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Local Etale Extensions And Normalizations

By

Mark S. McCormick

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ABSTRACT

Local Etale Extensions And Normalizations

By

Mark S. McCormick

An easy generalization of a result of C. Rotthaus yields that for A an excellent normal local domain with strict Henselization A^{hs} , any local normal intermediate ring $A \subseteq D \subseteq A^{hs}$ dominated by A^{hs} is a direct limit of local etale extensions of A . In a more general setting, we use this fact to characterize etaleness of intermediate rings in terms of properties of their normalizations. Specifically, let A be an excellent reduced local ring with strict Henselization A^{hs} , and let D be an intermediate ring $A \subseteq D \subseteq A^{hs}$ dominated by A^{hs} and having total quotient ring finitely generated over that of A . Then D is etale over A if and only if the normalization of D is semilocal and has an appropriate residual field structure.

Using analyses of normalizations, we study intersection rings under the Henselization and under the strict Henselization. We find that for A an excellent reduced local ring and $Q(A) \subseteq L \subseteq Q(A^h)$ an intermediate field obtained by adjoining idempotent elements to $Q(A)$, the intersection $L \cap A^h$ is a local etale extension of A . We present an example to show that this does not hold for A^{hs} in place of A^h . Then turning to the strict Henselization, we find that for A an excellent *Henselian* reduced local ring, $I \in \text{Min}(A^{hs})$ and $Q(A/(I \cap A)) \subseteq L \subseteq Q(A^{hs}/I)$ an intermediate field which

is finitely generated over $Q(A/(I \cap A))$, the intersection $L \cap (A^{hs}/I)$ is finite and unramified over $A/(I \cap A)$. We use this intersection theorem to characterize etaleness and Noetherianness for rings between a Henselian ring A and its strict Henselization in terms of the residual fields of their normalizations.

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Introduction

In this work we study the etaleness of intermediate rings $A \subseteq D \subseteq A^{hs}$ through an analysis of properties of their normalizations, where A is an excellent reduced local ring with strict Henselization A^{hs} and where D is a local ring dominated by A^{hs} . C. Rotthaus began the study of such rings, as part of a general project with W. Heinzer and S. Wiegand, in [Ro3] where she looked at intersections $L \cap A^h$ under the Henselization A^h of A , where L is an intermediate ring $Q(A) \subseteq L \subseteq Q(A^h)$ which is finitely generated over $Q(A)$. Her work utilizes some powerful connections between intermediate rings and their normalizations. In this work we generalize such techniques in the strict Henselization setting and prove that some nice properties of intermediate rings are determined by the residual field structure of their normalizations.

We prove in section 2.1 that the residual fields of the normalization $\tilde{A}$ of A determine the splitting of minimal prime ideals across the extension $A^h \rightarrow A^{hs}$ and observe that there is no splitting of minimal prime ideals if and only if the residual field of A is purely inseparable in each of the residual fields of $\tilde{A}$. While it is known that normalization and Henselization commute for A , $\tilde{A}^h \cong \widetilde{A^h}$. This is not so for strict Henselization. The most that can be said in general for an excellent reduced local ring A is that $\tilde{A}^{hs}$ is a localization of $\widetilde{A^{hs}}$ at certain maximal ideals (18.8.10 of EGA_{IV}). However, we observe in section 2.1 that $\tilde{A}^{hs} \cong \widetilde{A^{hs}}$ exactly when there is no splitting of minimal prime ideals across the extension $A^h \rightarrow A^{hs}$.

In section 3.1, we generalize a result of C. Rotthaus to state that for A an excel-

lent reduced normal local domain, *any* normal local intermediate ring $A \subseteq D \subseteq A^{hs}$ dominated by A^{hs} is a direct limit of local etale extensions of A (that is, of etale, local, and essentially of finite type extensions). This is an astonishingly powerful statement when one considers the strong hypotheses necessary in broader situations for an extension of normal domains to be etale, typically requiring at least unramifiedness in codimension one as in the body of work on purity of branch locus. We use this statement to prove that for A an excellent reduced local ring and $A \subseteq D \subseteq A^{hs}$ a local intermediate ring dominated by A^{hs} and having total quotient ring finitely generated over that of A , D is a local etale extension of A if and only if the normalization of D has an appropriate residual field structure.

In chapter 3 we return to the issue of the intersection rings of C. Rotthaus. In section 3.1 we prove that for $Q(A) \subseteq L \subseteq Q(A^h)$ an intermediate ring obtained by adjoining finitely many idempotent elements to the total quotient ring $Q(A)$ of A , the intersection $L \cap A^h$ is a local etale extension of A . This says that there is a unique minimal local etale extension $A \subseteq E \subseteq A^h$ having any possible splitting of minimal prime ideals. We present an example to show that this statement does not hold with A^{hs} in place of A^h . Thus the minimal prime ideal structure of an intermediate ring $A \subseteq D \subseteq A^{hs}$ can be an obstruction for such a ring to be etale. This suggests that the theory for A^{hs} is nicer upon reduction to the case of domains.

In section 3.2 we prove in a difficult theorem that for an intermediate ring $A^h \subseteq D \subseteq A^{hs}$ finitely generated over A^h and $p \in \text{Min}(D)$, D/p is birationally dominated by an unramified extension of A which is exhibited as an intersection. Specifically, we prove the following. Let A be an excellent Henselian reduced local ring, $P \in \text{Min}(A^{hs})$ and $p = P \cap A^h$. Then for $k(p) \subseteq L \subseteq k(Q)$ an intermediate field which is finitely generated over $k(p)$, the intersection $k(p) \cap (A^{hs}/Q)$ is a local domain which is finite and unramified over A .

We use this intersection theorem to characterize etaleness and Noetherianness of

intermediate rings $A^h \subseteq D \subseteq A^{hs}$ and give an easy application to homomorphic images of power series rings over $\mathbb{Q}$.

Intersections of the form studied in chapter 3 are closely related to a broad class of rings studied by W. Heinzer, C. Rotthaus, J. Sally and S. Wiegand. They observe that many of the well-known examples of rings with bad properties are intersections of the form $Q(S) \cap (\hat{S}/\mathfrak{a})$, where S is a localization of a polynomial ring over a field and where $\mathfrak{a} \subseteq \hat{S}$ is an ideal. Indeed, Nagata's first examples of non-excellent rings and Ogama's example of a noncatenary pseudogeometric normal domain are obtained in this way ([N], [O1]), as well as many other other examples ([Ro1], [Ro2], [BR1], [BR2], [W]). In light of this, the four are developing an extensive theory of such intersections in their recent papers [HR], [HRS], [HRW1], [HRW2] and [R4].

Conventions and Notations:

All rings are commutative with unity. For any ring R , $\text{Min}(R)$ represents the set of minimal prime ideals of R and $\text{Max}(R)$ represents the set of maximal ideals of R . For any prime ideal P of R , $k(P)$ is the quotient field of R/P . If S is an overring of R then $\text{IntClos}_S(R)$ represents the integral closure of R in S . If R is a reduced ring with finitely many minimal prime ideals, K_R is used to represent the total quotient ring of R , and $\tilde{R}$ represents the integral closure of R in K_R . Note that K_R is a finite product of fields. If R is a local ring, then m_R is the maximal ideal and $k_R := R/m_R$ (where the symbol $:=$ always means *is defined to be* or *is by definition*).

A morphism $R \xrightarrow{\phi} S$ of rings is said to be *essentially of finite type* if S is isomorphic as an R -algebra to a localization of a finitely generated R -algebra. If R and S are semilocal rings, then ϕ is said to be a *semilocal morphism* if for every maximal ideal N of S , $\phi^{-1}(N)$ is a maximal ideal of R , and if every maximal ideal of R can be obtained in this way.

For a semilocal ring $(R, (m_1, \dots, m_n), (k_1, \dots, k_n))$ we define the residual field

product to be

$$k_R := \prod_{i=1}^n k_i \cong R/\text{Rad}(R)$$

A semilocal morphism $R \longrightarrow S$ of semilocal rings induces a canonical morphism $k_R \hookrightarrow k_S$ of residual field products given by

$$R/\text{Rad}(R) \hookrightarrow S/\text{Rad}(S).$$

CHAPTER 1

Preliminaries

Definition 1.1

- i)* A morphism $A \longrightarrow B$ is said to be *etale* if it is 0-unramified (unramified) and 0-smooth (smooth) as in chapter 9, section 25 of [M]. This definition imposes no requirement on the manner in which B is generated over A .
- ii)* A local morphism $A \longrightarrow B$ of local rings is said to be a *local etale extension* if it is etale and essentially of finite type. If in addition $A \longrightarrow B$ is residually trivial, we say that B is an *etale neighborhood* of A .
- iii)* A semilocal morphism $A \longrightarrow B$ of semilocal rings is said to be a *semilocal etale extension* if it is etale and essentially of finite type.
- iv)* A local morphism $A \longrightarrow B$ of Noetherian local rings is said to be *regular* if it is flat and if for each prime ideal $p \subseteq A$ and each finite field extension L of $Q(A/p)$, the ring $B \otimes_A L$ is regular. In particular, the fibers of a regular morphism $A \longrightarrow B$ are regular.

Remark 1.2

- i)* There is little consistency in the literature about the definitions of *unramified* and *etale*, some authors requiring additionally that B be of finite type over A .

The above definition *i)* of Matsumura is the most general and makes no such assumptions.

- ii)* Unramifiedness is defined in [M] by the uniqueness of lifts of certain morphisms and this definition is equivalent to the vanishing of the module of differentials. That is a morphism $A \longrightarrow C$ of rings is unramified if and only if $\Omega_{C/A} = 0$. If an unramified morphism $A \longrightarrow C$ factors through an A -algebra B , then the morphisms $A \longrightarrow B \longrightarrow C$ induce an exact sequence

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

of modules of differentials (Thm. 25.1 of [M]). Then since $\Omega_{C/A} = 0$, we also have $\Omega_{C/B} = 0$. Therefore C is unramified over B . Thus we may always lift unramifiedness to unramifiedness over intermediate rings.

- iii)* An unramified, essentially of finite type morphism of Noetherian rings is etale if and only if it is flat (Prop. III.2.3 of [I]).
- iv)* For a morphism $A \longrightarrow B$ of Noetherian local rings smoothness is a stronger condition than regularity, but if in addition B is essentially of finite type over A then the notions of smoothness and regularity are equivalent. (Thm. 28.7 of [M] and Prop. III.3.3 of [I]).
- v)* For a semilocal etale extension $A \longrightarrow B$, A is normal (reduced) if and only if B is normal (reduced). To see this observe that for each maximal ideal N of B , the morphism $A_{N \cap A} \longrightarrow B_N$ is regular and then apply Theorem 32.2 of [M].

Let A be a ring and let

$$B = \left(\frac{A[X]}{(f)} \right)_{f'}$$

where $f \in A[X]$ is a monic polynomial and f' denotes its formal derivative. Then B is an étale A -algebra (ch. 2, Prop. 8 of [Ra]). The following important theorem states that all local étale extensions are localizations of étale A -algebras of this form. In particular, local étale extensions are all essentially finite extensions. The proof relies on Zariski's Main Theorem and can be found in [Ra].

Theorem 1.3 (Local Structure Theorem) *Let (A, m) be a local ring and let $A \rightarrow B$ be an essentially of finite type morphism of local rings. Then B is a local étale extension of A if and only if there is an isomorphism of A -algebras*

$$B \cong \left(\frac{A[X]}{(f)} \right)_Q,$$

where $f \in A[X]$ is a monic polynomial and where Q is a maximal ideal of $A[X]$ containing m but not containing f' . More generally, B is unramified over A if and only if B is a homomorphic image of such a structure.

Definition 1.4

- i) A local ring (A, m, k) is said to be *Henselian* if it satisfies Hensel's lemma, that is if whenever a monic polynomial $\bar{f} \in k[X]$ factors $\bar{f} = \bar{g}\bar{h}$ as a product of monic relatively prime polynomials $\bar{g}$ and $\bar{h}$, then there exist monic lifts f, g , and $h \in A[X]$ of $\bar{f}, \bar{g}$, and $\bar{h}$ respectively such that $f = gh$.
- ii) A local ring is said to be *strictly Henselian* if it is Henselian and if its residual field is separably closed.
- iii) A semilocal ring is said to be *Henselian* (resp. *strictly Henselian*) if it is a finite product of local Henselian rings (resp. local strictly Henselian rings).

Remark 1.5 A definition equivalent to i) above says that a local ring A is *Henselian* if every finite A -algebra decomposes as a product of local rings (ch. I of [Ra]).

If A is a Henselian local ring, B a finite A -algebra and B_i one of the local factors of B , then any finite B_i -algebra is also a finite A -algebra and so decomposes as a product of local rings since A is Henselian. It follows that each local factor B_i of B is Henselian and thus that B is a Henselian semilocal ring.

If A is an excellent Henselian local domain, then its normalization $\tilde{A}$ is a finite extension of A and thus a product of Henselian local rings by the above discussion. Since $\tilde{A}$ is also a domain, it must be local.

We give a brief summary of the construction and basic properties of Henselizations and strict Henselizations. For a more detailed treatment, see [Ra].

A local ring is Henselian if and only if it has no nontrivial etale neighborhoods, and strictly Henselian if and only if it has no nontrivial local etale extensions. Thus to construct a canonical Henselian (resp. strictly Henselian) ring from a local ring (A, m, k) , it is reasonable to try to close it with respect to etale neighborhoods (resp. local etale extensions). This is possible because the Local Structure Theorem gives a system of representatives of the etale neighborhoods (resp. local etale extensions) of A .

It can be shown that the system of representatives of etale neighborhoods forms a direct system whose limit is a Henselian local ring. This ring is called the *Henselization* of A and is denoted A^h . The extension $A \rightarrow A^h$ is local, flat, regular (if A is Noetherian), residually trivial, and m generates the maximal ideal of A^h . It follows that A and A^h have the same completion $\hat{A}$.

To speak of a direct limit of local etale extensions of A , we must consider the representatives of the local etale extensions together with morphisms of residual fields. Let Ω be a separable closure of k in some algebraic closure. Then the system of couples $(E_\lambda, \bar{\alpha}_\lambda)_{\lambda \in \Lambda}$, where E_λ is a local etale extension of A of the form given in the Local

Structure Theorem and having residual field k_{E_λ} , and where

$$\bar{\alpha}_\lambda : k_{E_\lambda} \longrightarrow \Omega$$

is a k -morphism, is a direct system whose limit is a local, strictly Henselian ring with residual field Ω , called the *strict Henselization* of A and denoted A^{hs} . Note that for each λ the canonical morphism $E_\lambda \longrightarrow A^{hs}$ induces the k -morphism $\bar{\alpha}_\lambda$ on residual fields. For an arbitrary local etale extension E of A there is no canonical morphism $E \longrightarrow A^{hs}$. Indeed, by Proposition 1.10 we have an isomorphism

$$\mathrm{Hom}_{\mathrm{loc}A}(E, A^{hs}) \cong \mathrm{Hom}_k(k_E, \Omega).$$

Thus to specify a A -morphism $E \longrightarrow A^{hs}$ one must choose a morphism of residual fields.

Though the construction of A^{hs} seems to depend on the choice of Ω , any two such constructions are isomorphic. The morphism $A \longrightarrow A^{hs}$ is similarly local, flat, regular (if A is Noetherian), and m generates the maximal ideal of A^{hs} . We also have that A and A^h have the same strict Henselization A^{hs} , so that $A \longrightarrow A^{hs}$ factors through A^h .

We now turn to some properties of local etale extensions which will be very useful in proving the results of the following sections.

Proposition 1.6 *Let A be a reduced semilocal ring and E be a semilocal etale extension of A . Then*

i) The total quotient ring K_E of E is finite over K_A and there is an isomorphism

$$K_E \cong E \otimes_A K_A.$$

ii) The normalization $\tilde{E}$ of E is given by $\tilde{E} \cong E \otimes_A \tilde{A}$.

