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Toeplitz Operators on Harmonic Bergman Spaces

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TOEPLITZ OPERATORS ON HARMONIC BERGMAN SPACES

By

Jie Miao

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ABSTRACT

TOEPLITZ OPERATORS ON HARMONIC BERGMAN SPACES

By

Jie Miao

In this dissertation, we study Toeplitz operators on harmonic Bergman spaces of the unit ball in $\mathbf{R}^n$ for $n \geq 2$. We give characterizations for Toeplitz operators with positive symbols to be bounded, compact, and in Schatten classes. We obtain compactness criteria for Toeplitz operators with continuous symbols and with bounded radial symbols. Our results are analogous to well known results on analytic Bergman spaces. However in $\mathbf{R}^n$ for $n > 2$, some methods that are effective in dealing with analytic Bergman spaces, such as using Möbius transformations, are not available. The reproducing kernels for harmonic Bergman spaces are also more complicated than those for analytic Bergman spaces. Our study focuses on reproducing kernels for harmonic Bergman spaces. We also give some applications of these reproducing kernels.

To my parents and wife

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Introduction

Toeplitz operators on analytic Bergman spaces have been well studied. McDonald and Sundberg [11], Luecking [9], Zhu [23], Korenblum and Zhu [7], Axler and Zheng [3] considered Toeplitz operators on analytic Bergman spaces and obtained criteria for Toeplitz operators to be bounded, compact, or in Schatten classes for different type of symbols such as positive, continuous, bounded, or bounded radial symbols.

We study Toeplitz (as well as Hankel operators) on harmonic Bergman spaces of the unit ball in $\mathbf{R}^n$ for $n \geq 2$. Compared to those on analytic Bergman spaces, Toeplitz and Hankel operators on harmonic Bergman spaces have not been as well studied and understood. Recently, Hankel operators on harmonic Bergman spaces of the unit ball in $\mathbf{R}^n$ for $n \geq 2$ were studied by Jovović [6], and Toeplitz and Hankel operators on harmonic Bergman spaces of the unit disk were studied by Wu [22]. We obtain results for Toeplitz and Hankel operators on harmonic Bergman spaces analogous to those for analytic Bergman spaces. Our results improve and extend the results in [6], [7], [9], and [22]. This dissertation is organized as follows.

In the first chapter we introduce the definitions for harmonic Bergman spaces, the reproducing kernel for harmonic Bergman spaces, and Toeplitz and Hankel operators on harmonic Bergman spaces. We also introduce some results that we will need for harmonic Bergman spaces such as duality results.

The second chapter is devoted to Toeplitz operators. We give characterizations for Toeplitz operators with positive symbols to be bounded, compact, and in Schatten classes. We obtain compactness criteria for the Toeplitz operators with continuous and bounded radial symbols.

In the third chapter we study reproducing kernels for weighted harmonic Bergman spaces. We obtain new properties for these reproducing kernels and give some applications of these properties. As one application, we extend the results for Toeplitz operators with positive symbols on harmonic Bergman spaces to weighted harmonic Bergman spaces.

Throughout this dissertation, all constants that depend only on n or other parameters and do not depend on functions and variables will be denoted by a single letter "C". The symbol "$\square$" will denote the end of a proof and "$\approx$" will indicate that the quotient of two positive quantities is bounded above and below by constants.

CHAPTER 1

Preliminaries

1.1 Harmonic Bergman Spaces

Let B denote the open unit ball in $\mathbf{R}^n$ for $n \geq 2$. Let V be Lebesgue volume measure on $\mathbf{R}^n$ and $L^p(B) = L^p(B, dV)$ for $1 \leq p \leq \infty$. For $1 \leq p < \infty$, the harmonic Bergman space $b^p(B)$ is the set of all complex-valued harmonic functions u on B such that

$$\|u\|_p = \left(\int_B |u|^p \, dV\right)^{1/p} < \infty.$$

As is well known, $b^p(B)$ is a closed subspace of $L^p(B)$. When $p = 2$, there is an orthogonal projection Q from the Hilbert space $L^2(B)$ onto $b^2(B)$.

For each $x \in B$, the map $u \mapsto u(x)$ is a bounded linear functional on $b^2(B)$. Thus there exists a unique function $R(x, \cdot) \in b^2(B)$ such that

$$u(x) = \int_B u(y) R(x, y) \, dV(y)$$

for every $u \in b^2(B)$. The function R on $B \times B$ is called the reproducing kernel of $b^2(B)$. For $f \in L^2(B, dV)$ and $x \in B$ we have

$$Qf(x) = \int_B f(y)R(x,y)\,dV(y).$$

In this section, we provide some basic results for harmonic Bergman spaces. These results are analogous to well known results for analytic Bergman spaces (see [4]) and they can be proved in a similar manner. We refer to a recent paper [21] by Stroethoff for Theorems 1.1-1.5 .

Theorem 1.1 *Let $1 < p < \infty$. Then Q is a bounded projection of $L^p(B)$ onto $b^p(B)$.*

The following duality result for $b^p(B)$ for $1 < p < \infty$ follows easily from Theorem 1.1. For $1 < p < \infty$, we use p' to denote the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.2 *Let $1 < p < \infty$. Then the dual of $b^p(B)$ can be identified with $b^{p'}(B)$. More precisely, every bounded linear functional on $b^p(B)$ is of the form*

$$f \mapsto \int_B f\bar{g}\,dV$$

for some unique $g \in b^{p'}(B)$. Furthermore, the norm of the linear functional on $b^p(B)$ induced by $g \in b^{p'}(B)$ is equivalent to $\|g\|_{p'}$.

A harmonic function u on B is said to be a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup\{(1 - |x|^2)|\nabla u(x)| : x \in B\} < \infty.$$

The harmonic Bloch space $\mathcal{B}$ is the set of all harmonic Bloch functions on B. If u is a constant function, then $\|u\|_{\mathcal{B}} = 0$, so the Bloch norm $\|\ \|_{\mathcal{B}}$ is not actually a norm on the harmonic Bloch space. However, $\|u\|_{\mathcal{B}} + |u(0)|$ does define a norm on $\mathcal{B}$. Whenever we refer to properties that require a norm for $\mathcal{B}$, it will be this norm that we have in mind.

Theorem 1.3 *Q maps $L^\infty(B)$ boundedly onto the harmonic Bloch space $\mathcal{B}$.*

The following duality result for $b^1(B)$ follows easily from Theorem 1.3.

Theorem 1.4 *The dual of $b^1(B)$ can be identified with the harmonic Bloch space $\mathcal{B}$. More precisely, every bounded linear functional on $b^1(B)$ is of the form*

$$f \mapsto \int_B f\bar{g}\, dV$$

for some unique $g \in \mathcal{B}$. Furthermore, the norm of the linear functional on $b^1(B)$ induced by $g \in \mathcal{B}$ is equivalent to $\|g\|_{\mathcal{B}} + |g(0)|$.

The harmonic little Bloch space $\mathcal{B}_0$ is the set of functions u harmonic on B such that

$$(1 - |x|^2)| \bigtriangledown u(x)| \to 0$$

as $|x| \to 1$. It is easy to see that $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$ and that all harmonic polynomials belong to $\mathcal{B}_0$. The following result shows that the harmonic little Bloch space is the pre-dual of the harmonic Bergman space $b^1(B)$.

Theorem 1.5 *The dual of the harmonic little Bloch space can be identified* *with $b^1(B)$. More precisely, every bounded linear functional on $\mathcal{B}_0$ is of the form*

$$f \mapsto \int_B f\bar{g}\,dV$$

for some unique $g \in b^1(B)$, and the norm of the linear functional on $\mathcal{B}_0$ induced by $g \in b^1(B)$ is equivalent to $\|g\|_1$.

Now we give a few more results that we will need.

Theorem 1.6 *Q maps $C(\bar{B})$ boundedly onto the harmonic little Bloch space $\mathcal{B}_0$.*

Proof. First we show that Q maps $C(\bar{B})$ boundedly into $\mathcal{B}_0$. By the Stone-Weierstrass Theorem, $C(\bar{B}) = L^\infty$-closure { polynomials on $\mathbf{R}^n$}. Thus we only need to show $Q(p) \in \mathcal{B}_0$ for any polynomial p, since $\mathcal{B}_0$ is closed in $\mathcal{B}$. By Theorem 8.14 of [2], $Q(p)$ is a polynomial of degree no more than that of p. Hence $Q(p) \in \mathcal{B}_0$. To show that Q maps $C(\bar{B})$ onto $\mathcal{B}_0$, we can use the same argument as for the proof of Theorem 2.11 in [4]. The details are omitted here. □

Let $1 \le p < \infty$ and let μ be a positive Borel measure on B. The Closed Graph Theorem shows that $b^p(B)$ is contained in $L^p(B, d\mu)$ if and only if the inclusion map from $b^p(B)$ to $L^p(B, d\mu)$ is a bounded linear operator. Furthermore we can ask when the inclusion map from $b^p(B)$ to $L^p(B, d\mu)$ is a compact linear operator. The following theorem gives a necessary and sufficient condition on μ for this to happen. First we introduce a covering lemma.

Fix $r \in (0,1)$. For $x \in B$, let $K_r(x) = \{y \in B : |y-x| < r(1-|x|)\}$. The following covering lemma says that we can cover B with $K_r(x)$'s that do not intersect too often. The proof of the following lemma is essentially the same as that for Lemma on Coverings of [13].

Lemma 1.7 *There exists a sequence* $\{x_i\}$ *in* B *such that*

(1) $\bigcup_{i=1}^{\infty} K_{\frac{r}{3}}(x_i) = B$;

(2) *There is a positive integer* N *such that each* $K_r(x_i)$ *intersects at most* N *spheres of* $\{K_r(x_j)\}$.

The number N depends on r for this lemma. We omit the details of the proof here.

We always assume that $\{x_i\}$ is a sequence given by Lemma 1.7 in this dissertation. If $\{x_i\}$ is such a sequence, then it is clear that $|x_i| \to 1$ as $i \to \infty$.

Theorem 1.8 *Let* $0 < r < 1$. *Let* $1 \leq p < \infty$ *and* μ *be a positive Borel measure on* B.

(i) *The inclusion map from* $b^p(B)$ *to* $L^p(B, d\mu)$ *is bounded if and only if* $\dfrac{\mu(K_r(x_i))}{V(K_r(x_i))}$ *is bounded for* $i = 1, 2, \cdots$;

(ii) *The inclusion map from* $b^p(B)$ *to* $L^p(B, d\mu)$ *is compact if and only if* $\dfrac{\mu(K_r(x_i))}{V(K_r(x_i))} \to 0$ *as* $i \to \infty$.

