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ABSTRACT

LONG MEMORY AND ASYMMETRY IN CONDITIONAL VARIANCE MODELS

By

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The dissertation introduces a new family of models for the conditional variance of economic time series. The new models allow for both asymmetries and long memory, whereas previous models had allowed for one or the other but not both. These models are applied to two different kinds of data, on stock returns and exchange rates. In each case, there is strong evidence of both asymmetry and long memory, and correspondingly the new models fit the data better than other simpler models.

Dedicated to my parents

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Any remaining errors are of course my own responsibility.

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Introduction

This dissertation proposes new models for the conditional variance of an economic time series. The first conditional variance model was the ARCH model of Engle (1982), which was followed by the GARCH model of Bollerslev (1986) and a large number of other models. These models are surveyed in Bollerslev, Engle and Nelson (1994).

Chapter 2 provides a general discussion of ARCH and GARCH models. It focuses on two distinct strands of this literature. First, in some empirical applications there is evidence of long memory in variance, in the sense that volatility is persistent. Standard ARCH, GARCH and related models cannot deal satisfactorily with long memory. The FIGARCH (Fractionally Integrated GARCH) model of Baillie, Bollerslev, and Mikkelsen (1996) was the first conditional variance model to allow long memory in variance. It was constructed by analogy to models of fractionally integration (long memory) in mean, which date back to Granger (1980) and Hosking (1981). Second, in some empirical applications there is evidence of asymmetry, which is usually taken to mean that negative innovations imply a different effect on variance than positive innovations of equal magnitude. A comprehensive treatment of asymmetry is given by Hentschel (1995), who defines the FGARCH (Family GARCH) models, a set of models that add parameters to represent asymmetries and also different power transformations in the basic GARCH equation. This family includes most previous models as special cases, but it does not allow for long memory. The main contribution of Chapter 2 is to combine these two strands of the literature.

We define FIFGARCH (Fractionally Integrated Family GARCH) models that basically combine the FGARCH models of Hentschel with the FIGARCH model of Baillie *et al.*, so as to allow simultaneously for both asymmetry and long memory in variance.

Chapter 2 also derives analytical results for the autocorrelations of the squared errors and of the conditional variances, for the special case of the asymmetric GARCH(1,1) model. These results generalize results of Ding and Granger (1996B), who derived the autocorrelations of the squared errors for the symmetric GARCH(1,1) model.

In Chapter 3 we apply the FIFGARCH model to data on daily stock returns. We use a very long data set of 17,582 daily returns from January 3, 1928 to September 30, 1993. With this abundance of data, there is hope of supporting a fairly intricate model. Preliminary analyses reveal evidence of both asymmetry and long memory, so the FIFGARCH model is a reasonable choice for these data. The model was consistently found to be better that other simpler models, according to standard statistical criteria including the value of the likelihood function, measures of the accuracy of prediction of squared errors, and closeness of the sample and theoretical correlations of squared errors. Simple models are also convincingly rejected by likelihood ratio tests. Thus we regard our application of the FIFGARCH model to these data as successful.

Chapter 4 is similar to Chapter 3, but now the FIFGARCH model is applied to daily data on exchange rate returns. For the case of the DM/$ exchange rate, there is evidence of long memory but not of asymmetry, while for the case of Yen/$ exchange rate, there is evidence of both asymmetry and long memory. A restricted version of the FIFGARCH model, which equates certain exponents in the power-transformed GARCH equation, fits the data well, and is superior to other simpler models.

The research in this dissertation could be continued in several ways. Further empirical work will be needed to understand how widely applicable the new models suggested here are. Theoretical research is also needed to establish rigorously the asymptotic properties of the estimates and inferences based on quasi-maximum likelihood estimation. We have followed much of the literature in simply assuming that the usual asymptotic properties of the quasi-MLE's apply. While there is no specific reason to doubt that this so, a rigorous investigation of this equation is called for, and remains to be done.

CHAPTER 2

CHAPTER 2

New family of fractionally integrated volatility models

I. Introduction

This chapter makes two contributions. The first is to propose a family of asymmetric, long-memory models for conditional variances. The second is to provide results on the correlations of squared observations and of the conditional variance for symmetric and asymmetric GARCH models.

There has recently been a large amount of econometric work on long-memory, fractionally integrated processes. These processes are associated with hyperbolically decaying impulse response weights, and therefore with long-memory persistence of shocks and of autocorrelations. They have been applied both to the level (mean) and to the volatility (variance) of economic time series. There is substantial evidence that long memory processes can be used to describe financial or macroeconomic data such as excess returns, inflation rates, forward premiums, interest rate differentials and exchange rates. Most recently, long memory models have been applied to the volatility of asset prices and to power transformations of returns. Specifically, the FIGARCH model of Baillie *et al.* (1996) allows for fractional integration of the conditional variance and thus provides a useful model for series for which the conditional variance is very persistent.

Another recent development has been the development of asymmetric models for conditional variances. There has long been evidence of asymmetries in financial data; for example, negative returns may have a different effect on volatility than positive returns of equal magnitude. Hentschel (1995) has defined a family of GARCH models that allow for such asymmetries. However, his models do not allow for long memory. This chapter defines models that allow for both long memory and asymmetry, thus joining two strands of the literature that had previously been largely separate. In a recent paper, McCurdy and Michaud (1997) combined the FIGARCH model with the asymmetric power ARCH model of Ding, Granger and Engle (1993). The basic idea is very similar to the idea of this chapter, but our models are more general than theirs, because Hentschel's model is more general than the model of Ding, Granger and Engle.

The second contribution of this chapter is to derive expressions for the autocorrelations of squared observations and conditional variances from symmetric and asymmetric GARCH models. Ding and Granger (1996B) have given the autocorrelations for the squared observations for the symmetric GARCH(1,1) model. In this chapter we provide similar expressions for the case of asymmetric GARCH. Note that, in discussing the notion of persistence in models of this type, one could focus on the degree of persistence either in the squared errors or in the conditional variance itself. The conditional variance is a random variable and it is perfectly reasonable to consider its autocorrelations. We derive expressions for the autocorrelations of the conditional variance for symmetric and asymmetric GARCH models.

The plan for the rest of the chapter is as follows. Section 2 establishes notation and presents a generic model of conditional heteroskedasticity. Section 3 discusses specific models, and proposes the *FIFGARCH* model, which allows for asymmetry and long memory in conditional variance. Section 4 presents results for the correlations of squared errors and conditional variances. These results are derived in detail in Appendices 1-3. The final section provides a brief review.

II. Basic model

Let y_t be an observed series. We specify its first two conditional moments:

$$E(y_t|\Omega_{t-1}) = g(\Omega_{t-1}, \theta_1), \tag{1}$$

$$\varepsilon_t \equiv y_t - E(y_t|\Omega_{t-1}) = y_t - g(\Omega_{t-1}, \theta_1), \tag{2}$$

$$VAR(y_t|\Omega_{t-1}) = E\left(\varepsilon_t^2\middle|\Omega_{t-1}\right) \equiv \sigma_t^2 = h(\Omega_{t-1}, \theta_2), \tag{3}$$

where Ω_{t-1} is the set of information available at time $t-1$, and θ_1 and θ_2 are sets of unknown parameters to be estimated. It is often assumed that $g(\Omega_{t-1}, \theta_1) = x_{t-1}\beta$; i.e. linearity is usually found adequate, assuming $x_{t-1} \in \Omega_{t-1}$. Different models correspond to different functional forms of g and h.

For constructing likelihoods, or predictions of anything other than mean and variance, we must assume a distribution for ε_t. We can write

$$\varepsilon_t = \sigma_t \omega_t, \quad \omega_t \sim i.i.d. \quad D(0,1), \tag{4}$$

where $D(0,1)$ represents some specific distribution with mean zero and variance one. Examples include normal, student's t, lognormal, or more flexible distributions. Although models of this form generate fat-tails in the unconditional distribution even under conditional normality, they do not fully account for excess-kurtosis present in many financial data. The student t-distribution with the number of degrees of freedom to be estimated has been used by several authors such as Bollerslev (1987) and Baillie and DeGennaro (1990). Other densities which have been used are the normal-Poisson mixture distribution of Jorion (1988) and Nieuwland *et al.* (1991), the normal-lognormal mixture distribution of Hsieh (1989), the generalized error distribution of Nelson (1991), the Bernoulli-normal mixture of Vlaar and Palm (1993), the power exponential of Baillie and Bollerslev (1989), and the stable distribution of De Vries (1991). The more flexible distributions include the stable, Pearson, generalized beta, exponential generalized beta of the second kind, and generalized t families of distributions. Each includes many common distributions as special cases.

So called "ARCH-M models", in which the conditional variance affects the mean of the series, can be specified as follows:

$$g(\Omega_{t-1}, \theta_1) = x_{t-1}\beta + \Upsilon\sigma_t^2. \tag{5}$$

Engle, Lilien and Robins (1987) introduced the ARCH in mean (ARCH-M) model in which the conditional mean is a function of the conditional variance, and the conditional variance follows an ARCH process. This model generalizes easily to more complicated models for σ_t^2. It arises in a natural way in mean-variance analysis where $\Upsilon\sigma_t^2$ could denote the risk premium for some asset with σ_t^2 being a measure of risk. Pagan and Ullah (1988) refer to these models as models with risk returns. For the usual ARCH model, the information matrix is block diagonal, with blocks for the mean and variance parameters. Therefore the regression coefficients and the ARCH parameters can be estimated separately without loss of asymptotic efficiency. Also, their variances can be obtained separately. These results do not hold for the ARCH-M model as the parameters of the conditional variance process affect the conditional mean of the series.

Given the density of ω, the likelihood can be formed. Suppose that the density of ω_t is $d(\omega_t)$. Then the density of ε_t is $\sigma_t^{-1} d\left(\sigma_t^{-1}\varepsilon_t\right)$ and the density of y_t is $\sigma_t^{-1} d\left(\sigma_t^{-1}(y_t - \mu_t)\right)$; as before, we assume $\mu_t = g(\Omega_{t-1}, \theta_1)$ and $\sigma_t^2 = h(\Omega_{t-1}, \theta_2)$. So the log likelihood function is

$$\ln L = -\frac{1}{2} \sum_{t=1}^{T} \ln\sigma_t^2 + \sum_{t=1}^{T} \ln d\left(\sigma_t^{-1}(y_t - \mu_t)\right). \tag{6}$$

The maximum likelihood estimates (MLEs) of θ_1 and θ_2 are then typically found by numerical maximization.

Consistency of the MLEs generally requires that the density of ω_t be specified correctly. An important exception is the assumption of normality. The (quasi) MLE obtained by maximizing the normal log likelihood is consistent even if the normality assumption is violated, as long as μ_t and σ_t^2 are correctly specified. This point is discussed in more detail in section II of chapter 3.

III. Specific models for σ_t^2

1. ARCH process

Engle (1983) that considers the discrete ARCH process, $\{\varepsilon_t\}$.

$$\sigma_t^2 = k + \alpha(L)\varepsilon_t^2 \tag{7}$$

$$\varepsilon_t = \omega_t \sigma_t \tag{8}$$

where ω_t is $iid\ (0, 1)$, $E_{t-1}(\varepsilon_t/\sigma_t) = 0$, $VAR_{t-1}(\varepsilon_t/\sigma_t) = 1$, L denotes the lag or backward shift operator, $\alpha(L) \equiv \alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^p$, σ_t is a positive time-varying and measurable function with respect to the information set available at time $t-1$, and $E_{t-1}(\ldots)$ and $VAR_{t-1}(\ldots)$ refer to the conditional expectation and variance with respect to this same information set. $\{\varepsilon_t\}$ is serially uncorrelated with mean zero, but the conditional variance of the process, σ_t^2, is changing over time.

2. GARCH process

The symmetric GARCH(p,q) specification of Bollerslev (1986) added flexibility to the ARCH(p) model of Engle(1983). The model is defined by

$$\sigma_t^2 = k + \alpha(L)\varepsilon_t^2 + \delta(L)\sigma_t^2 \quad (9)$$

where $\alpha(L) \equiv \alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^q$, and $\delta(L) \equiv \delta_1 L + \delta_2 L^2 + \ldots + \delta_p L^p$. An important special case is the GARCH(1,1) model in which $p = q = 1$, so that

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2 .$$

A main attraction of the GARCH model is that low-order models, like the (1,1) model, have often been found to be empirically adequate.

3. FGARCH process

There have been a large number of efforts to study asymmetries in ARCH and GARCH models. In the standard (symmetric) ARCH and GARCH models, only squared values of ε effect the conditional variance, so the sign of ε is unimportant. Models that allow negative errors to have different effects than positive errors will be called *asymmet-*

ric models. Examples include Pagan and Schwert (1990), Campbell and Hentschel (1992), Nelson (1991), Zakoïan (1994), Rabemananjara and Zakoïan (1993), Ding *et al.* (1993), Glosten, Jagannathan, and Runkle (1993), Harvey *et al.* (1994), Harvey and Shephard (1993), Hentschel (1995), and Fornari and Mele (1997). Negative equity returns are thought to be followed by larger increases in volatility than equally large positive returns, due to leverage effects. The economic explanation for this asymmetry, given by Black (1976), is that negative excess returns make the equity value less, increasing the leverage ratio of a given firm, thus raising its riskiness and the future volatility of its assets. This is called the leverage effect. For example, models such as exponential GARCH of Nelson (1991), quadratic GARCH of Sentana (1991) and Engle (1990), and threshold GARCH of Zakoïan (1994) allow for asymmetry. Volatility switching was added by Fornari and Mele (1997) to the sign switching developed by Granger and Teräsvirta (1993). The latter allows the drift term in the GARCH equation to change according to the sign of previous shocks, while the former captures asymmetries via the impact of past shocks on the level of the volatility.

A systematic attempt to capture asymmetry in the GARCH model is given by Hentschel (1995). He defined a family of asymmetric GARCH models (Family GARCH, or *FGARCH*) by allowing functions of ε_t other than ε_t^2 in the GARCH equation, and by considering power transformations. The FGARCH model is given by

$$\sigma_t^\lambda = \kappa + \alpha\sigma_{t-1}^\lambda f^\nu(\varepsilon_{t-1}) + \delta\sigma_{t-1}^\lambda , \tag{10}$$

$$f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right), \ |c| \leq 1 . \quad (11)$$

where eq. (11) is the news impact curve introduced by Pagan and Schwert (1990). Here $f^{\nu}(\varepsilon_{t-1}) = [f(\varepsilon_{t-1})]^{\nu}$, and λ and ν are parameters. Equation (10) essentially gives a Box-Cox transformation of the GARCH equation. The usual GARCH(1,1) model corresponds to $b = c = 0$ and $\lambda = \nu = 2$. Many other models in the literature are special cases of the FGARCH model. Table 1 lists some of these, along with the corresponding restrictions on b, c, ν, λ. Exponential GARCH of Nelson (1990), Threshold or Absolute GARCH of Zakoïan (1990), symmetric GARCH of Bollerslev (1986), Absolute Power GARCH of Engle and Ng (1993), and Family GARCH of Hentschel (1995) are representative examples. The asymmetry of eq. (11) is displayed in Figures 1 and 2.