Proof:

i) By Remark 1.2, E is reduced. By flatness of E over A , every regular element of A is regular in E . Thus $E \subseteq E \otimes_A K_A$. By base change $E \otimes_A K_A$ is etale over K_A and hence reduced. Let Q be a prime ideal of $E \otimes_A K_A$ and set $q := Q \cap K_A$. Then ring $(E \otimes_A K_A)_Q$ is a local etale extension of the field $(K_A)_q$. Since local etale extensions are essentially finite extensions, $(E \otimes_A K_A)_Q$ has dimension zero. It follows that $E \otimes_A K_A$ is zero-dimensional and thus isomorphic to K_E .

ii) Now $E \rightarrow E \otimes_A \tilde{A}$ is injective by faithful flatness of E over A . Since $\tilde{A}$ is an integral extensions of A , $E \otimes_A \tilde{A}$ is an integral extension of E . But since $E \otimes_A \tilde{A}$ is etale and essentially of finite type over the normal ring $\tilde{A}$ (ch. II, Prop. 2 of [Ra]), $E \otimes_A \tilde{A}$ is normal (ch. VII, sect. 2, Prop. 2 of [Ra]). Thus since by *i)* we have injections

$$E \hookrightarrow E \otimes_A \tilde{A} \hookrightarrow E \otimes_A K_A \cong K_E,$$

we must have $E \otimes_A \tilde{A} \cong \tilde{E}$. $\square$

Remark 1.7

i) It follows from *i)* of the above proposition that for A a reduced semilocal ring and B a direct limit of semilocal etale extensions of A having finitely many minimal prime ideals, the extension of total quotient rings $K_A \rightarrow K_B \cong B \otimes_A K_A$ is integral. Furthermore, for any intermediate ring $A \subseteq D \subseteq B$, the induced morphism $K_A \rightarrow D \otimes_A K_A \rightarrow B \otimes_A K_A \cong K_B$ is an integral extension of reduced zero dimensional semilocal rings. Hence the total quotient ring K_D of D is given by $K_D \cong D \otimes_A K_A$ and is also an integral extension of K_A .

ii) By ii) of the above proposition, it follows that for A a reduced semilocal ring, the normalizations of A^h and A^{hs} are obtained by application of $\otimes_A \tilde{A}$. If in addition A is Noetherian, then $\tilde{A}$ is semilocal by the Mori-Nagata integral closure theorem (sect. 33 of [M]). Then by 18.6.8 of [EGA_{IV}],

$$\widetilde{A^h} \cong A^h \otimes_A \tilde{A} \cong \tilde{A}^h.$$

It follows that there is a bijection

$$\text{Min}(A^h) \longleftrightarrow \text{Max}(\tilde{A}).$$

Proposition 1.8 *Let A be a Henselian local ring. Then every local etale extension E of A is finitely generated as an A -module.*

Proof:

By Theorem 1.3, E is a localization at a maximal ideal of a finite extension F of A , $E = F_Q$. Since A is Henselian, F is a finite product of local rings, $F = \prod_{i=1}^n F_i$. Thus $E = F_Q = F_i$ for some i . Then since E is a summand of a finite extension of A , E is finite over A . $\square$

Remark 1.9 Let (A, m, k) be a Noetherian local ring and let E be a local etale extension of A . Since

$$(A^h \otimes_A E) \otimes_A k \cong E \otimes_A k \cong k_E$$

is a field, m generates a maximal ideal Q of $A^h \otimes_A E$. Then $(A^h \otimes_A E)_Q$ is a local etale extension of A^h and hence by the previous proposition, a finite extension of A^h . Thus $(A^h \otimes_A E)_Q$ is a Henselian ring. Since it is also a direct limit of etale

neighborhoods of E , we must have

$$E^h \cong (A^h \otimes_A E)_Q.$$

In particular, E^h is etale and finite over A^h .

The following useful proposition (ch. VIII, sect. 1, Prop. 1 of [Ra]) is used again and again in this work to obtain morphisms from local etale extensions of a Henselian local ring to other finite extensions, or to prove that two such morphisms are isomorphic.

Proposition 1.10 *Let A be a Noetherian local ring, and let E and C be local A -algebras dominating A and with E a direct limit of local etale extensions of A . Let k_A , k_E and k_C be the respective residual fields. Then the canonical map*

$$\Phi : \text{Hom}_{\text{loc}A}(E, C) \longrightarrow \text{Hom}_{k_A}(k_E, k_C)$$

is injective. Furthermore, if C is Henselian, then Φ is an isomorphism.

Proposition 1.11 *Let (A, m, k) be a Noetherian local ring. Then for any finite separable field extension l of k , there is a finite local etale extension E of A with residual field $k_E \cong l$ as k -algebras. Furthermore, if A is Henselian then E is unique up to isomorphism.*

Proof:

Write $l \cong k[X]/(\bar{f})$ where $\bar{f} \in k[X]$ is a monic, irreducible and separable polynomial. Let $f \in A[X]$ be a monic preimage of $\bar{f}$. Set $E := A[X]/(f)$. Since E is finite over A , every maximal ideal of E lies over m . But by construction, $E \otimes_A k \cong l$. Thus E is local with residual field l .

Let Q be the preimage in $A[X]$ of the maximal ideal of E so that

$$E := \frac{A[X]}{(f)} \cong \left(\frac{A[X]}{(f)} \right)_Q.$$

Since $\bar{f}$ is separable, $\bar{f}$ and $\bar{f}'$ are relatively prime in $k[X]$. Thus for some $\bar{u}$ and $\bar{v}$ in $k[X]$ we have $\bar{u}\bar{f} + \bar{v}\bar{f}' = 1$. Then taking arbitrary preimages u and v in $A[X]$ of $\bar{u}$ and $\bar{v}$ respectively, there exists $w \in mA[X]$ such that

$$uf + vf' = 1 + w$$

Since w and f are contained in Q , $1 - vf' \in Q$. Thus f' is not in Q . By the Local Structure Theorem, E is a local etale extension of A .

Now suppose A is Henselian and that F is another local etale extension of A with residual field $k_F \cong l$ as k -algebras. Then there is a k -isomorphism

$$k_E \xrightarrow{\bar{\phi}} k_F.$$

By Proposition 1.10, since E and F are Henselian and local etale over A , we have isomorphisms

$$\mathrm{Hom}_{locA}(E, F) \cong \mathrm{Hom}_k(k_E, k_F) \text{ and}$$

$$\mathrm{Hom}_{locA}(F, E) \cong \mathrm{Hom}_k(k_F, k_E).$$

Thus $\bar{\phi}$ lifts to a local A -morphisms $\phi : E \rightarrow F$, and $\bar{\phi}^{-1}$ lifts to a local A -morphism $\psi : F \rightarrow E$. Then by construction, the composite $\psi\phi : E \rightarrow E$ induces the identity on k_E , and $\phi\psi : F \rightarrow F$ induces the identity on k_F . But since again by

Proposition 1.10

$$\mathrm{Hom}_{\mathrm{loc}A}(E, E) \cong \mathrm{Hom}_k(k_E, k_E) \text{ and}$$

$$\mathrm{Hom}_{\mathrm{loc}A}(F, F) \cong \mathrm{Hom}_k(k_F, k_F)$$

it follows that $\psi\phi$ is the identity on E , and $\phi\psi$ is the identity on F . Therefore ϕ is an isomorphism. $\square$

Remark 1.12 More generally, one can argue in a similar fashion that two direct limits of local etale extensions of a Henselian Noetherian local ring (A, m, k) having k -isomorphic residual fields are isomorphic.

The following proposition uses the module of differentials and André homology to characterize certain etale extensions.

Proposition 1.13 *Let $R \xrightarrow{\phi} S$ be an essentially of finite type morphism of Noetherian rings. Then ϕ is etale if and only if $\Omega_{S/R} = 0$ and $H_1(R, S, M) = 0$ for all S -modules M .*

Proof:

Suppose S is etale over R . Then since S is unramified over R , $\Omega_{S/R} = 0$ so that for all S -modules M we have

$$0 = \mathrm{Hom}_R(\Omega_{S/R}, M) \cong \mathrm{Der}_R(S, M) \cong H^0(R, S, M)$$

(ch. VI, Prop. 3 of [A1]). By smoothness of S over R we also have that $H^1(R, S, M) = 0$ for all S modules M (ch. XVI, Prop. 17 of [A1]). Since S is essentially of finite type over R , so is $S \otimes_R S$ and hence is Noetherian. Then since

$$0 = H^0(R, S, M) = H^1(R, S, M)$$

for all S -modules M , we may apply (ch. XV, Prop. 21, $2 \Rightarrow 4$ of [A1]) to obtain that $H_1(R, S, M) = 0$ for all S -modules M .

Conversely, assume $\Omega_{S/R} = 0$ and $H_1(R, S, M) = 0$ for all S -modules M . The former condition is equivalent to the unramifiedness of S over R . The latter condition implies that S is flat over R (ch. XV, Cor. 20 of [A1]). Thus S is etale over R by Remark 1.2. $\square$

Proposition 1.14 *Let A be a Noetherian ring, and let E and F be two etale, essentially of finite type A -algebras. Then any morphism $E \longrightarrow F$ of A -algebras is etale.*

Proof:

We consider the Zariski-Jacobi sequence induced by the morphism of A -algebras $E \longrightarrow F$ and by an arbitrary F -module M :

$$\begin{aligned} H_1(A, F, M) &\longrightarrow H_1(E, F, M) \longrightarrow \Omega_{E/A} \otimes_E M \longrightarrow \\ &\Omega_{F/A} \otimes_F M \longrightarrow \Omega_{F/E} \otimes_F M \longrightarrow 0 \end{aligned}$$

(ch. V, Thm. 1 of [A1]). By unramifiedness of F over A , we have $\Omega_{F/A} = 0$. Thus $\Omega_{F/E} \otimes_F M = 0$ for all F -modules M , implying that $\Omega_{F/E} = 0$. Using the previous proposition, by etaleness of F over A and of E over A we have $H_1(A, F, M) = 0$ and $\Omega_{E/A} = 0$. Thus

$$0 \longrightarrow H_1(E, F, M) \longrightarrow 0$$

is exact and so $H_1(E, F, M) = 0$ for all F -modules M . Then since

$$0 = \Omega_{F/E} = H_1(E, F, M),$$

$E \longrightarrow F$ is etale by the previous proposition. $\square$

Remark 1.15 Let (A, m, k) be a complete Noetherian local ring. A *coefficient ring* $R \subseteq A$ is a field or a homomorphic image of a complete p -ring (a DVR whose maximal ideal is generated by a prime number p), which satisfies $A = R + m$ and such that the local morphism $R \hookrightarrow A$ induces an isomorphism on residual fields. Every complete Noetherian local ring A has a coefficient ring. For a detailed discussion of this material, see section 29 of [M].

Proposition 1.16 *Let (A, m, k) be a complete Noetherian local ring and let R be a coefficient ring for A . Then for any local etale extension (E, m_E, k_E) of A there is a local etale extension S_E of R with residual field k_E and an isomorphism*

$$E \cong A \otimes_R S_E.$$

Under this isomorphism, S_E is identified with a coefficient ring of E .

Proof:

Fix a local etale extension E of A . By Proposition 1.11, there is a local etale extension S_E of R with residual field k_E . Since R is also Henselian, S_E is a finite R -module by Proposition 1.8. Thus $A \otimes_R S_E$ is finite and etale over A . In particular, every maximal ideal of $A \otimes_R S_E$ lies over m . Since R has residual field k and since the maximal ideal of R generates that of S_E , we have

$$k \otimes_A (A \otimes_R S_E) \cong k \otimes_R S_E \cong k_E.$$

Thus $A \otimes_R S_E$ is a *local* etale extension of A with residual field k_E . Then by Proposition 1.11, $E \cong A \otimes_R S_E$. $\square$

CHAPTER 2

Etale Extensions and Their Normalizations

2.1 The Splitting of Minimal Prime Ideals Across

$$A^h \hookrightarrow A^{hs}$$

Lemma 2.1.1 *Let k be a field, and let K and L be two algebraic field extensions of k with K purely inseparable over k and L finite separable over k . Then $K \otimes_k L$ is a field of degree over k given by*

$$[K \otimes_k L : k] = [L : k][K : k]$$

Proof:

First we argue that $K \otimes_k L$ is a field. We may assume neither K nor L is k . Since L is a finite separable field extension of k we may write $L = k[X]/(g)$, where $g \in k[X]$ is a monic, irreducible and separable polynomial of positive degree. Let F be a splitting field for g over k in some algebraic closure of K . Since K is purely inseparable over k we must have $K \cap F = k$.

Now if g factors over K , $g = hq$ where $h, q \in K[X]$ are monic polynomials of

positive degree, then the coefficients of h and q are in $K \cap F = k$, implying that g must factor over k , a contradiction. Therefore g is irreducible over K , and so

$$K \otimes_k L \cong \frac{K[X]}{(g)}$$

is a field.

Now to see the statement about degree, observe that

$$[K \otimes_k L : k] = [K \otimes_k L : K][K : k].$$

But by the above discussion $[K \otimes_k L : K] = \deg(g) = [L : k]$. $\square$

Lemma 2.1.2 *Let (A, m, k) be an excellent Henselian domain with normalization $(\tilde{A}, \tilde{m}, \tilde{k})$. The following are equivalent:*

- i) $\tilde{k}$ is purely inseparable over k .*
- ii) $\tilde{A}^{hs} \cong \widetilde{A^{hs}}$.*
- iii) A^{hs} is a domain.*

Proof:

$(i \Leftrightarrow iii)$: This is ch. IX, sect. 1, Cor. 1 of [Ra].

$(ii \Rightarrow iii)$: Since A is a Henselian local domain, $\tilde{A}$ is a normal local domain (ch. IX, sect. 1, Cor. 1 of [Ra]). Then $\tilde{A}^{hs}$ is also a normal local domain (ch. VII, sect. 2, Prop. 2 of [Ra]). So by *ii*), $\widetilde{A^{hs}}$ is a domain and hence A^{hs} is a domain.

(iii $\Rightarrow$ ii) : If A^{hs} is a domain, then $\widetilde{A^{hs}}$ is a local domain. By 18.8.10 of [EGA_{IV}], $\tilde{A}^{hs}$ is a localization of $\widetilde{A^{hs}}$ at some maximal ideals. Since $\widetilde{A^{hs}}$ is already local these rings are isomorphic. $\square$

Theorem 2.1.3 *Let (A, m, k) be an excellent reduced local ring with normalization $(\tilde{A}, (m_1, \dots, m_n), (k_1, \dots, k_n))$. The following are equivalent;*

- i) *For each i , k_i is purely inseparable over k .*
- ii) $\tilde{A}^{hs} \cong \widetilde{A^{hs}}$.
- iii) *The minimal prime ideals of A^{hs} are extended from A^h .*

Proof:

(ii $\Rightarrow$ iii) : Note that $\widetilde{A^{hs}}$ is a finite products of normal local domains and thus has the same number of maximal ideals as minimal prime ideals, which also correspond bijectively with the minimal prime ideals of A^{hs} . By definition of the strict Henselization of a semilocal ring, $\tilde{A}^{hs}$ and $\tilde{A}$ have the same number of maximal ideals. Thus by ii) we have the following bijections:

$$\text{Min}(A^{hs}) \longleftrightarrow \text{Max}(\widetilde{A^{hs}}) \longleftrightarrow \text{Max}(\tilde{A}^{hs}) \longleftrightarrow \text{Max}(\tilde{A}).$$

But then using the bijection $\text{Max}(\tilde{A}) \longleftrightarrow \text{Min}(A^h)$ (ch. IX, sect. 1, Cor. 1 of [Ra]), we obtain that A^{hs} and A^h have the same number of minimal prime ideals. Thus for P a minimal prime ideal of A^h , PA^{hs} is contained in exactly one minimal prime ideal of A^{hs} . But

$$\frac{A^{hs}}{PA^{hs}} \cong \left(\frac{A}{P} \right)^{hs}$$

is reduced (ch. VII, sect. 2, Prop.1 of [Ra]). Therefore PA^{hs} is a minimal prime ideal of A^{hs} .

(iii $\Rightarrow$ ii) : Arguing as above using iii) for the second bijection, we have

$$\begin{aligned} \text{Max}(\widetilde{A^{hs}}) &\longleftrightarrow \text{Min}(A^{hs}) \longleftrightarrow \text{Min}(A^h) \longleftrightarrow \\ &\text{Max}(\tilde{A}) \longleftrightarrow \text{Max}(\tilde{A}^{hs}). \end{aligned}$$

By 18.8.10 of [EGA_{IV}], $\tilde{A}^{hs}$ is a localization of $\widetilde{A^{hs}}$ at some maximal ideals. Since both rings have the same number of maximal ideals, they are isomorphic.

(i $\Leftrightarrow$ iii) : By Remark 1.7, $\widetilde{A^h} \cong \tilde{A}^h$. Thus the maximal ideals of $\widetilde{A^h}$ correspond bijectively with those of $\tilde{A}$ and furthermore, $\widetilde{A^h}$ has residual fields $k_1, \dots, k_n$. Thus we may assume A is Henselian.

Since A is Henselian, $\tilde{A}$ is a product of normal local domains. Thus there is a bijective correspondence $\text{Min}(A) \longleftrightarrow \text{Max}(\tilde{A})$. Let $\text{Min}(A) = \{P_1, \dots, P_n\}$. Then $\widetilde{A/P_i} \cong \tilde{A}_{m_i}$, and has residual field k_i . By Lemma 2.1.2, k_i is purely inseparable over k if and only if

$$(A/P_i)^{hs} \cong A^{hs}/P_i A^{hs}$$

is a domain, which holds exactly when $P_i A^{hs}$ is a minimal prime ideal of A^{hs} . The result follows. $\square$

Corollary 2.1.4 *Let (A, m, k) be an excellent Henselian local domain with normalization $(\tilde{A}, \tilde{m}, \tilde{k})$. Let k^s be the separable closure of k in $\tilde{k}$. Write $k^s = k[X]/(\bar{f})$, where $\bar{f} \in k[X]$ is a monic, irreducible and separable polynomial. Then*

i) A^{hs} has $d = \deg(\bar{f})$ minimal prime ideals.

ii) Let E be a local etale extension of A and fix a morphism $E \hookrightarrow A^{hs}$. Then the minimal prime ideals of A^{hs} are extended from E if and only if $\bar{f}$ splits into a product of linear factors over the residual field k_E of E . In this case, k_E is separably closed in each residual field of $\tilde{E}$.

Proof:

It is enough to show ii) and that such an E has d minimal prime ideals. Let E be a local etale extension of A such that $\bar{f}$ splits as a product of linear factors over k_E . Now $\tilde{E} \cong E \otimes_A \tilde{A}$ is a semilocal etale extension of $\tilde{A}$. Thus $\tilde{E}$ has residual field product given by

$$\begin{aligned} \tilde{E}/\tilde{m}\tilde{E} &\cong (\tilde{A} \otimes_A E) \otimes_{\tilde{A}} \tilde{k} \\ &\cong \tilde{k} \otimes_A E \\ &\cong \tilde{k} \otimes_k k_E \\ &\cong \tilde{k} \otimes_{k^s} k^s \otimes_k k_E \end{aligned}$$

But since $\bar{f}$ splits over k_E , we have

$$k^s \otimes_k k_E \cong (k_E)^d.$$

Thus the residual field product of $\tilde{E}$ is given by

$$\tilde{k} \otimes_{k^s} k^s \otimes_k k_E \cong \tilde{k} \otimes_{k^s} (k_E)^d \cong \left(\tilde{k} \otimes_{k^s} k_E \right)^d.$$

By Lemma 2.1.1, since $\tilde{k}$ is purely inseparable over k^s while k_E is separable over k^s , $\tilde{k} \otimes_{k^s} k_E$ is a field. Therefore $\tilde{E}$ has exactly d maximal ideals and the corresponding

residual fields are all isomorphic to $\tilde{k} \otimes_{k^s} k_E$. Since E is Henselian, $\tilde{E}$ is a product of local domains. Thus $\tilde{E}$ and hence E has d minimal prime ideals

Since $\tilde{k} \otimes_{k^s} k_E$ is purely inseparable over k_E , Theorem 2.1.3 applies for E so that the minimal prime ideals of $E^{hs} \cong A^{hs}$ are extended from E . This shows *i*).

