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NEW ZIG-ZAG COMPOSITE PLATE THEORIES AND 3-D  
FINITE ELEMENTS FOR STATIC AND EXPLICIT DYNAMIC  
ANALYSIS OF LAMINATED COMPOSITE AND SANDWICH PANELS

presented by

Yong Bae Cho

has been accepted towards fulfillment  
of the requirements for

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Ronald C. Arendt

Major professor

Date December 22, 1997



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**NEW ZIG-ZAG COMPOSITE PLATE THEORIES  
AND 3-D FINITE ELEMENTS FOR STATIC AND EXPLICIT DYNAMIC  
ANALYSIS OF LAMINATED COMPOSITE AND SANDWICH PANELS**

**By**

**Yong-Bae Cho**

**A DISSERTATION**

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## ABSTRACT

### NEW ZIG-ZAG COMPOSITE PLATE THEORIES AND 3-D FINITE ELEMENTS FOR STATIC AND EXPLICIT DYNAMIC ANALYSIS OF LAMINATED COMPOSITE AND SANDWICH PANELS

By

Yong-Bae Cho

Refined laminated plate theories based on linear or cubic plus zig-zag kinematics have been developed, and new  $C^0$  3-D finite elements have been developed on the basis of those theories for the purpose of static and dynamic analysis of sandwich and laminated composite panels. In the first order zig-zag plate theory, the in-plane displacement fields in each sublaminate are assumed to be piecewise linear and vary in a zig-zag fashion through the thickness of the sublaminate. In the third order zig-zag plate theory, the in-plane displacement fields in the laminate are assumed to be piecewise cubic and vary in a zig-zag fashion through the thickness of the laminate. In each case, the transverse displacement field is assumed to vary linearly through the thickness. The zig-zag functions are evaluated by enforcing the continuity of shear stress at each interface. This in-plane displacement field assumption accounts for discrete layer effects without increasing the number of degrees of freedom as the number of layers is increased. In order to maintain  $C^0$  continuity of all variables, new rotation degrees of freedom are introduced, and subsequently constrained via the penalty method. The transverse normal strain is improved by assuming a constant transverse normal stress through a sublaminate or the entire laminate, and hence Poisson's ratio stiffening in associated with finite element model is eliminated.

The proposed  $C^0$  finite elements have the topology of an eight-noded brick. Each node has five engineering degrees of freedom - three translations and two rotations. In-plane displacements and rotational degrees of freedom are approximated by the bilinear Lagrange interpolation functions. For transverse deflection degrees of freedom, an interdependent interpolation is utilized. The element stiffness coefficients are integrated exactly, yet the element exhibits no shear locking. This is achieved by using the consistent transverse shear strain fields as well as the edge-consistency of the tangential shear strain on any common inter-element edge. The elements are shown to be accurate, simple to use, and compatible with the requirements of commercial finite element codes. Based on the assumed displacement fields and the finite element approximations, consistent mass matrices has been obtained. Making use of the consistent mass matrices, lumped mass matrices have been derived according to HRZ lumping scheme. The developed elements have been implemented into the explicit finite element dynamics code, NEPTUNE, wherein the internal force vector is computed for the element.

Static, free vibration, and explicit structural dynamic analyses of sandwich panels and laminated composites are performed using the elements. The numerical results show that the elements are accurate, robust, and computationally efficient.

To my late father, Hahk-Kyu Cho,  
my mother, Eul-Soon Lee,  
and my family

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# **CHAPTER I**

## **INTRODUCTION**

### **1.1 Introduction**

Fiber reinforced laminated composites have been used in structural members due to better stiffness-to-weight and strength-to-weight ratios than those of conventional monolithic materials. The primary characteristics of laminated composites are that the material properties are orthotropic in the laminate plane, the ratio of transverse shear modulus to in-plane modulus is low and layers are laminated through the thickness. Therefore, unlike homogeneous isotropic materials, the behavior of laminated composites is unique and complicated at both the global and the local levels. Moreover, the early laminated composites were applied to thin structures and were designed to withstand mainly in-plane tension loads. But recently they are also employed for primary load-carrying structural members in automobile front aprons, bridge frames, submarine hulls and aircraft because of advances in their manufacturing techniques. Such recent applications of composite materials involve the use of thick-section laminates containing perhaps over one hundred layers and/or sandwich construction that contains a thick core between face sheets. Furthermore, such structures are typically designed to withstand complex mechanical loading states, e.g., static loadings and dynamic impact loading, in the presence of harsh environmental

conditions.

Because the importance of laminated composites in aerospace, automotive, civil and marine structures continues to increase, the development of accurate structural theories and efficient computational models for analyzing these structures is currently a critical area of research. Especially important is to develop accurate, efficient, and convenient models that can be used for static, dynamic, and progressive failure analysis of thick and thin laminated composite and sandwich structures. Thus, for the purpose of modeling the behavior of laminated composites efficiently under the various loading conditions in service, many refined plate theories have been developed and proposed on the basis of various kinematic assumptions.

The analytical solutions of such proposed theories are confined by the configurations of plate, boundary conditions and loading, and are usually not available for most real problems. The way of overcoming such obstacles is to make use of the finite element method, a prevalent tool used to obtain approximate solutions in complicated engineering problems. The quality of a finite element can be significantly affected by the kinematic assumptions made throughout its formulation. Thus, every aspect of the problems to be solved should be reflected in the kinematic assumptions of finite element formulation.

In the analysis of moderately thick laminated composites, local effects are so important that the three dimensional kinematics should be used on the development of finite element models. Moreover, when the loading is excessive, the composites undergo local damages, e.g., fiber breakage and/or fiber kinking, and local delamination. Thus, the element should have capabilities to capture more accurate distribution of local strains and stresses and predict damage in composite structures associated with proper failure criteria.

However, conventional three-dimensional finite elements are computationally too expensive to be used in the analysis of laminated composites to capture local effects of a composite structure, if one element per layer is used. Thus, a lot of research has been carried out to develop a robust finite element model that is more computationally efficient than the conventional displacement based continuum finite element and describes adequate local kinematics.

When external loading is varying with time or suddenly applied to a composite structure, e.g., seismic loading and impact loading, the response of the composite structure is time dependent and can be obtained using structural dynamics techniques. Structural dynamics is one of the rapidly expanding areas of application of the finite element method. Many problems in structural dynamics cannot be solved effectively by analytical methods, and the advances in both the digital computer and various associated numerical techniques of analysis have significantly enhanced solution capability of the finite element method. Therefore, the finite element method is as valuable a tool in dynamics as it is in the static problems.

In general structural dynamics problems, the governing equations are second order or fourth order partial differential equations in the space domain, and hyperbolic in the time domain. In this case, the solution is a function defined both on the space domain and on the time domain. Therefore, as a numerical solution procedure in explicit dynamics analysis, the partial differential equations in space and time are first discretized in space, yielding a system of decoupled ordinary differential equation in time. The discretization in space is usually achieved by employing the finite element method, and called a finite element semidiscretization, because though displacements are discrete functions of space, they are

still continuous functions in the time domain. To solve a system of decoupled ordinary differential equations the finite difference method is adopted in the time domain.

## **1.2 Goals and Objectives of Current Research**

To analyze static and dynamic responses of composite structures more accurately, first of all, it is required to develop a composite plate theory where the local distributions of stresses, strains and displacement fields within a laminate are taken into account. On the basis of the developed composite plate theory a robust finite element model that is more computationally efficient and accurate needs to be developed. The main objective of the current research is to develop a composite plate theory and the associated elements to provide static and dynamic analysis of composite laminate and sandwich structures. Thus the plate theory and element must possess the following features.

- Using 3-D kinematics, a zig-zag discrete layer-wise plate theory wherein local effects can be accounted for very well is to be developed.
- The developed plate theory should be  $C^0$  continuous in displacement fields. Thus, a  $C^0$  finite element should be developed.
- A shear correction factor should not be necessary in the developed plate theory.
- The number of degrees of freedom of the finite element model associated with the plate theory should be independent of the number of layers in the composites.
- The developed element should have only engineering degrees of freedom (translations and rotations) to facilitate implementation into commercial finite element code.
- The developed element should be free of shear locking and Poisson's ratio stiffening effects.

- To be used in structural dynamics problems, the consistent mass matrix should be developed. On the basis of the consistent mass matrix, the lumped mass matrix should be developed.
- In structural dynamics problems, the internal force vector is calculated for the developed element.
- The developed element should have a critical time step (related to natural frequencies) associated with explicit dynamic analysis that is as large as possible.

### **1.3 Literature Review: Plate Theories**

Many plate theories have been developed on the basis of various kinematic assumptions to predict the response of laminates more accurately. Comprehensive reviews of literature on the development of modern plate theories have recently been carried out by several authors [1-6]. They usually classify the plate theories by the assumptions of their displacement fields. Such models can be roughly divided into three categories: equivalent single layer theories, layerwise theories, and zig-zag in-plane displacement theories.

In the equivalent single layer approach, the displacement components are assumed to be  $C^1$  continuous through the thickness of the plate and functions of in-plane coordinates. Thus, the material properties of all the layers are “smeared,” and the laminate is modeled as an equivalent single anisotropic layer.

The classical laminated plate theory (CLPT) based on Kirchhoff - Love classical plate theory (CPT) was the first laminated plate theory employed in the early days of study. In this theory, it was assumed that the in-plane displacements vary linearly through the thickness, while the transverse displacement is constant through the thickness. The shortcom-



ing of this theory is that the predictions of displacements and stress become inaccurate when the aspect ratio of the plate becomes small (less than 20) or the transverse shear modulus is low. The shortcoming mainly comes from the fact that CPT is based on the Kirchhoff hypothesis that lines originally straight and normal to the reference surface remain straight and normal to the reference surface during deformation so that the transverse shear deformation is not allowed.

Timoshenko first recognized that rotary inertia and transverse shear deformation are of importance in the vibration analysis of elastic bodies, and considered the effects of rotary inertia and transverse shear deformation on the vibration of elastic bars [7]. Then Reissner [8] and Mindlin [9] expanded this concept to isotropic plate bending problems. On the basis of these theories, an improved composite plate theory was developed by Mindlin, called the first order shear deformation theory (FSDT) [10]. This theory assumes that a line originally straight and normal to the reference surface remains straight during deformation but not necessarily perpendicular to the reference surface. This theory yields good predictions of overall laminate behavior (e.g., deflections, natural frequencies, and buckling loads) and inplane stresses provided the plate is thin and the material properties of adjacent layers do not differ significantly. However, FSDT does not account for warpage of the cross section, which may be significant in laminated composites, because FSDT accounts for only average kinematics. Furthermore, FSDT yields constant transverse shear strains through the thickness of the plate, therefore a shear correction factor(s) is required to compute the shear strain energy accurately [11,12].

In order to reduce the inaccuracies of the FSDT, higher order shear deformation theories (HSDT) were proposed (e.g., [13-17]). In these models, it is assumed that the dis-

placements are of higher order polynomial form and are  $C^1$  continuous through the thickness. Using the shear traction boundary conditions on the top and bottom of a laminate, the higher-order undetermined functions in the displacement field are evaluated and are eliminated. The assumptions of displacement fields permit nonlinear variations of displacements, strains and stresses through the thickness, and thus warpage of the cross section is allowed. However, even though they often can predict well the gross behavior of thin and some moderately thick laminates, all equivalent single layer theories have a common shortcoming. Since they are unable to account for the sometimes severe discontinuity in transverse shear strains that occur at interfaces between two adjacent layers with drastically different stiffness properties, they preclude the satisfaction of continuity of transverse stresses at interfaces between adjacent layers with different material properties, and do not reflect the proper kinematics in laminates. Therefore, the local deformations and stresses, and sometimes even the overall laminate response, are not well predicted.

If the transverse shear stresses are needed, they can be recovered indirectly by integrating the equilibrium equations [18-22]. However, this procedure leaves controversies in its accuracy for the analysis of highly irregularly shaped plates since it involves higher order in-plane differentiation of the displacement functions. Moreover, the higher order in-plane differentiation requires to use at least quadratic finite element interpolation functions to obtain transverse shear stresses and cubic interpolation functions to capture transverse normal stresses.

In an effort to overcome the shortcomings of the equivalent single layer approach, discrete-layer (or layerwise) theories have been proposed [23-32]. These theories are based on a unique displacement field for each layer, and enforce interlaminar continuity of dis-

placements and sometimes of transverse stresses, as well. In addition, the through-layer variation of displacement is interpolated by one dimensional Lagrangian linear [23,24,27] or quadratic shape functions [30] or a high order hierarchical element [31]. Reddy [25] assumes that the displacement fields for each layer are expanded through the thickness direction using Lagrangian shape functions employed in the conventional finite element method. Therefore, displacement components are  $C^0$  continuous, so that transverse strains are piecewise continuous through the thickness. Toledano and Murakami assume independent fields for displacements and stresses in each layer, and employ Reissner's mixed variational principle in the formulation of model. They show that the in-plane displacement can be improved by this technique in the presence of transverse shear effects. In these theories, some of the dependent variables are eliminated in the development of the model by enforcing displacement continuity across the layer interfaces. In most of these theories, the continuity of transverse shear stresses is met in an integral sense through the potential energy formulation. When these theories are compared with the conventional 3-D displacement finite element model, the most significant feature of these theories is that it has a computational time - saving data structure while yielding exactly the same results for comparable meshes. These theories predict excellent global and local distributions of in-plane and out-of-plane displacements and stresses. The major shortcoming in these theories is their large computational expense, for as the number of layers increases, the number of degrees of freedom increases proportionally.

A new class of laminate theory, called here the first order zig-zag theory (FZZT), was developed by DiSciuva in the mid 1980's [33,34]. In this theory, in-plane displacements in a laminate are assumed to be piecewise (layerwise) linear and continuous through the

thickness, yet the total number of degrees of freedom is only five (not dependent on the number of layers). This is accomplished by analytically satisfying the transverse shear stress continuity conditions at each interface in the laminate. This theory was shown to be very accurate for many cases, especially symmetric laminates. On the basis of the concept introduced in [33,34], DiSciuva as well as other researchers made significant improvements to the FZZT [35–42]. The primary improvement was achieved by superimposing a piecewise linear variation of in-plane displacements on a continuous cubic function of the transverse coordinate [35,36,40–42], creating a displacement field that can better account for the warping that occurs during bending of unsymmetric laminates. These latter theories, denoted here as higher-order zig-zag theories (HZZT), also satisfy the homogeneous shear traction boundary conditions at the top and bottom surfaces of the laminate in order to maintain the number of degrees of freedom at five. This class of theories appears to have an ideal combination of accuracy and efficiency, making them well-suited for use in computational simulations. However, these theories have the unfortunate shortcoming that the transverse deflection degree of freedom  $w_0$  is required to be  $C^1$  continuous, so that Hermitian-type interpolation of  $w_0$  must be used within the finite element models [33,35]. Thus additional rotational degrees of freedom (gradients of  $w_0$ ) are present in the finite element models, making it inconvenient, if not impossible, to implement the finite element models based on these theories into commercial finite element software packages that allow only six degrees of freedom -- three translations and three rotations.

Averill proposed a generalized form of the first-order zig-zag theory [38] and a generalized form of the high-order zig-zag theory [40] for beams to alleviate the requirement of  $C^1$  continuity on the transverse deflection. A new variable was introduced along with an

associated constraint condition which was enforced by employing an interdependent (anisoparametric) element interpolation scheme and the penalty method. On the basis of the theories,  $C^0$  two-noded elements were developed. These elements were shown to be simple, robust and accurate for application to thick and thin laminated beams. However, these models still contained one additional rotational degree of freedom, and it was found that for very thick and complex laminates, more refinement through-the-thickness was needed.

Yip and Averill developed a model for laminated beams [41] and plates [42] that combined the benefits of the discrete-layerwise and high-order zig-zag theories. This model allows the laminate to be subdivided into a number of sublaminates, with each sublamine containing several, even many, physical layers. Within each sublamine, very accurate high-order zig-zag kinematics are assumed in which degrees of freedom describe displacements, rotations, and transverse shear stresses (tractions or interlaminar stresses) at the top and bottom surfaces of the sublamine. Each finite element represents one sublamine, and, if cast in the form of a four-noded quadrilateral (for beams) or an eight-noded brick (for plates), refinement of a model by through-thickness discretization can be achieved without the use of any special constraints. When only one sublamine is used through the entire thickness of the laminate, nodal degrees of freedom are three translations and two rotations. However, when multiple elements (sublaminates) are needed to model a laminate, additional shear stress degrees of freedom are present, making the element unsuitable for implementation into many of the current commercial finite element codes.

## 1.4 Literature Review: Three Dimensional Composite Shell/Plate Elements

Laminated composites are usually modeled by using plate and shell elements which are based on plate and shell theories. There have traditionally been three approaches to model general shell structures in finite element representation [43,44]:

- 1) The “*faceted*” model of flat plate element
- 2) The “*curved*” shell element formulated on the basis of curved shell theory
- 3) The “*degenerated*” isoparametric element

Among the above elements, the Ahmad type [45] degenerated isoparametric elements where the interpolation functions for rotational and translational d.o.f are independent are most popular. The degenerated elements, since the thickness of plate or shell is very small when compared with other dimensions, are obtained by constraining adjacent thickness-direction nodes on solid elements to have the same thickness-direction displacement and merging the top and bottom nodes to a mid-node.

Recently, a number of three dimensional composite plate/shell elements that have a thickness dimension have been proposed for the analysis of composite structures [47-53]. Such three dimensional elements are classified as “*regenerated*,” in contrast to the popular two dimensional “*degenerated*” shell elements. This regenerated approach has been employed successfully, and renders the very important advantage of allowing discretization in both inplane and through-thickness directions without imposing any special multi-point constraints. The regenerated elements must have just 3 translational degrees of freedom ( $u$ ,  $v$  and  $w$ ) to describe the deformations of a plate, whereas the degenerated element must have at least 5 degrees of freedom (3 translational d.o.f + 2 rotational d.o.f). Thus, as

far as the discretization through-thickness direction, the description of geometry and kinematics of deformation are concerned, the three dimensional regenerated element is preferred over the two dimensional degenerated element. The capability of discretizing in the through-thickness direction enables one to simulate the separation of the laminated composites due to delamination into two parts in the thickness direction.

Most proposed regenerated 3-D composite elements are similar to continuum elements that have only 3 translational degrees of freedom per node. Moreover, contrary to the structural element cases, in-plane displacement fields are approximated using the shape functions, while special functions in some cases are used for transverse displacement to improve the continuity of transverse stresses at the interface between two adjacent layers.

Adding composite analysis capability to Ausserer and Lee's isotropic shell solid element [46], Kim and Lee [47] presented an 18-node solid isoparametric element using assumed independent strains based on Hellinger-Reissner principle in an attempt to alleviate locking. However, the transverse strains on this element are continuous through thickness direction, so that the continuity of transverse stress is questionable. Kong and Cheong [52] developed an element incorporating the advantages of the layerwise plate theory and the equivalent single layer plate theory. Layerwise local shape functions in the regions were used where transverse shear stresses are of interest and importance, while a special through-thickness global-local interpolation is used in the zone where only global responses are of interest. The compatibility of the displacement between these two regions is satisfied by introducing a transition region. Wanthal and Yang [48] proposed a family of three quadrilateral isoparametric elements based on the layerwise modeling approach to include the effects of transverse shear and normal stress for the analysis of delamination,

holes and joints. The crucial shortcoming of these elements is that the total number of d.o.f depends on the number of layers in the laminate. Han and Hoa [50] developed a hybrid formulated plate element by taking three in-plane stresses and three transverse stresses as the basic variables. The continuity of the transverse stresses at the interface between two adjacent layers are satisfied by introducing a partial stress field parameter. The element is a super element composed of some three dimensional eight noded finite elements (layer element), so that the total number of d.o.f does not depend on the number of layers.

## **1.5 Literature Review: Mass Matrices and Lumping Schemes**

If a structure vibrates freely or excitation frequencies are higher than  $1/3$  of the lowest natural frequency of the structure, inertial effects must be included when investigating the response of the structure. When we try to analyze the dynamic response of a structure using the finite element method, the mass matrix accounts for inertia and represents discretely the continuous distribution of mass in a structure. Thus, in general structural dynamic response problems, the stiffness matrix is used to describe the elastic characteristics in terms of force - displacement relationships, and the mass matrix is exploited to define the inertial characteristics of the structure in terms of inertial force - acceleration relationships at a finite number of nodal points chosen to represent the behavior of the structure. In this case, the accuracy of the analysis is significantly affected by the accuracies of the stiffness matrix as well as the mass matrix.

Up to the early 1960s, the mass matrix was usually obtained by employing a lumped parameter method [54], in which the structural mass was physically grouped as a series of



concentrated mass at the nodes. The derived mass matrix is diagonal, and a simple technique is required to formulate and solve the equations. However, the computed natural frequencies and mode shapes might be greatly different from the exact solution, if a large number of elements are not used.

A consistent mass matrix was proposed by Archer in an attempt to improve the accuracy of the dynamic analysis [55]. The mass matrix was derived employing equations corresponding to the Rayleigh-Ritz approach, and the resulting mass matrix is “consistent” with the actual distribution of mass throughout the structure. Later on, Archer also a consistent mass matrix for Timoshenko beam element. The mass matrix was derived by superposing linear inertias on rotary inertias which were determined assuming that the mass angular acceleration is consistent with the angular motion of the transverse fibers of the beam [56].

Even though the consistent mass matrix represents a truer distribution of mass throughout the structure compared to a lumped mass matrix, it results in a coupled system of equations for explicit structural dynamic analysis as well as a need to construct a global mass matrix. Thus the solution procedure to solve the system of equations becomes complicated and computationally expensive. In addition, lumping lowers the highest wave speed in the finite element mesh, and increases the critical time step for explicit integration schemes. These facts motivate analysts to find rational ways to develop diagonal mass matrix by lumping appropriately. Several lumping schemes based on a consistent mass matrix have been proposed.

The easy-to-use and well-known lumping techniques are the row-sum lumping, the optimal lumping and proportional lumping (so called HRZ lumping) schemes. In row sum

lumping, the components in each row are added and lumped on the diagonal using the property of the interpolation functions that the sum of the interpolation functions is 1. This lumping scheme sometimes causes negative masses, especially for corner nodes of the eight-node serendipity element. On the other hand, in the optimal lumping scheme proposed by Fried and Malkus [57], the lumping of mass was accomplished by placing the location of nodes of an element at the integration point of a quadrature rule. This lumping scheme is practically useful only for low order Lagrange elements with only translational degrees of freedom, because this scheme produces a block diagonal lumped matrix for elements with rotational d.o.f, and zero or negative nodal masses for quadratic and higher order elements.

A special lumping technique was presented by Hinton, Rock and Zienkiewicz [58]. This lumping scheme is sometimes called as the proportional lumping scheme or the HRZ lumping scheme. In the HRZ (proportional) lumping scheme, lumping is achieved by calculating the diagonal elements of the lumped mass matrix to be proportional to the diagonal elements of the consistent mass matrix while preserving the total mass of the element. The most important feature of this lumping scheme is in that it always produces positive lumped masses without regard to the element type, because the diagonal terms in the consistent mass matrix must be positive by virtue of the positive-definiteness property. For low order elements, the accuracy of this lumping scheme in the prediction of natural frequencies often surpasses that of the consistent mass matrix. For Lagrange linear and quadratic elements with constant material density, the lumped mass matrix computed by the proportional lumping is the same as that obtained by the row-sum lumping.

The lumping scheme for two-dimensional degenerated general shell elements and axi-

symmetric shell elements with rotational d.o.f was developed by Surana [59]. In this lumping scheme, the rotary inertia terms are obtained by assuming that the masses in translational d.o.f at the top and bottom surfaces of the plate are rotating with respect to the mid surface, while the mass terms in lumped mass matrix for translational d.o.f. are derived by the same procedure as in the HRZ lumping scheme.

It was shown in [60,61] that the reduced integration rule to evaluate the element mass matrix may result in a singular mass matrix. The existence of a singular mass matrix implies that there are certain unacceptable modes of deformation associated with the element which produce zero kinetic energy. Thus, full Gauss quadrature rule is recommended to be used to evaluate the element mass matrix.

The the rates of convergence of eigenvalues and eigenfuctions between consistent mass and lumped mass formulations was studied in [62-64]. Tong, Pian and Bucciarelli [62] found that the lumped mass formulation suffers no loss in order of convergence for the membrane and rod cases employing simple interpolation functions. However, for problems with higher order equations such as the beam or plate, or for the second order equations with more accurate interpolation functions, the consistent mass formulation yields a better rate of convergence.

The effects of negative and zero masses and stability conditions reproduced by optimal lumping scheme in transient finite element analyses were investigated by Malkus and Plesha [65], and Malkus, Plesha and Liu [66].

## **1.6 The Present Study**

Refined plate theories based on linear or cubic plus zig-zag kinematics have been

developed, and new  $C^0$  3-D finite elements have been developed on the basis of those theories. The in-plane displacement fields in each sublaminate (for linear zig-zag) or in the laminate (for cubic zig-zag) are assumed to be piecewise linear or cubic functions and vary in a zig-zag fashion through the thickness of the sublaminate or the laminate. The transverse displacement field is assumed to vary linearly through the thickness direction. The zig-zag functions are evaluated by enforcing the continuity of shear stress at each interface. This in-plane displacement field assumption accounts for discrete layer effects without increasing the number of degrees of freedom as the number of layers is increased. The rotational variables in the in-plane displacement field are eliminated in favor of the translational variables, the inplane translations at the top and bottom surfaces of the sublaminate or laminate, in order to facilitate the development of convenient finite element models. For the cubic zig-zag model, high order rotational variables are removed by employing zero shear traction conditions at the top and bottom of the laminate. Moreover, since the derivatives of transverse deflections appear in the displacement field, their second derivatives will be present in the strain energy functional, so  $C^1$  continuity of transverse deflection is required. Such a requirement is alleviated by introducing new rotational degrees of freedom. The constraints between the derivatives of transverse deflections and new rotational degrees of freedom are imposed via the penalty method. To remove Poisson's ratio stiffening effect, the transverse normal strain is improved by assuming a constant transverse normal stress through a sublaminate or the entire laminate.

The equations of motion, the essential and natural boundary conditions, as well as the displacement-based finite element model are developed from Hamilton's principle. The new  $C^0$  finite element is developed with the topology of an eight-noded brick. Each node

has five engineering degrees of freedom - three translations and two rotations.

Inplane displacements and rotational degrees of freedom are approximated by the bilinear Lagrange interpolation functions. For transverse deflection degrees of freedom, an interdependent interpolation concept developed by Tessler and Hughes [75] is utilized.

The above interdependent interpolation scheme alleviates the shear locking problem but does not eliminate it totally. The element developed using this scheme still locks in the very thin regime. Thus, the consistent transverse shear strain fields as well as the edge-consistency of the tangential shear strain on any common inter-element edge were achieved utilizing Prathap and Somashekar's approach to prevent the element from locking [76,77]. The element stiffness coefficients are integrated exactly, yet the element exhibits no shear locking. The element is shown to be accurate, simple to use, and compatible with the requirements of commercial finite element codes.

Based on the displacement fields and the finite element model, a consistent mass matrix has been obtained. Making use of the consistent mass matrix, lumped mass matrices for explicit structural dynamic analyses have been derived by several lumping schemes.

The developed elements have been implemented into the explicit finite element dynamics code, NEPTUNE [78]. In the code the internal force vector is computed for the element. Using NEPTUNE, the dynamic analysis of composite plate was performed.

## **1.7 Organization of the Dissertation**

The dissertation will be focused on the development of robust and accurate composite beam and plate elements for the static and dynamic analyses of laminated composites and

sandwich structures. The dissertation is composed of 7 chapters, and the features and contents of each chapter are as follows.

In Chapter 2, first order layerwise beam theory and the two dimensional beam element associated with the beam theory is presented. Numerical results are obtained for cylindrical bending cases and compared with the exact solutions by Pagano [72].

A laminated plate theory with first-order zig-zag sublaminate approximations and a new 3-D finite element based on that theory are developed in Chapter 3. To alleviate locking, interdependent interpolation, consistent transverse shear strain fields, and edge-consistency of the tangential shear strain will be introduced. To remove Poisson's ratio stiffening effect, the transverse normal strain will be improved by assuming a constant transverse normal stress through each sublaminate. Numerical performance of the current element is investigated for a composite armored vehicle panel and a sandwich panel with low and high aspect ratios.

Chapter 4 will introduce a refined laminated plate theory based on cubic zig-zag kinematics and the associated element. The discretization in the through-thickness direction is not allowed in this element, because homogenous shear traction conditions at the top and bottom surfaces are assumed. The numerical experiments will be carried out by simulating a damaged composite plate under a double sinusoidal loading.

In Chapter 5, on the basis of the developed elements, the consistent mass matrices and the lumped mass matrices will be derived. Utilizing the derived stiffness and mass matrices, free vibration of a laminated plate will be analyzed.

The structural dynamic analyses of composite structures will be performed using the proposed elements in Chapter 6. Since the proposed elements have been implemented into

NEPTUNE, which utilizes the explicit time integration, the stability condition and critical time step for the explicit time integration will be also discussed.

The conclusions and recommendations of the current research will be present in Chapter 7.

# **CHAPTER II**

## **FIRST ORDER ZIG-ZAG BEAM THEORY**

### **BASED ON SUBLAMINATE APPROXIMATIONS**

#### **2.1 Introduction**

In this chapter, a new beam finite element based on a new discrete-layer laminated beam theory with sublaminates first-order zig-zag kinematic assumptions is presented and assessed for thick and thin laminated beams. The model allows a laminate to be represented as an assemblage of sublaminate in order to increase the model refinement through the thickness, when needed. Within each sublaminates, discrete-layer effects are accounted for via a modified form of DiSciuva's linear zig-zag laminate kinematics [33], in which continuity of interfacial transverse shear stresses is satisfied identically. In the computational model, each finite element represents one sublaminates. The finite element is developed with the topology of a four-noded rectangle, allowing the thickness of the beam to be discretized into several elements, or sublaminate, if necessary, to improve accuracy. Each node has three engineering degrees of freedom, two translations and one rotation. Thus, this element can be conveniently implemented into general purpose finite element codes. The element stiffness coefficients are integrated exactly, yet the element exhibits no shear locking due to the use of a consistent interdependent interpolation scheme. Numerical per-



formance of the current element is investigated for an arbitrarily layered beam, a symmetrically layered beam and a sandwich beam with low and high aspect ratios.

## 2.2 Kinematic Assumptions and Formulation of the Beam Theory

It is assumed that a laminate is composed of  $N$  perfectly bonded layers, with each layer being of independent thickness and having independent stiffness properties. The laminate will be modeled as  $M$  sublaminates ( $1 \leq M$ ), with each sublaminate containing  $N_m$  layers such that:

$$N = \sum_{m=1}^M N_m \quad (2.2.1)$$

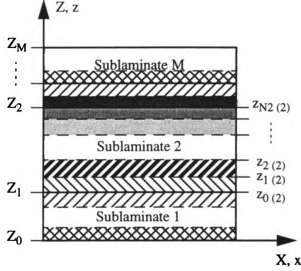
In order to facilitate the development of the finite element model, all expressions in the following derivation pertain to the  $m$ th sublaminate, and the sublaminate number designation is omitted for brevity hereafter.

The constitutive equation for a plane stress state in the  $k$ th layer of the  $m$ th sublaminate can be written as:

$$\begin{Bmatrix} \sigma_{11}^{(k)} \\ \sigma_{33}^{(k)} \\ \sigma_{13}^{(k)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(k)} & C_{13}^{(k)} & 0 \\ C_{13}^{(k)} & C_{33}^{(k)} & 0 \\ 0 & 0 & C_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_{11}^{(k)} \\ \epsilon_{33}^{(k)} \\ \gamma_{13}^{(k)} \end{Bmatrix} \quad (2.2.2)$$

where the principal material axes are assumed to coincide with the  $X, Z$  axes, and the subscripts 1,3 correspond to the  $X, Z$  or  $x, z$  directions, respectively. The global  $X$ - and local  $x$ -

directions coincide and are taken along the length of the beam. The global  $Z$ -axis is taken perpendicular to  $X$  and has its origin at the bottom of the laminate, while  $z$  is a local sublaminate coordinate in the direction of  $Z$  with its origin at the bottom of the sublaminate as shown in Figure 1.



