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The Product of a Composition Operator with the Adjoint of a Composition Operator

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John Howard Clifford

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Joel Shapiro
Major professor

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THE PRODUCT OF A COMPOSITION OPERATOR WITH THE ADJOINT OF A COMPOSITION OPERATOR

By

John Howard Clifford

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ABSTRACT

THE PRODUCT OF A COMPOSITION OPERATOR WITH THE ADJOINT OF A COMPOSITION OPERATOR

By

John Howard Clifford

We obtain an upper estimate for the essential norm of $C_\psi^* C_\varphi$ on the Hardy space H^2 as the upper bound of a quantity involving the product of the inducing maps' Nevanlinna counting functions. In the special case of univalent inducing maps we prove a complete function theoretic characterization of compactness in terms of the angular derivatives of the inducing maps.

We obtain necessary and sufficient conditions, under varied hypothesis on the inducing maps, for the operator $C_\varphi C_\psi^*$ to be compact on the Hardy space H^2. In the special case where one inducing map is boundedly valent we calculate a lower estimate for the essential norm as the upper bound of a quantity involving the product of the inducing maps' Nevanlinna counting functions.

To Joan

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Introduction

Let U denote the open unit disc of the complex plane and let φ and ψ be holomorphic self maps of the disk. The equation

$$C_\varphi f = f \circ \varphi$$

defines a composition operator C_φ on the space of holomorphic functions; φ is called the *inducing map* or *symbol* of C_φ. It is a consequence of Littlewood's subordination principle [8] that C_φ is bounded on H^2. This paper studies the compactness of the operators formed by multiplying a composition operator C_φ with the adjoint C_ψ^* of another composition operator to form either $C_\varphi C_\psi^*$ or $C_\psi^* C_\varphi$. Our goal is to give a function theoretic characterization of the essential norms of $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ in terms of the geometric properties of the inducing maps φ and ψ. This line of investigation has already been carried out for composition operators acting on the classical weighted Hardy and Bergman spaces. Let $||T||_e$ denote the essential norm of the operator T on H^2 (i.e the distance in the operator norm from T to the compact operators). Shapiro [16] gave the following expression for the essential norm of C_φ on H^2,

$$||C_\varphi||_e^2 = \limsup_{|w| \to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}}, \tag{1}$$

thus providing a complete function theoretic characterization of compact composition operators in terms of the inducing map's Nevanlinna counting function N_φ.

We make progress toward answering in the affirmative the following two conjectures:

Conjecture 1 *Suppose φ and ψ are holomorphic self maps of the disc. Then*

$$\|C_\psi^* C_\varphi\|_e^2 = \limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z))N_\psi(\psi(z))}{\log\frac{1}{|\varphi(z)|}\log\frac{1}{|\psi(z)|}}. \tag{2}$$

Conjecture 2 *Suppose φ and ψ are holomorphic self maps of the disc. Then*

$$\|C_\varphi C_\psi^*\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)N_\psi(w)}{\left(\log\frac{1}{|w|}\right)^2}. \tag{3}$$

The study of compact composition operators on H^2 first appeared in H. J. Schwartz's [14] thesis in the late sixties. He proved a necessary condition for a composition operator to be compact;

> *If C_φ is compact then $|\varphi^*| < 1$ a.e. on the unit circle.*

In other words, C_φ is not compact whenever the set $\{|\varphi^*| = 1\}$ has positive measure. We let φ^* denote the nontangential limit (when it exists) of φ. (By Fatou's theorem this limit exists at almost every point of ∂U). Schwartz also showed this necessary condition is not sufficient by showing the composition operator induced by

$$\varphi(z) = \frac{1+z}{2}$$

is not compact, even though φ maps only a single point of the unit circle onto the unit circle, $\varphi(1) = 1$. This work was carried on in [15] by Shapiro and Taylor who extended Schwartz's theorem as follows:

> *C_φ is not compact whenever φ has an angular derivative at some point of the unit circle.*

In [14], Schwartz also proved the following sufficient condition, which we will refer to as the L^1-condition, for a composition operator to be compact.

If $(1 - |\varphi^*(e^{it})|)^{-1} \in L^1(\partial U)$ *then* C_φ *is compact.*

Shapiro and Taylor [15] refined this result by showing that the L^1-condition characterizes the Hilbert-Schmidt composition operators on H^2. Moreover, they applied the L^1-condition to show that if the image of φ is contained inside a polygon then C_φ is Hilbert-Schmidt.

In [9] MacCluer and Shapiro studied the extent to which the angular derivative characterizes the compactness of a composition operator. They obtained a sharper result than we state (i.e. [9], Theorem 3.10) but for our purposes we highlight;

Suppose φ *is finitely valent. Then* C_φ *is compact if and only if* φ *has no angular derivative.*

In [9] it is shown that on the weighted Bergman spaces the angular derivative does tell the whole story, i.e. for any self map φ of the unit disc, C_φ is compact if and only if φ has no angular derivative.

In 1987, Shapiro [16] solved the compactness problem by deriving the essential norm formula,

$$||C_\varphi||_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}}.$$

Hence,

$$C_\varphi \text{ is compact if and only if } \lim_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0.$$

This paper is organized as follows. In the next chapter we consider the operators $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$ induced by linear fractional self maps of the disc. As motivation for more general results we completely characterize the compactness of these linear fractionaly induced operators in terms of the inducing maps. Chapter 3 consists of a sketch of background material. In Chapter 4 we develop the function theory needed to connect the compactness of the operators with the angular derivatives of φ and ψ. The next two chapters treat the operators $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$ respectively. In the first section of each these chapters we outline the main results about the operator in question and develop the connection between the compactness of that operator and the angular derivative of φ and ψ.

CHAPTER 1

Linear Fractional Maps

A composition operator C_φ induced by a linear fractional map φ is compact if and only if $\overline{\varphi(U)} \subset U$. Equivalently, C_φ is not compact if and only if φ maps a point of the unit circle onto the unit circle. This result can be deduced from the more general theorem ([18], page 57):

> *Suppose φ is univalent self map of the disc. Then C_φ is compact if and only if φ does not have an angular derivative.*

In this section we prove analogous results for the linear-fractionally induced operators $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$.

The proofs will require both an explicit computation of the adjoint of a composition operator induced by a linear fractional self map of the disc, and some properties of Toeplitz operators.

For g in $L^\infty(\partial U)$, the Toeplitz operator T_g is the operator on H^2 given by $T_g(f) = Pgf$ for f in H^2, where P is the orthogonal projection of L^2 onto H^2. (For more details on Toeplitz operators see [7] or [20]).

Cowen's Adjoint Theorem([5], Theorem 2, page 153) *Let $\psi(z) = (az+b)/(cz+d)$ be a linear fractional self map of U where $ad-bc \neq 0$. Then $\sigma(z) = (\bar{a}z-\bar{c})/(-\bar{b}z+\bar{d})$*

maps U into itself, $g(z) = (-\bar{b}z + \bar{d})^{-1}$ and $h(z) = cz + d$ are in H^∞, and

$$C_\psi^* = T_g C_\sigma T_h^*.$$

In [5] Theorem 2, Cowen requires that $ad - bc = 1$ but this is not necessary.

A comment about notation; throughout this chapter ψ and σ will be linear fractional self maps of the disc defined by the relationship,

$$C_\psi^* = T_g C_\sigma T_h^*.$$

A calculation shows that ψ and σ have the following nice properties:

$$C_\sigma^* = T_{\tilde{g}} C_\psi T_{\tilde{h}}^* \tag{1.1}$$

where $\tilde{g}(z) = (cz + d)^{-1}$ and $\tilde{h}(z) = -\bar{b}z + \bar{d}$,

$$\text{if for some } \eta \in \partial U, \quad \psi(\eta) = \omega \in \partial U \quad \text{then} \quad \sigma(\omega) = \eta, \tag{1.2}$$

and conversely,

$$\text{if for some } \omega \in \partial U, \quad \sigma(\omega) = \eta \in \partial U \quad \text{then} \quad \psi(\eta) = \omega. \tag{1.3}$$

Since $C_\sigma T_f = T_{f \circ \sigma} C_\sigma$ for any $f \in H^\infty$ we obtain another expression for the adjoint

$$C_\psi^* = C_\sigma T_f T_h^* \tag{1.4}$$

where $f(z) = g \circ \sigma^{-1}(z) = \bar{b}z + \bar{a}$.

A further comment about notation, by applying Cowen's Adjoint Theorem and (1.1) we will represent the operators $C_\psi^*, C_\sigma^*, C_\varphi^*$, and C_Σ^* as:

$$C_\psi^* = T_g C_\sigma T_h^*, \quad C_\sigma^* = T_{\tilde{g}} C_\psi T_{\tilde{h}}^*, \quad C_\varphi^* = T_G C_\Sigma T_H^*, \text{ and } C_\Sigma^* = T_{\widetilde{G}} C_\varphi T_{\widetilde{H}}^*.$$

Throughout this chapter we shall reserve the letters ψ, σ, g, h, $\tilde{g}$, $\tilde{h}$, φ, Σ, G, H, $\widetilde{G}$, and $\widetilde{H}$ for this meaning.

Theorem 1.1 *Suppose that φ and ψ are linear fractional self maps of U. Then $C_\varphi C_\psi^*$ is not compact if and only if there exist points η_1 and $\eta_2 \in \partial U$ such that $\varphi(\eta_1) = \psi(\eta_2) \in \partial U$.*

The key to the proof is the following lemma.

Lemma 1.2 *Suppose that φ and ψ are linear fractional self maps of U. Then $C_\varphi C_\psi^*$ is compact if and only if $C_\varphi C_\sigma$ is compact.*

Proof of Lemma: Suppose first that $C_\varphi C_\sigma$ is compact. Since

$$\begin{aligned} C_\varphi C_\psi^* &= C_\varphi T_g C_\sigma T_h^* && (\text{because} \quad C_\psi^* = T_g C_\sigma T_h^*) \\ &= T_{g\circ\varphi} C_\varphi C_\sigma T_h^* && (\text{because} \quad C_\varphi T_g = T_{g\circ\varphi} C_\varphi) \end{aligned}$$

we see that $C_\varphi C_\psi^*$ is compact.

Conversely, suppose $C_\varphi C_\psi^*$ is compact. Since

$$\begin{aligned} C_\varphi C_\sigma &= C_\varphi (C_\sigma^*)^* \\ &= C_\varphi T_{\tilde{h}} C_\psi^* T_{\tilde{g}}^* && (\text{by} \quad (1.1) \text{ and } C_\sigma^* = T_{\tilde{g}} C_\psi T_{\tilde{h}}^*) \\ &= T_{\tilde{h}\circ\varphi} C_\varphi C_\psi^* T_{\tilde{g}}^* && (\text{because} \quad C_\varphi T_{\tilde{h}} = T_{\tilde{h}\circ\varphi} C_\varphi). \end{aligned}$$

it follows that $C_\phi C_\sigma$ is compact.

Proof of Theorem 1.1: By Lemma 1.2 the operator $C_\varphi C_\psi^*$ is not compact if and only if $C_\varphi C_\sigma = C_{\sigma\circ\varphi}$ is not compact. Since $\sigma\circ\varphi$ is a linear fractional self map of U, $C_{\sigma\circ\varphi}$ is not compact if and only if $\sigma\circ\varphi$ maps a point of the unit circle onto the unit circle. So there exist points η_1 and $\eta_2 \in \partial U$ such that $\sigma\circ\varphi(\eta_1) = \eta_2$. Hence by (1.3) there exists $\omega \in \partial U$ such that $\varphi(\eta_1) = \omega = \psi(\eta_2)$, which completes the proof.

