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Residual Properties of Finitary Linear Groups

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Aflahiah Radford

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Richard E. Phillips
Major professor

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RESIDUAL PROPERTIES OF FINITARY LINEAR GROUPS

By

Aflahiah Radford

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ABSTRACT

RESIDUAL PROPERTIES OF FINITARY LINEAR GROUPS

By

Aflahiah Radford

In this thesis, we generalize a theorem of Bruno-Phillips describing certain locally finite finitary groups (modulo its unipotent subgroup) as a subdirect product of finite dimensional groups. Our only restriction is that such groups are residually finite and this forces all the composition factors of the module to be finite dimensional. Specifically we have Theorem 4 which states: Let $G \leq FGL(V, K)$ be such that G is both locally finite and residually finite. Then every G-composition factor of V is finite dimensional and G/unip(G) is a subdirect product of finite dimensional groups. We showed that any irreducible finitary linear group has the conjugate centralizer property with respect to any of its normal irreducible subgroup. In the case of primitive linear groups of infinite dimension, all normal subgroups are irreducible and if such groups are also locally finite then we find that their derived groups are simple (Theorem 3). Theorem 3 is used in the proof of Theorem 4 and the proof of Theorem 3 makes used of J. Hall's classification of simple, locally finite finitary linear groups. This allows us to describe certain subgroups as a direct product of two entities; a direct product of simple groups and subdirect product imprimitive groups, a crucial step in proving our main result.

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To my husband, Glenn Harley Radford

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Introduction

Let V be a vector space over a field K. An element $g \in GL(V, K)$ is **finitary** if $[V, g] = \{v(g-1) \mid v \in V\}$ has finite dimension. The **degree of g**, denoted by deg(g), is the dimension of [V, g]. Observe that g is finitary iff $C_v(g) = \{v \in V \mid vg = v\}$ has finite co-dimension in V.

The **group of finitary transformations of V**, denoted by FGL(V, K), is defined as the set $\{g \in GL(V, K) \mid g\ is\ finitary\}$. The group FGL(V, K) is a normal subgroup of GL(V, K) and seemed to have been first introduced by Dieudonne$'$ [2]. Rosenberg [14] used the concept of finitary linear transformation as an essential feature in classifying normal subgroups of general linear group.

A **K-finitary representation** of a group G is a homomorphism $\psi : G \to FGL(V, K)$, from G into FGL(V, K) for some K-vector space V. The representation is *faithful* if $ker\psi = 1$. Groups with faithful K-finitary representations are called **K-finitary linear groups** and if the associated vector space V is finite dimensional, then G is called finite dimensional .

Some examples of finitary groups are:

1. subgroups of GL(n,K).

2. groups of finitary permutations

3. $GL(n, K)wr_{\Omega}(G, \Omega)$ where (G, Ω) is a group of finitary permutations

4. Stable groups.

Obviously subgroups of GL(n,K) are finite dimensional groups. If Ω is a non-empty set, then $g \in Sym(\Omega)$ is *a finitary permutation* if $supp(g) = \{\alpha \in \Omega \mid \alpha^g \neq \alpha\}$ is a finite set. The set $\{g \in Sym(\Omega) \mid g\ is\ finitary\}$ is denoted by $Sym(\Omega, \aleph_0)$ and **finitary permutation groups** consist of all subgroups of $Sym(\Omega, \aleph_0)$. For any K, the finitary permutation groups (G, Ω) acts faithfully on the K-space $V = K\Omega$, the permutation module of G with basis $\{v_\alpha \mid \alpha \in \Omega\}$ by $v_\alpha^g = v_{\alpha^g}$. Results from the theory of finitary permutaton groups are used often in this thesis.

If V is a K-vector space, Ξ a basis of V, then let $stab(V, \Xi) =$ $\{g \in GL(V, K) \mid v^g = v\ for\ all\ but\ a\ finite\ number\ of\ v \in \Xi\}$. If $\Im$ is another basis, then $stab(V, \Xi, K)$ is conjugate to $stab(V, \Im, K)$ in GL(V, K). This common isomorphism type will be called the stable general linear group and denoted by Stab(GL(V, K)). Subgroups of Stab(GL(V, K)) are called **stable groups**. Obviously stable groups are finitary linear groups. Stable groups play an important role in the sequel. We prove in Chapter 1 that if $G \leq FGL(V, K)$ is an irreducible finitary linear group with dim(V) being countably infinite, then G has a stable basis. Thus G is a stable group. This fact is used in the proof of one of the main results of this thesis:

Theorem 2:

Let $G \leq FGL(V, K)$ be an irreducible, primitive and locally finite finitary group where V has infinite dimension. Then G' is simple.

In order to prove our result above, we made used of the following classification of locally finite, simple groups of FGL(V, K) by J. Hall [4] which states:

Let G be a locally finite, infinite, non-"finite dimensional ", simple subgroup of FGL(V, K) , where dimension V is infinite. Then G is either alternating, the derived subgroup of FSp(V, K, a), FU(V, K, h) or FO(V, K, q), or a group of type T(W, V), $W \subseteq V^*$ and $ann_V(W) = 0$.

Here a, h, q are respectively non-degenerate alternating, Hermitian and quadratic forms on V. The special transvection group $T(W,V) = < t(\alpha, x) \mid \alpha \in W,\ x \in ker(\alpha) >$ where $W \subseteq V^*$ and the map $t(\alpha, x) \in GL(V,K)$ is defined by $t(\alpha, x)(v) = v + ((v)\alpha)x$ where $v \in V$. In general T(W, V) is not simple but a sufficient condition for simplicity is $ann_V(W) = \{v \in V \mid (v)\alpha = 0$ for all $\alpha \in$ W $\} = 0$.

In the proof of Theorem 2 we show that G' is actually one of the simple groups listed above.

The proof of Theorem 2 makes substantial use of the **the conjugate centralizer property**. A group X has the *conjugate centralizer property* relative to its subgroup H if for every finitely generated subgroup F of G, there exist an element $h \in H$ such that $[F, F^h] = 1$. Specifically, we show that

(i) If $G \leq FGL(V,K)$ is primitive, then G has a unique component M, and

(ii) G has the conjugate centralizer property with respect to M.

From these two results, together with the consequence of the conjugate centralizer property, we are able to show that $M = G'$ and G' simple. It is in the proof of this that that we heavily use stable groups and the classification of J.Hall.

In addition, we will establish

Lemma 2:

Let $G \leq FGL(V,K)$ be an irreducible, infinite dimensional and imprimitive finitary linear group that is also locally finite. Let N be a normal irreducible subgroup of G. Then G has the conjugate centralizer property with respect to N.

A primary consequence of this result is:

Theorem 1:

Let $G \leq FGL(V,K)$ be an irreducible, infinite dimensional and imprimitive finitary linear group that is locally finite. Then G' is the unique minimal subnormal irreducible subgroup of G.

Our determination of the structure of primitive groups permits extension of results of Bruno and Phillips [1] regarding residually finite, finitary linear groups. In [1], it is shown that certain types of residually finite, finitary linear groups are subdirect products of finite dimensional groups. Here in Chapter 3, we are able to prove the following:

Theorem 4:

Let $G \leq FGL(V, K)$ be such that G is both locally finite and residually finite. Then

1. every G-composition factor of V is finite dimensional.
2. G/unip(G) is a subdirect product of finite dimensional groups.

Recall that a group X is *residually finite* if $1 = \cap\{H \trianglelefteq X \mid X/H \text{ is finite}\}$. This is equivalent to "if $1 \neq x \in X$, $\exists H \trianglelefteq X$ such that $|X/H| < \infty$ and $x \notin H$" .

Chapter 0

In this chapter, we list some definitions and significant results that will be used but are not stated in the introduction nor in the following chapters.

Definition 1 *Let $G \leq FGL(V, K)$ be a group and H a subgroup of G. Then the* **degree of H,** *denoted by deg(H) is the dimension of the vector space $[V, H]$.*

Definition 2 *Let $A \subseteq FGL(V, K)$, a non-empty subset of FGL(V, K)and $Y \subseteq V$, a non-empty subset of V. Then*

1. $< \mathrm{YA} >= < \mathrm{y^a} | \mathrm{y} \in \mathrm{Y}$ and $\mathrm{a} \in \mathrm{A} >$ *and*

2. $[\mathrm{Y}, \mathrm{A}] = < \mathrm{y^a} - \mathrm{y} | \mathrm{y} \in \mathrm{Y}$ and $\mathrm{a} \in \mathrm{A} >$.

Note that < YA > and [Y, A] are < A >-subspaces of V.

The following Lemma 1 from [15] is used in Chapter 1.

[15, Lemma 1]. Let G= $< \mathrm{T} > \subseteq \mathrm{FGL(V, K)}$ where T is a finite set. Then

1. $\dim(\mathrm{V}/\mathrm{C_V(G)})$ is finite.

2. $dim[V, G] \leq \sum\{dim[V, t] | t \in T\}$; thus dim[V, G] is finite.

3. If W is finite dimensional subspace of V, then $< WG >$ is finite dimensional.

4. There is a finite dimensional G subspace X of V such that dim(X) is finite, $[\mathrm{V}, \mathrm{G}] \subseteq \mathrm{X}$ and G acts faithfully on X; further there is a subspace Y of $\mathrm{C_V(G)}$ such that $V = X \oplus Y$.

Some significant results from finitary permutation groups and finitary linear groups used in this thesis will be discussed next.

Definition 3 *A set* $\mho = \{V_i | i \in I\}$ *of subspaces of* V *is a* **G-system of imprimitivity of V** *if*

1. $V = \oplus\{V_i | i \in I\}$*, and*

2. *for each* $g \in G$ *and* $i \in I$*, there is a* $j \in I$ *such that* $V_i^g = V_j$*.*

We will need the following properties of irreducible finitary linear groups.

[12, 2.2.3]. Suppose that $G \leq FGL(V, K)$ and that $\mho = \{V_i | i \in I\}$ is a system of imprimitivity of G with $|I| > 1$ on which G acts transitively. Then

1. If $i \in I$, then V_i is finite dimensional (and all of the V_i have the same dimension).

2. $G\pi$ is a group of finitary permutations.