4. FIGARCH process

Baillie *et al.* (1996) proposed the symmetric long memory fractionally integrated GARCH model for the long memory of the squared innovations. The ARMA(m,p) representation of ε_t^2 for the GARCH(p,q) process is:

$$(1 - \alpha(L) - \delta(L))\varepsilon_t^2 = \kappa + (1 - \delta(L))\left(\varepsilon_t^2 - \sigma_t^2\right) \quad (12)$$

where $m \equiv max\{p, q\}$, and $v_t \equiv \varepsilon_t^2 - \sigma_t^2$ is mean zero and serially uncorrelated. The fractionally integrated GARCH, *FIGARCH*, is defined by introducing the fractional differencing operator into the AR polynomial. Thus we obtain:

$$\phi(L)(1-L)^d \varepsilon_t^2 = \kappa + (1 - \delta(L))\left(\varepsilon_t^2 - \sigma_t^2\right), \qquad (13)$$

where $0 < d < 1$, and $\phi(L)$ and $\delta(L)$ are polynomials in the lag operator of orders *p* and *q* respectively. The fractional differencing operator, $(1-L)^d$, has a binomial expansion which is most conveniently expressed in terms of the hypergeometric function,

$$(1-L)^d = F(-d, 1, 1; L)$$

$$= \sum_{k=0}^{\infty} \Gamma(k-d)\Gamma(k+1)^{-1}\Gamma(-d)^{-1}L^k$$

$$= \sum_{k=0}^{\infty} \pi_k L^k, \qquad (14)$$

where Γ (.) denotes the Gamma function. The GARCH equation (9) for the FIGARCH(*p*,*d*,*q*) model is rewritten as:

$$[1-\delta(L)]\sigma_t^2 = \kappa + [1-\delta(L)-\phi(L)(1-L)^d]\varepsilon_t^2. \tag{15}$$

Thus, the conditional variance of ε_t can be expressed as follows:

$$\sigma_t^2 = \frac{\kappa}{1-\delta(1)} + [1 - \frac{\phi(L)(1-L)^d}{1-\delta(L)}]\varepsilon_t^2$$

$$= \frac{\kappa}{1-\delta(1)} + \lambda(L)\varepsilon_t^2, \tag{16}$$

where $\lambda(L) = \lambda_1 L + \lambda_2 L^2 + \dots$. We call this the reduced form or infinite ARCH version. It should be noted that the coefficients λ_k decay hyperbolically (λ_k is proportional to k^{d-1} for large k) rather than exponentially, as is true for the usual GARCH process. This slow decay generates long memory in σ_t^2. For the FIGARCH(p,d,q) model in eq. (15) to be well-defined and for the conditional variance to be positive almost surely for all t, all the coefficients in the infinite ARCH representation in equation (16) must be nonnegative. As for the GARCH(p, q) process analyzed by Nelson and Cao (1992), generalized conditions to ensure nonnegativity of all the lag coefficients in $\lambda(L)$ have proven elusive. Sufficient conditions are fairly easy to establish for low-order special cases.

The FIGARCH(p,d,q) process is strictly stationary and ergodic for $0 < d \leq 1$ with the roots of $\phi(L)$ and $\delta(L)$ outside the unit circle. Even though the cumulative impulse response function converges to zero for $0 \leq d < 1$, the fractional differencing parameter provides important information regarding the pattern and the speed with which shocks to the volatility process are propagated.

In most practical applications relatively simple low-order models such as GARCH(1,1) or GARCH(1,2) have often been found to be adequate. Similarly, the FIGARCH(1,d, 1) model may often be adequate to capture long memory in variance. The GARCH(1,1) model,

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2, \qquad (17)$$

is rewritten in ARMA(1,1) form as

$$(1 - \alpha L - \delta L)\varepsilon_t^2 = \kappa + (1 - \delta L)\left(\varepsilon_t^2 - \sigma_t^2\right). \qquad (18)$$

Similarly, the FIGARCH(1,d,1) is written as

$$\sigma_t^2 = \frac{\kappa}{1-\delta} + [1 - \frac{(1-\phi L)(1-L)^d}{1-\delta L}]\varepsilon_t^2, \qquad (19)$$

where $0 < d < 1$.

Under the assumption of conditional normality, the Maximum Likelihood Estimates (MLEs) for the parameters of the FIGARCH(p,d,q) process based on the sample $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T\}$ may be obtained by maximizing the expression

$$LogL(\Theta;\varepsilon_1,\varepsilon_2,...,\varepsilon_T) = -0.5 \cdot T \cdot \log(2\pi) - 0.5 \sum_{t=1}^{T} \left[\log\left(\sigma_t^2\right) + \varepsilon_t^2 \sigma_t^{-2}\right], \quad (20)$$

where $\Theta' \equiv (\kappa,\delta_1,...,\delta_p,\phi_1,...,\phi_q)$. The QMLE obtained by maximizing (20), say $\hat{\theta}_T$, is consistent and asymptotically normally distributed,

$$T^{1/2}\left(\hat{\theta}_T - \Theta_0\right) \to N\left(0, A(\Theta_0)^{-1} B(\Theta_0) A(\Theta_0)^{-1}\right), \quad (21)$$

where $A(.)$ and $B(.)$ represent the Hessian and the outer product of the gradient respectively, both evaluated at the true parameter, Θ_0. This so whether or not the normality assumption is correct. For different distributional assumptions, the likelihood can be constructed using the general expression (6) above.

For further discussion of Quasi MLE, see Bollerslev and Wooldridge (1992) and Brock and Lima (1996). The latter suggest that the asymptotic properties of the (quasi)-maximum likelihood estimation rely on the verification of a set of regularity conditions and that it is not yet known whether those are satisfied for FIGARCH.

5. FIFGARCH process

The Family FIGARCH, or *FIFGARCH*, model is a combination of the FGARCH and

FIGARCH models. Like the FGARCH model, it allows for asymmetric effects of ε_t on the conditional variance. Like the FIGARCH, it allows for long memory in the conditional variance process.

The FIFGARCH model modifies the FIGARCH model in the same way that FGARCH modifies GARCH. Thus ε_t^2 is replaced by $\sigma_t^{\lambda} f^{\nu}(\varepsilon_t)$ and σ_t^2 is replaced by σ_t^{λ}. Making these changes in the GARCH version of the FIGARCH model (equation (15) above), we obtain:

$$(1-\delta L)\sigma_t^{\lambda} = k + [1-\delta L - (1-\phi L)(1-L)^d]\sigma_t^{\lambda} f^{\nu}(\varepsilon_t), \qquad (22)$$

$$\text{where } f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right), \; |c| \leq 1. \qquad (23)$$

Alternatively, we can rewrite (22) as

$$\sigma_t^{\lambda} = \frac{\kappa}{1-\delta} + \left[1 - \frac{(1-\phi L)(1-L)^d}{1-\delta L}\right]\sigma_t^{\lambda} f^{\nu}(\varepsilon_t). \qquad (24)$$

This model nests existing short memory or long memory GARCH models in a general specification. It highlights the relations between those models and offers valuable opportunities for testing sequences of nested hypotheses regarding the functional form for conditional second order moments. Some special cases are discussed below.

6. Some special cases

Here we assume $p = q = 1$, for simplicity. The asymmetric short memory models can be embedded by a Box-Cox transformation of the absolute GARCH (AGARCH) model as follows

$$\frac{\sigma_t^{\lambda} - 1}{\lambda} = \kappa + \alpha f^{\nu}(\varepsilon_{t-1})\sigma_{t-1}^{\lambda} + \delta\frac{\sigma_{t-1}^{\lambda} - 1}{\lambda}, \qquad (25)$$

where $f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)$, $|c| \le 1$.

The exponential GARCH model arises from eq. (25) when $\lambda = 0$, $\nu = 1$, and $b = 0$. For $\lambda = \nu = 1$ and $|c| \le 1$, eq. (25) specializes to the AGARCH model. The model for the conditional standard deviation suggested by Taylor (1986) and Schwert (1989) arises when $\lambda = \nu = 1$ and $b = c = 0$. Zakoïan's (1994) TGARCH model for the conditional standard deviation is obtained when $\lambda = \nu = 1$, $b = 0$ and $|c| \le 1$. The GARCH model arises if $\lambda = \nu = 2$ and $b = c = 0$. Engle and Ng's (1993) nonlinear asymmetric GARCH corresponds to the values of $\lambda = \nu = 2$ and $c = 0$, whereas the GARCH model proposed by Glosten-Jagannathan-Runkle (1993) is obtained when $\lambda = \nu = 2$ and $b = 0$. The non-linear ARCH model of Higgins and Bera (1992) sets $\lambda = \nu$ with $b = c = 0$. The asymmetric power ARCH (APARCH) of Ding, Granger, and Engle (1993) sets $\lambda = \nu$ with $b = 0$ and $|c| \le 1$. The log likelihood and QMLE are assumed to follow eq. (20) and eq. (21) respectively. For the more complex details we include Table 1.

The models just listed are short-memory models that are special cases of Hentschel's FGARCH model. We could also modify them to allow for long-memory; that is, we could consider the corresponding special cases of the FIFGARCH model. For example, the FIA-PARCH model of McCurdy and Michaud combines the FIGARCH model with the APARCH model of Ding, Granger and Engle (1993), and thus corresponds to $b = 0$ and $\lambda = \nu$. We will consider some of these special cases in the empirical work in chapters 3 and 4.

IV. Autocorrelations of ε_t^2 and σ_t^2

We derive the autocorrelations of ε_t^2 and σ_t^2 in both the symmetric and asymmetric GARCH(1,1) models. Ding and Granger (1996B) gave the expression for the correlations of the ε_t^2 for the stationary symmetric GARCH(1,1) model. These autocorrelations may be useful for a variety of purposes. For example, a reasonable check of model specification would be to compare the sample autocorrelations of the ε_t^2 with the autocorrelations implied by the fitted model. We extend their results to the asymmetric GARCH(1,1) model. Also, we consider the autocorrelations of the conditional variance σ_t^2. Especially when considering questions of persistence or long memory, it is reasonable to ask whether one should think in terms of the autocorrelations of the ε_t^2 or the σ_t^2, and it is useful to have expressions for both.

1. Correlations of ε_t^2 for the symmetric GARCH(1,1) model

The correlations of ε_t^2 for symmetric GARCH(1,1) were derived by Ding and Granger (1996B). For the stationary $(\alpha + \delta < 1)$ GARCH(1,1) model, with $3\alpha^2 + 2\alpha\delta + \delta^2 < 1$ so that the fourth moment of ε_t exists, they show that

$$\rho_{k,\varepsilon_t^2} = \left[\alpha + \frac{\alpha^2\delta}{1 - 2\alpha\delta - \delta^2}\right](\alpha + \delta)^{k-1}. \tag{26}$$

In the case that the finite fourth moment condition does not hold, they derive the result:

$$\rho_{k,\varepsilon_t^2} \approx \left(\alpha + \tfrac{1}{3}\delta\right)(\alpha + \delta)^{k-1} \text{ for large } k. \tag{27}$$

2. Correlations of σ_t^2 for the symmetric GARCH(1,1) model

The autocorrelations ρ_{k,σ_t^2} of the conditional variances σ_t^2 for the stationary $(\alpha + \delta < 1)$ GARCH(1,1) model are as follows:

$$\rho_{k,\sigma_t^2} = (\alpha + \delta)^k. \tag{28}$$

This simple result does not depend on normality or on the finite fourth moment condition. Its derivation, which is straightforward, is given in Appendix 1.

Comparing equations (26) with (28), we see that the autocorrelations of ε_t^2 and σ_t^2 decay at the same rate. Both are proportional to $(\alpha + \delta)^k$. However, the factors of proportionality are different. From equations (26) and (28),

$$\rho_{k,\sigma_t^2} - \rho_{k,\varepsilon_t^2} = \delta\left[\frac{1-(\alpha+\delta)^2}{1-(\alpha+\delta)^2+\alpha^2}\right](\alpha+\delta)^{k-1} > 0 \tag{A.11}$$

For equal values of α and δ, the conditional variance is more strongly autocorrelated than the squared error in the symmetric model.

The situation is more complex when the condition $\alpha + \delta < 1$ is relaxed. See Appendix 1 for details.

3. Correlations of ε_t^2 for the asymmetric GARCH(1,1) model

We first concentrate on the asymmetric GARCH (1,1) model with b-shift ($\lambda = 2$, $\nu = 2$, and $c = 0$).

In Appendix 2 we derive the result

$$\rho^{(b)}_{k,\varepsilon_t^2} = \left(\alpha + \frac{\alpha^2\delta'(1+2b^2)}{1-2\alpha\delta'-\delta'^2+2\alpha^2b^2}\right)(\alpha+\delta')^{k-1}, \tag{29}$$

where $\delta' = \delta + \alpha b^2$ and $\rho^{(b)}_{k,\varepsilon_t^2} = corr\left(\varepsilon_t^2, \varepsilon_{t-k}^2\right)$ for the model with b-shift but $c = 0$.

We next consider the asymmetric GARCH (1,1) with c-rotation ($\lambda = 2$, $\nu = 2$, and $b = 0$).

We now obtain

$$\rho^{(c)}_{k,\varepsilon_t^2} = \left(\alpha' + \frac{\alpha'^2\delta+6\alpha^2c^2\delta}{1-2\alpha'\delta-\delta^2+6\alpha^2c^2}\right)(\alpha'+\delta)^{k-1}, \tag{30}$$

where $\alpha' = \alpha(1+c^2)$.

Finally, for the asymmetric GARCH(1,1) model with both b-shift and c-rotation, the correlations of the ε_t^2 depend in a complicated way on the nuisance parameter

$$\xi = 2\alpha c E(b-\omega_{t-1})|b-\omega_{t-1}|\sigma_{t-1}^2\left(\varepsilon_{t-1}^2-\sigma^2\right)$$

and no useful expression is derived. In contrast, in the next section we will see that a useful expression can be derived, in this model, for the autocorrelations of σ_t^2.

4. Correlations of σ_t^2 for the asymmetric GARCH(1,1) model

We derive the autocorrelations of σ_t^2 for the asymmetric GARCH(1,1) in Appendix 3. We first consider the asymmetric GARCH (1,1) model with b-shift ($\lambda = 2$, $\nu = 2$, and $c = 0$):

$$\rho^{(b)}_{k,\sigma_t^2} = (\alpha + \delta')^k, \tag{31}$$

where $\delta' = \delta(1 + \alpha b^2)$ and $\rho^{(b)}_{k,\sigma_t^2} = corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$ for the model with b-shift but $c = 0$. Comparing this result to the corresponding result for the symmetric GARCH(1,1) model, as given in equation (28) above, we have $\delta' = \delta(1 + \alpha b^2) > \delta$ if $\alpha > 0$, and hence $\rho^{(b)}_{k,\sigma_t^2} > \rho_{k,\sigma_t^2}$. For equal values of α and δ, the conditional variance is more strongly autocorrelated in the asymmetric model than in the symmetric model.

Next we consider the asymmetric GARCH(1,1) with c-rotation $(\lambda = 2, \nu = 2, b = 0)$. For this model we obtain the correlations:

$$\rho^{(c)}_{k,\sigma_t^2} = (\alpha' + \delta)^k, \tag{32}$$

where $\alpha' = \alpha(1+c^2)$. Since $\alpha' > \alpha$ for $c \neq 0$, we clearly have $\rho^{(c)}_{k,\sigma_t^2} > \rho_{k,\sigma_t^2}$. Once again, for equal values of α and δ, the conditional variance is more strongly autocorrelated in the asymmetric than in the symmetric model.

Finally, we consider the asymmetric GARCH(1,1) model with both b-shift and c-rotation $(\lambda = 2, \nu = 2)$. We obtain

$$\rho^{(b,c)}_{k,\sigma_t^2} = (\alpha' + \delta''')^k,$$

where $\alpha' = \alpha(1+c^2)$ as above, $\delta'' = \delta + \alpha' b^2$, $\tilde{\varphi}(\omega_t;b) = (b-\omega_t)|b-\omega_t|$, $\varphi \equiv E\tilde{\varphi}(\omega_t;b) = E(b-\omega_t)|b-\omega_t|$, and $\delta''' = \delta'' + 2\alpha c\varphi$.

For $\alpha > 0$, we have $\alpha' > \alpha$ and $\delta'' > \delta' > \delta$. $\varphi > 0$ for $b > 0$ and $\varphi(b) = -\varphi(-b)$ under the normality assumption. Therefore, for $b > 0$ and $c > 0$ or $b < 0$ and $c < 0$,

$$\rho^{(b,c)}_{k,\sigma_t^2} > \rho^{(b)}_{k,\sigma_t^2} > \rho_{k,\sigma_t^2} \text{ and } \rho^{(b,c)}_{k,\sigma_t^2} > \rho^{(c)}_{k,\sigma_t^2} > \rho_{k,\sigma_t^2}.$$

V. Conclusion

A new type of long memory Family GARCH, called the fractionally integrated family GARCH or FIFGARCH, has been proposed. It combines previous models that allowed for asymmetry or long memory, so as to allow for both at once. This model will be applied to stock returns and exchange rates in chapters 3 and 4.

The autocorrelations of squared errors and conditional variances were derived for symmetric and asymmetric GARCH(1,1) models. The autocorrelations of conditional variances are different from those of squared errors.