**Figure 1. Schematic of sublaminate and layer divisions in the laminated beam theory.**

Coefficients of the material stiffness matrix are given by:

$$\begin{aligned}
 C_{11}^{(k)} &= \frac{E_1^{(k)}}{1 - \nu_{13}^{(k)} \nu_{31}^{(k)}} , & C_{33}^{(k)} &= \frac{E_3^{(k)}}{1 - \nu_{13}^{(k)} \nu_{31}^{(k)}} \\
 C_{13}^{(k)} &= \frac{\nu_{13}^{(k)} E_1^{(k)}}{1 - \nu_{13}^{(k)} \nu_{31}^{(k)}} = \frac{\nu_{31}^{(k)} E_3^{(k)}}{1 - \nu_{13}^{(k)} \nu_{31}^{(k)}} , & C_{55}^{(k)} &= G_{13}^{(k)}
 \end{aligned} \tag{2.2.3}$$

where  $E_i$  are the Young's Moduli, and  $\nu_{ij}$  are the Poisson's ratios in  $k$ th layer, and  $G_{ij}$  are shear moduli.

The strain - displacement relationship for a plane stress state can be written as:

$$\varepsilon_{11}^{(k)} = \frac{\partial u_x^{(k)}}{\partial x}, \quad \varepsilon_{33}^{(k)} = \frac{\partial u_z^{(k)}}{\partial z}, \quad \gamma_{13}^{(k)} = \frac{\partial u_z^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial z} \quad (2.2.4)$$

The assumed displacement fields are initially assumed in the following form:

$$\begin{aligned} u_x^{(k)}(x, z) &= u_b(x) + z\psi + \sum_{i=1}^{k-1} (z - z_i)\xi_i \\ u_z^{(k)}(x, z) &= w_b(x)\left(1 - \frac{z}{h}\right) + w_t(x)\left(\frac{z}{h}\right) \end{aligned} \quad (2.2.5)$$

where  $h$  is the thickness of the sublaminate,  $u_b$  is the axial displacement at  $z = 0$ ,  $\psi$  is the rotation at  $z = 0$ , and  $w_b$  and  $w_t$  are the transverse deflections of the bottom and top surfaces, respectively, of the  $m$ th sublaminate. Thus, it is assumed that  $u_x^{(k)}$  varies in a piecewise linear fashion through the thickness of a sublaminate and  $u_z^{(k)}$  varies linearly through the thickness.

The parameters  $\xi_i$  in (2.2.5) are eliminated by enforcing the continuity of shear stress at each interface. The shear stress continuity condition between the  $k$ th layer and the  $k+1$ th layer can be expressed as follows:

$$\sigma_{zx}^{(k)} = \sigma_{zx}^{(k+1)} \quad \text{at } z = z_k \quad (2.2.6)$$

Substitution of the constitutive equation (2.2.2), the strain - displacement relationship (2.2.4) and the assumed displacement fields (2.2.5) into Eq. (2.2.6) gives:

$$\begin{aligned}
G_{13}^{(k)} \gamma_{zx}^{(k)} \Big|_{z=z_k} &= G_{13}^{(k+1)} \gamma_{zx}^{(k+1)} \Big|_{z=z_k} \\
&= G_{13}^{(k+1)} \left\{ \psi + \sum_{i=1}^k \xi_i + \frac{dw_b}{dx} \left( 1 - \frac{z_k}{h} \right) + \frac{dw_t}{dx} \left( \frac{z_k}{h} \right) \right\} \Big|_{z=z_k} \\
&= G_{13}^{(k+1)} \left( \gamma_{zx}^{(k)} \Big|_{z=z_k} + \xi_k \right)
\end{aligned} \tag{2.2.7}$$

From the above (2.2.7),  $\xi_k$  is:

$$\begin{aligned}
\xi_k &= \left( \frac{G_{13}^{(k)}}{G_{13}^{(k+1)}} - 1 \right) \gamma_{zx}^{(k)} \Big|_{z=z_k} \\
&= \left( \frac{G_{13}^{(k)}}{G_{13}^{(k+1)}} - 1 \right) \left\{ \psi + \sum_{i=1}^{k-1} \xi_i + \frac{dw_b}{dx} \left( 1 - \frac{z_k}{h} \right) + \frac{dw_t}{dx} \left( \frac{z_k}{h} \right) \right\}
\end{aligned} \tag{2.2.8}$$

Solving the recursive relationship for  $\xi_i$ , finally  $\xi_i$  is found to be:

$$\xi_i = \hat{a}_i \psi + \hat{b}_i \frac{dw_b}{dx} + \hat{c}_i \frac{dw_t}{dx} \tag{2.2.9}$$

where

$$\begin{aligned}
\hat{a}_i &= \left( \frac{G_{13}^{(k)}}{G_{13}^{(k+1)}} - 1 \right) \left( 1 + \sum_{j=1}^{i-1} \hat{a}_j \right) \\
\hat{b}_i &= \left( \frac{G_{13}^{(k)}}{G_{13}^{(k+1)}} - 1 \right) \left( 1 - \frac{z_i}{h} + \sum_{j=1}^{i-1} \hat{b}_j \right) \\
\hat{c}_i &= \left( \frac{G_{13}^{(k)}}{G_{13}^{(k+1)}} - 1 \right) \left( \frac{z_i}{h} + \sum_{j=1}^{i-1} \hat{c}_j \right)
\end{aligned} \tag{2.2.10}$$

From Eqs. (2.2.9) and (2.2.10) it can be seen that  $\xi_i$  depends on the ratios of shear proper-

ties between two adjacent layers and the shear deformation in each sublaminate. If either of these quantities is small, then discrete-layer effects will also be small.

The modified displacement field is given by:

$$\begin{aligned} u_x^{(k)}(x, z) &= u_b(x) + z\psi + \sum_{i=1}^{k-1} (z - z_i) \left( \hat{a}_i \psi + \hat{b}_i \frac{dw_b}{dx} + \hat{c}_i \frac{dw_t}{dx} \right) \\ u_z^{(k)}(x, z) &= w_b(x) \left( 1 - \frac{z}{h} \right) + w_t(x) \left( \frac{z}{h} \right) \end{aligned} \quad (2.2.11)$$

In order to facilitate the development of convenient finite element models, the rotational variable  $\psi$  is now eliminated in favor of the translational variable  $u_t$ , the inplane translation at the top surface of the sublaminate. Thus, rather than describing the inplane displacement field by a translation and rotation at one point, it can more conveniently be described here by the translation at two points. Introducing the new variable  $u_t$ ,  $\psi$  can be obtained as follows:

$$\psi = \frac{1}{\bar{a}} \left[ u_t - u_b - \bar{b} \frac{dw_b}{dx} - \bar{c} \frac{dw_t}{dx} \right] \quad (2.2.12)$$

where  $u_b$  and  $u_t$  are the axial displacement at the top and bottom of each sublaminate, and can be described mathematically as:

$$u_b = u_x^{(1)} \Big|_{z=0}, \quad u_t = u_x^{(N)} \Big|_{z=h} \quad (2.2.13)$$

The coefficients  $\bar{a}$ ,  $\bar{b}$  and  $\bar{c}$  are defined as:

$$\begin{aligned}
\bar{a} &= h + \sum_{i=1}^{N-1} (h - z_i) \hat{a}_i \\
\bar{b} &= \sum_{i=1}^{N-1} (h - z_i) \hat{b}_i \\
\bar{c} &= \sum_{i=1}^{N-1} (h - z_i) \hat{c}_i
\end{aligned} \tag{2.2.14}$$

Therefore,  $u_x^{(k)}$  can be written as follows:

$$\begin{aligned}
u_x^{(k)} &= u_b + \frac{z}{\bar{a}} \left\{ u_t - u_b - \bar{b} \frac{dw_b}{dx} - \bar{c} \frac{dw_t}{dx} \right\} \\
&\quad + \sum_{i=1}^{k-1} (z - z_i) \left\{ \frac{\hat{a}}{\bar{a}} \left( u_t - u_b - \bar{b} \frac{dw_b}{dx} - \bar{c} \frac{dw_t}{dx} \right) + \bar{b} \frac{dw_b}{dx} + \bar{c} \frac{dw_t}{dx} \right\}
\end{aligned} \tag{2.2.15}$$

Since the derivatives of transverse deflections,  $\frac{dw_b}{dx}$  and  $\frac{dw_t}{dx}$ , appear in the displacement field, their second derivatives will be present in the strain energy functional, so  $C^1$  continuity of  $w_b$  and  $w_t$  is required. It is desirable to alleviate such a requirement by introducing new rotational degrees of freedom as follows:

$$\frac{dw_b}{dx} = \theta_b, \quad \frac{dw_t}{dx} = \theta_t \tag{2.2.16}$$

The above relationship still must be enforced within the model, as discussed later. The final form of the axial displacement field can now be written as:

$$\begin{aligned}
u_x^{(k)} = & u_b \left\{ 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right\} + u_t \left\{ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right\} \\
& + \theta_b \left\{ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right\} + \theta_t \left\{ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right\}
\end{aligned} \tag{2.2.17}$$

where  $z_i$  is the coordinate of the  $i$ th interface in the sublamine. The coefficients

$a_0, b_0, c_0, a_i, b_i$  and  $c_i$  are given by:

$$\begin{aligned}
a_0 &= \frac{1}{\bar{a}}, & a_i &= a_0 \hat{a}_i \\
b_0 &= -\frac{\bar{b}}{\bar{a}} = -a_0 \bar{b}, & b_i &= \hat{b}_i - \frac{\hat{a}_i}{\bar{a}} \bar{b} = \hat{b}_i - a_0 \bar{b} \hat{a}_i \\
c_0 &= -\frac{\bar{c}}{\bar{a}} = -a_0 \bar{c}, & c_i &= \hat{c}_i - \frac{\hat{a}_i}{\bar{a}} \bar{c} = \hat{c}_i - a_0 \bar{c} \hat{a}_i
\end{aligned} \tag{2.2.18}$$

Thus, the strains are defined as:

$$\begin{aligned}
\varepsilon_{11}^{(k)} &= \frac{du_b}{dx} \left\{ 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right\} + \frac{du_t}{dx} \left\{ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right\} \\
&+ \frac{d\theta_b}{dx} \left\{ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right\} + \frac{d\theta_t}{dx} \left\{ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right\} \\
\varepsilon_{33}^{(k)} &= \frac{w_b - w_t}{h}
\end{aligned} \tag{2.2.19}$$

$$\begin{aligned}
\gamma_{13}^{(k)} &= \frac{dw_b}{dx} \left( 1 - \frac{z}{h} \right) + \frac{dw_t}{dx} \left( \frac{z}{h} \right) \\
&+ u_b \left\{ -a_0 - \sum_{i=1}^{k-1} a_i \right\} + u_t \left\{ a_0 + \sum_{i=1}^{k-1} a_i \right\} + \theta_b \left\{ b_0 + \sum_{i=1}^{k-1} b_i \right\} + \theta_t \left\{ c_0 + \sum_{i=1}^{k-1} c_i \right\}
\end{aligned}$$

The governing equations, the essential and natural boundary conditions, as well as the displacement-based finite element model can be derived from the principle of total poten-

tial energy. Again, attention will be limited to a single sublaminate. The potential energy functional is modified here to include the imposition of the constraints (2.2.16) via the penalty method [68-70]. The modified total potential energy for the  $m$ th sublaminate can be defined as follows:

$$\begin{aligned} \pi_p^{(m)} = & \sum_{k=1}^{N_m} \int_{V_k} \frac{1}{2} [\sigma_{xx}^{(k)} \epsilon_{xx}^{(k)} + \sigma_{zz}^{(k)} \epsilon_{zz}^{(k)} + \sigma_{zx}^{(k)} \gamma_{zx}^{(k)}] dV_k - \int_0^L (q_t w_t + q_b w_b) dx \\ & + \int_0^L \frac{\gamma}{2} \left( \theta_b - \frac{dw_b}{dx} \right)^2 dx + \int_0^L \frac{\gamma}{2} \left( \theta_t - \frac{dw_t}{dx} \right)^2 dx \end{aligned} \quad (2.2.20)$$

where  $N_m$  is the number of layers within the  $m$ th sublaminate,  $V_k$  represents the volume of the  $k$ th layer in the sublaminate,  $\gamma$  is the penalty parameter, and  $q_b(x)$  and  $q_t(x)$  are the distributed transverse loads applied on the surfaces of the sublaminate (if these surfaces also represent surfaces of the beam). As  $\gamma$  is assigned successively larger values, the constraints of (2.2.16) are satisfied more exactly in the least squares sense. Substituting Eqs. (2.2.2), (2.2.19) into (2.2.20), pre-integrating the internal strain energy terms through the thickness and through the width of the beam, taking the variation of  $\pi_p^{(m)}$ , and integrating by parts, (2.2.20) can be rewritten as:



$$\begin{aligned}
\delta\pi_p^{(m)} = & \int_0^L \left[ \delta u_b \left( Q_b - \frac{dN_{xb}}{dx} \right) + \delta u_t \left( Q_t - \frac{dN_{xt}}{dx} \right) + \delta w_b \left\{ N_{zb} - q_b - \frac{dT_b}{dx} + \gamma \left( \frac{d\theta_b}{dx} - \frac{d^2 w_b}{dx^2} \right) \right\} \right. \\
& + \delta w_t \left\{ N_{zt} - q_t - \frac{dT_t}{dx} + \gamma \left( \frac{d\theta_t}{dx} - \frac{d^2 w_t}{dx^2} \right) \right\} + \delta \theta_b \left\{ S_b - \frac{dM_{xb}}{dx} + \gamma \left( \theta_b - \frac{dw_b}{dx} \right) \right\} \\
& \left. + \delta \theta_t \left\{ S_t - \frac{dM_{xt}}{dx} + \gamma \left( \theta_t - \frac{dw_t}{dx} \right) \right\} \right] dx \quad (2.2.21) \\
& + \left[ N_{xb} \delta u_b + N_{xt} \delta u_t + M_{xb} \delta \theta_b + M_{xt} \delta \theta_t + \left\{ T_b - \gamma \left( \theta_b - \frac{dw_b}{dx} \right) \right\} \delta w_b \right. \\
& \left. + \left\{ T_t - \gamma \left( \theta_t - \frac{dw_t}{dx} \right) \right\} \delta w_t \right] \Big|_0^L = 0
\end{aligned}$$

where  $N_{xb}, N_{xt}, M_{xb}, M_{xt}, N_{zb}, N_{zt}, Q_b, Q_t, S_b, S_t, T_b, T_t$  are the beam sublaminar stress resultants defined as:

$$\begin{aligned}
N_{x(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} b^{(k)} \left\{ \left( 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right), \left( a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right) \right\} dx \\
M_{x(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} b^{(k)} \left\{ \left( b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right), \left( c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right) \right\} dx \\
N_{z(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} b^{(k)} \left\{ \left( -\frac{1}{h} \right), \left( \frac{1}{h} \right) \right\} dx \\
Q_{(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zx}^{(k)} b^{(k)} \left\{ \left( -a_0 - \sum_{i=1}^{k-1} a_i \right), \left( a_0 + \sum_{i=1}^{k-1} a_i \right) \right\} dx \\
S_{(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zx}^{(k)} b^{(k)} \left\{ \left( b_0 + \sum_{i=1}^{k-1} b_i \right), \left( c_0 + \sum_{i=1}^{k-1} c_i \right) \right\} dx \\
T_{(b,t)} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zx}^{(k)} b^{(k)} \left\{ \left( 1 - \frac{z}{h} \right), \left( \frac{z}{h} \right) \right\} dx
\end{aligned} \quad (2.2.22)$$

Substituting Eqs. (2.2.2), (2.2.4) into (2.2.22), the sublamine stress resultants are described in terms of the generalized displacement degrees of freedom by:

$$\begin{aligned}
N_{x(b,t)} &= (A, A^a) \frac{du_b}{dx} + (A^a, B) \frac{du_t}{dx} + (A^b, B^a) \frac{d\theta_b}{dx} + (A^c, B^b) \frac{d\theta_t}{dx} \\
&\quad + (C, C^a) w_b - (C, C^a) w_t \\
M_{x(b,t)} &= (A^b, A^c) \frac{du_b}{dx} + (D, B^b) \frac{du_t}{dx} + (D^a, D^b) \frac{d\theta_b}{dx} + (D^b, F) \frac{d\theta_t}{dx} \\
&\quad + (C^b, C^c) w_b - (C^b, C^c) w_t \\
N_{z(b,t)} &= (C, -C) \frac{du_b}{dx} + (C^a, -C^a) \frac{du_t}{dx} + (C^b, -C^b) \frac{d\theta_b}{dx} + (C^c, -C^c) \frac{d\theta_t}{dx} \\
&\quad + (G, -G) w_b + (-G, G) w_t \\
Q_{(b,t)} &= (H, -H) u_b + (-H, H) u_t + (H^a, -H^a) \theta_b + (H^b, -H^b) \theta_t \\
&\quad + (H^c, -H^c) \frac{dw_b}{dx} + (H^d, -H^d) \frac{dw_t}{dx} \\
S_{(b,t)} &= (H^a, H^b) u_b + (-H^a, -H^b) u_t + (I, I^a) \theta_b + (I^a, J) \theta_t \\
&\quad + (I^b, J^a) \frac{dw_b}{dx} + (I^c, J^b) \frac{dw_t}{dx} \\
T_{(b,t)} &= (H^c, H^d) u_b + (-H^c, -H^d) u_t + (I^b, I^c) \theta_b + (J^a, J^b) \theta_t \\
&\quad + (K, K^a) \frac{dw_b}{dx} + (K^a, L) \frac{dw_t}{dx}
\end{aligned} \tag{2.2.23}$$

where the coefficients in (2.2.23) are defined as follows:

$$\begin{aligned}
(A, A^a, A^b, A^c) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{11}^{(k)} b^{(k)} \left( 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right) \left( \left[ 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right] \right. \\
&\quad \left. , \left[ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right], \left[ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right], \left[ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right] \right) dx \\
(B, B^a, B^b) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{11}^{(k)} b^{(k)} \left( a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right) \bullet \left( \left[ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right], \right. \\
&\quad \left. \left[ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right], \left[ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right] \right) dx \\
(C, C^a, C^b, C^c) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{13}^{(k)} b^{(k)} \left( -\frac{1}{h} \right) \bullet \left( \left[ 1 - a_0 z - \sum_{i=1}^{k-1} (z - z_i) a_i \right], \right. \\
&\quad \left. \left[ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right], \left[ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right], \left[ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right] \right) dx \\
(D, D^a, D^b) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{11}^{(k)} b^{(k)} \left( b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right) \left( \left[ a_0 z + \sum_{i=1}^{k-1} (z - z_i) a_i \right], \right. \\
&\quad \left. \left[ b_0 z + \sum_{i=1}^{k-1} (z - z_i) b_i \right], \left[ c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right] \right) dx \\
(F) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{11}^{(k)} b^{(k)} \left( c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right) \bullet \left( c_0 z + \sum_{i=1}^{k-1} (z - z_i) c_i \right) dx \\
(G) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{33}^{(k)} b^{(k)} \left( \frac{1}{h} \right) \bullet \left( \frac{1}{h} \right) dx \\
(H, H^a, H^b, H^c, H^d) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{55}^{(k)} b^{(k)} \left( -a_0 - \sum_{i=1}^{k-1} a_i \right) \bullet \left( \left[ -a_0 - \sum_{i=1}^{k-1} a_i \right], \left[ b_0 + \sum_{i=1}^{k-1} b_i \right], \right. \\
&\quad \left. \left[ c_0 + \sum_{i=1}^{k-1} c_i \right], \left[ 1 - \frac{z}{h} \right], \left[ \frac{z}{h} \right] \right) dx \\
(I, I^a, I^b, I^c) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{55}^{(k)} b^{(k)} \left( b_0 + \sum_{i=1}^{k-1} b_i \right) \bullet \left( \left[ b_0 + \sum_{i=1}^{k-1} b_i \right], \left[ c_0 + \sum_{i=1}^{k-1} c_i \right], \right. \\
&\quad \left. \left[ 1 - \frac{z}{h} \right], \left[ \frac{z}{h} \right] \right) dx
\end{aligned} \tag{2.2.24}$$

$$\begin{aligned}
(J, J^a, J^b) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{55}^{(k)} b^{(k)} \left( c_0 + \sum_{i=1}^{k-1} c_i \right) \bullet \left( \left[ c_0 + \sum_{i=1}^{k-1} c_i \right], \left[ 1 - \frac{z}{h} \right], \left[ \frac{z}{h} \right] \right) dx \\
(K, K^a) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{55}^{(k)} b^{(k)} \left( 1 - \frac{z}{h} \right) \bullet \left( \left[ 1 - \frac{z}{h} \right], \left[ \frac{z}{h} \right] \right) dx \\
(L) &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} C_{55}^{(k)} b^{(k)} \left( \frac{z}{h} \right) \bullet \left( \frac{z}{h} \right) dx
\end{aligned}$$

The governing equations and boundary conditions for the  $m$ th sublaminate are obtained from (2.2.21). If the entire laminated beam is modeled using a single sublaminate, then the beam equilibrium and boundary conditions are:

$$\begin{aligned}
\delta u_b: \quad & -\frac{dN_{xb}}{dx} + Q_b = 0 \\
\delta u_t: \quad & -\frac{dN_{xt}}{dx} + Q_t = 0 \\
\delta w_b: \quad & N_{zb} - \frac{dT_b}{dx} - q_b(x) + \gamma \left( \frac{d\theta_b}{dx} - \frac{d^2 w_b}{dx^2} \right) = 0 \\
\delta w_t: \quad & N_{zt} - \frac{dT_t}{dx} - q_t(x) + \gamma \left( \frac{d\theta_t}{dx} - \frac{d^2 w_t}{dx^2} \right) = 0 \\
\delta \theta_b: \quad & -\frac{dM_{xb}}{dx} + S_b + \gamma \left( \theta_b - \frac{dw_b}{dx} \right) = 0 \\
\delta \theta_t: \quad & -\frac{dM_{xt}}{dx} + S_t + \gamma \left( \theta_t - \frac{dw_t}{dx} \right) = 0
\end{aligned} \tag{2.2.25}$$

| Essential B.C | Natural B.C.                                             |          |
|---------------|----------------------------------------------------------|----------|
| $u_b$         | $N_{xb}$                                                 |          |
| $u_t$         | $N_{xt}$                                                 |          |
| $\theta_b$    | $M_{xb}$                                                 |          |
| $\theta_t$    | $M_{xt}$                                                 | (2.2.26) |
| $w_b$         | $T_b - \gamma \left( \theta_b - \frac{dw_b}{dx} \right)$ |          |
| $w_t$         | $T_t - \gamma \left( \theta_t - \frac{dw_t}{dx} \right)$ |          |

### 2.3 Finite Element Formulation

From (2.2.23), it can be seen that the primary variables are functions of  $x$  only, thus we need to consider only one dimensional shape functions for the finite element model. The most obvious path of development would be toward a simple two-noded beam element with six degrees of freedom per node. However, such an element would not fall within the constraints of having three translational and three rotational degrees of freedom. Even more importantly, a major advantage of the present formulation would be lost if a traditional two-noded element were developed. Because the degrees of freedom represent quantities at the surfaces of a sublaminate, it is most convenient to cast the finite element model in the form of a four-noded element with three degrees of freedom per node as shown in Figure 2. This element is classified here as a “regenerated,” element. This approach has been taken by other investigators as well [41,42,67], and has the very important advantage of allowing discretization in both inplane and through-thickness directions without the need for any special multipoint constraints. Therefore, the element can be used to predict the global response of a laminate using only one (or a small number) of ele-

ments through the thickness, and local effects such as interlaminar stresses can be captured by refining the mesh in the thickness direction near the interface(s) of interest.

*Inplane displacement and rotation* degrees of freedom are approximated by the linear Lagrange interpolation functions:

$$u_{b,t} = \sum_{i=1}^2 u_{bi,ti} P_i \quad \theta_{b,t} = \sum_{i=1}^2 \theta_{bi,ti} P_i \quad (2.3.1)$$

where

$$P_1 = \frac{1}{2}(1 - \xi) \quad P_2 = \frac{1}{2}(1 + \xi) \quad (2.3.2)$$

and  $\xi$  is the element natural (local) coordinate.

The constraints between the derivatives of transverse deflection and the rotation degree of freedom is imposed in the strain energy functional via the penalty method. As mentioned, the rotation degrees of freedom are approximated by the linear Lagrange interpolation functions. Thus, if the linear Lagrange interpolation functions are employed for the transverse deflection degrees of freedom, the constraints in the strain energy functional enforce the relationships between the linear variations of rotation degrees of freedom and the constant part of the deflection approximation, because the first derivative of a linear function is constant. This causes locking of the element. In order to alleviate locking, transverse deflection degrees of freedom are approximated by an interdependent interpolation concept developed by Tessler and Dong [71]. Initially, the transverse deflections,  $w_b$  and  $w_r$ , are approximated by quadratic Lagrange interpolation functions.

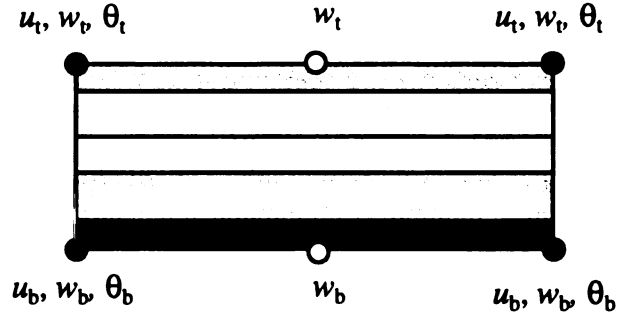
$$w_{b,t} = \sum_{i=1}^3 w_{bi,ti} N_i \quad (2.3.3)$$

where

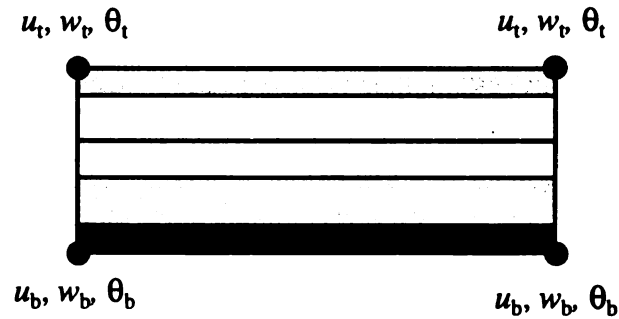
$$N_1 = \frac{1}{2}(\xi^2 - \xi), \quad N_2 = \frac{1}{2}(\xi^2 + \xi), \quad N_3 = 1 - \xi^2, \quad -1 \leq \xi \leq 1 \quad (2.3.4)$$

The virgin element is shown in Figure 2, and has a mid-node at the top and the bottom surfaces.

(a) virgin element



(b) constrained element



**Figure 2. Element topology and nodal degrees of freedom.**

The resulting mid-side deflection degrees of freedom can be eliminated by imposing the

high-order part of the constraints in (2.2.16):

$$\frac{d}{dx}\left(\theta_\alpha - \frac{dw_\alpha}{dx}\right) = 0, \quad \alpha = 1(\text{bottom}), 2(\text{top}) \quad (2.3.5)$$

Substituting Eqs. (2.3.1-2.3.4) into (2.3.5):

$$\begin{aligned} \frac{d}{dx}\left(\theta_\alpha - \frac{dw_\alpha}{dx}\right) &= 0 \\ &= \frac{1}{J_l d\xi} \frac{d\theta_\alpha}{d\xi} - \frac{1}{J_l^2 d\xi^2} \frac{d^2 w_\alpha}{d\xi^2} \\ &= \frac{2}{h_e} \left( \frac{dP_1}{d\xi} \theta_{\alpha 1} + \frac{dP_2}{d\xi} \theta_{\alpha 2} \right) - \frac{4}{h_e^2} \left( \frac{d^2 N_1}{d\xi^2} w_{\alpha 1} + \frac{d^2 N_2}{d\xi^2} w_{\alpha 2} + \frac{d^2 N_3}{d\xi^2} w_{\alpha 3} \right) \\ &= \left( \frac{-\theta_{\alpha 1} + \theta_{\alpha 2}}{h_e} \right) - \frac{4}{h_e^2} (w_{\alpha 1} + w_{\alpha 2} - 2w_{\alpha 3}) \end{aligned} \quad (2.3.6)$$

where  $J_l$  is line Jacobian and  $h_e$  is the element length.

If we solve for  $w_{\alpha 3}$  in (2.3.6),  $w_{\alpha 3}$  is:

$$w_{\alpha 3} = \frac{1}{2}(w_{\alpha 1} + w_{\alpha 2}) + \frac{h_e}{8}(\theta_{\alpha 1} - \theta_{\alpha 2}) \quad (2.3.7)$$

The substitution of  $w_{\alpha 3}$  into (2.3.3) results in the new interpolation functions for transverse deflection degrees of freedom:

$$w_{b,i} = \sum_{i=1}^2 w_{bi,ti} P_i + \sum_{i=1}^2 \theta_{bi,ti} \hat{N}_i \quad (2.3.8)$$

$$\text{where } \hat{N}_1 = \frac{h_e}{8}(1 - \xi^2), \quad \hat{N}_2 = -\frac{h_e}{8}(1 - \xi^2)$$

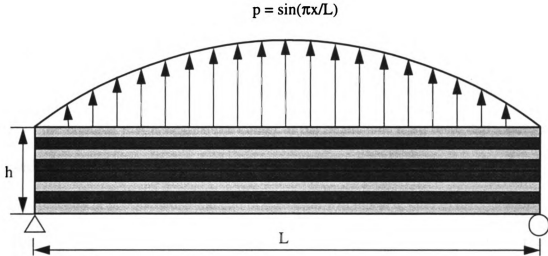


This approach improves the element behavior very effectively without introducing additional degrees of freedom. Since the interdependent interpolation scheme identically satisfies the linear part of the constraint in (2.2.25), the penalty parameter need only enforce the constant part of the constraint. It has been found that a penalty parameter on the order of the largest material shear stiffness is all that is needed to impose the constraint sufficiently, so the possibility of numerical ill-conditioning due to large penalty parameters is eliminated. Further, shear locking is obviated so that an exact order of integration can be used.

An interesting property of the present model is uncovered when it is used to model isotropic beams. Note that for the case of an isotropic beam the displacement field in (2.2.15) does not contain any derivatives of the transverse deflection degrees of freedom, since the coefficients of those terms vanish. The theory thus requires  $C^1$  transverse deflection degrees of freedom for laminated beams in which any two adjacent layers have different transverse shear stiffnesses, but requires only  $C^0$  transverse deflection degrees of freedom for isotropic beams. As formulated, the present finite element is  $C^0$ , with the constraints in (2.2.16) imposed via the interpolation scheme and the penalty method. When used to model isotropic beams, the current model yields predictions almost identical to those of the one-point-integrated four-noded isoparametric plane stress element. The present element may also be viewed as a planar element with drilling degrees of freedom, though the rotations defined in (2.2.16) are in terms of only deflection degrees of freedom. The present element is designed specifically for the analysis of beam bending, not for general plane elasticity problems.

## 2.4 Numerical results

The accuracy of the proposed laminated beam theory and the appropriateness of the finite element model were assessed by simulating the response of three different simply-supported laminated beams subjected to a transverse sinusoidal loading as shown in Figure 3. The elasticity solution of Pagano [72] was suitably modified to provide comparison with the current beam results. The material properties used in the analyses are listed in Table 1, and the lamination schemes studied are defined in Table 2.