3. $ker(\pi)$ is a subgroup of the direct product of groups $L_i \subseteq GL(V_i, K)$ and each L_i is an image of $ker(\pi)$; note that the groups L_i are finite dimensional K-linear groups of bounded degree.

Thus by (2) above $G/ker(\pi)$ is an infinite transitive group of finitary permutations. This connects the irreducible finitary linear groups with the transitive permutation groups. We will next list some crucial properties of transitive finitary permutation groups.

Let X be an infinite transitive group of finitary permutations on a set Ω. Since X is finitary, the X-blocks are finite sets. Furthermore either Ω is the union of an ascending chain of a countable number of blocks (in which case X is called **totally imprimitive**), or Ω has a maximal block (in which case X is called **almost primitive**). If X is totally imprimitive, then there is a chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_i \subseteq \ldots$ of normal subgroups of X such that

1. $\cup\{N_i\} = \mathrm{X}$, and

2. each N_i is a subdirect power of a finite group.

Therefore both Ω and X are countable.

On the other hand if X is an almost primitive group, then X has a normal subgroup N such that

1. N is a subdirect power of a finite group and

2. G/N is either the alternating group or the full group of finitary permutation on an infinite set.

Definition 4 *Suppose* $\mathrm{G} \leq \mathrm{FGL(V,K)}$, $\mho = \{V_i | i \in I\}$ *a G-system of imprimitivity and F a non-empty subset of G. Then the* **support of F**, *denoted by supp(F), is the set* $\{\mathrm{V_i} \mid \mathrm{i} \in \mathrm{I} \text{ and } \mathrm{V_i^f} \neq \mathrm{V_i} \text{ for some } \mathrm{f} \in \mathrm{F}\}$.

Note that the support of F depends on a given system of imprimitivity. A result involving the support of F from [12] which is used in this thesis is the following:

[12, Lemma 8]. Let $G \leq FGL(V,K)$ and $\mho = \{V_i | i \in I\}$ be a system of imprimitivity of V with $|I| > 1$. Further, let A be a subgroup of G and let
$\Im(A) = \Im(A : \mho) = \{i \in I | [V_i, A] \neq 0\}$.
For any element $g \in G$, define $\Im(\mathrm{A})\mathrm{g} = \{\mathrm{j} \in \mathrm{J} | \text{ for some } \mathrm{i} \in \Im(\mathrm{A}), \mathrm{V_i^g} = \mathrm{V_j}\}$. Then

1. for all $g \in G$, $\Im(A^g) = \Im(A)g$;

2. $|\Im(A)| \leq |supp(A)| + deg(A)$;

3. if there is a $g \in G$ such that $\Im(A)g \cap \Im(A) = \emptyset$, then $[V, A^g] \subseteq C_V(A)$ and $[V, A] \subseteq C_V(A^g)$.

Part (3) above implies $[A, A^g] = 1$ (Lemma 9 of [12]).

CHAPTER 1

Conjugate Centralizer Property of Imprimitive Groups and Stable Groups

Definition 5 *Let V be a vector space over the field K with basis χ and $G \leq FGL(V, K)$. Then χ is a G-stable basis of V if each element of G moves only a finite number of elements of χ.*

For countable irreducible groups $G \leq FGL(V, K)$, we can always find a stable basis for G. This is the content of the following lemma.

Lemma 1 *Let G be a countable subgroup of FGL(V,K). Then V contains a G-subspace W, with the following properties:*

1. *G acts faithfully on W.*

2. *W has a G-stable basis $\chi = \{v_\alpha \mid \alpha \geq 1\}$.*

Proof:

Enumerate the elements of G as $\{g_1, g_2, g_3, g_4, \ldots\}$ and for each n, let G_n be the group generated by $\{g_1, g_2, \ldots, g_n\}$. We then obtain an ascending sequence of finitely generated subgroups, $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$. Lemma 1 of [15] produces for each n, a finite dimensional subspace of V , X_n, such that G_n acts faithfully on X_n and [V,G_n] $\subseteq X_n$. Furthermore this lemma gives subspaces Y_n with the property that V=$X_n \oplus Y_n$ and $Y_n \subseteq C_V(G_n)$.

Define $S_1 = X_1$. For $n \geq 2$, let $S_n = X_1 + X_2 + \cdots + X_n$ and J_n be a direct compliment of $S_{n-1} \cap Y_{n-1}$ in Y_{n-1}.

Claim 1 There exist disjoint sets $\{A_n\}_{n\geq 1} \in V$ such that

(a) $A_1 \cup A_2 \cdots \cup A_k$ is a basis for $X_1 + X_2 + \cdots + X_k$ $\forall k \geq 1$ and

(b) If $1 \leq i < k$ then $[A_k,\ G_i] = 0$.

Proof:

Let A_1 be a basis of X_1 over K. Suppose $\forall\ 1 < k < n$, $\exists$ disjoint sets $A_1,\ A_2, \ldots, A_k \in V$ such that

1. $A_1 \cup A_2 \cdots \cup A_k$ is a basis of $X_1 + X_2 + \cdots + X_k$ and
2. If $1 \leq i < k$, then $[A_k, G_i] = 0$.

Recall that $S_n = X_1 + X_2 + \cdots + X_n$ and J_n is a direct compliment of $S_{n-1} \cap Y_{n-1}$ in Y_{n-1} and $Y_{n-1} \in C_V(G_{n-1})$. Choose A_n to be a basis of $S_n \cap J_n$ over K. Then

$$\begin{aligned} V &= X_{n-1} \oplus Y_{n-1} \\ &= (X_1 + X_2 + \cdots + X_{n-1}) + Y_{n-1} \\ &= S_{n-1} + Y_{n-1} \\ &= S_{n-1} \oplus J_n \quad (since\ Y_{n-1} = (S_{n-1} \cap Y_{n-1}) \oplus J_n) \\ &= (X_1 + X_2 + \cdots + X_{n-1}) \oplus J_n. \end{aligned}$$

Thus,

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$= (X_1 + \cdots + X_{n-1}) \oplus (X_n \cap J_n)$$

$$= (X_1 + \cdots + X_{n-1}) \oplus (S_n \cap J_n) \quad (since\ (X_1 + X_2 + \cdots + X_{n-1}) \cap J_n = 0)$$

$$= S_{n-1} \oplus (S_n \cap J_n).$$

Therefore, since $A_1, A_2, \ldots, A_{n-1}$ are disjoint sets and $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ is a basis for S_{n-1} and A_n is a basis of $S_n \cap J_n$, $A_1, A_2, \ldots, A_n$ are disjoint sets and $A_1 \cup A_2 \cup \cdots \cup A_n$ is a basis of $S_n = X_1 + X_2 + \cdots + X_n$.

Observe next that for $1 \leq i < n$, $[A_n,\ G_i] \subseteq [S_n \cap J_n, G_i] = 0$ since $J_n \subseteq Y_{n-1} \subseteq C_V(G_{n-1})$ and $G_i \subseteq G_{n-1}$.

□

Claim 2 $[V, G] \subseteq \cup_{n\geq 1} S_n$.

Proof:

Since $[V, G_n] \subseteq X_n \subseteq S_n$ and $G = \cup_{n\geq 1} G_n$, $[V, G] \subseteq \cup_{n\geq 1}[V, G_n] \subseteq \cup_{n\geq 1} S_n$.

□

Claim 3 The group G acts faithfully on $\cup_{n\geq 1} S_n$.

Proof:

Suppose not. Then there exist $k \geq 1$ and $g_1, g_2 \in G_k$ such that $g_1|_{\cup_{n\geq 1} S_n} = g_2|_{\cup_{n\geq 1} S_n}$. Thus the action of g_1 on X_k is the same as the action of g_2 on X_k. However G_k acts faithfully on X_k which leads to a contradiction.

□

Choose $W = \cup_{n\geq 1} S_n$. By Claim 2, W is a G-subspace and by Claim 3, G acts faithfully on W. By Claim 1, $\cup_{n\geq 1} A_n$ forms a basis of W over K and for each n, G_n moves at most $A_1, A_2, \ldots, A_n$ and fixes pointwise the set $\cup_{k>n} A_n$. Furthermore the A_n's are finite sets since the S_n's are finite dimensional. Therefore $\cup_{n\geq 1} A_n$ forms a G-stable basis for W .

□

Note that if $G \leq FGL(V, K)$ is irreducible, then W= V in the lemma above.

The concept of the conjugate centralizer property was introduced by P.M. Neumann in [8] as the tool for studying finitary permutation groups. It is also useful in working with finitary linear groups.

Definition 6 *A group G has the conjugate centralizer property relative to its subgroup H if for every finitely generated subgroup F of G , there exist an element h in H such that* $[F, F^h] = 1$

The significance of the conjugate centralizer property is realised in the statement below.

[1, 6.2]. Suppose that $H \trianglelefteq G$ and that G has the conjugate centralizer property relative to H. Then

1. $G' \subseteq H$ and

2. if $T \trianglelefteq H$, then $T \trianglelefteq G$.

In the case where G is locally finite, the subgroup F in the definition of the conjugate centralizer property above is a finite group. The ultimate goal here is to show that any irreducible finitary linear group that is locally finite has the conjugate centralizer property with respect to any **normal irreducible subgroup**.

Lemma 2 *Let* $G \leq FGL(V, K)$ *be an irreducible, infinite dimensional and imprimitive finitary linear group that is also locally finite. Let N be a normal irreducible subgroup of G. Then G has the conjugate centralizer property with respect to N.*

Proof:

Let $\mho = \{V_i \mid i \in I\}$ be a G -system of imprimitivity of V with $|I| > 1$ and let $\pi : G \to Sym(\{V_i\}_{i \in I})$ be the permutation representation of G on $\mho$ with $\pi(g)(V_i) = V_i^g$. The group $G/ker\pi$ is an infinite transitive group of finitary permutations on $\mho$ (refer to Chapter 0). We define $\tilde{N} = Nker\pi/ker\pi$. Since N is irreducible, $\tilde{N}$ is also an infinite transitive group of finitary permutations on $\mho$. Hence $\tilde{N}$ is either totally imprimitive or almost primitive. In the latter case $\tilde{N}$ contains an infinite alternating group as a section.