Table 1. Various short memory GARCH models with special names

λ	ν	b	c	Model
0	1	0	free	Exponential GARCH (Nelson)
1	1	0	$\lvert c\rvert \le 1$	Threshold GARCH (Zakoian)
1	1	free	$\lvert c\rvert \le 1$	Absolute value GARCH (Taylor/Schwert)
2	2	0	0	GARCH (Bollerslev)
2	2	free	0	Nonlinear-asymmetric GARCH (Engle, Ng)
2	2	0	free	GJR GARCH (Glosten, Jagannathan, Runkle)
free	λ	0	0	Nonlinear ARCH (Higgins, Bera)
free	λ	0	$\lvert c\rvert \le 1$	Asymmetric power ARCH (Ding, Granger, Engle)
free	free	free	$\lvert c\rvert \le 1$	FGARCH (Hentschel)

1. Exponential GARCH (Nelson)

$$\ln\sigma_t^2 = \kappa + \alpha[f(\varepsilon_{t-1}) - E(f(\varepsilon_{t-1}))] + \delta\ln\sigma_{t-1}^2$$

2. Threshold GARCH (Zakoian) and Absolute value GARCH (Taylor/Schwert)

$$\sigma_t = \kappa + \alpha\sigma_{t-1}f(\varepsilon_{t-1}) + \delta\sigma_{t-1}$$

3. GARCH (Bollerslev), Nonlinear-asymmetric GARCH (Engle, Ng), and GJR GARCH (Glosten, Jagannathan, Runkle)

$$\sigma_t^2 = \kappa + \alpha\sigma_{t-1}^2 f^2(\varepsilon_{t-1}) + \delta\sigma^2{}_{t-1}$$

4. Nonlinear ARCH (Higgins, Bera) and Asymmetric power ARCH (Ding, Granger, Engle)

$$\sigma_t^\lambda = \kappa + \alpha\sigma_{t-1}^\lambda f^\lambda(\varepsilon_{t-1}) + \delta\sigma_{t-1}^\lambda$$

5. FGARCH (Hentschel)

$$\sigma_t^\lambda = \kappa + \alpha\sigma_{t-1}^\lambda f^\nu(\varepsilon_{t-1}) + \delta\sigma_{t-1}^\lambda$$

Note that the following is assumed throughout

$$f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right),\ \lvert c\rvert \le 1 .$$

Figure 1. The asymmetric transformation of $f\left(\frac{\varepsilon_t}{\sigma_t}\right) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)$

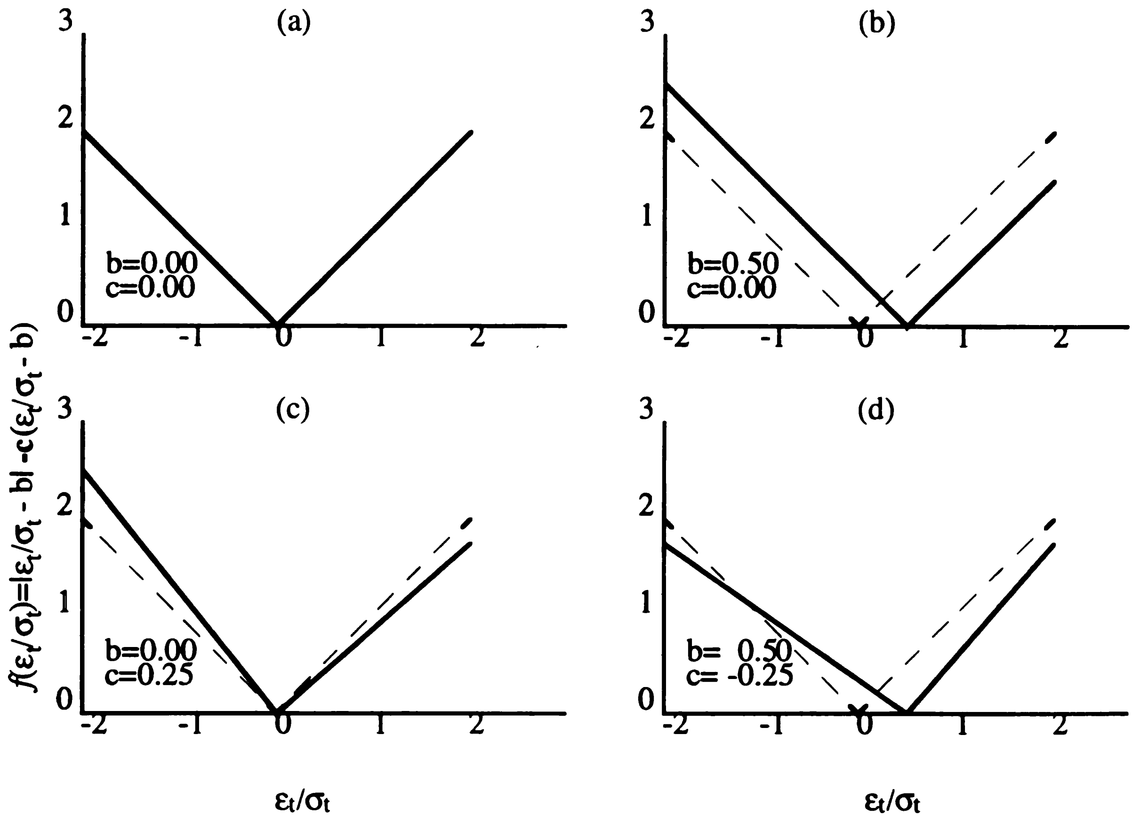
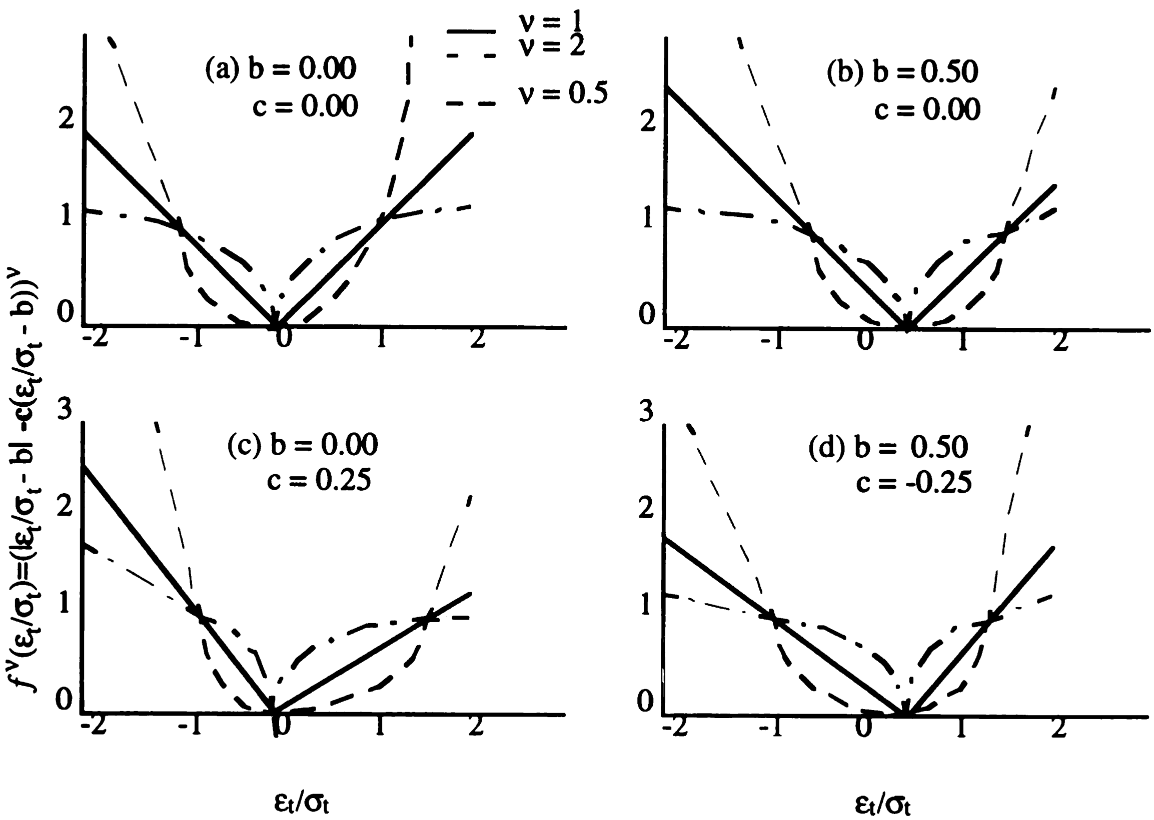


Figure 2. The transformation of $f^{\nu}\left(\frac{\varepsilon_t}{\sigma_t}\right) = \left(\left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)\right)^{\nu}$



Appendix 1.

We first derive the autocorrelation functions of conditional variances for the covariance-stationary GARCH (1,1) model. For notational simplicity we will write $\rho_k \equiv \rho_{k,\varepsilon_t^2}$ and $\rho_k^s \equiv \rho_{k,\sigma_t^2}$, and similarly for γ_k and γ_k^s.

When $\alpha + \delta < 1$, the GARCH (1,1) process can be represented as follows:

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2, \qquad \text{(A.1)}$$

$$\sigma^2 = \kappa/(1-\alpha-\delta), \qquad \text{(A.2)}$$

where σ^2 is the unconditional variance of ε_t. Substituting (A.2) to (A.1) one gets

$$\sigma_t^2 = \sigma^2(1-\alpha-\delta) + \alpha\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2. \qquad \text{(A.3)}$$

Rearranging the above equation one gets

$$\sigma_t^2 - \sigma^2 = (\alpha+\delta)\left(\sigma_{t-1}^2 - \sigma^2\right) + \alpha\left(\sigma_{t-1}^2\omega_{t-1}^2 - \sigma_{t-1}^2\right)$$

$$= (\alpha + \delta)\left(\sigma_{t-1}^2 - \sigma^2\right) + \alpha\sigma_{t-1}^2\left(\omega_{t-1}^2 - 1\right). \tag{A.4}$$

where as before $\varepsilon_t = \sigma_t\omega_t$ and the ω_t are *iid* $(0, 1)$. Now, for $k > 0$, multiply both sides by $\left(\sigma_{t-k}^2 - \sigma^2\right)$ and take expectations to obtain

$$\gamma_k^s = (\alpha + \delta)\gamma_{k-1}^s, \ k \geq 1. \tag{A.5}$$

In evaluating the required expectation, we note that

$$E\sigma_{t-1}^2\left(\omega_{t-1}^2 - 1\right)\left(\sigma_{t-k}^2 - \sigma^2\right) = E\left(\omega_{t-1}^2 - 1\right)\left[\sigma_{t-1}^2\left(\sigma_{t-k}^2 - \sigma^2\right)\right] = 0 \tag{A.6}$$

since, for $k \geq 1$, σ_{t-1}^2 and σ_{t-k}^2 are functions of ω_{t-j}, $j \geq 2$, which are uncorrelated with ω_{t-1}^2 in light of the *iid* assumption on the ω_t.

Clearly (A.5) implies that

$$\rho_k^s = (\alpha + \delta)^k \tag{A.7}$$

which is equation (32) of the main text.

To obtain an explicit expression for γ_k^s, we can further assume that $3\alpha^2 + 2\alpha\delta + \delta^2 < 1$, so that the fourth moment of ε_t exists. From the conditional variance equation one can get

$$E\sigma_{t-1}^4 = \frac{\sigma^4(1-\alpha-\delta)(1+\alpha+\delta)}{1-(3\alpha^2+2\alpha\delta+\delta^2)}. \tag{A.8}$$

Substituting (A.8) into $\gamma_0^s = E\left(\sigma_t^2-\sigma^2\right)^2 = E\sigma_t^4-\sigma^4$ and doing some simple algebra shows

$$\gamma_0^s = \frac{2\alpha^2\sigma^4}{1-(3\alpha^2+2\alpha\delta+\delta^2)}. \tag{A.9}$$

Then $\gamma_k^s = \gamma_0^s(\alpha+\delta)^k$ for $k \geq 1$. (A.10)

We compare the autocorrelation functions of squared errors ρ_k with those of conditional variances ρ_k^s. Note that equations (30) and (31) below for the autocorrelation functions of squared errors refer to those in the main text.

For the stationary ($\alpha+\delta<1$) GARCH(1,1) model, with $3\alpha^2+2\alpha\delta+\delta^2<1$ so that the fourth moment of ε_t exists,

$$\rho_k = \left[\alpha+\frac{\alpha^2\delta}{1-2\alpha\delta-\delta^2}\right](\alpha+\delta)^{k-1}. \tag{30}$$

In the case that the finite fourth moment condition does not hold,

$$\rho_k \approx \left(\alpha + \frac{1}{3}\delta\right)(\alpha + \delta)^{k-1} \text{ for large } k. \tag{31}$$

From equations (A.7) and (30),

$$\rho_k^s - \rho_k = \delta\left[\frac{1-(\alpha+\delta)^2}{1-(\alpha+\delta)^2+\alpha^2}\right](\alpha+\delta)^{k-1} > 0 \tag{A.11}$$

since $\alpha + \delta < 1$ so that both denominator and nominator are positive.

From equations (A.7) and (31) in the main text, obviously

$$\rho_k^s - \rho_k = \frac{2}{3}\delta(\alpha+\delta)^{k-1} > 0. \tag{A.12}$$

Of course $3\alpha^2 + 2\alpha\delta + \delta^2 < 1 \Leftrightarrow \delta\left[\frac{1-(\alpha+\delta)^2}{1-(\alpha+\delta)^2+\alpha^2}\right](\alpha+\delta)^{k-1} > \frac{2}{3}\delta(\alpha+\delta)^{k-1}$. This simplifies some other comparisons.

The situation is quite different when the covariance-stationary assumption is removed. We consider the IGARCH(1,1) case which $\alpha + \delta = 1$. Assume

$$\varepsilon_t = \sigma_t\omega_t, \quad \omega_t \sim iid\ \mathrm{N}(0,1), \quad \sigma_t^2 = \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2, \tag{A.13}$$

and $\sigma_0 = 1$, a constant. Then

$$\varepsilon_t^2 = \left(\alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2\right)\omega_t^2 \tag{A.14}$$

and it is not difficult to show that:

$$E\sigma_t^2 = 1,$$

$$E\sigma_t^4 = (1+2\alpha^2)^t E\sigma_0^4 = (1+2\alpha^2)^t,$$

$$E\varepsilon_t^2\varepsilon_{t-k}^2 = (1+2\alpha)E\sigma_{t-k}^4 = (1+2\alpha)(1+2\alpha^2)^{t-k},$$

$$E\varepsilon_t^2 E\varepsilon_{t-k}^2 = \left(E\sigma_{t-k}^2\right)^2 = 1,$$

$$V_{\varepsilon_t^2} = 3(1+2\alpha^2)E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2 = 3(1+2\alpha^2)^t - 1,$$

$$V_{\varepsilon_{t-k}^2} = 3E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2 = 3(1+2\alpha^2)^{t-k} - 1,$$

Wh

ρ_k^s

$$E\sigma_t^2\sigma_{t-k}^2 = E\sigma_{t-k}^4 = (1+2\alpha^2)^{t-k} ,$$

$$E\sigma_t^2 E\sigma_{t-k}^2 = \left(E\sigma_{t-k}^2\right)^2 = 1 ,$$

$$V_{\sigma_t^2} = E\sigma_t^4 - \left(E\sigma_t^2\right)^2 = (1+2\alpha^2)^t - 1 ,$$

$$V_{\sigma_{t-k}^2} = E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2 = (1+2\alpha^2)^{t-k} - 1 ,$$

$$\rho_{k,t} = \frac{(1+2\alpha)(1+2\alpha^2)^{t-k} - 1}{\sqrt{3(1+2\alpha^2)^t - 1}\sqrt{3(1+2\alpha^2)^{t-k} - 1}} , \tag{A.15}$$

$$\rho_{k,t}^s = \frac{(1+2\alpha^2)^{t-k} - 1}{\sqrt{(1+2\alpha^2)^t - 1}\sqrt{(1+2\alpha^2)^{t-k} - 1}} . \tag{A.16}$$

When $t \gg k > 0$ and $\alpha \neq 0$, one has approximately

$$\rho_k \approx \frac{1+2\alpha}{3}(1+2\alpha^2)^{-k/2} , \tag{A.17}$$

$$\rho_k^s \approx (1+2\alpha^2)^{-k/2} , \tag{A.18}$$

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that

$$\rho_k^s > \rho_k. \tag{A.19}$$

It is seen that the autocorrelation function decreases exponentially. In the extreme case $\alpha = 0$, so that $\sigma_t^2 = \sigma_{t-1}^2 = \ldots = \sigma_1^2 = \sigma^2$, i.e., the variance is constant over time and there is no heteroskedasticity, then (A.15) and (A.16) give $\rho_k = \rho_k^s = 0$. On the other extreme, if $\alpha = 1$, so that $\sigma_t^2 = \varepsilon_{t-1}^2$, then (A.15) and (A.16) give

$$\rho_{k,t} = \sqrt{(3^{t-k+1} - 1)/(3^{t+1} - 1)}, \tag{A.20}$$

$$\rho_{k,t}^s = \sqrt{(3^{t-k} - 1)/(3^t - 1)}, \tag{A.21}$$

When $t \gg k > 0$, (A.20) and (A.21) become

$$\rho_k = \rho_k^s \approx 3^{-k/2} \tag{A.22}$$

and again it is exponentially decreasing.