**Figure 3. Simply supported laminated composite beam subjected to sinusoidal loading.**

**Table 1. Material properties used in the analyses.**

|            | $E_{11}$ | $E_{33}$ | $G_{13}$ | $\nu_{13}$ |
|------------|----------|----------|----------|------------|
| Material 1 | 1.00e6   | 1.00e6   | 0.20e6   | 0.25       |
| Material 2 | 25.0e6   | 1.00e6   | 0.50e6   | 0.25       |
| Material 3 | 33.0e6   | 21.0e6   | 8.00e6   | 0.25       |
| Material 4 | 32.6e6   | 10.6e6   | 8.21e6   | 0.10       |
| Core       | 50.0e3   | 50.0e3   | 21.7e3   | 0.15       |

For a given location along the length of the beam, explicit expressions for the through-thickness variation of stresses are provided by the present model. Along the beam axis, stresses were evaluated at the centroid of each element, where stress is super-convergent [73]. The bending stresses presented in the forthcoming discussion were evaluated at the center of the element that is nearest the center of the beam. The transverse deflections are obtained at the top surface of the center of the beam where load is applied, while the in-plane displacement is obtained at the end of the beam where  $x = 0$ . Due to the symmetry of the problem, only half of the beam was modeled. Unless noted otherwise, ten elements were used along half the length of the beam, and the aspect ratio of the beam was set to four.

An aspect ratio of four was chosen in the present examples in order to highlight the ability of the present model to obtain accurate results for very thick laminates via through-thickness refinement of the mesh. Analyses of beams with the same lamination schemes but with an aspect ratio of ten were also performed. In most of these cases, the predictions using one sublaminates were nearly indistinguishable from the exact elasticity solution. These results are omitted for brevity.

**Table 2. Ply stacking tables for the three laminates analyzed.**

| <b>Laminate</b> | <b>Layer</b> | <b>Material</b> | <b>Thickness</b> |
|-----------------|--------------|-----------------|------------------|
| <b>1</b>        | <b>1</b>     | <b>2</b>        | <b>0.125</b>     |
|                 | <b>2</b>     | <b>1</b>        | <b>0.125</b>     |
|                 | <b>3</b>     | <b>2</b>        | <b>0.125</b>     |
|                 | <b>4</b>     | <b>1</b>        | <b>0.125</b>     |
|                 | <b>5</b>     | <b>1</b>        | <b>0.125</b>     |
|                 | <b>6</b>     | <b>2</b>        | <b>0.125</b>     |
|                 | <b>7</b>     | <b>1</b>        | <b>0.125</b>     |
|                 | <b>8</b>     | <b>2</b>        | <b>0.125</b>     |
| <b>2</b>        | <b>1</b>     | <b>2</b>        | <b>0.300</b>     |
|                 | <b>2</b>     | <b>1</b>        | <b>0.200</b>     |
|                 | <b>3</b>     | <b>2</b>        | <b>0.150</b>     |
|                 | <b>4</b>     | <b>3</b>        | <b>0.250</b>     |
|                 | <b>5</b>     | <b>1</b>        | <b>0.100</b>     |
| <b>3</b>        | <b>1</b>     | <b>1</b>        | <b>0.010</b>     |
|                 | <b>2</b>     | <b>4</b>        | <b>0.025</b>     |
|                 | <b>3</b>     | <b>2</b>        | <b>0.015</b>     |
|                 | <b>4</b>     | <b>1</b>        | <b>0.020</b>     |
|                 | <b>5</b>     | <b>2</b>        | <b>0.030</b>     |
|                 | <b>6</b>     | <b>Core</b>     | <b>0.800</b>     |
|                 | <b>7</b>     | <b>2</b>        | <b>0.030</b>     |
|                 | <b>8</b>     | <b>1</b>        | <b>0.020</b>     |
|                 | <b>9</b>     | <b>2</b>        | <b>0.015</b>     |
|                 | <b>10</b>    | <b>4</b>        | <b>0.025</b>     |
|                 | <b>11</b>    | <b>1</b>        | <b>0.010</b>     |

### ***Laminate 1:***

The first example is denoted as Laminate 1 and has the lamination sequence (0/90/0/90),. In Figure 4, normalized predicted center deflection and normalized predicted in-plane stress is plotted as a function of the number of elements used in the mesh. In this set of results,  $L/h=100$ , and only one sublamine was used. The error was less than 1% when seven elements were used in half the length of the beam. The influence of aspect ratios on transverse deflection is shown in Figure 5. Nine different aspect ratios were considered to check a wide range of beam bending behavior ( $L/h= 4, 8, 10, 20, 30, 50, 100, 500, 1000$ , and 10000). When one sublamine was used, the error in predicted deflection was less than 5 percent even for the case  $L/h=4$ . The use of two sublaminae of equal thickness resulted in more accurate predictions, but the trends were almost the same as those when one sublamine was used. When eight sublaminae (one element per layer) were used, the error in the predicted deflection was less than one percent for all aspect ratios considered.

In Figure 6, the variation of in-plane displacement through the thickness of the laminate is plotted for a beam with aspect ratio of four. The exact distribution of in-plane displacement is approximately linear within each layer, and globally piece-wise linear and of zig-zag pattern due to the difference in the shear moduli of adjacent layers. For the model containing only one sublamine, the difference between the present model predictions and the elasticity solution at the top surface was about 13 percent. However, when eight sublaminae were used, the numerical predictions were in excellent agreement with the elasticity solution, with error of about 2.75 percent. The stress distribution through the thickness is illustrated in Figure 7. It can be observed that the prediction of in-plane stress

$\sigma_{xx}$  converges rapidly to the exact solution as more sublaminates are used.

***Laminate 2:***

Laminate 2 is a random five layer laminate in which the thickness and material properties vary considerably from layer to layer, a challenging problem for any model to solve accurately. In this case, most predicted results are locally relatively poorer than those for Laminate 1 which has a symmetric lamination sequence. Predictions of normalized transverse deflection versus aspect ratio are represented in Figure 8. It can be observed that when the aspect ratio is greater than 20, a one sublamine approximation gives quite accurate estimates of center deflection. However, five sublaminates (one element per layer) are needed to obtain accurate predictions of deflection for an aspect ratio of four. The in-plane displacements predicted by the present model and by elasticity are plotted versus the thickness of the beam in Figure 9. Even though the overall prediction of in-plane displacement by the current model agreed well with the elasticity solution, a rather large error in the prediction of in-plane displacement occurred in the first layer of the beam when either one or two sublaminates were used. This is an example of the shortcoming of first-order zig-zag theory in modeling very unsymmetric laminates. However, the prediction using five sublaminates was very close to that of the elasticity solution. The predicted distribution of in-plane stress as a function of thickness is plotted in Figure 10. Overall, the predictions of the current model are very good, with the maximum error occurring at the top of the laminate.

***Laminate 3:***

The last example is a sandwich panel, denoted here as Laminate 3, which consists of a

thick core between two sets of five face sheets. As can be seen in Table 2, the core occupies eighty percent of the thickness of the beam, while each set of face sheets contains five layers (with a lamination sequence very similar to that of Laminate 2) and occupies ten percent of the total thickness. Three types of through-thickness discretization were used in the analyses: (1) a single sublaminate, (2) three sublaminae -- one for the core and one each for the face sheets, and (3) five sublaminae -- three for the core and one each for the face sheets.

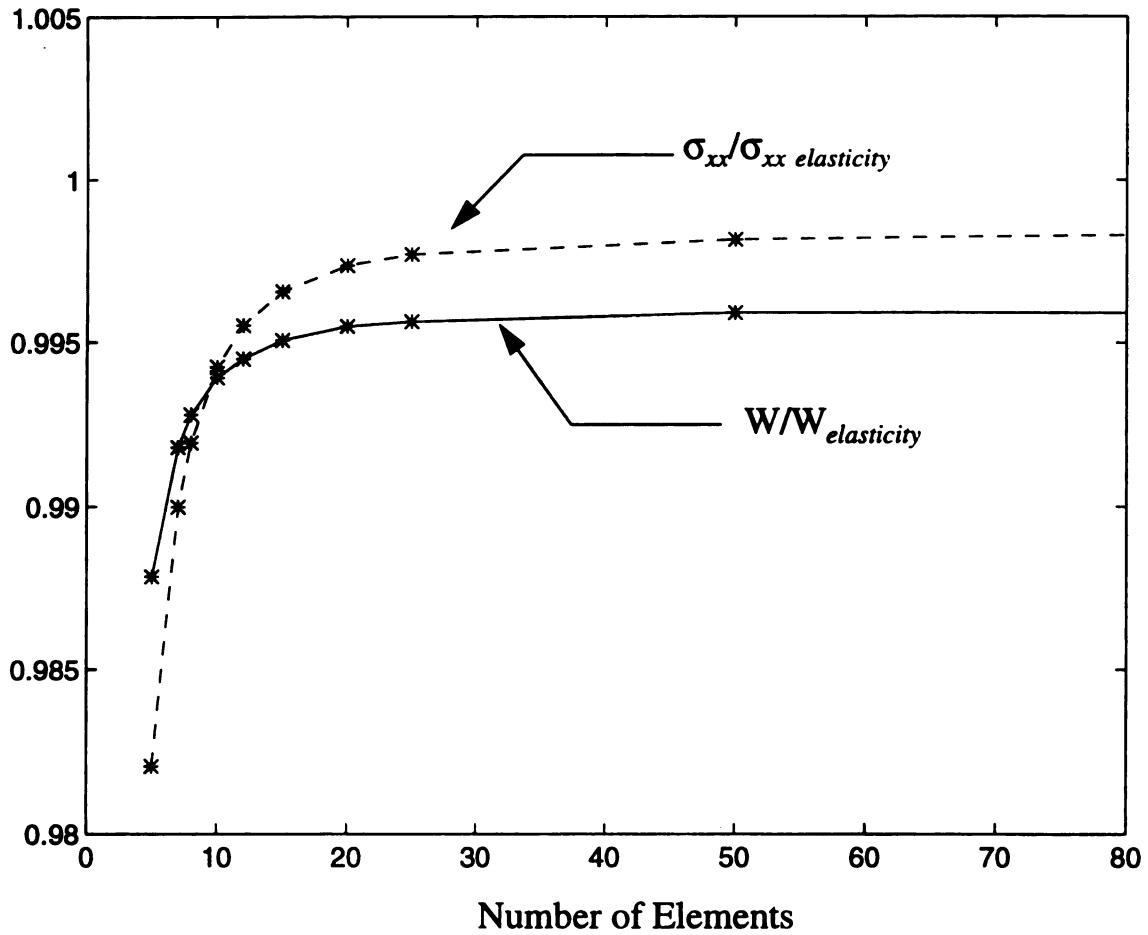
The predicted normalized deflections for various aspect ratios are plotted in Figure 11. When the aspect ratio was greater than 4, only one sublaminate was needed to yield accurate results. In Figure 12 the predicted through-thickness distributions of in-plane displacement are illustrated. All predictions were very accurate in the upper and lower face sheet regions, but there were significant errors in the core region when only one or three sublaminae were used. These errors were due to the very large transverse shear strains and transverse squashing in the core of this thick sandwich beam. The nonlinear variation of inplane displacement in this region could not be captured accurately by a single sublaminate in the core region. For thinner beams (i.e.,  $L/h$  greater than about 10), the inplane displacements throughout the thickness are predicted satisfactorily using a single sublaminate through the entire thickness of the laminate.

The bending stress predictions shown in Figure 13 are so accurate that it is difficult to distinguish between analytical and finite element results. To assess these more clearly, the stress distribution in the upper face sheet region is enlarged and shown in Figure 14. It can be observed that the finite element model using only one sublaminate predicts the stress distribution in the face sheets very accurately.

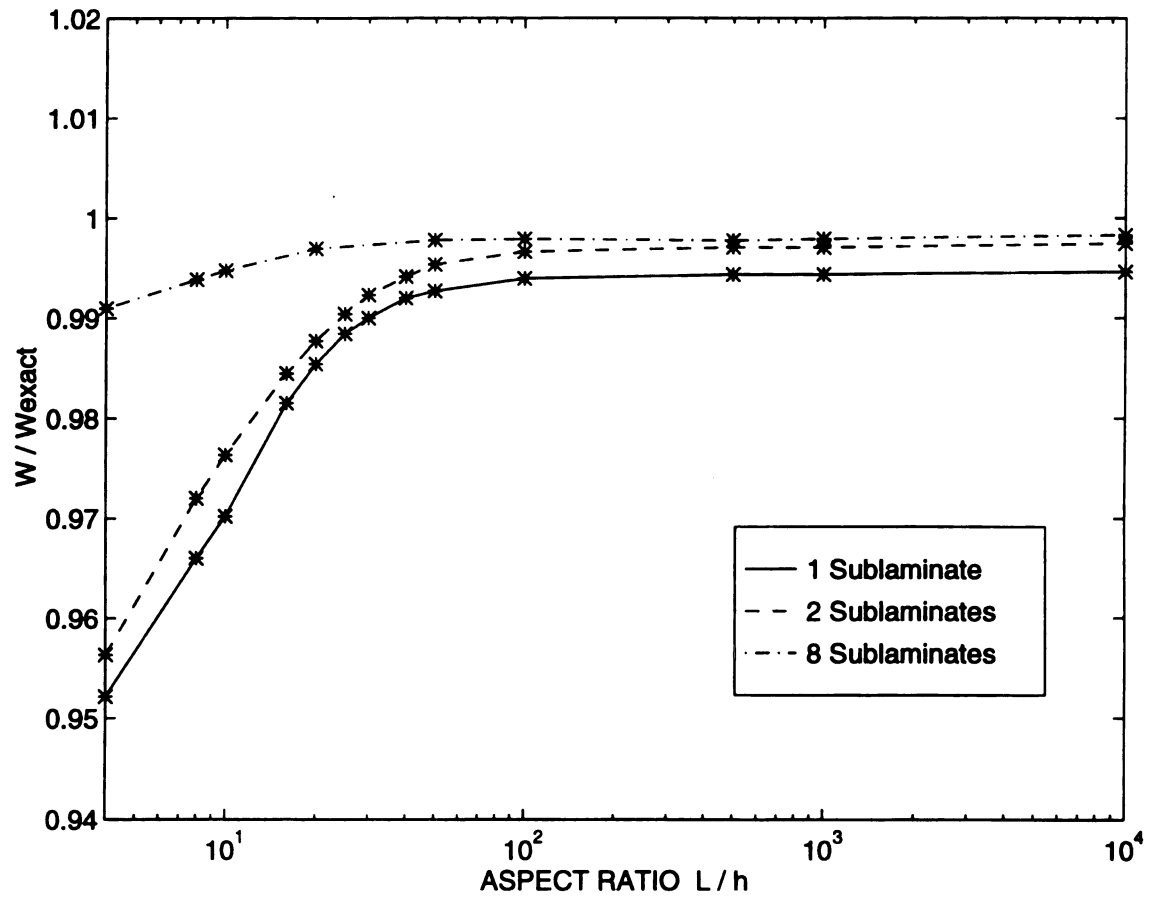
## 2.5 Summary

An improved first order zig-zag theory and an associated finite element model are presented. The model allows the representation of a laminate as an assemblage of sublaminates in order to increase the model refinement through the thickness, when needed. Within each sublaminate, an accurate first-order zig-zag sublaminate kinematic approximation is made which minimizes the need for multiple sublaminates for many problems of practical interest yet easily accommodates through-thickness refinement for predicting local variations of stress and deformation. The finite element model is “regenerated” in the form of a four-noded planar element with three degrees of freedom (two translations and one rotation) per element. Thus, it is suitable for implementation into commercial finite element codes. Numerical results demonstrate that the current model is accurate, efficient, and robust for analysis of a wide variety of thick or thin laminated beams.

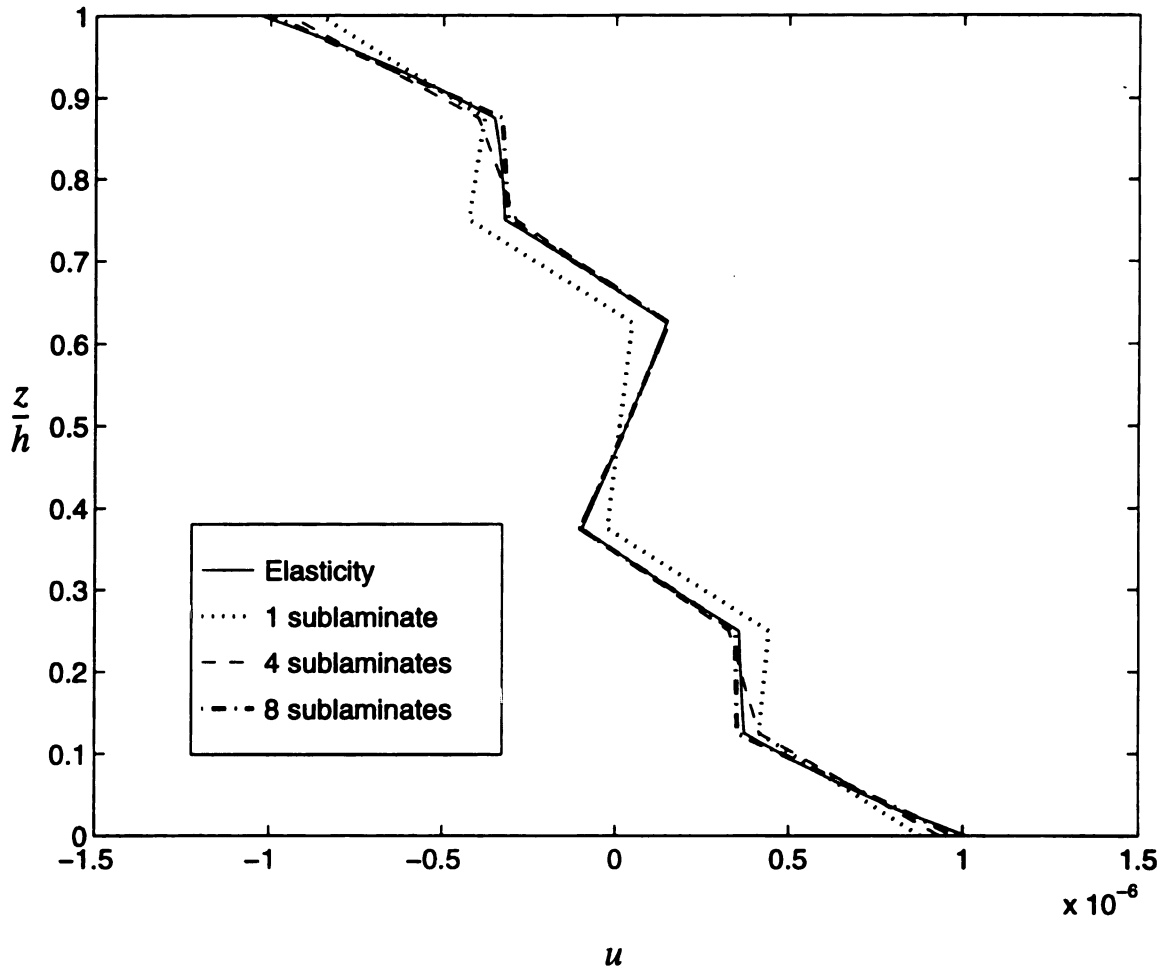




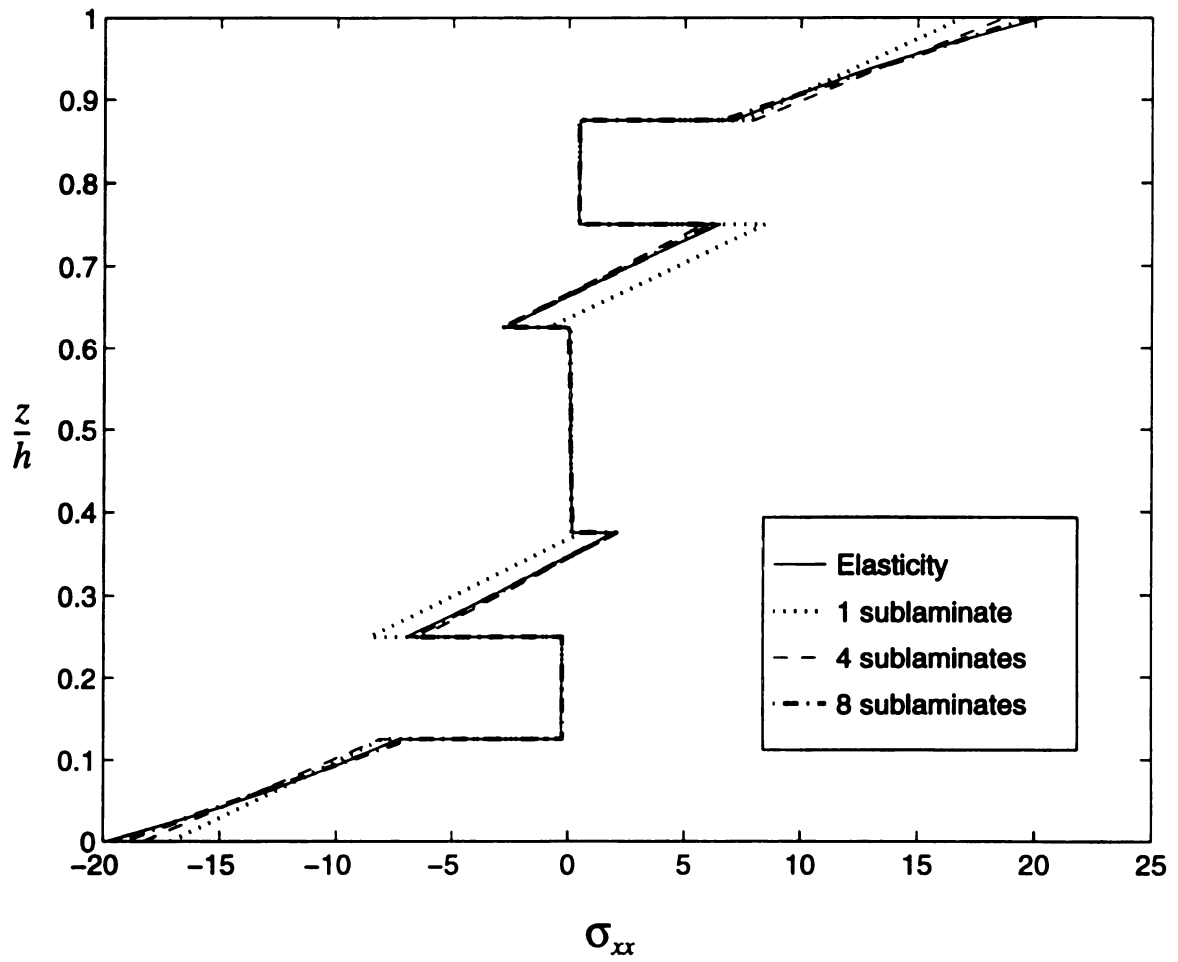
**Figure 4. Variation of normalized center deflection and maximum bending stress versus number of elements in the mesh (Laminate 1,  $L/h=100$ ).**



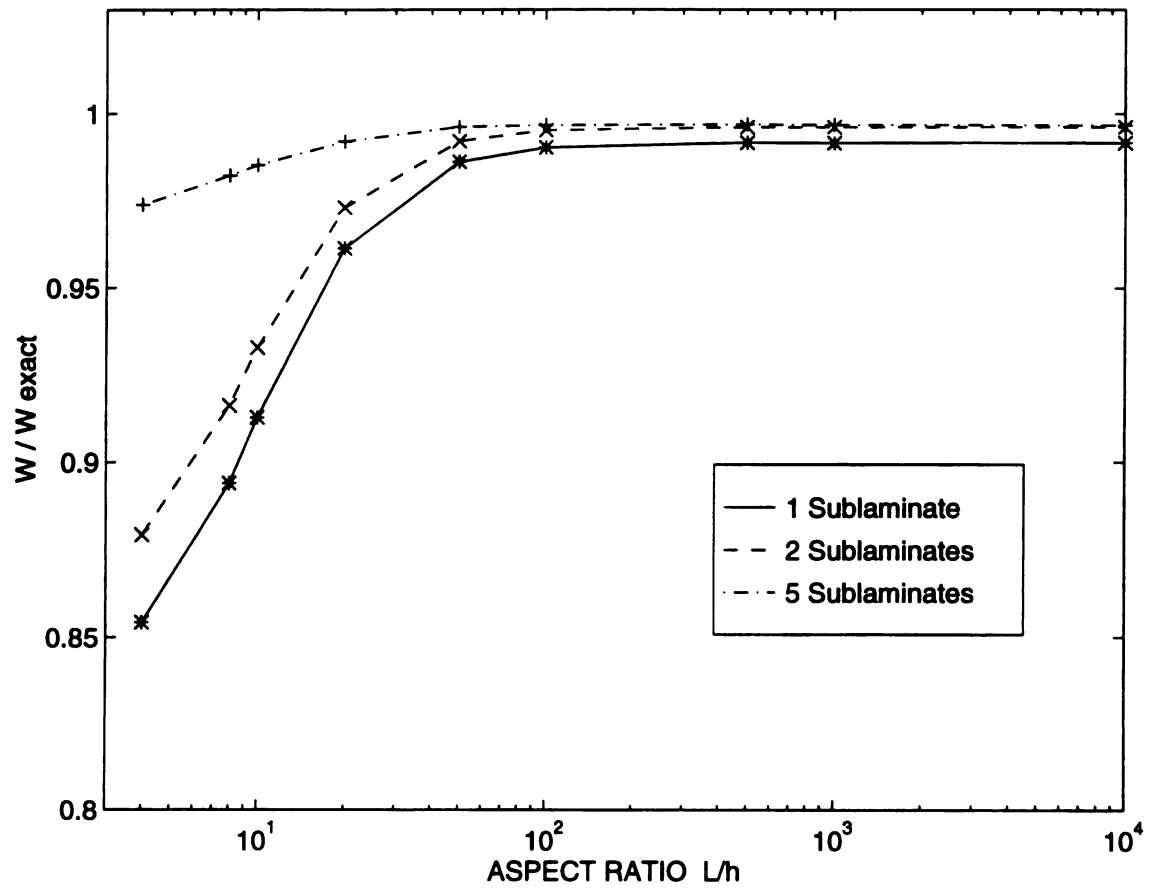
**Figure 5. Variation of normalized center deflection versus beam aspect ratio (Laminate 1).**



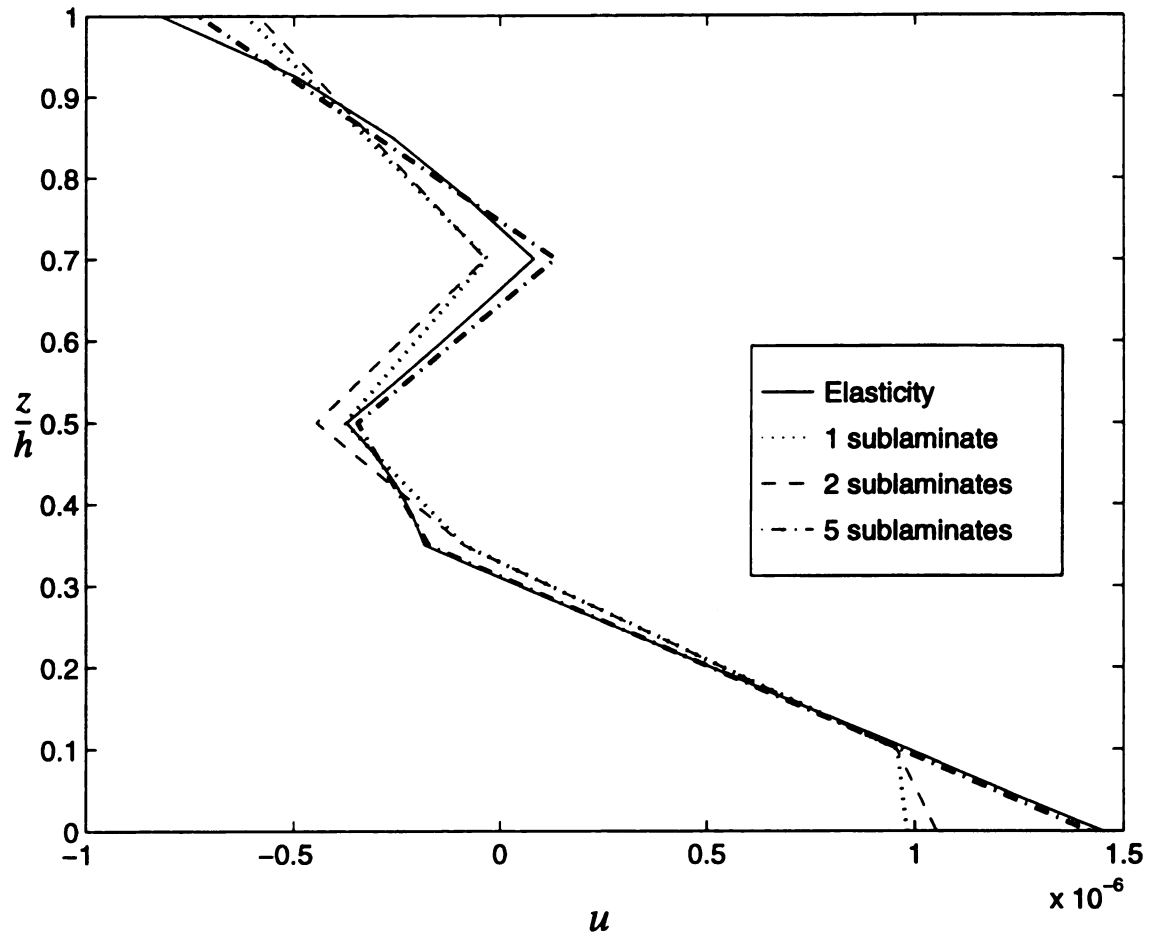
**Figure 6. Inplane displacement versus normalized thickness coordinate (Laminate 1,  $L/h=4$ ).**



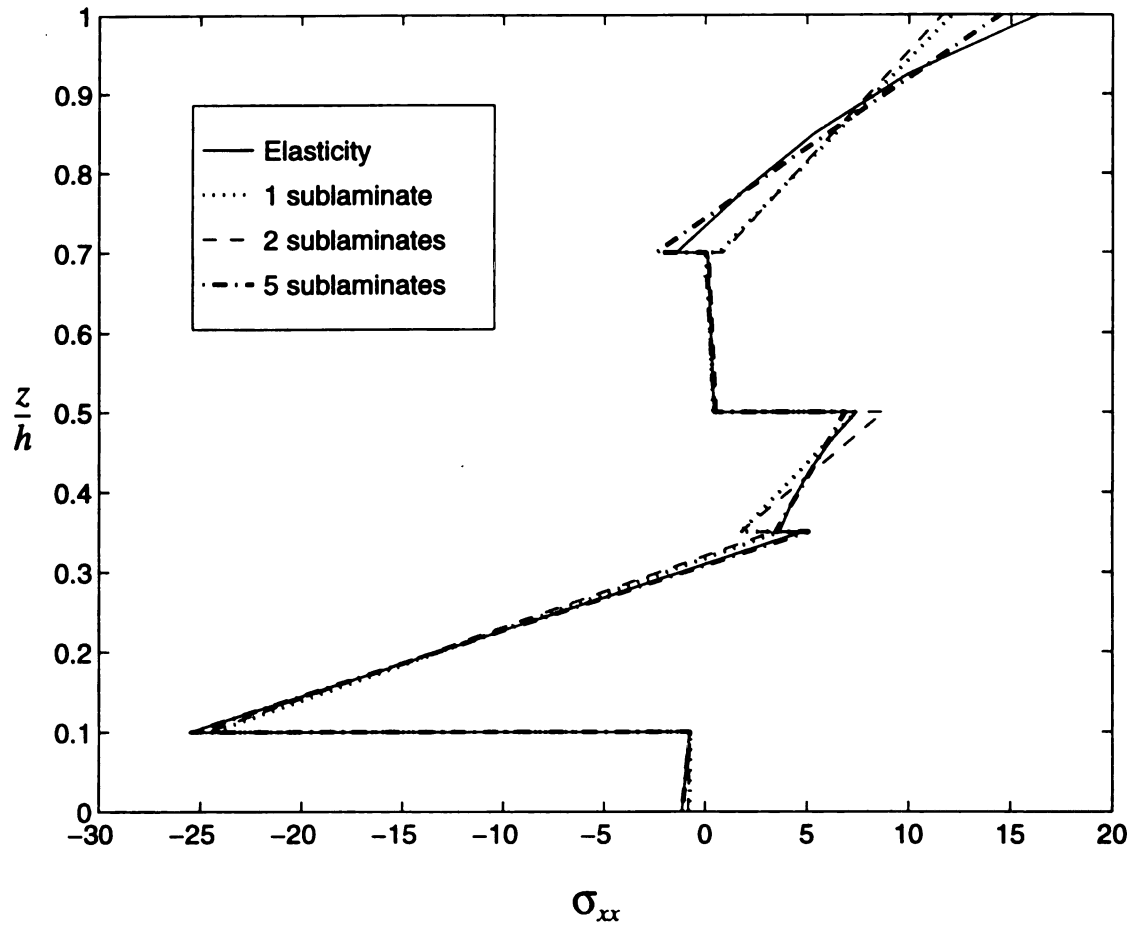
**Figure 7. Bending stress versus normalized thickness coordinate (Laminate 1,  $L/h=4$ ).**