Let $F \subseteq G$ be a finite subgroup of G. We denote the group $Fker\pi/ker\pi$ by $\tilde{F}$ and its elements $fker\pi$ by $\tilde{f}$. The support of $\tilde{F} = supp(\tilde{F}) = \{V_i \mid V_i^{\tilde{f}} \neq V_i \; for \; some \; f \in F\}$ is finite since F is a finite set. Let $\Im(F) = \{i \in I \mid [V_i, F] \neq 0\}$. Using Lemma 8(ii) of [1], $|\Im(F)| \leq |supp(F)| + deg(F) < \infty$. Define W= $\{V_i \mid i \in \Im(F)\}$.

Assume first that $\tilde{N}$ is totally imprimitive. Then $\mho$ is an ascending union of $\tilde{N}$-blocks. Since W is a finite set, there exist a block B which contains W. Furthermore $\exists$ an $n \in N$ such that if $\tilde{n} = nker\pi$, then $B^{\tilde{n}} \cap B = \emptyset$. However

$$\begin{aligned} \Im(F)n &= \{j \in I \mid V_j = V_i^n \; for \; some \; i \in \Im(F)\} \\ &= \{i \in I \mid V_i \in B^{\tilde{n}}\}. \end{aligned}$$

Thus $\Im(F) \cap \Im(F)n = \emptyset$ and by Lemmas 8(iii) and 9 of [12], $[F, F^n] = 1$.

Assume next that $\tilde{N}$ is almost primitive. Let B be a maximal $\tilde{N}$- block and $\Xi = \{B_i \mid i \in I\}$ be the block system of $\mho$ generated by B. The permutation representation of $\tilde{N}$ on Ξ, $\pi^* : \tilde{N} \to Sym(\Xi)$, is infinite, finitary and primitive. Hence $\tilde{N}/ker\pi^*$ is an infinite primitive finitary permutation group and so it contains the infinite alternating group (refer to Chapter 0). Define for each $i \in \Im(F)$, a block $B_i \in \Xi$ such that $V_i \subseteq B_i$. Let $\bar{W} = \{B_i \mid i \in \Im(F)\}$. We can write $\mho$ as a disjoint union of $\bar{W}$ and say another set W_1. Let $\bar{W}$= $\{B_1, B_2, \ldots, B_t\}$ and pick any subset of W_1 of t-elements say $\{U_1, U_2, \ldots, U_t\}$. Since $\tilde{N}/ker\pi^*$ contains an infinite alternating group and is therefore t-transitive, there exist $n \in N$ such that $B_i^{\bar{n}} = U_i$ for all $i = 1, 2, \ldots, t$. Again $\Im(F) \cap \Im(F)n = \emptyset$ and $[F, F^n] = 1$.

□

Theorem 1 *Let $G \leq FGL(V, K)$ be an irreducible, infinite dimensional and imprimitive finitary linear group that is locally finite. Then G' is the unique minimal subnormal irreducible subgroup of G.*

Proof:

The derived group G′ of G acts irreducibly on V [Lemma 1 of [1]]. Using Lemma 2 above and (6.2) of [1], the derived group G' is contained in every normal irreducible subgroup N of G. Furthermore if $H \trianglelefteq N$, then $H \trianglelefteq G$. So if M is an irreducible subnormal subgroup of G, then $M \trianglelefteq G$. Therefore $G' \subseteq M$.

□

We will prove a similar version of Lemma 2 for irreducible, *primitive* linear groups that are locally finite. However some preliminary results are required and so we will handle the primitive groups version in Chapter 2.

Recall that stable linear groups are finitary linear groups. We will prove next that certain stable linear groups have the conjugate centralizer property with respect to classical subgroups.

The stable classical groups consist of the stable general linear group, the stable special linear group, the stable symplectic group, the stable orthogonal groups and the stable unitary group. We will show next that if G is the stable general linear group whose associated vector space has countably infinite dimension and whose associated field is locally finite, then G has the conjugate centralizer property with respect to its subgroups, the special general linear group, the orthogonal groups (excluding the case where the characteristic of the field is 2 and the defect of the quadratic form is

1) and the unitary group. If the characteristic of the field is 2 and the defect of the quadratic form is 1, then we will show that the orthogonal group acts reducibly on the associated vector space.

Theorem 2 *Let V be a vector space of countable dimension over a field K where K is locally finite. Then the stable linear groups Stab(GL(V,K)) have the conjugate centralizer property with respect to each of the following:*

1. *the stable special linear group, Stab(SL(V, K));*
2. *the stable symplectic group, Stab(Sp(V, K));*
3. *the stable orthogonal groups, Stab(O(V,K)), except when char(K)= 2 and the defect of quadratic form is 1, and*
4. *the stable unitary group, Stab(U(V, K)).*

Proof:

Let $\{v_\alpha \mid \alpha \geq 1\}$ be the G-stable basis of the vector space V over the locally finite field K. The group G is a stable general linear group i.e.

$$\mathrm{G} = \left\{ \begin{bmatrix} A_m & 0 \\ 0 & I_\infty \end{bmatrix} | A_m \in Gl(m, K), m \geq 0 \right\}.$$

Here I_∞ is the countably infinite identity matrix.

Let $F \leq G$ be a finitely generated subgroup of G. There exist a positive integer n such that for all $A \in F$,

$$\mathrm{A} = \begin{bmatrix} A_n & 0 \\ 0 & I_\infty \end{bmatrix} \text{ where } A_n \in GL(V, K).$$

We can assume that n is even. We will need the following proposition.

Proposition 1 *Let $A \in Stab(GL(V,K))$ be such that* $A = \begin{bmatrix} A_n & 0 \\ 0 & I_\infty \end{bmatrix}$,

$B = \begin{bmatrix} B_n & 0 \\ 0 & I_\infty \end{bmatrix}$, $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ *and* $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$.

Then $[A, B^\Delta] = 1$ and $[A, B^\Lambda] = 1$ and so Stab(GL(V, K)) has the conjugate centralizer property with respect to any subgroup H which contains either Δ or Λ.

Proof of Proposition 1:

The matrix $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \Delta^{-1}$. Hence $B^\Delta = \Delta^{-1} B \Delta$

$$= \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} B_n & 0 \\ 0 & I_\infty \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_n & 0 \\ B_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix}.$$

The matrix $AB^\Delta = \begin{bmatrix} A_n & 0 \\ 0 & I_\infty \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \begin{bmatrix} A_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$, and

$$B^\Delta A = \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} A_n & 0 \\ 0 & I_\infty \end{bmatrix} = \begin{bmatrix} A_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = AB^\Delta.$$

Thus $[A, B^{\Delta}] = 1$.

The matrix $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ has inverse $\Lambda^{-1} = \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$.

So $B^{\Lambda} = \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} B_n & 0 \\ 0 & I_\infty \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$

$$= \begin{bmatrix} 0 & -I_n & 0 \\ B_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = B^{\Delta},$$

and as above, $[A, B^{\Lambda}] = [A, B^{\Delta}] = 1$.

$\square$

We will show in the next lemma that either Δ or Λ is contained in almost all of the classical linear groups.

Lemma 3 *Let the vector space V over the locally finite field K have countably infinite dimension. Then all the classical linear groups , Stab(GL(V, K)), Stab(SL(V, K)), Stab(Sp(V, K)), Stab(O(V, K ,f)) and Stab(U(V, K)), except for Stab(O(V, K, f)) when the characteristic of K is 2 and the quadratic form f has defect 1, contain (relative to some basis)*

$$\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \text{ or } \Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \text{ for all } n \text{ even.}$$

Proof:

case (i) :

Let Y = Stable general linear group. Obviously both $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ and

$\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ are in Stab(GL(V,K)).

case (ii) :

Let Y = Stable special linear group, that is

$$\mathrm{Y} = \left\{ \begin{bmatrix} A_n & 0 \\ 0 & I_\infty \end{bmatrix} | A_n \in SL(n,K), n \geq 1 \right\}.$$

Since $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in SL(2n,K)$, $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \in$ Stab(SL(V,K)).

□

case (iii) :

Let Y = Stable Symplectic Group.

The skew symmetric form on V can be represented with respect to a suitable basis $\chi = \{v_\alpha \mid \alpha \geq 1\}$ by the matrix B= $\begin{bmatrix} J & 0 & 0 & \cdots \\ 0 & J & 0 & \cdots \\ 0 & 0 & J & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ where J = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then the stable symplectic group Y = $\{A \in GL(V,K) \mid A^tBA = A \text{ and } A(v_\alpha) = v_\alpha$ for all but a finite number of $v_\alpha \in \chi\}$.

Note that A^t is the transpose of matrix A . We will show that $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \in Y$ where n is even.

The transpose of $\Lambda = \Lambda^t = \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$.

Hence $\Lambda^t B \Lambda = \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \begin{bmatrix} J & 0 & 0 & \cdots \\ 0 & J & 0 & \cdots \\ 0 & 0 & J & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$

$= \begin{bmatrix} 0 & -J_n & 0 \\ J_n & 0 & 0 \\ 0 & 0 & J_\infty \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$

where $J_n = \begin{bmatrix} J & 0 & 0 & \cdots \\ 0 & J & 0 & \cdots \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & 0 & J \end{bmatrix} \in Sp(n, K)$ and $J_\infty = \begin{bmatrix} J & 0 & 0 & \cdots \\ 0 & J & 0 & \cdots \\ 0 & 0 & J & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

Thus $\Lambda^t B \Lambda = \begin{bmatrix} J & 0 & 0 & \cdots \\ 0 & J & 0 & \cdots \\ 0 & 0 & J & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ = B.

Also since the matrix Λ has the property that $\Lambda(v_\alpha) \neq v_\alpha$ only for $\alpha \in \{1, 2, 3, \ldots, 2n\}$, $\Lambda \in Y$.

□

case (iv) :

Y = the stable orthogonal group.

The proof of this case is split into two subcases; in the first we assume $char(K) \neq 2$ and in the second we assume $char(K) = 2$.