Similar results can be derived for the IGARCH (1,1) process with a drift. Assume now that

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2 \tag{A.23}$$

and $\sigma_0^2 = \kappa$, a constant. Then

$$E\sigma_t^2 = (t+1)\kappa . \tag{A.24}$$

When $\alpha \neq 0$ and t is large, $E\sigma_t^4$ is approximately as follows

$$E\sigma_t^4 \approx \frac{\kappa^2}{2\alpha^2}\left(1+\frac{1}{\alpha^2}\right)(1+2\alpha^2)^{t+1}, \tag{A.25}$$

$$\gamma_{k,t} = (1+2\alpha)E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2, \tag{A.26}$$

$$\gamma_{k,t}^s = E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2, \tag{A.27}$$

$$\rho_k = \frac{(1+2\alpha)E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2}{\sqrt{\left[3E\sigma_t^4 - E\left(\sigma_t^2\right)^2\right]\left[3E\sigma_{t-k}^4 - E\left(\sigma_{t-k}^2\right)^2\right]}}, \tag{A.28}$$

$$\rho_k^s = \frac{E\sigma_{t-k}^4 - \left(E\sigma_{t-k}^2\right)^2}{\sqrt{\left[E\sigma_t^4 - E\left(\sigma_t^2\right)^2\right]\left[E\sigma_{t-k}^4 - E\left(\sigma_{t-k}^2\right)^2\right]}}. \tag{A.29}$$

When $t \gg k > 0$, one has approximately

$$\rho_k \approx \frac{1+2\alpha}{3}(1+2\alpha^2)^{-k/2}, \tag{A.30}$$

$$\rho_k^s \approx (1+2\alpha^2)^{-k/2}. \tag{A.31}$$

Comparing (A.17) with (A.30), it is seen that the autocorrelation functions for IGARCH (1,1) models with or without a drift are the same.

Appendix 2.

We derive the autocorrelation functions of squared innovations for the covariance-stationary asymmetric GARCH (1,1) model under the assumption of conditional normality. The algebra is very similar to that in Ding and Granger (1996B, Appendix).

When $\alpha + \delta < 1$, the asymmetric Family GARCH (1,1) with b-shift and c-rotation can be represented as follows:

$$\sigma_t^{\lambda} = \kappa + \alpha f^{\nu}(\varepsilon_{t-1})\sigma_{t-1}^{\lambda} + \delta\sigma_{t-1}^{\lambda}, \tag{A.1}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b| - c(\varepsilon_t/\sigma_t - b)$, $-1 < c < 1$. (A.2)

We first concentrate on the asymmetric GARCH (1,1) with b-shift ($\lambda = 2$, $\nu = 2$, and $c = 0$),

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2, \tag{A.3}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b|$, $\omega_t = \varepsilon_t/\sigma_t$, $\omega_t \sim iid\ D(0,1)$.

We rewrite (A.3)

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + (\delta + \alpha b^2)\sigma_{t-1}^2 - 2\alpha b\omega_{t-1}\sigma_{t-1}^2, \tag{A.4}$$

Define $\delta' = \delta + \alpha b^2$, and $\sigma^2 = \kappa/(1-\alpha-\delta')$, where σ^2 is the unconditional variance of ε_t.

Substituting $\kappa = \sigma^2(1-\alpha-\delta')$ to (A.4) one gets

$$\sigma_t^2 = \sigma^2(1-\alpha-\delta') + \alpha\varepsilon_{t-1}^2 + \delta'\sigma_{t-1}^2 - 2\alpha b\omega_{t-1}\sigma_{t-1}^2. \tag{A.5}$$

Rearranging the above equation one gets

$$\varepsilon_t^2 - \sigma^2 = (\alpha+\delta')\left(\varepsilon_{t-1}^2 - \sigma^2\right) + (1-\delta' L)\left(\sigma_t^2\omega_t^2 - \sigma_t^2\right) - 2\alpha b\omega_{t-1}\sigma_{t-1}^2. \tag{A.6}$$

Multiplying both sides of the above equation by $\left(\varepsilon_{t-1}^2 - \sigma^2\right)$ and taking expectations one has

$$\gamma_1 = (\alpha+\delta')\gamma_0^b - 2\delta' E\sigma_{t-1}^4, \tag{A.7}$$

where $\gamma_1 = E\left(\varepsilon_t^2 - \sigma^2\right)\left(\varepsilon_{t-1}^2 - \sigma^2\right)$ is the covariance between ε_t^2 and ε_{t-1}^2 while $\gamma_0 = E\left(\varepsilon_{t-1}^2 - \sigma^2\right)^2$ is the variance of ε_{t-1}^2.

Dividing both sides of (A.7) by γ_0, which is finite, gives

$$\rho_1 = \alpha + \delta' - 2\delta' E\frac{\sigma_{t-1}^4}{\gamma_0}. \tag{A8}$$

Also by definition

$$\gamma_0 = E\left(\varepsilon_{t-1}^2 - \sigma^2\right)^2 = 3E\sigma_{t-1}^4 - \sigma^4, \tag{A.9}$$

$$E\sigma_{t-1}^4 = \frac{1}{3}\left(\gamma_0 + \sigma^4\right). \tag{A.10}$$

If it is further assumed $3\alpha^2 + 2\alpha\delta' + \delta'^2 + 4\alpha^2 b^2 < 1$, so that the fourth moment of ε_t exists, from the conditional variance equation one can get

$$E\sigma_{t-1}^4 = \frac{\sigma^4(1 - \alpha - \delta')(1 + \alpha + \delta')}{1 - (3\alpha^2 + 2\alpha\delta' + \delta'^2 + 4\alpha^2 b^2)}. \tag{A.11}$$

Substituting this into (A.9) and some simple algebra shows

$$\gamma_0 = \frac{2\sigma^4(1 - 2\alpha\delta' - \delta'^2 + 2\alpha^2 b^2)}{1 - (3\alpha^2 + 2\alpha\delta' + \delta'^2 + 4\alpha^2 b^2)}. \tag{A.12}$$

Combining (A.8) with $\rho_k = \rho_1(\alpha + \delta')^{k-1}$, $k \geq 2$, one has the autocorrelation function for the asymmetric GARCH (1,1) process as follows:

$$\rho_k = \left(\alpha + \frac{\alpha^2\delta'(1+2b^2)}{1 - 2\alpha\delta' - \delta'^2 + 2\alpha^2 b^2}\right)(\alpha + \delta')^{k-1}. \tag{A.13}$$

We next consider the asymmetric GARCH(1,1) model with c-rotation ($\lambda = 2$, $\nu = 2$, and $b = 0$):

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2, \tag{A.14}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t| - c(\varepsilon_t/\sigma_t)$, $-1 < c < 1$.

We rewrite (A.14) as

$$\sigma_t^2 = \kappa + \alpha'\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2 - 2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2, \tag{A.15}$$

where $\alpha' = \alpha(1 + c^2)$.

$$\sigma^2 = \kappa/(1 - \alpha' - \delta), \tag{A.16}$$

where σ^2 is the unconditional variance of ε_t. Substituting (A.16) to (A.15) one gets

$$\sigma_t^2 = \sigma^2(1-\alpha'-\delta)+\alpha'\varepsilon_{t-1}^2+\delta\sigma_{t-1}^2-2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2. \quad \text{(A.17)}$$

Rearranging the above equation one gets

$$\varepsilon_t^2-\sigma^2 = (\alpha'+\delta)\left(\varepsilon_{t-1}^2-\sigma^2\right)+(1-\delta L)\left(\sigma_t^2\omega_t^2-\sigma_t^2\right)-2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2. \quad \text{(A.18)}$$

Multiplying both sides of the above equation by $\left(\varepsilon_{t-1}^2-\sigma^2\right)$ and taking expectations one has

$$\gamma_1 = (\alpha'+\delta)\gamma_0-2\delta E\sigma_{t-1}^4, \quad \text{(A.19)}$$

where $\gamma_1 = E\left(\varepsilon_t^2-\sigma^2\right)\left(\varepsilon_{t-1}^2-\sigma^2\right)$ is the covariance between ε_t^2 and ε_{t-1}^2 while $\gamma_0 = E\left(\varepsilon_{t-1}^2-\sigma^2\right)^2$ is the variance of ε_{t-1}^2.

Dividing both sides of (A.19) by γ_0, which is finite, gives

$$\rho_1 = \alpha'+\delta-2\delta E\frac{\sigma_{t-1}^4}{\gamma_0}. \quad \text{(A.20)}$$

Also by definition

$$\gamma_0 = E\left(\varepsilon_{t-1}^2 - \sigma^2\right)^2 = 3E\sigma_t^4 - \sigma^4, \tag{A.21}$$

$$E\sigma_t^4 = \frac{1}{3}\left(\gamma_0 + \sigma^4\right). \tag{A.22}$$

If it is further assumed $3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2 < 1$, so that the fourth moment of ε_t exists, from the conditional variance equation one can get

$$E\sigma_{t-1}^4 = \frac{\sigma^4(1-\alpha'-\delta)(1+\alpha'+\delta)}{1-(3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2)}. \tag{A.23}$$

Substituting this into (A.21) and some simple algebra shows

$$\gamma_0 = \frac{2\sigma^4(1-2\alpha'\delta - \delta^2 + 6\alpha^2 c^2)}{1-(3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2)}. \tag{A.24}$$

Combining (A.20) with $\rho_k = \rho_1(\alpha' + \delta)^{k-1}$, $k \geq 2$, one has the autocorrelation function for GARCH (1,1) process as follows:

$$\rho_k = \left(\alpha' + \frac{\alpha'^2\delta + 6\alpha^2 c^2 \delta}{1 - 2\alpha'\delta - \delta^2 + 6\alpha^2 c^2}\right)(\alpha' + \delta)^{k-1} . \tag{A.25}$$

Finally we consider the asymmetric GARCH (1,1) with b-shift and c-rotation ($\lambda = 2$, $\nu = 2$),

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2 , \tag{A.26}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b| - c(\varepsilon_t/\sigma_t - b)$, $-1 < c < 1$.

We rewrite (A.26) as

$$\sigma_t^2 = \kappa + \alpha'\varepsilon_{t-1}^2 + \delta''\sigma_{t-1}^2 - 2\alpha' b\omega_{t-1}\sigma_{t-1}^2 - 2\alpha c|\omega_{t-1} - b|(\omega_{t-1} - b)\sigma_{t-1}^2 , \tag{A.27}$$

where $\alpha' = \alpha(1 + c^2)$, $\delta'' = \delta + \alpha' b^2$. Define

$$\tilde{\varphi}(\omega_t;b) = (b - \omega_t)|b - \omega_t| ,$$

$$\varphi \equiv E\tilde{\varphi}(\omega_t;b) = E(b - \omega_t)|b - \omega_t|$$

$$= b^2 W^0 - 2bW^1 + W^2 .$$

where $W^i = \int_{-\infty}^{b} \omega^i f(\omega)d\omega - \int_{b}^{\infty} \omega^i f(\omega)d\omega$, $i = 0, 1, 2$. Under the assumption of normality,

$$\varphi = (b^2 + 1)(2\Phi(b) - 1) + 2b\phi(b)$$

where $\phi(.)$ and $\Phi(.)$ are the p.d.f. and c.d.f. under the normality assumption respectively. Set

$$\delta''' = \delta'' + 2\alpha c\varphi \text{ and } \tilde{\eta} = \tilde{\varphi} - \varphi .$$

Define

$$\sigma^2 = \kappa/(1 - \alpha' - \delta'''), \tag{A.28}$$

where σ^2 is the unconditional variance of ε_t. Substituting (A.28) to (A.27) one gets

$$\sigma_t^2 = \sigma^2(1 - \alpha' - \delta''') + \alpha'\varepsilon_{t-1}^2 + \delta'''\sigma_{t-1}^2 - 2\alpha' b\omega_{t-1}\sigma_{t-1}^2 + 2\alpha c\tilde{\eta}\sigma_{t-1}^2 . \tag{A.29}$$

Rearranging the above equation one gets

$$\varepsilon_t^2 - \sigma^2 = (\alpha' - \delta''')\left(\varepsilon_{t-1}^2 - \sigma^2\right) + (1 - \delta'''L)\left(\sigma_t^2\omega_t^2 - \sigma_t^2\right) - 2\alpha' b\omega_{t-1}\sigma_{t-1}^2 + 2\alpha c\tilde{\eta}\sigma_{t-1}^2 . \quad \text{(A.30)}$$

Multiplying both sides of the above equation by $\left(\varepsilon_{t-1}^2 - \sigma^2\right)$ and taking expectations, one has

$$\gamma_1 = (\alpha' + \delta''')\gamma_0 - 2\delta''' E\sigma_{t-1}^4 + \xi , \quad \text{(A.31)}$$

where $\gamma_1 = E\left(\varepsilon_t^2 - \sigma^2\right)\left(\varepsilon_{t-1}^2 - \sigma^2\right)$ is the covariance between ε_t^2 and ε_{t-1}^2 while $\gamma_0 = E\left(\varepsilon_{t-1}^2 - \sigma^2\right)^2$ is the variance of ε_{t-1}^2 and $\xi = E2\alpha c\tilde{\eta}\sigma_{t-1}^2\left(\varepsilon_{t-1}^2 - \sigma^2\right)$. The presence of the nuisance parameter ξ prevents us from obtaining a useful expression for higher-order autocovariances or autocorrelations.

Appendix 3.

We derive the autocorrelation functions of conditional variances for the covariance-stationary asymmetric GARCH (1,1) model.

When $\alpha + \delta < 1$, the asymmetric Family GARCH (1,1) process with b-shift and c-rotation can be represented as follows:

$$\sigma_t^{\lambda} = \kappa + \alpha f^{\nu}(\varepsilon_{t-1})\sigma_{t-1}^{\lambda} + \delta\sigma_{t-1}^{\lambda}, \tag{A.1}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b| - c(\varepsilon_t/\sigma_t - b)$, $-1 < c < 1$. (A.2)

We first concentrate on the asymmetric GARCH (1,1) process with b-shift ($\lambda = 2$, $\nu = 2$, and $c = 0$):

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2, \tag{A.3}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b|$, $\omega_t = \varepsilon_t/\sigma_t$, $\omega_t \sim iid\ D(0,1)$.

We rewrite (A.3) as

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + (\delta + \alpha b^2)\sigma_{t-1}^2 - 2\alpha b\omega_{t-1}\sigma_{t-1}^2. \tag{A.4}$$

Define $\delta' = \delta + \alpha b^2$ and $\sigma^2 = \kappa/(1 - \alpha - \delta')$, where σ^2 is the unconditional variance of ε_t.

Substituting $\kappa = \sigma^2(1 - \alpha - \delta')$ to (A.4) one gets

$$\sigma_t^2 = \sigma^2(1 - \alpha - \delta') + \alpha\varepsilon_{t-1}^2 + \delta'\sigma_{t-1}^2 - 2\alpha b\omega_{t-1}\sigma_{t-1}^2. \tag{A.5}$$

Rearranging the above equation one gets

$$\sigma_t^2 - \sigma^2 = (\alpha + \delta')\left(\sigma_{t-1}^2 - \sigma^2\right) + \alpha\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) - 2\alpha b\omega_{t-1}\sigma_{t-1}^2. \tag{A.6}$$

Multiplying both sides of the above equation by $\left(\sigma_{t-k}^2 - \sigma^2\right)$, for $k \geq 1$, and taking expectations one has

$$\gamma_k^{bs} = (\alpha + \delta')\gamma_{k-1}^{bs}, \tag{A.7}$$

where $\gamma_k^{bs} = E\left(\sigma_t^2 - \sigma^2\right)\left(\sigma_{t-k}^2 - \sigma^2\right)$ is the covariance between σ_t^2 and σ_{t-k}^2. Superscript "b" indicates b-shift, while "s" represents a covariance for the conditional variance. This implies

$$\rho_k^{bs} = (\alpha + \delta')^k. \tag{A.8}$$

To proceed further, assumed that $3\alpha^2 + 2\alpha\delta' + \delta'^2 + 4\alpha^2 b^2 < 1$, so that the fourth moment of ε_t exists. From the conditional variance equation one can get

$$E\sigma_{t-1}^4 = \frac{\sigma^4(1-\alpha-\delta')(1+\alpha+\delta')}{1-(3\alpha^2+2\alpha\delta'+\delta'^2+4\alpha^2 b^2)}. \tag{A.9}$$

Substituting this into $\gamma_0^{bs} = E\sigma_t^4 - \sigma^4$ and doing some simple algebra show

$$\gamma_0^{bs} = \frac{2\sigma^4\alpha^2(1+2b^2)}{1-(3\alpha^2+2\alpha\delta'+\delta'^2+4\alpha^2 b^2)}. \tag{A.10}$$

Then $\gamma_k^{bs} = \gamma_0^{bs} \cdot \rho_k^{bs} = \gamma_0^{bs}(\alpha+\delta')^k$.