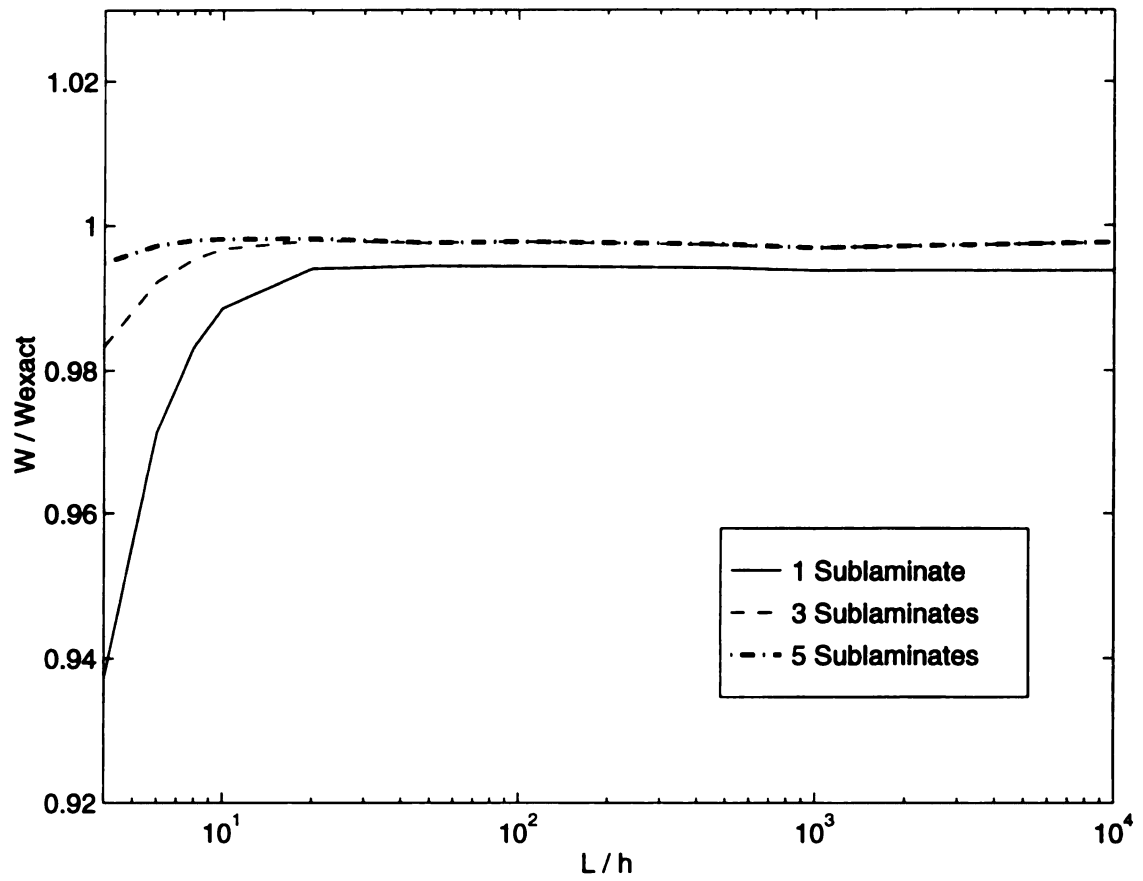
**Figure 8. Variation of normalized center deflection versus beam aspect ratio (Laminate 2).**



**Figure 9. Inplane displacement versus normalized thickness coordinate (Laminate 2,  $L/h=4$ ).**

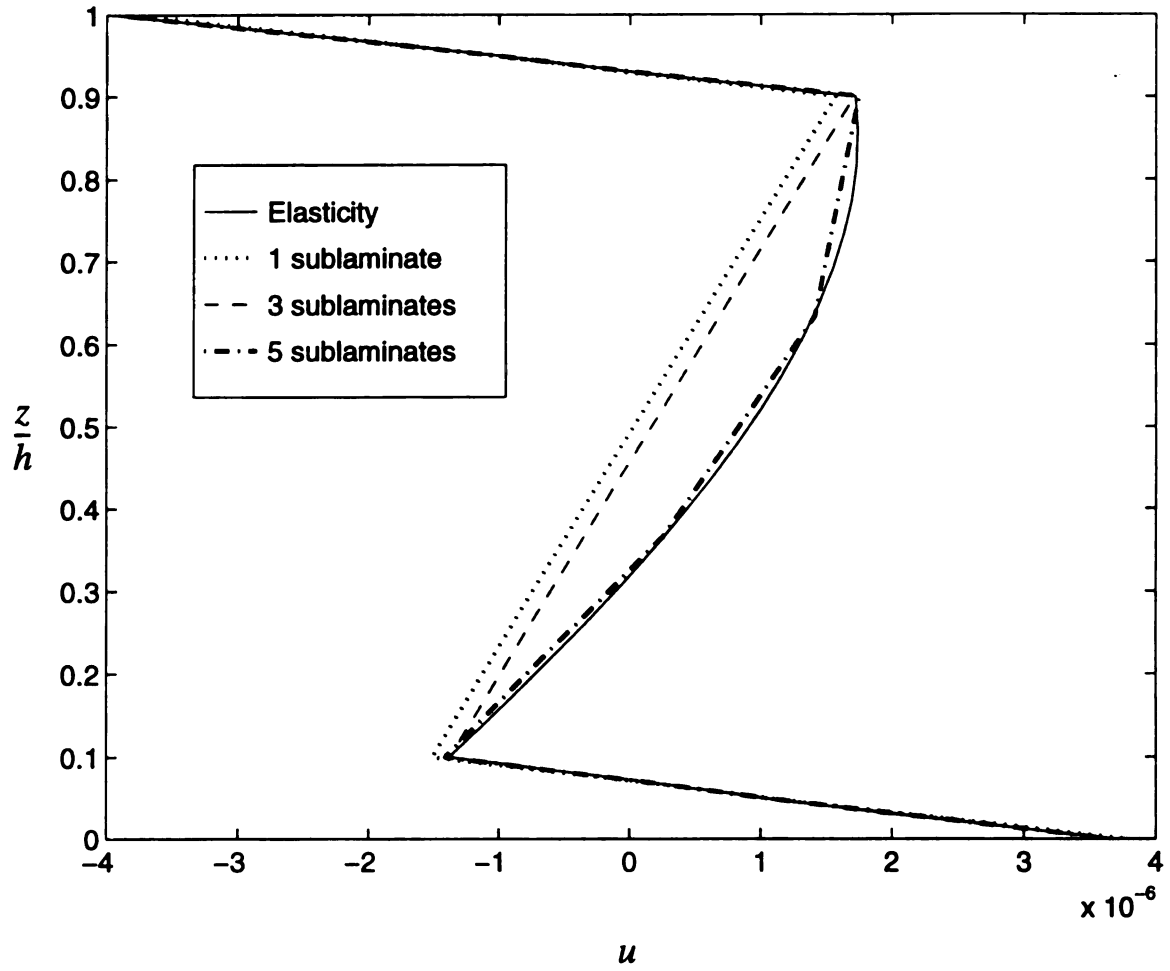


**Figure 10. Bending stress versus normalized thickness coordinate  
(Laminate 2,  $L/h=4$ ).**

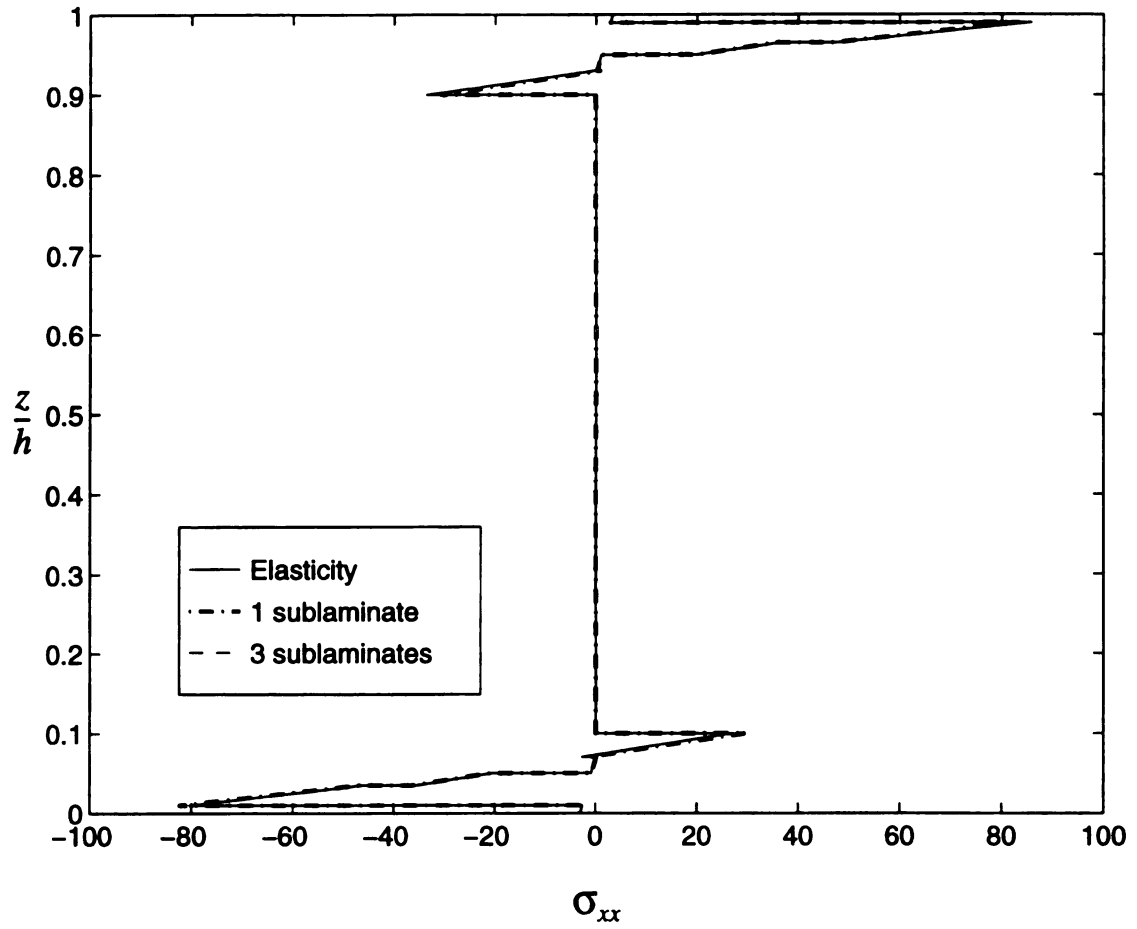


**Figure 11. Variation of normalized center deflection versus sandwich beam aspect ratio (Laminate 3).**

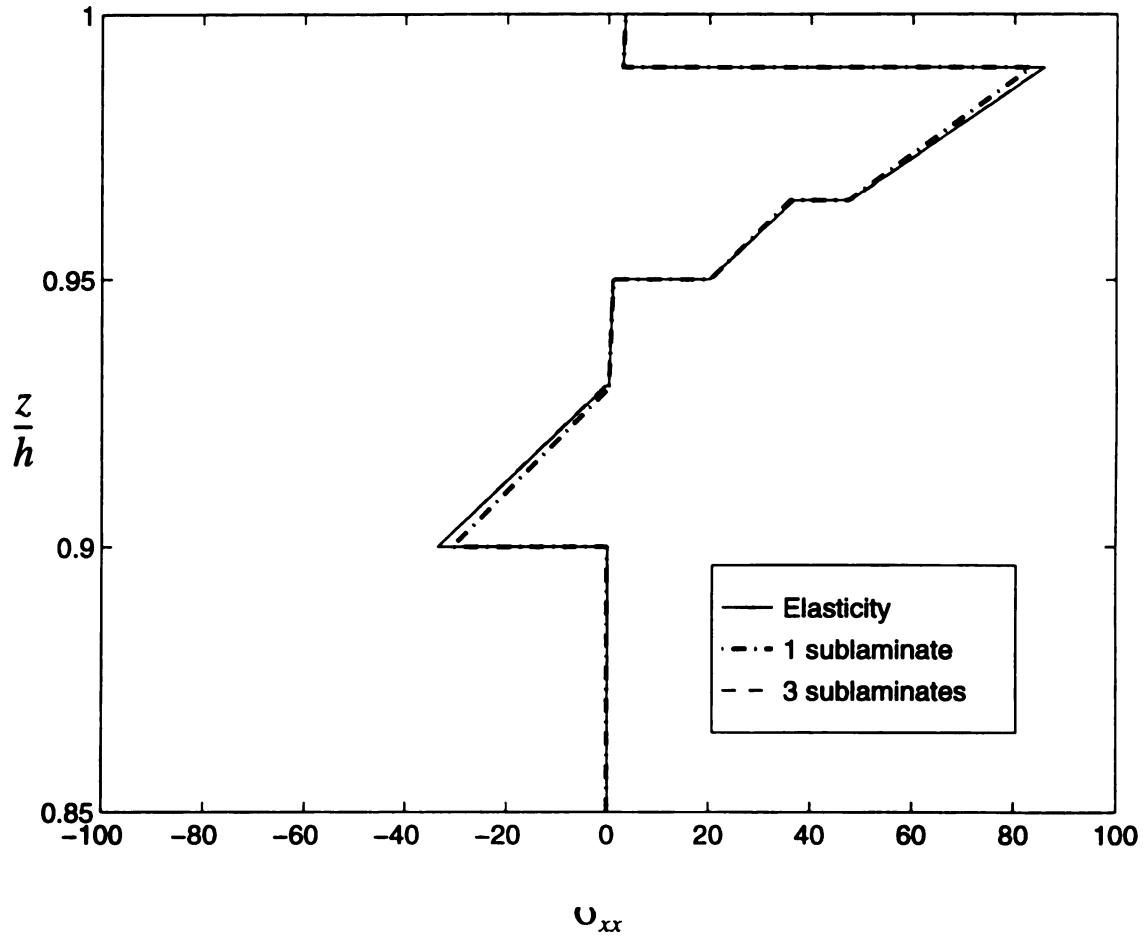




**Figure 12. Inplane displacement versus normalized thickness coordinate (Laminate 3,  $L/h=4$ ).**



**Figure 13. Bending stress versus normalized thickness coordinate (Laminate 3,  $L/h=4$ ).**



**Figure 14. Bending stress in face sheets versus normalized thickness coordinate (Laminate 3,  $L/h=4$ ).**

# **CHAPTER III**

## **PLATE ELEMENT BASED ON**

### **FIRST ORDER ZIG-ZAG SUBLAMINATE THEORY**

#### **3.1 Introduction**

In this chapter, after estimating the shortcomings and merits of the first order zig-zag beam theory and the beam element, a refined plate theory based on sublaminate linear zig-zag kinematics and a new 3-D finite element based on that theory are developed. The in-plane displacement fields in each sublaminate are assumed to be piecewise linear functions and vary in a zig-zag fashion through the thickness of the sublaminate. The zig-zag functions are obtained by satisfying the continuity of transverse shear stresses at layer interfaces. This in-plane displacement field assumption accounts for discrete layer effects without increasing the number of degrees of freedom as the number of layers is increased. The transverse normal strain predictions are improved by assuming a constant variation of transverse normal stress in each sublaminate. In the computational model, each finite element represents one sublaminate. The finite element is developed with the topology of an eight-noded brick, allowing the thickness of the plate to be discretized into several elements, or sublaminate, where each sublaminate can contain more than one physical layer. Each node has five engineering degrees of freedom, three translations and two rotations.

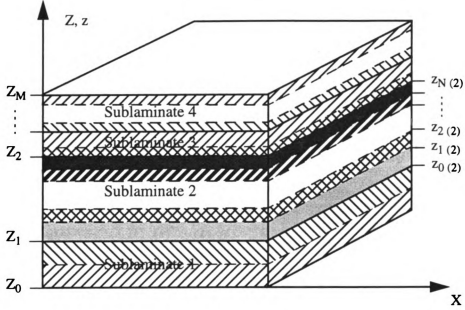
Thus, this element can be conveniently implemented into general purpose finite element codes. The element stiffness coefficients are integrated exactly, yet the element exhibits no shear locking due to the use of an interdependent interpolation scheme and consistent shear strain fields. Numerical performance of the current element is investigated for a composite armored vehicle panel and a sandwich panel with low and high aspect ratios. Comparison of numerical results with elasticity solutions shows that the element is very accurate and robust.

### 3.2 Kinematic Assumptions

In this plate theory, a laminate will be modeled as  $M$  sublaminates, and each sublamine is assumed to consist of  $N_m$  perfectly bonded layers which have independent thickness and independent stiffness properties. Mathematically, the total number of layers,  $N$ , in the laminate is written as:

$$N = \sum_{m=1}^M N_m \quad (3.2.1)$$

The global  $Z$ -axis is taken perpendicular to the inplane  $X, Y$  coordinate axes and has its origin at the bottom of the laminate, while  $z$  is a local sublamine coordinate in the direction of  $Z$  with its origin at the bottom of the sublamine (see Figure 15). In the following derivation all expressions are related to the  $m$ th sublamine in order to facilitate the development a computationally convenient finite element model. All indices indicating the sublamine number are omitted for brevity.



**Figure 15. Schematic of sublaminate and layer divisions in the laminated plate theory.**

The constitutive relations for a three dimensional stress state in the  $k$ th layer of the  $m$ th sublaminate can be written as:

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \sigma_{zz}^{(k)} \\ \sigma_{yz}^{(k)} \\ \sigma_{zx}^{(k)} \\ \sigma_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(k)} & C_{12}^{(k)} & C_{13}^{(k)} & 0 & 0 & C_{16}^{(k)} \\ C_{12}^{(k)} & C_{22}^{(k)} & C_{23}^{(k)} & 0 & 0 & C_{26}^{(k)} \\ C_{13}^{(k)} & C_{23}^{(k)} & C_{33}^{(k)} & 0 & 0 & C_{36}^{(k)} \\ 0 & 0 & 0 & C_{44}^{(k)} & C_{45}^{(k)} & 0 \\ 0 & 0 & 0 & C_{45}^{(k)} & C_{55}^{(k)} & 0 \\ C_{16}^{(k)} & C_{26}^{(k)} & C_{36}^{(k)} & 0 & 0 & C_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \epsilon_{zz}^{(k)} \\ \gamma_{yz}^{(k)} \\ \gamma_{zx}^{(k)} \\ \gamma_{xy}^{(k)} \end{Bmatrix} \quad (3.2.2)$$

For monoclinic materials, 13 coefficients of the material stiffness matrix are independent, but for most practical laminated composites, nine coefficients of the material stiffness

matrix are independent and the others are obtained through a transformation law [79].

These nine coefficients are given as follows:

$$\begin{aligned}
 \bar{C}_{11}^{(k)} &= E_1^{(k)} \frac{1 - \nu_{23}^{(k)} \nu_{32}^{(k)}}{\Delta^{(k)}}, \quad \bar{C}_{12}^{(k)} = E_1^{(k)} \frac{\nu_{21}^{(k)} + \nu_{31}^{(k)} \nu_{23}^{(k)}}{\Delta^{(k)}}, \quad \bar{C}_{13}^{(k)} = E_1^{(k)} \frac{\nu_{31}^{(k)} + \nu_{21}^{(k)} \nu_{32}^{(k)}}{\Delta^{(k)}} \\
 \bar{C}_{22}^{(k)} &= E_2^{(k)} \frac{1 - \nu_{13}^{(k)} \nu_{31}^{(k)}}{\Delta^{(k)}}, \quad \bar{C}_{23}^{(k)} = E_2^{(k)} \frac{\nu_{32}^{(k)} + \nu_{12}^{(k)} \nu_{31}^{(k)}}{\Delta^{(k)}}, \quad \bar{C}_{33}^{(k)} = E_3^{(k)} \frac{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}{\Delta^{(k)}} \\
 \bar{C}_{44}^{(k)} &= G_{23}^{(k)}, \quad \bar{C}_{55}^{(k)} = G_{13}^{(k)}, \quad \bar{C}_{66}^{(k)} = G_{12}^{(k)} \\
 \Delta^{(k)} &= 1 - \nu_{12}^{(k)} \nu_{21}^{(k)} - \nu_{23}^{(k)} \nu_{32}^{(k)} - \nu_{31}^{(k)} \nu_{13}^{(k)} - 2\nu_{21}^{(k)} \nu_{32}^{(k)} \nu_{13}^{(k)} \\
 \frac{\nu_{ij}^{(k)}}{E_i^{(k)}} &= \frac{\nu_{ji}^{(k)}}{E_j^{(k)}} \quad i, j = 1, 2, 3
 \end{aligned} \tag{3.2.3}$$

where  $E_i^{(k)}$  are the Young's moduli in  $x, y, z$  directions and  $\nu_{ij}^{(k)}$  are the Poisson's ratios associated with transverse strain in the  $j$  direction due to axial strain in the  $i$  direction in the  $k$ th layer. From the transformation law of a fourth order tensor, 13 coefficients are derived using the transformation law of a fourth order tensor:

$$C_{ijkl}^{(k)} = a_{im} a_{jn} a_{kp} a_{lq} \bar{C}_{mnpq}^{(k)} \tag{3.2.4}$$

where  $a_{im}$  stands for the direction cosine associated with the material and the global axes.

The infinitesimal strain - displacement relationships are defined as:

$$\begin{aligned}
 \epsilon_{xx}^{(k)} &= \frac{\partial u_x^{(k)}}{\partial x}, \quad \epsilon_{yy}^{(k)} = \frac{\partial u_y^{(k)}}{\partial y}, \quad \epsilon_{zz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial z} \\
 \gamma_{yz}^{(k)} &= \frac{\partial u_z^{(k)}}{\partial y} + \frac{\partial u_y^{(k)}}{\partial z}, \quad \gamma_{xz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial z}, \quad \gamma_{xy}^{(k)} = \frac{\partial u_y^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial y}
 \end{aligned} \tag{3.2.5}$$

### 3.3 Formulation of the Plate Theory

The displacement fields in a sublaminate are initially assumed in the following form:

$$\begin{aligned}
 u_x^{(k)}(x, y, z, t) &= u_b(x, y, t) + z\psi_x(x, y, t) + \sum_{i=1}^{k-1} (z - z_i)\xi_i \\
 u_y^{(k)}(x, y, z, t) &= v_b(x, y, t) + z\psi_y(x, y, t) + \sum_{i=1}^{k-1} (z - z_i)\eta_i \\
 u_z^{(k)}(x, y, z, t) &= w_b(x, y, t)\left(1 - \frac{z}{h}\right) + w_t(x, y, t)\left(\frac{z}{h}\right)
 \end{aligned} \tag{3.3.1}$$

where  $h$  is the thickness of the sublaminate,  $u_b$  and  $v_b$  are the axial displacements in  $x$  and  $y$  directions, respectively, at  $z = 0$ ,  $\psi_x$  and  $\psi_y$  are rotations of the normal at  $z = 0$ , and  $w_b$  and  $w_t$  are the transverse deflections of the bottom and top surfaces, respectively, of the  $m$ th sublaminate. Thus, it is assumed that  $u_x^{(k)}$  and  $u_y^{(k)}$  vary in a piecewise linear fashion through the thickness of a sublaminate and  $u_z^{(k)}$  varies linearly through the thickness.

The parameters  $\xi_i$  and  $\eta_i$  in Eq. (3.3.1) are eliminated by enforcing the continuity of shear stress at each interface [33, 34]. The transverse shear stress continuity conditions at the  $k$ th interface can be expressed as:

$$\sigma_{yz}^{(k)} = \sigma_{yz}^{(k+1)}, \quad \sigma_{zx}^{(k)} = \sigma_{zx}^{(k+1)}. \tag{3.3.2}$$

Assuming that infinitesimal strain theory holds, the form of the assumed displacement fields in (3.3.1) is such that the inplane displacements  $u_x^{(k)}$  and  $u_y^{(k)}$  contribute only constant (i.e., no variation in  $z$ ) terms to the transverse shear stresses,  $\sigma_{yz}^{(k)}$  and  $\sigma_{xz}^{(k)}$ . For consistency, then, the transverse displacement  $u_z^{(k)}$  should also contribute only constant terms to these stress components. In order to achieve this consistency, terms in the trans-



verse shear strain expressions that involve the transverse displacement variables are evaluated at the mid-plane of the sublaminate, thereby taking a thickness-averaged value of these terms. This approach is equivalent to ignoring the effect of transverse normal strain on the transverse shear strains, a very good assumption for most laminates. Thus, the transverse shear strains in the  $k$ th layer are:

$$\begin{aligned}\gamma_{zx}^{(k)} &= \psi_x + \sum_{i=1}^{k-1} \xi_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \\ \gamma_{yz}^{(k)} &= \psi_y + \sum_{i=1}^{k-1} \eta_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_t}{\partial y} \right)\end{aligned}\tag{3.3.3}$$

From (3.3.3), the relationships between transverse shear strains in the  $(k+1)$ th layer and the  $k$ th layer are:

$$\begin{aligned}\gamma_{zx}^{(k+1)} &= \psi_x + \sum_{i=1}^k \xi_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \\ &= \psi_x + \sum_{i=1}^{k-1} \xi_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) + \xi_k \\ &= \gamma_{zx}^{(k)} + \xi_k\end{aligned}\tag{3.3.4}$$

By the same token,

$$\gamma_{yz}^{(k+1)} = \gamma_{yz}^{(k)} + \eta_k\tag{3.3.5}$$

From the constitutive equation in (3.2.2), the transverse shear stress-strain relationships are:

$$\begin{aligned}\sigma_{yz}^{(k)} &= C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)} \\ \sigma_{xz}^{(k)} &= C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)}\end{aligned}\quad (3.3.6)$$

Making use of (3.3.4), (3.3.5) and (3.3.6), the transverse shear stress continuity conditions at the  $k$ th interface can be recast as:

$$\begin{aligned}(C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=z_k} &= \{C_{44}^{(k+1)} (\gamma_{yz}^{(k)} + \eta_k) + C_{45}^{(k)} (\gamma_{xz}^{(k)} + \xi_k)\} \Big|_{z=z_k} \\ (C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=z_k} &= \{C_{45}^{(k+1)} (\gamma_{yz}^{(k)} + \eta_k) + C_{55}^{(k)} (\gamma_{xz}^{(k)} + \xi_k)\} \Big|_{z=z_k}\end{aligned}\quad (3.3.7)$$

Solving (3.3.7) for  $\eta_k$  and  $\xi_k$ , and using (3.3.3), we can develop the following recursive equations.

$$\begin{aligned}\eta_k &= \hat{a}_k \left\{ \psi_y + \sum_{i=1}^{k-1} \eta_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_t}{\partial y} \right) \right\} + \hat{b}_k \left\{ \psi_x + \sum_{i=1}^{k-1} \xi_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \right\} \\ \xi_k &= \hat{c}_k \left\{ \psi_y + \sum_{i=1}^{k-1} \eta_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_t}{\partial y} \right) \right\} + \hat{d}_k \left\{ \psi_x + \sum_{i=1}^{k-1} \xi_i + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \right\}\end{aligned}\quad (3.3.8)$$

where

$$\begin{aligned}\hat{a}_k &= \frac{1}{\Delta_{k+1}} (C_{55}^{(k+1)} C_{44}^{(k)} - C_{45}^{(k+1)} C_{45}^{(k)}) - 1 \\ \hat{b}_k &= \frac{1}{\Delta_{k+1}} (C_{55}^{(k+1)} C_{45}^{(k)} - C_{45}^{(k+1)} C_{55}^{(k)}) \\ \hat{c}_k &= \frac{1}{\Delta_{k+1}} (C_{44}^{(k+1)} C_{45}^{(k)} - C_{45}^{(k+1)} C_{44}^{(k)}) \\ \hat{d}_k &= \frac{1}{\Delta_{k+1}} (C_{44}^{(k+1)} C_{55}^{(k)} - C_{45}^{(k+1)} C_{45}^{(k)}) - 1 \\ \Delta_{k+1} &= C_{44}^{(k+1)} C_{55}^{(k)} - C_{45}^{(k+1)^2}\end{aligned}\quad (3.3.9)$$

Solving the recursive relation, (3.3.8),  $\xi_i$  and  $\eta_i$  are then found to be:

$$\begin{aligned}\xi_i &= c_i \left\{ \psi_y + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_t}{\partial y} \right) \right\} + d_i \left\{ \psi_x + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \right\} \\ \eta_i &= a_i \left\{ \psi_y + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_t}{\partial y} \right) \right\} + b_i \left\{ \psi_x + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_t}{\partial x} \right) \right\}\end{aligned}\quad (3.3.10)$$

where

$$\begin{aligned}a_i &= \hat{a}_i \left( 1 + \sum_{j=1}^{i-1} a_j \right) + \hat{b}_i \sum_{j=1}^{i-1} c_j \\ b_i &= \hat{b}_i \left( 1 + \sum_{j=1}^{i-1} d_j \right) + \hat{a}_i \sum_{j=1}^{i-1} b_j \\ c_i &= \hat{c}_i \left( 1 + \sum_{j=1}^{i-1} a_j \right) + \hat{d}_i \sum_{j=1}^{i-1} c_j \\ d_i &= \hat{d}_i \left( 1 + \sum_{j=1}^{i-1} d_j \right) + \hat{c}_i \sum_{j=1}^{i-1} b_j\end{aligned}\quad (3.3.11)$$

From Eqs. (3.3.9-3.3.11) it can be seen that  $\xi_i$  and  $\eta_i$  depend on the ratios of shear properties between adjacent layers and the shear deformation in each sublaminate. If either of these quantities is small, then discrete-layer effects will also be small.

The rotational variables  $\psi_x$  and  $\psi_y$  in the displacement fields are now eliminated by introducing translational variables  $u_t$  and  $v_t$ , the inplane translations at the top surface of the sublaminate [41,42] in order to expedite the development of versatile finite element models. Thus, rather than describing the inplane displacement field by a translation and rotation at one point, it can more conveniently be described here by the translation at two points.