The same argument as in [23] (see pages 338, 342, and 343) can be used to prove Theorem 1.8, so we will not give the details. Note that the subharmonicity of $|u|^p$ for a harmonic function u on B and the decomposition from Lemma 1.7 are needed for the proof.

1.2 The Reproducing Kernel

In this section we give an introduction of the reproducing kernel for $b^2(B)$. We need to introduce zonal harmonics first.

Let $\mathcal{H}_m(\mathbf{R}^n)$ denote the space of all homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree m. A spherical harmonic of degree m is the restriction to S, the unit sphere, of an element of $\mathcal{H}_m(\mathbf{R}^n)$. The collection of all spherical harmonics of degree m is denoted by $\mathcal{H}_m(S)$. For every $\eta \in S$, there exists a unique $Z_m(\eta, \cdot) \in \mathcal{H}_m(S)$ such that

$$p(\eta) = \int_S p(\zeta) Z_m(\eta, \zeta)\, d\sigma(\zeta)$$

for all $p \in \mathcal{H}_m(S)$, where σ is the normalized surface-area measure on S. The spherical harmonic $Z_m(\eta, \cdot)$ is called the zonal harmonic of degree m. One can extend the zonal harmonic to a function on $\mathbf{R}^n \times \mathbf{R}^n$ by making Z_m homogeneous of degree m in the second variable as well as in the first. Let h_m denote the dimension (over $\mathbf{C}$) of the vector space $\mathcal{H}_m(S)$. One can compute h_m explicitly (see Exercise 5.5 of [1]):

$$h_m = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2}, \tag{1.1}$$

for $m > 0$. Also, $h_0 = 1$.

The following lemma states some properties of zonal harmonics that we will need. For more information on zonal harmonics, see Chapter 5 of [2].

Lemma 1.9 *Let m be a non-negative integer.*

(i) *If* $\zeta, \eta \in S$, *then* $Z_m(\zeta,\zeta) = Z_m(\eta,\eta) = h_m$*;*

(ii) *If* $\zeta \in S$, *then* $\max_{\eta\in S} |Z_m(\zeta,\eta)| = Z_m(\zeta,\zeta) = h_m$.

We have the following representation for R (Theorem 8.9 of [2]).

Theorem 1.10 *If* $x,\ y \in B$, *then*

$$R(x,y) = \frac{1}{nV(B)} \sum_{m=0}^{\infty} (n+2m) Z_m(x,y).$$

The series converges absolutely and uniformly on $K \times B$ *for every compact* $K \subset B$.

Since Z_m is real valued for each m (see Theorem 5.24 of [2]), we see that R is real valued.

For $x, y \in B$, let $P(x,y)$ be the "extended Poisson kernel" for B. Then (see pages 156 and 157 of [2])

$$P(x,y) = \sum_{m=0}^{\infty} Z_m(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{\frac{n}{2}}}, \quad x,y\in B. \tag{1.2}$$

From the equation above and Theorem 1.10, we have the following beautiful equation

$$R(x,y) = \frac{1}{nV(B)}(nP(x,y) + \frac{d}{dt}P(tx,ty)|_{t=1}).$$

This simple representation gives us a formula in closed form for $R(x,y)$ (Theorem 8.13 of [2]).

Theorem 1.11 *Let* $x, y \in B$. *Then*

$$R(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1-2x\cdot y+|x|^2|y|^2)^{1+n/2}}.$$

It follows easily from Theorem 1.11 that

$$R(x,y) = R(y,x), \quad R(x,ry) = R(rx,y) \tag{1.3}$$

for $x, y \in B, r \in (0,1)$. It is also clear that $R(x,y)$ is bounded for $|x| \leq r < 1, |y| < 1$ for fixed $r < 1$.

A simple computation gives

$$R(x,y) = \frac{n(1-|x|^2|y|^2)^2 - 4|x|^2|y|^2((1-|x|^2)(1-|y|^2) + |x-y|^2)}{nV(B)((1-|x|^2)(1-|y|^2) + |x-y|^2)^{1+n/2}}. \tag{1.4}$$

It follows easily from Theorem 1.11 that

$$|R(x,y)| \leq \frac{4}{(1-2x\cdot y + |x|^2|y|^2)^{n/2}}. \tag{1.5}$$

Note that the reproducing kernel is much more complicated than that for analytic Bergman spaces (see Chapter 3 of [16]). One of our tasks is to establish properties for R analogous to those for the analytic Bergman kernel.

For $f \in L^1(B)$ and $x \in B$, we now see that $Qf(x) = \int_B f(y)R(x,y)\,dV(y)$ is well defined. We end this section with the following lemma (see Lemma 2.1 of [6]).

Lemma 1.12 *For all $u \in b^1(B)$ we have*

$$u(x) = \int_B u(y)R(x,y)\,dV(y).$$

1.3 Toeplitz and Hankel Operators

Let $1 \leq p < \infty$. For a function $f \in L^1(B, dV)$, the Toeplitz and the Hankel operators with symbol f are densely defined on $b^p(B)$ by

$$T_f(u) = Q(fu), \qquad H_f(u) = (I - Q)(fu)$$

for $u \in b^p(B) \cap L^\infty(B)$ (noting that harmonic polynomials are dense in $b^p(B)$ and $Q(g)$ is well defined for $g \in L^1(B)$). If T_f is a bounded operator (when we put the L^p norm on $b^p(B) \cap L^\infty(B)$), then T_f extends to a bounded operator from $b^p(B)$ to $b^p(B)$, also denoted by T_f. We do the same for H_f.

In this section we give some preliminary results for Toeplitz and Hankel operators.

For $f \in L^\infty(B)$, it is easy to see that T_f and H_f are bounded linear operators on $b^p(B)$ with

$$\|T_f\| \leq \|f\|_\infty, \qquad \|H_f\| \leq \|f\|_\infty.$$

In contrast to Hardy space Toeplitz operators, Toeplitz operators with unbounded symbols can be bounded on Bergman spaces. For $1 < p < \infty$, $f \in L^p(B), g \in L^{p'}(B)$, define

$$\langle f, g \rangle = \int_B f\bar{g}\, dV.$$

Since $b^{p'}(B)$ is the dual of $b^p(B)$ with respect to the pairing $\langle \cdot, \cdot \rangle$, define the adjoint of T_f: $b^p(B) \mapsto b^p(B)$ to be the operator $T_f^* : b^{p'}(B) \mapsto b^{p'}(B)$ such that

$$\langle T_f^* u, v \rangle = \langle u, T_f v \rangle$$

for $u \in b^{p'}(B)$ and $v \in b^p(B)$.

Similarly, define the adjoint of $H_f : b^p(B) \mapsto L^p(B)$ to be the operator H_f^* mapping $L^{p'}(B)$ to $b^{p'}(B)$ such that

$$\langle H_f^* u, v \rangle = \langle u, H_f v \rangle$$

for $u \in L^{p'}(B)$ and $v \in b^p(B)$.

The following lemmas are standard (see Lemma 3.1 and 3.2 of [6])

Lemma 1.13 *Let $p \in (1, \infty)$, a, b scalars, and $f, g \in L^1(B)$. Then*

(i) $T_{af+bg} = aT_f + bT_g$ *on* $b^p(B)$.

(ii) $T_f^* = T_{\bar{f}}$ *on* $b^{p'}(B)$.

The connection between Toeplitz and Hankel operators is provided by the following lemma.

Lemma 1.14 *Let $p \in (1, \infty)$ and either f or $g \in L^\infty(B)$. Then $T_{fg} - T_f T_g = H_{\bar{f}}^* H_g$ on $b^p(B)$.*

We end this section with the following result that was proved in [6].

Theorem 1.15 *Let $p \in (1, \infty)$ and let f be a continuous function on $\bar{B}$. Then the Hankel operator H_f: $b^p(B) \to L^p(B)$ is compact.*

CHAPTER 2

Toeplitz Operators

2.1 Introduction

In this chapter we study Toeplitz operators on harmonic Bergman spaces $b^p(B)$ for $1 < p < \infty$. We will look at three special classes of symbols. For positive symbols, we give characterizations for Toeplitz operators to be bounded, compact, and in Schatten classes. For continuous symbols, a compactness criteria is obtained. In fact, the essential spectrum of a Toeplitz operator with a continuous symbol is found. We also obtain a compactness criteria for Toeplitz operators with bounded radial symbols. A sufficient condition for Hankel operators to be compact, which improves Theorem 1.15, is given along the way.

2.2 Teoplitz Operators with Positive Symbols

First we give three lemmas for the reproducing kernel R.

Lemma 2.1 *Let* $x \in B$. *Then* $R(x,y) \approx \dfrac{1}{(1-|x|)^n}$ *for* $y \in K_r(x)$.

Proof. We use the following formula (1.4) for $R(x,y)$:

$$R(x,y) = \frac{n(1-|x|^2|y|^2)^2 - 4|x|^2|y|^2((1-|x|^2)(1-|y|^2)+|x-y|^2)}{nV(B)((1-|x|^2)(1-|y|^2)+|x-y|^2)^{1+n/2}} = \frac{I_1(x,y)}{I_2(x,y)}.$$

If $y \in K_r(x)$, then $(1-r)(1-|x|) < 1-|y| < (1+r)(1-|x|)$. It is clear that $I_2(x,y) \approx (1-|x|)^{2+n}$ when $y \in K_r(x)$. It is also clear that $I_1(x,y) \leq C(1-|x|)^2$ if $y \in K_r(x)$. Next we will try to find a lower bound for $I_1(x,y)$. Since $n \geq 2$,

$$I_1(x,y) \geq 2(1+|x||y|)^2(1-|x||y|)^2 - 4|x|^2|y|^2(1-|x|^2)(1-|y|^2) - 4|x|^2|y|^2|x-y|^2.$$

Let

$$I_3(x,y) = 2(1+|x||y|)^2(1-|x||y|)^2 = 2(1+|x||y|)^2(1-|y|+|y|(1-|x|))^2.$$

So we have

$$\begin{aligned} I_3(x,y) &= 2(1+|x||y|)^2(1-|y|)^2 + 4|y|(1+|x||y|)^2(1-|y|)(1-|x|) \\ &\quad +2|y|^2(1+|x||y|)^2(1-|x|)^2. \end{aligned}$$

Since

$$2(1+|x||y|)^2(1-|y|)^2 > 2(1-r)^2(1-|x|)^2$$

and

$$4|y|(1+|x||y|)^2(1-|y|)(1-|x|) > 4|x|^2|y|^2(1-|x|^2)(1-|y|^2),$$

we will have $I_1(x,y) > 2(1-r)^2(1-|x|)^2$ if we can show the following inequality:

$$2|y|^2(1+|x||y|)^2(1-|x|)^2 \geq 4|x|^2|y|^2|x-y|^2$$

for $y \in K_r(x)$. This can be reduced to showing that

$$(1+|x||y|)^2 \geq 2|x|^2$$

for $y \in K_r(x)$. If $|x| < 0.7$, then $|x|^2 < 0.49$ and the above inequality is trivial. If $|x| \geq 0.7$, then $|y-x| < 1-|x| \leq 0.3$. So $|y| > |x| - 0.3$. So we have

$$(1+|x||y|)^2 \geq 4|x||y| > 4|x|(|x|-0.3) > 2|x|^2,$$

for $y \in K_r(x)$. This finishes the proof of Lemma 2.1. □

Now we estimate the L^p-norm of the reproducing kernel.