For the symmetric GARCH(1,1) model, we showed in Appendix 1 that $\rho_k^s = (\alpha+\delta)^k$. In the asymmetric case, $\rho_k^{bs} = (\alpha+\delta')^k$ where $\delta' = \delta + \alpha b^2$. If $\alpha > 0$, $\delta' > \delta$ and $\rho_k^{bs} > \rho_k^s$; the conditional variance is more strongly autocorrelated (for equal values of α and δ) in the asymmetric case.

Next we consider the asymmetric GARCH (1,1) process with c-rotation ($\lambda = 2$, $\nu = 2$, and $b = 0$):

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2, \tag{A.11}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t| - c(\varepsilon_t/\sigma_t)$, $-1 < c < 1$.

We rewrite (A.11) as:

$$\sigma_t^2 = \kappa + \alpha'\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2 - 2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2,$$

where $\alpha' = \alpha(1 + c^2)$ and $\sigma^2 = \kappa/(1 - \alpha' - \delta)$, where σ^2 is the unconditional variance of ε_t. Substituting $\kappa = \sigma^2(1 - \alpha' - \delta)$ into this equation one gets

$$\sigma_t^2 = \sigma^2(1 - \alpha' - \delta) + \alpha'\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2 - 2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2. \quad \text{(A.12)}$$

Rearranging the above equation one gets

$$\sigma_t^2 - \sigma^2 = (\alpha' + \delta)\left(\sigma_{t-1}^2 - \sigma^2\right) + \alpha'\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) - 2\alpha c|\omega_{t-1}|\omega_{t-1}\sigma_{t-1}^2. \quad \text{(A.13)}$$

Proceeding as before, we now multiply both sides of the equation by $\left(\sigma_{t-1}^2 - \sigma^2\right)$, for $k \geq 1$, and take expectations. We obtain

$$\gamma_k^{cs} = (\alpha' + \delta)\gamma_{k-1}^{cs} - 2\alpha c E\omega_{t-1}|\omega_{t-1}|\sigma_{t-1}^2\left(\sigma_{t-k}^2 - \sigma^2\right), \quad \text{(A.14)}$$

where $\gamma_k^{cs} = E\left(\sigma_t^2 - \sigma^2\right)\left(\sigma_{t-k}^2 - \sigma^2\right)$ for the model with c-rotation. As before, $\omega_{t-1}|\omega_{t-1}|$ is uncorrelated with $\sigma_{t-1}^2(\sigma_{t-k}^2 - \sigma^2)$ by virtue of the *iid* assumption. However, for $E\omega_{t-1}|\omega_{t-1}| = 0$ we require the symmetry (e.g. normality) of ω. Under this further assumption, we obtain:

$$\gamma_k^{cs} = (\alpha' + \delta)\gamma_{k-1}^{cs}.$$

which implies

$$\rho_k^{cs} = (\alpha' + \delta)^k.$$

We note that $\alpha' = \alpha(1 + c^2) > \alpha$ for $c \neq 0$, so that $\rho_k^{cs} > \rho_k^s$ for equal values of α and δ.

If it is further assumed that $3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2 < 1$, so that the fourth moment of ε_t exists, then from the conditional variance equation one can get

$$E\sigma_{t-1}^4 = \frac{\sigma^4(1 - \alpha' - \delta)(1 + \alpha' + \delta)}{1 - (3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2)}. \tag{A.15}$$

Substituting this into $\gamma_0^{cs} = E\sigma_t^4 - \sigma^4$ and doing some simple algebra show

$$\gamma_0^{cs} = \frac{2\sigma^4(\alpha'^2 + 6\alpha^2 c^2)}{1 - (3\alpha'^2 + 2\alpha'\delta + \delta^2 + 12\alpha^2 c^2)} . \qquad \text{(A.16)}$$

Then $\gamma_k^{cs} = \gamma_0^{cs} \rho_k^{cs}$.

Finally, we consider the asymmetric GARCH (1,1) model with b-shift *and* c-rotation ($\lambda = 2, \nu = 2$):

$$\sigma_t^2 = \kappa + \alpha f^2(\varepsilon_{t-1})\sigma_{t-1}^2 + \delta\sigma_{t-1}^2 , \qquad \text{(A.17)}$$

where $f(\varepsilon_t) = |\varepsilon_t/\sigma_t - b| - c(\varepsilon_t/\sigma_t - b)$, $-1 < c < 1$.

We rewrite (A.17) as (A.29) in Appendix 2

$$\sigma_t^2 = \sigma^2(1 - \alpha' - \delta''') + \alpha'\varepsilon_{t-1}^2 + \delta'''\sigma_{t-1}^2 - 2\alpha' b\omega_{t-1}\sigma_{t-1}^2 + 2\alpha c\tilde{\eta}\sigma_{t-1}^2 \qquad \text{(A.18)}$$

where as before $\alpha' = \alpha(1 + c^2)$, $\delta'' = \delta + \alpha' b^2$, $\tilde{\varphi}(\omega_t; b) = (b - \omega_t)|b - \omega_t|$,

$\varphi \equiv E\tilde{\varphi}(\omega_t; b) = E(b - \omega_t)|b - \omega_t|$, $\delta''' = \delta'' + 2\alpha c\varphi$, and $\tilde{\eta} = \tilde{\varphi} - \varphi$.

Rearranging the above equation one gets

$$\sigma_t^2 - \sigma^2 = (\alpha' + \delta''')\left(\sigma_{t-1}^2 - \sigma^2\right) + \alpha'\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) - 2\alpha' b\omega_{t-1}\sigma_{t-1}^2 + 2\alpha c\tilde{\eta}\sigma_{t-1}^2 . \quad \text{(A.19)}$$

Now multiply both sides of this equation by $\left(\sigma_{t-k}^2 - \sigma^2\right)$ and take expectations. This yields

$$\gamma_k^{bcs} = (\alpha' + \delta''')\gamma_{k-1}^{bcs} - \xi_k$$

where $\gamma_k^{bcs} \equiv E\left(\sigma_t^2 - \sigma^2\right)\left(\sigma_{t-k}^2 - \sigma^2\right)$ and $\xi_k = 2\alpha c E\tilde{\eta}\sigma_{t-1}^2\left(\sigma_{t-k}^2 - \sigma^2\right)$. Therefore

$$\gamma_k^{bcs} = (\alpha' + \delta''')\gamma_{k-1}^{bcs}$$

which implies

$$\rho_k^{bcs} = (\alpha' + \delta''')^k .$$

CHAPTER 3

CHAPTER 3

Asymmetric long memory in variance of U.S. stock returns

I. Introduction

In the previous chapter we introduced a new family of GARCH models that allow for both asymmetry and long memory in variance. In this chapter we apply these newly developed models to data on daily stock returns.

The remainder of the chapter is organized as follows. In this section we briefly describe the data and the mean equation specification. Simple examples of the need to allow for asymmetry and long memory are presented. Section 2 lists the models employed and discusses measures of fit. Section 3 discusses the estimation of the asymmetric family short memory or long memory GARCH models and gives the empirical results. The final section concludes.

1. Data

The stock returns are daily returns of Standard & Poor's 500 index, defined as the first differenced logarithms of the index:

$$y_t = \ln(p_t/p_{t-1}).$$

The data span the period from January 3, 1928 to September 30, 1993, and contain 17,582 observations. The source of the data is Compustat.

2. Mean equation specification

The conditional mean equation is an MA(1) Model:

$$y_t = cnst + \varepsilon_t + \theta_1 \varepsilon_{t-1} . \tag{1}$$

As in chapter 2, let Ω_t be the set of information available at the time t. Then we assume $E(\varepsilon_t | \Omega_{t-1}) = 0$ so that ε_t is a martingale-difference process. However, it may be conditionally heteroskedastic. As in chapter 2 we assume

$$\varepsilon_t = \sigma_t \omega_t , \quad \omega_t \sim i.i.d. \quad D(0,1) , \tag{2}$$

where σ_t is a positive time-varying and measurable function with respect to the information set available at time $t-1$, and $VAR(\varepsilon_t | \Omega_{t-1}) = \sigma_t^2$. We considered higher-order ARMA specifications for the mean equation, but the MA(1) process seemed adequate. This is also reasonable theoretically, since we do not expect levels of returns to be very predictable.

3. Simple evidence of the need for asymmetric models

The results of this section are based on the residuals ε_t of an MA(1)-symmetric GARCH(1,1) model. There has not been much discussion in the literature of how to easily detect the existence or measure the degree of asymmetry in stock returns without actually fitting an asymmetric specifications (even though the importance of asymmetries is widely acknowledged). We consider the very simple method of computing the average values of ε_t^2 following positive and negative shocks, respectively. These results are given below.

	Frequency	$\sum\varepsilon_{t+1}^2$	Average of ε_{t+1}^2
Negative ε_t	8,769	1.251553	0.000143
Positive ε_t	8,810	1.029605	0.000117
Difference	-41	0.221947	0.000026

There are 8,769 negative ε_t, and the average value of ε_{t+1}^2 given $\varepsilon_t < 0$ is 0.000143. Similarly there are 8,810 positive ε_t, and the average value of ε_{t+1}^2 given $\varepsilon_t > 0$ is 0.000117. That is, negative shocks tend to be followed by larger squared errors than positive shocks.

4. Simple evidence of the need for long memory models

Persistent autocorrelations of absolute returns have been much discussed in the literature. For example, Ding and Granger(1996) and Ding, Granger, and Engle (1993) describe the long memory property of S&P daily 500 stock market absolute returns. The absolute values of returns have significantly positive serial correlations up to 2,700 lags. In our example, we are more interested in persistent autocorrelations of the squared errors. This would be in line with the stylized fact that volatility (variance) shows mean-reverting long memory while returns are stationary.

As before, let ε_t be based on the residual from the fitted MA(1)-symmetric GARCH(1,1) model, based on 17,582 daily observations as above. The table below gives the autocorrelations of our ε_t and ε_t^2. The sample autocorrelations of the squared errors are consistently positive until 2,683 lags. They decrease rather quickly for very small lags, but then decay very, very slowly. For example, the 10-period autocorrelation of the squared errors is 0.115, and by 300 periods it has decreased only to 0.075. This very slow decay at long lags is persuasive evidence of long memory and suggests the applicability of a fractionally integrated model for the variance.

In contrast, autocorrelations of the ε_t are very small at all lags, and are sometimes positive and sometimes negative. This is as expected since returns themselves should be unforecastable.

Figures 1 and 2 give a graphical display of the autocorrelations of ε_t^2 and ε_t, and also support the conclusions given above.

Order \ ρ	ρ of ε_t^2	ρ of ε_t
1	0.249	0.002
10	0.115	0.008
20	0.094	-0.017
30	0.093	0.011
50	0.068	-0.001
100	0.049	-0.008
200	0.064	0.007
300	0.075	0.001
1,000	0.014	0.002
5,000	0.001	0.003

II. Methodology

1. List of Models

The conditional variance equation is assumed to follow one of the following short or long memory family GARCH models:

$$\sigma_t^{\lambda} = \kappa + \alpha\sigma_{t-1}^{\lambda} f^{\nu}(\varepsilon_{t-1}) + \delta\sigma_{t-1}^{\lambda}\text{, for the asymmetric family GARCH;} \quad (3)$$

$$\sigma_t^{\lambda} = \frac{\kappa}{1-\delta} + \left[1 - \frac{(1-\phi L)(1-L)^d}{1-\delta L}\right]\sigma_t^{\lambda} f^{\nu}(\varepsilon_t)\text{, for the asymmetric family FIGARCH;} \quad (4)$$

where in either case

$$f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right),\ |c| \le 1\,. \tag{5}$$

As in chapter 2, we have $\varepsilon_t = \sigma_t\omega_t$, $\omega_t \sim i.i.d.\ \ D(0,1)$. The members of the short memory family of eq. (3) are listed in Table 1 of the previous chapter. For the "short memory" family members we have $d = 0$, whereas the "long memory" models have $d \neq 0$. Similarly, the "symmetric" models have $b = c = 0$, whereas "asymmetric" models have b and/or $c \neq 0$. Some special symmetric models that are of interest are as follows:

$$\sigma_t^2 = \kappa + \alpha\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2\text{, for GARCH(1,1) } (\lambda = \nu = 2,\ d = 0) \tag{6}$$

$$\sigma_t^\lambda = \kappa + \alpha\sigma_{t-1}^\lambda|\varepsilon_{t-1}/\sigma_{t-1}|^\lambda + \delta\sigma_{t-1}^\lambda\text{, for NGARCH(1,1) } (\lambda = \nu,\ d = 0) \tag{7}$$

$$\sigma_t^\lambda = \kappa + \alpha\sigma_{t-1}^\lambda|\varepsilon_{t-1}/\sigma_{t-1}|^\nu + \delta\sigma_{t-1}^\lambda\text{, for FGARCH(1,1) } (d = 0) \tag{8}$$

$$\sigma_t^2 = \kappa/(1-\delta) + [1-(1-\phi L)(1-L)^d/(1-\delta L)]\varepsilon_t^2\text{, for FIGARCH(1,}d\text{,1) } (\lambda = \nu = 2,\ d \text{ unrestricted}) \tag{9}$$

$$\sigma_t^\lambda = \kappa/(1-\delta) + [1-(1-\phi L)(1-L)^d/(1-\delta L)]|\varepsilon_t/\sigma_t|^\lambda\sigma_t^\lambda\text{, for FINGARCH(1,}d\text{,1) } (\lambda = \nu,\ d \text{ unrestricted}) \tag{10}$$

$$\sigma_t^{\lambda} = \kappa/(1-\delta) + [1-(1-\phi L)(1-L)^d/(1-\delta L)]|\varepsilon_t/\sigma_t|^{\nu}\sigma_t^{\lambda}\text{, for FIFGARCH}(1,d,1)\ (\lambda,\nu,\ d \text{ unrestricted}) \quad (11)$$

We can also consider the asymmetric versions of each of the models. These are models of the general form of equation (4) above, but with the various restrictions given above. We will denote these with the same names just used, preceded by "ASYMMETRIC", so that, for example, ASYMMETRIC NGARCH(1,1) corresponds to $\lambda = \nu$, $d = 0$, but $b \neq 0$ and/or $c \neq 0$.

2. Measures of Fit for Comparing Different Models

(1) Log likelihood values

One measure of fit is simply the maximized value of the log likelihood. If we compare two different models with the same number of parameters (e.g. NGARCH (1,1) vs. FIGARCH(1,d,1)), it is fair to say that the model with the higher log likelihood value fits the data better. If we compare different models with different numbers of parameters, comparisons are less clear because extra parameters will tend to improve the fit. However, nested models can be compared simply using likelihood ratio tests. All of our models are special cases of the ASYMMETRIC FIFGARCH model, and we can use likelihood ratio tests to test the restrictions that they impose. This statement assumes that the regularity conditions necessary for standard inference from the QMLE are satisfied, so we will now give a brief review of the literature on the QMLE in GARCH-type models.

