}{\int_{J_\star} \log|T'|\, d\lambda} \tag{4.2.16}$$

Then there exists some n_δ such that for all $n > n_\delta$ the following holds:

$$\frac{1}{n} H_\lambda(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \ldots \vee T^{-n+1}\mathcal{C}) \leq (1+\delta_0) h_\lambda(T). \tag{4.2.17}$$

Let $\epsilon > 0$. Let $n > n_\epsilon$ be large such that $\text{diam}\underline{A} \leq \epsilon$ for all $\underline{A} \in \mathcal{W}_n(\mathcal{C})$.

Let $\mathcal{F}_n$ be the family of all n-generation cylinders having positive λ mass and let

$$X = \bigcap_{n>0} \bigcup_{\underline{A}\in\mathcal{F}_n} \underline{A}.$$

Then $\lambda(X) = 1$ and if

$$H_{\alpha,\epsilon}(X) \stackrel{def}{=} \inf\{\sum (\text{diam}B)^\alpha;\ \text{the sets B cover X and } \text{diam}(B) < \epsilon\ \}$$

we get :

$$H_{\alpha,\epsilon}(X) \leq \sum_{\underline{A}\in\mathcal{F}_n} (\text{diam}\underline{A})^\alpha \asymp \sum_{\underline{A}\in\mathcal{F}_n} \frac{1}{|(T^{n\prime}(x_{\underline{A}})|^\alpha} =$$

$$= \sum_{\underline{A}\in\mathcal{F}_n} e^{-\alpha S_n(\log|T'|)(x_{\underline{A}})} = \sum_{\underline{A}\in\mathcal{F}_n} \lambda(\underline{A}) e^{-\alpha S_n(\log|T'|)(x_{\underline{A}}) + \log 1/\lambda(\underline{A})} \leq$$

$$\leq \exp \sum_{\underline{A}\in\mathcal{F}_n} [-\alpha\lambda(\underline{A}) S_n(\log|T'|(x_{\underline{A}}) + \lambda(\underline{A}) \log 1/\lambda(\underline{A})\ ]$$

By Lemma 3.8

$$|S_n(\log|T'|)(x_{\underline{A}}) - S_n(\log|T'|(z))| = |\frac{(T^n)'(x_{\underline{A}})}{(T^n)'(z)}| \leq c_3 \quad \text{for all } z \in \underline{A} \tag{4.2.18}$$

Therefore

$$H_{\alpha,\epsilon}(X) \leq \exp\Big(\sum_{\underline{A} \in \mathcal{F}_n} [-\alpha \int_{\underline{A}} S_n(\log|T'|)\, d\lambda \;+\; \lambda(\underline{A}) \log \frac{1}{\lambda(\underline{A})}] \Big)\, e^{c_3} \;\leq$$

$$\leq c_4 \,(\exp[\frac{-\alpha}{n} \int_X S_n(\log|T'|)\, d\lambda \;+\; \frac{1}{n} \sum_{\underline{A}} \lambda(\underline{A}) \log \frac{1}{\lambda(\underline{A})}])^n \;\leq$$

$$\leq c_4 (\exp[-\alpha \int_X \log(|T'|)\, d\lambda \;+\; (1+\delta)h_\lambda]\,)^n$$

By (4.2.16)

$$-\alpha \int_X \log(|T'|)\, d\lambda \;+\; (1+\delta_0)h_\lambda \;=\; -\alpha \int_X \log(|T'|)\, d\lambda \;+\; (1+\delta_0)\alpha_0 \int_X \log(|T'|)\, d\lambda$$

$$= -[\alpha - \alpha_0(1+\delta_0)] \int \log|T'|\, d\lambda$$

The jump transformation being expanding and the way δ_0 was selected imply that $-[\alpha - \alpha_0(1+\delta_0)] \int \log|T'|\, d\lambda$ is negative, say equal to $-\delta_1$. We get

$$H_{\alpha,\epsilon}(X) \leq c_4\, e^{-n\delta_1} \;\text{ for all } n \geq n_\epsilon. \tag{4.2.19}$$