Lemma 2.2 *If* $1 < p < \infty$, *then* $\|R(x,\cdot)\|_p \approx (1-|x|)^{-\frac{n(p-1)}{p}}$.

Proof. $\|R(x,\cdot)\|_p^p \leq C(1-|x|)^{-n(p-1)}$ follows from Lemma 3.2 (c) in [5]. On the other hand, by Lemma 2.1 we have

$$\begin{aligned} \|R(x,\cdot)\|_p^p &\geq \int_{K_r(x)} |R(x,y)|^p \, dV(y) \\ &\geq C \frac{1}{(1-|x|)^{np}} \int_{K_r(x)} dV(y) \end{aligned}$$

$$\approx \frac{(1-|x|)^n}{(1-|x|)^{np}}.$$

This proves the lemma. □

Lemma 2.2 was known when $p = 2$ (see [2], Exercise 8.15).

Lemma 2.3 *If* $1 < p < \infty$, *then* $\frac{R(x,\cdot)}{\|R(x,\cdot)\|_p} \to 0$ *weakly in* $b^p(B)$ *as* $|x| \to 1$.

Proof. Let $v \in b^{p'}(B)$. By Lemma 2.2

$$|\langle \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p}, v\rangle| \approx (1-|x|)^{\frac{n}{p'}} |v(x)|.$$

Using Exercise 8.2 of [2], we see that the quantity above has limit 0 as $|x| \to 1$. □

Now we extend the notation of Toeplitz operators to the case where we allow measures as symbols. Let μ be a finite complex Borel measure on B. We densely define the Toeplitz operator with symbol μ on $b^p(B)$ by

$$T_\mu u(x) = \int_B R(x,y) u(y)\, d\mu(y)$$

for $u \in b^p(B) \cap L^\infty(B, dV)$. If $d\mu(y) = f(y)\, dV(y)$, then $T_\mu = T_f$.

Let μ be a finite positive Borel measure. For our purpose we will consider the following two cases:

(1) Suppose u, v are both harmonic on B and continuous on $\bar{B}$. If $T_\mu u \in b^1(B)$, then we have

$$\begin{aligned} \langle T_\mu u, v\rangle &= \lim_{r\to 1} \int_{rB} T_\mu u\, \bar{v}\, dV \\ &= \lim_{r\to 1} \int_B \int_{rB} R(y,x) \bar{v}(x)\, dV(x) u(y)\, d\mu(y) \end{aligned}$$

$$\begin{aligned} &= \lim_{r\to 1}\int_B r^n\bar{v}(r^2y)u(y)\,d\mu(y) \\ &= \int_B u\bar{v}\,d\mu, \end{aligned}$$

where we used Fubini's Theorem in the second step ($R(x,y)$ is bounded for $|x| \le r < 1$) and the following properties of R (see (1.3)):

$$R(x,y) = R(y,x), \qquad R(y,rz) = R(ry,z)$$

for x, y, $z \in B$.

(2) Suppose μ satisfies (i) of Theorem 1.8 and $u \in b^p(B)$ for $1 < p < \infty$. Then Theorem 1.8 shows that $T_\mu u$ is well defined. By the proof of Lemma 3.2 of [5], one can show that $\|R(x,\cdot)\|_1 \le C\ln\dfrac{1}{1-|x|} + C$ for any $x \in B$. Thus $\|R(x,\cdot)\|_1 \le \dfrac{C}{(1-|x|)^\alpha}$ for each $\alpha > 0$. So we have

$$\begin{aligned} \|T_\mu u\|_1 &\le \int_B\int_B |R(x,y)||u(y)|\,d\mu(y)\,dV(x) \\ &= \int_B\int_B |R(x,y)|\,dV(x)|u(y)|\,d\mu(y) \\ &\le C\int_B |u(y)|\frac{1}{(1-|y|)^\alpha}\,d\mu(y) \end{aligned}$$

for some $\alpha > 0$ to be decided later. The same argument mentioned after Theorem 1.8 and Hölder's inequality give

$$\begin{aligned} \|T_\mu u\|_1 &\le C\int_B |u(y)|\frac{1}{(1-|y|)^\alpha}\,dV(y) \\ &\le C\|u\|_p\left[\int_B \frac{1}{(1-|y|)^{\alpha p'}}\,dV(y)\right]^{\frac{1}{p'}} \end{aligned}$$

$$< \infty,$$

if α is small enough.

Hence if μ satisfies (1) of Theorem 1.7, $u \in b^p(B)$ for $1 < p < \infty$, and v is a bounded harmonic function on B, then Fubini's Theorem gives

$$\langle T_\mu u, v\rangle = \int_B u\bar{v}\, d\mu.$$

Now we can characterize the boundedness and compactness of positive Toeplitz operators.

Theorem 2.4. *Let $1 < p < \infty$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *T_μ is bounded on $b^p(B)$;*

(ii) $\dfrac{\mu(K_r(x))}{V(K_r(x))}$ *is bounded for $x \in B$;*

(iii) $\dfrac{\mu(K_r(x_i))}{V(K_r(x_i))}$ *is bounded for $i = 1, 2, \cdots$.*

Proof. (i)⇒(ii). By Lemma 2.1 and 2.2, and (1) mentioned just before the theorem we have

$$\begin{aligned}\left|\langle T_\mu \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p}, \frac{R(x,\cdot)}{\|R(x,\cdot)\|_{p'}}\rangle\right| &= \frac{1}{\|R(x,\cdot)\|_p\|R(x,\cdot)\|_{p'}}\,|\langle T_\mu R(x,\cdot), R(x,\cdot)\rangle| \\ &\approx (1-|x|)^n \int_B |R(x,y)|^2\, d\mu(y) \\ &\geq (1-|x|)^n \int_{K_r(x)} |R(x,y)|^2\, d\mu(y) \\ &\approx \frac{\mu(K_r(x))}{V(K_r(x))}\end{aligned}$$

for $x \in B$. This shows that (ii) follows from (i).

(ii)$\Rightarrow$(iii). This direction is trivial.

(iii)$\Rightarrow$(i). Let $u \in b^p(B)$ and v be a bounded harmonic function on B. Then by Hölder's inequality,

$$\begin{aligned} |\langle T_\mu u, v\rangle| &= \left|\int_B u\bar{v}\, d\mu\right| \\ &\le \left(\int_B |u|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_B |v|^{p'}\, d\mu\right)^{\frac{1}{p'}} \\ &\le C\|u\|_p \|v\|_{p'} \end{aligned}$$

using (i) of Theorem 1.8 in the last inequality. Since the set of harmonic polynomials is dense in $b^{p'}(B)$, the duality argument shows that T_μ is bounded on $b^p(B)$. This completes the proof of the theorem. □

The following theorem is the little o version of Theorem 2.4.

Theorem 2.5 *Let $1 < p < \infty$ and μ be a finite Borel measure on B. Then the following conditions are equivalent:*

(i) *T_μ is compact on $b^p(B)$;*

(ii) $\dfrac{\mu(K_r(x))}{V(K_r(x))} \to 0$ *as* $|x| \to 1$;

(iii) $\dfrac{\mu(K_r(x_i))}{V(K_r(x_i))} \to 0$ *as* $i \to \infty$.

Proof. (i)$\Rightarrow$(ii). It follows from Lemma 2.3 and the proof of (i)$\Rightarrow$(ii) of the previous theorem.

(ii)$\Rightarrow$(iii). This direction is trivial.

(iii)⇒(i). Let $u_i \to 0$ weakly in $b^p(B)$ as $i \to \infty$. For any bounded harmonic function v on B, we have

$$|\langle T_\mu u_i, v\rangle| \le C\left(\int_B |u_i|^p \, d\mu\right)^{\frac{1}{p}} \|v\|_{p'}.$$

It follows that

$$\|T_\mu u_i\|_p \le C\left(\int_B |u_i|^p \, d\mu\right)^{\frac{1}{p}}.$$

So $\|T_\mu u_i\|_p \to 0$ by (ii) of Theorem 1.8. This shows that T_μ is compact on $b^p(B)$ and completes the proof of the theorem. □

When $p = 2$, the equivalence of (i) and (ii) for both Theorem 2.4 and 2.5 can be deduced from Theorem 1 of [14].

In the rest of this section, we will prove a trace ideal criteria for the positive Toeplitz operators on $b^2(B)$. The techniques used here were developed in [9] as well as in [23], however the approach in [23] will be used for our purpose.

If T is a compact operator on a separable Hilbert space H, then there exist numbers $s_0(T) \ge s_1(T) \ge \cdots$, called the singular numbers of T, and orthonormal vectors $\{e_i\}$ and $\{f_i\}$ such that

$$Tx = \sum_{i=0}^{\infty} s_i(T)\langle x, e_i\rangle f_i$$

for $x \in H$. For $1 \le p < \infty$, the Schatten ideal $S_p(H)$ is defined to be the set of all compact operators T for which $\|T\|_{S_p} = (\sum_{i=0}^{\infty} s_i(T)^p)^{\frac{1}{p}} < \infty$. As is well known, $S_p(H)$ is a Banach space with the norm $\|\cdot\|_{S_p}$ and is a two-sided ideal in the space of bounded linear operators on H. If $T \in S_1(H)$ and $\{e_i\}$ is an orthonormal basis for H, then

$$tr(T) = \sum_{i=0}^{\infty} \langle Te_i, e_i \rangle,$$

where the series is convergent and independent of $\{e_i\}$. If $T \in S_p(H)$ and $T \geq 0$, then $\|T\|_{S_p} = [tr(T^p)]^{\frac{1}{p}}$ for $1 \leq p < \infty$.

We need the following three lemmas.

Lemma 2.6 *If T is either in $S_1(b^2(B))$ or positive, then*

$$tr(T) = \int_B \langle TR(x,\cdot), R(x,\cdot) \rangle \, dV(x).$$

The proof of Lemma 2.6 is similar to Proposition 3.5 of [1] or Lemma 13 of [23]. We omit the details here.