Theorem 1.3 *Suppose that φ and ψ are linear fractional self maps of U. Then $C_\psi^* C_\varphi$ is not compact if and only if there exist points ω_1 and $\omega_2 \in \partial U$ such that $\varphi^{-1}(\omega_1) = \psi^{-1}(\omega_2) \in \partial U$.*

Proof: First we will reduce the compactness of $C_\psi^* C_\varphi$ to that of the composition operator $C_\sigma C_\varphi = C_{\varphi\circ\sigma}$, where $C_\psi^* = T_g C_\sigma T_h^*$. Suppose $C_\psi^* C_\varphi$ is compact.

First we will show that $C_\psi^* C_\Sigma^*$ is compact. By (1.4), C_Σ^* can be written in the form $C_\Sigma^* = C_\varphi T_{\tilde{F}} T_{\tilde{H}}^*$ where $\tilde{F}(z) = \tilde{G}\circ\varphi(z)$. Thus

$$C_\psi^* C_\Sigma^* = C_\psi^* C_\varphi T_{\tilde{F}} T_{\tilde{H}}^*.$$

Thus $C_\psi^* C_\Sigma^*$ is compact.

Second we will show $C_\sigma C_\Sigma^*$ compact. We can apply Lemma 1.2 to conclude $C_\psi^* C_\Sigma^*$ is compact if and only if $C_\sigma C_\Sigma^*$ is compact; in more detail: $C_\psi^* C_\Sigma^* = (C_\Sigma C_\psi)^*$ is compact if and only if (by Lemma 1.2) $(C_\Sigma C_\sigma^*)^* = C_\sigma C_\Sigma^*$ is compact.

Finally we show $C_\sigma C_\varphi$ is compact by observing that

$$C_\sigma C_\varphi = C_\sigma (C_\varphi^*)^* = C_\sigma T_H C_\Sigma^* T_G^* = T_{H\circ\sigma} C_\sigma C_\Sigma^* T_G^*.$$

Thus $C_\sigma C_\varphi$ is compact.

For the other direction suppose $C_\sigma C_\varphi$ is compact, and note that by Lemma 1.2 $C_\sigma C_\varphi$ is compact if and only if $C_\sigma C_\Sigma^*$ is compact. Now we will show that if $C_\sigma C_\varphi$ and $C_\sigma C_\Sigma^*$ are both compact, then $C_\psi^* C_\varphi$ is compact.

By Cowen's Adjoint Theorem, $C_\psi^* = T_g C_\sigma T_h^*$ where $h(z) = cz + d$, thus

$$C_\psi^* C_\varphi = T_g C_\sigma T_h^* C_\varphi = \bar{c} T_g C_\sigma T_z^* C_\varphi + \bar{d} T_g C_\sigma C_\varphi. \tag{1.5}$$

By hypothesis $C_\sigma C_\varphi$ is compact so the second term in expression (1.5) is compact. In order to conclude $C_\psi^* C_\varphi$ is compact it suffices to show that the factor $C_\sigma T_z^* C_\varphi$ of the first term in expression (1.5) is compact. The key to this is the following calculation:

$$\begin{aligned} C_\sigma T_z^* C_\varphi &= C_\sigma T_z^* (C_\varphi^*)^* \\ &= C_\sigma T_z^* T_H C_\Sigma^* T_G^* \\ &= C_\sigma T_{\bar{z}H} C_\Sigma^* T_G^* \\ &= C_\sigma T_b^* C_\Sigma^* T_G^* \end{aligned}$$

where $b(z) = z\overline{H(z)}$. By Cowen's Adjoint Theorem we know that $H(z)$ has the form $Az + B$. Thus $b(z) = z\overline{H(z)} \in H^\infty$, which implies that $T_b^* C_\Sigma^* = C_\Sigma^* T_{b\circ\Sigma}^*$. Hence,

$$C_\sigma T_z^* C_\varphi = C_\sigma C_\Sigma^* T_{b\circ\Sigma}^* T_G^*.$$

By hypothesis $C_\sigma C_\Sigma^*$ is compact, hence $C_\psi^* C_\varphi$ is compact as desired.

Since $\varphi \circ \sigma$ is a linear fractional map of U, $C_{\varphi\circ\sigma}$ is not compact if and only if there exist points ω_1 and ω_2 on the unit circle such that $\varphi(\sigma(\omega_2)) = \omega_1$. Thus there exists $\eta \in \partial U$ such that $\varphi(\omega_1) = \eta$ and $\sigma(\omega_2) = \eta$. By (1.3) $\psi(\eta) = \omega_2$, which completes the proof.

We will see that both of these linear fractional map results are prototypes for more general theorems. Theorem 1.1 illustrates that the compactness of $C_\varphi C_\psi^*$ depends on the behavior of the range of φ and ψ. It illustrates the intuitive principle that if the sets $\varphi(U)$, $\psi(U)$, and ∂U are close then $C_\varphi C_\psi^*$ is not compact. Similarly, Theorem 1.3

points to the fact that the compactness of $C_\psi^* C_\varphi$ depends on the behavior in the domain of φ and ψ.

We now consider three examples that illustrate some differences between the compactness of the operators $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$. In all three examples φ and ψ are linear fractional self maps of the disc and the composition operators C_φ and C_ψ are not compact.

Example (1) $C_\psi^* C_\varphi$ **is not compact but** $C_\varphi C_\psi^*$ **is compact.**

Consider the operators induced by,

$$\varphi(z) = \frac{1+z}{2} \quad \text{and} \quad \psi(z) = \frac{-1-z}{2}.$$

The important points to notice are $\varphi(1) = 1$ and $\psi(1) = -1$. Since $\varphi(1) \neq \psi(1)$, $C_\varphi C_\psi^*$ is not compact by Theorem 1.1. On the other hand, $\varphi^{-1}(1) = \psi^{-1}(-1)$, so $C_\psi^* C_\varphi$ is compact by Theorem 1.3.

Example (2) $C_\psi^* C_\varphi$ **is compact but** $C_\varphi C_\psi^*$ **is not compact.**

Consider the operators induced by,

$$\varphi(z) = \frac{1+z}{2} \quad \text{and} \quad \psi(z) = \frac{1-z}{2},$$

where $\varphi(1) = 1$ and $\psi(-1) = 1$. Since $\varphi(1) = \psi(-1)$, $C_\varphi C_\psi^*$ is not compact by Theorem 1.1. On the other hand, $\varphi^{-1}(1) \neq \psi^{-1}(1)$, so $C_\psi^* C_\varphi$ is compact by Theorem 1.3.

Example (3) **Both** $C_\psi^* C_\varphi$ **and** $C_\varphi C_\psi^*$ **are compact.**

Consider the operators induced by,

$$\varphi(z) = \frac{1+z}{2} \quad \text{and} \quad \psi(z) = \frac{-1+z}{2},$$

where $\varphi(1) = 1$ and $\psi(-1) = -1$. Since $\varphi(1) \neq \psi(-1)$, $C_\varphi C_\psi^*$ is compact by Theorem 1.1. On the other hand, $\varphi^{-1}(1) \neq \psi^{-1}(1)$, so $C_\psi^* C_\varphi$ is compact by Theorem 1.3.

1.1 Comparison principle.

We will now use Theorem 1.1, the linear fractional theorem for $C_\varphi C_\psi^*$, to significantly enlarge the set of operators $C_\varphi C_\psi^*$ that we know are not compact. This is done in the lemma below, but the key idea is the following theorem.

Theorem 1.4 (Comparison Principle for the Compactness of $C_\varphi C_\psi^*$.) *Suppose φ and ψ are univalent holomorphic self maps of U, and α and β are holomorphic self maps of U such that $\alpha(U) \subset \varphi(U)$ and $\beta(U) \subset \psi(U)$. If $C_\varphi C_\psi^*$ is compact then so is $C_\alpha C_\beta^*$.*

Proof: Because φ and ψ are univalent, and the range of φ and ψ contains the range of α and β respectively, we can form

$$\begin{aligned} \tau_1(z) &= \varphi^{-1} \circ \alpha(z), \quad \text{and} \\ \tau_2(z) &= \psi^{-1} \circ \beta(z) \end{aligned}$$

both of which take U holomorphically into itself. Thus, $\alpha(z) = \varphi \circ \tau_1(z)$ and $\beta(z) = \psi \circ \tau_2(z)$, and at the operator level,

$$C_\alpha C_\beta^* = C_{\varphi \circ \tau_1} C_{\psi \circ \tau_2}^* = C_{\tau_1} C_\varphi C_\psi^* C_{\tau_2}^*.$$

Hence $C_\alpha C_\beta^*$ is compact.

Corollary 1.5 *Suppose φ and ψ are univalent holomorphic self maps of U, and*

$\varphi(U) \cap \psi(U)$ contains a disc that is tangent to the unit circle. Then $C_\varphi C_\psi^*$ is not compact.

Proof: Let Δ be a disc that is contained in $\varphi(U) \cap \psi(U)$ and is tangent to the unit circle. Then there exists a linear fractional map α that maps U onto Δ. Thus $\alpha(U) \subset \varphi(U) \cap \psi(U)$. Since $C_\alpha C_\alpha^*$ is not compact we conclude by the Comparison Principle that $C_\varphi C_\psi^*$ is not compact, which completes the proof.

Most theorems seem to come in pairs: one for the operator $C_\varphi C_\psi^*$ and one for the operator $C_\psi^* C_\varphi$ but we are unable to find a comparison principle for the operator $C_\psi^* C_\varphi$.

CHAPTER 2

Preliminaries

In this reference section we introduce our notation, and sketch the prerequisites for the rest of the paper.

2.1 An equivalent inner product on H^2.

The Hardy space H^2 is the collection of functions that are holomorphic in the unit disc U and whose Taylor coefficients in the expansion about the origin are square summable. H^2 is a Hilbert space where the inner product is defined by:

$$< f, g > \;=\; \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)},$$

with $\hat{f}(n)$ and $\hat{g}(n)$ denoting the n-th Taylor coefficient of f and g respectively. The *Littlewood-Paley identity* for the H^2 inner product is

$$< f, g >= f(0)\overline{g(0)} + \int_U f'(z)\overline{g'(z)} \log \frac{1}{|z|^2} dA(z) \tag{2.1}$$

with dA representing normalized Lebesgue area measure, $A(U) = 1$. A calculation with the Taylor series of f and g proves that these inner products are the same.

Moreover, when $f = g$ we obtain the Littlewood-Paley identity for the H^2 norm

$$||f||^2 = |f(0)|^2 + \int_U |f'(z)|^2 \log \frac{1}{|z|^2} dA(z). \tag{2.2}$$

2.2 Reproducing kernels and essential norm.

Let $K_a(z)$ be the reproducing kernel at the point $a \in U$ and let k_a be K_a divided by its norm,

$$k_a(z) = \frac{K_a(z)}{||K_a||} = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}, \qquad (z \in U).$$

Let $||T||_e$ denote the essential norm of the operator T on H^2 (recall that this is the distance in the operator norm from the compact operators). Since k_a converges to zero uniformly on compact subsets of U as $|a| \to 1^-$ and $||k_a|| = 1$ for all $a \in U$, it converges weakly to zero as $|a| \to 1^-$. Thus, $||Ak_a|| \to 0$ for every compact operator A on H^2, hence

$$\begin{aligned} ||T||_e &\geq ||T + A|| \\ &\geq ||(T+A)k_a||. \end{aligned}$$

Hence,

$$||T||_e \geq \limsup_{|a|\to 1} ||Tk_a||.$$

A result that follows immediately from the proof in [16] of Shapiro's essential norm formula is,

$$\limsup_{|a|\to 1} ||C_\varphi k_a||^2 = \limsup_{|a|\to 1} \frac{N_\varphi(a)}{\log \frac{1}{|a|}}.$$

Hence a composition operator's action on the normalized reproducing kernels of H^2

completely determines the essential norm.