(a) $char(K) \neq 2$:

Let B be a non-degenerate symmetric bilinear form on V over K. We will prove the following:

(i) : There exist a basis $\{u_1, u_2, u_3, \ldots\}$ and field elements $\{b_1, b_2, b_3, \ldots\}$ such that $B(u_i, u_j) = b_i\delta_{ij},\ b_i \neq 0$ for all $i = 1, 2, 3, \ldots$.

(ii) : If K is a finite field, then B has the matrix of the form $diag\{1, 1, 1, \ldots\}$.

(iii) : If K is a locally finite field, the conclusions of (ii) still hold.

Proof of (i) :

There exist $v \in V$ such that $B(v, v) \neq 0$. Otherwise for all $u \in V$, $0 = B(v, v) = B(u + v, u + v) - B(u, u) - 2B(u, v)$. Thus B(u, v)= 0 for all $u, v \in V$ and B is degenerate. Let $u_1 = v$.

Let $\{u_1, u_2, \ldots\} \subseteq V$ be a linearly independent set with $B(u_i, u_j) = b_i\delta_{ij},\ b_i \neq 0\ \forall\ i, j \in \{1, 2, \ldots, k\}$. Define $V_k = < u_1, u_2, \ldots, u_k >$. We will show that $V = V_k \oplus V_k{}^{\perp}$ where $V_k{}^{\perp}$ is the subspace $\{v \in V \mid B(v, u_i) = 0\ \forall\ i = 1, 2, 3, \ldots, k\}$.

Let $x \in V$. Choose $y = x - \sum_{i=1}^{k} b_i^{-1} B(x, u_i)u_i$. We now have

$$\begin{aligned} B(u_j, y) &= B(x, u_j) - b_j^{-1}B(x, u_j)B(u_j, u_j) \\ &= B(x, u_j) - B(x, u_j) \\ &= 0 \text{ for all j} = 1, 2, 3, \ldots, \text{k}. \end{aligned}$$

Thus $y \in V_k^\perp$ and $x = \sum_{i=1}^k b_i^{-1} B(x, u_i) u_i + y \ \in V_k + V_k^\perp$.

Since B is non-degenerate, B restricted to the space $V_k^\perp$ is non-degenerate. So there exist a vector $u_{k+1} \in V_k^\perp$ such that $B(u_{k+1}, u_{k+1}) \neq 0$. Let $b_j = B(u_{k+1}, u_{k+1})$. Since $u_{k+1} \in V_k^\perp$, $B(u_i, u_j) = b_i \delta_{ij}$ for all $i, j \in \{1, 2, 3, \ldots, k+1\}$.

□

Proof of (ii) :

Choose $\{u_1, u_2\}$ to be linearly independent vectors of V such that $B(u_1, u_2) = b_i \delta_{ij}$ and $b_i \neq 0$, $i, j \in \{1, 2\}$. The symmetric form B restricted to $V_2 =< u_1, u_2 >$ is thus non-degenerate. From [5, p. 360], any non-degenerate symmetric bilinear form B on a vector space V of *dimension* ≥ 2 over a field of *characteristic* $\neq 2$ is universal i.e. B(v, v)= b has a solution for every $0 \neq b \in K$. Choose $\tilde{u}_1 \in V_2$ to be such that $B(\tilde{u}_1, \tilde{u}_1) = 1$.

Suppose there exist linearly independent vectors $\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k\}$ such that $B(\tilde{u}_i, \tilde{u}_j) = \delta_{ij} \ \forall \ i, j \in \{1, 2, \ldots, k\}$. As in the proof of (i), $V = \tilde{V}_k \oplus \tilde{V}_k^\perp$ where $\tilde{V}_k =< \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k >$. The symmetric bilinear form, B, restricted to $\tilde{V}_k^\perp$ is non-degenerate so by (i), there exist vectors of $\tilde{V}_k^\perp$, u_1^k and u_2^k, such that $B(u_i^k, u_j^k) = b_i \delta_{ij} \ \forall \ i, j \in \{1, 2\}$ and $b_i \neq 0$. We can once again choose $\tilde{u}_{k+1} \in< u_1^k, u_2^k >$ such that $B(\tilde{u}_{k+1}, \tilde{u}_{k+1}) = 1$. Thus from this construction, we obtain $\{\tilde{u}_1, \tilde{u}_2, \ldots\}$ such that $B(\tilde{u}_i, \tilde{u}_j) = \delta_{ij} \ \forall \ i, j \geq 1$.

□

Proof of (iii) :

The field K is locally finite and so can be written as $K = \cup_{\alpha \in A} K_\alpha$ where the K_α's are finite fields. Let V_K^2 be the vector space over the field K spanned by $\{u_1, u_2\}$ and for any $\alpha \in A$, let $V_{K_\alpha}^2$ be the vector space over the field K_α spanned by the same

vectors. From (i), we can assume that $B(u_i, u_j) = b_i\delta_{ij},\ b_i \neq 0$ and $i, j \in \{1, 2\}$. We have $V^2_{K_\alpha} \subseteq V^2_K$ as subsets. We will show that B is universal on V_K.

Since $B(u_i, u_j) = b_i\delta_{ij}$, B restricted to $V^2_{K_\alpha}$ is non-degenerate and is therefore universal. Let $0 \neq f \in K$. There exist $\beta \in A$ such that $f \in K_\beta$. Choose $v \in V^2_{K_\beta}$ such that $B(v, v) = f$. But $v \in V^2_{K_\beta} \subseteq V_K$ and B is therefore universal on V_K.

Then using similar methods as in (ii), we can construct a basis $\{v_1, v_2, \ldots\}$ of V over K such that $B(v_i, v_j) = \delta_{ij}\ \forall\ i, j \geq 1$.

□

Let $A \in GL(V, K)$ be such that $A^tBA = B$. By (iii), we can choose a basis $\chi = \{u_1, u_2, \ldots\}$ of V such that B = I with rspect to this basis. Hence $A^tA = I$ where I is the identity matrix. Therefore Y =

$\{A \in GL(V)\ |A^tA = I$ and $A(u_\alpha) = u_\alpha\ for\ all\ but\ a\ finite\ number\ of\ \alpha's \geq 1\}$.

Obviously both $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \in Y$.

□

(b)$char(K) = 2$:

For any $x, y \in V$ let B(x,y) be a symplectic scalar product on V, not necessarily non-singular. A quadratic form f on V is a function with values in K satisfying : $f(\lambda x + \mu y) = \lambda^2 f(x) + \mu^2 f(y) + \lambda\mu B(x, y)$, for all $x, y \in V$ and $\lambda, \mu \in K$. Assume f is non-degenerate. Since K is locally finite, K is perfect ie. $K = K^2$ so the defect of f is 0 or 1.

Let the *defect of f be 0* . The matrix of B with respect to some basis

$\Xi = \{e_1, e_2, e_3, \ldots\}$ is of the form $\begin{bmatrix} 0 & 1 & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & 1 & \cdots \\ \cdots & \cdots & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ and therefore $Y \leq$ $Stable(Sp(V, \Xi, K))$.

We will show that there exist a basis $\Upsilon = \{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \ldots\}$ that preserves B and such that $f(e_i) = f(e_{-i}) = 0 \; \forall \; i \geq 1$.

If for any i, $f(e_i) = 0$, we will first show that we can also assume $f(e_{-i}) = 0$. Suppose $f(e_i) = 0$ and $f(e_{-i}) \neq 0$. Let $e_i^{'} = c_i e_i$ and $e_{-i}^{'} = c_i e_i + c_i^{-1} e_{-i}$ where $c_i \in K$ is such that $c_i^2 = (f(e_{-i}))$. We have

$$\begin{aligned}
f(e_i^{'}) &= c_i^2 f(e_i) \\
&= 0 \; and \\
f(e_{-i}^{'}) &= c_i^2 f(e_i) + (c_i^{-1})^2 f(e_{-i}) + B(e_i, e_{-i}) \\
&= 0 + 1 + 1 \\
&= 0. \\
Also \; B(e_i^{'}, e_{-i}^{'}) &= B(c_i e_i, \; c_i e_i + c_i^{-1} e_{-i}) \\
&= c_i c_i^{-1} B(e_i, e_{-i}) \\
&= 1, \\
B(e_i^{'}, e_i^{'}) &= c_i^2 B(e_i, e_i) \\
&= 0, \\
and \; B(e_{-i}^{'}, e_{-i}^{'}) &= B(c_i e_i + c_i^{-1} e_{-i}, \; c_i e_i + c_i^{-1} e_{-i}) \\
&= 1 + 1
\end{aligned}$$

$$= \quad 0.$$

Consider e_1 and e_{-1}. If $f(e_1) = 0$, we can also assume that e_{-1} is also 0. Then let $\tilde{e}_1 = e_1$ and $\tilde{e}_{-1} = e_{-1}$. Suppose $f(e_1) \neq 0$. Wlog we can also assume $f(e_{-1}) \neq 0$. Let $V_1 =< e_1, e_{-1} >$ and $V_1^\perp = \{x \in V \mid B(e_i, x) = 0 \ for \ i = \pm 1\}$. We will show that there exist $x \in V_1^\perp$ such that $f(x) \neq 0$.

Suppose $f(x) = 0$ for all $x \in V_1^\perp$. Since e_2, e_{-2} and hence $e_2 + e_{-2} \in V_1^\perp$, then $f(e_2)$, $f(e_{-2})$ and $f(e_2 + e_{-2})$ are all $0's$. On the other hand,

$$\begin{aligned} f(e_2 + e_{-2}) &= f(e_2) + f(e_{-2}) + B(e_2, e_{-2}) \\ &= 1. \end{aligned}$$

Let $x \in V_1^\perp$ such that $f(x) \neq 0$. Let c and $c_1 \in K$ be such that $c^2 = (f(x))^{-1}$ and $c_1^2 = (f(e_1))^{-1}$. Let $e_1' = c_1 e_1 + cx$. We have

$$\begin{aligned} f(e_1') &= c_1^2 f(e_1) + c^2 f(x) + cc_1 B(e_1, x) \\ &= 1 + 1 + 0 \\ &= 0. \end{aligned}$$

Let $e_{-1}' = c_1^{-1} e_{-1}$. We have $B(e_1',\ e_{-1}') = B(c_1 e_1 + cx,\ c_1{}^{-1} e_{-1}) = 1$, $f(e_1') = 0$ and $f(e_{-1}') = (c_1^2)^{-1} f(e_{-1}) \neq 0$. Now use the technique above to produce $\tilde{e}_1, \tilde{e}_{-1}$ such that $f(\tilde{e}_1) = f(\tilde{e}_{-1})$= 0 i.e. let $\tilde{e}_1 = de_1'$ and $\tilde{e}_{-1} = de_1' + d^{-1}e_{-1}'$ where $d^2 = (f(e_{-1}'))$.