Therefore $\mathrm{Hdim}(\lambda) \leq \alpha$ for all $\alpha \geq \frac{h_\lambda}{\int \log|T'|\, d\lambda}$. This is enough to conclude that $\mathrm{Hdim}(\lambda) \leq \frac{h_\lambda}{\int \log|T'|\, d\lambda}$

Lemma 4.12 *If μ is the T-invariant measure equivalent to the δ conformal measure m, then*

$$\sum_{n>0} \mu(C_n) \log \frac{1}{\mu(C_n)} < \infty$$

Proof: Without loss of generality we can assume that $\inf J = 0$ and that $\sup J = 1$.

The measures m and μ being boundedly equivalent, it is enough to show that

$$\sum_{n>0} m(C_n) \log \frac{1}{m(C_n)} < \infty$$

Use the fact that $|T'|^\delta$ is the Jacobian of m to get that $\int_{C_n} |T'|^\delta dm = 1$. By

$$M_1^{-1} \leq |\frac{T'(x)}{T'(x')}| \leq M_1 \quad \text{for } x, x' \text{ in the same cylinder} \tag{4.2.20}$$

we get that $|T'(x)|^\delta \asymp \frac{1}{m(C_n)}$ for all $x \in C_n$.

On the other hand if the cylinder C_n is $C_n = [a_n, b_n] \bigcap J$, then

$$\int_{a_n}^{b_n} |T'(x)| dx = \int_{a_n}^{b_n} T'(x) dx = T(b_n) - T(a_n) = 1 - 0 = 1$$

so according to (4.2.20) we get $|T'(x)| \asymp \frac{1}{|C_n|}$ for $x \in C_n$. (The notation $|E|$ means the diameter of the set E.)

By construction $|C_n| = |f_1^{-K_n} I|$ for some interval I that is mapped by f univalently onto [0,1], where K_n is the integer part of $\frac{n}{deg(f)-1}$. Using the estimates in the Appendix we get

$$|T'(x)| \asymp \frac{1}{|C_n|} \asymp \frac{1}{n^2} \quad \text{for } x \in C_n$$

so $m(C_n) \asymp \frac{1}{n^{2\delta}}$. The measure m being finite we get that $1 \asymp \sum_{n>0} \frac{1}{n^{2\delta}}$, which implies that $2\delta > 1$. Therefore

$$\sum_{n>0} m(C_n) \log \frac{1}{m(C_n)} \leq \text{ const } + \sum_{n>0} \frac{1}{n^{2\delta}} \log \frac{1}{n^{2\delta}} < \infty$$

for $2\delta > 1$.

Lemma 4.13 *If ν is the T-invariant measure equivalent to the harmonic measure* *ω, then*

$$\sum_{n>0} \nu(C_n) \log \frac{1}{\nu(C_n)} < \infty$$

Proof: It is enough to show that $\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} < \infty$.

As in the previous proof we will use the fact that $|C_n| \asymp \frac{1}{n^2}$ and $|x| \asymp \frac{1}{n}$ for $x \in C_n$ to estimate the harmonic measure $\omega(C_n)$. Let $\inf C_n = a_n$, $\sup C_n = b_n$, $c_n = \frac{a_n+b_n}{2}$, $r_n = \frac{b_n-a_n}{2}$

Let us consider the auxiliary domain Ω_n contained in $C\backslash[0,\infty)$ whose boundary coincides with

$$\partial\Omega_n \bigcap B(c_n, r_n) = J \bigcap B(c_n, r_n)$$

$$\partial\Omega_n \bigcap ([0,\infty) \setminus B(c_n, r_n)) = [0,\infty) \setminus B(c_n, r_n) \tag{4.2.21}$$