Moreover, the displacement fields still need to be manipulated to develop a  $C^0$  finite element, because the derivatives of transverse deflections,

$$\frac{\partial w_b}{\partial x}, \frac{\partial w_t}{\partial x}, \frac{\partial w_b}{\partial y} \text{ and } \frac{\partial w_t}{\partial y} \quad (3.3.12)$$

appear in the displacement field. The existence of the derivatives of transverse deflections in the displacement fields results in their second derivatives appearing in the strain energy functional, so  $C^1$  continuity of  $w_b$  and  $w_t$  is required. It is desirable to alleviate such a requirement by introducing new rotational degrees of freedom as follows:

$$\frac{\partial w_b}{\partial x} = -\theta_{yb}, \frac{\partial w_t}{\partial x} = -\theta_{yt}, \frac{\partial w_b}{\partial y} = \theta_{xb}, \frac{\partial w_t}{\partial y} = \theta_{xt} \quad (3.3.13)$$

These relationships will be constrained in the strain energy via the penalty method. Therefore, the displacement fields can be written as follows:

$$\begin{aligned} u_x^{(k)} &= \Phi_{11}^{(k)} u_b + \Phi_{12}^{(k)} u_t + \Phi_{21}^{(k)} v_b + \Phi_{22}^{(k)} v_t + \Phi_{31}^{(k)} \theta_{xb} + \Phi_{32}^{(k)} \theta_{xt} + \Phi_{41}^{(k)} \theta_{yb} + \Phi_{42}^{(k)} \theta_{yt} \\ u_y^{(k)} &= \Psi_{11}^{(k)} u_b + \Psi_{12}^{(k)} u_t + \Psi_{21}^{(k)} v_b + \Psi_{22}^{(k)} v_t + \Psi_{31}^{(k)} \theta_{xb} + \Psi_{32}^{(k)} \theta_{xt} \\ &\quad + \Psi_{41}^{(k)} \theta_{yb} + \Psi_{42}^{(k)} \theta_{yt} \\ u_z^{(k)} &= w_b \Omega_1 + w_t \Omega_2 \end{aligned} \quad (3.3.14)$$

Employing the indicial notation, the modified displacement fields can be expressed in more concise forms:

$$u_x^{(k)} = \bar{u}_{\alpha\beta} \Phi_{\alpha\beta}^{(k)}, \quad u_y^{(k)} = \bar{u}_{\alpha\beta} \Psi_{\alpha\beta}^{(k)}, \quad u_z^{(k)} = w_\beta \Omega_\beta, \quad \alpha = 1 \dots 4; \beta = 1, 2 \quad (3.3.15)$$

where the index  $\beta$  is used to denote the top and bottom of the sublaminates with 1: bottom

and 2: top, and  $\Phi_{\alpha\beta}^{(k)}$ ,  $\Psi_{\alpha\beta}^{(k)}$  and  $\Omega_{\beta}$  are shape functions in the thickness direction, and will be defined in Appendix A. Unless noted otherwise, summation on repeated indices is implied. The variables  $\bar{u}_{\alpha\beta}$  are:

$$\bar{u}_{1\beta} = u_{\beta}, \bar{u}_{2\beta} = v_{\beta}, \bar{u}_{3\beta} = \theta_{x\beta}, \bar{u}_{4\beta} = \theta_{y\beta} \quad (3.3.16)$$

Since through-the-thickness shape functions,  $\Phi_{\alpha\beta}^{(k)}$ ,  $\Psi_{\alpha\beta}^{(k)}$  and  $\Omega_{\beta}$ , which are only functions of  $z$ , are included in displacement field, nodal variables in the finite element can be approximated by two dimensional shape functions in  $x, y$ . Thus, the strains are obtained from Eq. (3.3.15) as:

$$\begin{aligned} \epsilon_{xx}^{(k)} &= \bar{u}_{\alpha\beta, x} \Phi_{\alpha\beta}^{(k)} & \epsilon_{yy}^{(k)} &= \bar{u}_{\alpha\beta, y} \Psi_{\alpha\beta}^{(k)} & \epsilon_{zz}^{(k)} &= w_{\beta} \Omega_{\beta, z} \\ \gamma_{yz}^{(k)} &= \bar{u}_{\alpha\beta} \Psi_{\alpha\beta, z}^{(k)} + w_{\beta, y} \Omega_{\beta} \Big|_{z=\frac{h}{2}} & \gamma_{xz}^{(k)} &= \bar{u}_{\alpha\beta} \Phi_{\alpha\beta, z}^{(k)} + w_{\beta, x} \Omega_{\beta} \Big|_{z=\frac{h}{2}} \\ \gamma_{xy}^{(k)} &= \bar{u}_{\alpha\beta, y} \Phi_{\alpha\beta}^{(k)} + \bar{u}_{\alpha\beta, x} \Psi_{\alpha\beta}^{(k)} & \alpha &= 1 \dots 4; \beta = 1, 2 \end{aligned} \quad (3.3.17)$$

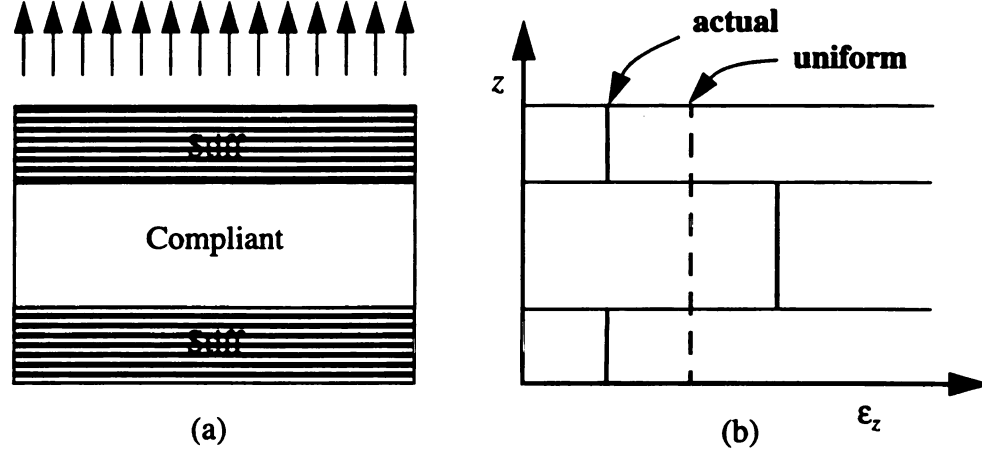
where tensor notations are employed and a comma in the subscript implies partial differentiation. In (3.3.17), the contribution of the transverse deflection to the transverse shear strains  $\gamma_{yz}^{(k)}$  and  $\gamma_{xz}^{(k)}$  are evaluated at the middle of the sublaminate to ensure constant shear stresses through-the-thickness of a sublaminate (as discussed previously).

From (3.3.17), it can be seen that the transverse normal strain,

$$\epsilon_{zz}^{(k)} = w_{\beta} \Omega_{\beta, z} = \frac{w_t - w_b}{h} \quad (3.3.18)$$

is constant in a sublaminate. Such a uniform normal strain may give rise to substantial errors in composites which have a soft core or layer. Let's consider a composite which is

composed of 3 layers, one of which is very compliant. When the composite is under loading as shown in Figure 16 (a), the uniform transverse normal strain distribution through the thickness is predicted from (3.3.18) as shown in Figure 16 (b).



**Figure 16.** A composite under transverse tension load (a) and the corresponding transverse strain distribution through the thickness (b)

However, the actual distribution is not uniform, so that we can see that there is a big deviation between the predicted strain and the actual one. Thus, it is desirable to improve the distribution of the transverse normal strain through the thickness. If the transverse strain field is improved, the Poisson's ratio locking can be prevented as well (to be discussed later). The improvement can be achieved by instead assuming a constant transverse normal stress,  $\overline{\sigma_{zz}}$ , through-the-thickness of a sublaminate. Tessler and Seather attempted to improve the transverse strain, but they used a field consistent polynomial [80].  $\overline{\sigma_{zz}}$  can be determined using Reissner's mixed variational principle [81]. From (3.2.2), the transverse normal strain is obtained as:

$$\bar{\epsilon}_{zz}^{(k)} = -\frac{C_{13}^{(k)}}{C_{33}^{(k)}}\epsilon_{xx}^{(k)} - \frac{C_{23}^{(k)}}{C_{33}^{(k)}}\epsilon_{yy}^{(k)} - \frac{C_{36}^{(k)}}{C_{33}^{(k)}}\gamma_{xy}^{(k)} + \frac{1}{C_{33}^{(k)}}\bar{\sigma}_{zz} \quad (3.3.19)$$

$\bar{\sigma}_{zz}$  is then determined by analytically solving the following relation:

$$\begin{aligned} 0 &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} (\bar{\epsilon}_{zz}^k - \epsilon_{zz}^k) dz \\ &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \left\{ \frac{w_t - w_b}{h} - \left( -\frac{C_{13}^{(k)}}{C_{33}^{(k)}}\epsilon_{xx}^{(k)} - \frac{C_{23}^{(k)}}{C_{33}^{(k)}}\epsilon_{yy}^{(k)} - \frac{C_{36}^{(k)}}{C_{33}^{(k)}}\gamma_{xy}^{(k)} + \frac{1}{C_{33}^{(k)}}\bar{\sigma}_{zz} \right) \right\} dz \end{aligned} \quad (3.3.20)$$

Substituting the strain - displacement relations in (3.3.17) into (3.3.20), the constant transverse normal stress  $\bar{\sigma}_{zz}$  is found to be:

$$\begin{aligned} \bar{\sigma}_{zz} &= \frac{\partial u_b}{\partial x}(P_1 + S_6) + \frac{\partial u_t}{\partial x}(P_2 - S_6) + \frac{\partial v_b}{\partial x}(P_3 + S_7) + \frac{\partial v_t}{\partial x}(-P_3 + S_8) + \frac{\partial \theta}{\partial x}^{yb}(P_4 + S_9) \\ &+ \frac{\partial \theta}{\partial x}^{yt}(P_4 + S_9) + \frac{\partial \theta}{\partial x}^{xb}(P_5 + S_{10}) + \frac{\partial \theta}{\partial x}^{xt}(P_5 + S_{10}) + \frac{\partial u_b}{\partial y}(Q_1 + S_1) + \frac{\partial u_t}{\partial y}(-Q_1 + S_2) \\ &+ \frac{\partial v_b}{\partial y}(Q_2 + S_3) + \frac{\partial v_t}{\partial y}(Q_3 - S_3) + \frac{\partial \theta}{\partial y}^{yb}(Q_4 + S_4) + \frac{\partial \theta}{\partial y}^{yt}(Q_4 + S_4) + \frac{\partial \theta}{\partial y}^{xb}(Q_5 + S_5) \\ &+ \frac{\partial \theta}{\partial y}^{xt}(Q_5 + S_5) + w_b(-T_1) + w_t(T_1) \end{aligned} \quad (3.3.21)$$

where the coefficients are defined in the Appendix A. The newly defined transverse normal strain can now be written as:

$$\epsilon_{zz}^{(k)} = \bar{u}_{\alpha\beta, x}\lambda_{x\alpha\beta}^{(k)} + \bar{u}_{\alpha\beta, y}\lambda_{y\alpha\beta}^{(k)} + w_\beta\chi_\beta \quad \alpha = 1 \dots 4; \beta = 1, 2 \quad (3.3.22)$$

where the functions  $\lambda_{x\alpha\beta}^{(k)}$  and  $\chi_\beta$  are defined in the Appendix A.

The important properties of the present laminate theory are:

- 1) The laminate thickness may be subdivided into several sublaminae for refined analysis, if needed.
- 2) Discrete-layer effects are captured within each sublaminate by allowing a piecewise (layerwise) continuous through-the-thickness variation of the in-plane displacements.
- 3) Continuity of transverse shear stresses through the thickness of each sublaminate is satisfied.
- 4) Transverse normal strain distribution through the thickness of sublaminate was improved by assuming constant transverse normal stress.
- 5) Accuracy and efficiency of the theory are adaptable, depending on the number of sublaminae used.
- 6) In the displacement fields, only five engineering d.o.f. (three translations + two rotations) are present.

### **3.4 Governing Equations and Boundary Conditions**

The equations of motion, the essential and natural boundary conditions, as well as the displacement-based finite element model can be developed from Hamilton's principle. Again, attention will be limited to a single sublaminate. The constraints between the derivatives of transverse deflection and the rotational d.o.f in (3.3.13) should be imposed in the energy functional. There are two ways to impose the constraints - either Lagrange multiplier method or penalty method. Since the Lagrange multiplier method introduces additional d.o.f., the penalty method has been used to maintain computational efficiency and convenience. Therefore, the energy functional is modified here to include the imposition of the constraints via the penalty method. Hamilton's principle for the  $m$ th sublaminate



can be defined as follows:

$$\delta \int_{t_0}^{t_1} \{U - K - W + P\} dt = 0 \quad (3.4.1)$$

where  $U$  is the internal strain energy:

$$U = \sum_{k=1}^{N_m} \int_{V_k} \frac{1}{2} [\sigma_{xx}^{(k)} \epsilon_{xx}^{(k)} + \sigma_{yy}^{(k)} \epsilon_{yy}^{(k)} + \sigma_{zz}^{(k)} \epsilon_{zz}^{(k)} + \sigma_{yz}^{(k)} \gamma_{yz}^{(k)} + \sigma_{zx}^{(k)} \gamma_{zx}^{(k)} + \sigma_{xy}^{(k)} \gamma_{xy}^{(k)}] dV_k \quad (3.4.2)$$

$V_k$  is the volume of the  $k$ -th layer.  $K$  is the kinetic energy:

$$K = \sum_{k=1}^{N_m} \int_{V_k} \frac{1}{2} \rho^{(k)} [\dot{u}_x^{(k)^2} + \dot{u}_y^{(k)^2} + \dot{u}_z^{(k)^2}] dV_k \quad (3.4.3)$$

$\rho^{(k)}$  is the density of the  $k$ -th layer and a superposed dot indicates differentiation with respect to time.  $W$  is the work of external forces:

$$W = \int_{\Omega} w_{\alpha} P_{\alpha} d\Omega + \int_{\Gamma} t_i u_i d\Gamma \quad (3.4.4)$$

$$t_i = \sigma_{ij} n_j; \alpha = 1, 2; i = 1, 2$$

$P$  is the penalty constraint:

$$P = \frac{\gamma}{2} \int_{\Omega} \left[ \sum_{\beta=1}^2 \left\{ \left( u_{3\beta} - \frac{\partial w_{\beta}}{\partial y} \right)^2 + \left( u_{4\beta} + \frac{\partial w_{\beta}}{\partial x} \right)^2 \right\} \right] d\Omega \quad (3.4.5)$$

where  $\gamma$  is the penalty parameter. As  $\gamma$  is assigned successively larger values, the constraints of Eq. (3.3.13) are satisfied more exactly in the least squares sense. In the internal

strain energy expression, (3.2.4), we need to pay attention to the transverse normal stress terms. The constant  $\overline{\sigma_{zz}}$  leads to an asymmetric stiffness matrix. Thus,  $\sigma_{zz}$  from the constitutive equation, (3.2.2), instead of the constant transverse normal stress should be employed.

Substituting Eqs. (3.2.2, 3.3.15, 3.3.17, 3.3.22, 3.4.2, 3.4.3, 3.4.4, and 3.4.5) into (3.4.1), taking the variation of (3.4.1), and integrating by parts, (3.4.1) can be rewritten as:

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{\Omega} \left\{ \delta \bar{u}_{\alpha\beta} \left( \frac{\partial N_{x\alpha\beta}}{\partial x} - \frac{\partial N_{y\alpha\beta}}{\partial y} - \frac{\partial N_{z\alpha\beta}}{\partial z} - \frac{\partial Q_{z\alpha\beta}}{\partial y} + Q_{y\alpha\beta} + Q_{x\alpha\beta} - \frac{\partial R_{xy\alpha\beta}}{\partial y} - \frac{\partial R_{yx\alpha\beta}}{\partial x} \right. \right. \\
& \quad \left. \left. - \gamma \left( \bar{u}_{\alpha\beta} - \frac{\partial w_\beta}{\partial y} \right) \delta_{3\alpha} - \gamma \left( \bar{u}_{\alpha\beta} + \frac{\partial w_\beta}{\partial x} \right) \delta_{4\alpha} + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^u + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^v \right\} \right. \\
& \quad \left. + \delta w_\beta \left\{ R_{z\beta} - \frac{\partial R_{y\beta}}{\partial y} - \frac{\partial R_{x\beta}}{\partial x} - P_\beta - \gamma \left( \frac{\partial \bar{u}_{3\beta}}{\partial y} - \frac{\partial^2 w_\beta}{\partial y^2} \right) + \gamma \left( \frac{\partial \bar{u}_{4\beta}}{\partial x} + \frac{\partial^2 w_\beta}{\partial x^2} \right) + \frac{\partial^2 w_p}{\partial t^2} I_{\beta p}^w \right\} \right\} d\Omega \\
& \quad + \oint_s \left\{ \delta \bar{u}_{\alpha\beta} \left\{ N_{x\alpha\beta} n_x + N_{y\alpha\beta} n_y + N_{z\alpha\beta} n_z + Q_{z\alpha\beta} n_y + R_{xy\alpha\beta} n_y + R_{yx\alpha\beta} n_x \right\} \right. \\
& \quad \left. + \delta w_\beta \left\{ R_{y\beta} n_y + R_{x\beta} n_x + \gamma n_y \left( \bar{u}_{3\beta} - \frac{\partial w_\beta}{\partial y} \right) - \gamma n_x \left( \bar{u}_{4\beta} + \frac{\partial w_\beta}{\partial x} \right) \right\} \right\} ds \\
& \quad + \int \sum_{s=1}^{N_m} \int_{z_{k-1}}^{z_k} \left( \sigma_{1j}^{(k)} n_j \delta \bar{u}_{\alpha\beta} \Phi_{\alpha\beta}^{(k)} + \sigma_{2j}^{(k)} n_j \delta \bar{u}_{\alpha\beta} \Psi_{\alpha\beta}^{(k)} \right) dz ds \Big] dt = 0 \\
& \quad (\alpha, r = 1 \dots 4; \beta, p = 1, 2)
\end{aligned} \tag{3.4.6}$$

where  $N_{x\alpha\beta}, N_{y\alpha\beta}, N_{z\alpha\beta}, Q_{z\alpha\beta}, Q_{y\alpha\beta}, Q_{x\alpha\beta}, R_{xy\alpha\beta}, R_{yx\alpha\beta}, R_{z\beta}, R_{y\beta}, R_{x\beta}$  are the sublaminate stress resultants shown in the Appendix A,  $n_x, n_y$  are the directional cosines,  $I_{\alpha\beta rp}^u, I_{\alpha\beta rp}^v, I_{\beta p}^w$  are the plate inertias defined in the Appendix A, and  $\delta_{\alpha\beta}$  is the Kronecker delta. Considering that variations of the generalized displacement degrees of freedom are independent for  $t_0 < t < t_1$  and zero for  $t = t_0, t_1$ , the equations of motion are

derived as:

$$\begin{aligned}
 \delta \bar{u}_{\alpha\beta}: \quad & \frac{\partial N_{x\alpha\beta}}{\partial x} + \frac{\partial N_{y\alpha\beta}}{\partial y} + \frac{\partial N_{z\alpha\beta}}{\partial x} + \frac{\partial Q_{z\alpha\beta}}{\partial y} - Q_{y\alpha\beta} - Q_{x\alpha\beta} + \frac{\partial R_{xy\alpha\beta}}{\partial y} + \frac{\partial R_{yx\alpha\beta}}{\partial x} \\
 & + \gamma \left( \bar{u}_{\alpha\beta} - \frac{\partial w_{\beta}}{\partial y} \right) \delta_{3l} + \gamma \left( \bar{u}_{\alpha\beta} + \frac{\partial w_{\beta}}{\partial x} \right) \delta_{4l} = \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^u + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^v \quad (3.4.7) \\
 \delta w_{\beta}: \quad & \frac{\partial R_{y\beta}}{\partial y} + \frac{\partial R_{x\beta}}{\partial x} - R_{z\beta} + P_{\beta} + \gamma \left( \frac{\partial \bar{u}_{3\beta}}{\partial y} - \frac{\partial^2 w_{\beta}}{\partial y^2} \right) - \gamma \left( \frac{\partial \bar{u}_{4\beta}}{\partial x} + \frac{\partial^2 w_{\beta}}{\partial x^2} \right) = \frac{\partial^2 w_p}{\partial t^2} I_{\beta p}^w
 \end{aligned}$$

The corresponding boundary conditions are:

Essential B.C

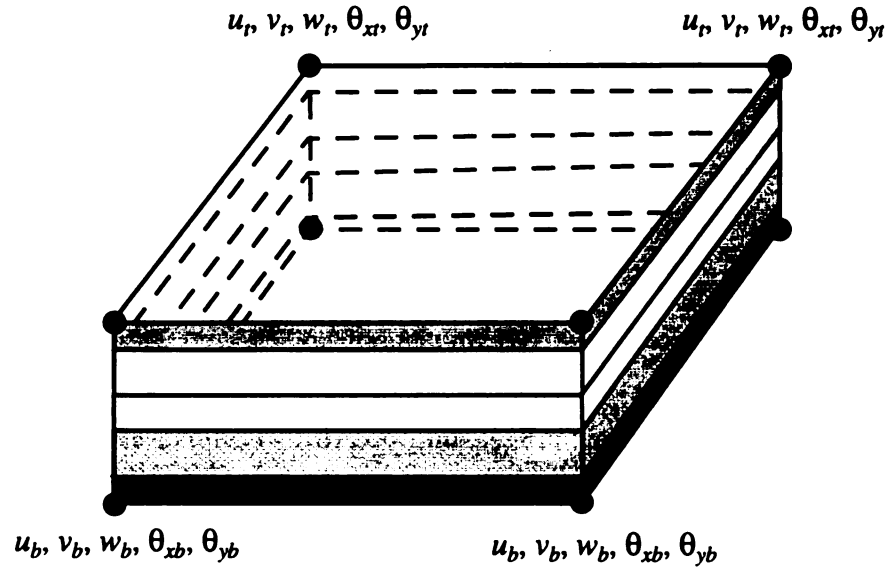
Natural B.C.

$$\begin{aligned}
 \bar{u}_{\alpha\beta} \quad & (N_{x\alpha\beta} n_x + N_{y\alpha\beta} n_y + N_{z\alpha\beta} n_z + Q_{z\alpha\beta} n_y + R_{xy\alpha\beta} n_y + R_{yx\alpha\beta} n_x) \\
 & + \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \left( \sigma_{1j}^{(k)} n_j \Phi_{\alpha\beta}^{(k)} + \sigma_{2j}^{(k)} n_j \Psi_{\alpha\beta}^{(k)} \right) dz \quad (3.4.8) \\
 w_{\beta} \quad & R_{y\beta} n_y + R_{x\beta} n_x + \gamma n_y \left( \bar{u}_{3\beta} - \frac{\partial w_{\beta}}{\partial y} \right) - \gamma n_x \left( \bar{u}_{4\beta} + \frac{\partial w_{\beta}}{\partial x} \right)
 \end{aligned}$$

### 3.5 Finite Element Formulation

Since the primary variables are functions of  $x$ ,  $y$  and  $t$  only in (3.4.6), we need to consider only two dimensional shape functions for the finite element model. Thus we can consider that the most obvious element to be developed would be a simple four-noded Ahmad-type degenerated plate element with 10 degrees of freedom per node (5 bottom surface d.o.f + 5 top surface d.o.f). However, such an element would not fall within the constraints of having three translational and three rotational degrees of freedom. (Most available commercial codes do not allow more than six d.o.f per node). Even more impor-

tantly, a major advantage of the present formulation would be lost if a traditional four-noded element were developed. Because the degrees of freedom represent quantities at the surfaces of a sublaminate (this was not by chance), it is most reasonable to develop the finite element model in the form of an eight-noded element with five degrees of freedom per node (see Figure 17). Such elements are classified here as “*regenerated*,” compared to the popular “*degenerated*” shell elements obtained by imposing constraints on solid elements.



**Figure 17. Element topology and nodal degrees of freedom.**

This approach has been used with great success [41,42], and has the very important advantage of allowing discretization in both inplane and through-thickness directions without the need for any special multi-point constraints. Elements are assembled in the standard way. Therefore, the element can be used to predict the global response of a laminate using only one (or a small number) of elements through the thickness, and local effects such as interlaminar stresses can be captured by refining the mesh in the thickness direction near

the interface(s) of interest. The capability of discretization in the through-thickness direction enables one to simulate the separation of the laminated composites due to delamination into two parts in the thickness direction.

Inplane displacement and rotation degrees of freedom are approximated by the bilinear Lagrange interpolation functions:

$$\begin{aligned}\bar{u}_{1\beta} &= \sum_{i=1}^4 u_{\beta i} P_i & \bar{u}_{2\beta} &= \sum_{i=1}^4 v_{\beta i} P_i \\ \bar{u}_{3\beta} &= \sum_{i=1}^4 \theta_{x\beta i} P_i & \bar{u}_{4\beta} &= \sum_{i=1}^4 \theta_{y\beta i} P_i\end{aligned}\tag{3.5.1}$$

where

$$P_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta)\tag{3.5.2}$$

and  $\xi, \eta$  are the element natural (local) coordinates. For transverse deflection degrees of freedom, an interdependent interpolation concept similar to that developed by Tessler and Hughes [75] is utilized.

### ***Interdependent interpolation***

One of objectives in the current research is to develop a  $C^0$  element. However, the first derivatives of the transverse deflection are present in the displacement fields due to evaluating the zig-zag functions. In consequence a  $C^1$  element needs to be developed unless we relax such a requirement. The requirement of  $C^1$  continuity was eliminated by introducing new rotational d.o.f. as discussed previously. Then the constraints between the derivatives

of transverse deflection and the rotation degrees of freedom are imposed in the strain energy functional using the penalty method. Recall that the rotational degrees of freedom were approximated using bilinear shape functions. If we employ bilinear shape functions to approximate the transverse deflection d.o.f. the constraints in the strain energy functional must enforce the relationships between the bilinear variations of rotation degrees of freedom and linear variations of transverse degrees of freedom. As a result, the element locks. Thus, the order of shape functions to be used for transverse deflection d.o.f. is biquadratic. The simplest shape functions to be used is the quadratic Serendipity shape functions which have mid-side nodes. However, such a finite element is not of convenient form. Thus, the interdependent interpolation scheme introduced by Tessler and Hughes [75] is utilized in the current research. This approach improves the element behavior very effectively without introducing additional degrees of freedom.

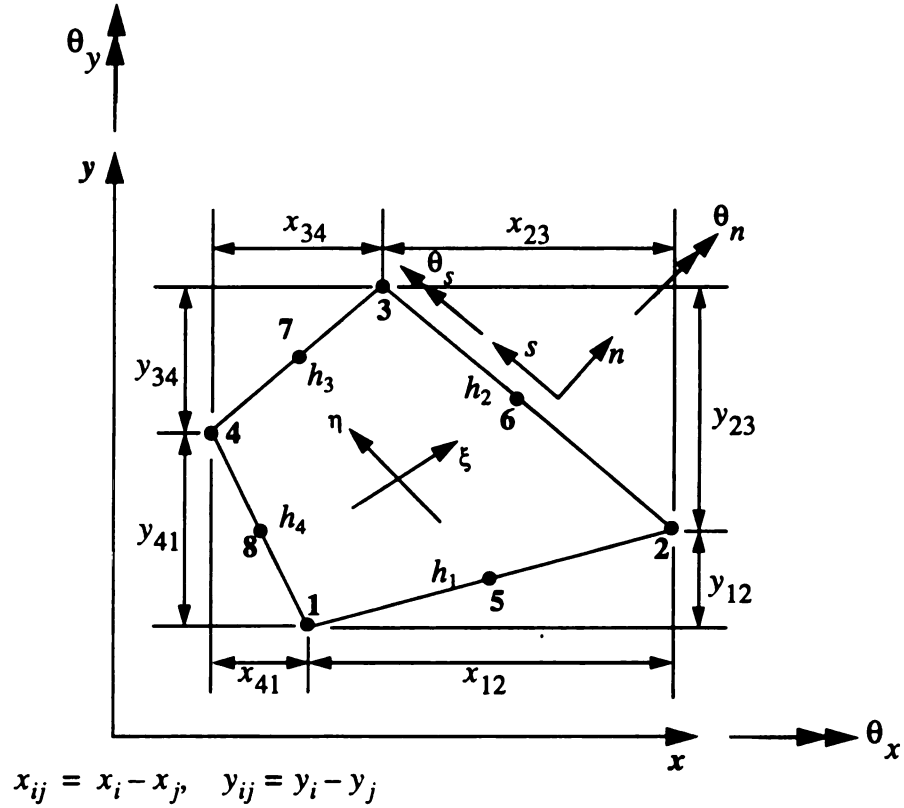
The virgin element is assumed to have mid-side nodes with only deflection degrees of freedom at the top and bottom surfaces (see Figure 18), and then the mid-side nodes are removed by writing the deflection degrees of freedom in terms of rotational degrees of freedom according to interdependent interpolation scheme. The interdependent interpolation scheme enforces the linear part of the constraints of the form:

$$\frac{\partial w_{b,t}}{\partial x} + \theta_{yb,t} = 0, \quad \frac{\partial w_{b,t}}{\partial y} - \theta_{xb,t} = 0 \quad (3.5.3)$$

where the comma in the subscript does not imply any differentiation. Generally, in quadrilateral element coordinates, the constraint (3.5.3) is recast into:

$$\left( \frac{\partial w}{\partial s} - \theta_n \right) = 0 \quad (3.5.4)$$

Let  $s$  designate the coordinate running along an element edge and denote  $n$  the coordinate running normal to an element edge as shown in Figure 18.



**Figure 18. Quadrilateral element coordinates and definition of rotation**

The  $xy$  coordinate and the  $\xi\eta$  coordinate systems are referred to as the “global” and “local” coordinate systems, respectively. Initially, a Serendipity (eight-node) transverse deflection distribution is assumed:

$$w_{b,t} = \sum_{i=1}^8 N_i(\xi, \eta) w_{bi,ti} \quad (3.5.5)$$

$$x = \sum_{i=1}^4 P_i(\xi, \eta) x_i \quad y = \sum_{i=1}^4 P_i(\xi, \eta) y_i$$

where  $P_i(\xi, \eta)$ , and  $N_i(\xi, \eta)$  are bi-linear and Serendipity shape functions respectively, and  $w_{bi, ii}$  denote transverse nodal degrees-of-freedom at the bottom and top surfaces. The biquadratic serendipity functions are [82]:

$$N_i = \frac{1}{4}[(1 + \xi_i \xi)(1 + \eta_i \eta) - (1 - \xi^2)(1 + \eta_i \eta) - (1 - \eta^2)(1 + \xi_i \xi)]\xi_i^2 \eta_i^2 + \frac{1}{2}(1 - \xi^2)(1 + \eta_i \eta)(1 - \xi_i^2)\eta_i^2 + \frac{1}{2}(1 - \eta^2)(1 + \xi_i \xi)(1 - \eta_i^2)\xi_i^2 \quad (3.5.6)$$

where  $-1 \leq \xi, \eta \leq 1$ , and  $\eta_i, \xi_i$  are the location of node in the local coordinate. Rotation degrees of freedom are approximated by the bilinear Lagrange interpolation functions as in (3.5.1).