Suppose we have $\{\tilde{e}_1, \tilde{e}_{-1}, \tilde{e}_2, \tilde{e}_{-2}, \ldots, \tilde{e}_k, \tilde{e}_{-k}\}$, linearly independent vectors of V such that $f(\tilde{e}_i) = 0$, $B(\tilde{e}_i, \tilde{e}_j) = \begin{cases} 0 & \text{if } i \neq -j, \\ 1 & \text{if } i = -j \end{cases}$ for all $i, j \in \{\pm 1, \pm 2, \ldots, \pm k\}$. Let $V_k =< \tilde{e}_{\pm 1}, \tilde{e}_{\pm 2}, \ldots, \tilde{e}_{\pm k} >$ and $V_k^\perp = \{x \in V \mid B(x, \tilde{e}_i) = 0 \ for \ all \ i, j \in \{\pm 1, \pm 2, \ldots, \pm k\}\}$. Restricting f to $V_k^\perp$, f is non-degenerate. As before there exist $\bar{e}_1$ and $\bar{e}_{-1} \in V_k^\perp$ such that $f(\bar{e}_1) = f(\bar{e}_{-1}) = 0$ and for all $i, j \in \{1, -1\}$,

$$B(\bar{e}_i, \bar{e}_j) = \begin{cases} 1 & \text{if } i = -j, \\ 0 & \text{if } i \neq -j \end{cases}$$. Now, let $\tilde{e}_{k+1} = \bar{e}_1$ and $\tilde{e}_{-(k+1)} = \bar{e}_{-1}$.

□

Once we have bases $\{\tilde{e}_{\pm 1}, \tilde{e}_{\pm 2}, \ldots\}$ that preserves B and $f(\tilde{e}_{\pm i}) = 0$ for all $i \geq 1$, the form f can then be described by $f(x) = x_1 x_{-1} + x_2 x_{-2} + \cdots + x_l x_{-l}$ if $x = \sum_{i=-l}^{i=l} x_i \tilde{e}_i,\ x_i \in K, l \geq 1$. Thus

$$\begin{aligned} Y &= \{T \in GL(V, K) | f(Tx) = f(x)\ \forall \mathrm{x} \in \mathrm{V} \text{ and } \mathrm{T}(\tilde{\mathrm{e}}_\mathrm{i}) = \tilde{\mathrm{e}}_\mathrm{i} \text{ for all but a finite } i's\} \\ &= \{T \in Stable(Sp(V, \Upsilon, K)) \mid f(T\tilde{e}_i) = 0\ \forall i = \pm 1, \pm 2, \ldots\} \end{aligned}$$

Consider $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$ where n is even.

Notice that

$$\begin{aligned} f(\Delta\tilde{e}_i) &= f(\tilde{e}_{n+i}) = 0, \\ f(\Delta\tilde{e}_{n+i}) &= f(\tilde{e}_i) = 0\ \forall\ i = \pm 1, \ldots, \pm\frac{n}{2}, \\ f(\Delta\tilde{e}_i) &= f(\tilde{e}_i) = 0\ \forall\ |i| > 2n, \\ and\ B(\Delta\tilde{e}_i, \Delta\tilde{e}_j) &= B(\tilde{e}_i, \tilde{e}_j)\ \forall\ i, j \in Z^* \end{aligned}$$

Thus $\Delta \in Y$. Also since char(K) = 2, $\Delta = \Lambda$.

□

case (iv) :

Y = Stable Unitary Group.

Let $\sigma : \lambda \mapsto \bar{\lambda}$ be an automorphism of K of order 2 and H(x, y) be a non-singular hermitian product on $V \times V$. We will show that there exist a basis $\chi = \{e_1, e_2, \ldots\}$ such that $H(e_i, e_j) = \delta_{ij},\ i, j = 1, 2, 3, \ldots$

There exist an $x \in V$ and $y \in V$ such that $H(x,x) \neq 0$. Otherwise if $x, y \in V$,

$$\begin{aligned} 0 &= H(x+y, x+y) \\ &= H(x,x) + H(y,y) + H(x,y) + H(y,x) \\ &= H(x,y) + \overline{H(x,y)}. \end{aligned}$$

Let $\alpha = H(x,y)$. Since $0 = \alpha + \bar{\alpha}$, $\alpha = -\bar{\alpha}$. Furthermore,

$$\begin{aligned} \sigma(\alpha\bar{\alpha}) &= \sigma(\alpha)\sigma(\bar{\alpha}) \\ &= \bar{\alpha}\alpha \\ &= \alpha\bar{\alpha} \end{aligned}$$

Thus $\alpha\bar{\alpha} \in K_0$, the fixed field of involution of K. We obtain $-\alpha^2 \in K_0$, and since K is locally finite and therefore perfect, $\alpha \in K_0$. Thus $H(x,y) \in K_0$ for all $x, y \in V$. However if $\beta \in K \backslash K_0$, $H(\beta x, y) = \beta H(x,y) \in K \backslash K_0$ which is a contradiction.

The field K is perfect so wlog we can assume H(x, x)= 1 for some $x \in V$. Let $e_1 = x$. Assume next there exist linearly independent vectors in V, $\{e_1, e_2, \ldots, e_k\}$ such that $H(e_i, e_j) = \delta_{ij}$, $i, j \in \{1, 2, \ldots, k\}$. Let $V_k =< e_1, e_2, \ldots, e_k >$. As before $V = V_k \oplus V^\perp$ where $V_k^\perp = \{v \in V \mid H(v, e_i) = 0 \;\forall\; i = 1, 2, \ldots, k\}$. The hermitian product H restricted to $V_k^\perp$ is again non-singular and one obtains as above $e_{k+1} \in V_k^\perp$ such that $H(e_{k+1}, e_{k+1}) = 1$. Since $e_{k+1} \in V_k^\perp$, $H(e_{k+1}, e_i) = 0 \;\forall\; i = 1, 2, \ldots, k$. Thus one obtains the basis as described above.

The stable unitary group G can now be described as the set $\{T \in GL(V,K) \mid Te_i = e_i \textit{ for all but a finite number of } e_i^\prime s \textit{ and } H(Te_i, Te_j) = H(e_i, e_j) = \delta_{ij},\ i, j \geq 1\}$. Consider $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$. Since for any i and j,

$\Delta e_i = e_{i_1}$ and $\Delta e_j = e_{j_1}$ for some i_1 and j_1,

$$\begin{aligned} H(\Delta e_i, \Delta e_j) &= H(e_{i_1}, e_{j_1}) \\ &= \delta_{i_1,\, j_1} \\ &= \delta_{i,j} \\ &= H(e_i, e_j). \end{aligned}$$

Thus $\Delta \in Y$. It is also easy to check that $\Lambda \in Y$.

□

Thus by Proposition 2, Stab(GL(V, K)) has the conjugate centralizer property with respect to each subgroup, Stab(SL(V,K)), Stab(Sp(V,K), Stab (O(V, K, f)) (except for when char(K) = 2 and defect of f = 1) and Stab(U(V, K)). This completes the proof of our Theorem 2.

□

On the other hand the stable orthogonal group, O(V, K, f), with char(K) = 2 and defect f= 1 is shown in Lemma 4 below to act reducibly on the vector space V. With this result, if M is a simple component of a group G, where G is a countable, primitive, locally finite finitary linear group of contable dimension, then M cannot be a group of this type (see chapter 2).

Lemma 4 *Let V be a vector space of countable dimension over a locally finite field of characteristic 2. Let $G \leq FGL(V, K)$ be a countable group, $G \leq$ Stab$(GL(V, K))$, and $H \leq G$ be the stable orthogonal group, H=O(V, K, f), where the quadratic form, f, has defect 1. Then H acts reducibly on V.*

Proof:

We will first show that there exist basis $\Lambda = \{e_1, e_2, \ldots\}$ such that $f(e_1) = 1$ and $f(e_i) = 0$ for all $i \geq 2$. By a suitable choice of basis, say $\chi = \{\tilde{e}_1, \tilde{e}_2, \ldots\}$, the matrix of B can be represented by $\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 1 & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. The orthogonal group Stable O(V,χ, K, f)=

$\{T \in GL(V) |\ f(Tx) = f(x)\ \forall x \in V\ and\ T(\tilde{e}_i) = \tilde{e}_i\ for\ all\ but\ a\ finite\ \tilde{e}_i's\ \}$.

Since F is non-denegerate, $f(\tilde{e}_1) \neq 1$. Let $e_1 = \alpha\tilde{e}_1$ where $\alpha^2 = (f(\tilde{e}_1))^{-1}$. Since $B(\tilde{e}_1, x) = 0\ \forall x \in V$, then $B(e_1, x) = 0\ \forall\ x \in V$. In addition, $f(e_1) = \alpha^2 f(\tilde{e}_1) = 1$.

Use the proof from the case of the stable orthogonal group whose associated field has characteristic 2 and the the quadratic form is of defect 0 to produce basis elements $\{e_2, e_3, \ldots\}$ of the subspace $V_1 =< \tilde{e}_2, \tilde{e}_3, \ldots >$ that preserves the scalar product $B|_{V_1}$ and $f(e_i) = 0\ \forall\ i \geq 2$.