Let ω_{Ω_n} be the harmonic measure in the domain Ω_n. It is clear that

$$\omega(C_n, \cdot) \geq \omega_{\Omega_n}(C_n, \cdot) \tag{4.2.22}$$

The Julia set J being uniformly perfect gives

$$\omega_{\Omega_n}(J \bigcap B(c_n, r_n), y) \geq c(\beta) \quad \text{for } \forall y \in B(c_n, \frac{1}{2} r_n)$$

for some constant $c(\beta)$ depending only on the constant β from the definition of uniformly perfectness of J. In particular if w denotes the harmonic function in $C \setminus [0,\infty)$ that vanishes on $[0,\infty)\backslash B(c_n, \frac{1}{2}r_n)$ and which equals 1 on $[p,\infty) \cap B(c_n, \frac{1}{2}r_n)$ we conclude that for every $y \in C$ we have:

$$\omega_{\Omega_n}(J \bigcap B(c_n, r_n), y) \geq c(\beta)\, w(y) \tag{4.2.23}$$

The inequality (4.2.22) becomes

$$\omega(C_n, \infty) \geq \omega_{\Omega_n}(C_n, \infty) \geq c(\beta)\, \omega(\infty) \qquad (4.2.24)$$

Using the change of coordinates $z \longrightarrow z^2$ the upper half plane is mapped onto the domain $C \setminus [0, \infty)$ so one can easily estimate

$$w(\infty) \geq c r_n c_n^{-1/2} \asymp n^{-3/2}$$

Finally this computation together with (4.2.24) gives

$$\omega(C_n) \geq c n^{-3/2} \geq c' |x|^{3/2} \ \ \forall x \in C_n \qquad (4.2.25)$$

which implies:

$$\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} \leq \sum_{n>0} \omega(C_n) \inf_{x \in C_n} \log \frac{1}{|x|^{3/2}} + c.$$

So

$$\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} \leq c'' \int_J \log \frac{1}{|x|} d\omega(x) + c' = c'' \mathcal{U}_\omega(0) + c'$$

where $\mathcal{U}_\omega$ is the potential of the harmonic measure. The Julia set being uniformly perfect, it is regular for the Dirichlet problem; as a consequence the potential $\mathcal{U}_\omega$ is finite everywhere on J. In particular $\mathcal{U}_\omega(0) < \infty$.

Applying Theorems 4.8, 4.10, 4.11 we get that if the invariant harmonic measure and the invariant conformal measure are orthogonal, then the dimension of the harmonic measure is less than the dimension of the Julia set.

CHAPTER 5

Appendix

The next proposition is a particular case of Proposition 8.3 of [ADU], where all necessary references are made to restore its proof. But we decided to include the proof here to make the thesis as self contained as possible for the convenience of the reader.

Proposition Let f be a rational function with parabolic point $p = 0$ whose Julia set J_f is contained in R and whose only Fatou component is the parabolic basin.

Let f_1^{-1} denote the inverse branch of f that sends the parabolic point to itself. Then for any closed interval $I \subset J_f$ close to 0

$$|f_1^{-n}I| \asymp \frac{1}{n^2} \text{ and } \forall x \in f_1^{-n}I \;\; |x| \asymp \frac{1}{n} \tag{5.0.1}$$

Proof: It is easier to carry out the estimates if we change the coordinates such that the parabolic point becomes ∞. If we denote the new function g without loss of generality we can assume that $(0, \infty) \subset F_g$, $0 \in J_g$ and that the expansion of g near infinity is

$$g(z) = z + 1 + \frac{A}{z} + \frac{B}{z^2} + \dots$$

or simply $g(z) = z + 1 + \frac{A}{z} + \theta_1(z) = z + 1 + \frac{A}{z} + \frac{B}{z^2} + \theta_1(z)$ where $|\theta_1(z)| \le M_1 \frac{1}{z^2}$ and $|\theta_2(z)| \le M_2 \frac{1}{z^3}$ for large values of z.