Lemma 2.7 *If T_1 and T_2 are compact and $0 \leq T_1 \leq T_2$, then $s_i(T_1) \leq s_i(T_2)$ for $i = 0, 1, \cdots$.*

This is Lemma 14 of [23].

Lemma 2.8 *If $\mu \geq 0$, then there exists a constant C depending only on r such that*

$$\mu(K_r(x)) \leq \frac{C}{V(K_r(x))} \int_{K_r(x)} \mu(K_r(y)) \, dV(y)$$

for all $x \in B$.

Proof. For $x \in B$, we have

$$\begin{aligned} \int_{K_r(x)} \mu(K_r(y)) \, dV(y) &= \int_{K_r(x)} dV(y) \int_B \mathcal{X}_{K_r(y)}(z) \, d\mu(z) \\ &= \int_B d\mu(z) \int_{K_r(x)} \mathcal{X}_{K_r(y)}(z) \, dV(y). \end{aligned}$$

If $|y-z| < \frac{r}{1+r}(1-|z|)$, then it is easy to see that $|y-z| < r(1-|y|)$. This gives that $\mathcal{X}_{K_r(y)}(z) \geq \mathcal{X}_{K_{\frac{r}{1+r}}(z)}(y)$. So we have

$$\int_{K_r(x)} \mu(K_r(y))\, dV(y) \geq \int_{K_r(x)} d\mu(z) \int_{K_r(x)\cap K_{\frac{r}{1+r}}(z)} dV(y).$$

If $z \in K_r(x)$, then $1-|z| > (1-r)(1-|x|)$. It is a clear geometric fact that

$$CV(K_r(x) \cap K_{\frac{r}{1+r}}(z)) \geq V(K_r(x))$$

for all $z \in K_r(x)$. Combining the above two inequalities we prove the lemma. □

Now we can characterize the positive Toeplitz operators that lie in the Schatten p-class.

Theorem 2.9 *Let $1 \leq p < \infty$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) $T_\mu \in S_p(b^2(B))$;

(ii) $\frac{\mu(K_r(x))}{V(K_r(x))} \in L^p(B, (1-|x|)^{-n} dV(x))$;

(iii) $\sum_{i=1}^{\infty} \left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p < \infty$.

Proof. (i)⇒(ii). Suppose $T_\mu \in S_p(b^2(B))$. We have $\|T_\mu\|_{S_p}^p = tr(T_\mu^p)$ since $T_\mu \geq 0$. Lemma 2.6 and Lemma 2.2 together with 6.4 of [1] give

$$\begin{aligned} \|T_\mu\|_{S_p}^p &= \int_B \langle T_\mu^p R(x,\cdot), R(x,\cdot)\rangle\, dV(x) \\ &= \int_B \|R(x,\cdot)\|_2^2 \left\langle T_\mu^p \frac{R(x,\cdot)}{\|R(x,\cdot)\|_2}, \frac{R(x,\cdot)}{\|R(x,\cdot)\|_2}\right\rangle dV(x) \end{aligned}$$

$$\geq C\int_B (1-|x|)^{-n}\left[\langle T_\mu \frac{R(x,\cdot)}{\|R(x,\cdot)\|_2}, \frac{R(x,\cdot)}{\|R(x,\cdot)\|_2}\rangle\right]^p dV(x).$$

By Lemma 2.1 we get

$$\begin{aligned}
\|T_\mu\|_{S_p}^p &\geq C\int_B \left[(1-|x|)^n \int_B |R(x,y)|^2\, d\mu(y)\right]^p (1-|x|)^{-n}\, dV(x) \\
&\geq C\int_B \left[(1-|x|)^n \int_{K_r(x)} |R(x,y)|^2\, d\mu(y)\right]^p (1-|x|)^{-n}\, dV(x) \\
&\geq C\int_B \left[\frac{\mu(K_r(x))}{V(K_r(x))}\right]^p (1-|x|)^{-n}\, dV(x).
\end{aligned}$$

(ii)$\Rightarrow$(iii). Suppose $\int_B \left[\frac{\mu(K_r(x))}{V(K_r(x))}\right]^p (1-|x|)^{-n}\, dV(x) < \infty$. Then we have

$$\sum_{i=1}^{\infty}\int_{K_r(x_i)} \frac{[\mu(K_r(x))]^p}{(1-|x|)^{np+n}}\, dV(x) < \infty.$$

It follows that

$$\sum_{i=1}^{\infty}\frac{1}{(1-|x_i|)^{np+n}}\int_{K_r(x_i)} [\mu(K_r(x))]^p\, dV(x) < \infty.$$

By Lemma 2.8 and Hölder's inequality we get

$$\sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p \approx \sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{(1-|x_i|)^n}\right]^p < \infty.$$

(iii)$\Rightarrow$(i). We prove this direction by complex interpolation.

First consider the case where $p=1$. We have

$$\|T_\mu\|_{S_1} = tr(T_\mu)$$

$$
\begin{aligned}
&= \int_B \langle T_\mu R(x,\cdot), R(x,\cdot)\rangle \, dV(x) \\
&= \int_B \int_B |R(x,y)|^2 \, d\mu(y) \, dV(x) \\
&= \int_B \int_B |R(x,y)|^2 \, dV(x) \, d\mu(y) \\
&\approx \int_B (1-|y|)^{-n} \, d\mu(y) \\
&\le \sum_{i=1}^{\infty} \int_{K_r(x_i)} (1-|y|)^{-n} \, d\mu(y) \\
&\le C \sum_{i=1}^{\infty} \frac{\mu(K_r(x_i))}{V(K_r(x_i))}.
\end{aligned}
$$

Now consider the case where $1 < p < \infty$. We will show $\|T_\mu\|_{S_p}^p \le C \sum_{i=1}^{\infty} \left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p$. For a complex number ζ with $0 \le Re\zeta \le 1$, we can define a finite Borel measure on B by

$$
\mu_\zeta(y) = \sum_{i=1}^{\infty} \left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^{p\zeta - 1} \mathcal{X}_{K_r(x_i)}(y)\mu(y)
$$

and the Toeplitz operator on $b^2(B)$ by

$$
T_{\mu_\zeta} u(x) = \int_B R(x,y) u(y) \, d\mu_\zeta(y).
$$

It is easy to see that both T_μ and $T_{\mu_{\frac{1}{p}}}$ are compact and $T_{\mu_{\frac{1}{p}}} \ge T_\mu \ge 0$. Thus complex interpolation and Lemma 2.7 give

$$
\|T_\mu\|_{S_p} \le \|T_{\mu_{\frac{1}{p}}}\|_{S_p} \le M_0^{1-\frac{1}{p}} M_1^{\frac{1}{p}},
$$

where $M_0 = \{\|T_{\mu_\zeta}\| : Re\zeta = 0\}$ and $M_1 = \{\|T_{\mu_\zeta}\|_{S_1} : Re\zeta = 1\}$.

Let $Re\zeta = 0$. Then we have

$$|\mu_\zeta|(K_r(x_k)) \leq C\sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^{-1}\mu(K_r(x_k)\cap K_r(x_i))$$

for $k = 1, 2, \cdots$. Suppose $K_r(x_k)\cap K_r(x_i)$ is not empty, it is easy to see that $(1-|x_i|) < \dfrac{1+r}{1-r}(1-|x_k|)$. Thus by Lemma 1.7 one can show $|\mu_\zeta|(K_r(x_k)) \leq CV(K_r(x_k))$ for $k = 1, 2, \cdots$. Hölder's inequality and Theorem 1.8 give

$$\begin{aligned}|\langle T_{\mu_\zeta}u, v\rangle| &\leq \left(\int_B |u|^2\, d|\mu_\zeta|\right)^{\frac{1}{2}}\left(\int_B |v|^2\, d|\mu_\zeta|\right)^{\frac{1}{2}}\\ &\leq C\|u\|_2\|v\|_2\end{aligned}$$

for all $u, v \in b^2(B)$. This shows that $\|T_{\mu_\zeta}\| \leq C$ for all $Re\zeta = 0$. So $M_0 \leq C$.

Let $Re\zeta = 1$ and $\{u_i\}, \{v_i\}$ be two orthonormal bases for $b^2(B)$. It can be shown in the exactly same way as in [23] (see page 351) that

$$\sum_{i=1}^{\infty}|\langle T_{\mu_\zeta}u_i, v_i\rangle| \leq C\sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p,$$

which implies that

$$M_1 \leq C\sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p.$$

Hence we have

$$\|T_\mu\|_{S_p} \leq C\left\{\sum_{i=1}^{\infty}\left[\frac{\mu(K_r(x_i))}{V(K_r(x_i))}\right]^p\right\}^{\frac{1}{p}}.$$

This finishes the proof of the theorem. □

2.3 Toeplitz Operators with Continuous Symbols

In this section we will find the essential spectra of Toeplitz operators with continuous symbols.

The first theorem of this section gives a sufficient condition for Hankel operators to be bounded or compact on $b^p(B)$ for $1 < p < \infty$. The techniques used to prove this theorem were established in [10] and when $n = 2$ this condition was proved in [22]. Since the same method in [22] can be applied to the case $n \geq 2$, we will only outline the proof.

Let $1 < p < \infty$ and $(b^{p'}(B))^{\perp} = \{u \in L^p(B, dV) : \langle u, v \rangle = 0 \ \ \forall v \in b^{p'}(B)\}$.

Lemma 2.10 $(b^{p'}(B))^{\perp} = L^p$-closure $\{\triangle h : h \in C_0^{\infty}(B)\}$.

Proof. Let $u \in L^{p'}(B, dV)$. Then $u \in b^{p'}(B)$ if and only if

$$\langle \triangle u, h \rangle = \langle u, \triangle h \rangle = 0$$

for all $h \in C_0^{\infty}(B)$. This implies the conclusion of the lemma. □

By Lemma 2.10 and the boundedness of Q in L^p norm, we have $L^p(B, dV) = b^p(B) \oplus (b^{p'}(B))^{\perp} = b^p(B) \oplus L^p$-closure $\{\triangle h : h \in C_0^{\infty}(B)\}$.

Lemma 2.11. *Let* $1 < p < \infty$. *Then*

$$\int_B \frac{|h(x)|^p}{(1-|x|)^{2p}}\, dV(x) \leq C \int_B \frac{|\nabla h(x)|^p}{(1-|x|)^p}\, dV(x)$$

and

$$\int_B \frac{|\nabla h(x)|^p}{(1-|x|)^p}\, dV(x) \le C \int_B |\triangle h(x)|^p \, dV(x)$$

for all $h \in C_0^\infty(B)$.