2.3 Nevanlinna counting function.

For a holomorphic self map φ of the open disc U, we define the Nevanlinna counting function of φ by:

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|}, \qquad w \in U\setminus\{\varphi(0)\}$$

where $\varphi^{-1}(w)$ denotes the set of φ-preimages of w counting the multiplicity, and $N_\varphi(w) = 0$ if $w \notin \varphi(U)$. One of the main ingredients in the proof of the essential norm formula of a composition operator is the the following property of N_φ:

Sub-mean-value property ([16], Theorem 4.6, page 390) *If Δ is a disk in U not containing $\varphi(0)$, with center a, then*

$$N_\varphi(a) \leq \frac{1}{|\Delta|}\int_\Delta N_\varphi(w)dA(w)$$

where $|\Delta|$ is normalized area measure of Δ.

The sub-mean-value property is used in [16] to establish the lower estimate,

$$||C_\varphi||_e^2 \geq \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log\frac{1}{|w|}}.$$

The next proposition follows directly from the proof of the lower estimate in [16] and is a consequence of the counting function's sub-mean-value property. Set $c_r(a) = \dfrac{2r^2|a|^2}{(1+|a|)}\dfrac{\log\frac{1}{|a|}}{(1-|a|)}$ and recall k_a is the normalized reproducing kernel for the point $a \in U$.

Proposition 1 *For* $0 < r < 1$ *there exists* $\delta > 0$ *such that,*

$$\|C_\varphi k_a\|^2 \geq c_r(a)\frac{N_\varphi(a)}{\log\frac{1}{|a|}} \quad \text{for all} \quad 1-\delta \leq |a| \leq 1.$$

Note that $\lim_{|a|\to 1^-} c_r(a) = r^2$.

2.4 Change of variables formula.

The following non-univalent change of variable formula was first used in the study of composition operators by Shapiro in [16].

Theorem([16], Theorem 4.3, page 389) *If* F *is a positive measurable function on* U *and* φ *is an holomorphic self map of* U, *then:*

$$\int_U (F\circ\varphi)|\varphi'|^2 \log\frac{1}{|z|^2}dA(z) = 2\int_U F(w)N_\varphi(w)dA(w). \tag{2.3}$$

The following calculation establishes the connection between composition operators and the Nevanlinna counting function. By applying the Littlewood-Paley identity(2.2) for the H^2 norm to $C_\varphi f = f\circ\varphi$ we obtain,

$$\begin{aligned}
\|f\circ\varphi\|^2 &= \int_U |(f\circ\varphi)'(z)|^2 \log\frac{1}{|z|^2}dA(z) + |f(\varphi(0))|^2 \\
&= \int_U |f'\circ\varphi(z)|^2|\varphi'(z)|^2 \log\frac{1}{|z|^2}dA(z) + |f(\varphi(0))|^2 \\
&= 2\int_U |f'(w)|^2 N_\varphi(w)dA(w) + |f(\varphi(0))|^2
\end{aligned}$$

where the last line follows from the change of variables formula (2.3), with $g = |f'|^2$. Hence,

$$\|f\circ\varphi\|^2 = 2\int_U |f'(w)|^2 N_\varphi(w)dA(w) + |f(\varphi(0))|^2 \qquad f\in H^2. \tag{2.4}$$

2.5 Littlewood's Inequality.

Littlewood [8] in 1925 established the boundedness of composition operators on the Hardy Spaces. The key to the proof is a result called Littlewood's inequality, proofs and development of this result can be found in [8], [16], and [18].

Littlewood's Inequality.([16], Theorem 2.2, page 380) *If φ is a holomorphic self map of the disc, then for each $z \in U \setminus \{\varphi(0)\}$,*

$$N_\varphi(z) \leq \log\left|\frac{1 - \overline{z}\varphi(0)}{z - \varphi(0)}\right|. \tag{2.5}$$

In the case when $\varphi(0) = 0$, Littlewood's inequality simplifies to

$$N_\varphi(z) \leq \log\frac{1}{|z|} \qquad \text{for} \qquad z \in U \setminus \{0\}. \tag{2.6}$$

An immediate observation from Littlewood's inequality and the boundedness of $\log|z|$ near ∂U is that the Nevanlinna counting function is bounded near the boundary of the unit disc. More precisely for each $|\varphi(0)| < r < 1$ there exists a positive constant C such that

$$N_\varphi(z) \leq C \quad \text{for all} \quad |\varphi(0)| < r < 1. \tag{2.7}$$

We now prove a lemma that we will use in the proof of Theorem 5.5.

Lemma 2.1 *If φ is a holomorphic self map of U, then*

$$\int_U N_\varphi(z)dA(z) \leq \frac{1}{2}(1 - |\varphi(0)|^2).$$

Proof: Set $\alpha_z(w) = \dfrac{w - z}{1 - \overline{w}z}$, and notice that we can write Littlewood's inequality as

$$N_\varphi(z) \le \log\left|\frac{1}{\alpha_{\varphi(0)}(z)}\right| \quad \text{for all} \quad z \in U \setminus \varphi(0).$$

By applying successively Littlewood's inequality, the change of variables $w = \alpha_{\varphi(0)}(z)$, the Littlewood-Paley indentity (2.2) for the H^2 norm, we obtain the desired result,

$$\begin{aligned}
\int_U N_\varphi(z)dA(z) &\le \int_U \log\left|\frac{1}{\alpha_{\varphi(0)}(z)}\right| dA(z) \\
&\le \int_U \log\frac{1}{|w|}|\alpha'_{\varphi(0)}(w)|^2 dA(w) \\
&= \frac{1}{2}\int_U |\alpha'_{\varphi(0)}(w)|^2 \log\frac{1}{|w|^2} dA(w) \\
&= \frac{1}{2}\left(\|\alpha_{\varphi(0)}\|^2 - |\alpha_{\varphi(0)}(0)|^2\right) \\
&= \frac{1}{2}\left(1 - |\varphi(0)|^2\right).
\end{aligned}$$

2.6 Angular derivative.

We say φ has a *finite angular derivative* at a point $\zeta \in \partial U$ if there is a point $\omega \in \partial U$ such that the difference quotient

$$\frac{\varphi(z) - \omega}{z - \zeta}$$

has a finite limit as z tends non tangentially to ζ. The connection between composition operators and angular derivative depends heavily on the following classical theorem of Julia and Caratheódory.

Julia-Carathеódory Theorem.([18], Section 4.2, page 57) *For $\zeta \in \partial U$, the following conditions are equivalent:*

1. *φ has a finite angular derivative at ζ.*

2. *φ has a nontangential limit of modulus 1 at ζ, and the complex derivative φ' has a finite limit at ζ. In this case the limit of φ' is $\varphi'(\zeta)$.*

3. $\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = d < \infty$. *In this case, $\varphi'(\zeta) = \varphi(\zeta)\bar{\zeta}d$.*

(For more information on the Julia-Carathеódory Theorem and its connection with composition operators see [16], Section 3 or [18], Chapter 4)

The Julia-Carathеódory Theorem allows us to think of $|\varphi'|$ as a function mapping the unit circle to $(0,\infty]$. In the case when φ is univalent it is shown in [3] that the essential norm of C_φ can be computed explicitly in terms of the angular derivative of φ. We reproduce part of the proof below. The argument relies on the fact that if φ is univalent, then $|\varphi'|$ is lower semicontinuous, a proof of which can be found in [3].

Theorem A ([3]) *Suppose φ is univalent. Then*

$$||C_\varphi||_e^2 = \left[\min_{\zeta\in\partial U} |\varphi'|\right]^{-1}$$

Proof: Applying Shapiro's essential norm formula equation (1) of Chapter 1, and noting that for univalent functions the Nevanlinna counting function simplifies to $\log(1/|z|)$ where $z = \varphi^{-1}(w)$ (with the understanding that $\log(1/|z|)$ is zero if w is not in the image of φ) we obtain

$$||C_\varphi||_e^2 = \limsup_{|w|\to 1^-} \frac{\log\frac{1}{|z|}}{\log\frac{1}{|w|}} = \limsup_{|z|\to 1^-} \frac{1-|z|}{1-|\varphi(z)|} = \left[\liminf_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|}\right]^{-1}.$$

Upon applying the Julia-Carathеódory Theorem to the term on the right, and noting that by the lower semicontinuity $|\varphi'|$ obtains its infimum on ∂U, we see

$$\|C_\varphi\|_e^2 = \left[\min_{\zeta\in\partial U} |\varphi'|\right]^{-1}.$$

CHAPTER 3

The Angular Derivative and the Essential Norm

We now develop the link between the angular derivatives of φ and ψ and the conjectured essential norm formulas, equations (2) and (3) of the Introduction.

We use the following notation for nontangential approach regions: For $0 < \rho < 1$, let $A_\rho(\zeta)$ be the convex hull of the disc ρU and the point ζ. For $0 < r < 1$, let $A_{\rho,r}(\zeta) = A_\rho(\zeta) \setminus rU$. Let $\varphi^*(\zeta)$ denote the nontangential limit (when it exists) of $\varphi(z)$. By Fatou's Theorem this limit exists for a.e. $\zeta \in \partial U$.

For $\omega \in \partial U$ we define

$$E(\varphi, \omega) = \{\zeta \in \partial U, \varphi^*(\zeta) = \omega\}$$

with the understanding that this set is empty if ω is not a nontangential limiting value of φ. Now we define for $\omega \in \partial U$:

$$\delta(\varphi, \omega) = \left\{ \sum \frac{1}{|\varphi'(\zeta)|} \ : \ \zeta \in E(\varphi, \omega) \right\},$$

where $1/|\varphi'(\zeta)| = 0$ if φ does not have a finite angular derivative at ζ, and $\delta(\varphi, \omega) = 0$ if $E(\varphi, \omega)$ is empty.

Our goal in this chapter is to generalize the following result;

Theorem ([16], Theorem 3.3) *Suppose φ is a holomorphic self map of U. Then*

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} \geq \sup \{\delta(\varphi, \omega) : \omega \in \partial U\}.$$

We will give two different generalizations of this result, the first of which is:

Theorem 3.1 *Suppose that φ and ψ are holomorphic self maps of the disc. Then*

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w) N_\psi(w)}{\left(\log \frac{1}{|w|}\right)^2} \geq \sup \{\delta(\varphi, \omega)\delta(\psi, \omega) \ : \ \omega \in \partial U\}.$$

Proof: Fix $\omega \in \partial U$ a nontangential limiting value of both φ and ψ. Suppose $\{\zeta_k\}_{k=1}^n \subset E(\varphi, \omega)$ and $\{\eta_k\}_{k=1}^m \subset E(\psi, \omega)$ such that φ has a finite angular derivative at ζ_k, $1 \leq k \leq n$ and ψ has a finite angular derivative at η_k, $1 \leq k \leq m$. Fix $0 < \rho < 1$, and choose $0 < t < 1$ so that the angular regions $A_k = A_{\rho,t}(\zeta_k)$ are disjoint for $1 \leq k \leq n$, and similarly $B_k = A_{\rho,t}(\eta_k)$, $1 \leq k \leq m$. Corollary 3.2 of [16] insures that

$$\left(\bigcap \{\varphi(A_k) \ : \ 1 \leq k \leq n\}\right) \bigcap \left(\bigcap \{\psi(B_k) \ : \ 1 \leq k \leq m\}\right)$$

contains a nontangential approach region A with vertex ω.