Conversely, suppose the minimal prime ideals of A^{hs} are extended from E . Since E is Henselian, $\tilde{E}$ is a product of local domains. So there are bijections

$$\text{Max}(\tilde{E}) \longleftrightarrow \text{Min}(E) \longleftrightarrow \text{Min}(A^{hs}).$$

Thus $\tilde{E}$ has exactly d maximal ideals.

Arguing as before, $\tilde{E}$ has residual field product $\tilde{k} \otimes_{k^s} k^s \otimes_k k_E$. Since $\tilde{k}$ is purely inseparable over k^s , for any separable field extension l of k^s , $\tilde{k} \otimes_{k^s} l$ is a field by Lemma 2.1.1. Thus since $k^s \otimes_k k_E$ is a product of finite separable field extensions of k^s , the number of fields in the product $\tilde{k} \otimes_{k^s} (k^s \otimes_k k_E)$ is the same as the number of fields in the product $k^s \otimes_k k_E$. Since $\tilde{E}$ has d maximal ideals, $k^s \otimes_k k_E$ is a product of d fields. But this can only happen if $\bar{f}$ splits as a product of linear factors over k_E .

□

2.2 Characterizing Etale Intermediate Rings

The following lemma relates some of the hypotheses which appear in this section.

Lemma 2.2.1 *Let A be a reduced local ring, and let $A \subseteq D \subseteq A^{hs}$ be local intermediate ring dominated by A^{hs} . Then the total quotient ring K_D of D is finitely generated over K_A if and only if the containment $D \subseteq A^{hs}$ factors through some local etale extension of A . Furthermore, the above equivalent conditions hold if D is essentially of finite type over A .*

Proof:

Suppose that K_D is finitely generated over K_A . Then since $K_{A^{hs}} \cong A^{hs} \otimes_A K_A$ is a direct limit of the total quotient rings of local etale extensions, K_D is contained in the total quotient ring of some local etale extension E of A . But then by faithful flatness of $E \longrightarrow A^{hs}$, we have

$$D \subseteq K_E \cap A^{hs} = E.$$

On the other hand, suppose D is contained in some local etale extension E of A . Then we have containments $K_A \subseteq K_D \subseteq K_E$ of total quotient rings where K_E is finite over K_A by Proposition 1.6. Thus K_D is also finite over K_A .

Suppose that D is essentially of finite type over A . Let $d_1, \dots, d_r \in D$ be such that D is a localization of $A[d_1, \dots, d_r]$, and let E be a local etale extension of A such that $A[d_1, \dots, d_r] \subseteq E$. Then since D is dominated by A^{hs} , if we localize $A[d_1, \dots, d_r]$ at the preimage of the maximal ideal of E , we obtain D . Therefore, $D \subseteq E$. $\square$

Remark 2.2.2 Let A be a reduced local ring and $A \subseteq D \subseteq A^{hs}$ an intermediate ring with total quotient ring K_D finitely generated over K_A . Since by Remark 1.7, $K_D \cong K_A \otimes_A D$ is integral over K_A , we may write

$$K_D = K_A[d_1, \dots, d_r]$$

where the $d_i \in D$ are integral over A . In particular it follows that the integral closure of A in D contains the elements d_i and hence has total quotient ring K_D .

The following proposition is a generalization of (1.3) of [Ro3], and although the statement below seems much more general, the proof is essentially the same.

Proposition 2.2.3 *Let (A, m, k) be an excellent normal local domain, and let $A \subseteq D \subseteq A^{hs}$ be a normal local intermediate ring dominated by A^{hs} . Then D is a direct limit of local etale extensions of A . Furthermore, if in addition the quotient field K_D of D is finitely generated over K_A , then D is a local etale extension of A .*

Proof:

First we assume that K_D is finitely generated over K_A .

Let C_0 be the integral closure of A in K_D . Then by Remark 2.2.2, C_0 is a normal domain with quotient field K_D . Since D is normal, we also have $C_0 \subseteq D$ and so we may let $C \subseteq D$ be the localization of C_0 at the maximal ideal which lies under the maximal ideal of D . Then C also has quotient field K_D . Since A is excellent, C_0 is a finite A -module and hence C is a local normal domain which is essentially finite over A .

By Lemma 2.2.1, D is contained in some local etale extension E of A . So we now have containments

$$A \subseteq C \subseteq D \subseteq E.$$

Set $r := \dim(A)$. Since C is essentially finite over A , $\dim(C) \leq r$. On the other hand, since E is essentially finite over A , E is also essentially finite over C . Thus $\dim(C) \geq \dim(E) = r$ and so $\dim(C) = r$.

Considering the canonical surjection $C \otimes_A E \rightarrow E$, we localize at the preimage Q of the maximal ideal of E to obtain a surjection

$$(C \otimes_A E)_Q \xrightarrow{\phi} E.$$

Note that since E is etale over A , $(C \otimes_A E)_Q$ is a local etale extension of C . Then since C is a normal domain of dimension r , $(C \otimes_A E)_Q$ is also a normal domain of

dimension r . Thus ϕ is a surjection between local domains of the same dimension r and so must be an isomorphism. So $E \cong (C \otimes_A E)_Q$ is a local etale extension of C .

Finally observe that by faithful flatness of $C \rightarrow E$ and since C has quotient field K_D , we have

$$C = K_D \cap E \supseteq D.$$

Therefore $C = D$, concluding the proof in the case where K_D is finitely generated over K_A .

For the general case, making no assumption about how K_D is generated over K_A , we write D as a direct limit of local, essentially of finite type A -subalgebras dominated by D ,

$$D = \varinjlim_{\lambda \in \Lambda} D_\lambda = \bigcup_{\lambda \in \Lambda} D_\lambda.$$

Since A is excellent, so are the D_λ . Hence the normalization $\widetilde{D}_\lambda$ is a finite extension of D_λ , and so is also essentially of finite type over A . Since D is normal we have

$$D = \varinjlim_{\lambda \in \Lambda} \widetilde{D}_\lambda.$$

Then localizing each $\widetilde{D}_\lambda$ at the maximal ideal m_λ which lies under the maximal ideal of D , we have

$$D = \varinjlim_{\lambda \in \Lambda} \left(\widetilde{D}_\lambda \right)_{m_\lambda}.$$

Now for each λ , $\left(\widetilde{D}_\lambda \right)_{m_\lambda}$ is essentially of finite type over A and thus by Lemma 2.2.1 has quotient field finitely generated over K_A . Thus by the previous case, since the

$(\widetilde{D}_\lambda)_{m_\lambda}$ are also normal and dominated by A^{hs} , these rings are local etale extensions of A . $\square$

We now generalize to the semilocal case to obtain a result which can be applied to normal intermediate rings $\tilde{A} \subseteq \tilde{D} \subseteq \widetilde{A^{hs}}$, where A is an excellent reduced local ring with normalization $\tilde{A}$. Note that since $\widetilde{A^{hs}} \cong A^{hs} \otimes_A \tilde{A}$ with A^{hs} a direct limit of local etale extensions of A , $\widetilde{A^{hs}}$ is a direct limit of semilocal etale extensions of $\tilde{A}$.

Proposition 2.2.4 *Let A be an excellent normal semilocal ring, let B be a semilocal ring which is a direct limit of semilocal etale extensions of A , and let $A \subseteq D \subseteq B$ be a normal semilocal intermediate ring such that the inclusion $D \hookrightarrow B$ is a semilocal morphism. Then D is a direct limit of semilocal etale extensions of A . Furthermore, if in addition the total quotient ring K_D of D is finitely generated over K_A , then D is an essentially finite, semilocal etale extension of A .*

Proof:

As with Proposition 2.2.3, it suffices to prove the case where we assume that K_D is finitely generated over K_A . The general statement then follows using an analogous direct limiting argument.

Let C_0 be the integral closure of A in K_D . By Remark 2.2.2, C_0 is a normal ring having total quotient ring K_D . Since D is normal, $C_0 \subseteq D$.

Now for M any maximal ideal of B , set $n := M \cap C_0$ and $m := M \cap A$. Then B_M is a direct limit of local etale extensions of A_m and so has strict Henselization

$$(B_M)^{hs} \cong (A_m)^{hs}.$$

Thus $(C_0)_n$ is a normal local intermediate ring

$$A_m \subseteq (C_0)_n \subseteq (A_m)^{hs}$$

dominated by $(A_m)^{hs}$. By Proposition 2.2.3, $(C_0)_n$ is a local etale extension of A_m .

Let C be the localization of C_0 at the maximal ideals lying under the maximal ideals of B (hence also at the maximal ideals lying under D), so that

$$A \longrightarrow C \longrightarrow D \longrightarrow B$$

are semilocal morphisms. Then for any maximal ideal n of C , C_n is a local etale extension of $A_{n \cap A}$. Thus C is a semilocal etale extension of A . Since for any maximal ideal M of B , $C_{M \cap C}$ and B_M have the same strict Henselization, $C_{M \cap C} \longrightarrow B_M$ is faithfully flat. Hence $C \longrightarrow B$ is faithfully flat. Since C has total quotient ring K_D , we have

$$C = K_D \cap B \supseteq D.$$

Therefore $D = C$ is a semilocal etale extension of A . $\square$

Lemma 2.2.5 *Let A be a reduced local ring and E be a local etale extension of A . Let L be an intermediate ring $K_A \subseteq L \subseteq K_E$. Then*

- i) The intersection $D := L \cap E$ is a local ring with maximal ideal lying under that of E and having total quotient ring $K_D = L$.*
- ii) The intersection $\tilde{C} := L \cap \tilde{E}$ is a normal semilocal ring with maximal ideals the preimages of those of $\tilde{E}$ and having total quotient ring $K_{\tilde{C}} = L$.*

Proof:

Note that since $K_A \subseteq K_E$ is finite by Proposition 1.6, so is $K_A \subseteq L$. Thus L is zero dimensional.

To prove *i)*, let $l \in L$. Since $K_E \cong E \otimes_A K_A$, we may write $l = e/s$ where $e \in E$ and s is a nonzerodivisor of A . Then $e \in L \cap E = D$. Thus $l = e/s \in K_D$ and therefore $K_D = L$. If $d \in D$ is a unit of E , then $d^{-1} \in L \cap E = D$. Therefore every nonunit of D is a nonunit of E , so that every nonunit of D is in the prime ideal $m_E \cap D$. Hence this ideal is the unique maximal ideal of D .

To prove *ii)*, observe that since we also have $K_E \cong \tilde{E} \otimes_{\tilde{A}} K_A$ the argument that $K_{\tilde{C}} = L$ follows just as above. If $c \in \tilde{C}$ is a unit of $\tilde{E}$, then $c^{-1} \in L \cap \tilde{E} = \tilde{C}$. Thus every nonunit of $\tilde{C}$ is a nonunit of $\tilde{E}$ and so any maximal ideal N of $\tilde{C}$ is contained in the union $(m_1 \cap \tilde{C}) \cup \dots \cup (m_s \cap \tilde{C})$ where $\text{Max}(E) = \{m_1, \dots, m_s\}$. Then by prime avoidance we must have $N = m_i \cap \tilde{C}$ for some i . The statement follows. $\square$

The following theorem, which generalizes some ideas of C. Rotthaus, exhibits the strong connection between intermediate rings $A \subseteq D \subseteq A^{hs}$ and their normalizations. Under certain circumstances the intersection rings of the previous lemma are etale extensions. The local etale extension of *iii)* in the following theorem is obtained by such an intersection. The issue of etale intersections is explored in greater detail Chapter 3.

Theorem 2.2.6 *Let (A, m, k) be an excellent reduced local ring, E be a local etale extension of A , and let $A \subseteq D \subseteq E$ be a local intermediate ring dominated by E . Suppose $\tilde{D}$ is a semilocal ring and $\tilde{D} \hookrightarrow \tilde{E}$ is a semilocal morphism. Then*

i) $\tilde{D}$ is an essentially finite, semilocal etale extension of $\tilde{A}$.

ii) D is essentially of finite type over A .

iii) If $D \hookrightarrow E$ is residually trivial, then D is birationally dominated by a local etale extension of A which has the same normalization $\tilde{D}$.

Proof:

i) This follows immediately by Lemma 2.2.1 and Proposition 2.2.4.

ii) Let C be the integral closure of A in D localized at the preimage of the maximal ideal m_D of D . By Remark 2.2.2, C has total quotient ring K_D . Then since normalization and localization commute, $\tilde{C}$ is a localization of $\text{IntClos}_{K_D}(A)$. Since $\tilde{D}$ is essentially finite over $\tilde{A}$, $\tilde{D}$ is also a localization of $\text{IntClos}_{K_D}(A)$. It follows that $\tilde{D}$ is a localization of $\tilde{C}$ at certain maximal ideals. Let

$$\text{Max}(\tilde{C}) = \{m_1, \dots, m_t\}$$

where for some $1 \leq s \leq t$ and

$$S := \tilde{C} - (m_1 \cup \dots \cup m_s)$$

we have $\tilde{D} = S^{-1}\tilde{C}$. Then for each $s+1 \leq i \leq t$ there is an element $c_i \in m_i$ with $c_i^{-1} \in \tilde{D}$. Thus the c_i^{-1} are integral over D and we may let $\{d_{ij}\}$ be the coefficients of the corresponding integral equations. Set

$$C_1 := C[\{d_{ij}\}]_M \subseteq D$$

where M is the preimage of m_D . Then c_i^{-1} is in $\tilde{C}_1$ for $s+1 \leq i \leq t$, so that considering the containment $\tilde{C} \subseteq \tilde{C}_1$, no maximal ideals of $\tilde{C}_1$ lie over $m_{s+1}, \dots, m_t$. Therefore $S^{-1}\tilde{C}_1 = \tilde{C}_1$ and thus we have

$$S^{-1}\tilde{C} = \tilde{C}_1 = \tilde{D}.$$

Since C_1 is excellent, $C_1 \hookrightarrow \tilde{C}_1 = \tilde{D}$ is finite. Since this map factors through D , D is finite over C_1 which is itself essentially of finite type over A . Thus D is also essentially of finite type over A .

iii) Set $F = K_D \cap E$. Then F is a local ring with maximal ideal lying under that of E , and F has total quotient ring K_D by Lemma 2.2.5. Since $\tilde{D}$ and $\tilde{E}$ are both semilocal etale extensions of $\tilde{A}$ and since $\tilde{D} \rightarrow \tilde{E}$ is semilocal, $\tilde{E}$ is a semilocal etale extension of $\tilde{D}$, by Proposition 1.14. Hence by faithful flatness we have $\tilde{D} = K_D \cap \tilde{E}$. Thus $D \subseteq F \subseteq \tilde{D}$. Since D is excellent by ii), $\tilde{D}$ is finite over D . Hence F is finite over D and has normalization $\tilde{F} = \tilde{D}$. Since D is essentially of finite type over A and F is finite over D , F is also essentially of finite type over A . In particular, F is excellent.

Since by hypothesis $D \rightarrow F \rightarrow E$ is residually trivial and since the maximal ideal m_F of F generates the maximal ideal of E , we have a surjection on completions $\hat{F} \rightarrow \hat{E}$. To see that this surjection is also injective, consider the commutative diagram

$$\begin{array}{ccc} \hat{\tilde{F}} = \hat{\tilde{D}} & \longrightarrow & \hat{\tilde{E}} \\ \uparrow & & \uparrow \\ \hat{F} & \longrightarrow & \hat{E}. \end{array}$$

Since the morphisms $\tilde{D} \rightarrow \tilde{E}$ is a semilocal etale extension and hence faithfully flat, the induced morphism of completions (with respect to Jacobson radicals) is injective. Hence the top morphism in the diagram is injective. Since F is excellent and reduced, normalization and completion commute for F and thus the left morphism

$$\hat{F} \longrightarrow \hat{\tilde{F}} \cong \tilde{\hat{F}}$$

is injective. Then by commutativity, $\hat{F} \longrightarrow \hat{E}$ is also injective and thus an isomorphism.