$\square$

Let $T \in O(V, \Lambda, K, f)$. We will next show that $T(e_1) = e_1$. Suppose $T(e_1) = \sum_{i=1}^{n} c_i e_i$. We have

$$\begin{aligned} f(Te_1) &= f(\sum_{i=1}^{n} c_i e_i) \\ &= f(e_1) \end{aligned}$$

For the next step in the argument we need to observe that B(x, y)= B(Tx,Ty) for all $x, y \in V$. This is because,

$$\begin{aligned} f(x+y) &= f(x)+f(y)+B(x,y) \\ &= f(T(x+y)) \\ &= f(Tx+Ty) \\ &= f(Tx)+f(Ty)+B(Tx,Ty) \\ &= f(x)+f(y)+B(Tx,Ty), \end{aligned}$$

so B(x, y) = B(Tx, Ty) $\forall\ x, y \in V$. Since $B(x, e_1) = 0\ \forall x \in V$,

$$\begin{aligned} f(\sum_{i=1}^{n} c_i e_i + e_1) &= f(\sum_{i=1}^{n} c_i e_i) + f(e_1) \\ &= f(e_1)+f(e_1) \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} B(\sum_{i=1}^{n} c_i e_i + e_1, x) &= B(T(e_1)+e_1, x) \\ &= B(T(e_1,x)+B(e_1,x) \\ &= B(Te_1, x) \\ &= B(Te_1, TT^{-1}x) \\ &= B(e_1, T^{-1}x) \ \mathit{since}\ B(Tx,Ty)=B(x,y) \\ &= 0 \end{aligned}$$

Since f is non-degenerate, this implies $\sum_{i=1}^{n} c_i e_i + e_1 = 0$ and $c_2 = c_3 = \cdots = c_n = 0$, and $c_1 = 1$.

□

Thus Ke_1 is an H-subspace and so H acts reducibly on V.

□

CHAPTER 2

Simplicity of the Derived Groups of Primitive Groups

It has been observed that many results in finitary permutation groups have analogues in finitary linear groups. Here we show two such correspondences. From [8, p.10] we have the fact that every infinite transitive group of finitary permutation has the conjugate centralizer property with respect to every **transitive** normal subgroup of G. In this thesis, Lemma 2 of Chapter 1 and Lemma 5 below yield the following result: If $G \leq FGL(V, K)$ is locally finite and dimension of V is infinite, then G has the conjugate centralizer property with respect to every **irreducible** normal subgroup of G.

Secondly, from 4.1.3 [10] we have that if Ω is infinite and $G \leq FSym(\Omega, \aleph_0)$ is primitive, then every normal subgroup of G must be transitive. It follows that G′ must be simple. Observe that if $G \leq FGL(V, K)$ is primitive and dim(V) is infinite then by 4.1 of [1], every normal subgroup of G must be irreducible. Furthermore if in addition G is locally finite, Theorem 3 below produces the result that G' is simple.

Definition 7 *Let G be a locally finite group. A subgroup M of G is a* **component** *of G if*

(a) $M \lhd \lhd G$

(b) $M' = M$

(c) $M/Z(M)$ *is simple, where Z(M) is the center of M.*

In the following lemma, we will prove that if N is a simple subgroup of a locally finite and primitive finitary linear group and N is a stable group with respect to some basis Λ of V, then G is also stable with respect to basis Λ. The proof uses ideas from a recent paper by Leinen and Puglisi [7]. First we will prove the following:

Proposition 2 *Let $G \leq FGL(V,K)$ be locally finite and primitive, N a simple subgroup of G and $N \leq Stab(V,\Lambda,K)$ for some basis Λ of V. If V_0 is a finite dimensional subspace of V, then $\exists$ a finite subset Λ_0 of Λ and a finite subgroup N_0 of N such that the following holds:*

1. $V_0 \subseteq K\Lambda_0$
2. $[\Lambda\backslash\Lambda_0, N_0] = 0$,
3. *for every* $v \in \Lambda_0$, $\exists\ g \in N_0$ *such that* $g(v) \neq v$ *and*
4. N_0 *acts faithfully on* $K\Lambda_0$.

Proof:

The subspace V_0 is finite dimensional so $\exists$ a finite set $A \subseteq \Lambda$ such that $V_0 \subseteq KA$. Let $A = \{V_i \mid i = 1, 2, \ldots, k\}$. Since N is simple and G is primitive, N acts irreducibly on V. Thus for each $i \in \{1, 2, \ldots, k\}$, $\exists\ g_i \in N$ such that $g(v_i) \neq v_i$. Let $N_0 =< g_i \mid i = 1, 2, \ldots, k >$.

Since G is locally finite, N_0 is a finite subgroup of N. Let $\Lambda_0 = \{v \in \Lambda \mid g(v) \neq v \textit{ for some } g \in N_0\}$. Obviously $V_0 \subseteq KA \subseteq K\Lambda_0$, and since $N_0 \leq N \leq Stab(V,\Lambda,K)$, Λ_0 is finite. Furthermore, by the definition of Λ_0, $[\Lambda\backslash\Lambda_0]=$ 0, for each $v \in \Lambda_0$, $g(v) \neq v$ for some $g \in N_0$ and N_0 acts faithfully on KΛ_0.

$\square$

Lemma 5 *Let $G \leq FGL(V,K)$ be locally finite and primitive, N a simple subgroup of G such that $N \leq Stab(V,\Lambda,K)$ for some basis Λ of V. Then $G \leq Stab(V,\Lambda,K)$.*

Proof:

Let $g \in G \leq FGL(V, K)$. By Proposition 1, $[V, g] \subseteq K\Lambda_0$ for some finite subset $\Lambda_0 \subseteq \Lambda$. In addition $\exists$ a finite subgroup $N_0 \subseteq N$ such that $[\Lambda \backslash \Lambda_0, N_0] = 0$, N_0 acts faithfully on $K\Lambda_0$ and for each $v \in \Lambda_0$, $\exists\ g \in N_0$ such that $g(v) \neq v$.

Suppose $\exists\ g \in G$ such that $g \notin Stab(V, \Lambda, K)$. Then $\exists$ an infinite set $\{v_i \mid i \in I\} \subseteq \Lambda \backslash \Lambda_0$ such that for each $i \in I$, $g(v_i) \neq v_i$. We will show that for each $i \in I$, $\exists\ h_i \in N_0$ such that $g^{-1}h_ig_i(v_i) \neq v_i$.

Since $[V, g] \subseteq K\Lambda_0$, we can write $g(v_i) = v_i + w_i$ for some $w_i \in K\Lambda_0$. Choose $h_i \in N_0$ such that $h_i(w_i) \neq w_i$. Therefore

$$\begin{aligned}
g^{-1}h_ig(v_i) &= g^{-1}h_i(v_i + w_i) \\
&= g^{-1}h_i(v_i) + g^{-1}h_i(w_i) \\
&= g^{-1}(v_i) + g^{-1}h_i(w_i) \textit{ since } v_i \in \Lambda \backslash \Lambda_0 \\
&= g^{-1}(v_i + h_i(w_i)) \\
&\neq g^{-1}(v_i + w_i) \textit{ since } h_i(w_i) \neq w_i \\
&= v_i.
\end{aligned}$$

Thus for each $i \in I$, $g^{-1}h_ig(v_i) \neq v_i$ for some $h_i \in N_0$. But $N_0 \leq N$ is a finite subgroup and so $g^{-1}N_0g$ is also finite. Since $N \trianglelefteq G$, $g^{-1}N_0g \leq N \leq Stab(V, \Lambda, K)$ and so $g^{-1}N_0g$ centralizers all but a finite subset of Λ. However I is infinite and this leads to a contradiction.

□

Lemma 6 *Let $G \leq FGL(V, K)$ be countable, locally finite and primitive. Furthermore let the dimension of V be countably infinite and K be a locally finite field. Then G has the conjugate centralizer property with respect to a simple normal subgroup of G.*

Proof:

The group $G \leq FGL(V, K)$ is locally finite and primitive and the dimension of V is infinite. Thus from 10.12 of [10], G has a unique component M and $C_G(M) = 1$. Therefore M is simple and $M \trianglelefteq G$. J. Hall's theorem [4] provides a complete list of locally finite simple subgroups of FGL(V, K) under certain restrictions. This list is crucial in describing the simple subgroup M.

J.Hall's theorem states the following;

Let G be a locally finite, infinite, non- "finite dimensional", simple subgroup of FGL(V,K) where dim(V) is infinite. Then G is either alternating , the derived subgroup of FSp(V,K,a), FO(V, K, q) or FU(V, K, h) or a group of type T(W, V), $W \subseteq V^*$ where $ann_V(W) = 0$. (See the introduction for a more thorough description of these groups.) Here a,h,q are respectively non-degenerate alternating, Hermitian and quadratic forms on V.

If V has countably infinite dimension , the last group mentioned above in J. Hall's theorem is just the group FSL(V, K). The group $G \leq FGL(V, K)$ is countable and irreducible so by Lemma 1 from Chapter 1, G has a G-stable basis and thus we can regard G as a stable linear group. Furthermore if $M \leq Stab(V, \Lambda, K)$ for some basis Λ of V, then the previous lemma showed that $G \leq Stab(V, \Lambda, K)$.

The derived group of GL(V, K) is SL(V, K) and the derived subgroups of FSp(V, K, a), FU(V, K, h) or FO(V, K, q) are nothing more then the intersection of SL(V, K) with each subgroup FSp(V, K, a), FU(V, K, h) or FO(V, K,q).

Therefore the simple group M is one of the following;

- the alternating group
- stable SL(V, K)
- $SL(V, K) \cap Stable(Sp(V, K, a))$

- $SL(V,K) \cap Stable(U(V,K,h))$

- $SL(V,K) \cap Stable(O(V,K,q))$

We will now show that G has the conjugate centralizer property with respect to M.

case (i) : M = Stable SL(V,K)

This follows from Theorem 2 of Chapter 1.

□

case (ii) : $M = SL(V,K) \cap stable(Sp(V,K,a)), SL(V,K) \cap stable(U(V,K,h)) or$ $SL(V,K) \cap stable((O,K,q))$. In the case of stable((O, K, q) we assume $char(K) \neq 2$

Refering back to Lemma 3 in Chapter 1, observe that $\Lambda = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \in M.$

Thus by Proposition 1 of Chapter 1, G has the conjugate centralizer property with respect to M .