For side 1-2, the constraint along this side ( $\eta = -1$ ) is:

$$\left( \frac{\partial w_\alpha}{\partial s} - \theta_{n\alpha} \right) \Big|_{\eta = -1} = 0 \quad \alpha = 1, 2 \quad (3.5.7)$$

where  $\alpha = 1, 2$  imply bottom(1) and top(2) of the sublaminate. Since the different shape functions are used, the number of d.o.f at the corner nodes is not same as those at mid-side nodes. In order to have a uniform four-node topology at the top and bottom surfaces, the four mid-side node transverse deflection degrees-of-freedom at edges are condensed out by using four differential constraints at the edges:

$$\frac{\partial}{\partial s} \left( \frac{\partial w_\alpha}{\partial s} - \theta_{n\alpha} \right) \Big|_{\xi, \eta = \pm 1} = 0 \quad (0 \leq t \leq 1) \quad (3.5.8)$$

where  $h_k$  is the edge length. For the edge 1-2, the differential constraint is given by:



$$\left. \frac{\partial}{\partial s} \left( \frac{\partial w_\alpha}{\partial s} - \theta_{n\alpha} \right) \right|_{\eta = -1} = 0 \quad (3.5.9)$$

Due to the property of the shape functions, the evaluation of the derivatives of transverse deflection at the edge leads to:

$$w_\alpha|_{\eta = -1} = \sum_{i=1}^8 N_i w_{\alpha i} \Big|_{\eta = -1} \quad (3.5.10)$$

since  $N_3|_{\eta = -1} = N_4|_{\eta = -1} = N_6|_{\eta = -1} = N_7|_{\eta = -1} = N_8|_{\eta = -1} = 0$ ,

$$\begin{aligned} w_\alpha|_{\eta = -1} &= \frac{1}{2}\xi(\xi - 1)w_1 + \frac{1}{2}\xi(\xi + 1)w_2 + (1 - \xi^2)w_5 \\ \frac{\partial w_\alpha}{\partial s} \Big|_{\eta = -1} &= \left\{ \left( -\frac{1}{2} + \xi \right) w_1 + \left( \frac{1}{2} + \xi \right) w_2 - 2\xi w_5 \right\} \frac{1}{J_l} \Big|_{\eta = -1} \\ \frac{\partial^2 w_\alpha}{\partial s^2} \Big|_{\eta = -1} &= (w_1 + w_2 - 2w_5) \frac{1}{J_l^2} \Big|_{\eta = -1} \end{aligned} \quad (3.5.11)$$

where  $ds = J_l|_{\eta = -1} d\xi$ , and  $J_l$  is the line Jacobian. The normal rotation at the edge is approximated using bilinear shape functions. Then the derivatives of the normal edge rotation along side 1-2 due to the property of the shape functions is given by:

$$\theta_n|_{\eta = -1} = \sum_{i=1}^4 P_i \theta_{ni} \Big|_{\eta = -1} \quad (3.5.12)$$

since  $P_3|_{\eta = -1} = P_4|_{\eta = -1} = 0$ ,

$$\begin{aligned}
\theta_n|_{\eta=-1} &= \sum_{i=1}^4 P_i \theta_{ni}|_{\eta=-1} \\
&= \frac{1}{2}[(1-\xi)\theta_{n1} + (1+\xi)\theta_{n2}]
\end{aligned} \tag{3.5.13}$$

$$\left. \frac{\partial \theta_n}{\partial s} \right|_{\eta=-1} = \frac{1}{2}(\theta_{n2} - \theta_{n1}) \left. \frac{1}{J_l} \right|_{\eta=-1}$$

Substituting (3.5.11) and (3.5.13) into (3.5.9), solving for the mid-side  $w_5$  degree-of-freedom, then  $w_5$  is found to be:

$$w_5 = \frac{1}{2}(w_1 + w_2) - \frac{1}{4}(\theta_{n2} - \theta_{n1})J_l \tag{3.5.14}$$

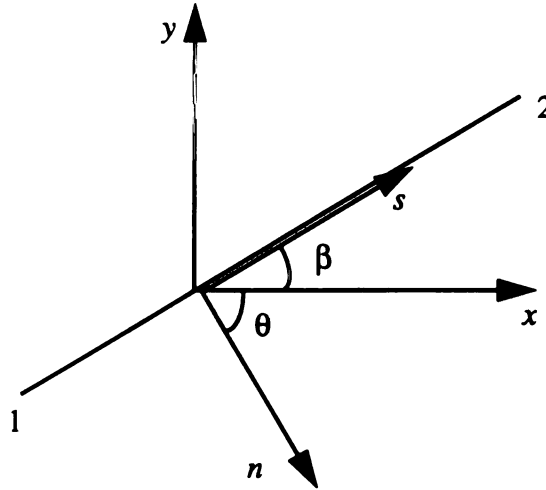
Since the geometry of the element can be approximated by bilinear shape functions, the line jacobian,  $J_l$ , is obtained as follows:

$$\begin{aligned}
x|_{\eta=-1} &= \sum_{i=1}^4 P_i x_i \Big|_{\eta=-1}, & \therefore \left. \frac{\partial x}{\partial \xi} \right|_{\eta=-1} &= \frac{1}{2}(x_2 - x_1) \\
y|_{\eta=-1} &= \sum_{i=1}^4 P_i y_i \Big|_{\eta=-1}, & \therefore \left. \frac{\partial y}{\partial \xi} \right|_{\eta=-1} &= \frac{1}{2}(y_2 - y_1) \\
J_l|_{\eta=-1} &= \sqrt{\left( \left. \frac{\partial x}{\partial \xi} \right|_{\eta=-1} \right)^2 + \left( \left. \frac{\partial y}{\partial \xi} \right|_{\eta=-1} \right)^2} \\
&= \sqrt{\left( \frac{x_2 - x_1}{2} \right)^2 + \left( \frac{y_2 - y_1}{2} \right)^2} \\
&= \frac{h_1}{2}
\end{aligned} \tag{3.5.15}$$

The other mid-side  $w_6, w_7, w_8$  degrees-of-freedom can easily be obtained by the same procedure, and are found to be:

$$\begin{aligned}
w_6 &= \frac{1}{2}(w_2 + w_3) - \frac{1}{4}(\theta_{n3} - \theta_{n2})\frac{h_2}{2} \\
w_7 &= \frac{1}{2}(w_3 + w_4) - \frac{1}{4}(\theta_{n4} - \theta_{n3})\frac{h_3}{2} \\
w_8 &= \frac{1}{2}(w_4 + w_1) - \frac{1}{4}(\theta_{n1} - \theta_{n4})\frac{h_4}{2}
\end{aligned} \tag{3.5.16}$$

Now we need to transform the rotations at the edges into ones in global coordinates. Taking the edge 1 - 2, we can derive the transformation law.



**Figure 19. Coordinate transformation between edge and global coordinates**

From Figure 19, the transformation law is derived as:

$$\begin{bmatrix} n \\ s \end{bmatrix} = \begin{bmatrix} \cos(n, x) & \cos(n, y) \\ \cos(s, x) & \cos(s, y) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.5.17}$$

where  $(n, x)$  is the angle between the two axes. Then on the 1 - 2 edge,

$$\begin{bmatrix} \theta_{n1} \\ \theta_{s1} \end{bmatrix} = \begin{bmatrix} \cos(n, x) & \cos(n, y) \\ \cos(s, x) & \cos(s, y) \end{bmatrix} \begin{bmatrix} \theta_{x1} \\ \theta_{y1} \end{bmatrix} \quad (3.5.18)$$

where

$$\begin{aligned} \cos(n, x) &= \cos \theta = \cos(-\beta + 90) = \sin \beta = \frac{y_2 - y_1}{h_1} \\ \cos(n, y) &= \cos(\theta + 90) = -\sin \theta = -\cos \beta = \frac{x_1 - x_2}{h_1} \end{aligned} \quad (3.5.19)$$

Thus, (3.5.18) becomes:

$$\theta_{n1} = \frac{y_2 - y_1}{h_1} \theta_{x1} + \frac{x_1 - x_2}{h_1} \theta_{y1}, \quad \theta_{n2} = \frac{y_2 - y_1}{h_1} \theta_{x2} + \frac{x_1 - x_2}{h_1} \theta_{y2} \quad (3.5.20)$$

By the same procedures, the coordinate transformations of the other sides are obtained:

on the 2 - 3 edge,

$$\theta_{n2} = \frac{y_3 - y_2}{h_2} \theta_{x2} + \frac{x_2 - x_3}{h_2} \theta_{y2}, \quad \theta_{n3} = \frac{y_3 - y_2}{h_2} \theta_{x3} + \frac{x_2 - x_3}{h_2} \theta_{y3} \quad (3.5.21)$$

on the 3 - 4 edge,

$$\theta_{n3} = \frac{y_4 - y_3}{h_3} \theta_{x3} + \frac{x_3 - x_4}{h_3} \theta_{y3}, \quad \theta_{n4} = \frac{y_4 - y_3}{h_3} \theta_{x4} + \frac{x_3 - x_4}{h_3} \theta_{y4} \quad (3.5.22)$$

on the 4 - 1 edge,

$$\theta_{n4} = \frac{y_1 - y_4}{h_4} \theta_{x4} + \frac{x_4 - x_1}{h_4} \theta_{y4}, \quad \theta_{n1} = \frac{y_1 - y_4}{h_4} \theta_{x1} + \frac{x_4 - x_1}{h_4} \theta_{y1} \quad (3.5.23)$$

Substitution of Eqs. (3.5.20-3.5.23) into (3.5.14) and (3.5.16) yields the following expressions for the mid-side node transverse deflection degrees-of-freedom in global coordinates:

$$\begin{aligned}
 w_5 &= \frac{1}{2}(w_1 + w_2) - \frac{1}{8}[(y_2 - y_1)(\theta_{x2} - \theta_{x1}) + (x_1 - x_2)(\theta_{y2} - \theta_{y1})] \\
 w_6 &= \frac{1}{2}(w_2 + w_3) - \frac{1}{8}[(y_3 - y_2)(\theta_{x3} - \theta_{x2}) + (x_2 - x_3)(\theta_{y3} - \theta_{y2})] \\
 w_7 &= \frac{1}{2}(w_3 + w_4) - \frac{1}{8}[(y_4 - y_3)(\theta_{x4} - \theta_{x3}) + (x_3 - x_4)(\theta_{y4} - \theta_{y3})] \\
 w_8 &= \frac{1}{2}(w_4 + w_1) - \frac{1}{8}[(y_1 - y_4)(\theta_{x1} - \theta_{x4}) + (x_4 - x_1)(\theta_{y1} - \theta_{y4})]
 \end{aligned} \tag{3.5.24}$$

Finally, interdependent interpolation functions are derived by substituting the mid-side node transverse deflection degrees-of-freedom expressions, (3.5.24), back into (3.5.4):

$$w = \sum_{i=1}^4 (P_i w_i + N_{xi} \theta_{xi} + N_{yi} \theta_{yi}) \tag{3.5.25}$$

where  $N_{xi}$  and  $N_{yi}$  depend on  $N_{i+4}$  and the projections of each edge onto x and y axes and are given by:

$$\begin{aligned}
 N_{x1} &= -\frac{1}{8}(y_{12}N_5 - y_{41}N_8), & N_{y1} &= \frac{1}{8}(x_{12}N_5 - x_{41}N_8) \\
 N_{x2} &= -\frac{1}{8}(y_{23}N_6 - y_{12}N_5), & N_{y2} &= \frac{1}{8}(x_{23}N_6 - x_{12}N_5) \\
 N_{x3} &= -\frac{1}{8}(y_{34}N_7 - y_{23}N_6), & N_{y3} &= \frac{1}{8}(x_{34}N_7 - x_{23}N_6) \\
 N_{x4} &= -\frac{1}{8}(y_{41}N_8 - y_{34}N_7), & N_{y4} &= \frac{1}{8}(x_{41}N_8 - x_{34}N_7)
 \end{aligned} \tag{3.5.26}$$

where

$$\begin{aligned}
N_5 &= \frac{1}{2}(1 - \xi^2)(1 - \eta), & N_6 &= \frac{1}{2}(1 - \eta^2)(1 + \xi) \\
N_7 &= \frac{1}{2}(1 - \xi^2)(1 + \eta), & N_8 &= \frac{1}{2}(1 - \eta^2)(1 - \xi) \\
x_{ij} &= x_i - x_j, & y_{ij} &= y_i - y_j
\end{aligned} \tag{3.5.27}$$

Furthermore, the mid-side transverse deflection d.o.f. in terms of covariant rotations is readily obtained from Eqs. (3.5.14, 3.5.16) as:

$$\begin{aligned}
w_5 &= \frac{1}{2}(w_1 + w_2) - \frac{1}{4}(\theta_{\eta 1} - \theta_{\eta 2}) \\
w_6 &= \frac{1}{2}(w_2 + w_3) - \frac{1}{4}(\theta_{\xi 3} - \theta_{\xi 2}) \\
w_7 &= \frac{1}{2}(w_3 + w_4) - \frac{1}{4}(\theta_{\eta 4} - \theta_{\eta 3}) \\
w_8 &= \frac{1}{2}(w_4 + w_1) - \frac{1}{4}(\theta_{\xi 4} - \theta_{\xi 1})
\end{aligned} \tag{3.5.28}$$

The interdependent interpolation functions in terms of covariant rotations can be written as in the following expression by substituting (3.5.28) into (3.5.5) (these interdependent interpolation functions will be used to derive field consistent shear strains):

$$w = \sum_{i=1}^4 (P_i w_i + N_{\xi i} \theta_{\xi i} + N_{\eta i} \theta_{\eta i}) \tag{3.5.29}$$

where  $N_{\xi i}$  and  $N_{\eta i}$  are given by:

$$\begin{aligned}
N_{\xi 1} &= \frac{1}{4}N_8, & N_{\eta 1} &= -\frac{1}{4}N_5 \\
N_{\xi 2} &= \frac{1}{4}N_6, & N_{\eta 2} &= \frac{1}{4}N_5 \\
N_{\xi 3} &= -\frac{1}{4}N_6, & N_{\eta 3} &= \frac{1}{4}N_7 \\
N_{\xi 4} &= -\frac{1}{4}N_8, & N_{\eta 4} &= -\frac{1}{4}N_7
\end{aligned} \tag{3.5.30}$$

Since the interdependent interpolation scheme is employed, the transverse deflection d.o.f. is coupled with the rotational d.o.f. Therefore, the transverse distributed load acting on the top and bottom surfaces should be applied to the appropriate nodal d.o.f.

### ***Field Consistent Shear Strain***

The locking phenomenon has been a big issue in the development and application of the finite element method. In an attempt to alleviate shear locking, the interdependent interpolation scheme was adopted. However, it turned out that it did not remove the shear locking completely, and the developed element still locks in the very thin regime.

To avoid shear locking, a lot of works have been accomplished. One popular method employed is reduced/selective integration [83-88]. However, reduced/selective integration has had limited success. In some elements the locking phenomenon is not completely avoided. Moreover, when the order of integration is reduced enough to alleviate locking, spurious kinematic or zero strain energy modes arise in the element stiffness matrix. These kinematic modes can be suppressed by adding a stabilization matrix to the stiffness matrix [89,90]. Thus, some researchers try to alleviate the locking problem by special treatment of the transverse shear strain components [91-94], while using full integration to preserve the correct rank of the element stiffness matrix.

Prathap and Somashekar found that the source of the shear locking phenomenon comes from inconsistency of the transverse shear strain fields with respect to the inplane coordinates,  $x$  and  $y$  [72,73]. There are also inconsistencies in the thickness coordinate,  $z$ , but terms associated with the thickness coordinate are small when the aspect ratio is large, so that the transverse interpolation inconsistencies are not so crucial in locking problems compared to the in-plane interpolation inconsistencies [95]. Thus, in order to prevent the shear locking phenomenon, we need to make only the transverse shear strain fields be field consistent with respect to only the in-plane coordinates.

Since a general quadrilateral element is non-uniformly mapped from a Cartesian (global) system into a natural (local) coordinate system, it is difficult to see the consistent form of the transverse shear strains clearly. We need to transform them to the natural coordinate system. The transverse shear strains in the natural coordinate system are called covariant shear strains. Any fields can be transformed from the covariant to Cartesian coordinate systems by the following expression:

$$\begin{Bmatrix} \vartheta_{\xi z} \\ \vartheta_{\eta z} \end{Bmatrix} = [J] \begin{Bmatrix} \vartheta_{xz} \\ \vartheta_{yz} \end{Bmatrix} \quad (3.5.31)$$

where  $[J]$  is the Jacobian matrix and is defined as:

$$[J] = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (3.5.32)$$

The inverse Jacobian is obtained as:



$$[J]^{-1} = \begin{bmatrix} \bar{j}_{11} & \bar{j}_{12} \\ \bar{j}_{21} & \bar{j}_{22} \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \quad (3.5.33)$$

where  $|J|$  is the determinant of the Jacobian matrix.

The transverse shear strain in the Cartesian coordinates,  $\gamma_{xz}$  is mapped into the natural coordinates as:

$$\begin{aligned} \gamma_{xz}(\xi, \eta) &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ &= \frac{\partial u_x}{\partial z} + \left( \frac{\partial u_z}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u_z}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \end{aligned} \quad (3.5.34)$$

Considering that the inverse Jacobian terms are included in Eq. (3.5.34), some new covariant in-plane displacement fields,  $u_\xi$  and  $u_\eta$  are introduced in order to derive the covariant shear strains, so that the in-plane displacement field,  $u_x$  can be rewritten as:

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} \quad (3.5.35)$$

As the modified displacement fields can be expressed in more concise forms in Eq. (3.3.15), covariant in-plane displacement fields,  $u_\xi$  and  $u_\eta$  can be expressed by:

$$u_\xi = \bar{u}_{\xi\alpha\beta} \Phi_{\xi\alpha\beta}^{(k)}, \quad u_\eta = \bar{u}_{\eta\alpha\beta} \Phi_{\eta\alpha\beta}^{(k)} \quad \alpha = 1, 2; \beta = 1, 2 \quad (3.5.36)$$

where the new set of degrees-of-freedom will be represented as:

$$\begin{aligned} \bar{u}_{\xi\alpha\beta}: \bar{u}_{\xi 1\beta}, \bar{u}_{\xi 2\beta} &= u_{\xi\beta}, \theta_{\xi\beta} \\ \bar{u}_{\eta\alpha\beta}: \bar{u}_{\eta 1\beta}, \bar{u}_{\eta 2\beta} &= u_{\eta\beta}, \theta_{\eta\beta} \end{aligned} \quad \beta = 1(\text{bottom}), 2(\text{top}) \quad (3.5.37)$$

Thus,  $u_\xi$  and  $u_\eta$  can be defined in terms of their new set of degrees-of-freedom,  $(\bar{u}_{\xi\alpha\beta}, \bar{u}_{\eta\alpha\beta})$  in the covariant frame as:

$$u_x = \bar{u}_{\xi\alpha\beta} \Phi_{\xi\alpha\beta}^{(k)} \frac{\partial \xi}{\partial x} + \bar{u}_{\eta\alpha\beta} \Phi_{\eta\alpha\beta}^{(k)} \frac{\partial \eta}{\partial x} \quad \alpha = 1, 2; \beta = 1, 2 \quad (3.5.38)$$

where  $\Phi_{\xi\alpha\beta}^{(k)}$  and  $\Phi_{\eta\alpha\beta}^{(k)}$  are still shape functions of the thickness coordinate ( $z$ ) only and are given in detail in the Appendix A. Substituting Eq. (3.5.35) into Eq. (3.5.34), the Cartesian based shear strain,  $\gamma_{xz}$ , can be obtained in terms of the covariant shear strains,  $\bar{\gamma}_{\xi z}, \bar{\gamma}_{\eta z}$  as:

$$\begin{aligned} \gamma_{xz}(\xi, \eta) &= \left( \frac{\partial u_\xi}{\partial z} + \frac{\partial u_z}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \left( \frac{\partial u_\eta}{\partial z} + \frac{\partial u_z}{\partial \eta} \right) \frac{\partial \eta}{\partial x} \\ &= \bar{\gamma}_{\xi z} \bar{j}_{11} + \bar{\gamma}_{\eta z} \bar{j}_{12} \end{aligned} \quad (3.5.39)$$

where

$$\bar{\gamma}_{\xi z} = \left( \frac{\partial u_\xi}{\partial z} + \frac{\partial u_z}{\partial \xi} \right), \quad \bar{\gamma}_{\eta z} = \left( \frac{\partial u_\eta}{\partial z} + \frac{\partial u_z}{\partial \eta} \right) \quad (3.5.40)$$

Now, the covariant shear strain is derived in terms of covariant degrees of freedom by substituting Eq. (3.3.15) and Eq. (3.5.38) into Eq. (3.5.40):

$$\begin{aligned} \bar{\gamma}_{\xi z} &= u_{\xi 1\beta} \Phi_{\xi 1\beta, z} + \theta_{\xi 2\beta} \Phi_{\xi 2\beta, z} + w_{\beta, \xi} \Omega_\beta = \bar{u}_{\xi\alpha\beta} \Phi_{\xi\alpha\beta, z} + w_{\beta, \xi} \Omega_\beta \quad \alpha = 1, 2 \\ \bar{\gamma}_{\eta z} &= u_{\eta 1\beta} \Phi_{\eta 1\beta, z} + \theta_{\eta 2\beta} \Phi_{\eta 2\beta, z} + w_{\beta, \eta} \Omega_\beta = \bar{u}_{\eta\alpha\beta} \Phi_{\eta\alpha\beta, z} + w_{\beta, \eta} \Omega_\beta \quad \beta = 1, 2 \end{aligned} \quad (3.5.41)$$

The shear strain field,  $\gamma_{yz}$ , can be expressed in terms of covariant degrees of freedom by the same procedure as  $\gamma_{xz}$ :

$$\begin{aligned}\gamma_{yz}(\xi, \eta) &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ &= \frac{\partial u_y}{\partial z} + \left( \frac{\partial u_z}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u_z}{\partial \eta} \frac{\partial \eta}{\partial y} \right)\end{aligned}\quad (3.5.42)$$

$u_y$  can be written as:

$$u_y = \hat{u}_\xi \frac{\partial \xi}{\partial y} + \hat{u}_\eta \frac{\partial \eta}{\partial y} \quad (3.5.43)$$

Covariant in-plane displacement fields,  $\hat{u}_\xi$  and  $\hat{u}_\eta$  can be expressed as:

$$u_y = \bar{u}_{\xi\alpha\beta} \Psi_{\xi\alpha\beta} \bar{j}_{21} + \bar{u}_{\eta\alpha\beta} \Psi_{\eta\alpha\beta} \bar{j}_{22} \quad \alpha = 1, 2; \beta = 1, 2 \quad (3.5.44)$$

where through-the-thickness shape functions,  $\Psi_{\xi\alpha\beta}$  and  $\Psi_{\eta\alpha\beta}$  are given in the Appendix A. Then the shear strain,  $\gamma_{yz}$ , can be expressed in terms of covariant shear strains,  $\hat{\gamma}_{\xi z}$  and  $\hat{\gamma}_{\eta z}$ . Mathematically, this is as shown below:

$$\begin{aligned}\gamma_{yz}(\xi, \eta) &= \left( \frac{\partial \hat{u}_\xi}{\partial z} + \frac{\partial u_z}{\partial \xi} \right) \bar{j}_{21} + \left( \frac{\partial \hat{u}_\eta}{\partial z} + \frac{\partial u_z}{\partial \eta} \right) \bar{j}_{22} \\ &= \hat{\gamma}_{\xi z} \bar{j}_{21} + \hat{\gamma}_{\eta z} \bar{j}_{22}\end{aligned}\quad (3.5.45)$$

where

$$\begin{aligned}\hat{\gamma}_{\xi z} &= \bar{u}_{\xi\alpha\beta} \Psi_{\xi\alpha\beta, z}^{(k)} + w_{\beta, \xi} \Omega_\beta \Big|_{z=\frac{h}{2}} \\ \hat{\gamma}_{\eta z} &= \bar{u}_{\eta\alpha\beta} \Psi_{\eta\alpha\beta, z}^{(k)} + w_{\beta, \eta} \Omega_\beta \Big|_{z=\frac{h}{2}}\end{aligned} \quad \alpha = 1, 2; \beta = 1, 2 \quad (3.5.46)$$

$\bar{u}_{\xi\alpha\beta}$  and  $\bar{u}_{\eta\alpha\beta}$  of Eq. (3.5.37) can be approximated by bilinear Lagrangian interpolation functions,  $P_i(\xi, \eta)$  which are used to interpolate  $\bar{u}_{k\beta}$  degrees-of-freedom in Eq. (3.5.1):

$$\begin{aligned}\bar{u}_{\xi\alpha\beta} &= P_i \bar{u}_{\xi\alpha\beta i}; & \bar{u}_{\eta\alpha\beta} &= P_i \bar{u}_{\eta\alpha\beta i} \\ i &= 1, \dots, 4; \alpha = 1, 2, ; \beta = 1, 2\end{aligned}\quad (3.5.47)$$

For  $w_\beta$ , the interdependent interpolation functions in the covariant frame as in Eq. (3.5.29) should be used. Using Eqs. (3.5.25), (3.5.41), (3.5.47), the covariant shear strain component,  $\bar{\gamma}_{\xi z}$  in  $\gamma_{xz}$  of Eq. (3.5.39) can be recast as:

$$\begin{aligned}\bar{\gamma}_{\xi z} &= \left( \frac{\partial u_\xi}{\partial z} + \frac{\partial u_z}{\partial \xi} \right) \\ &= P_i (u_{\xi\beta i} \Phi_{\xi 1\beta, z} + \theta_{\xi i \beta} \Phi_{\xi 2\beta, z}) + \Omega_\beta \Big|_{z=\frac{h}{2}} (P_{i, \xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi \beta i} + N_{\eta i, \xi} \theta_{\eta \beta i}) \\ &= \frac{(1 - \eta^2)}{8} \{ -\theta_{\eta 1\beta} + \theta_{\eta 2\beta} - \theta_{\eta 3\beta} + \theta_{\eta 4\beta} \} \Omega_\beta \Big|_{z=\frac{h}{2}} \\ &\quad + \frac{\xi}{4} \left[ (1 - \eta) \left\{ \Phi_{\xi 1\beta, z} (u_{\xi 2\beta} - u_{\xi 2\beta}) + (\theta_{\xi 2\beta} - \theta_{\xi 1\beta}) \Phi_{\xi 2\beta 2, z} + (\theta_{\eta 1\beta} - \theta_{\eta 2\beta}) \Omega_\beta \Big|_{z=\frac{h}{2}} \right\} \right. \\ &\quad \left. + (1 + \eta) \left\{ \Phi_{\xi 1\beta, z} (u_{\xi 3\beta} - u_{\xi 4\beta}) + (\theta_{\xi 3\beta} - \theta_{\xi 4\beta}) \Phi_{\xi 2\beta, z} + (\theta_{\eta 3\beta} - \theta_{\eta 4\beta}) \Omega_\beta \Big|_{z=\frac{h}{2}} \right\} \right] \\ &\quad + \frac{(1 - \eta)}{4} \left\{ \Phi_{\xi 1\beta, z} (u_{\xi 1\beta} + u_{\xi 2\beta}) + \Phi_{\xi 2\beta, z} (\theta_{\xi 1\beta} + \theta_{\xi 2\beta}) + \Omega_\beta \Big|_{z=\frac{h}{2}} (w_{2\beta} - w_{1\beta}) \right\} \\ &\quad + \frac{(1 + \eta)}{4} \left\{ \Phi_{\xi 1\beta, z} (u_{\xi 3\beta} + u_{\xi 4\beta}) + \Phi_{\xi 2\beta, z} (\theta_{\xi 3\beta} + \theta_{\xi 4\beta}) + \Omega_\beta \Big|_{z=\frac{h}{2}} (w_{3\beta} - w_{4\beta}) \right\}\end{aligned}\quad (3.5.48)$$

Since the coefficients associated with the quadratic term  $(1 - \eta^2)$  in Eq. (3.5.48) include only  $\theta_{\eta i}$  rotational degrees-of-freedom, they apparently yields an inconsistent shear strain field,  $\bar{\gamma}_{\xi z}$ . Those terms do not vanish in the thin regime where  $\bar{\gamma}_{\xi z}$  tends to zero, leading to spurious constraints and shear locking. It was identified that the inconsistent terms came

from the interdependent interpolation scheme for degrees-of-freedom  $w_\beta$ . Eliminating the inconsistent terms, a consistent  $\bar{\gamma}_{\xi z}$  can then be obtained in a concise form:

$$\bar{\gamma}_{\xi z} = P_i \bar{u}_{\xi\alpha\beta i} \Phi_{\xi\alpha\beta, z}^{(k)} + (P_{i,\xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi\beta i}) \Omega_\beta \Big|_{z=\frac{h}{2}} \quad (3.5.49)$$

$$i = 1, \dots, 4; \quad \alpha = 1, 2; \quad \beta = 1, 2$$

Thus, we can obtain the consistent transverse shear strain,  $\bar{\gamma}_{\xi z}$  of Eq. (3.5.49) by approximating  $w_\beta$  by the modified interpolation function as follows:

$$\bar{w}_\beta = P_i w_{\beta i} + N_{\xi i} \theta_{\xi\beta i} \quad i = 1, \dots, 4, \quad \beta = 1, 2 \quad (3.5.50)$$

The other consistent covariant strain,  $\bar{\gamma}_{\eta z}$  can be derived from the inconsistent shear strain,  $\bar{\gamma}_{\eta z}$  in Eq. (3.5.39), in a similar way as:

$$\bar{\gamma}_{\eta z} = P_i \bar{u}_{\eta\alpha\beta i} \Phi_{\eta\alpha\beta, z}^{(k)} + (P_{i,\eta} w_{\beta i} + N_{\eta i, \eta} \theta_{\eta\beta i}) \Omega_\beta \Big|_{z=\frac{h}{2}} \quad (3.5.51)$$

with interpolation function for the  $w_\beta$  degrees-of-freedom now given by:

$$\bar{w}_\beta = P_i w_{\beta i} + N_{\eta i} \theta_{\eta\beta i} \quad i = 1, \dots, 4, \quad \beta = 1, 2 \quad (3.5.52)$$

The other transverse strain  $\gamma_{yz}$  of Eq. (3.5.46) can be written in terms of the other two covariant consistent transverse shear strains,  $\hat{\gamma}_{\xi z}$  and  $\hat{\gamma}_{\eta z}$  using the inconsistent covariant shear strains,  $\hat{\gamma}_{\xi z}$  and  $\hat{\gamma}_{\eta z}$  of Eq. (3.5.45). The other two covariant consistent transverse shear strains are:

$$\begin{aligned}
\hat{\gamma}_{\xi z} &= P_i \bar{u}_{\xi\alpha\beta i} \Psi_{\xi\alpha\beta, z}^{(k)} + (P_{i,\xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi\beta i}) \Omega_{\beta} \Big|_{z=\frac{h}{2}} \\
\hat{\gamma}_{\eta z} &= P_i \bar{u}_{\eta\alpha\beta i} \Psi_{\eta\alpha\beta, z} + (P_{i,\eta} w_{\beta i} + N_{\eta i, \eta} \theta_{\eta\beta i}) \Omega_{\beta} \Big|_{z=\frac{h}{2}} \\
i &= 1, \dots, 4; \quad \alpha = 1, 2; \quad \beta = 1, 2
\end{aligned} \tag{3.5.53}$$

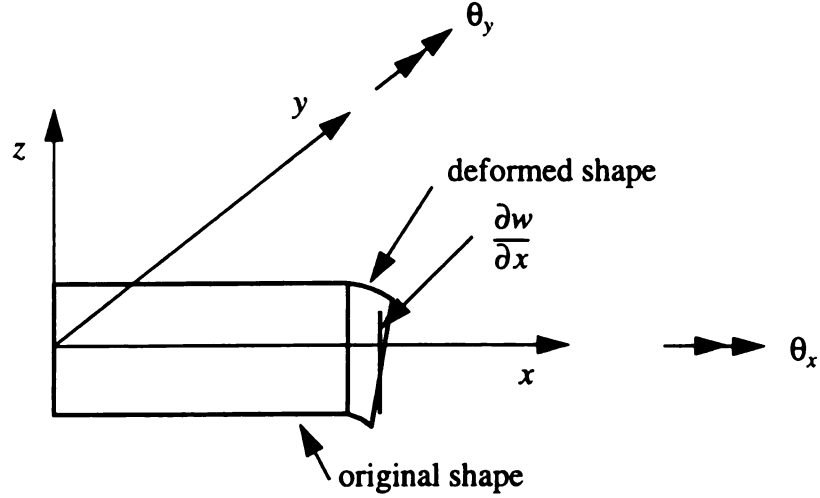
The derived consistent transverse strain field is based on the concept that all transverse shear strain fields should vanish in the thin regime of plate due to Kirchhoff constraints, and the inconsistent fields which don't vanish should be eliminated. This method can be viewed as one of the assumed shear strain fields approaches [91-94], and it is analogous to the selective/reduced integration concept except that it only removes the inconsistent terms. However, selective/reduced integration removes even necessary terms in the strain fields and causes spurious zero energy modes.