Since $\hat{E} \cong \hat{F}$ is faithfully flat over both E and F , we also have that E is faithfully flat over F . But E is also essentially of finite type and unramified over F by *ii*) of Remark 1.2. Thus E is a local etale extension of F . Since E is etale over both F and A and since F is essentially of finite type over A , F is a local etale extension of A by (1.4) of [Ro3]. $\square$

Theorem 2.2.7 *Let (A, m, k) be an excellent reduced local ring, and let $A \subseteq D \subseteq A^{hs}$ be a local intermediate ring dominated by A^{hs} . Then the following are equivalent:*

- i) The Henselization D^h of D is a direct limit of local etale extensions of A (so that A and D have the same strict Henselization A^{hs} , and hence $A \longrightarrow D$ is a regular morphism.)*
- ii) The normalization $\tilde{D}$ of D is a semilocal ring and the canonical morphisms $k_{\tilde{A}} \hookrightarrow k_{\tilde{D}}$ and $k_D \hookrightarrow k_{\tilde{D}}$ of residual field products induce an isomorphism*

$$k_{\tilde{A}} \otimes_k k_D \xrightarrow{\cong} k_{\tilde{D}}.$$

Proof:

(i $\Rightarrow$ ii) : Suppose that D^h is a direct limit of local etale extensions of A . Then D^h is excellent and by faithful flatness of $D \longrightarrow D^h$, D is Noetherian. Also, since D^h is faithfully flat over both D and A , D is faithfully flat over A . Hence we have injections

$$D \hookrightarrow \tilde{A} \otimes_A D \hookrightarrow K_A \otimes_A D \cong K_D.$$

Since D^h is a direct limit of local etale extensions of A , D^h has normalization given by $\widetilde{D^h} \cong \tilde{A} \otimes_A D^h$. But $\tilde{A} \otimes_A D^h$ is a direct limit of semilocal etale extensions of $\tilde{A} \otimes_A D$. Thus $\tilde{A} \otimes_A D \longrightarrow \tilde{A} \otimes_A D^h$ is regular and so $\tilde{A} \otimes_A D$ is normal. Then we must have

$$\tilde{A} \otimes_A D \cong \tilde{D},$$

and so *ii*) follows.

(*ii* $\Rightarrow$ *i*) : Applying $\otimes_D k_D$ to $D \otimes_A \tilde{A}$ we obtain

$$\begin{aligned} (D \otimes_A \tilde{A}) \otimes_D k_D &\cong k_D \otimes_A \tilde{A} \\ &\cong k_D \otimes_k k \otimes_A \tilde{A} \\ &\cong k_D \otimes_k (\tilde{A}/m\tilde{A}). \end{aligned}$$

Similarly, since m_D generates the maximal ideal of A^{hs} , applying $\otimes_D k_D$ to $A^{hs} \otimes_A \tilde{A}$ we obtain

$$\begin{aligned} (A^{hs} \otimes_A \tilde{A}) \otimes_D k_D &\cong k_{A^{hs}} \otimes_A \tilde{A} \\ &\cong k_{A^{hs}} \otimes_k k \otimes_A \tilde{A} \\ &\cong k_{A^{hs}} \otimes_k (\tilde{A}/m\tilde{A}). \end{aligned}$$

Thus applying $\otimes_D k_D$ to the morphism

$$D \otimes_A \tilde{A} \longrightarrow A^{hs} \otimes_A \tilde{A} \cong \widetilde{A^{hs}},$$

we obtain a commutative diagram

$$\begin{array}{ccc}
 D \otimes_A \tilde{A} & \xrightarrow{\alpha} & A^{hs} \otimes_A \tilde{A} \cong \widetilde{A^{hs}} \\
 \phi \downarrow & & \downarrow \psi \\
 k_D \otimes_k (\tilde{A}/m\tilde{A}) & \xrightarrow{\beta} & k_{A^{hs}} \otimes_k (\tilde{A}/m\tilde{A})
 \end{array}$$

Since $D \otimes_A \tilde{A}$ is finite over D , every maximal ideal of $D \otimes_A \tilde{A}$ contains the maximal ideal m_D of D . Since the kernel of ϕ is generated by m_D , no maximal ideals are lost under the surjection ϕ . So ϕ is a semilocal morphism between rings with the same number of maximal ideals. An analogous statement holds for ψ . Since β is a faithfully flat morphism of semilocal rings, hence also semilocal, α is a semilocal morphism by commutativity.

Let $r = |\text{Max}(D \otimes_A \tilde{A})|$. Since α factors through $\tilde{D}$ to yield morphisms $D \otimes_A \tilde{A} \rightarrow \tilde{D} \rightarrow \widetilde{A^{hs}}$, it follows that the maximal ideals of $\widetilde{A^{hs}}$ lie over at least r distinct maximal ideals of $\tilde{D}$. But by hypothesis,

$$|\text{Max}(\tilde{D})| = |\text{Max}(k_{\tilde{A}} \otimes_k k_D)| = |\text{Max}(\tilde{A} \otimes_A D)| = r.$$

Therefore $\tilde{D} \rightarrow \widetilde{A^{hs}}$ is semilocal. By Proposition 2.2.4, $\tilde{D}$ is a direct limit of semilocal etale extensions of $\tilde{A}$. In particular, $\tilde{D}$ is excellent.

Now the residual field k_D of D is a direct limit of finite separable field extensions of k . Let $\{(F_\lambda, \bar{\rho}_\lambda) : \lambda \in \Lambda\}$ be a system of representatives of all couples (F, ρ) , where F is a local etale extension of A and $\rho : k_F \rightarrow k_D$ is a k -morphism. Then exactly as in the construction of the strict Henselization [Ra], $\{F_\lambda\}_{\lambda \in \Lambda}$ is a direct system. Let F^h be the limit of this system. Note that F^h is Henselian since it may also be obtained by taking an analogous direct limit of local etale extensions of A^h . Since F^h is a direct limit of local etale extensions of A it has strict Henselization A^{hs} , so that the canonical morphism $F^h \rightarrow A^{hs}$ is faithfully flat.

Now there exists canonical a k -morphism $k_{F^h} \longrightarrow k_D$ induced by the morphisms $\bar{\rho}_\lambda$. Since k_D is a separable algebraic over k , for any $\bar{d} \in k_D$, $k[\bar{d}]$ is a finite separable field extension of k . By Proposition 1.11, there is a local etale extension F of A with residual field isomorphic to $k[\bar{d}]$. Thus letting $\bar{\rho}$ be the inclusion $k[\bar{d}] \subseteq k_D$, $(F, \bar{\rho})$ is then represented in the above system. Thus $\bar{d}$ is in the image of k_{F^h} in k_D . It follows that the canonical k -morphism $k_{F^h} \longrightarrow k_D$ is an isomorphism. By Proposition 1.10, there is a morphism $F^h \xrightarrow{\phi} D^h$ of A^h -algebras which is injective since the composite

$$F^h \xrightarrow{\phi} D^h \longrightarrow A^{hs}$$

is faithfully flat. To complete the proof, we argue that ϕ is an isomorphism.

We consider the following commutative diagram

$$\begin{array}{ccccccc} \widetilde{A^h} & \longrightarrow & \widetilde{F^h} & \longrightarrow & \widetilde{D^h} & \longrightarrow & \widetilde{A^{hs}} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A^h & \longrightarrow & F^h & \xrightarrow{\phi} & D^h & \longrightarrow & A^{hs} \end{array}$$

By 18.8.6 of EGA_{IV}, since $\tilde{D}$ is semilocal and integral over D , $\tilde{D}$ has Henselization given by

$$\tilde{D}^h \cong \tilde{D} \otimes_D D^h \cong \widetilde{D^h}.$$

In particular $\widetilde{D^h}$ is a product of finitely many local domains. Since $\tilde{D}$ is a direct limit of semilocal etale extensions of A , its Henselization $\tilde{D}^h$ is a direct limit of semilocal etale extensions of $\tilde{A}^h \cong \widetilde{A^h}$ (see for example ch. VIII, sect. 3, Prop. 5 of [Ra]). Thus $\widetilde{D^h}$ is a direct limit of semilocal etale extensions of $\tilde{A}^h \cong \widetilde{A^h}$ and by the above isomorphisms has the same residual field product as $\tilde{D}^h$, hence also the same as $\tilde{D}$. But by hypothesis $\tilde{D}$ has residual field product $k_{\tilde{A}} \otimes_k k_D$.

Making similar considerations for F^h we have

$$\widetilde{F^h} \cong \widetilde{A^h} \otimes_{A^h} F^h,$$

and thus $\widetilde{F^h}$ is a direct limit of semilocal etale extensions of $\widetilde{A^h}$. Since $\widetilde{A^h} \cong \widetilde{A^h}$ which has residual field product given by $k_{\widetilde{A}}$, it follows from the above that $\widetilde{F^h}$ also has residual field product $k_{\widetilde{A}} \otimes_k k_D$. Therefore the morphism

$$\widetilde{F^h} \longrightarrow \widetilde{D^h}$$

induces an isomorphism of residual field products. In particular both rings are products of the same number of local normal domains.

Let M be a maximal ideal of $\widetilde{D^h}$, Q its preimage in $\widetilde{F^h}$, and P its preimage in $\widetilde{A^h}$. Then $(\widetilde{D^h})_M$ and $(\widetilde{F^h})_Q$ are both direct limits of local etale extensions of $(\widetilde{A^h})_P$. But by Proposition 1.10, two Henselian direct limits of local etale extensions are isomorphic if their residual fields are isomorphic. Thus

$$(\widetilde{F^h})_Q \cong (\widetilde{D^h})_M.$$

Then since $\widetilde{F^h}$ and $\widetilde{D^h}$ are products of the same number of local domains, they are isomorphic. In particular, $\widetilde{D^h} \longrightarrow \widetilde{A^{hs}}$ is injective and so every morphism in the previous commutative diagram is injective by commutativity.

Now the isomorphism $\widetilde{F^h} \cong \widetilde{D^h}$ yields an isomorphism of total quotient rings upon application of $K_{A \otimes A}$. Thus applying $K_{A \otimes A}$ to the top row of the previous commutative diagram, we obtain the following commutative diagram of injective

morphisms

$$\begin{array}{ccccccc}
 K_{A^h} & \longrightarrow & K_{F^h} & \xrightarrow{\cong} & K_{D^h} & \longrightarrow & K_{\widetilde{A^{hs}}} \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 A^h & \longrightarrow & F^h & \xrightarrow{\phi} & D^h & \longrightarrow & \widetilde{A^{hs}}.
 \end{array}$$

Thus considered as subrings of $K_{A^{hs}}$, F^h and D^h have the same total quotient ring.

Hence by faithful flatness of $F^h \rightarrow A^{hs}$ we have

$$F^h = K_{F^h} \cap A^{hs} = K_{D^h} \cap A^{hs} \supseteq D^h.$$

Therefore ϕ must be an isomorphism. $\square$

Remark 2.2.8 From the proof of the previous theorem it follows that under the equivalent conditions *i*) and *ii*), the induced morphism of normalizations $\tilde{D} \rightarrow \widetilde{A^{hs}}$ is semilocal.

One would like to say that for A an excellent reduced local ring and $A \hookrightarrow D \hookrightarrow A^{hs}$ local regular morphisms, D is a direct limit of local etale extensions of A . Thus the previous theorem would give a characterization of when local intermediate rings $A \subseteq D \subseteq A^{hs}$ are direct limits of local etale extensions of A . However the author does not know how to prove this.

General Neron Desingularization ([P1], [P2], [O2], [Sp], [Sw]) gives that in the above situation D is a direct limit of smooth A -algebras of finite type. If in fact D is a direct limit of smooth A -subalgebras of finite type,

$$D = \varinjlim_{\lambda \in \Lambda} S_\lambda = \bigcup_{\lambda \in \Lambda} S_\lambda,$$

then we can conclude that D is a direct limit of local etale extensions of A as desired. To see this, we only need to observe that in this situation each localization $(S_\lambda)_{m_\lambda}$ at the preimage of the maximal ideal of D is a local etale extension of A . Indeed, since $(S_\lambda)_{m_\lambda} \subseteq A^{hs}$ is essentially of finite type over A , it is contained in some local etale extension E of A . Then by regularity of the morphisms $(S_\lambda)_{m_\lambda} \rightarrow D \rightarrow A^{hs}$ and $E \rightarrow A^{hs}$, the inclusion $(S_\lambda)_{m_\lambda} \hookrightarrow E$ is local and regular, thus faithfully flat. Since E is also unramified over $(S_\lambda)_{m_\lambda}$ by Remark 1.2, E is etale over $(S_\lambda)_{m_\lambda}$. Thus $(S_\lambda)_{m_\lambda}$ is etale over A by (1.4) of [Ro3].

It is currently unknown whether a regular morphism $A \rightarrow D$ is a direct limit of smooth *subalgebras*, even in the algebraic situation of this work, where $A \rightarrow D$ induces an integral extension of total quotient rings. Spivakovsky has a proof in his preprint that under certain circumstances a regular morphism is a direct limit of smooth *subalgebras* of finite type, but this preprint is rumored to contain errors. So this author is uncertain of the veracity of that statement.

The only statement we currently make is the following characterization of etaleness for intermediate rings $A \subseteq D \subseteq A^{hs}$ which are contained in a local etale extension of A or which equivalently have total quotient ring finitely generated over A .

Corollary 2.2.9 *Let (A, m, k) be an excellent reduced local ring, and let $A \subseteq D \subseteq A^{hs}$ be a local intermediate ring dominated by A^{hs} and whose total quotient ring K_D is finitely generated over K_A . Then D is a local etale extension of A if and only if $\tilde{D}$ is a semilocal ring and the canonical morphisms $k_{\tilde{A}} \hookrightarrow k_{\tilde{D}}$ and $k_D \hookrightarrow k_{\tilde{D}}$ of residual field products induce an isomorphism*

$$k_{\tilde{A}} \otimes_k k_D \xrightarrow{\cong} k_{\tilde{D}}.$$

Proof:

($\Rightarrow$): This is trivial since if D is a local etale extension of A then the normalization $\tilde{D}$ of D is given by $\tilde{D} \cong D \otimes_A \tilde{A}$.

($\Leftarrow$): By Lemma 2.2.1, the containment $D \subseteq A^{hs}$ factors through some local etale extension E of A . By the Theorem 2.2.7 and the subsequent Remark, the morphism $\tilde{D} \longrightarrow \widetilde{A^{hs}}$ is semilocal. Thus since the composite

$$\tilde{D} \longrightarrow \tilde{E} \longrightarrow \widetilde{A^{hs}}$$

is semilocal and since $\tilde{E} \longrightarrow \widetilde{A^{hs}}$ is semilocal, so is $\tilde{D} \longrightarrow \tilde{E}$. By Theorem 2.2.6, D is essentially of finite type over A .

Now by Theorem 2.2.7, D has strict Henselization A^{hs} . Since E also has strict Henselization A^{hs} , A^{hs} is faithfully flat over both D and E . Thus E is faithfully flat over D . Since E is essentially of finite type over A , E is also essentially of finite type over D . Then since E is unramified over D by Remark 1.2, E is etale over D . Thus E is etale over both D and A , and so D is also etale over A by (1.4) of [Ro3]. $\square$

We conclude this section with another easy characterization of etaleness for intermediate rings, which does not rely on an analysis of normalizations.

Proposition 2.2.10 *Let (A, m, k) be a Noetherian local ring, and let $A \subseteq D \subseteq A^{hs}$ be a local intermediate ring dominated by A^{hs} and whose total quotient ring K_D is finitely generated over K_D . Suppose m generates the maximal ideal m_D of D and that either of the following conditions holds.*

- i) D is essentially of finite type over A .*
- ii) A is excellent reduced and D is Noetherian.*

Then D is a local etale extension of A .

Proof:

By Lemma 2.2.1, the containment $D \subseteq A^{hs}$ factors through some local etale extension E of A . The morphisms $A \rightarrow D \rightarrow E$ induce morphisms $\hat{A} \hookrightarrow \hat{D} \rightarrow \hat{E}$ of completions. Let R be a coefficient ring for $\hat{A}$. By Proposition 1.11, since R is complete and hence Henselian, there is a unique local etale extension S of R with residual field k_D . Then $\hat{A} \otimes_R S$ is a local etale extension of $\hat{A}$ with residual field k_D . By Proposition 1.10, since $\hat{D}$ is Henselian, we have an isomorphism

$$\mathrm{Hom}_{loc\hat{A}}(\hat{A} \otimes_R S, \hat{D}) \cong \mathrm{Hom}_k(k_D, k_D).$$

Hence the identity morphism on residual fields lifts to a morphism $\hat{A} \otimes_R S \rightarrow \hat{D}$. Since $\hat{E}$ is a localization of $E \otimes_A \hat{A}$, $\hat{E}$ is a local etale extension of $\hat{A}$. Thus since the composite

$$\hat{A} \otimes_R S \rightarrow \hat{D} \rightarrow \hat{E}$$

is etale, $\hat{A} \otimes_R S \hookrightarrow \hat{D}$ is injective.

Now by hypothesis the maximal ideal of $\hat{A} \otimes_R S$ generates that of $\hat{D}$. Thus since $\hat{A} \otimes_R S \hookrightarrow \hat{D}$ is a residually trivial morphism of complete rings, it follows that $\hat{A} \otimes_R S \xrightarrow{\cong} \hat{D}$ is an isomorphism. Hence $\hat{D} \rightarrow \hat{E}$ is etale. Note that with either of the hypotheses *i*) or *ii*) we have that D is Noetherian, and so $D \rightarrow \hat{D} \rightarrow \hat{E}$ is faithfully flat. Since $\hat{E}$ is faithfully flat over both E and D , E is faithfully flat over D . Since E is also unramified and essentially of finite type over D , $D \rightarrow E$ is a local etale extension.

Assume that *i*) holds. Since E is etale over both A and D , D is a local etale extension of A by (1.4) of [Ro3].

Now suppose *ii*) holds. Since E is étale over D , we have $\tilde{E} \cong E \otimes_D \tilde{D}$. So $\tilde{E}$ is a semilocal étale extension of $\tilde{D}$. By Theorem 2.2.6, D is essentially of finite type over A . Thus again by (1.4) of [Ro3], D is a local étale extension of A . $\square$

CHAPTER 3

Etale and Unramified Intersections

3.1 Etale Intersections Using Idempotents

Let A be an excellent reduced local ring. In this section we study intersections $D := L \cap A^{hs}$ where $K_A \subseteq L \subseteq K_{A^{hs}}$ is an intermediate ring obtained by adjoining finitely many idempotent elements of $K_{A^{hs}}$ to K_A . We find that if $L \subseteq K_{A^h}$ then the intersection D is a local etale extension of A , but without this hypothesis, it need not be the case that D is etale. The difference arises in whether $A \rightarrow D$ is residually trivial.