One can use the same argument as for Lemma 3 of [10] or Lemma 5.2 of [22] to prove the above two inequalities. Note that we need

$$|\nabla h(x)|^p \le \max\{n^{\frac{p}{2}-1}, 1\} \sum_{i=1}^{n} |h_{x_i}(x)|^p$$

and Proposition 3 of [21] (page 59) for the second inequality. The details are omitted here.

Theorem 2.12 *Let* $1 < p < \infty$ *and* $f \in L^p(B, dV)$. *Suppose* $f = f_1 + f_2$ *with* $f_1 \in C^1(B)$.

(1) *If both* $|\nabla f_1(x)|(1-|x|)$ *and* $\frac{1}{V(K_r(x))}\int_{K_r(x)} |f_2|^p \, dV$ *are bounded for* $x \in B$, *then* H_f *is bounded on* $b^p(B)$;

(2) *If both* $|\nabla f_1(x)|(1-|x|)$ *and* $\frac{1}{V(K_r(x))}\int_{K_r(x)} |f_2|^p \, dV$ *approach* 0 *as* $|x| \to 1$, *then* H_f *is compact on* $b^p(B)$.

Sketch of the proof. We have $H_f = H_{f_1} + H_{f_2}$ and $H_{f_2} = (I - Q)M_{f_2}$, where M_{f_2} is the multiplication by f_2. If f_2 satisfies the condition of (1) or (2), then M_{f_2} is bounded or compact on $b^p(B)$, respectively, according to Theorem 1.8. Thus H_{f_2} is bounded or compact on $b^p(B)$, respectively.

So we only need to deal with H_{f_1}. By Lemma 2.10 we only need to show

$$|\langle H_{f_1}(u), \triangle h\rangle| = |\langle f_1 u, \triangle h\rangle| \le C\|u\|_p \|\triangle h\|_{p'}$$

for any $u \in b^p(B) \cap L^\infty(B, dV), h \in C_0^\infty(B)$ in order to prove the boundedness of H_{f_1}. The above inequality will follow from the following identity (which is from integration by parts) and Lemma 2.11

$$\langle f_1 u, \triangle h \rangle = -\langle u \bigtriangledown f_1, \bigtriangledown h \rangle + \langle \bigtriangledown f_1 \cdot \bigtriangledown u, h \rangle,$$

where $\langle u \bigtriangledown f_1, \bigtriangledown h \rangle = \sum_{i=1}^{n} \int_B u(f_1)_{x_i} \bar{h}_{x_i} dV$. Similarly if $\{u_i\}$ is a sequence tending to 0 weakly in $b^p(B)$, one can show that $\|H_{f_1} u_i\|_p \to 0$ as $i \to \infty$. This gives the compactness of H_{f_1}. □

Now we show that Theorem 1.15 follows from Theorem 2.12.

Corollary 2.13 *Let $1 < p < \infty$ and $f \in C(\bar{B})$. Then H_f is compact on $b^p(B)$.*

Proof. We have

$$f = P(f|_S) + (f - P(f|_S) = f_1 + f_2,$$

where $P(f|_S)$ is the Poisson integral of $f|_S, f_1 = P(f|_S)$, and $f_2 = f - P(f|_S)$. Since $f|_S \in C(S)$, we have $f_2 \to 0$ as $|x| \to 1$. On the other hand, f_1 is harmonic on B and $f_1 \in C(\bar{B})$. By Theorem 1.6, $f_1 = Q(f_1) \in \mathcal{B}_0$. So f_1 and f_2 satisfy the conditions in (2) of Theorem 2.12. So H_f is compact. □

The proof above only requires that f be continuous on S. Hence this corollary and the remaining results of this section are valid for a larger class of symbols than the continuous functions on $\bar{B}$.

We need two more lemmas.

Lemma 2.14 *Let $1 < p < \infty$. If $f, g \in C(\bar{B})$, then both $T_{fg} - T_f T_g$ and $T_f T_g - T_g T_f$ are compact on $b^p(B)$.*

This is a consequence of Corollary 4.5 of [6].

Lemma 2.15 *Let $1 \le p < \infty$. If $f \in C(\bar{B})$ and $f = 0$ on S, then T_f is compact on $b^p(B)$.*

Proof. It is easy to see that there exists $f_i \in C(\bar{B})$ such that each $f_i = 0$ on a neighborhood of S and $\|f_i - f\|_\infty \to 0$ as $i \to \infty$. Theorem 1.8 shows that each M_{f_i} is compact on $b^p(B)$; thus so is each T_{f_i}. Since $T_{f_i} \to T_f$, T_f is compact. □

For $1 < p < \infty$, let $B(b^p(B))$ be the set of bounded linear operators on $b^p(B)$, and let $\sigma_e(T)$ denote the essential spectrum of $T \in B(b^p(B))$.

Theorem 2.16 *If $f \in C(\bar{B})$, then $\sigma_e(T_f) = f(S)$.*

Proof. First we show $f(S) \subset \sigma_e(T_f)$. Without loss of generality we assume $f(\eta) = 0$ for some $\eta \in S$. We need to show T_f is not a Fredholm operator. We prove this by contradiction. Suppose T_f is a Fredholm operator. Then by Atkinson's Theorem, there exists $P \in B(b^p(B))$ such that $PT_f - I$ is compact on $b^p(B)$. By Lemma 2.2,

$$PT_f \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p} - \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p} \to 0$$

as $|x| \to 1$.

On the other hand, we have

$$\|PT_f \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p}\|_p^p \;\le\; C\|M_f \frac{R(x,\cdot)}{\|R(x,\cdot)\|_p}\|_p^p$$

$$= C\int_B |f(y)|^p \frac{|R(x,y)|^p}{\|R(x,\cdot)\|_p^p}\, dV(y)$$
$$= C(I_1 + I_2),$$

where I_1 is the integral over $A = \{y \in B : |y-\eta| < \delta\}$ and I_2 is the integral over $B \setminus A$ for $\delta > 0$. Given $\epsilon > 0$, since $f(x) \to 0$ as $x \to \eta$, we have

$$I_1 \leq \epsilon \int_B \frac{|R(x,y)|^p}{\|R(x,\cdot)\|_p^p}\, dV(y) = \epsilon$$

if δ is small enough. It is easy to get that $|R(x,y)| \leq \dfrac{C}{|x-y|^n}$ for any $x, y \in B$ by (1.5). So we have

$$I_2 \leq C(1-|x|)^{n(p-1)} \int_{|y-\eta|\geq\delta} \frac{dV(y)}{|x-y|^n} < C\epsilon$$

if $|x-\eta|$ is small enough. This gives a contradiction.

Now we show $\sigma_e(T_f) \subset f(S)$. Without loss of generality we assume 0 is not in $f(S)$. We need to show that T_f is a Fredholm operator. Let $g \in C(\bar{B})$ be such that $g = \dfrac{1}{f}$ on S. By Lemma 2.15, $T_{fg} - I = T_{fg-1}$ is compact. Thus by Lemma 2.14, $T_fT_g - I$ and $T_gT_f - I$ are both compact. Again by Atkinson's Theorem we conclude that T_f is a Fredholm operator. □

Theorem 2.16 implies the following corollary.

Corollary 2.17 *Let $1 < p < \infty$. If $f \in C(\bar{B})$, then T_f is compact on $b^p(B)$ if and only if $f = 0$ on S.*

2.4 Toeplitz Operators with Bounded Radial Symbols

Let D denote the open unit disk in the complex plane $\mathbf{C}$ and let A be the normalized area measure on D. The analytic Bergman space on D, denoted $L_a^2(D)$, consists of the analytic functions f on D with

$$\|f\|_2^2 = \int_D |f|^2 \, dA < \infty.$$

Let P denote the orthogonal projection from $L^2(D)$ onto its closed subspace $L_a^2(D)$. For $f \in L^\infty(D)$, the Toeplitz operator T_f is defined on $L_a^2(D)$ by $T_f g = P(fg)$. It is easy to see that T_f is bounded with $\|T_f\| \le \|f\|_\infty$.

Since point evaluation is a bounded linear functional on $L_a^2(D)$, for each $z \in D$ there exists a unique $K_z \in L_a^2(D)$ such that

$$f(z) = \langle f, K_z \rangle$$

for all $f \in L_a^2(D)$. The functions K_z ($z \in D$) are called reproducing kernels for $L_a^2(D)$; they have the explicit form

$$K_z(w) = \frac{1}{(1 - \bar{z}w)^2}, \quad w \in D.$$

For every $z \in D$, let $k_z(w) = K_z(w)/\|K_z\|_2$. Then k_z ($z \in D$) are called normalized reproducing kernels for $L_a^2(D)$. For $f \in L^\infty(D)$, the function $\tilde{f}$, called the Berezin transform of f, is defined on D by

$$\tilde{f}(z) = \langle T_f k_z, k_z \rangle = \int_D f(w)|k_z(w)|^2\, dA(w).$$

Recently, the following compactness characterization for Toeplitz operators with bounded radial symbols on the Bergman space of the unit disk was proved in [7]:

Theorem 2.18 *Let f be a bounded radial function on D. Then the following conditions are equivalent:*

(i) $T_f : L^2_a(D) \to L^2_a(D)$ *is compact;*

(ii) $\tilde{f}(z) \to 0$ *as* $|z| \to 1^-$*;*

(iii) $\dfrac{1}{1-r}\displaystyle\int_r^1 f(t)\, dt \to 0$ *as* $r \to 1^-$.

More recently, Axler and Zheng [3] showed that (i) and (ii) above are equivalent even for nonradial bounded functions on the disk. The purpose of this section is to extend Theorem 2.18 to spaces of harmonic functions in higher dimensions. We consider Toeplitz operators on the harmonic Bergman space of the unit ball in $\mathbf{R}^n$ for $n \geq 2$. We use the same basic approach as in [7], but our context of harmonic functions and higher dimensions requires new estimates. Although there appears to be no canonical choice for an orthonormal basis for the harmonic Bergman space, and reproducing kernels for the harmonic Bergman space appear to be quite different from analytic Bergman kernels when $n > 2$, it turns out that a similar approach can be used.

For $f \in L^\infty(B)$, the Berezin transform $\tilde{f}$ is defined on B by

$$\tilde{f}(x) = \langle T_f r(x,\cdot), r(x,\cdot)\rangle = \int_B f(y)|r(x,y)|^2\, dV(y),$$

where $r(x,\cdot) = R(x,\cdot)/\|R(x,\cdot)\|_2$. Although a formula for $R(x,y)$ in closed form is available, we will not use it. We will use Theorem 1.10 instead.

We need two lemmas from [7].