For $w \in A \setminus \{\varphi(0), \psi(0)\}$ choose a set of preimages of w for each inducing map φ and ψ, $\{z_k(w)\}_{k=1}^n$ and $\{u_k(w)\}_{k=1}^m$, such that

$$\varphi(z_k(w)) = w \quad \text{and} \quad z_k(w) \in A_k, \quad k = 1, ..., n$$

$$\psi(u_k(w)) = w \quad \text{and} \quad u_k(w) \in B_k, \quad k = 1, ..., m.$$

By the definition of the Nevanlinna counting function we see:

$$N_\varphi(w) \geq \sum_{k=1}^{n} \log \frac{1}{|z_k(w)|} \qquad \text{and} \qquad N_\psi(w) \geq \sum_{k=1}^{m} \log \frac{1}{|u_k(w)|}. \tag{3.1}$$

For fixed k, we know by the Schwarz Lemma that $z_k(w) \to \zeta_k$ and $u_k(w) \to \eta_k$ through A_k and B_k respectively as $w \to \omega$ through A. Thus by the Julia-Carathеódory theorem:

$$\lim_{w\to\omega,\ w\in A} \frac{\log \frac{1}{|z_k(w)|}}{\log \frac{1}{|w|}} = |\varphi'(\zeta_k)|^{-1} \tag{3.2}$$

$$\lim_{w\to\omega,\ w\in A} \frac{\log \frac{1}{|u_k(w)|}}{\log \frac{1}{|w|}} = |\psi'(\eta_k)|^{-1}. \tag{3.3}$$

Applying (3.1), (3.2), and (3.3) we obtain:

$$\begin{aligned}
\limsup_{|w|\to 1^-} \frac{N_\varphi(w)N_\psi(w)}{\left(\log \frac{1}{|w|}\right)^2} &\geq \limsup_{w\to\omega,\ w\in A} \sum_{k=1}^{n} \frac{\log \frac{1}{z_k(w)}}{\log \frac{1}{|w|}} \sum_{k=1}^{m} \frac{\log \frac{1}{u_k(w)}}{\log \frac{1}{|w|}} \\
&= \sum_{k=1}^{n} \lim_{w\to\omega,\ w\in A} \frac{\log \frac{1}{z_k(w)}}{\log \frac{1}{|w|}} \sum_{k=1}^{m} \lim_{w\to\omega,\ w\in A} \frac{\log \frac{1}{u_k(w)}}{\log \frac{1}{|w|}} \\
&= \sum_{k=1}^{n} \frac{1}{|\varphi'(\zeta_k)|} \sum_{k=1}^{m} \frac{1}{|\psi'(\eta_k)|}.
\end{aligned}$$

Now take the supremum over m and n and then the supremum over $\omega \in \partial U$ to finish the proof.

An immediate consequence of the proof of Theorem 3.1 is;

Corollary 3.2 *If there exist three points ζ, η, and ω on the unit circle such that $\varphi(\zeta) = \psi(\eta) = \omega$ and $\varphi'(\zeta)$ and $\psi'(\eta)$ exist, then*

$$\limsup_{w\to\omega} \frac{N_\varphi(w)N_\psi(w)}{\left(\log \frac{1}{|w|}\right)^2} \geq \frac{1}{|\varphi'(\zeta)\psi'(\eta)|}.$$

Our second generalization is:

Theorem 3.3 *Suppose that φ and ψ are holomorphic self maps of the disc. Then*

$$\limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z))N_\psi(\psi(z))}{\log\frac{1}{|\varphi(z)|}\log\frac{1}{|\psi(z)|}} \geq \sup\{\delta(\varphi,\varphi(\zeta))\delta(\psi,\psi(\zeta)) \ : \ \zeta \in \partial U\}.$$

Proof: Fix $\zeta \in \partial U$ such that both φ and ψ have a finite angular derivative at ζ. Let $\omega_1 = \varphi(\zeta)$ and $\omega_2 = \psi(\zeta)$ be the nontangential limiting values of φ and ψ as $z \to \zeta$ nontangentially. Suppose $\{\zeta_k\}_{k=1}^n \subset E(\varphi,\omega_1)$ and $\{\eta_k\}_{k=1}^m \subset E(\psi,\omega_2)$ are such that φ has a finite angular derivative at ζ_k, $1 \leq k \leq n$, and ψ has a finite angular derivative at η_k, $1 \leq k \leq m$.

Fix $0 < \rho < 1$, and choose $0 < t < 1$ so that the angular regions $A_k = A_{\rho,t}(\zeta_k)$ are disjoint for $1 \leq k \leq n$, and similarly for $B_k = A_{\rho,t}(\eta_k)$, $1 \leq k \leq m$. Corollary 3.2 of [16] insures that the set

$$\bigcap\{\varphi(A_k) \ : \ 1 \leq k \leq n\}$$

contains a nontangential approach region A_φ with vertex ω_1, and that the set

$$\bigcap\{\psi(B_k) \ : \ 1 \leq k \leq m\}$$

contains a nontangential approach region B_ψ with vertex ω_2. For a point $z \in U$ such that $\varphi(z) \in A_\varphi$ and $\psi(z) \in B_\psi$, choose a set of φ-preimages $\{v_k(\varphi(z))\}_{k=1}^n$ for φ, and choose a set of ψ-preimages $\{u_k(\psi(z))\}_{k=1}^m$ of $\psi(z)$, so that

$$\varphi(v_k(\varphi(z))) = \varphi(z) \quad \text{and} \quad v_k(\varphi(z)) \in A_k, \quad k = 1, ..., n$$

$$\psi(u_k(\psi(z))) = \psi(z) \quad \text{and} \quad u_k(\psi(z)) \in B_k, \quad k = 1, ..., m.$$

By the definition of the Nevanlinna counting function we see:

$$N_\varphi(\varphi(z)) \geq \sum_{k=1}^{n} \log \frac{1}{v_k(\varphi(z))} \quad \text{and} \quad N_\psi(\psi(z)) \geq \sum_{k=1}^{m} \log \frac{1}{u_k(\psi(z))}. \tag{3.4}$$

For fixed k, we know by Schwarz Lemma that $v_k(\varphi(z)) \to \zeta_k$ and $u_k(\psi(z)) \to \eta_k$ through A_k and B_k respectively as $z \to \zeta$ through $A = \varphi^{-1}(A_\varphi) \cap \psi^{-1}(B_\psi)$. Thus by the Julia-Caratheódory theorem:

$$\lim_{z\to\zeta,\ z\in A} \frac{\log \frac{1}{|v_k(\varphi(z))|}}{\log \frac{1}{|\varphi(z)|}} = |\varphi'(\zeta_k)|^{-1} \tag{3.5}$$

$$\lim_{z\to\zeta,\ z\in A} \frac{\log \frac{1}{|u_k(\psi(z))|}}{\log \frac{1}{|\psi(z)|}} = |\psi'(\eta_k)|^{-1}. \tag{3.6}$$

Applying (3.4), (3.5), and (3.6) we obtain:

$$\begin{aligned}
\limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z))N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} &\geq \limsup_{z\to\zeta, z\in A} \sum_{k=1}^{n} \frac{\log \frac{1}{v_k(\varphi(z))}}{\log \frac{1}{|\varphi(z)|}} \sum_{k=1}^{m} \frac{\log \frac{1}{u_k(\psi(z))}}{\log \frac{1}{|\psi(z)|}} \\
&= \sum_{k=1}^{n} \lim_{z\to\zeta, z\in A} \frac{\log \frac{1}{v_k(\varphi(z))}}{\log \frac{1}{|\varphi(z)|}} \sum_{k=1}^{m} \lim_{z\to\zeta, z\in A} \frac{\log \frac{1}{u_k(\psi(z))}}{\log \frac{1}{|\psi(z)|}} \quad (z \in A) \\
&= \sum_{k=1}^{n} \frac{1}{|\varphi'(\zeta_k)|} \sum_{k=1}^{m} \frac{1}{|\psi'(\eta_k)|}.
\end{aligned}$$

Now take the supremum over n and m, and then the supremum over $\zeta \in \partial U$ to finish the proof.

An immediate corollary of the proof of Theorem 3.3 is

Corollary 3.4 *If φ and ψ have an angular derivative at the point $\zeta \in \partial U$ then*

$$\limsup_{z\to\zeta} \frac{N_\varphi(\varphi(z))N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} \geq \frac{1}{|\varphi'(\zeta)\psi'(\zeta)|}.$$

CHAPTER 4

The Operator $C_\psi^* C_\varphi$ on H^2

This chapter is broken up into three parts: In the first section we outline the main results for the operator $C_\psi^* C_\varphi$, and using Theorem 3.3 we develop the connection between the compactness of $C_\psi^* C_\varphi$ and the angular derivative of the inducing maps. In the second section we establish an upper bound on the essential norm of $C_\psi^* C_\varphi$, and in the third section we prove a necessary condition for the operator to be compact.

4.1 Main results for $C_\psi^* C_\varphi$.

Theorem 4.1 *Suppose that φ and ψ are holomorphic self maps of U. Then*

$$\|C_\psi^* C_\varphi\|_e \;\leq\; \limsup_{|z|\to 1^-} \left(\frac{N_\varphi(\varphi(z)) N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} \right)^{1/2}.$$

Corollary 4.2 *Suppose that φ and ψ are holomorphic self maps of U, and*

$$\limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z)) N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} = 0.$$

Then $C_\psi^* C_\varphi$ is compact on H^2.

Remark. In the case when $\psi = \varphi$, Shapiro's essential norm formula tells us that,

$$\|C_\psi^* C_\varphi\|_e = \|C_\varphi\|_e^2 = \limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z))}{\log \frac{1}{|\varphi(z)|}}.$$

hence the estimate of Theorem 4.1 is sharp. We now consider the form that Theorem 4.1 takes when the inducing maps are univalent. If φ is univalent then the Nevanlinna counting function of φ, evaluated at $\varphi(z)$, simplifies to $\log \frac{1}{|z|}$, i.e. $N_\varphi(\varphi(z)) = \log \frac{1}{|z|}$. Thus if φ and ψ are univalent the above upper bound on the essential norm simplifies to;

$$\begin{aligned} \limsup_{|z|\to 1^-} \frac{N_\varphi(\varphi(z))N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} &= \limsup_{|z|\to 1^-} \frac{\left(\log \frac{1}{|z|}\right)^2}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} \\ &= \limsup_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)}. \end{aligned}$$

Hence an immediate corollary of Theorem 4.1 is;

Corollary 4.3 *Suppose φ and ψ are univalent self maps of the disc, and*

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0.$$

Then $C_\psi^* C_\varphi$ is compact.

The next theorem is stronger than the converse of Corollary 4.3 in that we only assume one of the inducing maps is univalent.

Theorem 4.4 *Suppose φ and ψ are self maps of the disc and one of φ or ψ is univalent. If $C_\psi^* C_\varphi$ is compact on H^2 then*

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0.$$

By Theorem 3.3 this says that if $C_\psi^* C_\varphi$ is compact then φ and ψ do not have an angular derivative at the same point.

Putting together Theorem 4.1 and Theorem 4.4 we obtain the following complete function theoretic characterization of compactness when the inducing maps are univalent.

Corollary 4.5 *Suppose φ and ψ are univalent self maps of the disc. Then $C_\psi^* C_\varphi$ is compact on H^2 if and only if*

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0.$$

In terms of the angular derivative of φ and ψ we obtain the equivalent statement:

Corollary 4.6 *Suppose φ and ψ are univalent self maps of the disc. Then $C_\psi^* C_\varphi$ is not compact if and only if φ and ψ have an angular derivative at the same point.*

To connect Corollaries 4.5 and 4.6 with the work we did in Chapter 1, note that if φ is a linear fractional self map of the disc then the existence of the angular derivative is equivalent to φ mapping a point of the unit circle onto the unit circle. Thus if φ and ψ are linear fractional self maps of U, Corollary 4.6 implies that $C_\psi^* C_\varphi$ is not compact if and only if there exists a point $\eta \in \partial U$ such that $\varphi(\eta) \in \partial U$ and $\psi(\eta) \in \partial U$. Equivalently, there exist points ω_1 and $\omega_2 \in \partial U$ such that $\varphi^{-1}(\omega_1) = \psi^{-1}(\omega_2) \in \partial U$, which is Theorem 1.3.