### ***Consistent Penalty Constraints***

To develop a  $C^0$  laminate theory and the finite element associated with the theory, four constraints between rotation d.o.f and the first derivatives of transverse deflection were imposed in the energy functional by employing the penalty method. The constraints are given as follows:

$$C_{x\beta} = \left( \frac{\partial w_{\beta}}{\partial y} - \theta_{x\beta} \right) \quad C_{y\beta} = \left( \frac{\partial w_{\beta}}{\partial x} + \theta_{y\beta} \right) \quad \beta = 1, 2 \tag{3.5.54}$$

These constraints are very similar to the form of the shear strains in first order shear deformation plate theory. This can be simply verified. If we consider the following rotation convention and deformed shape,



**Figure 20. Schematic and definition of rotation and deformed shape**

then displacement field becomes:

$$\begin{aligned}
 u &= u_0 - z \frac{\partial w}{\partial x} = u_0 + z \theta_y \\
 v &= v_0 - z \frac{\partial w}{\partial y} = v_0 - z \theta_x \\
 w &= w_0
 \end{aligned}
 \tag{3.5.55}$$

Thus, the transverse shear strains are given by:

$$\begin{aligned}
 \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w_0}{\partial x} + \theta_y \\
 \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w_0}{\partial y} - \theta_x
 \end{aligned}
 \tag{3.5.56}$$

We can see clearly that the constraints of Eq. (3.5.54) are exactly the same as the transverse shear strains in first order shear deformation plate theory. Therefore, just like the above transverse shear strain fields, we need to check if any of these constraints are field

inconsistent, and hence causes any spurious constraints. Using the relation from Eq. (3.5.31), the constraints in Eq. (3.5.54) can be written in terms of their covariant form as:

$$\begin{aligned}
 C_{x\beta} &= \left( \frac{\partial w_\beta}{\partial y} - \theta_{x\beta} \right) \\
 &= \bar{j}_{11} \frac{\partial w_\beta}{\partial \xi} + \bar{j}_{12} \frac{\partial w_\beta}{\partial \eta} - \bar{j}_{12} \theta_{\xi\beta} + \bar{j}_{11} \theta_{\eta\beta} \\
 &= \bar{j}_{11} \left( \frac{\partial w_\beta}{\partial \xi} + \theta_{\eta\beta} \right) + \bar{j}_{12} \left( \frac{\partial w_\beta}{\partial \eta} - \theta_{\xi\beta} \right) \\
 &= \bar{j}_{11} C_{\xi\beta} + \bar{j}_{12} C_{\eta\beta}
 \end{aligned} \tag{3.5.57}$$

After substitution of shape functions into the covariant penalty constraints,  $C_{\xi\beta}$  can be recast as:

$$\begin{aligned}
 C_{\xi\beta} &= \left( \frac{\partial w_\beta}{\partial \xi} + \theta_{\eta\beta} \right) \\
 &= \frac{(1 - \eta^2)}{8} (\theta_{\xi\beta 2} - \theta_{\xi\beta 1} + \theta_{\xi\beta 4} - \theta_{\xi\beta 3}) \\
 &\quad + \frac{(1 - \eta)}{4} (w_{\beta 2} - w_{\beta 1} + \theta_{\eta\beta 1} + \theta_{\eta\beta 2}) + \frac{(1 + \eta)}{4} (w_{\beta 3} - w_{\beta 4} + \theta_{\xi\beta 3} + \theta_{\xi\beta 4})
 \end{aligned} \tag{3.5.58}$$

If we examine Eq. (3.5.58), the coefficients of the quadratic term  $(1 - \eta^2)$  are found to be inconsistent because they are composed of only the rotational degrees-of-freedom  $\theta_{\xi\beta i}$ . Thus, they must be removed to make all the fields consistently vanish without inducing spurious or locking constraints. The consistent covariant penalty component,  $\bar{C}_{\xi\beta}$  can be obtained by interpolating the  $w_\beta$  d.o.f. in a similar form as in Eq. (3.5.50) above instead of Eq. (3.5.25) while keeping the interpolation scheme for the rotational degrees-of-freedom unchanged.  $\bar{C}_{\xi\beta}$  can then be expressed in a succinct form,



$$\bar{C}_{\xi\beta} = P_{i,\xi} w_{\beta i} + (N_{\eta i,\xi} + P_i) \theta_{\xi\beta i} \quad (3.5.59)$$

Similarly, the other covariant penalty constraint  $\bar{C}_{\eta\beta}$  can be derived as:

$$\bar{C}_{\eta\beta} = P_{i,\eta} w_{\beta i} + (N_{\xi i,\eta} - P_i) \theta_{\eta\beta i} \quad (3.5.60)$$

with the  $w_\beta$  degrees-of-freedom interpolated as in Eq. (3.5.52).

### ***Edge Consistency***

Now we have developed field consistent shear strains by examining transverse shear strains in the covariant frame and eliminating inconsistent terms. However, even though transverse shear strain fields are field-consistent, the element still locks if edge-consistency of transverse shear strain on any common inter-element edge is not met, especially for general quadrilateral elements. It is crucial to ensure matching of the tangential shear strain on any common inter-element edge. Mismatch of tangential shear strain will give rise to spurious constraints on the edges when the shape of elements is a general quadrilateral. This stems from the fact that when the field consistent covariant transverse shear strain is mapped back into global cartesian coordinates from the covariant frame, jacobian transformations of two adjoining elements at their own integration points will redefine this consistency, and cause a mismatch of tangential shear strain at the common edge. Thus, the choice of sampling points of the jacobian transformations plays a very important role in edge consistency of tangential shear strain at the common edge. Prathap and Somashekar [72] showed that the consistency of tangential shear strain along a common edge between two elements is maintained by transforming field consistent shear strains in covariant frame into global cartesian fields at each node of the element instead of at Gauss

integration points. In this way, edge consistency in the field consistent covariant penalty constraint fields can be treated as well.

Replacement of Eqs. (3.5.49) and (3.5.51) with Eq. (3.5.39) gives the field consistent shear strain in the covariant frame as:

$$\begin{aligned}
 \bar{\gamma}_{xz}(\xi, \eta) &= \bar{\gamma}_{\xi z} \bar{j}_{11} + \bar{\gamma}_{\eta z} \bar{j}_{12} \\
 &= (P_i \bar{u}_{\xi\alpha\beta i} \Phi_{\xi\alpha\beta, z}^{(k)} + (P_{i,\xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi\beta i}) \bar{\Omega}_\beta) \bar{j}_{11} \\
 &\quad + (P_i \bar{u}_{\eta\alpha\beta i} \Phi_{\eta\alpha\beta, z}^{(k)} + (P_{i,\eta} w_{\beta i} + N_{\eta i, \eta} \theta_{\eta\beta i}) \bar{\Omega}_\beta) \bar{j}_{12} \\
 &\quad \text{where } i = 1, \dots, 4; \alpha = 1, 2; \beta = 1, 2; \bar{\Omega}_\beta = \Omega_\beta \Big|_{z = \frac{h}{2}}
 \end{aligned} \tag{3.5.61}$$

To maintain the edge consistency on the common inter-element edge, the covariant nodal d.o.f,  $\bar{u}_{\xi\beta i}$ ,  $\bar{u}_{\eta\beta i}$  of Eq. (3.5.37), should be mapped into their Cartesian coordinate definitions,  $\bar{u}_{\alpha\beta i}$  of Eq. (3.3.16) at each node [42,76], i.e.:

$$\xi = -1, \eta = -1; \quad \xi = 1, \eta = -1; \quad \xi = 1, \eta = 1; \quad \xi = -1, \eta = 1 \tag{3.5.62}$$

Then, using Eqs. (3.5.32), (3.5.61), after the nodal transformation is accomplished, the consistent Cartesian based shear strain,  $\bar{\gamma}_{xz}$  is given by:

$$\begin{aligned}
 \bar{\gamma}_{xz}(\xi, \eta) &= \sum_{i=1}^4 \left[ u_{\beta i} \left\{ (j_{11})_i P_i \Phi_{\xi 1\beta, z}^{(k)} \bar{j}_{11} + (j_{21})_i P_i \Phi_{\eta 1\beta, z}^{(k)} \bar{j}_{12} \right\} \right. \\
 &\quad + v_{\beta i} \left\{ (j_{12})_i P_i \Phi_{\xi 1\beta, z}^{(k)} \bar{j}_{11} + (j_{22})_i P_i \Phi_{\eta 1\beta, z}^{(k)} \bar{j}_{12} \right\} + w_{\beta i} \left( P_{i,\xi} \bar{j}_{11} + P_{i,\eta} \bar{j}_{12} \right) \bar{\Omega}_\beta \\
 &\quad + \theta_{x\beta i} \left\{ (j_{22})_i (P_i \Phi_{\xi 2\beta, z}^{(k)} \bar{j}_{11} + N_{\xi i, \eta} \bar{\Omega}_\beta \bar{j}_{12}) - (j_{12})_i (P_i \Phi_{\eta 2\beta, z}^{(k)} \bar{j}_{12} + N_{\eta i, \xi} \bar{\Omega}_\beta \bar{j}_{11}) \right\} \\
 &\quad \left. + \theta_{y\beta i} \left\{ (j_{11})_i (P_i \Phi_{\eta 2\beta, z}^{(k)} \bar{j}_{12} + N_{\eta i, \xi} \bar{\Omega}_\beta \bar{j}_{11}) - (j_{21})_i (P_i \Phi_{\xi 2\beta, z}^{(k)} \bar{j}_{11} + N_{\xi i, \eta} \bar{\Omega}_\beta \bar{j}_{12}) \right\} \right]
 \end{aligned} \tag{3.5.63}$$

$\bar{j}_{ij}$  are used to map Cartesian d.o.f into the covariant frame d.o.f. Therefore, it must be evaluated at the integration points.  $j_{ij}$  are employed to map covariant frame d.o.f back to Cartesian d.o.f, and must be assessed at the nodal points. Similarly,

$$\begin{aligned}
 \bar{\gamma}_{yz}(\xi, \eta) = & \sum_{i=1}^4 \left[ u_{\beta i} \left\{ (j_{11})_i P_i \Psi_{\xi 1 \beta}^{(k)} \bar{j}_{21} + (j_{21})_i P_i \Psi_{\eta 1 \beta}^{(k)} \bar{j}_{22} \right\} \right. \\
 & + v_{\beta i} \left\{ (j_{12})_i P_i \Psi_{\xi 1 \beta}^{(k)} \bar{j}_{21} + (j_{22})_i P_i \Psi_{\eta 1 \beta}^{(k)} \bar{j}_{22} \right\} + w_{\beta i} \left( P_{i,\xi} b_{21} + P_{i,\eta} \bar{j}_{22} \right) \bar{\Omega}_{\beta} \\
 & + \theta_{x\beta i} \left\{ (j_{22})_i (P_i \Psi_{\xi 2 \beta}^{(k)} \bar{j}_{21} + N_{\xi i, \eta} \bar{\Omega}_{\beta} \bar{j}_{22}) - (j_{12})_i (P_i \Psi_{\eta 2 \beta}^{(k)} \bar{j}_{22} + N_{\eta i, \xi} \bar{\Omega}_{\beta} \bar{j}_{21}) \right\} \\
 & \left. + \theta_{y\beta i} \left\{ (j_{11})_i (P_i \Psi_{\eta 2 \beta}^{(k)} \bar{j}_{22} + N_{\eta i, \xi} \bar{\Omega}_{\beta} \bar{j}_{21}) - (j_{21})_i (P_i \Psi_{\xi 2 \beta}^{(k)} \bar{j}_{21} + N_{\xi i, \eta} \bar{\Omega}_{\beta} \bar{j}_{22}) \right\} \right] \quad (3.5.64)
 \end{aligned}$$

Field consistent penalty constraint  $\bar{C}_{x\beta}$  is obtained by expressing Eq. (3.5.57) in terms of the field consistent covariant penalty constraints,  $\bar{C}_{\xi\beta}$  and  $\bar{C}_{\eta\beta}$  from Eqs. (3.5.59), (3.5.60):

$$\begin{aligned}
 \bar{C}_{x\beta}(\xi, \eta) = & \bar{C}_{\xi\beta} \bar{j}_{11} + \bar{C}_{\eta\beta} \bar{j}_{12} \\
 = & \{ P_{i,\xi} w_{\beta i} + (N_{\eta i, \xi} + P_i) \theta_{\xi\beta i} \} \bar{j}_{11} + \{ P_{i,\eta} w_{\beta i} + (N_{\eta i, \eta} - P_i) \theta_{\eta\beta i} \} \bar{j}_{12} \quad (3.5.65) \\
 & i = 1, \dots, 4; \beta = 1, 2
 \end{aligned}$$

By nodal transformation, the nodal covariant rotations,  $\theta_{\xi i}$  and  $\theta_{\eta i}$  are transformed back to the Cartesian coordinate definitions of the rotations,  $\theta_{xi}$  and  $\theta_{yi}$ . Thus, the consistent covariant penalty constraint,  $\bar{C}_{x\beta}$  can be written in terms of  $\theta_{xi}$  and  $\theta_{yi}$  explicitly as:

$$\begin{aligned}
\bar{C}_{x\beta}(\xi, \eta) = \sum_{i=1}^4 [\theta_{x\beta i} \{ -(j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{11} + (j_{22})_i (N_{\xi i, \eta} - P_i) \bar{j}_{12} \} \\
+ \theta_{y\beta i} \{ (j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{11} - (j_{21})_i (N_{\xi i, \eta} - P_i) \bar{j}_{12} \} \\
+ w_{\beta i} (P_{i, \xi} \bar{j}_{11} + P_{i, \eta} \bar{j}_{12}) ] \quad \beta = 1, 2
\end{aligned} \tag{3.5.66}$$

By the similar procedure, the consistent covariant penalty constraint,  $\bar{C}_{y\beta}$  are:

$$\begin{aligned}
\bar{C}_{y\beta}(\xi, \eta) = \sum_{i=1}^4 [\theta_{x\beta i} \{ -(j_{12})_i (N_{\eta i, \xi} + P_i) \bar{j}_{21} + (j_{22})_i (N_{\xi i, \eta} - P_i) \bar{j}_{22} \} \\
+ \theta_{y\beta i} \{ (j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{21} - (j_{21})_i (N_{\xi i, \eta} - P_i) \bar{j}_{22} \} \\
+ w_{\beta i} \{ P_{i, \xi} \bar{j}_{21} + P_{i, \eta} \bar{j}_{22} \} ] \quad \beta = 1, 2
\end{aligned} \tag{3.5.67}$$

The consistent shear strain fields,  $\bar{\gamma}_{xz}$  and  $\bar{\gamma}_{yz}$  in Cartesian coordinates as well as the consistent penalty constraints,  $\bar{C}_{x\beta}$  and  $\bar{C}_{y\beta}$  derived in this method ensure to alleviate locking phenomenon, therefore an exact numerical integration rule can be used to evaluate the shear strain energy.

### 3.6 Poisson's Ratio Stiffening Effect

The element with uniform transverse normal strain predicts deflections that are about 10 % too stiff when one sublaminate (one element through thickness) is used in the thin regime, depending on Poisson's ratio, even though the element is developed using field consistent shear fields and edge consistency to prevent shear locking. This phenomenon was identified as Poisson's ratio stiffening effect [77,96,97].

From the assumption of transverse deflection displacement, Eq. (3.3.1), the original form of transverse normal strain is constant through-the-thickness as:

$$\epsilon_{zz} = \frac{w_t - w_b}{h} \quad (3.6.1)$$

This transverse normal strain can be also obtained from the constitutive equation Eq. (3.2.2) and is given by:

$$\epsilon_{zz}^{(k)} = -\frac{C_{13}^{(k)}}{C_{33}^{(k)}}\epsilon_{xx}^{(k)} - \frac{C_{23}^{(k)}}{C_{33}^{(k)}}\epsilon_{yy}^{(k)} - \frac{C_{36}^{(k)}}{C_{33}^{(k)}}\gamma_{xy}^{(k)} + \frac{1}{C_{33}^{(k)}}\sigma_{zz} \quad (3.6.2)$$

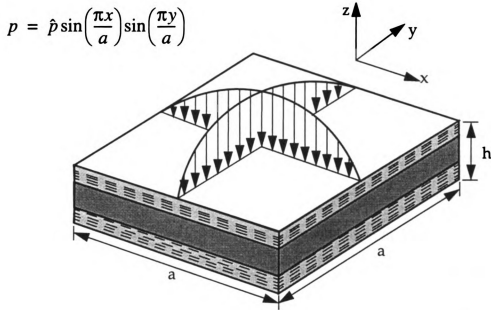
When the laminate is thin, the laminate is in a plane stress state, and the transverse normal stress approaches to zero. Eq. (3.6.2) becomes:

$$\epsilon_{zz}^{(k)} = -\frac{C_{13}^{(k)}}{C_{33}^{(k)}}\epsilon_{xx}^{(k)} - \frac{C_{23}^{(k)}}{C_{33}^{(k)}}\epsilon_{yy}^{(k)} - \frac{C_{36}^{(k)}}{C_{33}^{(k)}}\gamma_{xy}^{(k)} \quad (3.6.3)$$

Due to the assumption of displacement field, Eq. (3.3.1),  $\epsilon_{xx}^{(k)}$ ,  $\epsilon_{yy}^{(k)}$  and  $\gamma_{xy}^{(k)}$  in Eq. (3.6.3) vary linearly in the thickness direction, while  $\epsilon_{zz}^{(k)}$  in Eq. (3.6.1) is constant through the thickness by the definition of transverse strain. Thus, the element can not capture the variation of  $\epsilon_{zz}^{(k)}$  in the thickness direction. The difference in the through-thickness variation between transverse normal strain by Eq. (3.6.1) and by Eq. (3.6.3) produces a stiffening effect of the order of 10%. This effects goes away rapidly when more than one element is used through the thickness. In the current element, this stiffening effect is removed by using the assumed transverse normal stress approach as discussed previously.

### 3.7 Numerical Results

The accuracy of the proposed laminated plate theory and the appropriateness of the finite element model were assessed by simulating the response of two different simply-supported laminated plates subjected to a transverse double sinusoidal loading as shown in Figure 21. The elasticity solution of Burton and Noor [98] and the predictions based on first-order shear deformation theory (FSDT) [10] were used to provide comparison with the current plate results (FZZ3D). For a given location in the plane of the plate, explicit expressions for the through-thickness variation of stresses are provided by the present model. In the plane of the plate, stresses were evaluated at the centroid of each element, where stress is super-convergent [73]. The bending stresses presented in the forthcoming discussion were evaluated at the center of the element that is nearest the center of the plate. The transverse deflections are obtained at the top surface of the center of the plate where load is applied. The in-plane displacements are obtained at the edges of the plate where  $x = 0$  and  $y = a/2$  (for  $u_x$  displacement) or  $y = 0$  and  $x = a/2$  (for  $u_y$  displacement), and  $a$  is the length of each side of the plate. Due to the symmetry of the panels, only a quarter of the plate was modeled in each case. A uniform mesh of 8x8 elements was utilized in the plane of a quarter of the plate, and the through-thickness discretization was varied to assess the utility of the present models. A length-to-thickness aspect ratio of four was chosen in the present examples in order to highlight the ability of the present model to obtain accurate results for very thick laminates via through-thickness refinement of the mesh.



**Figure 21. Simply supported laminated composite plate subjected to sinusoidal loading.**

***Example 1: CAV Panel***

The first example is a panel from the US Army TACOM's Composite Armored Vehicle (CAV). Material properties used in the analysis are given in Table 3. The laminate consists of 55 layers and is nearly two inches thick. The laminate can be divided into three sections, as described below:

- Inner Shell: Four plies of S-2 Glass/Phenolic fabric with stacking sequence  $\{[90_2/0_2]\}$  (ply thickness = 0.01 in.). Thirty-seven plies of S-2 Glass/8553-40 Epoxy tows with stacking sequence  $\{[0/90], [45/-45/0/90]_4, [45/0/45], [90/0/-45/45]_4\}$  (ply thickness = 0.021 in.).
- Armor Core. One layer of EPDM rubber (ply thickness = 0.0625 in.) and one layer of ceramic tile with inserts (ply thickness = 0.7 in.).
- Outer Shell. Twelve plies of S-2 Glass/8553-40 Epoxy fabric with stacking sequence

$[0/90/45/-45/0/90]_s$  (ply thickness = 0.01 in.).

To allow comparison of predictions of the current model with the elasticity solution, all layers are oriented at either  $90^\circ$  or  $0^\circ$  from the reference axis by replacing all  $45^\circ$  plies with  $90^\circ$  plies and  $-45^\circ$  plies with  $0^\circ$  plies.

Two types of through-thickness refinement were used in the analyses: (1) a single sublaminate, and (2) three sublaminae with a sublaminate for the inner shell, a sublaminate for the rubber layer, and a sublaminate for the ceramic tile layer and the outer shell.

The influence of aspect ratios on transverse deflection is shown in Figure 22. Eight different aspect ratios were considered to check a wide range of plate bending behavior ( $L/h = 4, 8, 10, 20, 30, 50, 100, 500, \text{ and } 1000$ ). When one sublaminate was used, the error in predicted deflection was about 15 percent for the case  $a/h=4$ . The use of three sublaminae resulted in more accurate predictions, with an error of less than one percent. It was observed that the prediction of transverse deflection converges rapidly to the exact solution as more sublaminae are used. In Figure 23, the variation of in-plane displacement,  $u_x$ , through the thickness of the laminate is plotted at  $x = 0$  and  $y = a/2$  for the case of aspect ratio of four. The exact distribution of in-plane displacement is approximately linear within each layer, and globally piece-wise linear and of zig-zag pattern due to the difference in the shear moduli of adjacent layers. It is shown that the current model containing only one sublaminate captures the trends of the distribution of displacement very well, while FSDT is unable to simulate such behavior. The bending stress distribution through the thickness is illustrated in Figure 24. The stress was evaluated at the Gauss point nearest to the center of the plate. It can be observed that the prediction of in-plane normal stress by the current model agrees well with the elasticity solution.



**Table 3. Material properties for TACOM's composite armored vehicle (CAV) panel**

|            | <b>S-2 Glass/<br/>Phenolic<br/>Fabric</b> | <b>S-2 Glass/<br/>8553-40<br/>Tow</b> | <b>S-2 Glass/<br/>8553-40<br/>Fabric</b> | <b>EPDM<br/>Rubber</b> | <b>Ceramic</b> |
|------------|-------------------------------------------|---------------------------------------|------------------------------------------|------------------------|----------------|
| $E_{11}$   | 3.0e6                                     | 6.2e6                                 | 3.0e6                                    | 3.0e3                  | 5.0e6          |
| $E_{22}$   | 3.0e6                                     | 1.0e6                                 | 3.0e6                                    | 3.0e3                  | 5.0e6          |
| $E_{33}$   | 1.2e6                                     | 1.0e6                                 | 1.1e6                                    | 3.0e3                  | 1.25e5         |
| $\nu_{12}$ | 0.13                                      | 0.29                                  | 0.13                                     | 0.45                   | 0.15           |
| $\nu_{23}$ | 0.18                                      | 0.37                                  | 0.18                                     | 0.45                   | 0.15           |
| $\nu_{13}$ | 0.18                                      | 0.29                                  | 0.18                                     | 0.45                   | 0.15           |
| $G_{12}$   | 1.0e6                                     | 0.3e6                                 | 1.0e6                                    | 1.0e3                  | 2.5e6          |
| $G_{23}$   | 4.6e5                                     | 0.3e6                                 | 3.9e5                                    | 1.0e3                  | 8.5e3          |
| $G_{13}$   | 4.6e5                                     | 0.3e6                                 | 3.9e5                                    | 1.0e3                  | 8.5e3          |

The prediction of in-plane shear stress is shown in Figure 25. There is significant error in the armor core region, but predictions of the current model are very good in other regions of the laminate. The distribution of transverse shear stress,  $\sigma_{xz}$  through thickness is shown in Figure 26. As assumed, this component of stress is constant in each sublaminate. It is shown that the use of multiple sublaminae allows the current model to capture the trends of the through-thickness distribution of shear stress. The transverse normal strain,  $\epsilon_{zz}$ , is predicted in Figure 27. When three sublaminae are used, the distribution of transverse strain is nearly indistinguishable from the elasticity solution. In Figure 28, transverse normal stress,  $\sigma_{zz}$ , versus normalized thickness coordinate at the center of the plate is illustrated. As assumed, the model captures about the average value of the exact distribution in a sublaminate.

### ***Example 2: Sandwich Panel***

The second example is a sandwich panel which consists of a thick core between two sets of five face sheets. The material properties used in the analysis are listed in Table 4., and the lamination scheme is defined in Table 5. The core occupies eighty percent of the thickness of the plate, while each set of face sheets contains five layers and occupies ten percent of the total thickness. Two types of through-thickness discretization were used in the analyses: (1) a single sublaminate, (2) three sublaminae with equal thickness. In Figure 29, the predicted through-thickness distributions of in-plane displacement are illustrated. All predictions were very accurate in the upper and lower face sheet regions, but there were significant errors in the core region when only one sublaminate was used. These errors were due to the very large transverse shear strains and transverse squashing in the core of this thick sandwich plate. The nonlinear variation of inplane displacement in this region could not be captured accurately by a single sublaminate in the core region. For thinner plates (i.e.,  $a/h$  greater than about 10), the inplane displacements throughout the thickness are predicted satisfactorily using a single sublaminate through the entire thickness of the laminate. To assess the bending stress,  $\sigma_{xx}$ , more clearly, the stress distribution in the upper face sheet region is enlarged and shown in Figure 30. It can be observed that the current finite element model using only one sublaminate predicts the stress distribution in the face sheets fairly accurately, and even greater accuracy is achieved with three sublaminae. In Figure 31, in-plane shear stress in the face sheets are predicted. It is shown in Figure 32 that the average transverse shear stress is captured well by the current model. Figure 33 depicts the distribution of transverse normal strain,  $\epsilon_{zz}$ , through the thickness of the laminate. When three sublaminae were employed, the strain distribution through the

thickness is predicted relatively well, capturing the trend very well in the core region. Transverse normal stress,  $\sigma_{zz}$ , versus normalized thickness coordinate at the center of the plate is shown in Figure 34. It can be observed that it takes about the average value of the exact distribution in a sublaminate, and captures the distribution trends as more elements are used through thickness.

### 3.8 Summary

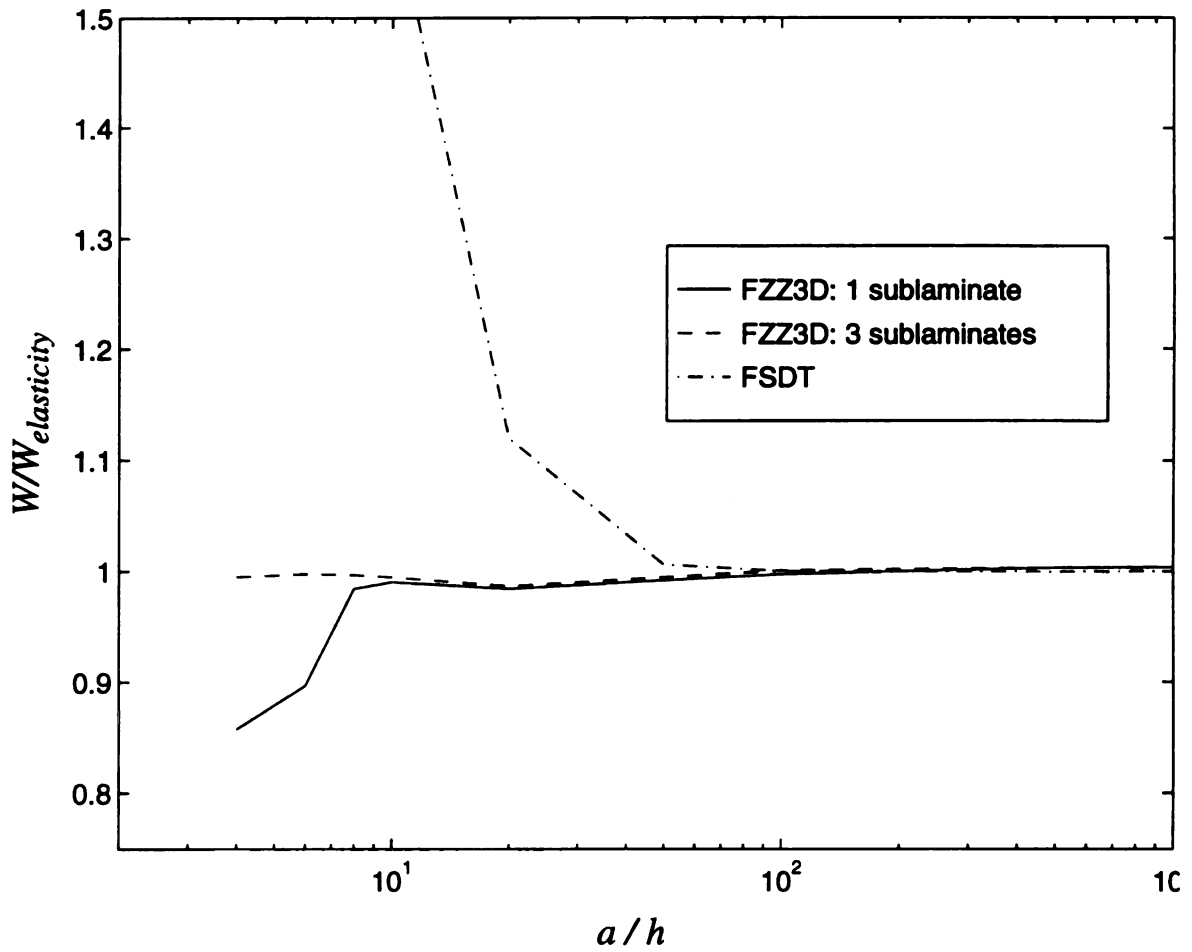
An improved first order zig-zag theory and an associated finite element model are presented. The model allows the representation of a laminate as an assemblage of sublaminates in order to increase the model refinement through the thickness, when needed. Within each sublaminate, an accurate first-order zig-zag sublaminate kinematic approximation is made which minimizes the need for multiple sublaminates for many problems of practical interest yet easily accommodates through-thickness refinement for predicting local variations of stress and deformation. The finite element model is “regenerated” in the form of an eight-noded three dimensional element with five degrees of freedom (three translations and two rotations) per node. Thus, it is suitable for implementation into commercial finite element codes. Numerical results demonstrate that the current model is accurate, efficient, and robust for analysis of a wide variety of thick or thin laminated plates.