We begin with some elementary results which are mostly due to C. Rotthaus.

Proposition 3.1.1 *Let (A, m, k) be a Noetherian reduced local ring which is not Henselian and let F be an etale neighborhood of A . Then there exists a regular element $a \in A$ such that for all $n \in \mathbb{N}$ we have*

$$\frac{A}{a^n A} \cong \frac{F}{a^n F}.$$

In particular, A and F have the same a -adic completion.

Proof:

By Theorem 1.3, there exists $\alpha \in A^h$ such that F is a localization of $A[\alpha]$. By (1.8) of [Ro3], there is an etale neighborhood

$$E \cong \left(\frac{A[X]}{(f)} \right)_Q$$

of A containing α and satisfying:

a) $x \in Q$,

b) $a := f(0) \in m$ is a regular element,

c) $f'(0) \notin m$.

Then modulo $aA[X]$, $f \equiv Xg$ for some $g \notin Q$. It follows that $E/aE \cong A/aA$. Also, localizing the inclusion

$$A[\alpha] \hookrightarrow E$$

at the preimage of the maximal ideal of E , we obtain $F \hookrightarrow E$. Then since the morphisms $A \hookrightarrow F \hookrightarrow E$ are etale and hence faithfully flat, we obtain morphisms

$$A/aA \hookrightarrow F/aF \hookrightarrow E/aE \cong A/aA.$$

Therefore $A/aA \cong F/aF$. It follows that F is given by

$$F = A + aF = A + aA + a^2F, \text{ etc.}$$

Thus

$$\frac{A}{a^n A} \cong \frac{F}{a^n F}. \quad \square$$

Corollary 3.1.2 *Let (A, m, k) be an excellent reduced local ring, and let $K_A \subseteq L \subseteq K_{A^h}$ be an intermediate ring which is finitely generated over K_A . Set $D := L \cap K_{A^h}$. If D is Noetherian, then D is an etale neighborhood of A .*

Proof:

If A is Henselian, the result is trivial. So assume A is not Henselian. By Lemma 2.2.5, D is a local ring with maximal ideal lying under that of A^h , and D has total quotient ring L . By Lemma 2.2.1, the containment $D \subseteq A^h$ factors through some etale neighborhood F of A . Then by faithful flatness of $F \rightarrow A^h$, we have

$$D = L \cap A^h = L \cap K_F \cap A^h = L \cap F.$$

As in Proposition 3.1.1, let $a \in A$ be a regular element such that for all $n \in \mathbb{N}$ we have $A/a^n A = F/a^n F$. Since $D = L \cap F$, it follows that for all $n \in \mathbb{N}$, $a^n F \cap D = a^n D$. Thus we have inclusions

$$A/a^n A \hookrightarrow D/a^n D \hookrightarrow F/a^n F = A/a^n A,$$

so that $A/a^n A = D/a^n D = F/a^n F$. Therefore F and D have the same a -adic completion $(A, a)^\wedge$. Since $(A, a)^\wedge$ is faithfully flat over both F and D , F is also faithfully flat over D (and hence, similarly, D is faithfully flat over A). Since F is also unramified over D by Remark 1.2, $D \rightarrow F$ is etale. Then

$$D \otimes_A \tilde{A} \longrightarrow F \otimes_A \tilde{A} \cong \tilde{F}$$

is a semilocal etale extension and hence is faithfully flat. Thus $D \otimes_A \tilde{A}$ is a normal ring. Since we have inclusions

$$D \hookrightarrow D \otimes_A \tilde{A} \hookrightarrow D \otimes_A K_A \cong K_D,$$

we must have $\tilde{D} \cong D \otimes_A \tilde{A}$, and so $\tilde{D} \hookrightarrow \tilde{F}$ is etale and thus semilocal. By Theorem 2.2.6, D is essentially of finite type over A . Then by (1.4) of [Ro3], since F is etale over both D and A , D is etale over A . $\square$

Proposition 3.1.3 *Let (A, m, k) be an excellent reduced local ring and let $\epsilon_1, \dots, \epsilon_r \in K_{A^h}$ be idempotent elements. Set $L := K_A[\epsilon_1, \dots, \epsilon_r]$ and $D := L \cap A^h$. Then D is etale over A .*

Proof:

It suffices to show this for $r = 1$ and we may assume that $\epsilon = \epsilon_1$ is different from 1.

Write

$$\text{Min}(A) = \{P_1, \dots, P_n\},$$

and for each i let $k(P_i)$ be the quotient field of A/P_i . Also write

$$\text{Min}(A^h) = \{P_{11}, \dots, P_{1d}, \dots, P_{n1}, \dots, P_{nd_n}\},$$

where $P_{ij} \cap A = P_i$, and similarly for each i and j let $k(P_{ij})$ be the quotient field of A^h/P_{ij} . Then

$$K_{A^h} \cong \prod_{i=1}^n \prod_{j=1}^{d_i} k(P_{ij}),$$

and we may identify $\epsilon \in K_{A^h}$ with an element $(e_{11}, \dots, e_{1n}, \dots, e_{n1}, \dots, e_{nd_n})$, where for each i and j , e_{ij} is either 1 or 0. Under this identification, $e_{ij} = 0$ if and only if $k(P_{ij})\epsilon = 0$. We may adjust the indexing if necessary, so that for each $1 \leq i \leq n$

there exists an s_i with $0 \leq s_i \leq d_i$ such that

$$e_{i1} = \cdots = e_{is_i} = 1 \text{ and } e_{i,s_i+1} = \cdots = e_{id_i} = 0.$$

Then $K_{A^h} \cong K_{A^h}\epsilon \times K_{A^h}(1 - \epsilon)$ where we now have

$$K_{A^h}\epsilon \cong \prod_{i=1}^n \prod_{j=1}^{s_i} k(P_{ij})$$

and

$$K_{A^h}(1 - \epsilon) \cong \prod_{i=1}^n \prod_{j=s_i+1}^{d_i} k(P_{ij}).$$

Now

$$\begin{aligned} L &:= K_A[\epsilon] \\ &\cong K_A\epsilon \times K_A(1 - \epsilon) \\ &\cong \prod_{i=1}^n [k(P_i)\epsilon \times k(P_i)(1 - \epsilon)]. \end{aligned}$$

It is possible in the above product that $k(P_i)\epsilon = 0$, that $k(P_i)(1 - \epsilon) = 0$ or that neither are zero. The case where $k(P_i)\epsilon = 0$ occurs if we have $s_i = 0$ so that $e_{i1} = \cdots = e_{id_i} = 0$, and analogously the case where $k(P_i)(1 - \epsilon) = 0$ occurs if $s_i = d_i$ so that $e_{i1} = \cdots = e_{id_i} = 1$. Note that the inclusion $L \hookrightarrow K_{A^h}$ is given by the product of all the diagonal maps

$$\begin{aligned} k(P_i)\epsilon &\longrightarrow \prod_{i=1}^{s_i} k(P_{ij}) \\ x\epsilon &\longmapsto (x, x, \dots, x), \end{aligned}$$

and

$$\begin{aligned} k(P_i)(1 - \epsilon) &\longrightarrow \prod_{j=s_i+1}^{d_i} k(P_{ij}) \\ x(1 - \epsilon) &\longmapsto (x, x, \dots, x). \end{aligned}$$

Since the minimal prime ideals of A^h correspond bijectively with the maximal ideals of the normalization $\tilde{A}$ of A by Remark 1.7, we may write

$$\text{Max}(\tilde{A}) = \{m_{11}, \dots, m_{1d_1}, \dots, m_{n1}, \dots, m_{nd_n}\},$$

where m_{ij} corresponds to the minimal prime ideal P_{ij} of A^h so that

$$\left(\tilde{A}_{m_{ij}}\right)^h \cong (A^h/P_{ij})^\sim.$$

Note that $\widetilde{A/P_i}$ has maximal ideals generated by the images of $m_{i1}, \dots, m_{id_i}$, so that the maximal ideal m_{ij} of $\tilde{A}$ contains only that minimal prime ideal of $\tilde{A}$ which lies over $P_i \subseteq A$. Now set

$$\mathcal{S}_i := \tilde{A} - \bigcup_{j=1}^{s_i} m_{ij}$$

and

$$\mathcal{T}_i := \tilde{A} - \bigcup_{j=s_i+1}^{d_i} m_{ij}.$$

If $s_i \neq 0$ then $\mathcal{S}_i^{-1}\tilde{A}$ is a normal domain with quotient field $k(P_i)$ and Henselization

$$\left(\mathcal{S}_i^{-1}\tilde{A}\right)^h \cong \prod_{j=1}^{s_i} \left(\tilde{A}_{m_{ij}}\right)^h \cong \prod_{j=1}^{s_i} \widetilde{A^h/P_{ij}}.$$

Also, if $s_i \neq 0$ then the image of the quotient field $k(P_i)$ of $\mathcal{S}_i^{-1}\tilde{A}$ in the quotient field of its Henselization is given by the diagonal map

$$\begin{aligned} k(P_i) &\longrightarrow \prod_{j=1}^{s_i} k(P_{ij}) \\ x &\longmapsto (x, x, \dots, x). \end{aligned}$$

Similarly if $s_i \neq d_i$ then $\mathcal{T}_i^{-1}\tilde{A}$ is a normal domain with quotient field $k(P_i)$ and Henselization

$$\left(\mathcal{T}_i^{-1}\tilde{A}\right)^h \cong \prod_{j=s_i+1}^{d_i} \left(\tilde{A}_{m_{ij}}\right)^h \cong \prod_{j=s_i+1}^{d_i} \widetilde{A^h/P_{ij}}.$$

Also, if $s_i \neq d_i$ then the image of the quotient field $k(P_i)$ of $\mathcal{T}_i^{-1}\tilde{A}$ in the quotient field of its Henselization is given by the diagonal map

$$\begin{aligned} k(P_i) &\longrightarrow \prod_{j=s_i+1}^{d_i} k(P_{ij}) \\ x &\longmapsto (x, x, \dots, x). \end{aligned}$$

Define

$$\tilde{C} := \prod_{i=1}^n \mathcal{S}_i^{-1}\tilde{A} \times \mathcal{T}_i^{-1}\tilde{A}.$$

Then $\tilde{C}$ is an $\tilde{A}$ -algebra via the diagonal map, $\tilde{C}$ has Henselization $\widetilde{\tilde{C}}^h \cong \tilde{A}^h$ and furthermore, by construction the image of the total quotient ring of $\tilde{C}$ in K_{A^h} is the same as the image of L in K_{A^h} . Thus C has total quotient ring L . Since $\tilde{C}$ is a product of localizations of $\tilde{A}$ which loses no maximal ideals, $\tilde{C}$ is faithfully flat, unramified and essentially finite over $\tilde{A}$. Thus $\tilde{C}$ is étale over $\tilde{A}$.

Suppose that $\tilde{D} = \tilde{C}$. Then by Proposition 2.2.6, D is essentially of finite type

over A , and then by Corollary 3.1.2, D is etale over A . Therefore, it suffices to show that $\tilde{D} = \tilde{C}$.

Since K_{A^h} is a direct limit of the total quotient rings of etale neighborhoods of A , we may choose an etale neighborhood E of A whose total quotient ring K_E contains L . Then by faithful flatness of $E \rightarrow A^h$,

$$D := L \cap A^h = L \cap K_E \cap A^h = L \cap E.$$

By Proposition 3.1.1 there exists $a \in A$ be a regular element such that $A/aA \cong E/aE$.

Since $D = L \cap E$, it follows that $aE \cap D = aD$. Thus we have injections

$$A/aA \hookrightarrow D/aD \hookrightarrow E/aE \cong A/aA.$$

So $A/aA \cong D/aD \cong E/aE$.

Consider $D[\tilde{A}] \subseteq \tilde{D} \subseteq \tilde{C}$. Since $D[\tilde{A}]$ is a homomorphic image of $D \otimes_A \tilde{A}$, we have morphisms

$$\tilde{A} \longrightarrow D \otimes_A \tilde{A} \xrightarrow{\text{surj.}} D[\tilde{A}] \longrightarrow \widetilde{A^h},$$

with the composite being faithfully flat (since $\widetilde{A^h} \cong \tilde{A}^h$). Since $D/aD \cong A/aA$, applying $\otimes_A A/aA$ we obtain morphisms

$$\frac{\tilde{A}}{a\tilde{A}} \xrightarrow{\cong} \frac{\tilde{A}}{a\tilde{A}} \xrightarrow{\text{surj.}} \frac{D[\tilde{A}]}{aD[\tilde{A}]} \longrightarrow \frac{\widetilde{A^h}}{a\widetilde{A^h}}$$

with the composite being faithfully flat. Thus the surjection

$$\frac{\tilde{A}}{a\tilde{A}} \xrightarrow{\text{surj.}} \frac{D[\tilde{A}]}{aD[\tilde{A}]}$$

is also injective, so that we have an isomorphism

$$\frac{D[\tilde{A}]}{aD[\tilde{A}]} \cong \frac{\tilde{A}}{a\tilde{A}}.$$

Since $D[\tilde{A}]$ is also finite over D so that all of its maximal ideals contain the element a , it follows that the maximal ideals of $D[\tilde{A}]$ correspond bijectively with those of $\tilde{A}$ (and hence also with those of $\tilde{C}$). Since $D[\tilde{A}]$ also has total quotient ring L , the minimal prime ideals of $D[\tilde{A}]$ and $\tilde{C}$ also correspond bijectively.

For $p, p' \in \text{Min}(D[\tilde{A}])$, if $(p + p')D[\tilde{A}]$ is contained in some maximal ideal N of $D[\tilde{A}]$, then $(p + p')\tilde{C}$ is contained in the maximal ideal of $\tilde{C}$ which lies over N . However, $p\tilde{C}$ is primary to some minimal prime ideal q of $\tilde{C}$ and $p'\tilde{C}$ is primary to some minimal prime ideal q' of $\tilde{C}$. Thus any prime ideal of $\tilde{C}$ which contains $(p + p')\tilde{C}$ also contains the minimal prime ideals q and q' . But since $\tilde{C}$ is a product of domains, $q + q'$ generates the ring, a contradiction. Therefore $(p + p')D[\tilde{A}]$ is contained in no maximal ideal of $D[\tilde{A}]$ and so by the Chinese Remainder Theorem, $D[\tilde{A}]$ is a product of domains.

Write

$$\text{Max}(D[\tilde{A}]) = \{N_{11}, \dots, N_{1d_1}, \dots, N_{n1}, \dots, N_{nd_n}\},$$

where $N_{ij} = m_{ij}\tilde{C} \cap D[\tilde{A}]$. Then by the above discussion and the correspondences between the maximal and minimal prime ideals of $D[\tilde{A}]$ and $\tilde{C}$, we must have

$$D[\tilde{A}] = \prod_{i=1}^n \mathcal{U}_i^{-1} D[\tilde{A}] \times \mathcal{V}_i^{-1} D[\tilde{A}],$$

where as with $\mathcal{S}_i$ and $\mathcal{T}_i$ we have

$$\mathcal{U}_i := D[\tilde{A}] - \bigcup_{i=1}^{s_i} N_{ij} \quad \text{and} \quad \mathcal{V}_i := D[\tilde{A}] - \bigcup_{i=s_i+1}^{d_i} N_{ij}.$$

Therefore we have injections

$$\mathcal{S}_i^{-1} \tilde{A} \hookrightarrow \mathcal{U}_i^{-1} D[\tilde{A}] \hookrightarrow \mathcal{U}_i^{-1} \tilde{C} = \mathcal{S}_i^{-1} \tilde{A}, \text{ and}$$

$$\mathcal{T}_i^{-1} \tilde{A} \hookrightarrow \mathcal{V}_i^{-1} D[\tilde{A}] \hookrightarrow \mathcal{V}_i^{-1} \tilde{C} = \mathcal{T}_i^{-1} \tilde{A}.$$

Thus $D[\tilde{A}] = \tilde{C}$ and so $D[\tilde{A}] = \tilde{D} = \tilde{C}$. $\square$

Example 3.1.4 For (A, m, k) a complete Noetherian local ring and $K_A \subseteq L \subseteq K_A^h$, an intermediate ring obtained by adjoining idempotent elements to K_A , it need not be true that the local ring $D := L \cap A^{hs}$ is etale over A .

Proof:

Let A be the power series ring in two variables, X and Y , over $\mathbb{Q}$ modulo the ideal generated by $X^3 - 3XY^2 + Y^3$,

$$A := \frac{\mathbb{Q}[[X, Y]]}{(X^3 - 3XY^2 + Y^3)}.$$

If $X^3 - 3XY^2 + Y^3$ factors in the polynomial ring $\mathbb{Q}[X, Y]$, then substituting $Y = 1$ we obtain a factorization of $X^3 - 3X + 1$ over $\mathbb{Q}$. However $X^3 - 3X + 1$ is easily seen to be irreducible over $\mathbb{Q}$. Thus $X^3 - 3XY^2 + Y^3$ is an irreducible element of $\mathbb{Q}[X, Y]$ and so $(X^3 - 3XY^2 + Y^3)\mathbb{Q}[X, Y]$ is a homogeneous prime ideal. It follows that $X^3 - 3XY^2 + Y^3$ generates a prime ideal of $\mathbb{Q}[[X, Y]]$. Therefore A is a domain.