We now turn our attention to the proofs of the theorems.

4.2 Upper estimate on the essential norm of $C_\psi^* C_\varphi$.

The proof of Theorem 4.1 is based on the approach Shapiro used in [16] to find the upper estimate of the essential norm of a composition operator. We will use the following general formula for the essential norm of a linear operator on a Hilbert space.

Proposition 2 *Suppose T is a bounded linear operator on a Hilbert space H. Let $\{K_n\}$ be a sequence of compact self-adjoint operators on H, and write $R_n = I - K_n$. Suppose $\|R_n\| = 1$ for each n, and $\|R_n x\| \to 0$ for each $x \in H$. Then $\|T\|_e = \lim_n \|R_n T R_n\|$.*

The proof of Proposition 2 follows directly from the proof of Proposition 5.1 in [16]. We also require the following estimates on H^2 functions, whose proof can be found in [16].

Proposition 3 *Suppose $f \in H^2$ has a zero of order at least n at the origin. Then for each $z \in U$:*

1. $|f(z)| \leq |z|^n(1 - |z|^2)^{-\frac{1}{2}}\|f\|_2,$ *and*

2. $|f'(z)| \leq \sqrt{2}n|z|^{n-1}(1 - |z|^2)^{-\frac{3}{2}}\|f\|_2.$

Proof of Theorem 4.1: We will show that

$$\|C_\psi^* C_\varphi\|_e \leq \limsup_{|z|\to 1} \left(\frac{N_\varphi(\varphi(z)) N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} \right)^{1/2}.$$

This will be done by applying Proposition 2 with K_n the operator that takes f to the nth partial sum of its Taylor series:

$$K_n f(z) = \sum_{k=0}^{n} \hat{f}(k) z^k, \quad \text{where} \quad f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in H^2.$$

Since K_n is the orthogonal projection of H^2 onto the closed subspace spanned by the monomials $1, z, ..., z^n$, it is self-adjoint and compact. Since $R_n = I - K_n$ is the complementary projection, its norm is 1. Thus the hypotheses of Proposition 2 are fulfilled, so that

$$\|C_\psi^* C_\varphi\|_e \le \lim_n \|R_n C_\psi^* C_\varphi R_n\| = \lim_n \sup_{f,g \in (H^2)_1} | < C_\varphi R_n f, C_\psi R_n g > | \tag{4.1}$$

where $(H^2)_1$ is the unit ball of H^2. To estimate the inner product on the right hand side of equation (4.1) it is enough to consider the supremum over the unit ball of H_0^2 because $R_n f$ and $R_n g$ are in the unit ball of H_0^2 for all $n \ge 1$ and for all f and g in $(H^2)_1$. Therefore, to estimate the inner product, fix the functions f and g in the unit ball of H_0^2, and a positive integer n, and use the Littlewood-Paley identity for the H^2 inner product to obtain,

$$| < C_\varphi R_n f, C_\psi R_n g > | \le |R_n f(\varphi(0)) R_n g(\psi(0))| \tag{4.2}$$

$$+ \int_U |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z). \tag{4.3}$$

Since $R_n f$ and $R_n g$ are in the H^2 unit ball and both have a zero of order n at the origin, Proposition 3 implies that

$$|R_n f(\varphi(0))| \le |\varphi(0)|^n / \sqrt{(1 - |\varphi(0)|^2)}, \tag{4.4}$$

and

$$|(R_n f)'(z)|^2 \le 2n^2 |z|^{2(n-1)} / (1 - |z|^2)^3, \tag{4.5}$$

and similarly for $|R_n g(\psi(0))|$ and $|(R_n g)'(z)|$. Now fix $0 < r < 1$ and split the integral on the right side of equation (4.3) into two parts: one over the disc rU and the other over its complement. Use estimate (4.4) on the first term of equation (4.3) to obtain,

$$\begin{aligned} |< C_\varphi R_n f, C_\psi R_n g >| &\leq \frac{|\varphi(0)|^n |\psi(0)|^n}{\sqrt{(1-|\varphi(0)|^2)(1-|\psi(0)|^2)}} \\ &+ \int_{rU} |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z) \qquad (4.6) \\ &+ \int_{U \setminus rU} |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z). \end{aligned}$$

The first term on the right hand side has limit zero as n tends to infinity, so we need only be concerned with the two integrals. Let I represent the integral over rU. Set $\rho = \sup \{\max(|\varphi(z)|, |\psi(z)|) : z \in rU\}$, which is clearly less than one. To estimate I, use the Cauchy-Schwartz inequality, the change of variables formula (2.3), and estimate (4.5) to obtain

$$\begin{aligned} I &= \int_{rU} |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z) \\ &\leq \left(\int_{rU} |(C_\varphi R_n f)'(z)|^2 \log \frac{1}{|z|^2} dA(z)\right)^{1/2} \left(\int_{rU} |(C_\psi R_n g)'(z)|^2 \log \frac{1}{|z|^2} dA(z)\right)^{1/2} \\ &\leq 2 \left(\int_{\rho U} |(R_n f)'(w)|^2 N_\varphi(w) dA(w)\right)^{1/2} \left(\int_{\rho U} |(R_n g)'(w)|^2 N_\psi(w) dA(w)\right)^{1/2} \\ &\leq \frac{4n^2 \rho^{2(n-1)}}{(1-\rho^2)^3} \left(\int_U N_\varphi(w) dA(w)\right)^{1/2} \left(\int_U N_\psi(w) dA(w)\right)^{1/2} \\ &\leq \frac{2n^2 \rho^{2(n-1)}}{(1-\rho^2)^3}. \end{aligned}$$

The last inequality follows from Littlewood's inequality (Lemma 2.1, Section 2.5). Thus the supremum over f and g in the unit ball of H_0^2 of the integral I is bounded by an expression whose limit is zero as n tends to infinity.

A note about notation; an unadorned "sup" will mean the supremum over f and g in the unit ball of H_0^2 throughout this section.

We have

$$\sup \int_{rU} |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z) \leq \frac{2n^2 \rho^{2(n-1)}}{(1-\rho^2)^3}. \tag{4.7}$$

Therefore

$$\begin{aligned} \|R_n C_\psi^* C_\varphi R_n\| &\leq \sup \int_{U \setminus rU} |(C_\varphi R_n f)'(z)(C_\psi R_n g)'(z)| \log \frac{1}{|z|^2} dA(z) \\ &+ \frac{2n^2 \rho^{2(n-1)}}{(1-\rho^2)^3} + \frac{|\varphi(0)|^n |\psi(0)|^n}{\sqrt{(1-|\varphi(0)|^2)(1-|\psi(0)|^2)}}. \end{aligned}$$

We may replace $R_n f$ by f and $R_n g$ by g in the above integral because f and g range over a larger set than their projections $R_n f$ and $R_n g$. Hence,

$$\begin{aligned} \|R_n C_\psi^* C_\varphi R_n\| &\leq \sup \int_{U \setminus rU} |(C_\varphi f)'(z)(C_\psi g)'(z)| \log \frac{1}{|z|^2} dA(z) \\ &+ \frac{2n^2 \rho^{2(n-1)}}{(1-\rho^2)^3} + \frac{|\varphi(0)|^n |\psi(0)|^n}{\sqrt{(1-|\varphi(0)|^2)(1-|\psi(0)|^2)}}. \end{aligned}$$

Now let n tend to infinity. Because ρ, $|\psi(0)|$, and $|\varphi(0)|$ are less than one, we conclude

$$\|C_\psi^* C_\varphi\|_e \leq \sup \int_{U \setminus rU} |(f \circ \varphi)'(z)(g \circ \psi)'(z)| \log \frac{1}{|z|^2} dA(z). \tag{4.8}$$

To finish the proof set $h_\varphi(z) = \dfrac{N_\varphi(z)}{\log \frac{1}{|z|}}$ and

$$H(z) = (h_\varphi(\varphi(z)) h_\psi(\psi(z)))^{1/2} = \left(\frac{N_\varphi(\varphi(z)) N_\psi(\psi(z))}{\log \frac{1}{|\varphi(z)|} \log \frac{1}{|\psi(z)|}} \right)^{1/2}.$$

We have

$$\|C_\psi^* C_\varphi\|_e \leq \sup \int_{U \setminus rU} |(f \circ \varphi)'(g \circ \psi)'| \log \frac{1}{|z|^2} dA(z)$$

$$
\begin{aligned}
&= 2\sup\int_{U\setminus rU} |(f\circ\varphi)'(g\circ\psi)'|H(z)\frac{1}{H(z)}\log\frac{1}{|z|}dA(z)\\
&\leq 2\sup_{r\leq|z|<1}\{H(z)\}\sup\int_{U\setminus rU} |(f\circ\varphi)'(g\circ\psi)'|\frac{1}{H(z)}\log\frac{1}{|z|}dA(z)\\
&\leq 2\sup_{r\leq|z|<1}\{H(z)\}\sup\int_U \frac{|(f\circ\varphi)(z)'|}{(h_\varphi(\varphi(z)))^{\frac{1}{2}}}\frac{|(g\circ\psi)'(z)|}{(h_\psi(\psi(z)))^{\frac{1}{2}}}\log\frac{1}{|z|}dA(z)\\
&\leq 2\sup_{r\leq|z|<1}\{H(z)\}\sup\left(\int_U |(f\circ\varphi)'|^2\frac{\log\frac{1}{|\varphi(z)|}}{N_\varphi(\varphi(z))}\log\frac{1}{|z|}dA(z)\right)^{1/2}\\
&\left(\int_U |(g\circ\psi)'|^2\frac{\log\frac{1}{|\psi(z)|}}{N_\psi(\psi(z))}\log\frac{1}{|z|}dA(z)\right)^{1/2},
\end{aligned}
$$

where the last line follows from the Cauchy-Schwarz inequality. Now we will calculate the two integrals in the last expression above; because the calculations are similar we will only explicitly compute the first integral. To do this, use the change of variables formula (2.3) and the Littlewood-Paley identity (2.2) for the norm to obtain ,

$$
\begin{aligned}
&\int_U |(f\circ\varphi)'|^2\frac{\log\frac{1}{|\varphi(z)|}}{N_\varphi(\varphi(z))}\log\frac{1}{|z|}dA(z)\\
&= \int_U |f'\circ\varphi(z)|^2|\varphi'(z)|^2\frac{\log\frac{1}{|\varphi(z)|}}{N_\varphi(\varphi(z))}\log\frac{1}{|z|}dA(z) \qquad \text{let} \quad w=\varphi(z)\\
&= \int_U |f'(w)|^2\frac{\log\frac{1}{|w|}}{N_\varphi(w)}N_\varphi(w)dA(w)\\
&= \int_U |f'(w)|^2\log\frac{1}{|w|}dA(w)\\
&= \frac{1}{2}\|f\|^2.
\end{aligned}
$$

Similarly,

$$
\int_U |(g\circ\psi)'(z)|^2\frac{\log\frac{1}{|\psi(z)|}}{N_\psi(\psi(z))}\log\frac{1}{|z|}dA(z) \;=\; \frac{1}{2}\|g\|^2.
$$

Since $\|f\| = \|g\| = 1$, we get

$$\|C_\psi^* C_\varphi\|_e \leq \sup\{H(z) : r \leq |z| < 1\},$$

and the desired result follows upon letting r tend to 1^-.

4.3 Necessary condition for $C_\psi^* C_\varphi$ to be compact.

We will prove the contrapositive form of Theorem 4.4:

Suppose φ or ψ is univalent. If φ and ψ have a finite angular derivative at the same point of ∂U then $C_\psi^* C_\varphi$ is not compact on H^2.