Maximum likelihood (ML) or quasi-maximum likelihood (QML) are often employed for the estimation of various GARCH models, while the generalized method of moments (GMM)[1] is generally utilized for stochastic volatility models. Several authors have recently developed Bayesian methods for GARCH[2] and stochastic volatility models.[3] The simplicity of MLE is an attractive advantage over other methodologies. Under the assumption of conditional normality, the log likelihood is as given in equation (20) of chapter 2. It is well known that under certain regularity conditions, the normal (Q)MLE of the GARCH(1,1) model, say $\hat{\Theta}_T$, is consistent and asymptotically normally distributed:

$$T^{1/2}\left(\hat{\Theta}_T - \Theta_0\right) \rightarrow N(0, Var(\hat{\Theta}_T)) . \qquad (12)$$

Lumsdaine (1992) provides a proof of the consistency and asymptotic normality of the ML-estimator for the GARCH(1,1) and IGARCH(1,1) models under the condition that $E\left[\ln\left(\alpha\omega_t^2 + \delta\right)\right] < 0$. Unlike models with a unit root in the conditional mean, the ML estimator has the same limiting distribution in models with and without a unit root in the conditional variance. Bollerslev and Wooldridge (1992) and Gouriéroux (1992) showed that the quasi-MLE of Θ for the GARCH model, obtained by maximizing the normal log likelihood function even though the true probability density function is non-normal, is consistent and asymptotically normal. Weiss (1986) showed this earlier for the ARCH model.

1. See Glosten *et al.* (1993).
2. See Jacquier *et al.* (1994).
3. See Geweke (1994).

Lee and Hansen (1994) prove the consistency and asymptotic normality of the QMLE of the Gaussian GARCH(1,1) model, where $\omega_t = \varepsilon_t / \sigma_t$ need neither be normally distributed nor independent over time. A simulation study by Bollerslev and Wooldridge (1992) found that the QMLE is close to the exact MLE for symmetric departures from conditional normality, in finite samples. However, for nonsymmetric conditional distributions, both in small and large samples the loss of efficiency of QML compared to exact ML can be quite substantial. Palm (1996) argues that semi-parametric density estimation as proposed by Engle and Gonzalez-Rivera (1991) using a linear spline with smoothness priors would be an attractive alternative to QMLE. Palm (1996) also suggests that indirect inference as in Gouriéroux and Monfort (1993) and the efficient method of moments of Gallant *et al.* (1994) would be good alternatives when (Q)MLE is difficult to apply.

How good the normal QMLE is depends firstly on the distribution assumptions of the errors. Although GARCH combined with conditional normality generates fat-tails in the unconditional distribution, it does not fully account for the excess-kurtosis present in many financial data. The t-distribution, normal-Poisson mixture, the normal-lognormal mixture, generalized error distribution, Bernoulli-normal mixture, and stable distribution are used by numerous authors in this context.

Secondly, asymmetric errors raise questions that have not been addressed rigorously in the literature. Not much is known about the properties of the normal QMLE in the presence of asymmetry, nor has anyone checked whether Hentschel's FGARCH models satisfy standard regularity conditions for MLE.

Thirdly, the long memory mean reverting features present in financial data are not included in short memory GARCH, while the FIGARCH model has no terms to represent asymmetry. Brock and Lima (1996) suggest that the asymptotic properties of the (quasi)-maximum likelihood estimators discussed by Baillie *et al.* (1996) rely on the verification of a set of conditions put forward by Bollerslev and Wooldridge (1992) and that it is not yet known whether those conditions are satisfied for FIGARCH or FIEGARCH processes. Proceeding to the ASYMMETRIC FIFGARCH model (and its special cases), it is similarly true that, while there is no specific reason to doubt that the required regularity conditions hold, this has not been verified.

(2) Fit of σ_t^2 to ε_t^2

Since $E\left(\varepsilon_t^2 \middle| \Omega_{t-1}\right) = \sigma_t^2$, any measure of the closeness of ε_t^2 to σ_t^2 can be a reasonable measure of the fit of the model. For example, the normal log likelihood value is proportional to $\sum_t \left[\log\left(\sigma_t^2\right) + \varepsilon_t^2/\sigma_t^2\right]$. A more direct measure of fit might be the sum of squared differences:

$$SSD = \sum_t \left(\varepsilon_t^2 - \sigma_t^2\right)^2$$

or the sum of absolute differences:

$$SAD = \sum_t \left|\varepsilon_t^2 - \sigma_t^2\right| .$$

The SAD measure puts less weight on extreme observations.

(3) Comparison of actual and theoretical autocorrelation of ε_t^2

Let ρ_k be the correlation between ε_t^2 and ε_{t-k}^2 $(k = 1,2,\dots)$ implied by a particular model. The form of these correlations was derived in the previous chapter for some of the models we consider. For more complicated models, these correlations could be calculated by simulation, given specified values of the parameters. Now let $\hat{\rho}_k$ be the sample autocorrelation between ε_t^2 and ε_{t-k}^2. If the model is correct, the $\hat{\rho}_k$ should be close to the ρ_k, and this is a basis for measures of model adequacy. We will consider the measure

$$SSRHO = \sum_{i=1}^{n} (\rho_i - \hat{\rho}_i)^2$$

where "n" indicates the highest order autocorrelation that we seek to match.

III. Empirical Results

In this section and Tables 1-6, we report our empirical results.

Table 1 gives the quasi-MLE estimates of the parameters, with robust standard errors in parentheses; the log likelihood values, denoted "Likelihood"; and the measure $SAD = \sum_t |\varepsilon_t^2 - \sigma_t^2|$ of the quality of the one-period-ahead forecasts of ε_t^2.

1. Log likelihood values

The ASYMMETRIC FIFGARCH model has the highest log likelihood value, as it must because it nests all of the other models. The differences in log likelihood values are quite large. There is very strong evidence of asymmetry, since each asymmetric model (b, c unrestricted) has a log likelihood value that is more than 100 larger than the value for the corresponding symmetric model ($b = c = 0$). There is also strong evidence of long memory. For example, the log likelihood for FIGARCH is about 80 larger than the log likelihood for GARCH; and similarly the log likelihood for ASYMMETRIC FIGARCH is about 80 larger than the log likelihood for ASYMMETRIC GARCH.

Broadly speaking, the results in Table 1 favor the ASYMMETRIC FIFGARCH model. All of its parameters except the intercept are statistically significant at usual levels. Furthermore, likelihood ratio tests would reject the restrictions that lead to any of the simpler models. We will discuss these tests in more detail below, but for now we simply note that the ASYMMETRIC FIFGARCH model may be said to have a "significantly" higher log likelihood value than the other models.

2. Fit of σ_t^2 to ε_t^2

We consider $SAD = \sum_t \left|\varepsilon_t^2 - \sigma_t^2\right|$, given in the last line of Table 1.

(1) Symmetric family

More complicated models generally have smaller values of SAD than simpler models. The exception is that NGARCH has a larger value of SAD than GARCH. The FIFGARCH model has the smallest value, 2.4217, and therefore fits best in the sense of SAD.

(2) Asymmetric family

The SAD values of 2.3823 of FIFGARCH indicates that it has the best predictive fit among all of the models, symmetric or asymmetric, long memory or short memory, power transformed or not. ASYMMETRIC FIGARCH is not favored over symmetric FIGARCH. Otherwise, models with more parameters are better in prediction than simpler ones. This is further evidence that relatively complicated models, such as ASYMMETRIC FIFGARCH, are supported by the data.

In Table 2 we present some additional information on the prediction of ε_t^2. Notably, we present $SSD = \sum_t \left(\varepsilon_t^2 - \sigma_t^2\right)^2$ as well as $SAD = \sum_t \left|\varepsilon_t^2 - \sigma_t^2\right|$. The comparison of SSD is much the same as the comparison of SAD. The ASYMMETRIC FIFGARCH model has the smallest value of SSD among all models considered. More generally, asymmetric models fit better than the corresponding symmetric models, and in fact more complicated models essentially always fit better than simpler ones. As before, we conclude that a data set of over 17,000 observations will support a fairly complex parameterization.

3. Comparison of actual and theoretical autocorrelations of ε_t^2

In this section we compare the sample and theoretical autocorrelations of ε_t^2, as a measure of the adequacy of our fitted models. For a given model, let $\hat{\rho}_j$ be the jth sample autocorrelation of the ε_t^2 and ρ_j be the jth theoretical autocorrelation, evaluated at the estimated parameter values. Our summary statistic is

$$\sum_{j=1}^{m} (\hat{\rho}_j - \rho_j)^2$$

where "m" is the highest-order autocorrelation considered. Because we have long memory in variance, large values of m may be relevant. We display results in Table 3 for values of m from 1 to 5,000.

We provide this measure for these models: GARCH, ASYMMETRIC GARCH, and ASYMMETRIC FIFGARCH. For GARCH and asymmetric GARCH the theoretical autocorrelations were calculated using results from chapter 2. For ASYMMETRIC FIFGARCH, they were obtained from a simulation. The simulation used $T = 17,582$ (as in the sample), with artificially generated data from the fitted ASYMMETRIC FIFGARCH model, with conditional normality assumed. 6,000 observations were generated and discarded (for purposes of initialization) before each artificial sample was drawn. The number of replications was 100; given the large sample, more replications did not change the results perceptibly.

We see in Table 3 that, according to this criterion, ASYMMETRIC GARCH is better than GARCH, and ASYMMETRIC FIFGARCH is better than ASYMMETRIC GARCH. This is more evidence of the relevance of allowing for both asymmetry and long memory.

4. Tests of hypotheses and further discussion of results

(1) Introduction

In this section we test various hypotheses concerning the parameters in our models. We are especially interested in testing the restrictions that convert our more complicated models into simpler ones.

We can test hypotheses in two ways. First, we can construct tests based on the (asymptotic) standard normal or chi-squared distributions, using the estimated variance matrix of the estimates. The usual t-tests fit this category of tests. An advantage is that the estimated asymptotic variance matrix is robust to violation of the assumption of conditional normality. Second, we can use likelihood ratio tests based on the maximized values of the likelihood functions. These tests are very simple, but depend on conditional normality for their validity. We will consider both types of tests.

We begin with some general remarks about the results for the ASYMMETRIC FIF-GARCH model, our most general and best-fitting model. All of the parameter estimates, except for the scale parameter κ, are significantly different from zero by the usual t statistics. The exponents λ and ν are significantly different from zero, one, two, and each other. The value of the long memory parameter d, 0.279, is significantly different from zero and from one-half, so that it supports the finding of long-memory in variance, but does not lead to nonstationarity or a failure of mean reversion. The values of b and c are significantly different from zero, and support the relevance of asymmetry.

(2) Asymmetry tests

The significant estimates of b and c support the idea that asymmetry is an important feature of daily U.S. stock returns. For the ASYMMETRIC FIFGARCH model, the "*shift*" parameter b is significantly different from zero based on its asymptotic t-statistic of 5.97. The "*rotation*" parameter c is significantly different from zero based on its asymptotic t-statistic of 2.31. The joint hypothesis $b = c = 0$ is decisively rejected with a value of χ_2^2 of 393.

Similar results are obtained in other models. In virtually every model, the null hypothesis of symmetry ($b = c = 0$) is rejected. See Table 4 for the relevant likelihood ratio statistics.

(3) Functional form tests

Family models provide easy step-by-step hypotheses test procedures because they imply sequentially nested functional forms. We begin by discussing the most general model, ASYMMETRIC FIFGARCH. Here we have $\hat{\lambda} = 1.295$, $\hat{\nu} = 1.629$. The ASYMMETRIC FINGARCH model imposes the restriction $\lambda = \nu$, and yields $\hat{\lambda} = \hat{\nu} = 1.548$. This restriction is rejected by the likelihood ratio test, with a test statistic $\left(\chi_1^2\right)$ of 5.60. Similarly, the ASYMMETRIC FIGARCH model imposes $\lambda = \nu = 2$, and this model is decisively rejected by the likelihood test, with a test statistic $\left(\chi_2^2\right)$ of 86.3. The restriction $\lambda = \nu = 2$ is also rejected in the ASYMMETRIC FINGARCH model, with a statistic $\left(\chi_1^2\right)$ of 80.71.

Similar results hold for the symmetric long memory models. See Table 5 for more details.

We conclude that the power transformations in the FIFGARCH and ASYMMETRIC FIFGARCH model are clearly supported for these data.

(4) Tests of short versus long memory

In the ASYMMETRIC FIFGARCH model, the long memory parameter d is very significantly different from zero. With $\hat{d} = 0.279$ and an asymptotic standard error of 0.034, we have an asymptotic t-statistic of 8.20. The estimate of d is in fact very significantly different from zero in every model, symmetric or asymmetric, in which it is estimated.

Long memory in variance is obviously a strong feature of these data.

Table 6 gives the results of more likelihood ratio tests, in which short memory models (symmetric and asymmetric) are tested against long memory alternatives. In all cases, short memory is rejected decisively.

VI. Conclusions

In this chapter we have applied the conditional variance models of chapter 2 to a long series of daily stock returns. The results support the empirical relevance of both asymmetry and long memory, as embodied in the ASYMMETRIC FIFGARCH model. This model was consistently found to be better than other simpler models according to log likelihood values, predictions of squared errors, and closeness of sample and theoretical autocorrelations of squared errors. Simpler models are also clearly rejected by likelihood ratio tests.

Further research will be needed to see whether the ASYMMETRIC FIFGARCH model performs well in other similar problems.

Table 1. Family GARCH and Family FIGARCH: 17,582 daily returns of the Standard & Poor Index, 1928/01/03-1993/09/30

	Specification											
Parameter	GARCH	NGARCH	FGARCH	FIGARCH	FINGAR	FIFGAR	ASYMM. GARCH	ASYMM. NGARCH	ASYMM. FGARCH	ASYMM. FIGARCH	ASYMM. FINGAR	ASYMM. FIFGAR
cnst	4.27E-04	4.30E-04	4.27E-04	4.56E-04	4.57E-04	4.56E-04	1.96E-04	1.92E-04	1.79E-04	2.53E-04	2.11E-04	1.98E-04
	(6.11E-05)	(6.09E-05)	(6.08E-05)	(6.14E-05)	(6.14E-05)	(6.13E-05)	(6.20E-05)	(6.19E-05)	(6.20E-05)	(6.12E-05)	(6.16E-05)	(6.20E-05)
θ	0.140	0.139	0.140	0.146	0.146	0.147	0.144	0.143	0.144	0.144	0.148	0.149
	(0.008)	(0.008)	(0.008)	(0.009)	(0.009)	(0.009)	(0.008)	(0.008)	(0.008)	(0.009)	(0.008)	(0.008)
κ	8.29E-07	3.09E-06	1.26E-04	2.30E-06	2.75E-06	2.15E-05	9.74E-07	5.92E-06	1.40E-04	1.39E-06	2.21E-05	9.70E-05
	(8.87E-08)	(1.25E-06)	(1.52E-04)	(2.72E-07)	(7.97E-07)	(1.69E-05)	(1.03E-07)	(2.41E-06)	(1.38E-04)	(3.43E-07)	(7.42E-06)	(6.16E-05)
δ	0.908	0.910	0.942	0.605	0.609	0.589	0.893	0.900	0.932	0.470	0.506	0.482
	(0.004)	(0.004)	(0.011)	(0.028)	(0.028)	(0.027)	(0.006)	(0.006)	(0.011)	(0.046)	(0.035)	(0.035)
α	0.087	0.095	0.056				0.078	0.085	0.054			
	(0.004)	(0.005)	(0.013)				(0.005)	(0.005)	(0.010)			
b							0.471	0.364	0.392	0.771	0.313	0.400
							(0.061)	(0.064)	(0.063)	(0.096)	(0.056)	(0.067)
c							0.048	0.127	0.107	(0.220)	0.206	0.134
							(0.039)	(0.048)	(0.047)	(0.044)	(0.055)	(0.058)
d				0.450	0.455	0.375				0.410	0.352	0.279
				(0.025)	(0.026)	(0.037)				(0.031)	(0.022)	(0.034)
ϕ				0.281	0.281	0.316				0.233	0.279	0.301
				(0.022)	(0.021)	(0.024)				(0.037)	(0.026)	(0.028)
λ	2.000	1.711	0.955	2.000	1.973	1.613	2.000	1.630	0.981	2.000	1.548	1.295
		(0.081)	(0.248)		(0.040)	(0.134)		(0.081)	(0.203)		(0.048)	(0.100)
ν	2.000	1.711	1.569	2.000	1.973	1.979	2.000	1.630	1.526	2.000	1.548	1.629
		(0.081)	(0.085)		(0.040)	(0.047)		(0.081)	(0.079)		(0.048)	(0.066)
Likelihood	58,737.14	58,743.74	58,747.08	58,818.34	58,818.57	58,821.77	58,893.17	58,901.58	58,905.57	58,975.32	59,015.67	59,018.47
SAD	2.478687	2.484656	2.464315	2.435937	2.428104	2.421737	2.456815	2.436221	2.416121	2.471729	2.39181	2.382273