Lemma 2.19 *Let $\lambda \geq 1$. Suppose $|a_{m+1} - a_m| \leq C(m+1)^{\lambda-2}$ for some positive constant C and all $m \geq 0$. Then $\lim_{m\to\infty} a_m/(m+1)^{\lambda-1} = 0$ if and only if*

$$\lim_{t\to 1^-} (1-t)^\lambda \sum_{m=0}^{\infty} a_m t^m = 0.$$

Proof. This can be proved using Lemma 1 of [7] and the same proof as for Theorem 2 of [7]. □

Lemma 2.20 *Let k be a nonnegative integer. Suppose $f \in L^\infty[0,1)$. Then*

$$\lim_{r\to 1^-} \frac{1}{1-r}\int_r^1 f(t)\, dt = 0$$

if and only if

$$\lim_{m\to\infty} m \int_0^1 f(t) t^{2m+k}\, dt = 0.$$

Proof. Note that the boundedness of f implies $\lim_{m\to\infty} \int_0^1 f(t)t^{2m+k}\, dt = 0$, and the condition $\lim_{m\to\infty} m\int_0^1 f(t)t^{2m+k}\, dt = 0$ implies $\lim_{s\to\infty} s\int_0^1 f(t)t^s\, dt = 0$. Thus the lemma follows from Theorem 4 of [7]. □

For a radial function f on B, we define a function f^* on $[0,1)$ to be the function such that $f^*(|x|) = f(x)$. Now we can prove an analogue of Theorem 2.18 for the harmonic Bergman space.

Theorem 2.21 *Let f be a bounded radial function on B. Then the following conditions are equivalent:*

(i) $T_f : b^2(B) \to b^2(B)$ *is compact;*

(ii) $\tilde{f}(x) \to 0$ *as* $|x| \to 1^-$;

(iii) $\dfrac{1}{1-r}\displaystyle\int_r^1 f^*(t)\,dt \to 0$ *as* $r \to 1^-$.

Proof. For $x \in B$, by Theorem 1.10 and using the fact that spherical harmonics of different degrees are mutually orthogonal to each other, we have

$$\begin{aligned}
\tilde{f}(x) &= \|R(x,\cdot)\|_2^{-2}\int_B f(y)|R(x,y)|^2\,dV(y)\\
&\approx (1-|x|)^n\sum_{m=o}^{\infty}(n+2m)^2h_m|x|^{2m}\int_0^1 f^*(t)t^{2m+n-1}\,dt\\
&= (1-|x|)^n\sum_{m=0}^{\infty}a_m(f)|x|^{2m},
\end{aligned}$$

where $a_m(f) = (n+2m)^2h_m\int_0^1 f^*(t)t^{2m+n-1}\,dt$. In order to apply Lemma 2.19, we need to estimate $|a_{m+1}(f) - a_m(f)|$ for $m \geq 0$. We have

$$\begin{aligned}
a_{m+1}(f) - a_m(f) &= (n+2m+2)^2h_{m+1}\int_0^1 f^*(t)t^{2m+n+1}\,dt\\
&\quad -(n+2m)^2h_m\int_0^1 f^*(t)t^{2m+n-1}\,dt\\
&= [(n+2m+2)^2h_{m+1} - (n+2m)^2h_m]\int_0^1 f^*(t)t^{2m+n+1}\,dt\\
&\quad +(n+2m)^2h_m\int_0^1 f^*(t)(t^{2m+n+1} - t^{2m+n-1})\,dt\\
&= I_1(f) + I_2(f),
\end{aligned}$$

where $I_1(f)$ denotes the first term and $I_2(f)$ the second term.

Since $h_m \le C(m+1)^{n-2}$ for all $m \ge 0$, we have $|I_2(f)| \le C(m+1)^{n-2}$ for all $m \ge 0$. Thus if we can show that

$$|(n+2m+2)^2 h_{m+1} - (n+2m)^2 h_m| \le C(m+1)^{n-1}, \tag{2.1}$$

we will have $|I_1(f)| \le C(m+1)^{n-2}$, and consequently $|a_{m+1}(f) - a_m(f)| \le C(m+1)^{n-2}$ for all $m \ge 0$. Clearly

$$(n+2m+2)^2 h_{m+1} - (n+2m)^2 h_m = (n+2m)^2(h_{m+1} - h_m) + 4(n+2m)h_{m+1} + 4h_{m+1}.$$

It follows easily from (1.1) that

$$h_{m+1} - h_m = \binom{n+m-2}{n-3} + \binom{n+m-3}{n-3}.$$

Combining the identities above we have the desired inequality (2.1). By Lemma 2.19, the condition (ii) holds if and only if $\lim_{m\to\infty} a_m(f)/(m+1)^{n-1} = 0$. So (ii) holds if and only if $\lim_{m\to\infty} m \int_0^1 f^*(t) t^{2m+n-1} dt = 0$. From Lemma 2.20, we see conditions (ii) and (iii) are equivalent.

On the other hand, every function in $b^2(B)$ is a sum of homogeneous harmonic polynomials. For $m \ge 0$, let $p_{m,1}, \cdots, p_{m,h_m}$ be an orthonormal basis for $\mathcal{H}_m(S)$. Then

$$\bigcup_{m=0}^{\infty} \{c_m p_{m,1}, \cdots, c_m p_{m,h_m}\}$$

is an orthonormal basis for $b^2(B)$, where $c_m = \sqrt{(n+2m)/nV(B)}$. It is easy to see that T_f is a diagonal operator with respect to this basis since f is a radial function. For each $j \in \{1, \cdots, h_m\}$ we have

$$\begin{aligned} \langle T_f c_m p_{m,j}, c_m p_{m,j} \rangle &= (n+2m) \int_0^1 f^*(t) t^{2m+n-1}\, dt \\ &= \frac{a_m(f)}{(n+2m)h_m}. \end{aligned}$$

Thus T_f is compact on $b^2(B)$ if and only $\lim_{m\to\infty} a_m(f)/(n+2m)h_m = 0$. It is clear that $(n+2m)h_m \approx (m+1)^{n-1}$. Again by Lemma 2.19, the condition (ii) holds if and only if $\lim_{m\to\infty} a_m(f)/(n+2m)h_m = 0$. This finishes the proof of the equivalence of (i) and (ii) and the proof of the theorem. □

The equivalence of (i) and (ii) in Theorem 2.18 was extended to higher dimensions in [20] for the analytic Bergman spaces of the unit ball in $\mathbf{C}^n$.

CHAPTER 3

Weighted Harmonic Bergman Spaces

3.1 Introduction

The harmonic Bergman space $b^2_\alpha(B)$, with $\alpha > -1$, is the set of all complex-valued harmonic functions u on B with

$$\|u\|_{2,\alpha} = \left(\int_B |u(x)|^2(1-|x|^2)^\alpha \, dV(x)\right)^{\frac{1}{2}} < \infty.$$

Point evaluation is a bounded linear functional on $b^2_\alpha(B)$. Hence for every $x \in B$, there exists a unique $R_\alpha(x,\cdot) \in b^2_\alpha(B)$ such that

$$u(x) = \int_B u(y)R_\alpha(x,y)(1-|y|^2)^\alpha \, dV(y)$$

for all $u \in b^2_\alpha(B)$. The functions $R_\alpha(x, \cdot)$ are called reproducing kernels for $b^2_\alpha(B)$. Obviously $R_0 = R$. We will see that each R_α is real valued for $\alpha > -1$ in Section 3.3.

The purpose of this chapter is to study these reproducing kernels. These reproducing kernels have been studied by different authors in [2], [5], [8], and [15]. While reproducing kernels for (analytic) Bergman spaces of the unit ball in $\mathbf{C}^n$ have simple formulas in closed form, those for harmonic Bergman spaces are much more complicated, and it appears to be impossible to find formulas in closed form for R_α in general, except when $n = 2$. In Section 2.2, we point out how harmonic reproducing kernels behave differently from analytic ones on the unit disk. In Section 3.3, we give a representation for R_α in terms of zonal harmonics in higher dimensions and establish some properties for R_α. We use an estimate on R_α given recently in [15] to prove one property for R_α. In the last two sections, we give some applications of these properties.

3.2 Reproducing Kernels on the Unit Disk

We consider R_α when $n = 2$ in this section. Let D denote the open unit disk in the complex plane $\mathbf{C}$ and A be Lebesgue area measure on D. For $\alpha > -1$, the analytic Bergman space $L^{2,\alpha}_a(D)$ is the set of all analytic functions in $L^2(D, (1-|z|^2)^\alpha dA(z))$. Let K_α be the reproducing kernel for $A^2_\alpha(D)$, i.e.,

$$f(z) = \int_D f(w)\bar{K}_\alpha(z, w)(1 - |w|^2)^\alpha \, dA(w), \quad z \in D,$$

for all $f \in A^2_\alpha(D)$. We know that

$$\bar{K}_\alpha(z,w) = \frac{\alpha+1}{\pi}\frac{1}{(1-z\bar{w})^{2+\alpha}}, \quad z,w \in D.$$

The reproducing kernels for $b^2_\alpha(D)$ are closely related to $\bar{K}_\alpha(z,w)$. We have (see page 357 of [22])

$$R_\alpha(z,w) = \frac{\alpha+1}{\pi}\left(2\,\mathrm{Re}\frac{1}{(1-z\bar{w})^{2+\alpha}} - 1\right), \quad z,w \in D.$$

For $z \in D, r \in (0,1)$, let $D_r(z) = \{w \in \mathbf{C} : |w-z| < r(1-|z|)\}$. An important property for $K_\alpha(z,w)$ is that

$$|K_\alpha(z,w)| \approx 1/(1-|z|)^{2+\alpha}, \quad w \in D_r(z).$$

For the unit disk, one usually uses the pesudo-hyperbolic disk instead of $D_r(z)$ because of its connection with Möbius transformations; see [4] for example. However we will use the obvious extension of $D_r(z)$ for higher dimensions in the next section.

We find that $R_\alpha(z,w)$ behaves quite differently from $K_\alpha(z,w)$, which never vanishes.