Suppose φ is univalent and let $\eta \in \partial U$ be such that $\varphi'(\eta)$ and $\psi'(\eta)$ exist. We may assume the following;

(1) $\eta = 1$ (by rotations)

(2) $\varphi(1) = 1$ (by rotations)

(3) $\varphi'(1) = 1$ (by hyperbolic automorphism)

(4) $\|C_\varphi\|_e = 1$ (by parabolic non-automorphism)

The first three modifications above will be obtained by multiplying $C_\psi^* C_\varphi$ by an invertible composition operator or by the adjoint of an invertible composition operator thus not changing the compactness of $C_\psi^* C_\varphi$. The fourth modification will involve multiplying $C_\psi^* C_\varphi$ by a non-invertible composition operator, this is not a problem because if the product of $C_\psi^* C_\varphi$ with any operator is not compact then $C_\psi^* C_\varphi$ can not be compact.

To see why we may assume conditions (1) and (2), let α be a rotation of the disc that takes the point 1 to η. The induced composition operator is unitary, i.e.

$C_\alpha^{-1} = C_\alpha^*$. Let β be the rotation of the disc that takes η to 1, note $\beta = \alpha^{-1}$. Then $C_\psi^* C_\varphi$ is not compact if and only if $C_\psi^* C_\alpha^* C_\alpha C_\varphi C_\beta = C_{\psi\circ\alpha}^* C_{\beta\circ\varphi\circ\alpha}$ is not compact. Since both $\psi \circ \alpha$ and $\beta \circ \varphi \circ \alpha$ have angular derivatives at 1 and $\beta \circ \varphi \circ \alpha(1) = 1$ we will assume $\eta = 1$ and $\varphi(1) = 1$.

To see why we may assume condition (3), we know by (1) and (2) that 1 is a boundary fixed point of φ, so $s = \varphi'(1) > 0$. Let τ be a hyperbolic automorphism of the unit disc that has fixed point, with $\tau'(1) = 1/s$. Then $(\tau\circ\varphi)'(1) = 1$, $\tau\circ\varphi(1) = 1$, it suffices to show $C_\psi^* C_{\tau\circ\varphi} = C_\psi^* C_\varphi C_\tau$ is not compact inorder to conclude $C_\psi^* C_\varphi$ is not compact. Thus we may assume that $\varphi'(1) = 1$.

To show that we may assume (4), recall from Theorem A in Section 2.6, that the essential norm of a univalently induced composition operator C_φ is given by,

$$||C_\varphi||_e^2 = \max\left\{\frac{1}{|\varphi'(\eta)|} : \eta \in \partial U\right\}. \tag{4.9}$$

Let β be a linear fractional self map of the unit disc, not an automorphism, with 1 its only fixed point, i.e. β is a parabolic non-automorphism of the disc with fixed point 1. It can be shown that the derivative of β at 1 is one, thus the angular derivative is one. Also, since β is a non-automorphism, 1 is the only point for which the angular derivative exists. So $\beta \circ \varphi$ has boundary fixed point 1, angular derivative one at 1, and 1 is the only point for which the angular derivative of $\beta \circ \varphi$ exists. This implies, by (4.9), that the essential norm of $C_{\beta\circ\varphi}$ is one. It suffices to show that $C_\psi^* C_{\beta\circ\varphi} = C_\psi^* C_\varphi C_\beta$ is not compact, and $C_\psi^* C_{\beta\circ\varphi}$ is an operator with $\beta \circ \varphi$ having all the desired conditions (1),(2),(3), and (4). Thus we may assume that $||C_\varphi||_e = 1$.

We continue the proof of Theorem 4.4 under the assumptions (1)-(4) on the univalent map φ. Consider the family of normalized reproducing kernels $\{k_r(z)\}$ for $0 < r < 1$, where

$$k_r(z) = \frac{K_r(z)}{||K_r||} = \frac{\sqrt{1-r^2}}{1-rz}.$$

Since $\{k_r\}$ converges weakly to zero as $r \to 1$, it will suffice to show that

$$\limsup_{r\to 1} \|C_\psi^* C_\varphi k_r\| > 0, \tag{4.10}$$

thus showing $C_\psi^* C_\varphi$ is not compact.

Set $g_r = C_\varphi k_r - k_r$, which implies

$$\begin{aligned} \|C_\psi^* C_\varphi k_r\| &= \|C_\psi^* k_r + C_\psi^* g_r\| \\ &\geq \|C_\psi^* k_r\| - \|C_\psi\|\|g_r\|. \end{aligned} \tag{4.11}$$

We will now show $\lim_{r\to 1} \|C_\psi^* k_r\| > 0$ and $\lim_{r\to 1} \|g_r\| = 0$, which will prove inequality (4.10).

Since $C_\psi^* K_w = K_{\psi(w)}$ and $\psi'(1)$ exists, upon applying the Julia-Carathеódory Theorem we obtain,

$$\begin{aligned} \lim_{r\to 1} \|C_\psi^* k_r\|^2 &= \lim_{r\to 1}(1-r^2)\|K_{\psi(r)}\| \\ &= \lim_{r\to 1} \frac{1-r^2}{1-|\psi(r)|^2} \\ &= \lim_{r\to 1} \frac{1-r}{1-|\psi(r)|} \\ &= \frac{1}{|\psi'(1)|} \\ &> 0. \end{aligned}$$

Thus $\lim_{r\to 1} \|C_\psi^* k_r\| = \frac{1}{|\psi'(1)|} > 0$, so we have reduced the problem to showing that $\lim_{r\to 1} \|g_r\| = 0$.

$$\begin{aligned} \|g_r\|^2 &= \|C_\varphi k_r - k_r\|^2 \\ &= \|C_\varphi k_r\|^2 + \|k_r\|^2 - 2Re\frac{< K_r \circ \varphi, K_r >}{\|K_r\|^2} \end{aligned}$$

$$= \quad ||C_\varphi k_r||^2 + 1 - 2Re\frac{1-r^2}{1-r\varphi(r)}. \qquad (4.12)$$

The lim sup as $r \to 1$ of the first term is bounded by the essential norm of the operator C_φ which by hypothesis is one, i.e.

$$\lim_{r\to 1} ||C_\varphi k_r||^2 \leq \lim \sup_{|w|\to 1} ||C_\varphi k_w||^2 \leq ||C_\varphi||_e^2 = 1.$$

Now to finish the proof we have to deal with the third term in equation (4.12), which we do in the following calculation,

$$\begin{aligned} \frac{1-r\varphi(r)}{1-r^2} &= \frac{1}{1+r}\left(\frac{1-\varphi(r)+\varphi(r)-r\varphi(r)}{1-r}\right) \\ &= \frac{1}{1+r}\left(\frac{1-\varphi(r)}{1-r}+\varphi(r)\right). \end{aligned}$$

And since $\lim_{r\to 1}\frac{1-\varphi(r)}{1-r} = \varphi'(1) = 1$ and $\varphi(1) = 1$, we conclude that

$$\lim_{r\to 1}\frac{1-r^2}{1-r\varphi(r)} = 1.$$

Hence $\lim_{r\to 1}||g_r|| = 0$, which completes the proof of Theorem 4.4.

CHAPTER 5

The Operator $C_\varphi C_\psi^*$ on H^2

This chapter is broken up into three parts: In the first section we outline the main results for the operator $C_\varphi C_\psi^*$, and using Theorem 3.1 we develop the connection between the compactness of the operator $C_\varphi C_\psi^*$ and the angular derivative of the inducing maps. In the second section we establish a lower bound on the essential norm of $C_\varphi C_\psi^*$, and in the third section we prove a sufficient condition for the operator to be compact.

5.1 Main results for $C_\varphi C_\psi^*$.

We establish the following lower estimates on the essential norm, which provides a necessary condition for $C_\varphi C_\psi^*$ to be compact.

Theorem 5.1 *Suppose φ and ψ are holomorphic self maps of the disc. Then*

1. $$\|C_\varphi C_\psi^*\|_e^2 \geq \limsup_{|z|\to 1^-} \frac{\log \frac{1}{|z|}}{\log \frac{1}{|\psi(z)|}} \frac{N_\varphi(\psi(z))}{\log \frac{1}{|\psi(z)|}}, \quad \textit{and}$$

2. $$\|C_\varphi C_\psi^*\|_e^2 \geq \limsup_{|z|\to 1^-} \frac{\log \frac{1}{|z|}}{\log \frac{1}{|\varphi(z)|}} \frac{N_\psi(\varphi(z))}{\log \frac{1}{|\varphi(z)|}}.$$

Remark. In the special case when one of the inducing maps is univalent Theorem 5.1 reduces to the lower estimate on the essential norm of $C_\varphi C_\psi^*$ conjectured in equation (3) of the Introduction. To see this suppose ψ is univalent, and apply the change of variables $w = \psi(z)$ to the first lower estimate in Theorem 5.1 to obtain,

$$\|C_\varphi C_\psi^*\|_e^2 \geq \limsup_{|w|\to 1^-} \frac{N_\varphi(w) N_\psi(w)}{\left(\log \frac{1}{|w|}\right)^2}.$$

With a little more care we obtain the following,

Corollary 5.2 *Suppose φ and ψ are holomorphic self maps of the disc and either φ or ψ is boundedly valent. Then*

$$\|C_\varphi C_\psi^*\|_e^2 \geq M \limsup_{|w|\to 1^-} \frac{N_\psi(w) N_\varphi(w)}{\left(\log \frac{1}{|w|}\right)^2}. \tag{5.1}$$

where M is a positive constant.

The next corollary is a sufficient condition for noncompactness of $C_\varphi C_\psi^*$ in terms of the angular derivatives of the inducing maps φ and ψ and is a generalization of the angular derivative criterion for a composition operator. The corollary follows directly from Theorem 5.1 and Theorem 3.1.

Corollary 5.3 *Suppose ζ_1, ζ_2, and ω are three points on the unit circle such that*

1. *$\varphi(\zeta_1) = \psi(\zeta_2) = \omega$ and,*

2. *$\varphi'(\zeta_1)$ and $\psi'(\zeta_2)$ exist.*

Then $C_\varphi C_\psi^*$ is not compact.

The two lower estimates on the essential norm in Theorem 5.1 are not sufficient for compactness of a composition operator. This can by seen by considering the inducing map φ defined in [18] page 185: an inner function which does not have an angular derivative at any point. By the Julia-Caratheodory Theorem the non-existence of an angular derivative is equivalent to,

$$\limsup_{|z|\to 1^-} \frac{\log|z|}{\log|\varphi(z)|} = 0,$$

and since φ is inner, its counting function approaches zero slowly in the sense

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log\frac{1}{|w|}} > 0.$$

The next theorem is a sufficient condition for $C_\varphi C_\psi^*$ to be compact. Roughly the sufficient condition says that if $\varphi(U)$, $\psi(U)$, and ∂U are not too close then $C_\varphi C_\psi^*$ is compact. We use the following notation,

$$E_\varphi = \left\{\zeta \in \partial U : \ \limsup_{w\to\zeta} \frac{N_\varphi(w)}{\log(\frac{1}{|w|})} > 0\right\}. \tag{5.2}$$

Theorem 5.4 *Suppose φ and ψ are holomorphic self maps of the disc. If* $\mathrm{dist}(E_\varphi, E_\psi) > 0$ *then $C_\varphi C_\psi^*$ is compact.*

It is a straightforward calculation to show $\mathrm{dist}(E_\varphi, E_\psi) > 0$ implies

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)N_\psi(w)}{\left(\log\frac{1}{|w|}\right)^2} = 0.$$

Thus Theorem 5.4 is a partial converse to Theorem 5.1.