□

case (iii a) : M = $SL(V,K) \cap stable(O(V,K,q))$, char(K)= 2 and defect of q= 0

Since char(K) = 2, $\Delta = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_\infty \end{bmatrix} = \Lambda \in M$. Thus again by Proposition 1, G has the conjugate centralizer property with respect to M.

□

case (iii b) : M $= SL(V,K) \cap stable(O(V,K,q))$, char(K)= 2 and defect of q= 1

The subgroup M is the unique component of G and thus acts irreducibly on the vector space V. But Lemma 4 from the previous chapter showed that stab(O(V, K, q)) acts reducibly on V when char(K)=2 and defect q= 1. Therefore M cannot be a group of this type.

$\square$

case (iv) : M= the infinite alternating group.

The vector space V has a G-stable basis say $\Upsilon = \{u_1, u_2, \ldots\}$. The subgroup $M \leq G$ is irreducible and so the unique finitary representation of M is the natural module [12, Prop 3]. Thus with respect to M, V has the basis of the form $u_i - u_i^g$ where $g \in M$. Call this basis $\chi = \{v_i \mid i \geq 1\}$. We can then describe M =

$$\left\{ \begin{bmatrix} A_n & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix} \mid \textit{where } A_n \textit{ is a permutation matrix on } \{v_1, v_2, \ldots, v_n\},\ n \geq 1 \right\}.$$

Let $F \leq G$ be a finite subgroup. Then F moves at most $\{v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_k}\}$ for some $k \geq 1$. Let $\{v_{\beta_1}, v_{\beta_2}, \ldots, v_{\beta_k}\}$ be disjoint from support F. The group M is k-transitive and so there exist $S \in M$ such that $Sv_{\alpha_i} = v_{\beta_i}$ $\forall i = 1, 2, \ldots, k$. Let $A, B \in F$ and let $Av_{\alpha_i} = v_{\alpha_{i'}}$ for $i = 1, 2, \ldots, k$. Therefore,

$$\begin{aligned} B^S(v_{\alpha_i}) &= S^{-1}BS(v_{\alpha_i}) \\ &= S^{-1}B(v_{\beta_i}) \\ &= S^{-1}(v_{\beta_i}) \textit{ since } v_{\beta_i} \textit{ is not in support}(F) \\ &= v_{\alpha_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} AB^S(v_{\alpha_i}) &= v_{\alpha_{i'}} \text{ and} \\ B^S A(v_{\alpha_i}) &= B^S(v_{\alpha_{i'}}) \\ &= S^{-1}BS(v_{\alpha_{i'}}) \\ &= S^{-1}B(v_{\beta_{i'}}) \\ &= S^{-1}(v_{\beta_{i'}}) \\ &= v_{\alpha_{i'}} \\ &= AB^S(v_{\alpha_i}) \end{aligned}$$

Thus $[F, F^S] = 1$.

□

Observe that case (iii) above does not use the assumption that the field K is locally finite. We will show next that Lemma 5 is also true for all $G \leq FGL(V, K)$ where G is locally finite and primitive, and V has infinite dimension.

Proposition 3 *Let $G \leq FGL(V, K)$ be locally finite and $M \trianglelefteq G$ be a simple irreducible subgroup of G. Then if $F \leq G$ is a finite subgroup, $\exists$ a simple countable subgroup of M, R, such that*

1. $R^F = R$ *and*

2. $< R, F >$ *acts irreducibly on a countable subspace $V_{R,F}$ of V over a countable subfield $K_{R,F}$ of K.*

Proof:

We have $M \trianglelefteq G$ simple and irreducible. Thus utilizing Therem 4.1 of [9] and Lemma 12 of [12], there are countably directed systems $\wp$ and $\Re$ of M such that

(a) If $C \in \wp$, C is a countable simple group;

(b) If $D \in \Re$, D is countable and there is a countable subfield K_D of K and a countable dimensional K_D-subspace V_D of V such that V_D is a faithful and irreducible D-module. Further, if $D, E \in \Re$ with $D \subseteq E$, then $K_D \subseteq K_E$ and $V_D \subseteq V_E$.

Let F be a finite subgroup of G, $D \in \Re$ and $C \in \wp$. Then there is a $C_1 \in \wp$ and a $D_1 \in \Re$, such that $C^F \subseteq C_1 \subseteq D_1$. Inductively define $C_n \in \wp$ and $D_n \in \Re$ by

$$C_n^F \subseteq C_{n+1} \subseteq D_{n+1}.$$

Then, if $R = \cup D_n = \cup C_n$, we see that $C \subseteq R$, R is countable, R^F is simple, R acts irreducibly on a countable dimensional subspace of V, and $R^F = R$. If Y is a countable dimensional subspace on which R acts faithfully and irreducibly, then $V_{R,F} = < YF >$ is still of countable dimension; further $< R, F >$ acts irreducibly on $V_{R,F}$ and this completes the proof of the proposition.

$\square$

Lemma 7 *Let $G \leq FGL(V, K)$ be locally finite and primitive. Also let the dimension of V be infinite. Then G has the conjugate centralizer property with respect to a simple subgroup of G.*

Proof:

Let M be the unique simple component of G. We show that G has the conjugate centralizer property with respect to M.

Let $F \leq G$ be a finite group. From Proposition 2 above, $\exists$ a countable subgroup R of M such that $R^F = R$, R is simple and $< R, F >$ acts irreducibly on some countable subspace, $V_{R,F}$ of V over a countable subfield, $K_{R,F}$ of K.

Suppose $< F, R >$ is primitive on $V_{F,R}$. We will first show that there exist an infinite simple subgroup N such that $< F, R >$ has the conjugate centralizer property

with respect to N. If char(K) is 0, then let $M_{R,F}$ be the unique simple component of $< F, R >$. Another theorem by J. Hall [3, Theorem 1] states:

Suppose G is a locally finite simple subgroup of FGL(V, K) where char(K)= 0 and dimension V is infinite. Then G is the infinite alternating group.

Thus $M_{R,F}$ is an infinite alternating group and so case (iii) of Lemma 5 implies that $< F, R >$ has the conjugate centralizer property with respect to $M_{R,F}$ (Recall that case (iii) in Lemma 5 does not use the assumption that K is locally finite). Let N = $M_{R,F}$.

Suppose $char(K) \neq 0$. Then a theorem of Leinen [6] shows that we may assume that K is locally finite. Since the dimension of $V_{R,F}$ is countably infinite, Lemma 5 implies that $< F, R >$ has the conjugate centralizer property with respect to a simple infinite subgroup, say H. Let N = H.

We will show next that N = R. Since R is simple, $R \cap N = 1$ or $R \cap N = R$. If $R \cap N = 1$, then $NR/R \cong N$ which is infinite. However, $NR/R \subseteq < F, R > /R \cong F/F \cap R$ which is finite. Thus $R \cap N = R$ and since N is simple, R = N. Hence $[F, F^g] = 1$ for some $g \in R$. But $R \subseteq M$ and so G has the conjugate centralizer property with respect to M.

Suppose $< F, R >$ is imprimitive on $V_{F,R}$. Proposition 2 implies that $R \trianglelefteq < F, R >$ acts irreducibly on $V_{F,R}$ and by Lemma 2 of Chapter 1, $< F, R >$ has the conjugate centralizer property with respect to R. Hence G has the conjugate centralizer property with respect to M.

□

Theorem 3 *Let $G \leq FGL(V, K)$ be locally finite and primitive. Then G' is simple*

Proof:

Again as in Lemma 6 above, let M be the unique simple component of G. Since G has the conjugate centralizer property with respect to M, we have [12, 6.2] $G' \subseteq M$. However M is simple, so $G' = M$.

$\square$

CHAPTER 3

Composition Factors of Residually Finite Groups

Theorem 4 below is the main result of this thesis. It is a generalization of a theorem by Bruno and Phillips [1, Theorem A]. The theorem by Bruno-Phillips states the following:

Let $G \leq FGL(V, K)$ with unip(G)= 1 and suppose that G satisfies either one of the following conditions.

(i) G is both residually solvable and locally solvable

(ii) G is residually finite, locally finite, and either char(K) $= 0$, or $char(K) = p > 0$ and G is a p'-group.

Then

(a) every G-composition factor of V is finite dimensional;

(b) G is a subdirect product of finite dimensional groups;

(c) If the hypothesis (i) holds, G is a subdirect product of solvable groups.

Our Theorem 4 generalizes the assumption of (ii). In it we only assume that G is residually finite and locally finite, with no restrictions being made on the field K. The conclusions (a) and (b) are shown to still hold.

Definition 8 *An element $g \in FGL(V,K)$ is* **unipotent** *if $(g-1)^n = 0$ for some n. A subgroup $H \leq FGL(V,K)$ is unipotent if each of its elements is unipotent. The subgroup* **unip(H)** *is the largest normal unipotent subgroup of H.*

Definition 9 *A* **G-composition system** *of subspaces of V is a collection $\Im = \{V_i \mid i \in I\}$ such that:*

1. *$\emptyset$ and $V \in \Im$;*
2. *$\Im$ is closed under unions and intersection;*
3. *$\Im$ is a chain i.e. if $i, j \in I$, then either $V_i \subseteq V_j$ or $V_j \subseteq V_i$;*
4. *If the pair (V_i, V_j) is a jump in $\Im$ (i.e. $V_i \subseteq V_j$, $V_i \neq V_j$ and $V_i \subseteq V_k \subseteq V_j \Rightarrow k \in \{i, j\}$), then V_j/V_i is an irreducible G-module.*

For a G-composition system $\Im$, the **factors of V** *are elements of the collection $\mho = \{V_j/V_i \mid (V_i, V_j)$ is a jump $\in \Im\}$ (see [12, 7.1 and Lemma 13]).*

Theorem 4 *Let $G \leq FGL(V,K)$ be such that G is both locally finite and residually finite. Then*

1. *every G-composition factor of V is finite dimensional and*
2. *$G/unip(G)$ is a subdirect product of finite dimensional groups.*

Proof of (1):

The group G is residually finite. A recent paper by O. Puglisi [13, *Homomorphic Images of Finitary Linear Groups*, Arch. Math. 60 (1993), 497-504] showed that G/unip(G) is also residually finite. Thus wlog assume unip(G)= 1. Let $L = \{V_i \mid i \in I\}$ be a G-composition system of V. Let ψ_i be the representation of G on V_i and S=$\cap\{ker\varphi_i \mid V_i \; is \; finite \; dimensional\}$. We will first prove the following lemma:

Lemma 8 *The quotient group G/S is a subdirect product of finite dimensional groups.*

Proof: Let $J = \{i \in I \mid V_i \textit{ is finite dimensional}\}$. The representation $\varphi_j : G \to FGL(V_j, K)$ is irreducible and finite dimensional. Thus $G/ker\varphi_j \simeq Im\varphi_j \leq FGL(V_j, K)$ is finite dimensional. Define $\eta : G \to \pi_{j\in J}(G/ker\varphi_j)$ by $\eta(g) = (gker\varphi_j)_{j\in J}$. From [1, 2.2], $\varphi_j(g) = 1$ for all but a finite number of $j \in J$.