Let g_1^{-1} denote the inverse branch of g that sends infinity to itself. In this setting the statement we have to prove is: if $I \subset (-\infty, -R]$ for some large R, then

$$|g_1^{-n} I| \asymp 1 \quad \text{and} \quad \forall x \in g_1^{-n} I \;\; |x| \asymp n \tag{5.0.2}$$

We will start by proving the second estimate. For $z \in (-\infty, -R]$ let $g_1^{-1}(z) = w$. We get

$$w + 1 + \frac{A}{w} + \theta_1(w) = z \Rightarrow \; w^2 + w(z-1) + A = w\theta_1(w) \in [-1, 1]$$

Solving the double inequality we get that

$$w \le \frac{z - 1 - \sqrt{(z-1)^2 - 4(A-1)}}{2} \text{ and } w \ge \frac{z - 1 - \sqrt{(z-1)^2 - 4(A+1)}}{2}$$

We will show that $\frac{z-1-\sqrt{(z-1)^2-4(A-1)}}{2} \le z - \frac{1}{2}$ and $\frac{z-1-\sqrt{(z-1)^2-4(A+1)}}{2} \ge x - \frac{3}{2}$. These inequalities are equivalent to :

$$4(A-1) - 1 \le -2z \quad \text{and} \quad -2z \ge 1 - 4(A+1)$$

which are true for $z \in (-\infty, -R]$. Applying n times $z - \frac{3}{2} \le g_1^{-1}(z) \le z - \frac{1}{2}$ we get

$$x - \frac{3n}{2} \le g_1^{-n}(z) \le x - \frac{n}{2} \tag{5.0.3}$$

To get the first estimate of (5.0.2), let $I = [-x_0 - a_0, -x_0]$ and $g_1^{-n} I = [-x_n - a_n, -x_n]$. We have that:

$$g(-x_{n+1} - a_{n+1}) = -x_n - a_n = g(-x_{n+1}) - a_n$$

$$x_{n+1}+a_{n+1}-1+\frac{A}{x_{n+1}+a_{n+1}}-\frac{B}{(x_{n+1}+a_{n+1})^2}-\theta_3(x_{n+1}+a_{n+1})=$$

$$=a_n+x_{n+1}-1+\frac{A}{x_{n+1}}-\frac{B}{x_{n+1}^2}-\theta_3(x_{n+1})$$

$$\Rightarrow \quad a_{n+1}[1-\frac{A}{x_{n+1}(x_{n+1}+a_{n+1})}-\frac{B(2x_{n+1}+a_{n+1})}{x_{n+1}^2(x_{n+1}+a_{n+1})^2}]\geq a_n-\frac{B'}{x_{n+1}^3}$$

Using the inequality (5.0.3) we get that

$$a_n+\frac{B''}{n^3}\geq a_{n+1}[1+\frac{A(n)}{n^2}]\geq a_n-\frac{B''}{n^3} \tag{5.0.4}$$

for some positive constant B'' and for some $A(n)$ bounded away from zero and infinity. Let $c_n=\alpha\prod_{k=1}^{n-1}\frac{1}{1+A(k)/k^2}$ where α small enough to guarantee $a_i\geq\frac{c_i}{\sqrt{i}}$ for $i=1,\dots n_0$. Notice that the product that defines c_n is bounded away from zero. We will prove by induction that $a_n\geq\frac{c_n}{\sqrt{n}}$.

Assume the inequality holds for n, then by (5.0.4)

$$a_{n+1}\geq\frac{\frac{c_n}{\sqrt{n}}-\frac{B''}{n^3}}{1+\frac{A(n)}{n^2}}$$

It is enough to show that the last ratio is greater than $\frac{c_{n+1}}{\sqrt{n+1}}$. This is equivalent to :

$$\frac{1}{\sqrt{n}}-\frac{B''}{c_n n^3}\geq\frac{1}{\sqrt{n+1}}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{n(n+1)(\sqrt{n}+\sqrt{n+1})}\geq\frac{B''}{c_n n^3}$$

which is true for $n\geq n_0$ if n_0 was selected large enough.

Use $a_n\geq\frac{c_n}{\sqrt{n}}$ in the inequality (5.0.4) to get

$$a_{n+1}[1+\frac{A(n)}{n^2}]\geq a_n-\frac{c_0 a_n B''}{n^{2.5}}$$

therefore

$$a_{n+1} \geq a_n[1 - \frac{A'}{n^2} - \frac{c_0 B''}{n^{2.5}}].$$

This defines a convergent product, so $\inf a_n > 0$.

From the first inequality of (5.0.4) we immediately get $\sup a_n < \infty$.

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