5.2 Lower estimate for the essential norm of $C_\varphi C_\psi^*$.

Theorem 5.1 *Suppose φ and ψ are holomorphic self maps of the disc. Then*

1. $\|C_\varphi C_\psi^*\|_e^2 \geq \limsup_{|a|\to 1^-} \frac{\log\frac{1}{|a|}}{\log\frac{1}{|\psi(a)|}} \frac{N_\varphi(\psi(a))}{\log\frac{1}{|\psi(a)|}}$

2. $\|C_\varphi C_\psi^*\|_e^2 \geq \limsup_{|a|\to 1^-} \frac{\log\frac{1}{|a|}}{\log\frac{1}{|\varphi(a)|}} \frac{N_\psi(\varphi(a))}{\log\frac{1}{|\varphi(a)|}}.$

Proof: Let $K_a(z)$ be the reproducing kernel at the point $a \in U$ and let k_a be K_a divided by its norm,

$$k_a(z) = \frac{K_a(z)}{||K_a||} = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}, \qquad (z \in U). \tag{5.3}$$

Since k_a converges to zero uniformly on compact subsets of U as $|a| \to 1^-$ and $||k_a|| = 1$, it converges weakly to zero as $|a| \to 1^-$. Hence,

$$||C_\varphi C_\psi^*||_e \geq \limsup_{|a|\to 1^-} ||C_\varphi C_\psi^* k_a||.$$

Using the identity $C_\psi^* K_a = K_{\psi(a)}$ and normalizing $K_{\psi(a)}$ we obtain,

$$||C_\varphi C_\psi^* k_a||^2 = (1-|a|^2)||C_\varphi K_{\psi(a)}||^2 = \frac{1-|a|^2}{1-|\psi(a)|^2}||C_\varphi k_{\psi(a)}||^2.$$

Therefore,

$$\limsup_{|a|\to 1^-} ||C_\varphi C_\psi^* k_a||^2 = \limsup_{|a|\to 1^-} \frac{\log \frac{1}{|a|}}{\log \frac{1}{|\psi(a)|}} ||C_\varphi k_{\psi(a)}||^2$$

Now fix $0 < r < 1$. By Proposition 1 of Section 2.3 we obtain,

$$||C_\varphi k_{\psi(a)}||^2 \geq \frac{N_\varphi(\psi(a))}{\log \frac{1}{|\psi(a)|}} c_r(a)$$

for $\psi(a)$ sufficiently close to ∂U. Thus,

$$\begin{aligned} \limsup_{|a|\to 1^-} ||C_\varphi C_\psi^* k_a||^2 &\geq \limsup_{|a|\to 1^-} c_r(a) \frac{\log \frac{1}{|a|}}{\log \frac{1}{|\psi(a)|}} \frac{N_\varphi(\psi(a))}{\log \frac{1}{|\psi(a)|}} \\ &= r^2 \limsup_{|a|\to 1^-} \frac{\log \frac{1}{|a|}}{\log \frac{1}{|\psi(a)|}} \frac{N_\varphi(\psi(a))}{\log \frac{1}{|\psi(a)|}}. \end{aligned}$$

Since r can be chosen arbitrarily close to one, this completes the proof for the first lower estimate.

Applying the above calculation to the adjoint of $C_\varphi C_\psi^*$, which is $C_\psi C_\varphi^*$, we obtain the second lower estimate, thus finishing the proof.

Corollary 5.2 *Suppose either φ or ψ is boundedly valent. Then*

$$\|C_\varphi C_\psi^*\|_e^2 \;\geq\; M \limsup_{|w|\to 1^-} \frac{N_\varphi(w) N_\psi(w)}{\left(\log \frac{1}{|w|}\right)^2}$$

where M is a positive constant.

Proof: Assume ψ is boundedly valent. Let $k_a(z) = k_a(z)/\|k_a\|$ be the normalized reproducing kernel at the point $a \in U$. For $w \in U$ set $\{a_i\}_1^n = \psi^{-1}(w)$, and

$$F_{\psi,w}(z) = \sum_{i=1}^n c_i k_{a_i}(z), \qquad \text{where} \quad c_i = \left(\sum_{j=1}^n \|K_{a_j}\|^{-2}\right)^{-\frac{1}{2}} \|K_{a_i}\|^{-1}.$$

Since ψ is boundedly valent $\|F_{\psi,w}\|$ is uniformly bounded for all $w \in U$ and $F_{\psi,w}$ converges to zero uniformly on compact subsets of U as $|w| \to 1^-$. Thus $F_{\psi,w}$ converges weakly to zero as $|w| \to 1^-$. Let $1/M$ be an upper bound on $\|F_{\psi,w}\|$ for all $w \in U$. Hence

$$\|C_\varphi C_\psi^*\|_e \geq \limsup_{|w|\to 1^-} \frac{\|C_\varphi C_\psi^* F_{\psi,w}\|}{\|F_{\psi,w}\|} \geq M \limsup_{|w|\to 1^-} \|C_\varphi C_\psi^* F_{\psi,w}\|. \tag{5.4}$$

Using the fact that $C_\psi^* K_{a_i} = K_w$ we see

$$C_\psi^* F_{\psi,w}(z) = \sum_{i=1}^n \frac{c_i}{\|k_{a_i}\|} C_\psi^* K_{a_i}(z) = K_w(z) \sum_{i=1}^n \frac{c_i}{\|k_{a_i}\|}. \tag{5.5}$$

A short calculation shows

$$\sum_{i=1}^n \frac{c_i}{\|K_{a_i}\|} = \left(\sum_{i=1}^n \frac{1}{\|K_{a_i}\|^2}\right)^{1/2}. \tag{5.6}$$

Thus using equation (5.5), equation (5.6), and normalizing K_w, we obtain,

$$\begin{aligned}
\|C_\varphi C_\psi^* F_{\psi,w}\| &= \|C_\varphi K_w\| \sum_{i=1}^{n} \frac{c_i}{\|K_{a_i}\|} \\
&= \|C_\varphi K_w\| \left(\sum_{i=1}^{n} \frac{1}{\|K_{a_i}\|^2} \right)^{1/2} \\
&= \|C_\varphi k_w\| \left(\sum_{i=1}^{n} \frac{\|K_w\|^2}{\|K_{a_i}\|^2} \right)^{1/2} .
\end{aligned}$$

Therefore, by equation (5.4) and Proposition 1 of Section 2.3 we obtain,

$$\begin{aligned}
\|C_\varphi C_\psi^*\|_e &\geq M \limsup_{|w|\to 1^-} \|C_\varphi C_\psi^* F_{\psi,w}\| \\
&\geq \limsup_{|w|\to 1^-} \left(\frac{N_\varphi(w)}{\log \frac{1}{|w|}} \right)^{1/2} \left(\sum_{i=1}^{n} \frac{\|K_w\|^2}{\|K_{a_i}\|^2} \right)^{1/2} \\
&= \limsup_{|w|\to 1^-} \left(\frac{N_\varphi(w)}{\log \frac{1}{|w|}} \frac{N_\psi(w)}{\log \frac{1}{|w|}} \right)^{1/2}
\end{aligned}$$

thus finishing the proof.

5.3 Sufficient condition for $C_\varphi C_\psi^*$ to be compact.

This section is broken into three parts: We start with a technical lemma. We then prove a theorem that is weaker than Theorem 5.4 and, using this result, we prove Theorem 5.4.

Lemma 5.5 *Suppose ψ is a holomorphic self map of the disc and $f \in H^2$ of norm one. Then*

$$|\left(C_\psi^* f\right)'(z)|^2 \leq 4 \int_U \frac{N_\psi(w)}{|1-\bar{z}w|^6} dA(w) + \frac{|\psi(0)|^2}{|1-\bar{z}\psi(0)|^4}.$$

Proof: Let $F_z(w)$ be the reproducing kernel for the derivative of an H^2 function at the point $z \in U$, i.e.

$$F_z(w) = \frac{w}{(1-\bar{z}w)^2} \quad \text{so that}$$

$$f'(z) = < f, F_z > \quad \text{for all} \quad f \in H^2.$$

In particular,

$$\left(C_\psi^* f\right)'(z) = < C_\psi^* f, F_z > = < f, F_z \circ \psi > . \tag{5.7}$$

Fix $f \in H^2$ such that $||f|| = 1$. Then use the inner product representation in equation (5.7) to estimate $|\left(C_\psi^* f\right)'(z)|$. Apply successively, the Cauchy-Schwartz inequality, the fact that $||f|| = 1$, and the change of variables formula (2.4), to obtain:

$$\begin{aligned}
|(C_\psi^* f)'(z)|^2 &= |< f, F_z \circ \psi >|^2 \\
&\leq ||f||^2 ||F_z \circ \psi||^2 \\
&= ||F_z \circ \psi||^2 \\
&= 2\int_{\psi(U)} |F_z'(w)|^2 N_\psi(w) dA(w) + |F_z \circ \psi(0)|^2 \\
&\leq 4\int_U \frac{N_\psi(w)}{|1-\bar{z}w|^6} dA(w) + |F_z \circ \psi(0)|^2,
\end{aligned}$$

which is the desired result.

Theorem 5.6 *If* $dist(E_\varphi, \psi(U)) > 0$ *then* $C_\varphi C_\psi^*$ *is compact.*

Proof: Let (f_n) be a sequence in H^2 that converges uniformly on compact subsets of U to zero and $||f_n|| = 1$ for all n. Thus (f_n) converges weakly to zero. We will show,

$$\lim_{n\to\infty} ||C_\varphi C_\psi^* f_n|| = 0.$$

To estimate $||C_\varphi C_\psi^* f_n||$, use the Littlewood-Paley identity(2.2) for the H^2 norm and the change of variable formula (2.4) to obtain,

$$||C_\varphi C_\psi^* f_n||^2 = 2\int_U |(C_\psi^* f_n)'|^2 N_\varphi dA + |C_\varphi C_\psi^* f_n(0)|^2.$$

Since (f_n) converges to zero weakly and $C_\varphi C_\psi^*$ is bounded operator, $(C_\varphi C_\psi^* f_n)$ converges to zero weakly. Thus $|C_\varphi C_\psi^* f_n(0)|$ converges to zero as $n \to \infty$. Hence as $n \to \infty$ we obtain,

$$||C_\varphi C_\psi^* f_n||^2 = 2\int_U |(C_\psi^* f_n)'|^2 N_\varphi dA + o(1). \tag{5.8}$$

Now temporarily fix $0 < r < 1$, and split the integral on the right side of (5.8) into two parts: one over the disc rU, and the other over its complement. Since $C_\psi^* f_n(z)$ converges weakly to zero it follows that $|(C_\psi^* f_n)'(z)|$ converges uniformly to zero on the relatively compact set rU as $n \to \infty$. Thus we obtain,

$$\begin{aligned} ||C_\varphi C_\psi^* f_n||^2 &= 2\int_{U\setminus rU} + 2\int_{rU} |(C_\psi^* f_n)'|^2 N_\varphi dA + o(1) \\ &= 2\int_{U\setminus rU} |(C_\psi^* f_n)'|^2 N_\varphi dA + o(1) \end{aligned}$$

as $n \to \infty$. We have reduced the proof to estimating the integral,

$$I_{r,n} = \int_{U\setminus rU} |(C_\psi^* f_n)'|^2 N_\varphi dA(z).$$

More precisely, we need to show that given $\epsilon > 0$ there exists an $0 < r < 1$ and a positive integer N such that

$$I_{r,n} \leq \epsilon \qquad \text{for all} \quad n \geq N.$$

Since $\mathrm{dist}(E_\varphi, \psi(U)) > 0$ (where E_φ is defined by equation (5.2)) there exists a subset S of the unit disc such that

$$E_\varphi \subset \overline{S} \quad \text{and,} \tag{5.9}$$

$$\mathrm{dist}(S, \psi(U)) > 0. \tag{5.10}$$

Now split the integral $I_{r,n}$ into two integrals: one over $S_r = S \cap (U \setminus rU)$ and the other over $S^c \cap (U \setminus rU)$ to obtain,

$$I_{r,n} = \int_{S_r} + \int_{S^c \cap (U \setminus rU)} |(C_\psi^* f_n)'|^2 N_\varphi dA.$$