Also from definition of S, $ker(\eta)$ is S. Thus G/S is a (restricted) subdirect product of finite dimensional groups.

□

Since G is locally finite, G/S is also locally finite. From Lemma 7, G/S is a subdirect product of finite dimensional groups. Using Proposition 1 of [1], we have

1. every transitive finitary permutation representation of G/S has finite degree.
2. every irreducible finitary representation of G is finite dimensional.

The two conclusions above are exactly the hypotheses of Proposition 2 of [1] and so we conclude that S acts irreducibly on all infinite dimensional factors of L.

Let $\tilde{J} = \{i \in I \mid V_i \textit{ is infinite dimensional}\}$. Let $X = \oplus_{j\in\tilde{J}} V_j$ and φ_j be the representation of G on V_j. The group S acts faithfully and finitarily on the K-space X by $s(v_j)_{j\in\tilde{J}} = (\varphi_j(s)(v_j))_{j\in\tilde{J}}$ for $s \in S$ and $v_j \in V_j$.

For each $l \in \tilde{J}$, let $W_l = \oplus\{V_j \mid V_j \leq V_l, j \in \tilde{J}\}$. Define $\Im_S = \{W_l \mid l \in \tilde{J}\}$. Observe now that every jump (W_l, W_k) of L_S is such that $W_k/W_j \simeq V_k$ and since S acts irreducibly on V_k, S must also act irreducibly on the jump W_k/W_l. Therefore $\Im_S$ forms an S-composition of subspaces of X and $L_S = \{V_i \mid i \in \tilde{J}\}$ is an S-composition series of X.

Define $S_j = \varphi_j(S)$ for each $j \in \tilde{J}$, and note that $S_j \leq FGL(V_j, K)$. Let the homomorphism $\eta : S \to \prod_{j\in\tilde{J}} S_j$ be defined by $\eta(s) = (\varphi_j(s))_{j\in\tilde{J}}$. Since unip(G)=1 and $S = \cap\{ker\varphi_j \mid j \in J\} \trianglelefteq G$, unip(S)=1. Hence η is an embedding and S is a subdirect product of groups $\{S_j \mid j \in \tilde{J}\}$. Note that when we say direct product here, we mean the restricted direct product.

Let $J_1 = \{j \in \tilde{J} \mid S_j \textit{ is imprimitive}\}$ and $J_2 = \{j \in \tilde{J} \mid S_j \textit{ is primitive}\}$. Let M be a subdirect product of $\{S_j \mid j \in J_1\}$ and N be a subdirect product of $\{S_j \mid j \in J_2\}$. Obviously S is a subdirect product of M and N.

The derived group of S, S′, is a subdirect product of M' and N'. If S_j is imprimitive, then S_j' is also imprimitive and if S_j is primitive, then S_j' is simple by Theorem 3 of the previous chapter. Thus M' is a subdirect product of imprimitive groups and N' is a subdirect product of simple groups. We will show that S' is in fact the direct product of M' and N'.

Proposition 4 *Let G be a subdirect product of $\{A_i \mid i \in I\}$ and $\{B_i \mid j \in J\}$ where the A_i's are simple groups and the B_j's are imprimitive groups. Then G is a direct product of A and B where $A = Dr\{A_i \mid i \in I\}$, the direct product of the A_i's, and B is a subdirect product of $\{B_j \mid j \in J\}$.*

Proof:

Let $\alpha : G \to Dr\{A_i \mid i \in I\} \times Dr\{B_j \mid j \in J\}$ be the canonical monomorphism and $\alpha_i : G \to A_i$, $i \in I$ and $\beta_j : G \to B_j$, $j \in J$ be the projection maps of G onto A_i and B_j respectively. Fix $k \in I$. Choose $x \in G$ such that $x \notin ker\alpha_k$. Then $\exists$ a minimal set $I_1 = \{k, i_1, i_2, \ldots, i_{t_1}\} \subseteq I$ and $J_1 = \{j_1, j_2, \ldots, j_{t_2}\} \subseteq J$ such that $x^G \subseteq ker\alpha_i$ for all $i \in I \backslash I_1$, $x^G \in ker\beta_j$ for all $j \in J \backslash J_1$ and $\bar{\alpha} : x^G \to A_k \times A_{i_1} \times A_{i_2} \times \cdots \times A_{i_{t_1}} \times B_{j_1} \times \cdots \times B_{j_{t_2}}$ is the induced monomorphism map of α. Define $M_k = (\cap_{i \in I_1 \backslash \{k\}} ker\alpha_i) \cap (\cap_{j \in J_2} ker\beta_j)$. By minimality of I_1 and J_1, $M_k \neq 1$.

Since A_k is simple , $M_k \trianglelefteq G$ and α_k is onto, $\alpha_k(M_k) = 1$ or $\alpha_k(M_k) = A_k$. Suppose $\alpha_k(M_k) = 1$. Then by definition of M_k, $M_k \subseteq ker(\alpha)$. Thus $1 \neq M_k \subseteq ker(\alpha) = 1$ which is a contradiction. Hence $\alpha_k(M_k) = A_k$. This implies for every $a_k \in A_k$, $\exists g \in M_k$ such that $\beta_j(g) = 1$ for all $j \in J$ and $\alpha_i(g) = 1$ $\forall$ $i \in I \backslash \{k\}$ and $\alpha_k(g) = a_k$. Thus $A_k \subseteq x^G \subseteq G$ $\forall k \in I$ and $Dr\{A_k | k \in I\} \subseteq G$.

Next, we will show that $B = SDP\{B_j \mid j \in J\} \subseteq G$. Let $g \in G$. Then

$(\alpha_i(g))_{i\in I} \times (\beta_j(g))_{j\in J} \in G$. Since $Dr\{A_i \mid i \in I\} \subseteq G$, $\alpha_i(g^{-1})_{i\in I} \times 1 \in G$ and so $1 \times (\beta_j(g))_{j\in J} \subseteq G$. Therefore $B \subseteq G$.

□

With the proposition above, $S' = M' \times N'$. Furthermore, we conclude from the proposition that N′ is in fact a direct product of simple groups.

We will next show that N=1. Since G is residually finite, then $S \subseteq G$ is residually finite, and so is S'. Since $S = M' \times N'$, then N' is also residually finite. But N' is a direct product of simple groups, $\{S_j \mid j \in J_2\}$ Therefore $N' = 1$ and N is abelian. But an irreducible representation of an abelian group is finite dimensional and the S_j's, $j \in J_2$, are infinite dimensional finitary groups. Thus $N = 1$.

We conclude now that S = M, a subdirect product of imprimitive groups. We will show next that S = 1 also. To do this we will need a definition below.

Definition 10 *Let V be a finitary KG-module where dim(V) is infinite. A finitary KG- module W_V is called an imprimitive cover of V if*

- $V \subseteq W_V$, *and*
- W_V *has a system of imprimitivity* $\Gamma(W_V)$ *on which G acts transitively.*

The KG-module V is of imprimitive type if V has imprimitive cover.

Recall that S = M is a subdirect product of $\{S_j \mid j \in J_1\}$ where $S_j' s$ are imprimitive groups. Thus every factor of L_S is of imprimitive type.

Furthermore recall that unip(S) = 1 . Using Proposition 3 of [1], S' is the unique minimal subnormal subgroup of S which acts irreducibly on every factor of L_S .

Suppose $S' = 1$; then S is abelian . But irreducible representations of abelian groups are finite dimensional. Thus $J_1 = \emptyset$ and S = 1

Suppose $S' \neq 1$. The subgroup S is residually finite and so is its subgroup S'. Let $x \in S'$. There then exist a normal subgroup $H \trianglelefteq S'$ and $H \neq S'$ such that $x \notin H$

and S'/H is finite. By minimality of S', H acts reducibly on some factor V_k of L_S. Using (4.1) of [1], $\exists$ an irreducible H-submodule D of $V_k \ni [D,H] \neq 0$ and the set $\Im = \{D_i \mid i \in \bar{I}\}$ of H-homogenous components determined by D is a system of imprimitivity for S'. Furthermore the D_i's are finite dimensional; thus $|\bar{I}|$ is infinite since V_k is infinite dimensional.

Let $\pi : S' \to Sym(\Im)$ be the permutation representation of of S' on $\Im$. Then $\pi(S')$ is a transitive group of finitary permutation and $H \subseteq ker\pi$. So we have the induced map $\tilde{\pi} : S'/H \to Sym(\Im)$, the permutation representation of S'/H on $\Im$. Since S'/H is finite, $|\bar{I}|$ must also be finite and this is a contradiction.

Hence S = 1. Recall that $S = \cap_{i\in I}\{ker\varphi_i \mid V_i \; is \; finite \; dimensional\}$. Since S = 1, every G-composition factor V_i, $i \in I$ is finite dimensional.

□

Proof of (2):

From 2.2 of [1], G/unip(G) is a subdirect product of $\varphi_i(G)$, where $\varphi_i : G \to FGL(V_i)$ is the representation of G on V_i, $i \in I$. By part (1), all factors V_i, $i \in I$, are finite dimensional. Thus G/unip(G) is a subdirect product of finite dimensional groups.

□

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