We show that the normalization $\tilde{A}$ of A is given by $\tilde{A} = A[X/Y]$. Set

$$B := \left(\frac{\mathbb{Q}[X, Y]}{(X^3 - 3XY^2 + Y^3)} \right)_{(X, Y)}$$

Then $A \cong \hat{B}$ and $\tilde{A} \cong A \otimes_B \tilde{B}$. Thus it suffices to show that $\tilde{B}$ is given by $\tilde{B} = B[X/Y]$.

Now in K_B we have

$$(X/Y)^3 - 3(X/Y) + 1 = 0.$$

Thus X/Y is integral over B and so $B[X/Y] \subseteq \tilde{B}$. Thus we need only show that $B[X/Y]$ is normal. For this we consider the surjection

$$C := \frac{B[Z]}{(Z^3 - 3Z + 1, YZ - X)} \xrightarrow{\phi} B[X/Y], \text{ given by}$$

$$Z \longrightarrow X/Y.$$

Now

$$C \cong \frac{\mathbb{Q}[X, Y]_{(X, Y)}[Z]}{(Z^3 - 3Z + 1, YZ - X)} \cong \frac{\mathbb{Q}[Y]_{(Y)}[Z]}{(Z^3 - 3Z + 1)}.$$

Since C is finite over $\mathbb{Q}[Y]_{(Y)}$, all of its maximal ideals contain Y . However

$$\frac{C}{YC} \cong \frac{\mathbb{Q}[Z]}{(Z^3 - 3Z + 1)}$$

is a field. Therefore C is a DVR with maximal ideal generated by Y . Since ϕ is a surjection and since both C and $B[X/Y]$ are one-dimensional domains, ϕ is an isomorphism, and thus $B[X/Y]$ is normal.

Therefore $\tilde{A}$ is given by $\tilde{A} = A[X/Y]$. Note that since A is an excellent Henselian local domain, $\tilde{A}$ is a local ring. Hence $\tilde{A}$ has residual field

$$k_{\tilde{A}} \cong \frac{\mathbb{Q}[\xi]}{(\xi^3 - 3\xi + 1)}.$$

Since, using a discriminant argument, the Galois group of $\xi^3 - 3\xi + 1$ is cyclic of order three, $k_{\tilde{A}}$ is a normal field extension of $k = \mathbb{Q}$. In particular, the polynomial $\xi^3 - 3\xi + 1$ splits as a product of linear factors over $k_{\tilde{A}}$, so that

$$k_{\tilde{A}} \otimes_{\mathbb{Q}} k_{\tilde{A}} \cong (k_{\tilde{A}})^3.$$

Define $E := A \otimes_{\mathbb{Q}} k_{\tilde{A}}$. Then E is a local etale extension of A with residual field $k_E = k_{\tilde{A}}$. Since every maximal ideal of $\tilde{E}$ contains m , the residual field product of $\tilde{E} \cong E \otimes_A \tilde{A}$ is given by

$$(E \otimes_A \tilde{A}) \otimes_A \mathbb{Q} \cong k_E \otimes_{\mathbb{Q}} k_{\tilde{A}} = k_{\tilde{A}} \otimes_{\mathbb{Q}} k_{\tilde{A}} \cong (k_{\tilde{A}})^3.$$

Since E is an excellent Henselian reduced local ring, $\tilde{E}$ is a product of local domains. Therefore $\tilde{E}$ has three minimal prime ideals and hence so does E .

Write

$$\text{Min}(E) = \{P_1, P_2, P_3\}.$$

Denoting the quotient field of E/P_i by $k(P_i)$, we have

$$K_E \cong k(P_1) \times k(P_2) \times k(P_3).$$

Let $\eta := (1, 0, 0) \in K_E$ and put $L := K_A[\eta]$. Note that $L \cong (K_A)^2$ and maps into

K_E via

$$\begin{aligned} L \cong (K_A)^2 &\longrightarrow k(P_1) \times k(P_2) \times k(P_3) \cong K_E \\ (\alpha, \beta) &\longmapsto (\alpha, \beta, \beta). \end{aligned}$$

In particular L has two prime ideals.

Define $D := L \cap E$. Then, fixing an inclusion $E \hookrightarrow A^{hs}$, we also have by faithful flatness

$$D := L \cap E = L \cap K_E \cap A^{hs} = L \cap A^{hs},$$

Since D has total quotient ring L , D has two minimal prime ideals.

If D is etale over A , then D is excellent Henselian and so $\tilde{D}$ is a product of local domains. Since D has two minimal prime ideals, $\tilde{D} \cong D \otimes_A \tilde{A}$ is a product of two local domains and hence has residual field product

$$(D \otimes_A \tilde{A}) \otimes_A \mathbb{Q} \cong k_D \otimes_{\mathbb{Q}} k_{\tilde{A}}$$

which is then a product of two fields, $k_D \otimes_A k_{\tilde{A}} \cong k_1 \times k_2$. It follows that k_D is neither $\mathbb{Q}$ nor $k_{\tilde{A}}$ and hence that the containments

$$\mathbb{Q} \subseteq k_D \subseteq k_E = k_{\tilde{A}}$$

are proper. This is a contradiction since by construction there are no intermediate fields between $\mathbb{Q}$ and $k_{\tilde{A}}$. Therefore D cannot be etale over A . $\square$

Remark 3.1.5 In the previous example, while the intersection ring D is not etale over A , it is contained in a finite extension E of A . Thus D is finite over A and so is excellent. Nevertheless, it seems difficult to exhibit D explicitly as a finite A -algebra

without a fair amount of computation. One wonders if there is a general technique for exhibiting such intersection rings.

3.2 Intersections over Henselian Rings

Let A be an excellent reduced local ring, $K_A \subseteq L \subseteq K_{A^{hs}}$ be an intermediate ring which is finitely generated over K_A , and let D be the intersection ring given by $D := L \cap A^{hs}$. In section 3.1, we saw that the minimal prime structure of D can be an obstruction to D being etale over A . However, if A is Henselian, then the extension $A \hookrightarrow A^{hs}$ is integral. Thus $\tilde{A} \hookrightarrow \tilde{D} \hookrightarrow \widetilde{A^{hs}}$ are integral extensions and hence are semilocal. Then we know by Theorem 2.2.4 that $\tilde{D}$ is a direct limit of semilocal etale extensions of $\tilde{A}$. So D already has a great deal of structure through its normalization.

In Theorem 3.2.1 we avoid the minimal prime obstruction through modding out by minimal prime ideals and consider corresponding intersections under a homomorphic image of the strict Henselization. Making use of the integrality of the morphisms of normalizations we conclude in this theorem that the resulting intersection rings are as close to etale over A as we can hope for: they are unramified extensions. Specifically, we give the following theorem.

Theorem 3.2.1 *Let (A, m, k) be an excellent Henselian reduced local ring.*

- i) Let $A \subseteq D \subseteq A^{hs}$ be an intermediate ring which is module finite over A . Then for every minimal prime ideal p of D , there exists a local, finite and unramified extension D' of A with $D/p \subseteq D' \subseteq k(p) = Q(D/p)$. More generally, the following is true.*
- ii) Let $Q \in \text{Min}(A^{hs})$, $q := Q \cap A$ and let $k(q) \subseteq L \subseteq k(Q)$ be an intermediate*

field which is finitely generated over $k(q)$. Then the ring

$$D' := L \cap (A^{hs}/Q)$$

is finite and unramified over A (equivalently over A/q) and has strict Henselization A^{hs}/Q .

In the situation where the base ring A is Henselian, we use this theorem to give criteria for certain intermediate rings $A \subseteq D \subseteq A^{hs}$ to be etale over A . As another interesting application, we characterize Noetherianness of arbitrary local intermediate rings $A \subseteq D \subseteq A^{hs}$ dominated by A^{hs} completely in terms of the residual extensions arising from the morphism $D \rightarrow \tilde{D}$.

Before proving Theorem 3.2.1 we need a few technical lemmas.

Lemma 3.2.2 *Let (A, m, k) be an excellent Henselian reduced local ring with strict Henselization A^{hs} . Let $Q \in \text{Min}(A^{hs})$, $q := Q \cap A$, and let $k(q)$ and $k(Q)$ be the quotient fields of A/q and A^{hs}/Q respectively. Let $k(q) \subseteq L \subseteq k(Q)$ be an intermediate field which is finitely generated over $k(q)$. Define*

$$D' := L \cap (A^{hs}/Q).$$

Then there is a local etale extension $E \subseteq A^{hs}$ of A satisfying

- i) The minimal prime ideals of $A^{hs}/qA^{hs} \cong (A/q)^{hs}$ are extended from E/qE .*
- ii) For $P := Q \cap E$ we have*

$$D' = L \cap (E/P).$$

Proof:

Set $B := A/q$. Then $B = (B, mB, k)$ is an excellent Henselian local domain,

and so its normalization $\tilde{B}$ is a local domain. Let $k_{\tilde{B}}$ be the residual field of $\tilde{B}$ and k^s be the separable closure of k in $k_{\tilde{B}}$. Write $k^s = k[X]/(\bar{f})$, where $\bar{f} \in k[X]$ is a monic, irreducible and separable polynomial. Let G be a local etale extension of A such that $\bar{f}$ splits as a product of linear factors over k_G . Then G/qG is a local etale extension of B and by Corollary 2.1.4, the minimal prime ideals of $B^{hs} = A^{hs}/qA^{hs}$ are extended from G/qG .

Since L is finitely generated over $k(q) = Q(A/q)$ we may write

$$L = k(q)[\bar{x}_1, \dots, \bar{x}_n] \subseteq k(Q). \quad (3.1)$$

Since $k(Q) \cong (A^{hs}/Q) \otimes_A k(q)$ we may assume the $\bar{x}_i$ are in A^{hs}/Q . Let $x_1, \dots, x_n$ be preimages of the $\bar{x}_i$ in A^{hs} , and let $E \subseteq A^{hs}$ be a local etale extension of G containing the x_i . Then E is also a local etale extension of A and has strict Henselization A^{hs} . Since

$$B \subseteq G/qG \subseteq E/qE \subseteq A^{hs}/qA^{hs} \quad (3.2)$$

with the minimal prime ideals of A^{hs}/qA^{hs} extended from G/qG , they are also extended from E/qE .

Let $P := Q \cap E$ and $k(P)$ be the quotient field of E/P . Then since Q contains qA^{hs} , $Q = PA^{hs}$ is extended from E . Thus E/P has strict Henselization A^{hs}/Q , so that the morphism $E/P \rightarrow A^{hs}/Q$ is faithfully flat. Therefore E/P can be obtained by intersecting its quotient field with A^{hs}/Q ,

$$E/P = k(P) \cap (A^{hs}/Q). \quad (3.3)$$

Since by construction the $\bar{x}_i$ are in $k(P) = Q(E/P)$, $L \subseteq k(P)$. Thus we have

$$D' := L \cap (A^{hs}/Q) = L \cap k(P) \cap (A^{hs}/Q) = L \cap (E/P) \quad (3.4)$$

□

Remark 3.2.3 By $\hat{A}^{hs}$ in the following lemma, we mean $(\hat{A})^{hs}$. For a Henselian local ring (A, m, k) ,

$$(\hat{A})^{hs} \cong \hat{A} \otimes_A A^{hs}.$$

This is so because writing

$$A^{hs} = \varinjlim_{\lambda \in \Lambda} A_\lambda$$

as a direct limit of local etale extensions of A , we then have

$$\hat{A} \otimes_A A^{hs} \cong \varinjlim_{\lambda \in \Lambda} \hat{A} \otimes_A A_\lambda.$$

Since A is Henselian, each A_λ is a finite extension of A . Thus $\hat{A} \otimes_A A_\lambda$ is a finite extension of $\hat{A}$, so that all of its maximal ideals contain $m\hat{A}$. On the other hand

$$\hat{A} \otimes_A A_\lambda \otimes_A (A/m) \cong k \otimes_k k_{A_\lambda} \cong k_{A_\lambda}$$

is a field. Thus each $\hat{A} \otimes_A A_\lambda$ is local and so $\hat{A} \otimes_A A^{hs}$ is a direct limit of local etale extensions of $\hat{A}$. Since it is Henselian and has a separably closed residual field, it must be the strict Henselization of $\hat{A}$.

The ring $\hat{A}^{hs}$ need not be complete. If A is Noetherian or excellent then so is $\hat{A}^{hs}$.

Lemma 3.2.4 *Let (A, m, k) be an excellent Henselian reduced local ring with strict Henselization A^{hs} . Let $Q \in \text{Min}(A^{hs})$, $q := Q \cap A$ and let $k(q) \subseteq L \subseteq k(Q)$ be an intermediate field which is finitely generated over $k(q)$. Then*

$$D' := L \cap (A^{hs}/Q)$$

is an excellent Henselian local domain with quotient field L and completion given by

$$\hat{D}' \cong K_{\hat{D}'} \cap (\hat{A}^{hs}/\hat{Q})$$

where $\hat{Q}$ is a minimal prime ideal of $\hat{A}^{hs} \cong \hat{A} \otimes_A A^{hs}$ which is extended from $Q \subseteq A^{hs}$ along the canonical morphism.

Proof:

Replacing A by A/q we may assume that A is a domain, $q = (0)$ and $k(q) = K_A$.

Using Lemma 3.2.2 we may write

$$D' = L \cap (E/P), \tag{3.5}$$

where E is a local etale extension of A , $P := Q \cap E$, and such that the minimal prime ideals of A^{hs} are extended from E . By Theorem 2.1.3, the latter is equivalent to the residual field of E being separably closed in each residual field of its normalization.

Since E is a local etale extension of the Henselian ring A , it is a finite extension by Proposition 1.11. Thus $A \subseteq D' \subseteq E/P$ are finite extensions, so that D' must be an excellent Henselian local domain. To see that D' has quotient field L , write

$$L = K_A[\bar{x}_1, \dots, \bar{x}_n] \subseteq k(P). \tag{3.6}$$

Since $k(P) \cong (E/P) \otimes_A K_A$ we may assume the $\bar{x}_i$ are in E/P . Then the $\bar{x}_i$ are in

$L \cap (E/P) = D'$. Thus D' must have total quotient ring L .

Since the residual field of E is separably closed in each residual field of its normalization, the same is true of $\hat{E}$ in its normalization since completions are obtained merely by tensoring with $\otimes_A \hat{A}$. Thus again by Theorem 2.1.3, the minimal prime ideals of $\hat{E}^{hs}$ are extended from $\hat{E}$. Since $\hat{E} \cong E \otimes_A \hat{A}$ is a local etale extension of $\hat{A}$, it has the same strict Henselization,

$$\hat{E}^{hs} \cong \hat{A}^{hs}. \quad (3.7)$$

Furthermore, since E is excellent and Henselian, the minimal prime ideals of $\hat{E}$ are extended from E . It follows that $P \subseteq E$ extends to a minimal prime ideal in each ring of the following commutative diagram

$$\begin{array}{ccc} \hat{E} & \longrightarrow & \hat{E}^{hs} \cong \hat{A}^{hs} \\ \uparrow & & \uparrow \\ E & \longrightarrow & E^{hs} \cong A^{hs}. \end{array} \quad (3.8)$$

Thus $Q = PA^{hs}$. Set $\hat{P} := P\hat{E}$ and $\hat{Q} := \hat{P}\hat{A}^{hs} = P\hat{A}^{hs} = Q\hat{A}^{hs}$. Applying $\otimes_A \hat{A}$ to the injections

$$A \longrightarrow D' \longrightarrow E/P \longrightarrow A^{hs}/Q, \quad (3.9)$$

we thus obtain the following injections of domains,

$$\hat{A} \longrightarrow \hat{D}' \longrightarrow \hat{E}/\hat{P} \longrightarrow \hat{A}^{hs}/\hat{Q}. \quad (3.10)$$

We now argue that

$$\hat{D}' \cong K_{\hat{D}'} \cap (\hat{E}/\hat{P}) \cong K_{\hat{D}'} \cap \widehat{E/P}. \quad (3.11)$$

To see this, first observe that

$$\begin{aligned}
\hat{D}' &\cong D' \otimes_A \hat{A} \\
&= [L \cap (E/P)] \otimes_A \hat{A} \\
&\cong [L \otimes_A \hat{A}] \cap [(E/P) \otimes_A \hat{A}] \text{ by flatness} \\
&\cong [L \otimes_{D'} D' \otimes_A \hat{A}] \cap [(E/P) \otimes_A \hat{A}] \\
&\cong [L \otimes_{D'} \hat{D}'] \cap \widehat{E/P}
\end{aligned} \tag{3.12}$$

where the last isomorphism follows by finiteness of D' and E/P over A . Since for $\widetilde{\hat{D}'}$ the normalization of $\hat{D}'$ we have

$$\hat{D}' \subseteq \widetilde{\hat{D}'} \cong \tilde{D}' \otimes_{D'} \hat{D}' \subseteq L \otimes_{D'} \hat{D}', \tag{3.13}$$

we may replace $L \otimes_{D'} \hat{D}'$ in 3.12 with $\widetilde{\hat{D}'}$ to write

$$\hat{D}' \cong \widetilde{\hat{D}'} \cap \widehat{E/P}. \tag{3.14}$$

To establish 3.11 we need to make some observations about normalizations.

Since E is finite over A , $\hat{E} \cong E \otimes_A \hat{A}$ is a local etale extension of $\hat{A}$. Thus its normalization is given by

$$\widetilde{\hat{E}} \cong \hat{E} \otimes_{\hat{A}} \widetilde{\hat{A}} \tag{3.15}$$

and is finite and etale over $\widetilde{\hat{A}}$. Since $\widetilde{\hat{E}}$ is Henselian, semilocal and normal, it is a product of local normal domains with each factor finite and etale over the local normal domain $\widetilde{\hat{A}}$. Thus $\widehat{\widetilde{E/P}} \cong \widehat{\widetilde{E}/\hat{P}}$ is one of these factors and so is a local etale extension of $\widetilde{\hat{A}}$.