Table 2. Forecasts of ε_t^2 from **symmetric** and **asymmetric** models

Sum of	Specification											
	GARCH	NGARCH	FGARCH	FIGARCH	FINGARCH	FIFGARCH	ASYMM. GARCH	ASYMM. NGARCH	ASYMM. FGARCH	ASYMM. FIGARCH	ASYMM. FINGARCH	ASYMM. FIFGARCH
ε	-3.735650	-3.772166	-3.724224	-4.162452	-4.167470	-4.152293	-0.171473	-0.113495	0.088665	-1.050001	-0.396103	-0.200773
ε^2	2.281167	2.280892	2.280961	2.283199	2.283154	2.283536	2.281672	2.281365	2.281480	2.281577	2.282958	2.283278
σ^2	2.312048	2.323603	2.286509	2.269550	2.251812	2.242991	2.331265	2.287848	2.247749	2.408417	2.252049	2.240387
$\lvert\varepsilon^2-\sigma^2\rvert$	2.478687	2.484656	2.464315	2.435937	2.428104	2.421737	2.456815	2.436221	2.416121	2.471729	2.391810	2.382273
$(\varepsilon^2-\sigma^2)^2$	0.006253	0.006246	0.006250	0.006134	0.006133	0.006147	0.006130	0.006111	0.006103	0.006045	0.005964	0.005953
ε^3	-0.010583	-0.010620	-0.010594	-0.010609	-0.010615	-0.010582	-0.009096	-0.009096	-0.009009	-0.009450	-0.009089	-0.008990
Likelihood	58,737.14	58,743.74	58,747.08	58,818.34	58,818.57	58,821.77	58,893.17	58,901.58	58,905.57	58,975.32	59,015.67	59,018.47

Table 3. Comparison of sample and theoretical autocorrelations of ε_t^2

Entries in the table are $\sum_{j=1}^{m} (\hat{\rho}_j - \rho_j)^2$

m	GARCH	ASYMM. GARCH	ASYMM. FIFGARCH
1	0.0495	0.0275	0.0030
10	0.8735	0.5256	0.0501
20	2.0823	1.2150	0.1152
30	3.1669	1.7394	0.1564
50	5.0761	2.4692	0.2179
100	8.4068	3.1920	0.3411
200	10.8461	3.2827	0.4844
300	11.4038	3.4118	0.5972
1,000	11.6789	4.0499	1.3233
5,000	12.1544	4.5390	2.9024

Table 4. Likelihood ratio tests for asymmetry in volatility (with significance levels)

Maintained Hypothesis	H_0	H_A: $c = 0$, b free	H_A: $b = 0$, c free	H_A: b and c free
Short memory				
$\lambda = 2, \nu = 2$ GARCH	$b = c = 0$	310.50 (0.5%)	253.36 (0.5%)	312.06 (0.5%)
	$c = 0$, b free			1.56
	$b = 0$, c free			58.07 (0.5%)
$\lambda = \nu$ NGARCH	$b = c = 0$			315.68 (0.5%)
	$c = 0$, b free			
	$b = 0$, c free			
λ, ν free FGARCH	$b = c = 0$			316.98 (0.5%)
	$c = 0$, b free			
	$b = 0$, c free			
Long memory				
$\lambda = 2, \nu = 2$ FIGARCH	$b = c = 0$	106.01 (0.5%)	94.15 (0.5%)	113.96 (0.5%)
	$c = 0$, b free			7.85 (1.0%)
	$b = 0$, c free			19.81 (0.5%)
$\lambda = \nu$ FINGARCH	$b = c = 0$			394.67 (0.5%)
	$c = 0$, b free			
	$b = 0$, c free			
λ, ν free FIFGARCH	$b = c = 0$			393.41 (0.5%)
	$c = 0$, b free			
	$b = 0$, c free			

Table 5. Likelihood ratio tests of functional form in long memory models (with significance levels)

H_0 Maintained Hypothesis	H_A: $\lambda = \nu$	H_A: λ, ν free
Symmetric		
FIGARCH $\lambda = 2, \nu = 2$	0.46	6.85*** (<0.05)
FINGARCH $\lambda = \nu$		6.39** (<0.025)
Asymmetric		
FIGARCH $\lambda = 2, \nu = 2$	80.71*** (<0.005)	86.30*** (<0.005)
FINGARCH $\lambda = \nu$		5.59** (<0.025)

Table 6. Likelihood ratio tests of long memory and asymmetry (with significance levels)

H_0 / Maintained Hypothesis	H_A: *d* free, symmetric FIG	*d* free, symmetric FIN	*d* free, symmetric FIF	*d* free, asymmetric with *b* and *c* FIG	*d* free, asymmetric with *b* and *c* FIN	*d* free, asymmetric with *b* and *c* FIF
Symmetric						
GARCH $\lambda = 2, \nu = 2$	162.40* (<0.005)	162.86* (<0.005)	169.25* (<0.005)	476.36* (<0.005)	557.07* (<0.005)	562.66* (<0.005)
NGARCH $\lambda = \nu$		149.66* (<0.005)	156.05* (<0.005)		543.87* (<0.005)	549.46* (<0.005)
FGARCH λ, ν free			142.98* (<0.005)			542.78* (<0.005)
Asymmetric with *b* and *c*						
GARCH $\lambda = 2, \nu = 2$				164.30* (<0.005)	235.01* (<0.005)	250.60* (<0.005)
NGARCH $\lambda = \nu$					228.19* (<0.005)	233.80* (<0.005)
FGARCH λ, ν free						225.80* (<0.005)

Figure 1. Autocorrelations of squared errors

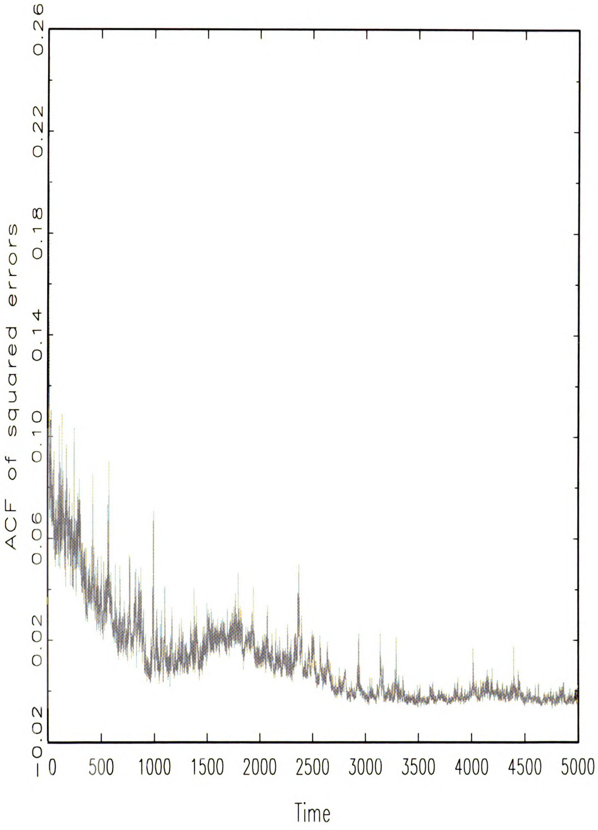
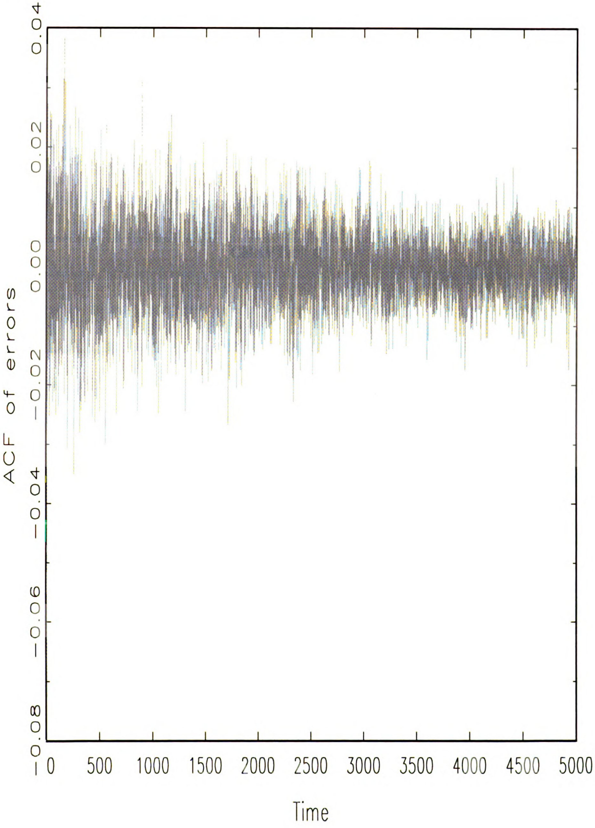


Figure 2. Autocorrelations of errors



CHAPTER 4

CHAPTER 4

Long memory in variance of exchange rates

I. Introduction

The long memory or persistence of autocorrelations of volatility has been documented in numerous articles on exchange rate returns as well as on stock returns. We apply the asymmetric long memory process in the conditional variance to exchange rate returns. We use daily data on DM/$ and Yen/$ exchange rates. In both cases, there is strong evidence of long memory, but asymmetric models were not really necessary for the DM/$ case.

II. The asymmetric long memory FIFGARCH model

The asymmetric FIFGARCH(1,d,1) model is defined as in chapter 2:

$$\sigma_t^{\lambda} = \frac{\kappa}{1-\delta} + \left[1 - \frac{(1-\phi L)(1-L)^d}{1-\delta L}\right] f^{\nu}(\varepsilon_t)\sigma_t^{\lambda}, \tag{1}$$

$$= \frac{\kappa}{1-\delta} + \lambda(L) f^{\nu}(\varepsilon_t)\sigma_t^{\lambda}, \tag{2}$$

where $f(\varepsilon_t) = \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)$, $|c| \leq 1$.

We name the following special cases of the FIFGARCH model:

FIEGARCH: $\lambda = 0, \nu = 1$, FITGARCH: $\lambda = 1, \nu = 1$, FIGARCH: $\lambda = 2, \nu = 2$, and FINGARCH: $\lambda = \nu$ but otherwise unrestricted.

III. Results and analysis

We consider daily data on both German Mark and Japanese Yen spot exchange rate returns from April 2, 1973 through February 13, 1998; which realizes a sample of $T = 6,241$ observations. The returns are defined as $\ln(p_t/p_{t-1})$ where p_t is the exchange rate in terms of foreign currency per dollar (e.g. DM/$).

1. Some relevant descriptive statistics

DM-$

Data\Stat.	Average	Std. dev.	Skewness	Kurtosis
p_t	2.055	0.453	0.517	-0.745
$\ln(p_t/p_{t-1})$	-7E-5	0.007	0.0738	3.553

Yen-$

Data\Stat.	Mean	Std. dev.	Skewness	Kurtosis
p_t	190.16	67.55	0.1507	-1.429
$\ln(p_t/p_{t-1})$	-0.00012	0.00623	5.6267	14.281

As in chapter 3, we calculate the average squared errors ε_{t+1}^2 (from a fitted MA(1) model), conditional on ε_t being positive or negative, as a simple measure of the need for an asymmetric model. The results are given below.

DM-$

	Frequency	$\sum\varepsilon_{t+1}^2$	Average of ε_{t+1}^2
Negative ε_t	3,089	0.135891	0.000044
Positive ε_t	3,149	0.141719	0.000045
Difference	-60	-0.005828	-0.000001

Yen-$

	Frequency	$\sum \varepsilon_{t+1}^2$	Average of ε_{t+1}^2
Negative ε_t	2,960	0.123525	0.000042
Positive ε_t	3,278	0.118329	0.000036
Difference	-318	0.005196	0.000006

Comparing the respective averages of squared errors for the DM/$ exchange rate (0.00044 vs. 0.00045) following negative and positive shocks, we see that they are very similar. There is certainly no strong evidence of the need for an asymmetric model. For the Yen/$ exchange rate, there is a larger difference (0.000042 vs. 0.000036, or about 17%), which is a slightly smaller percentage difference than was found in chapter 3 for stock returns. Thus there is some evidence of the need for an asymmetric model.

2. Germany (1973/04/02 - 1998/02/13 - 6,241 observations)

Table 1 gives the results from various models fitted to the DM/$ exchange rate return series. It gives the quasi-MLE estimates, robust standard errors, log likelihood values, and the sum of the absolute values of the one-period forecast errors of ε_t^2 (SAD). Table 2 gives more detail on the forecast errors, including the sum of squared forecast errors (SSD).

(1) Short memory GARCH models

In the GARCH(1,1) model, the constant term *cnst* (-0.000034) is insignificantly negative. The MA term θ (0.024) is insignificantly positive. The sum of $\delta+\alpha$ is 0.991, which is very close to unity, implying considerable persistence of volatility. The asymmetric GARCH(1,1) model gives similar results. This is not surprising, since the asymmetry parameters b (0.104) and c (-0.020) are small and statistically insignificant. The symmetric GARCH model fits better than the asymmetric GARCH model by the SAD criterion.

(2) Long memory GARCH models

Consider first the symmetric FIGARCH model. The differencing parameter d is estimated as 0.520 and is very significantly different from zero, so there is strong evidence of long memory. The FIGARCH model is also better (though not much better) than the GARCH model in terms of SAD and SSD.

Power transformation also seem to be supported by the data. The FIFGARCH model gives a significantly higher likelihood value than the FIGARCH model ($\chi_2^2 = 6.20$ is significant at the 5% level) and also yields smaller values of SAD and SSD. The FINGARCH model (FIFGARCH with the restriction $\lambda = \nu$) yields an insignificantly smaller log likelihood value ($\chi_1^2 = 1.36$) than FIFGARCH, and still smaller values of SAD and SSD; therefore FINGARCH may be preferred to FIFGARCH or FIGARCH. The FITGARCH model

($\lambda = \nu = 1$) is more controversial. It yields a much lower log likelihood value than FIGARCH, FIFGARCH or FINGARCH, but is superior in terms of SAD and SSD.

The results from asymmetric long memory models are generally quite similar to those from the corresponding symmetric models. For example, in the asymmetric FINGARCH model the asymmetry parameters b and c are individually and jointly insignificant. The same is true for the asymmetric FIFGARCH and FITGARCH models. The asymmetric models typically have very slightly smaller values of SAD and SSD than the corresponding symmetric models. Overall, there is just not much evidence of asymmetry.

Comparing the sample and theoretical autocorrelations of ε_t^2 (Table 3), the best models are FITGARCH and asymmetric FITGARCH.

(3) General conclusions for Germany

General conclusions for the DM/$ exchange rate returns are as follows. There is strong evidence of long memory in variance, but little evidence of asymmetry. In the general FIFGARCH model, the FINGARCH restriction $\lambda = \nu$ is supported by the data. The FINGARCH model (with $\lambda = \nu = 1.774$) is better than the FIGARCH model ($\lambda = \nu = 2$), and is better in some ways and worse in other ways than the FITGARCH model ($\lambda = \nu = 1$). Measures of fit other than log likelihood value would lead to the choice of the FITGARCH model as "best".

3. Japan (1973/04/02 - 1998/02/13)-Daily 6,241 observations

Tables 4A, 5A and 6 give the same results for the Yen/$ exchange rate as Table 1, 2 and 3 gave for the DM/$ rate. In addition, Tables 4B and 5B give some results for the FINGARCH and FIFGARCH $(1, d, 0)$ models

(1) Short memory GARCH models

The symmetric GARCH(1,1) model yields $\alpha + \delta = 0.993$, again indicating strong persistence of the variance. The MA(1) term θ is 0.046, which is higher than for Germany and significantly different from zero. In the ASYMMETRIC GARCH model, the asymmetry parameters b and c are very significant, and the log likelihood value is much higher than for the symmetric GARCH model.

(2) Long memory GARCH models

There is strong evidence of long memory in variance. For example, the differencing parameter d is estimated as 0.363 in the FIGARCH model and 0.312 in the ASYMMETRIC FIGARCH model. Alternative power transformations are possibly helpful. For example, the ASYMMETRIC FIFGARCH (λ, ν unrestricted) and ASYMMETRIC FINGARCH ($\lambda = \nu$ but otherwise unrestricted) models do not seem to improve on the ASYMMETRIC FIGARCH model; their log likelihood values are insignificantly higher and their SAD and SSD values are not much better. Interestingly, the ASYMMETRIC FITGARCH model ($\lambda = \nu = 1$) offers a clear improvement in terms of SAD and SSD even though its restriction would be rejected by a likelihood ratio test. The ASYMMET-

RIC FITGARCH model is also favored in terms of closeness of the theoretical and sample autocorrelations of the ε_t^2.