Proposition 3.1 *For each $r \in (0,1)$, there exist $z \in D$ and $\alpha > -1$ such that $R_\alpha(z,w) = 0$ for some $w \in D_r(z)$.*

Proof. For $z, w \in D$, we have

$$R_\alpha(z,w) = \frac{\alpha+1}{\pi}\left(\frac{2\,\mathrm{Re}(1-\bar{z}w)^{2+\alpha}}{|1-z\bar{w}|^{4+2\alpha}} - 1\right).$$

Let $z = t \in (0,1)$ and $\frac{1}{t} - w = se^{i\theta}$, where $s > 0$. It is easy to see that for $w \in D_r(t)$,

$$-\frac{rt}{1+t} = -\frac{r(1-t)}{\frac{1}{t}-t} < \sin\theta < \frac{r(1-t)}{\frac{1}{t}-t} = \frac{rt}{1+t},$$

and the range of θ is $(-\arcsin\frac{rt}{1+t}, \arcsin\frac{rt}{1+t})$ when w ranges over $D_r(t)$. Since $|1-t\bar{w}| \leq (1-t)(1+t+rt)$ for $w \in D_r(t)$, we can choose t close enough to 1 such that $|1-t\bar{w}|^{2+\alpha} \leq \frac{1}{2}$. If we choose α large enough, then the range of $\cos(2+\alpha)\theta$ is $[-1,1]$ when w ranges over $D_r(t)$. Hence the conclusion follows from

$$R_\alpha(t,w) = \frac{\alpha+1}{\pi}\left(\frac{2\cos(2+\alpha)\theta - |1-t\bar{w}|^{2+\alpha}}{|1-t\bar{w}|^{2+\alpha}}\right).$$

□

It is not difficult to see from the proof above that we still have

$$R_\alpha(z,w) \approx 1/(1-|z|)^{2+\alpha}, \quad w \in D_r(z),$$

provided that r is small enough (depending only on α). In the next section we will prove this property for $R_\alpha(x,y)$ in higher dimensions.

3.3 Some Properties of the Reproducing Kernels

We can describe the reproducing kernels in terms of zonal harmonics.

We have the following representation for R_α.

Proposition 3.2 *Let* $\alpha > -1$. *If* $x, y \in B$, *then*

$$R_\alpha(x,y) = \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} Z_m(x,y).$$

The series converges absolutely and uniformly on $K \times B$ *for every compact* $K \subset B$.

Proof. This can be proved using the same argument as for the proof of Theorem 8.9 of [2]. □

Since Z_m is real valued for each m, we see that R_α is real valued.

Now we can give an estimate for R_α.

Proposition 3.3 *Let* $\alpha > -1$. *Then*

(i) $R_\alpha(x,x) \approx 1/(1-|x|)^{n+\alpha}$ *for* $x \in B$;

(ii) $\|R_\alpha(x,\cdot)\|_{2,\alpha}^2 \approx 1/(1-|x|)^{n+\alpha}$ *for* $x \in B$;

(iii) $|R_\alpha(x,y)| \leq C/(1-|x||y|)^{n+\alpha}$ *for* $x, y \in B$.

Proof. First we prove (i). By Proposition 3.2 and (i) of Lemma 1.9, we have

$$R_\alpha(x,x) = \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} h_m |x|^{2m}.$$

Since $h_m \approx (m+1)^{n-2}$, by Stirling's formula we see the coefficients in the series above are of order $m^{\alpha-1}$ as $m \to \infty$. This proves (i).

(ii) follows from $\|R_\alpha(x,\cdot)\|_{2,\alpha}^2 = R_\alpha(x,x)$.

To show (iii), for $x, y \in B$, let $x = |x|\zeta, y = |y|\eta$. Then by (ii) of Lemma 1.9

$$\begin{aligned} |R_\alpha(x,y)| &\le \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} (|x||y|)^m |Z_m(\zeta,\eta)| \\ &\le \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} (|x||y|)^m h_m \\ &\le \frac{C}{(1-|x||y|)^{n+\alpha}}. \end{aligned}$$

This finishes the proof. □

The following simple fact will be used:

$$1 - |y| \approx 1 - |x|, \quad y \in K_r(x).$$

We have the following lower bound estimate for the reproducing kernels.

Proposition 3.4 *Let $\alpha > -1$ and $x \in B$. Then there exists $r = r(\alpha) \in (0,1)$ depending only on α such that $R_\alpha(x,y) \approx 1/(1-|x|)^{n+\alpha}$ for $y \in K_r(x)$.*

Proof. It follows from Proposition 3.3 (iii) that $R_\alpha(x,y) \le C/(1-|x|)^{n+\alpha}$ for $y \in K_r(x)$.

To show the other direction, for $y \in K_r(x)$, by the mean value theorem we have

$$\begin{aligned} R_\alpha(x,y) &\ge R_\alpha(x,x) - \max_{u \in K_r(x)} |\nabla_u R_\alpha(x,u)||y-x| \\ &\ge \frac{C}{(1-|x|)^{n+\alpha}} - \max_{u \in K_r(x)} |\nabla_u R_\alpha(x,u)||y-x|. \end{aligned}$$

If $u \in K_{\frac{1}{2}}(x)$, then $1 - |u| > \frac{1}{2}(1-|x|)$. Thus for $u \in K_{\frac{1}{2}}(x)$, Cauchy's estimates

(2.4 of [2]) gives

$$|\nabla_u R_\alpha(x,u)| \le \frac{C}{(1-|u|)} \max_{v\in K_{\frac{1}{2}}(u)} |R_\alpha(x,v)| \le \frac{C}{(1-|x|)^{n+\alpha+1}}.$$

Thus if r is chosen small enough, for $y \in K_r(x)$, we get

$$R_\alpha(x,y) \ge \frac{C}{(1-|x|)^{n+\alpha}} - \frac{Cr}{(1-|x|)^{n+\alpha}} \ge \frac{C}{(1-|x|)^{n+\alpha}}.$$

This proves the proposition. □

When $\alpha = 0$, the proposition above was proved in Lemma 2.1 for any $r \in (0,1)$ using the explicit formula for $R_0(x,y)$.

For $x, y \in B$, let $P(x,y)$ be the "extended Poisson kernel" for B given by (1.2). If α is a non-negative integer, then

$$\begin{aligned} R_\alpha(x,y) &= \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} (m+\frac{n}{2}+\alpha)\cdots(m+\frac{n}{2}) Z_m(x,y) \\ &= \frac{2}{nV(B)\Gamma(\alpha+1)} \left(\frac{d}{dt}\right)^{\alpha+1} \left[t^{\frac{n}{2}+\alpha} P(tx,y)\right]_{t=1}. \end{aligned}$$

For $x \in B, x \neq 0$, let $\tilde{x} = x/|x|^2$ be the inversion of x. Notice that our reproducing kernels are slightly different from those in [5] and [15] because we choose $(1-|x|^2)^\alpha$ as weights. We have the following lemma.

Lemma 3.5 *Let* $\alpha > -1$.

(i) $|R_\alpha(x,y)| \le C|\tilde{x} - y|^{-n-\alpha}$ *for* $x, y \in B$ *with* $|x| > \frac{1}{2}$;

(ii) *If* $\alpha > n(\frac{1}{p} - 1) - \frac{1}{p}$, *then*

$$\int_S |R_\alpha(\zeta, y)|^p \, d\sigma(\zeta) \le C(1 - |y|)^{n-1-(n+\alpha)p}, \quad y \in B.$$

Proof. The same proof as for Lemma 2.3 of [15] yields (i) (although only the case when $\alpha > 0$ was considered in [15]). Now (ii) follows from (i) by the proof for Lemma 3.2 of [5]. □

In order to prove our next result, we need the following simple estimate (see page 291 of [17]).

Lemma 3.6 *If* $\beta > -1$ *and* $m > 1 + \beta$, *then for* $0 \le t < 1$,

$$\int_0^1 (1 - tr)^{-m} (1 - r)^\beta \, dr \le C(1 - t)^{1+\beta-m}.$$

The following is the last property for R_α in this section.

Proposition 3.7 *If* $p > \dfrac{n+\beta}{n+\alpha}$, $\beta > -1$, *and* $\alpha > -1$, *then*

$$\int_B |R_\alpha(x, y)|^p (1 - |y|)^\beta \, dV(y) \approx \frac{1}{(1 - |x|)^{(n+\alpha)p-(n+\beta)}}, \quad x \in B.$$

Proof. For $x \in B$, by Proposition 3.4, we have

$$\begin{aligned} \int_B |R_\alpha(x, y)|^p (1 - |y|)^\beta \, dV(y) &\ge \int_{K_r(x)} |R_\alpha(x, y)|^p (1 - |y|)^\beta \, dV(y) \\ &\ge \frac{C}{(1 - |x|)^{(n+\alpha)p-(n+\beta)}}. \end{aligned}$$

To show the other direction, using Lemma 3.5 (ii) and the fact that $R_\alpha(rx, y) =$

$R_\alpha(x, ry)$ for $x, y \in B, 0 < r < 1$ (which follows from Proposition 3.2), we have

$$\begin{aligned}
\int_B |R_\alpha(x,y)|^p (1-|y|)^\beta \, dV(y) &= nV(B) \int_0^1 (1-r)^\beta r^{n-1} \left(\int_S |R_\alpha(x, r\zeta)|^p \, d\sigma(\zeta) \right) dr \\
&= nV(B) \int_0^1 (1-r)^\beta r^{n-1} \left(\int_S |R_\alpha(rx, \zeta)|^p \, d\sigma(\zeta) \right) dr \\
&\leq C \int_0^1 (1-r)^\beta (1-r|x|)^{n-1-(n+\alpha)p} \, dr \\
&\leq C(1-|x|)^{n+\beta-(n+\alpha)p},
\end{aligned}$$

where we used Lemma 3.6 in the last step. □

3.4 Application to an Inequality for Harmonic Functions

The following result was proved in [8] and [18].

Theorem 3.8 *Let G be a measurable subset of B and $p > 0, \beta > -1$. Then the following conditions are equivalent:*

(i) *There is a constant $C > 0$ such that*

$$\int_B |f(y)|^p (1-|y|)^\beta \, dV(y) \leq C \int_G |f(y)|^p (1-|y|)^\beta \, dV(y)$$

for each harmonic function f on B for which the left-hand side of the inequality is finite;

(ii) *There is a constant $\delta > 0$ such that $V(G \cap K) \geq \delta V(B \cap K)$ for every ball K whose center lies on S.*

Luecking [8] proved (ii)$\Rightarrow$(i), and (i)$\Rightarrow$(ii) only when $p = 2, \beta = 0$. Later Sledd proved (i)$\Rightarrow$(ii) for all $p > 0, \beta > -1$ in [18] (I thank Professor William T. Sledd for this reference). To prove (i)$\Rightarrow$(ii) in the case when $p = 2, \beta = 0$, Luecking [8] used $R_0(x, y)$ and suggested the use of $R_\beta(x, y)$ for the case when $p = 2, \beta > -1$. Sledd [18] developed a different approach by constructing harmonic functions using the Poisson kernel.

We here provide another proof of (i)$\Rightarrow$(ii) for Theorem 3.9. Our method is similar to that in [8]. For $\beta \geq 0$, our proof is even shorter than that in [8], where the explicit formula for $R_0(x, y)$ was used. For $-1 < \beta < 0$, our proof uses a careful argument. We believe the reproducing kernels are natural candidates for this type of inequality.