Let $\epsilon > 0$. We will estimate each integral separately starting with the integral over S_r. Choose $0 < r < 1$ so that $\varphi(0) \notin U \setminus rU$ and $A(S_r) < \epsilon$ and let C be a constant such that

$$\sup_{r \leq |z| < 1} N_\varphi(z) = C.$$

Upon applying Lemma (5.5) we obtain,

$$\int_{S_r} |(C_\psi^* f_n)'|^2 N_\varphi dA(z)$$
$$\leq 4 \int_{S_r} \int_{\psi(U)} \frac{N_\psi(w) N_\varphi(z)}{|1 - \overline{z} w|^6} dA(w) dA(z) + \int_{S_r} \frac{N_\varphi(z)}{|1 - \overline{z}\psi(0)|^4} dA(z). \tag{5.11}$$

Since $\operatorname{dist}(S, \psi(U)) > 0$ and $S_r \subset S$ it is clear that $\operatorname{dist}(S_r, \psi(U)) > 0$. Thus there exists $\delta > 0$ such that

$$\inf\left\{|1 - \bar{z}w|^6 : \ z \in S_r, \ w \in \psi(U)\right\} > \delta. \tag{5.12}$$

In particular $|1 - \bar{z}\psi(0)|^4 > \delta$ for all $z \in S_r$. Hence we estimate the first term on the right side of expression (5.11) by applying successively inequality (5.12) and Lemma 2.1 of Section 2.5 to obtain,

$$\begin{aligned}
\int_{S_r}\int_{\psi(U)} \frac{N_\psi(w)N_\varphi(z)}{|1 - \bar{z}w|^6} dA(w)dA(z) &\leq \frac{1}{\delta}\int_{S_r}\int_U N_\varphi(z)N_\psi(w)dA(w)dA(z) \\
&\leq \frac{A(S_r)}{\delta} \sup_{r\leq|z|<1} N_\varphi(z)\int_U N_\psi(w)dA(w) \\
&\leq \epsilon\left(\frac{C}{2\delta}\right)(1 - |\psi(0)|^2).
\end{aligned}$$

We estimate the second term of expression (5.11) to obtain,

$$\begin{aligned}
\int_{S_r} \frac{N_\varphi(z)}{|1 - \bar{z}\psi(0)|^4} dA(z) &\leq \frac{1}{\delta}\int_{S_r} N_\varphi(z)dA(z) \\
&\leq A(S_r)\frac{C}{\delta} \\
&\leq \epsilon\frac{C}{\delta}.
\end{aligned}$$

Hence the integral over S_r is less than a constant multiple of ϵ, with the constant independent of n.

We now consider the integral over $S^c \cap (U \setminus rU)$, i.e.

$$\int_{S^c\cap(U\setminus rU)} |(C_\psi^* f_n)'|^2 N_\varphi dA.$$

Since $E_\varphi \subset \overline{S}$ we see from the definition of E_φ, equation (5.2), that

$$\limsup_{|z|\to 1, z\in S^c} \frac{N_\varphi(z)}{\log\frac{1}{|z|}} = 0.$$

Thus there exists $0 < r < 1$ such that

$$N_\varphi(z) \le \epsilon \log\frac{1}{|z|} \quad \text{for all} \quad r \le |z| < 1 \text{ and } z \in S^c. \tag{5.13}$$

Hence applying inequality (5.13) and the Littlewood-Paley identity (2.2) we obtain,

$$\begin{aligned}\int_{S^c\cap(U\setminus rU)} |(C_\psi^* f_n)'(z)|^2 N_\varphi(z)\,dA(z) &\le \epsilon \int_U |(C_\psi^* f_n)'(z)|^2 \log\frac{1}{|z|} dA(z) \\ &\le \frac{\epsilon}{2}\left(\|C_\psi^*\|^2 - |(C_\psi^* f_n)(0)|^2\right).\end{aligned}$$

Thus the integrals over S_r and $S^c \cap (U \setminus rU)$ are both less then a constant multiple of ϵ, with each constant independent of n. This finishes the proof.

Before starting the proof of Theorem 5.4 we introduce the definition of a smooth sector. First by a sector we mean the interior of an angle with center at the origin.

Definition: A subset S of the unit disc is a *smooth sector* if S is contained in a sector of the unit disc and the boundary of S is smooth in the following sense: let τ be a Riemann map from U to S, then

$$\liminf_{|z|\to 1^-, z\in S} \frac{N_\tau(z)}{\log\frac{1}{|z|}} > 0.$$

Theorem 5.4 *If* $\operatorname{dist}(E_\varphi, E_\psi) > \delta > 0$ *then* $C_\varphi C_\psi^*$ *is compact.*

Proof: Without loss of generality we may assume that $\varphi(0) = 0$. Let (f_n) be a sequence in H^2 that converges weakly to zero. Using the same argument as in the proof of Theorem 5.6 we reduce this proof to estimating the integral,

$$I_{r,n} = \int_{U \setminus rU} |(C_\psi^* f_n)'|^2 N_\varphi dA(z).$$

More precisely, we need to show that given $\epsilon > 0$ there exists an $0 < r < 1$ and a positive integer N such that

$$I_{r,n} \leq \epsilon \qquad \text{for all} \quad n \geq N.$$

Let $\epsilon > 0$.

For each point $\zeta \in E_\varphi$, let I_ζ be an arc of the unit circle with center ζ and arc length $\delta/2$. By hypothesis $\text{dist}(E_\varphi, E_\psi) > \delta$ and since the arc length of I_ζ is $\delta/2$, it is clear that

$$\text{dist}(I_\zeta, E_\psi) \geq \frac{\delta}{2} \quad \text{for all } \zeta \in E_\varphi. \tag{5.14}$$

Set

$$I = \bigcup_{\zeta \in E_\varphi} I_\zeta.$$

Since each arc I_ζ has a fixed length it is clear that there exist a finite number of pairwise disjoint arcs $\{I_1, ..., I_m\}$ of the unit circle such that,

$$I = \bigcup_{i=1}^{m} I_i.$$

Now for each arc I_i let S_i be a corresponding smooth sector such that

$$I_i = \overline{S_i} \bigcap \partial U.$$

Let τ_i be the Riemann map from U onto S_i. Since S_i is a smooth sector there exists $\delta_i > 0$ such that

$$\liminf_{|z|\to 1^-, z\in S_i} \frac{N_{\tau_i}(z)}{\log\frac{1}{|z|}} > \delta_i. \tag{5.15}$$

Set, $S = \bigcup_{i=1}^{m} S_i$ and $S^c = U \setminus S$. By construction we see

$$\limsup_{|z|\to 1^-, z\in S^c} \frac{N_\varphi(z)}{\log\frac{1}{|z|}} = 0. \tag{5.16}$$

Thus there exist an $0 < r < 1$ such that

$$N_\varphi(z) \le \epsilon \log\frac{1}{|z|} \qquad (z \in S^c \quad \text{and} \quad r < |z| < 1.) \tag{5.17}$$

Now split the integral $I_{r,n}$ into two integrals: one over $S_r^c = S\bigcap(U \setminus rU)$ and the other over S to obtain,

$$\int_{\varphi(U)_r} |(C_\psi^* f_n)'|^2 N_\varphi dA \le \int_{S_r^c} + \int_S |(C_\psi^* f_n)'|^2 N_\varphi dA.$$

We now consider each integral separately. In the estimate below of the integral over S_r^c, we apply inequality (5.17) and then the Littlewood-Paley identity (2.2) for the H^2 norm:

$$\begin{aligned}
\int_{S_r^c} |(C_\psi^* f_n)'|^2 N_\varphi dA(z) &\le \epsilon \int_U |(C_\psi^* f_n)'|^2 \log\frac{1}{|z|} dA(z) \\
&\le \epsilon \left(\frac{\|C_\psi^* f_n\| - |(C_\psi^* f_n)(0)|^2}{2} \right) \\
&\le \epsilon \|C_\psi^*\|.
\end{aligned}$$

Thus the integral over S_r^c is a constant multiple of ϵ, with the constant independent of n.

We now turn our attention to estimating the integral over S. Using the fact that $S = \bigcup_{i=1}^{m} S_i$ we see,

$$\int_S |(C_\psi^* f_n)'|^2 N_\varphi dA(z) = \sum_{i=1}^{m} \int_{S_i} |(C_\psi^* f_n)'|^2 N_\varphi dA(z). \tag{5.18}$$

By Littlewood's inequality (2.6) in Section 2.5 we see,

$$\int_{S_i} |(C_\psi^* f_n)'|^2 N_\varphi dA(z) \leq \int_{S_i} |(C_\psi^* f_n)'(z)|^2 \log \frac{1}{|z|} dA(z). \tag{5.19}$$

By inequality (5.15) there exist $0 < r' < 1$ and a constant C such that

$$\log \frac{1}{|z|} < C N_{\tau_i}(z) \quad (z \in S_i \quad \text{and} \quad r' < |z| < 1)$$

for all $0 \leq i \leq m$. Thus,

$$\int_{S_i} |(C_\psi^* f_n)'(z)|^2 \log \frac{1}{|z|} dA(z) \leq C \int_{S_i} |(C_\psi^* f_n)'|^2 N_{\tau_i} dA(z) \tag{5.20}$$

for all $0 \leq i \leq m$. To the right side of equation (5.20) using the fact that $\tau_i(U) = S_i$ and applying succesively the inequalities (5.19) and (5.20) we obtain,

$$\int_S |(C_\psi^* f_n)'|^2 N_\varphi dA(z) \leq C \sum_{i=1}^{m} \int_{\tau_i(U)} |(C_\psi^* f_n)'|^2 N_{\tau_i} dA(z). \tag{5.21}$$

Applying to each term on the right side of (5.21) the change of variables formula (2.4) and then the Littlewood-Paley identity (2.2) for the H^2 norm we obtain,

$$\begin{aligned}\int_{\tau_i(U)} |(C_\psi^* f_n)'|^2 N_\varphi dA(z) &\le \int_U |(C_{\tau_i} C_\psi^*)'(z)|^2 \log\frac{1}{|z|} dA(z)\\ &\le \frac{1}{2}\left(\|C_{\tau_i} C_\psi^* f_n\|^2 - |C_{\tau_i} C_\psi^* f_n(0)|^2\right).\end{aligned}$$

Hence

$$\int_S |(C_\psi^* f_n)'|^2 N_\varphi dA(z) \le \frac{C}{2}\sum_{i=1}^m \left(\|C_{\tau_i} C_\psi^* f_n\|^2 - |C_{\tau_i} C_\psi^* f_n(0)|^2\right).$$

Since $\operatorname{dist}(E_\psi, \tau_i(U)) = \operatorname{dist}(E_\psi, I_i) \ge \delta/2$ where $\overline{\tau_i(U)} \cap \partial U = I_i$, we see by Theorem 5.6 that $C_\psi C_{\tau_i}^*$ is compact. Since $C_\psi C_{\tau_i}^*$ is compact its adjoint $C_{\tau_i} C_\psi^*$ is compact. Hence there exist an integer N such that

$$\frac{C}{2}\sum_{i=1}^m \left(\|C_{\tau_i} C_\psi^* f_n\|^2 - |C_{\tau_i} C_\psi^* f_n(0)|^2\right) < \epsilon \qquad (n > N),$$

and this concludes the proof.

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