Applying $\otimes_A \hat{A}$ to the morphisms

$$A \hookrightarrow D' \hookrightarrow E/P, \quad (3.16)$$

we obtain morphisms of completions

$$\hat{A} \hookrightarrow \hat{D}' \hookrightarrow \widehat{E/P}. \quad (3.17)$$

Then considering normalizations we obtain

$$\widetilde{\hat{A}} \hookrightarrow \widetilde{\hat{D}'} \hookrightarrow \widetilde{\widehat{E/P}}, \quad (3.18)$$

with the composite being finite, local and etale. By Proposition 2.2.4, $\widetilde{\hat{D}'}$ is etale over $\widetilde{\hat{A}}$. So the second morphism above is etale by Proposition 1.14 and also faithfully flat. In particular, $\widetilde{\hat{D}'}$ can be obtained by intersecting its total quotient ring with $\widetilde{\widehat{E/P}}$,

$$\widetilde{\hat{D}'} = K_{\hat{D}'} \cap \widetilde{\widehat{E/P}}. \quad (3.19)$$

Substituting in 3.14 we obtain

$$\hat{D}' \cong K_{\hat{D}'} \cap \widetilde{\widehat{E/P}} \cap \widehat{E/P} = K_{\hat{D}'} \cap \widehat{E/P}, \quad (3.20)$$

the desired isomorphism 3.11.

To finish, recall that $\hat{E}$ has strict Henselization $\hat{E}^{hs} \cong \hat{A}^{hs}$ and that $P \subseteq E$ extends to the prime ideals $\hat{P} \subseteq \hat{E}$ and $\hat{Q} \subseteq \hat{A}^{hs}$. Therefore

$$\widehat{E/P} \cong \frac{\hat{E}}{\hat{P}} \longrightarrow \frac{\hat{E}^{hs}}{\hat{P}\hat{E}^{hs}} \cong \frac{\hat{A}^{hs}}{\hat{Q}} \quad (3.21)$$

is a faithfully flat morphism of domains. Thus $\widehat{E/P} \cong \hat{E}/\hat{P}$ can be obtained by

intersecting its quotient field $k(\hat{P})$ with the faithfully flat extension $\hat{A}^{hs}/\hat{Q}$,

$$\widehat{E/P} = k(\hat{P}) \cap \frac{\hat{A}^{hs}}{\hat{Q}} \quad (3.22)$$

From 3.11 we now have

$$\begin{aligned} \hat{D}' &\cong K_{\hat{D}'} \cap \widehat{E/P} \\ &\cong K_{\hat{D}'} \cap k(\hat{P}) \cap \frac{\hat{A}^{hs}}{\hat{Q}} \\ &\cong K_{\hat{D}'} \cap \frac{\hat{A}^{hs}}{\hat{Q}}, \end{aligned} \quad (3.23)$$

where the last isomorphism follows since $K_{\hat{D}'} \subseteq k(\hat{P})$ by 3.10. $\square$

Remark 3.2.5 By Proposition 1.8, every local etale extension of a Henselian local ring A is a finite extension. Since the strict Henselization A^{hs} is a direct limit of local etale extensions, A^{hs} is integral over A . So for any intermediate ring $A \subseteq D \subseteq A^{hs}$, A^{hs} is also integral over D . Since A^{hs} is local, the intermediate ring D is also local.

Proof of Theorem 3.2.1:

The statement *i*) follows from *ii*) by setting $L := k(p)$. To show *i*), fix $Q \in \text{Min}(A^{hs})$. Replacing A by A/q we may assume that A is a domain, $q = (0)$ and $k(q) = K_A$. Since A is a Henselian local domain, its normalization $\tilde{A}$ is a local domain. Let $k_{\tilde{A}}$ be the residual field of $\tilde{A}$.

By Lemma 3.2.2, we may write

$$D' = L \cap (E/P), \quad (3.24)$$

where E is a local etale extension of A , $P := Q \cap E$, and such that the minimal prime ideals of A^{hs} are extended from E . Since E is a local etale extension of the Henselian

ring A , it is a finite extension. Thus since the composite $A \hookrightarrow D' \hookrightarrow E/P$ is finite, D' is also finite over A .

Note that by Theorem 2.1.3, since the minimal prime ideals of $A^{hs} = E^{hs}$ are extended from E , the residual field of E is separably closed in each residual field of its normalization. Let k^s be the separable closure of k in $k_{\hat{A}}$. Write $k^s = k[X]/(\bar{f})$ where $\bar{f}$ is a monic, irreducible and separable polynomial of degree d . Then by Corollary 2.1.4, $\bar{f}$ splits as a product of linear factors over k_E .

A) We may assume that A is complete:

Knowing the result for complete rings we can conclude, since $\hat{D}'$ is an appropriate intersection by Lemma 3.2.4, that $\hat{D}'$ is unramified over $\hat{A}$. But since D' is finite over A , $\hat{D}' \cong D' \otimes_A \hat{A}$. Then D' is unramified over A by (ch. II, Prop. 4 of [Ra]).

B) Expressing $\widetilde{E/P}$ as a tensor product:

Henceforth we assume that A is complete and that D' is given as in the intersection 3.24.

Let R be a coefficient ring for A , R^s be the local etale extension of R with residual field k^s , and let S_E be the local etale extension of R with residual field k_E . Note that since R is complete and hence Henselian, R^s and S_E are finite extensions of R .

Now by choice of E , we have an k -morphism $k^s \hookrightarrow k_E$. By Proposition 1.10, since R^s is etale over R and S_E is Henselian, $k^s \hookrightarrow k_E$ lifts to an R -morphism $R^s \rightarrow S_E$. By Proposition 1.14, since both R^s and S_E are etale over R , $R^s \rightarrow S_E$ is etale.

By Proposition 1.16

$$E \cong A \otimes_R S_E, \tag{3.25}$$

and so

$$\begin{aligned}
\tilde{E} &\cong \tilde{A} \otimes_A E \\
&\cong \tilde{A} \otimes_A (A \otimes_R S_E) \\
&\cong \tilde{A} \otimes_R S_E.
\end{aligned} \tag{3.26}$$

Since $\tilde{A}$ is Henselian and R^s is etale over R , there is an isomorphism

$$\mathrm{Hom}_{loc\ R}(R^s, \tilde{A}) \cong \mathrm{Hom}_k(k^s, k_{\tilde{A}}). \tag{3.27}$$

Thus there is a canonical morphism $R^s \longrightarrow \tilde{A}$ induced by the inclusion $k^s \hookrightarrow k_{\tilde{A}}$. So we may write

$$\tilde{E} \cong \tilde{A} \otimes_R S_E \cong \tilde{A} \otimes_{R^s} R^s \otimes_R S_E. \tag{3.28}$$

Since $R^s \otimes_R S_E$ is a *finite* semilocal etale extension of the Henselian ring R^s , it is a product of local etale extensions of R^s . Recall that $k^s = k[X]/(\bar{f})$ where $\bar{f}$ is a monic, irreducible and separable polynomial of degree d and that by choice of E , $\bar{f}$ splits as a product of linear factors over k_E . Therefore the residual field product of $R^s \otimes_R S_E$ is given by

$$(R^s \otimes_R S_E) \otimes_{R^s} k^s \cong k^s \otimes_k k_E \cong (k_E)^d. \tag{3.29}$$

Thus $R^s \otimes_R S_E$ is a product of d local etale extensions of R^s , each with residual field k_E . Since $(S_E)^d$ is also a product of d local etale extensions of R^s , each having residual field k_E , and since local etale extensions of the Henselian local ring R^s are

uniquely determined by their residual fields, it follows that

$$R^s \otimes_R S_E \cong (S_E)^d, \quad (3.30)$$

an isomorphism of R^s -algebras, and arguing similarly, also an isomorphism of S_E -algebras. Substituting in 3.28 yields

$$\tilde{E} \cong \tilde{A} \otimes_{R^s} (S_E)^d \cong \left(\tilde{A} \otimes_{R^s} S_E \right)^d. \quad (3.31)$$

Since $\tilde{A} \otimes_{R^s} S_E$ is finite over $\tilde{A}$ and since

$$\left(\tilde{A} \otimes_{R^s} S_E \right) \otimes_{\tilde{A}} k_{\tilde{A}} \cong k_{\tilde{A}} \otimes_{k^s} k_E \quad (3.32)$$

is a field by Lemma 2.1.1, $\tilde{A} \otimes_{R^s} S_E$ is a local ring. Since E is excellent Henselian and reduced, $\tilde{E}$ is a product of local domains. Therefore 3.31 exhibits $\tilde{E}$ as a product of local domains. Thus $\widetilde{E/P}$ is given by

$$\widetilde{E/P} \cong \tilde{A} \otimes_{R^s} S_E. \quad (3.33)$$

Furthermore there is a commutative diagram of S_E -algebras

$$\begin{array}{ccc} E \cong A \otimes_R S_E & \longrightarrow & \tilde{E} \cong \left(\tilde{A} \otimes_{R^s} S_E \right)^d \\ \text{can} \downarrow & & \downarrow \text{can} \\ E/P & \longrightarrow & \widetilde{E/P} \cong \tilde{A} \otimes_{R^s} S_E. \end{array} \quad (3.34)$$

To see that the composite $S_E \longrightarrow A \otimes_R S_E \cong E \longrightarrow E/P$ is injective note that since A is reduced, its coefficient ring R is a field or a complete p -ring. If R is a field then so is S_E since it is a local etale extension of R . Thus in this case injectivity is trivial. Suppose R is a complete p -ring with uniformizing element

$t \in \mathbb{Z}$. Then so is S_E . Since $R \subseteq A \subseteq E/P$, no power of t is in P . Therefore since t also generates the maximal ideal of S_E , the kernel of $S_E \rightarrow E/P$ must be zero.

C) *Expressing $\tilde{D}'$ as a tensor product:*

Since $A \hookrightarrow D' \hookrightarrow E/P$ are finite extensions, so are the morphisms of normalizations $\tilde{A} \hookrightarrow \tilde{D}' \hookrightarrow \widetilde{E/P}$. In particular, the second morphism is local. By Theorem 2.2.6, $\tilde{D}'$ is a local etale extension of $\tilde{A}$.

Now $k_{\tilde{A}} \rightarrow k_{\tilde{D}'}$ is a separable field extension. Let l be the separable closure of k^s in $k_{\tilde{D}'}$ (equivalently the separable closure of k in $k_{\tilde{D}'}$, since k^s is by definition the separable closure of k in $k_{\tilde{A}}$). Then we have the following diagram of field extensions

$$\begin{array}{ccc}
 k_{\tilde{A}} & \xrightarrow{\text{sep}} & k_{\tilde{D}'} \\
 \text{purely insep} \uparrow & & \uparrow \text{purely insep} \\
 k^s & \xrightarrow{\text{sep}} & l.
 \end{array} \tag{3.35}$$

It follows that $[k_{\tilde{D}'} : k_{\tilde{A}}] = [l : k^s]$ (see for example ch. V, sect. 6, Cor. 13 of [Hu]).

Therefore by Lemma 2.1.1, since $k_{\tilde{A}}$ is purely inseparable over k^s and l is separable over k^s , $k_{\tilde{A}} \otimes_{k^s} l$ is a *field* of degree over k^s given by

$$\begin{aligned}
 [k_{\tilde{A}} \otimes_{k^s} l : k^s] &= [k_{\tilde{A}} : k^s][l : k^s] \\
 &= [k_{\tilde{D}'} : k_{\tilde{A}}][k_{\tilde{A}} : k^s] \\
 &= [k_{\tilde{D}'} : k^s].
 \end{aligned} \tag{3.36}$$

Thus the canonical morphism $k_{\tilde{A}} \otimes_{k^s} l \hookrightarrow k_{\tilde{D}'}$ is a k^s -morphism between fields of the same degree over k^s , hence an isomorphism

$$k_{\tilde{A}} \otimes_{k^s} l \cong k_{\tilde{D}'}.$$
(3.37)

Let T be the unique local etale extension of R^s with residual field l . Note that T

is also etale over R . Then $\tilde{A} \otimes_{R^s} T$ is a local etale extension of $\tilde{A}$ with residual field $k_{\tilde{A}} \otimes_{k^s} l \cong k_{\widetilde{D'}}$. Since $\widetilde{D'}$ is local etale over $\tilde{A}$, Proposition 1.11 gives

$$\widetilde{D'} \cong \tilde{A} \otimes_{R^s} T \quad (3.38)$$

D) We show that the inclusion

$$\tilde{A} \otimes_{R^s} T \cong \widetilde{D'} \hookrightarrow \widetilde{E/P} \cong \tilde{A} \otimes_{R^s} S_E \quad (3.39)$$

sends T into S_E :

We have the following diagram of finite injective morphisms of complete local domains

$$\begin{array}{ccccc} \tilde{A} & \xrightarrow{\text{etale}} & \widetilde{D'} \cong \tilde{A} \otimes_{R^s} T & \xrightarrow{\text{etale}} & \widetilde{E/P} \cong \tilde{A} \otimes_{R^s} S_E \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & D' & \longrightarrow & E/P \end{array}$$

The inclusion $k_{\widetilde{D'}} \hookrightarrow k_{\widetilde{E/P}}$ of residual fields induces a morphism between separable closures over k^s . The separable closure of k^s in $k_{\widetilde{D'}}$ is by definition l . By choice of E the minimal prime ideals of A^{hs} are extended from E . Thus by Corollary 2.1.4, k_E is separably closed in each of the residual fields of $k_{\widetilde{E}}$. Hence k_E is separably closed in $k_{\widetilde{E/P}}$. Therefore the morphism of separable closures gives

$$l \xrightarrow{\bar{\phi}} k_E. \quad (3.40)$$

Applying $k_{\tilde{A}} \otimes_{R^s}$ we obtain the following commutative diagram of fields

$$\begin{array}{ccc}
 k_{\tilde{A}} \otimes_{R^s} l & \xrightarrow{\text{id} \otimes_{R^s} \bar{\phi}} & k_{\tilde{A}} \otimes_{R^s} k_E \\
 \cong \uparrow & & \cong \uparrow \\
 k_{\widetilde{D'}} & \xrightarrow{\text{inc}} & k_{\widetilde{E/P}}.
 \end{array} \tag{3.41}$$

By Proposition 1.10, since T is local etale over R^s and S_E is Henselian, there is an isomorphism

$$\text{Hom}_{\text{loc} R^s}(T, S_E) \cong \text{Hom}_{k^s}(l, k_E) \tag{3.42}$$

Therefore $\bar{\phi}$ lifts to an etale morphism

$$T \xrightarrow{\phi} S_E \tag{3.43}$$

of R^s -algebras. Then applying $\tilde{A} \otimes_{R^s}$ we obtain a morphism

$$\begin{array}{ccc}
 \tilde{A} \otimes_{R^s} T & \xrightarrow{\text{id} \otimes_{R^s} \phi} & \tilde{A} \otimes_{R^s} S_E \\
 \uparrow \cong & & \uparrow \cong \\
 \widetilde{D'} & & \widetilde{E/P}
 \end{array} \tag{3.44}$$

which induces by 3.41 the inclusion map on residual fields

$$k_{\widetilde{D'}} \hookrightarrow k_{\widetilde{E/P}}. \tag{3.45}$$

Then since there is an isomorphism

$$\text{Hom}_{\text{loc} \tilde{A}}(\widetilde{D'}, \widetilde{E/P}) \cong \text{Hom}_{k_{\tilde{A}}}(k_{\widetilde{D'}}, k_{\widetilde{E/P}}), \tag{3.46}$$

the composite

$$\widetilde{D'} \xrightarrow{\cong} \tilde{A} \otimes_{R^s} T \xrightarrow{\text{id} \otimes_{R^s} \phi} \tilde{A} \otimes_{R^s} S_E \xrightarrow{\cong} \widetilde{E/P} \quad (3.47)$$

is the inclusion map. That is, there is a commutative diagram

$$\begin{array}{ccc} \tilde{A} \otimes_{R^s} T & \xrightarrow{\text{id} \otimes_{R^s} \phi} & \tilde{A} \otimes_{R^s} S_E \\ \cong \uparrow & & \cong \uparrow \\ \widetilde{D'} & \xrightarrow{\text{inc}} & \widetilde{E/P} \end{array} \quad (3.48)$$

This establishes the claim.

E) Concluding the proof:

By part *D)* and we have $T \subseteq S_E \subseteq \widetilde{E/P}$. But by 3.34, considering E/P as a subset of $\widetilde{E/P}$, $S_E \subseteq E/P$. Thus also $T \subseteq E/P$, so that

$$T \subseteq \widetilde{D'} \cap (E/P) = D'. \quad (3.49)$$

Then defining $F := A \otimes_R T$, there is a canonical morphism $F \rightarrow D'$. Let p_0 be the kernel of this morphism so that $F/p_0 \hookrightarrow D'$ is injective.

Note that F is a local etale extension of A with residual field

$$k_F \cong F \otimes_A k = A \otimes_R T \otimes_A k \cong k \otimes_k l \cong l. \quad (3.50)$$

Also $\tilde{F} \cong F \otimes_A \tilde{A}$ is a semilocal etale extension of $\tilde{A}$. Since F is Henselian, $\tilde{F}$ is a product of local domains with each factor being a local etale extension of the local domain $\tilde{A}$. Thus $\widetilde{F/p_0}$ is one of these factors and so a local etale extension of $\tilde{A}$.