Proof of (i)$\Rightarrow$(ii). By the argument in the proof of Lemma 3 of [8], we only need to show that given $\epsilon > 0$, there is a constant C_ϵ (depending on ϵ) such that for every ball K with its center on S, there exists a harmonic function f (depending on ϵ and K) on B such that

(1) $\int_B |f(y)|^p (1 - |y|)^\beta \, dV(y) \geq C$, where C does not depend on K, ϵ, and f;

(2) $\int_{B \setminus K} |f(y)|^p (1 - |y|)^\beta \, dV(y) < \epsilon$;

(3) $\int_{G \cap K} |f(y)|^p (1 - |y|)^\beta \, dV(y) \leq C_\epsilon (V(G \cap K)/V(K \cap B))^a$ for some $a > 0$, where a depends only on β.

Without loss of generality let K have radius $h < 1$ and center $u = (1, 0, \cdots, 0)$. Choose α large enough so that $p > \dfrac{n+\beta}{n+\alpha}$. Let

$$f(y) = R_\alpha(x_k, y)(1 - |x_k|)^{n+\alpha-\frac{n+\beta}{p}},$$

where $x_k = ru, r > 0$, and $1 - r = sh$ for small $s > 0$ to be chosen.

Condition (1) follows from Proposition 3.7.

The case $\beta \geq 0$ is easier to deal with in order to show (2) and (3). Let $\beta \geq 0$. If $y \in K$, then $1 - |y| < h$. By Proposition 3.4, for $y \in K$, we have

$$|f(y)|^p(1-|y|)^\beta \leq C\frac{(1-|y|)^\beta}{(1-|x_k|)^{n+\beta}} \leq C\frac{h^\beta}{(sh)^{n+\beta}} = C_s\frac{1}{V(K)}.$$

This implies (3) for $a = 1$.

By Lemma 3.5, we have $|R_\alpha(x_k, y)| \leq C/|\tilde{x}_k - y|^{n+\alpha}$ if $s < \frac{1}{2}$. Notice that $(1 - |y|) < |\tilde{x}_k - y|$, $y \in B$. We have

$$\begin{aligned}
\int_{B\setminus K} |f(y)|^p(1-|y|)^\beta \, dV(y) &\leq C(1-|x_k|)^{p(n+\alpha)-(n+\beta)} \int_{B\setminus K} \frac{1}{|\tilde{x}_k - y|^{(n+\alpha)p-\beta}} \, dV(y) \\
&\leq C(sh)^{p(n+\alpha)-(n+\beta)} \int_h^\infty \frac{r^{n-1}}{r^{p(n+\alpha)-\beta}} \, dr \\
&\leq C(s)^{p(n+\alpha)-(n+\beta)},
\end{aligned}$$

where we used the fact that $B \setminus K \subset \{y \in \mathbf{R}^n : |y - \tilde{x}_k| > h\}$ in the second step. If s is chosen small, then we have condition (2).

The case when $-1 < \beta < 0$ requires more work. First we choose $q > 1$ such that $q\beta > -1$. Let q' denote the conjugate of q. Hölder's inequality gives

$$\begin{aligned}
\int_{B\setminus K} |f(y)|^p(1-|y|)^\beta \, dV(y) &\leq (1-|x_k|)^{p(n+\alpha)-(n+\beta)} \\
&\quad \cdot \left(\int_{B\setminus K} |R_\alpha(x_k, y)|^{\frac{pq}{2}}(1-|y|)^{\beta q} \, dV(y)\right)^{\frac{1}{q}} \\
&\quad \cdot \left(\int_{B\setminus K} |R_\alpha(x_k, y)|^{\frac{pq'}{2}} \, dV(y)\right)^{\frac{1}{q'}}.
\end{aligned}$$

If $(n+\alpha)p > 2(\frac{n}{q}+\beta)$, then by Proposition 3.7

$$\left(\int_{B\setminus K} |R_\alpha(x_k,y)|^{\frac{pq}{2}}(1-|y|)^{\beta q}\,dV(y)\right)^{\frac{1}{q}} \le C\frac{1}{(1-|x_k|)^{(n+\alpha)\frac{p}{2}-(\frac{n}{q}+\beta)}}.$$

If $(n+\alpha)p > 2\frac{n}{q'}$, then we have

$$\begin{aligned}\left(\int_{B\setminus K} |R_\alpha(x_k,y)|^{\frac{pq'}{2}}\,dV(y)\right)^{\frac{1}{q'}} &\le C\left(\int_h^\infty \frac{r^{n-1}}{r^{(n+\alpha)\frac{pq'}{2}}}\,dr\right)^{\frac{1}{q'}} \\ &= C\frac{1}{h^{(n+\alpha)\frac{p}{2}-\frac{n}{q'}}}.\end{aligned}$$

Combining the inequalities above, we get

$$\int_{B\setminus K} |f(y)|^p(1-|y|)^\beta\,dV(y) \le C(s)^{(n+\alpha)\frac{p}{2}-\frac{n}{q'}},$$

provided that α is large enough. This gives (2) if s is small enough.

We now show (3). We have

$$\int_{G\cap K} |f(y)|^p(1-|y|)^\beta\,dV(y) = (1-|x_k|)^{p(n+\alpha)-(n+\beta)}\int_{G\cap K} |R_\alpha(x_k,y)|^p(1-|y|)^\beta\,dV(y).$$

By Hölder's inequality and Proposition 3.7, we get

$$\begin{aligned}\int_{G\cap K} |R_\alpha(x_k,y)|^p(1-|y|)^\beta\,dV(y) &\le \left(\int_B |R_\alpha(x_k,y)|^{qp}(1-|y|)^{q\beta}\,dV(y)\right)^{1/q} \\ &\quad\cdot (V(G\cap K))^{1/q'} \\ &\le \frac{C}{(1-|x_k|)^{p(n+\alpha)-(\frac{n}{q}+\beta)}}(V(G\cap K))^{1/q'}.\end{aligned}$$

Hence we obtain that

$$\int_{G\cap K} |f(y)|^p (1-|y|)^\beta \, dV(y) \le C \left(\frac{V(G\cap K)}{(1-|x_k|)^n} \right)^{1/q'} \le C_s \left(\frac{V(G\cap K)}{V(B\cap K)} \right)^{1/q'}.$$

Thus the condition (3) is satisfied with $a = 1/q'$. □

3.5 Application to Toeplitz operators on Harmonic Bergman Spaces

Let μ be a finite complex Borel measure on B. We densely define the Toeplitz operator on $b^2_\alpha(B)$ with symbol μ by

$$T_\mu u(x) = \int_B R_\alpha(x,y) u(y) \, d\mu(y)$$

for $u \in b^2_\alpha(B) \cap L^\infty(B, (1-|x|^2)^\alpha dV(x))$. If $d\mu(y) = f(y)(1-|y|^2)^\alpha dV(y)$, then we write $T_\mu = T_f$. Let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product for $L^2(B, (1-|x|^2)^\alpha dV(x))$. For bounded $u, v \in b^2_\alpha(B)$, it follows from Fubini's Theorem that

$$\langle T_\mu u, v \rangle_\alpha = \int_B u\bar{v} \, d\mu.$$

Suppose $\mu \ge 0$ and let I denote the inclusion map from $b^2_\alpha(B)$ to $L^2(B, d\mu)$. It is clear that T_μ is bounded (compact) on $b^2_\alpha(B)$ if and only if I is bounded (compact).

The characterization of boundedness and compactness for the inclusion map was given in [14], where more general domains in $\mathbf{R}^n$ and more general spaces were considered, except for $-1 < \alpha < 0$. We can extend the characterization to all $\alpha > -1$ in our case. From here on we always assume r is the number given in Proposition 3.4 (which depends only on α).

Proposition 3.9 *Let $\alpha > -1$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *I is bounded (compact);*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}}$ is bounded for $x \in B$ ($\to 0$ as $|x| \to 1$).*

Proof. Oleinik and Pavlov [14] proved that (ii)$\Rightarrow$(i). To prove the implication in the other direction, suppose I is bounded. Then

$$\int_B |u|^2\, d\mu \leq C \int_B |u(y)|^2 (1-|y|^2)^\alpha\, dV(y)$$

for all $u \in b^2_\alpha(B)$. For $x \in B$, let $u(y) = R_\alpha(x,y) \in b^2_\alpha(B)$. Then

$$\begin{aligned} \frac{\mu(K_r(x))}{(1-|x|)^{2(n+\alpha)}} &\leq C \int_{K_r(x)} |R_\alpha(x,y)|^2\, d\mu(y) \\ &\leq C \int_B |R_\alpha(x,y)|^2\, d\mu(y) \\ &\leq C \int_B |R_\alpha(x,y)|^2 (1-|y|^2)^\alpha\, dV(y) \\ &\leq \frac{C}{(1-|x|)^{n+\alpha}}, \end{aligned}$$

where we used Proposition 3.3 (ii) in the last step. A modification of this argument shows that compactness of I implies the little o condition; we omit the details. This proves (ii). □

Now we can state Proposition 3.9 in terms of Toeplitz operators. Although [14] only gives the continuous version, a discrete version can be easily obtained (see, for example, Theorem 1.8).

Proposition 3.10 *Let $\alpha > -1$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *T_μ is bounded (compact) on $b^2_\alpha(B)$;*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}}$ is bounded for $x \in B$ ($\to 0$ as $|x| \to 1$);*

(iii) *$\mu(K_r(x_i))/V(K_r(x_i))^{1+\frac{\alpha}{n}}$ is bounded for $i = 1, 2, \cdots$ ($\to 0$ as $i \to \infty$).*

Now we can establish a trace ideal criteria for positive Toeplitz operators on $b^2_\alpha(B)$. The case $\alpha = 0$ was proved by Theorem 2.9 using ideas from [9] and [23]. That result can be extended to all $\alpha > -1$.

Theorem 3.11 *Let $1 \le p < \infty, \alpha > -1$, and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *$T_\mu \in S_p(b^2_\alpha(B))$;*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}} \in L^p(B, (1-|x|^2)^{-n}dV(x))$;*

(iii) *$\sum_{i=1}^\infty \left(\mu(K_r(x_i))/V(K_r(x_i))^{1+\frac{\alpha}{n}}\right)^p < \infty$.*

The proof of the theorem above is entirely analogous to that for Theorem 2.9, so we will not give a proof for it. We remark that the two properties for the reproducing kernels needed for the proof are supplied by Proposition 3.3 and 3.4, and the S_p-norm of T_μ is related to the reproducing kernels by the following identity:

$$||T_\mu||^p_{S_p} = \int_B \langle T^p_\mu R_\alpha(x, \cdot), R_\alpha(x, \cdot)\rangle_\alpha (1-|x|^2)^\alpha dV(x).$$

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