(3) General conclusions for Japan

There is strong evidence of long memory in variance. There is reasonably strong evidence also of asymmetry. As was the case for Germany, the FINGARCH restriction $\lambda = \nu$ is supported by the data, and the FITGARCH model ($\lambda = \nu = 1$) is favored in terms of measures of fit other than the likelihood value.

VI. Conclusions

To capture the long memory properties and asymmetry features in exchange rate returns, the asymmetric long memory FIFGARCH model was employed. The Yen/$ exchange rate returns are found to have significantly negative asymmetries, while the DM/$ returns are not. In both cases there was significant evidence of long memory in variance.

The full ASYMMETRIC FIFGARCH model was not necessarily required in either case. The FINGARCH restriction ($\lambda = \nu$) seemed useful, and not rejected by the data. The FITGARCH restriction ($\lambda = \nu = 1$) typically led to better measures of fit, even though this null hypothesis could be rejected by a likelihood ratio test. Thus the ASYMMETRIC FIFGARCH family proved useful here, as it did in chapter 3 for stock returns, even though the most general member of the family was not needed.

Table 1. Family GARCH and Family FIGARCH: 6,241 daily exchange returns of German Mark per U.S. Dollar, 1973/04/02-1998/02/13

	Specification											
Paramete	GARCH	FIEGARCH	FITGARCH	FIGARCH	FINGARCH	FIFGARCH	ASYMM. GARCH	ASYMM. FIEGARCH	ASYMM. FITGARCH	ASYMM. FIGARCH	ASYMM. FINGARCH	ASYMM. FIFGARCH
cnst	-3.34E-05	-1.97E-05	-8.95E-05	-6.10E-05	-5.90E-05	-5.50E-05	-6.02E-05	-3.45E-05	-1.15E-04	-8.91E-05	-8.91E-05	-8.14E-05
	(7.06E-05)	(6.16E-05)	(6.95E-05)	(6.99E-05)	(6.97E-05)	(7.01E-05)	(7.21E-05)	(6.79E-05)	(7.27E-05)	(7.17E-05)	(7.17E-05)	(7.16E-05)
θ	0.024	0.027	0.022	0.022	0.022	0.022	0.023	0.026	0.021	0.022	0.022	0.022
	(0.014)	(0.013)	(0.013)	(0.014)	(0.014)	(0.014)	(0.014)	(0.013)	(0.013)	(0.014)	(0.014)	(0.014)
κ	6.68E-07	7.75E-03	4.58E-04	1.52E-06	6.47E-06	1.11E-07	6.64E-07	7.44E-03	4.66E-04	1.52E-06	6.67E-06	1.48E-07
	(1.22E-07)	(2.38E-03)	(7.71E-05)	(2.43E-07)	(4.74E-06)	(3.02E-07)	(1.22E-07)	(2.33E-03)	(7.81E-05)	(2.45E-07)	(4.80E-06)	(2.77E-07)
δ	0.893		0.729	0.617	0.652	0.638	0.892		0.725	0.610	0.646	0.634
	(0.009)		(0.044)	(0.045)	(0.044)	(0.047)	(0.009)		(0.044)	(0.046)	(0.045)	(0.047)
α	0.098	0.197					0.099	0.194				
	(0.008)	(0.029)					(0.009)	(0.028)				
b							0.104	3.85E-04	-0.015	0.024	0.053	0.053
							(0.095)	(0.006)	(0.055)	(0.103)	(0.085)	(0.088)
c							-0.020	0.076	0.076	0.026	0.014	0.016
							(0.053)	(0.028)	(0.051)	(0.063)	(0.053)	(0.050)
d		0.862	0.696	0.520	0.566	0.613		0.862	0.691	0.515	0.561	0.609
		(0.025)	(0.068)	(0.048)	(0.056)	(0.059)		(0.025)	(0.068)	(0.048)	(0.055)	(0.055)
ϕ		0.220	0.155	0.213	0.202	0.164		0.225	0.154	0.209	0.198	0.159
		(0.149)	(0.034)	(0.032)	(0.031)	(0.043)		(0.146)	(0.033)	(0.032)	(0.031)	(0.040)
λ	2.000	0.000	1.000	2.000	1.774	2.467	2.000	0.000	1.000	2.000	1.770	2.420
					(0.117)	(0.458)					(0.115)	(0.317)
ν	2.000	1.000	1.000	2.000	1.774	1.973	2.000	1.000	1.000	2.000	1.770	1.951
					(0.117)	(0.129)					(0.115)	(0.101)
Likelihood	22,938.08	22,891.53	22,933.95	22,947.46	22,949.88	22,950.56	22,939.75	22,895.50	22,935.89	22,948.82	22,951.39	22,952.11
SAD	0.308984	0.321425	0.301696	0.308090	0.303479	0.303516	0.309186	0.321654	0.301370	0.307986	0.303463	0.303399

Table 2. Forecasts of ε_t^2 from **symmetric** and **asymmetric** models in exchange returns of German Mark per U.S. Dollar

Sum of	Specification											
	GARCH	FIEGARCH	FITGARCH	FIGARCH	FINGARCH	FIFGARCH	ASYMM. GARCH	ASYMM. FIEGARCH	ASYMM. FITGARCH	ASYMM. FIGARCH	ASYMM. FINGARCH	ASYMM. FIFGARCH
ε	-0.242222	-0.324244	0.100135	-0.073985	-0.086022	-0.110653	-0.078676	-0.234938	0.256052	0.097464	0.097484	0.050418
ε^2	0.277614	0.277608	0.277615	0.277613	0.277616	0.277615	0.277607	0.277605	0.277634	0.277614	0.277618	0.277614
σ^2	0.292840	0.316445	0.281176	0.292609	0.284059	0.283852	0.293374	0.316593	0.280833	0.292846	0.284406	0.284051
$\lvert\varepsilon^2-\sigma^2\rvert$	0.308984	0.321425	0.301696	0.308090	0.303479	0.303516	0.309186	0.321654	0.301370	0.307986	0.303463	0.303399
$(\varepsilon^2-\sigma^2)^2$	6.54E-05	6.63E-05	6.45E-05	6.52E-05	6.48E-05	6.49E-05	6.54E-05	6.64E-05	6.45E-05	6.52E-05	6.48E-05	6.48E-05
ε^3	0.000102	0.000091	0.000148	0.000125	0.000123	0.000120	0.000124	0.000103	0.000169	0.000148	0.000148	0.000141
Likelihood	22,938.08	22,891.53	22,933.95	22,947.46	22,949.88	22,950.56	22,939.75	22,895.50	22,935.89	22,948.82	22,951.39	22,952.11

Table 3. Comparison of sample and theoretical autocorrelations of ε_t^2 in German exchange returns

Entries in the table are $\sum_{j=1}^{m} (\hat{\rho}_j - \rho_j)^2$

m	GARCH	ASYMM. GARCH	ASYMM. FIFGARCH	FITGARCH	ASYMM. FITGARCH
1	0.0762	0.0859	0.0173	0.0075	0.0078
10	0.8977	1.0071	0.2074	0.0838	0.0908
20	1.8339	2.0647	0.3939	0.1378	0.1497
30	2.6423	2.9929	0.5346	0.1657	0.1790
50	3.9094	4.4847	0.7423	0.1927	0.2058
100	5.7362	6.7484	1.1142	0.2348	0.2469
200	6.6327	8.0173	1.4974	0.2922	0.3081
300	6.7658	8.2436	1.7324	0.3535	0.3689
1,000	6.9022	8.4234	2.4547	0.6262	0.6477
5,000	7.5065	9.0278	5.1250	2.1360	2.1641

Table 4A. Family GARCH and Family FIGARCH: 6,241 daily exchange returns of Japanese Yen per U.S. Dollar, 1973/04/02-1998/02/13

Parameter	Specification											
	GARCH	FIEGARCH	FITGARCH	FIGARCH	FINGARCH	FIFGARCH	ASYMM. GARCH	ASYMM. FIEGARCH	ASYMM. FITGARCH	ASYMM. FIGARCH	ASYMM. FINGARCH	ASYMM. FIFGARCH
cnst	-5.21E-05	-1.89E-04	3.36E-05	-1.52E-05	-2.41E-05	-2.33E-05	-5.34E-05	-2.19E-04	-1.52E-04	-9.08E-05	-8.82E-05	-8.59E-05
	(6.33E-05)	(4.81E-05)	(6.34E-05)	(5.87E-05)	(5.94E-05)	(5.91E-05)	(6.21E-05)	(6.63E-05)	(6.62E-05)	(6.08E-05)	(6.04E-05)	(6.06E-05)
θ	0.046	0.046	0.042	0.048	0.046	0.046	0.040	0.044	0.060	0.058	0.052	0.053
	(0.014)	(0.015)	(0.014)	(0.015)	(0.015)	(0.015)	(0.014)	(0.015)	(0.014)	(0.015)	(0.015)	(0.015)
κ	5.12E-07	1.66E-02	9.63E-04	1.99E-06	1.91E-07	1.49E-07	3.27E-07	1.78E-02	7.46E-04	2.44E-07	4.89E-07	1.45E-07
	(8.14E-08)	(3.99E-03)	(1.44E-07)	(3.72E-07)	(8.89E-08)	(2.97E-07)	(5.05E-08)	(4.37E-03)	(1.23E-04)	(1.27E-07)	(2.54E-07)	(2.34E-07)
δ	0.894		0.473	0.324	0.159	0.158	0.895		0.547	0.783	0.152	0.182
	(0.010)		(0.072)	(0.089)	(0.027)	(0.031)	(0.009)		(0.066)	(0.086)	(0.255)	(0.312)
α	0.099	0.357					0.130	0.356				
	(0.010)	(0.020)					(0.013)	(0.020)				
b							0.532	0.027	0.298	0.533	0.480	0.463
							(0.068)	(0.051)	(0.035)	(0.061)	(0.069)	(0.071)
c							-0.321	-0.022	-0.037	-0.072	-0.078	-0.067
							(0.035)	(0.048)	(0.037)	(0.038)	(0.034)	(0.037)
d		0.701	0.468	0.363	0.316	0.320		0.704	0.471	0.312	0.283	0.300
		(0.016)	(0.037)	(0.023)	(0.021)	(0.036)		(0.018)	(0.036)	(0.030)	(0.026)	(0.036)
ϕ		0.302	0.187	0.115	-8.82E-08	0.001		0.307	0.278	0.718	0.071	0.110
		(0.046)	(0.057)	(0.085)	(2.21E-04)	(0.014)		(0.046)	(0.059)	(0.106)	(0.248)	(0.308)
λ	2.000	0.000	1.000	2.000	2.291	2.328	2.000	0.000	1.000	2.000	2.115	2.289
					(0.444)	(0.288)					(0.062)	(0.226)
ν	2.000	1.000	1.000	2.000	2.291	2.292	2.000	1.000	1.000	2.000	2.115	2.105
					(0.444)	(0.442)					(0.062)	(0.062)
Likelihood	23,383.26	23,291.80	23,431.46	23,473.67	23,488.19	23,488.19	23,421.94	23,293.55	23,448.76	23,513.61	23,513.98	23,514.31
SAD	0.286924	0.331921	0.276801	0.278558	0.288033	0.287806	0.301524	0.331299	0.275074	0.287744	0.288130	0.287176

Table 4B. Family GARCH and Family FIGARCH(1,d,0): 6,241 daily exchange returns of Japanese Yen per U.S. Dollar, 1973/04/02-1998/02/13

Parameter	Specification			
	FINGARCH (1,d,0)	FIFGARCH (1,d,0)	ASYMM.(1,d,0) FINGARCH	ASYMM.(1,d,0) FIFGARCH
cnst	-2.41E-05 (5.91E-05)	-2.33E-05 (6.03E-05)	-8.97E-05 (6.02E-05)	-8.79E-05 (6.03E-05)
θ	0.046 (0.015)	0.046 (0.015)	0.052 (0.015)	0.052 (0.015)
κ	1.91E-07 (8.89E-08)	1.49E-07 (3.07E-07)	5.16E-07 (2.48E-07)	1.69E-07 (2.61E-07)
δ	0.159 (0.027)	0.159 (0.027)	0.079 (0.029)	0.070 (0.033)
α				
b			0.481 (0.069)	0.466 (0.071)
c			-0.077 (0.033)	-0.067 (0.037)
d	0.316 (0.021)	0.320 (0.037)	0.279 (0.022)	0.294 (0.031)
ϕ				
λ	2.291 (0.044)	2.328 (0.297)	2.119 (0.060)	2.284 (0.225)
ν	2.291 (0.044)	2.292 (0.044)	2.115 (0.062)	2.109 (0.061)
Likelihood	23,488.19	23,488.19	23,513.94	23,514.24
SAD	0.288033	0.287795	0.28818049	0.287258

Table 5A. Forecasts of ε_t^2 from <u>symmetric</u> and <u>asymmetric</u> models in exchange returns of Japanese Yen per U.S. Dollar

	Specification											
Sum of	GARCH	FIEGARCH	FITGARCH	FIGARCH	FINGARCH	FIFGARCH	ASYMM. GARCH	ASYMM. FIEGARCH	ASYMM. FITGARCH	ASYMM. FIGARCH	ASYMM1. FINGARCH	ASYMM. FIFGARCH
ε	-0.420994	0.395619	-0.935122	-0.639422	-0.587867	-0.592676	-0.414820	0.578303	0.169992	-0.187870	-0.203818	-0.217774
ε^2	0.241858	0.241855	0.241953	0.241910	0.241886	0.241887	0.241833	0.241876	0.241974	0.241948	0.241888	0.241891
σ^2	0.255400	0.319518	0.239535	0.239861	0.256290	0.256061	0.279109	0.319309	0.238429	0.258741	0.259080	0.258103
$\lvert\varepsilon^2-\sigma^2\rvert$	0.286924	0.331921	0.276801	0.278558	0.288033	0.287806	0.301524	0.331299	0.275074	0.287744	0.288130	0.287176
$(\varepsilon^2-\sigma^2)^2$	7.17E-05	8.84E-05	7.03E-05	7.18E-05	7.31E-05	7.30E-05	7.53E-05	8.76E-05	7.02E-05	7.34E-05	7.33E-05	7.27E-05
ε^3	-0.000453	-0.000358	-0.000513	-0.000478	-0.000472	-0.000473	-0.000452	-0.000336	-0.000383	-0.000425	-0.000427	-0.000429
Likelihood	23,383.26	23,291.80	23,431.46	23,473.67	23,488.19	23,488.19	23,421.94	23,293.55	23,448.76	23,513.61	23,513.98	23,514.31

Table 5B. Forecasts of ε_t^2 from **symmetric** and **asymmetric** Family FIGARCH(1,d,0) models in exchange returns of Japanese Yen per U.S. Dollar

Sum of	FINGARCH (1,d,0)	FIFGARCH (1,d,0)	ASYMM.(1,d,0) FINGARCH	ASYMM.(1,d,0) FIFGARCH
	Specification			
ε	-0.587855	-0.592657	-0.195000	-0.205895
ε^2	0.241886	0.241887	0.241883	0.241885
σ^2	0.256290	0.256043	0.259083	0.258123
$\lvert\varepsilon^2-\sigma^2\rvert$	0.288033	0.287795	0.288180	0.287258
$(\varepsilon^2-\sigma^2)^2$	7.31E-05	7.30E-05	7.33E-05	7.27E-05
ε^3	-0.000472	-0.000473	-0.000426	-0.000427
Likelihood	23,488.19	23,488.19	23,513.94	23,514.24

Table 6. Comparison of sample and theoretical autocorrelations of ε_t^2 in Japan exchange returns

Entries in the table are $\sum_{j=1}^{m} (\hat{\rho}_j - \rho_j)^2$

m	GARCH	ASYMM. GARCH	ASYMM. FIFGARCH	FITGARCH	ASYMM. FITGARCH
1	0.1216	0.0584	0.0225	0.0183	0.0198
10	1.5083	0.7865	0.3191	0.1141	0.1127
20	2.8803	1.4785	0.5317	0.1336	0.1372
30	4.1302	2.0921	0.7029	0.1439	0.1494
50	6.1310	3.0416	0.9416	0.1555	0.1633
100	9.2555	4.5076	1.3387	0.1796	0.1876
200	11.5112	5.7681	1.7793	0.2209	0.2324
300	12.0518	6.2766	2.0497	0.2557	0.2681
1,000	12.2353	7.0891	2.7518	0.4962	0.5168
5,000	12.8318	10.6355	5.5943	1.8165	1.8408

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