Now since F has residual field l and since $\tilde{A} \hookrightarrow \widetilde{F/p_0}$, the residual field of $\widetilde{F/p_0}$

contains the field

$$k_{\tilde{A}} \otimes_{k^*} l \cong k_{\tilde{D}'}. \quad (3.51)$$

Thus

$$\widetilde{F/p_0} \hookrightarrow \widetilde{D'} \quad (3.52)$$

is a residually trivial morphism of local etale extensions of $\tilde{A}$ and also etale by Proposition 1.14. But since $\widetilde{F/p_0}$ is Henselian, it is closed with respect to residually trivial local etale extensions. Therefore this morphism is an isomorphism. In particular, F/p_0 and D' have isomorphic quotient fields, $K_{F/p_0} \cong L$.

By Proposition 1.10, since E is Henselian, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{loc}A}(F, E) \cong \mathrm{Hom}_k(l, k_E). \quad (3.53)$$

Therefore the composite $l \hookrightarrow k_{D'} \hookrightarrow k_E$ induced by $F/p_0 \longrightarrow D' \longrightarrow E/P$ lifts to an etale morphism $F \hookrightarrow E$ of local etale extensions of A . Then to see that the resulting diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \end{array} \quad (3.54)$$

$$F/p_0 \longrightarrow D' \longrightarrow E/P \quad (3.55)$$

is commutative, observe that by construction the two morphisms $F \longrightarrow E/P$ obtained in the diagram induce the same map on residual fields. Then use the fact that by

Proposition 1.10, since E/P is Henselian, we have

$$\mathrm{Hom}_{\mathrm{loc}A}(F, E/P) \cong \mathrm{Hom}_k(l, k_E), \quad (3.56)$$

so that the two morphisms must be the same.

Since F/p_0 has residual field l and $\widetilde{F/p_0}$ has residual field $k_{\bar{A}} \otimes_{k^s} l$,

$$F/p_0 \hookrightarrow \widetilde{F/p_0} \quad (3.57)$$

is residually purely inseparable. By Theorem 2.1.3, since F/p_0 is Henselian, we have that

$$(F/p_0)^{hs} = A^{hs}/p_0 A^{hs} \quad (3.58)$$

is a domain. Thus $p_0 A^{hs} = Q$ and $P := Q \cap E = p_0 E$. Therefore $F/p_0 \rightarrow E/P$ is etale. Then by faithful flatness we have

$$\begin{aligned} F/p_0 &= K_{F/p_0} \cap (E/P) \\ &= L \cap (E/P) \\ &= D', \end{aligned} \quad (3.59)$$

where the rings here are considered as subrings of $k(P) = Q(E/P)$. Since F is etale over A , this establishes that D' is unramified over A . Furthermore we have

$$(D')^{hs} = (F/p_0)^{hs} = A^{hs}/Q \quad (3.60)$$

as desired. $\square$

Corollary 3.2.6 *Let (A, m, k) be an excellent Henselian local domain such that A^{hs} is a domain (equivalently, by Theorem 2.1.3, such that the residual field of $\tilde{A}$ is purely inseparable over k). Then for any intermediate field $K_A \subseteq L \subseteq K_{A^{hs}}$ which is finitely generated over K_A ,*

$$D' := L \cap A^{hs}$$

is a local etale extension of A .

Proof:

By the previous theorem, D' is finite and unramified over A and has Henselization $(D')^{hs} = A^{hs}$. Since then A^{hs} is faithfully flat over both A and D' , D' is faithfully flat over A . Thus D' is a local etale extension of A by Remark 1.2. $\square$

Theorem 3.2.7 *Let (A, m, k) be an excellent Henselian reduced local ring, and let $A \subseteq D \subseteq A^{hs}$ be an intermediate ring which is module finite over A . Then the following are equivalent:*

- i) *D is etale over A and the minimal prime ideals of $A^{hs} = D^{hs}$ are extended from D .*
- ii) *The residual field k_D of D is separably closed in each residual field of the normalization $\tilde{D}$ of D .*
- iii) *For each minimal prime ideal Q of A^{hs} and $p := Q \cap D$ the morphism*

$$D \longrightarrow D/p \hookrightarrow k(p) \cap (A^{hs}/Q)$$

is residually trivial.

Proof:

(i $\Rightarrow$ ii) : This is immediate from Theorem 2.1.3.

(ii $\Rightarrow$ iii) : Set $D' := k(p) \cap (A^{hs}/Q)$. By Theorem 3.2.1, D' is finite over A and hence $D/p \hookrightarrow D'$ is also finite. Since D/p and D' have the same quotient field $k(p)$, we must have

$$\widetilde{D/p} = \widetilde{D'}. \quad (3.61)$$

Since $k_{D'}$ is a finite separable field extension of k_D and since by ii) k_D is separably closed in the residual field $k_{\widetilde{D/p}} = k_{\widetilde{D'}}$, it follows that $k_D = k_{D'}$.

(iii $\Rightarrow$ i) : Let F be the unique local etale extension of A with residual field k_D . Since D' is unramified over A by Theorem 3.2.1, D' is a homomorphic image of a local etale extension of A which has residual field $k_{D'} = k_D$. But by Proposition 1.11, all such local etale extensions of A are isomorphic to F . Thus D' is a homomorphic image of F via an A -morphism ψ which induces the identity map on residual fields. Furthermore by Proposition 1.10, ψ is the unique local A -morphism from F to D' which induces the identity on residual fields. Using Proposition 1.10 again, since D is finite over A and hence Henselian, the identity map $k_D \rightarrow k_D$ lifts to a morphism $F \rightarrow D$ which is injective since, for E some local etale extension of A containing D , the composite

$$F \rightarrow D \rightarrow E \quad (3.62)$$

is etale. But since the composite

$$F \hookrightarrow D \longrightarrow D/p \hookrightarrow D' \quad (3.63)$$

induces the identity map on residual fields, this composite must be the surjection ψ by uniqueness. Therefore we have isomorphisms

$$F/p_0 \cong D/p \cong D', \quad (3.64)$$

where

$$p_0 := p \cap F = Q \cap F. \quad (3.65)$$

By Theorem 3.2.1 we have isomorphisms

$$(F/p_0)^{hs} \cong (D')^{hs} \cong A^{hs}/Q. \quad (3.66)$$

Then since F is local etale over A , we must have $Q = p_0 A^{hs}$.

Since this argument is independent of the choice of the minimal prime ideal Q of A^{hs} , it follows that the minimal prime ideals of A^{hs} are all extended from F . Furthermore, for any minimal prime ideal $Q \in \text{Min}(A^{hs})$ we have

$$\frac{F}{Q \cap F} \cong \frac{D}{Q \cap D}. \quad (3.67)$$

Since there are injections

$$F \hookrightarrow D \hookrightarrow A^{hs}, \quad (3.68)$$

the rings F , D and A^{hs} all have the same number of minimal prime ideals. By 3.67

it follows that considered as subrings of $K_{A^{hs}}$, F and D have the same total quotient ring. So by faithful flatness of $F \longrightarrow A^{hs}$ we have

$$F = K_F \cap A^{hs} \supseteq D. \quad (3.69)$$

Therefore $F \longrightarrow D$ is an isomorphism. $\square$

Theorem 3.2.8 *Let (A, m, k) be an excellent Henselian reduced local ring, and let $A \subseteq D \subseteq A^{hs}$ be an intermediate ring. Then D is Noetherian if and only if for every residual field l of the normalization $\tilde{D}$ of D , the separable closure of k_D in l is a finite field extension of k_D .*

Proof:

$(\Rightarrow)$: This follows by the Mori-Nagata integral closure theorem (sect. 33 of [M]).

$(\Leftarrow)$: Suppose that for every residual field l of $\tilde{D}$, the separable closure of k_D in l is a finite field extension of k_D .

To show that D is Noetherian, it is enough to show that D/P is Noetherian for any $P \in \text{Min}(D)$. So fix $P \in \text{Min}(D)$. Replacing A by $A/(P \cap A)$, we may assume A is a domain.

Let Q be a minimal prime ideal of A^{hs} which lies over P , and set

$$D' := k(P) \cap (A^{hs}/Q) \quad (3.70)$$

so that we have containments

$$A \subseteq D/P \subseteq D' \subseteq A^{hs}/Q. \quad (3.71)$$

We argue that D' is excellent by exhibiting it as a homomorphic image of a direct limit of local etale extensions of A . We then argue that D' is finite over D/P , so that D/P is Noetherian by Eakin-Nagata (Thm. 3.7 of [M]).

Write D as a direct limit of finite type A -subalgebras,

$$D = \varinjlim_{\gamma \in \Gamma} D_\gamma = \bigcup_{\gamma \in \Gamma} D_\gamma. \quad (3.72)$$

Since $A \longrightarrow A^{hs}$ is integral and local, the D_γ are finite local extensions of A , hence Henselian.

Let A_γ be the unique local etale extension of A with residual field k_{D_γ} . By Proposition 1.10, since D_γ is Henselian, there is an isomorphism

$$\mathrm{Hom}_{locA}(A_\gamma, D_\gamma) \cong \mathrm{Hom}_k(k_{D_\gamma}, k_{D_\gamma}). \quad (3.73)$$

Hence there is a morphism $A_\gamma \longrightarrow D_\gamma$ inducing the identity on residual fields. Since the composite $A_\gamma \longrightarrow D_\gamma \longrightarrow A^{hs}$ is faithfully flat,

$A_\gamma \hookrightarrow D_\gamma$ is injective.

Now for any $\gamma_0, \gamma_1 \in \Gamma$ we have an isomorphism

$$\mathrm{Hom}_{locA}(A_{\gamma_0}, A_{\gamma_1}) \cong \mathrm{Hom}_k(k_{D_{\gamma_0}}, k_{D_{\gamma_1}}). \quad (3.74)$$

So the A_γ form a direct system with $\gamma_0 \leq \gamma_1$ if and only if $D_{\gamma_0} \subseteq D_{\gamma_1}$, in which case the structure morphism $A_{\gamma_0} \longrightarrow A_{\gamma_1}$ is induced by $k_{D_{\gamma_0}} \longrightarrow k_{D_{\gamma_1}}$. Set

$$F := \varinjlim_{\gamma \in \Gamma} A_\gamma. \quad (3.75)$$

Then F is a direct limit of local etale extensions of A and has residual field k_D .

Furthermore, we have

$$F \cong \varinjlim_{\gamma \in \Gamma} A_\gamma \subseteq \varinjlim_{\gamma \in \Gamma} D_\gamma = D. \quad (3.76)$$

Let $N := P \cap F$ so that there are containments

$$A \subseteq F/N \subseteq D/P \subseteq D' \subseteq A^{hs}/Q. \quad (3.77)$$

For each $\gamma \in \Gamma$, we let $P_\gamma := P \cap D_\gamma$ and set

$$D'_\gamma := k(P_\gamma) \cap (A^{hs}/Q). \quad (3.78)$$

Then by Theorem 3.2.1, the D'_γ are unramified over A . Furthermore, we have

$$D' = \varinjlim_{\gamma \in \Gamma} D'_\gamma. \quad (3.79)$$

Let A'_γ be the unique local etale extension of A with residual field $k_{D'_\gamma}$. By Proposition 1.10, there is a morphism $A'_\gamma \rightarrow D'_\gamma$. Since D'_γ is unramified over A , D'_γ is a homomorphic image of a local etale extension of A which has residual field $k_{D'_\gamma}$. But all such local etale extensions of A are isomorphic to A'_γ . It follows that $A'_\gamma \rightarrow D'_\gamma$ is surjective.

Set

$$F' := \varinjlim_{\gamma \in \Gamma} A'_\gamma. \quad (3.80)$$

Taking the direct limit of the surjections

$$A'_\gamma \rightarrow D'_\gamma, \quad (3.81)$$

we obtain a surjection

$$F' \longrightarrow D' \subseteq A^{hs}/Q. \quad (3.82)$$

Thus

$$D' \cong F'/N', \quad (3.83)$$

where $N' := Q \cap F'$, and therefore D' is a homomorphic image of a direct limit of local etale extensions of A . Hence D' is excellent.

We now have containments

$$F/N \subseteq D/P \subseteq F'/N' \cong D' \subseteq A^{hs}/Q. \quad (3.84)$$

Since $D/P \longrightarrow D'$ is integral and since these rings have the same quotient field $k(P)$, they must have the same normalization. Then by hypothesis the residual field $k_{D'}$ of D' must be finite and separable over k_D . Since

$$\mathrm{Hom}_{locA}(F, F') \cong \mathrm{Hom}_k(k_D, k_{D'}), \quad (3.85)$$

the inclusion $k_D \hookrightarrow k_{D'}$ induces a morphism $F \longrightarrow F'$ and we have a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & F' \\ \downarrow & & \downarrow \\ F/N & \longrightarrow & F'/N' = D'. \end{array}$$

Then to show that $F/N \longrightarrow F'/N' = D'$ is finite and hence that D' is finite over D/P , it suffices to show that $F \longrightarrow F'$ is finite.

Let $c \in k_{D'}$ be such that $k_{D'} = k_D[c]$ and let G be the unique local etale extension

of A with residual field $k[c]$. By Proposition 1.10 we have a morphism $G \rightarrow F'$ which induces $k[c] \rightarrow k_{D'}$ on residual fields. This yields a residually trivial morphism

$$(F \otimes_A G)_U \rightarrow F' \quad (3.86)$$

where U is the preimage of the maximal ideal of F' . By Remark 1.12, since $(F \otimes_A G)_U$ and F' are both direct limits of local etale extensions of A having isomorphic residual fields, these rings are isomorphic. Since $(F \otimes_A G)_U$ is local etale over the Henselian ring F , it is finite over F by Proposition 1.8. Hence F' is finite over F . $\square$

As a simple application of the previous theorem, let A be the power series ring over $\mathbb{Q}$ in n variables, $X_1, \dots, X_n$, modulo some ideal I ,

$$A := \frac{\mathbb{Q}[[X_1, \dots, X_n]]}{I}. \quad (3.87)$$

Then for $\bar{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$, the strict Henselization of A is given by $A^{hs} \cong A \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} = A[\bar{\mathbb{Q}}]$. Let $a \in A$ be a regular element and set $D := A[\bar{\mathbb{Q}}a]$. Since $A \rightarrow D \rightarrow A^{hs}$ is integral, D is a local ring. To see that D is nonNoetherian by the previous theorem, note that D has residual field $\mathbb{Q}$ while $\bar{\mathbb{Q}}$ is contained in any residual field of $\tilde{D}$.

More generally, for Λ an infinite index set, $\{a_\lambda\}_{\lambda \in \Lambda}$ a collection of regular elements of A , and $\{c_\lambda\}_{\lambda \in \Lambda}$ a collection of elements of $\bar{\mathbb{Q}}$ which is contained in no finite field extension of $\mathbb{Q}$, then the ring $D := A[\{c_\lambda a_\lambda\}_{\lambda \in \Lambda}]$ is nonNoetherian since again D has residual field $\mathbb{Q}$ whereas $\{c_\lambda\}_{\lambda \in \Lambda} \subseteq \tilde{D}$.

BIBLIOGRAPHY

BIBLIOGRAPHY

- [A1] M. André, *Homologie des Algèbres Commutatives*, Springer-Verlag, Berlin, 1974.
- [A2] M. André, *Cinq exposés sur la désingularisation*, handwritten manuscript.
- [BR1] M. Brodmann and C. Rotthaus, *Local rings with bad sets of formal prime divisors*, J. Algebra **75** (1982), 386-394.
- [BR2] _____, *A peculiar unmixed domain*, Proc. Amer. Math. Soc. **87** (1983), 596-600.
- [EGA_{IV}] A. Grothendieck, *Éléments de géométrie algébrique*, Publications Mathématiques IHES **11** (1961).
- [HR] W. Heinzer and C. Rothaus, *Formal fibers and complete homomorphic images*, Proc. Amer. Math. Soc. **120** (1994), 359-369.
- [HRS] W. Heinzer, C. Rotthaus and J. Sally, *Formal fibers and birational extensions*, Nagoya Math. J. **131** (1993), 1-38.
- [HRW1] W. Heinzer, C. Rotthaus and S. Wiegand, *Idealwise algebraic independence for elements of the completion of a local domain*, Illinois J. Math. **41**(2) (1997).
- [HRW2] _____, *Noetherian rings between a semilocal domain and its completion*, preprint.
- [Hu] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1980.
- [I] B. Iversen, *Generic local structure of the morphisms in commutative algebra*, Springer-Verlag, Berlin, 1973.
- [M] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
- [N] M. Nagata, *Local Rings*, John Wiley, New York, 1962.

- [O1] T. Ogoma, *Non-catenary pseudo-geometric normal rings*, Japan J. Math. **6** (1980), 147-163.
- [O2] ———, *General Néron desingularization based on an idea of Popescu*, J. Algebra **167** (1994), 57-84.
- [P1] D. Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985), 97-126.
- [P2] ———, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85-115.
- [Ra] M. Raynaud, *Anneaux Local Henséliens*, Lecture Notes in Math. **169**, Springer-Verlag, Berlin, 1970.
- [Ro1] . Rotthaus, *Nicht ausgezeichnete, universell Japanische Ringe*, Math. Z. **6** (1977), 147-163.
- [Ro2] ———, *Universell Japanische Ringe mit nicht offenen regulärem*, Nagoya Mat. J. **74** (1979), 123-135.
- [Ro3] ———, *Homomorphic images of regular local rings*, Comm. Algebra **24**(2) (1996), 445-476.
- [Ro4] ———, *Excellent rings, henselian rings, and the approximation property*, Rocky Mtn. J. Math. **27**(1) (1997), 317-334.
- [Sp] M. Spivakovsky, *Smoothing of ring homomorphisms, approximation theorems and the Bass-Quillen conjecture*, preprint.
- [Sw] R. G. Swan, *Néron-Popescu desingularization*, preprint.
- [W] D. Weston, *On descent in dimension two and non-split Gorenstein modules*, J. Algebra **118** (1988), 263-275.

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