**Table 4. Material properties for sandwich composite plate**

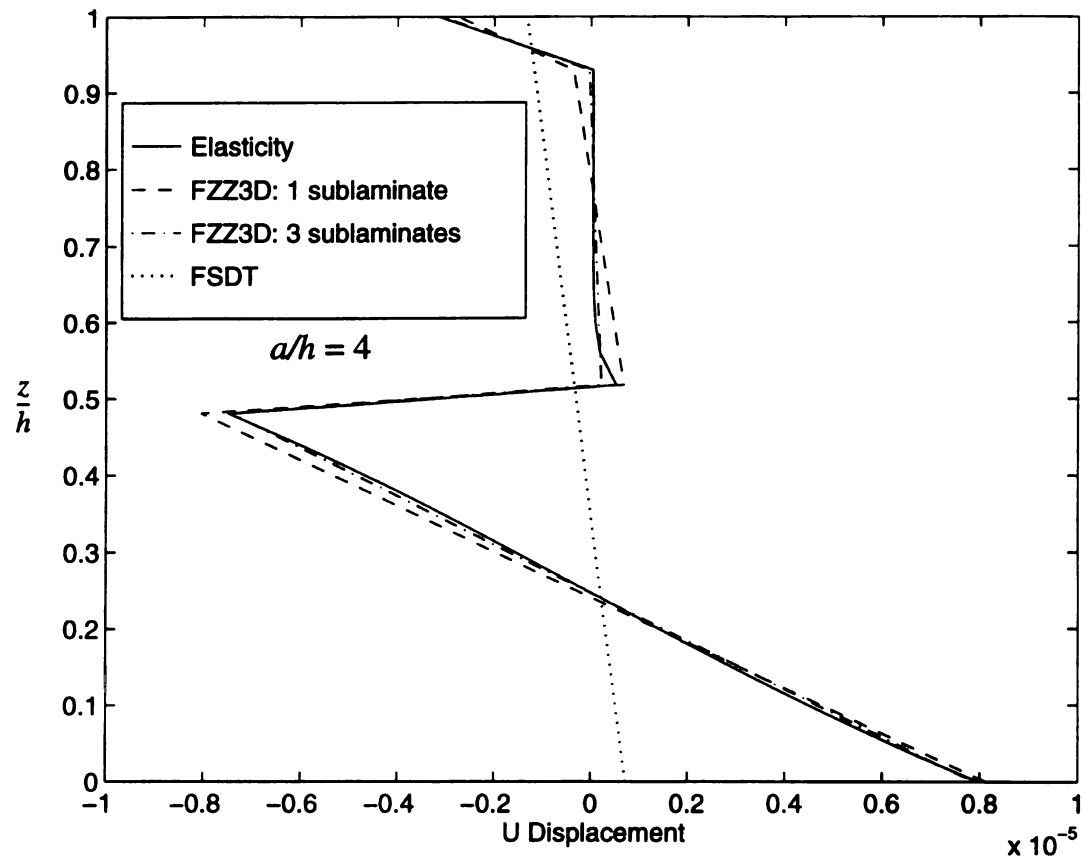
|            | material 1 | material 2 | material 3 | core   |
|------------|------------|------------|------------|--------|
| $E_{11}$   | 1.0e6      | 32.6e6     | 25.e6      | 50.e3  |
| $E_{22}$   | 25.0e6     | 1.0e7      | 1.0e6      | 150.e3 |
| $E_{33}$   | 1.0e6      | 1.0e7      | 1.0e6      | 50.e3  |
| $\nu_{12}$ | 0.01       | 0.10       | 0.25       | 0.01   |
| $\nu_{23}$ | 0.25       | 0.25       | 0.25       | 0.15   |
| $\nu_{13}$ | 0.25       | 0.10       | 0.25       | 0.15   |
| $G_{12}$   | 0.50e6     | 8.21e6     | 0.5e6      | 21.7e3 |
| $G_{23}$   | 0.50e6     | 0.5e6      | 0.2e6      | 42.0e3 |
| $G_{13}$   | 0.20e6     | 8.21e6     | 0.5e6      | 21.7e3 |

**Table 5. Lamination Scheme for Sandwich Panel**

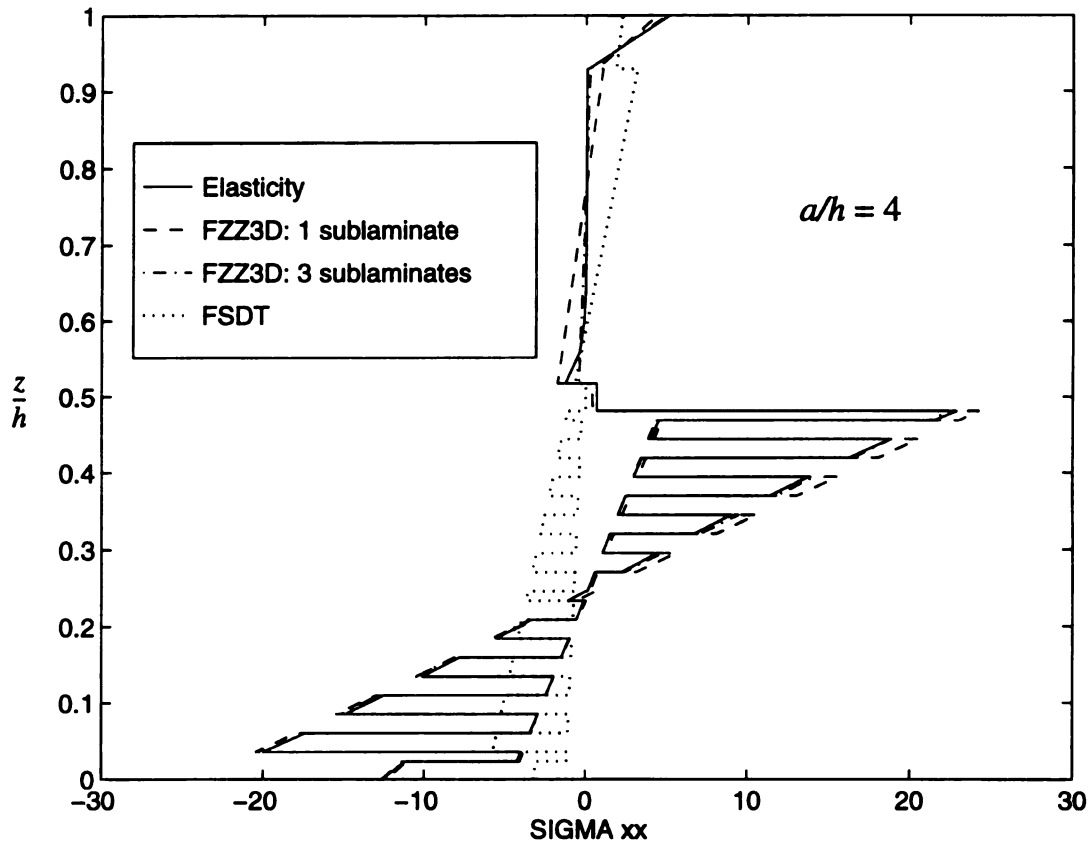
| Layer No. | Material | Thickness |
|-----------|----------|-----------|
| 1         | 1        | 0.010     |
| 2         | 2        | 0.025     |
| 3         | 3        | 0.015     |
| 4         | 1        | 0.020     |
| 5         | 3        | 0.030     |
| 6         | Core     | 0.800     |
| 7         | 3        | 0.030     |
| 8         | 1        | 0.020     |
| 9         | 3        | 0.015     |
| 10        | 2        | 0.025     |
| 11        | 1        | 0.010     |



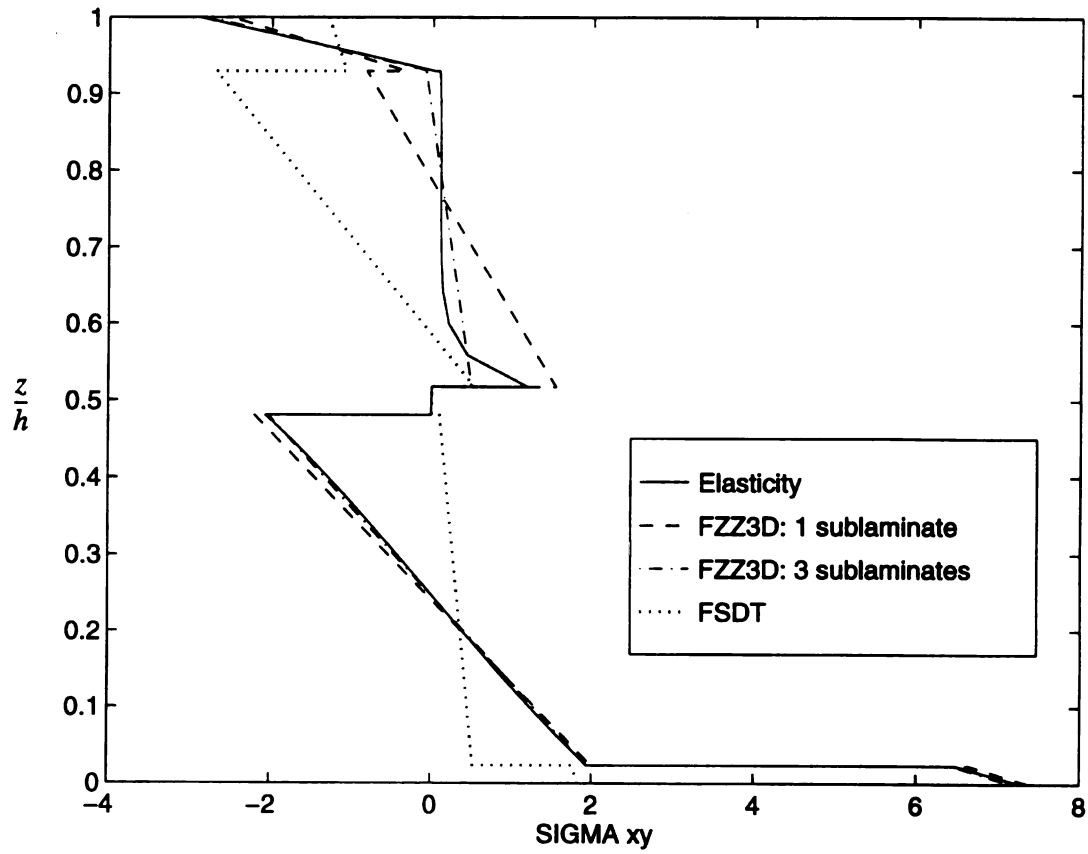
**Figure 22. Variation of normalized center deflection versus plate aspect ratio at the center of the plate (CAV Panel,  $a/h=4$ ).**



**Figure 23. In-plane displacement,  $u$ , versus normalized thickness coordinate (CAV Panel,  $a/h=4$ ).**

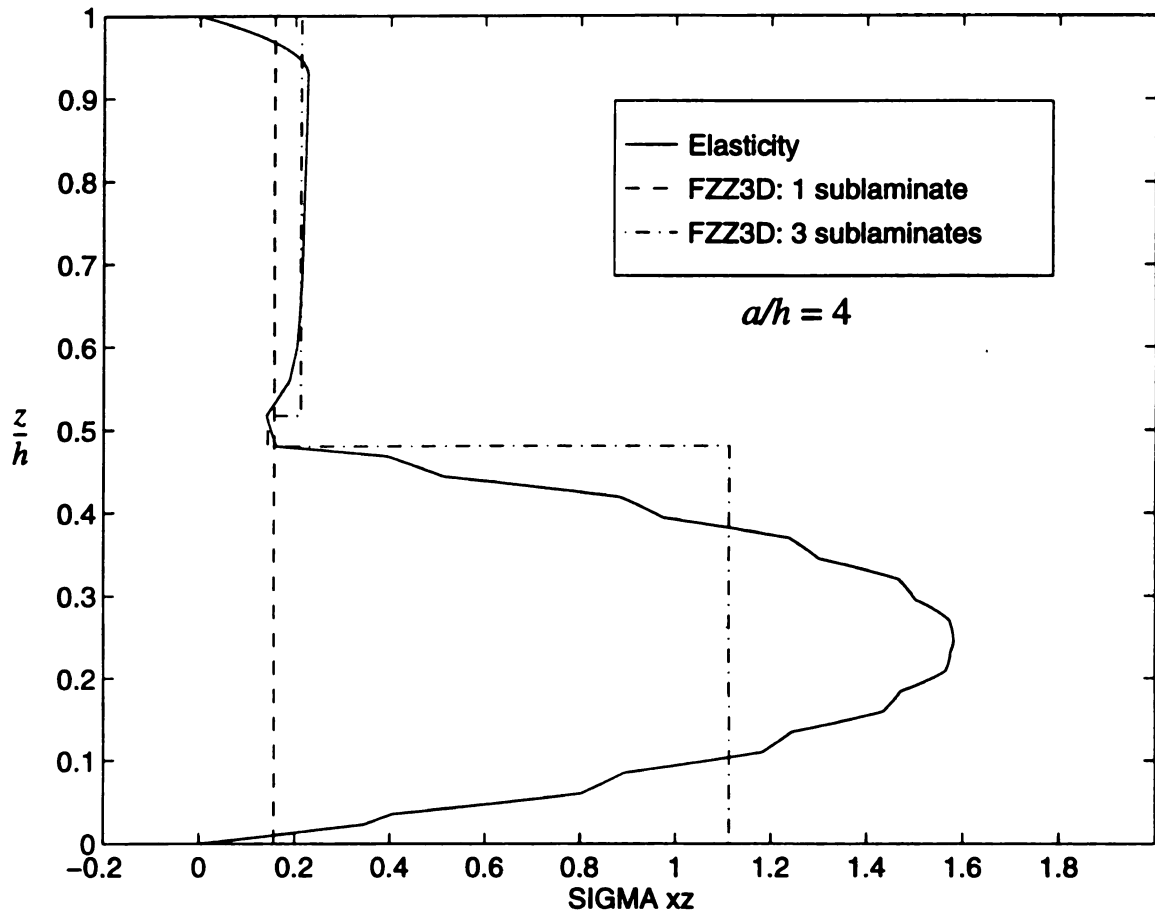


**Figure 24. Bending stress,  $\sigma_{xx}$ , versus normalized thickness coordinate at the center of the plate (CAV Panel,  $a/h=4$ ).**

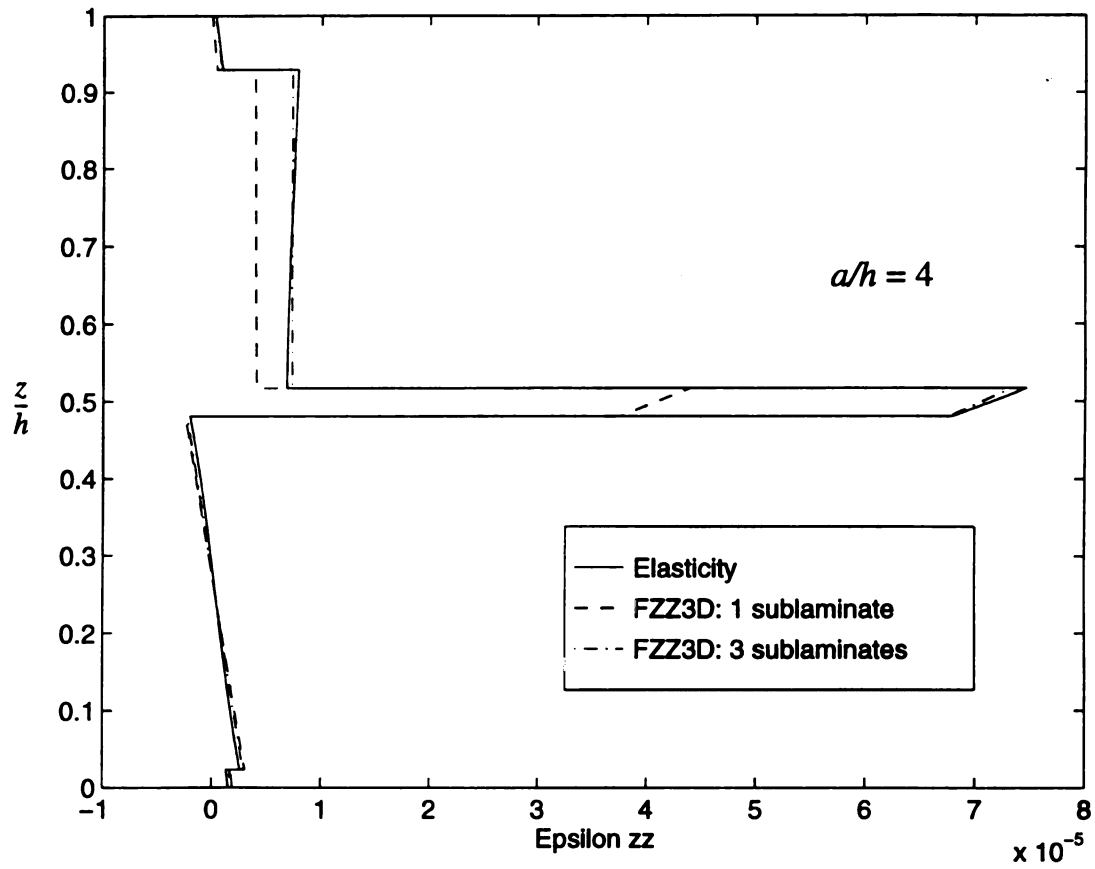


**Figure 25. In-plane Shear stress,  $\sigma_{xy}$ , versus normalized thickness coordinate at  $x, y = 0$ , (CAV Panel,  $a/h=4$ ).**

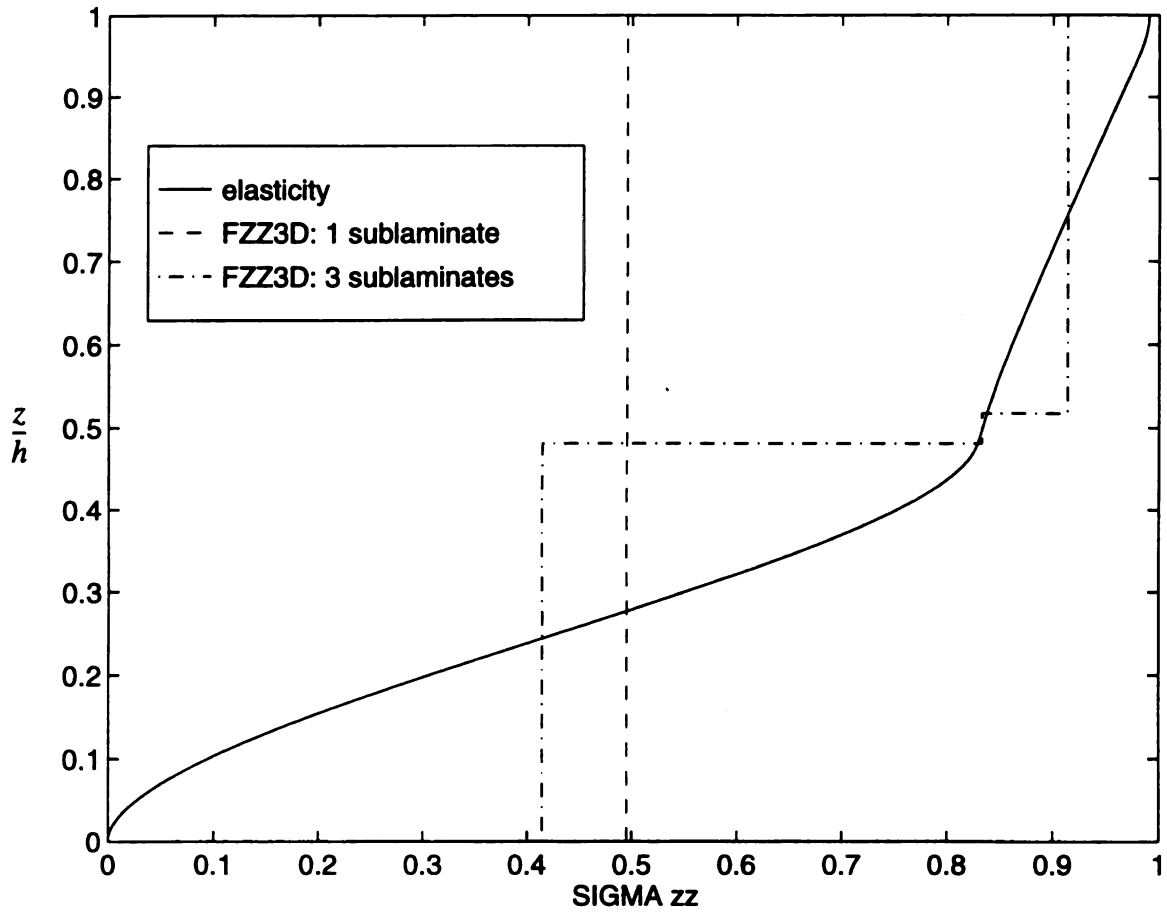




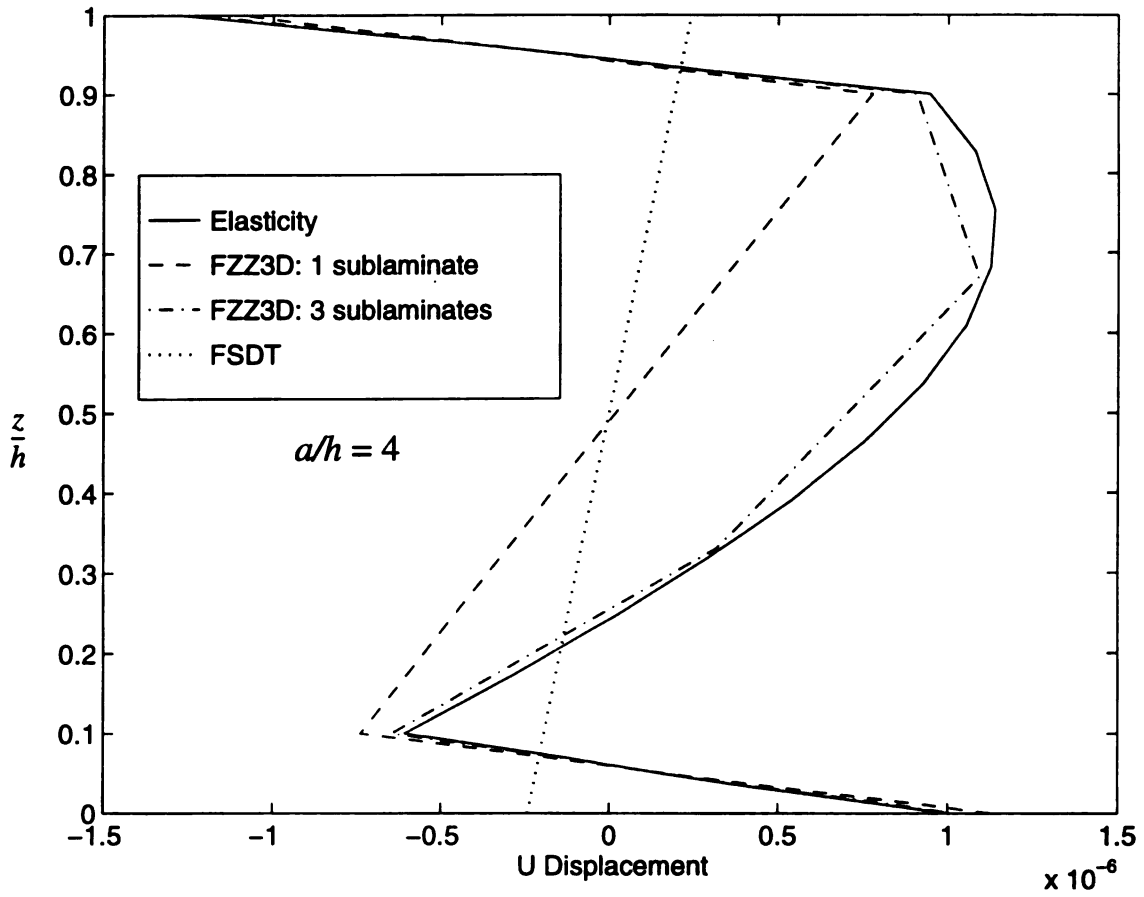
**Figure 26. Transverse shear stress,  $\sigma_{xz}$ , versus normalized thickness coordinate at  $x = 0, y = 2/a$  (CAV Panel,  $a/h=4$ ).**



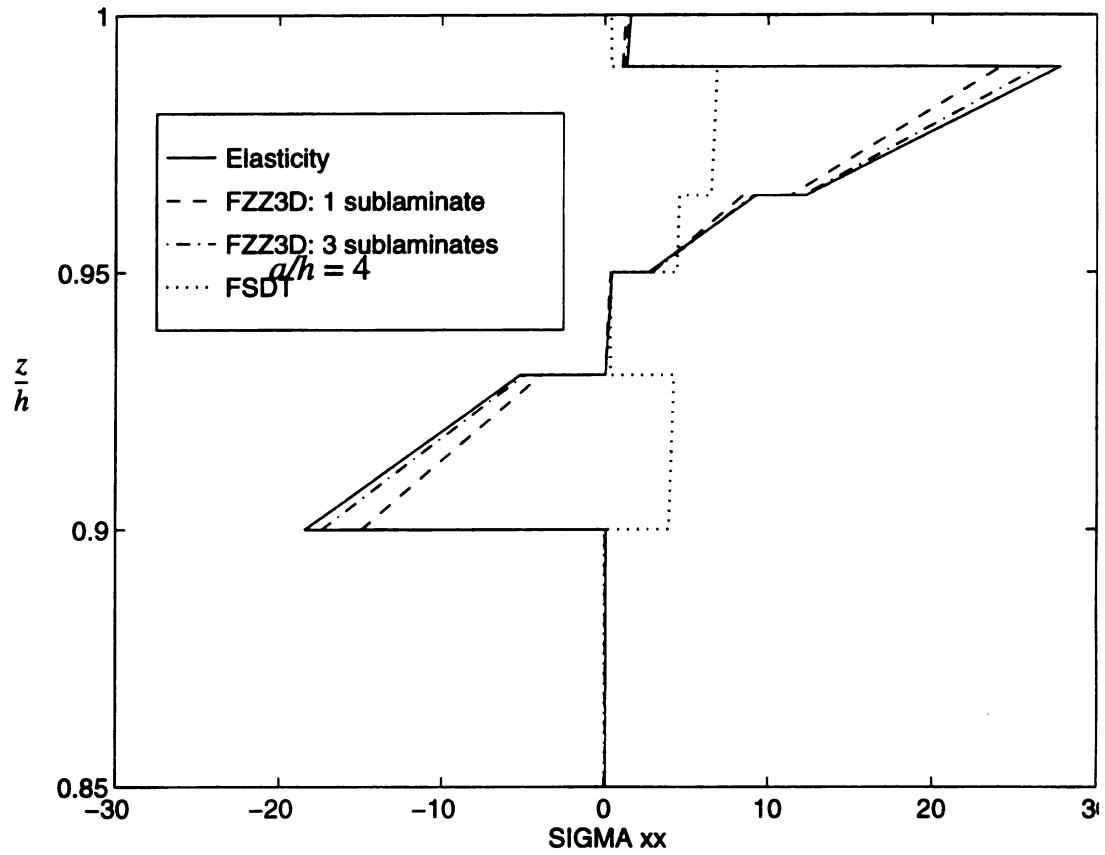
**Figure 27. Transverse normal strain,  $\epsilon_{zz}$ , versus normalized thickness coordinate at the center of the plate (CAV Panel,  $a/h=4$ ).**



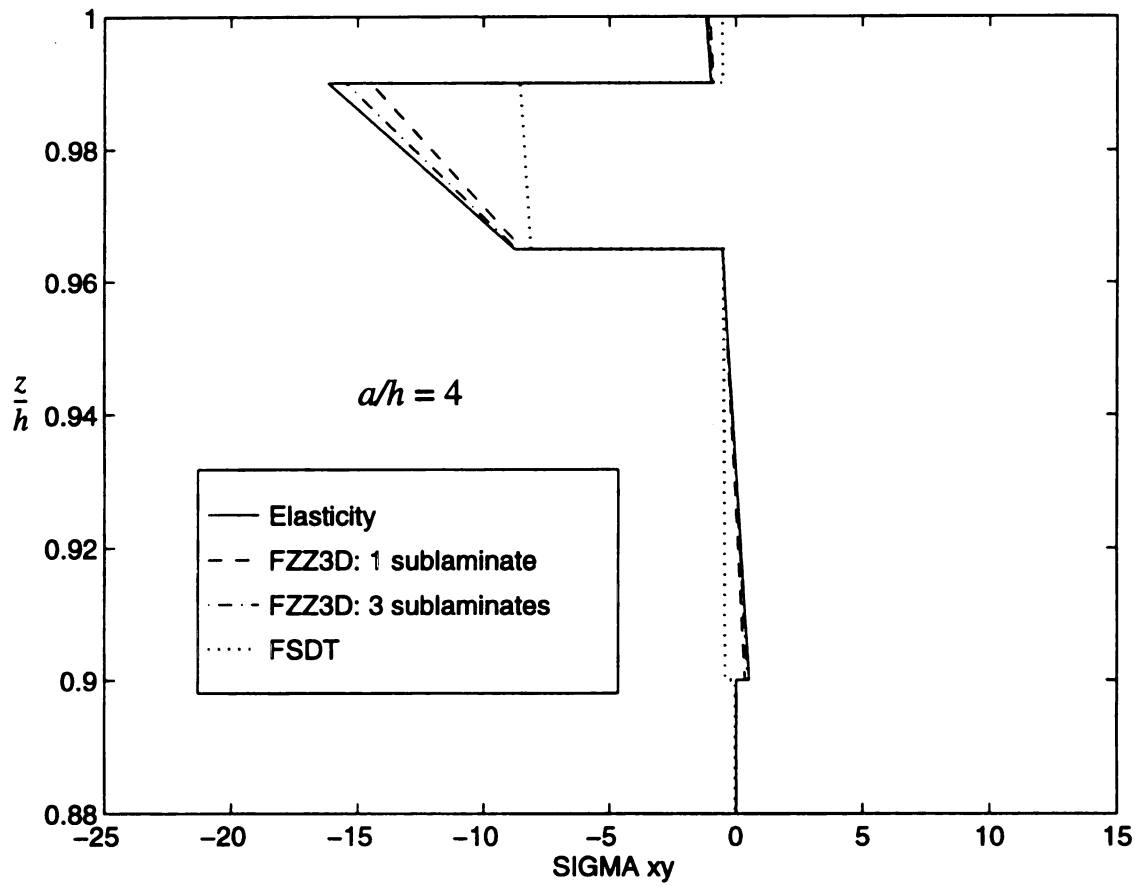
**Figure 28. Transverse normal stress,  $\sigma_{zz}$ , versus normalized thickness coordinate at the center of the plate (CAV Panel,  $a/h=4$ )**



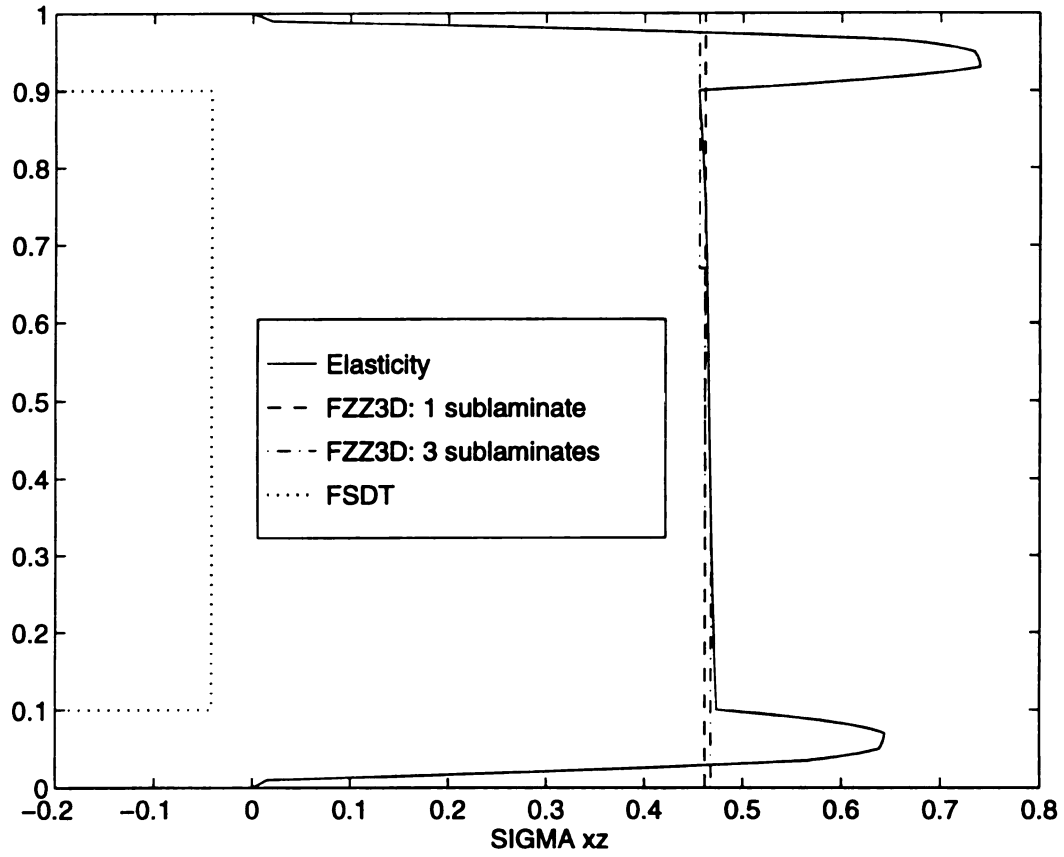
**Figure 29. In-plane displacement,  $u$ , versus normalized thickness coordinate (Sandwich Panel,  $a/h=4$ ).**



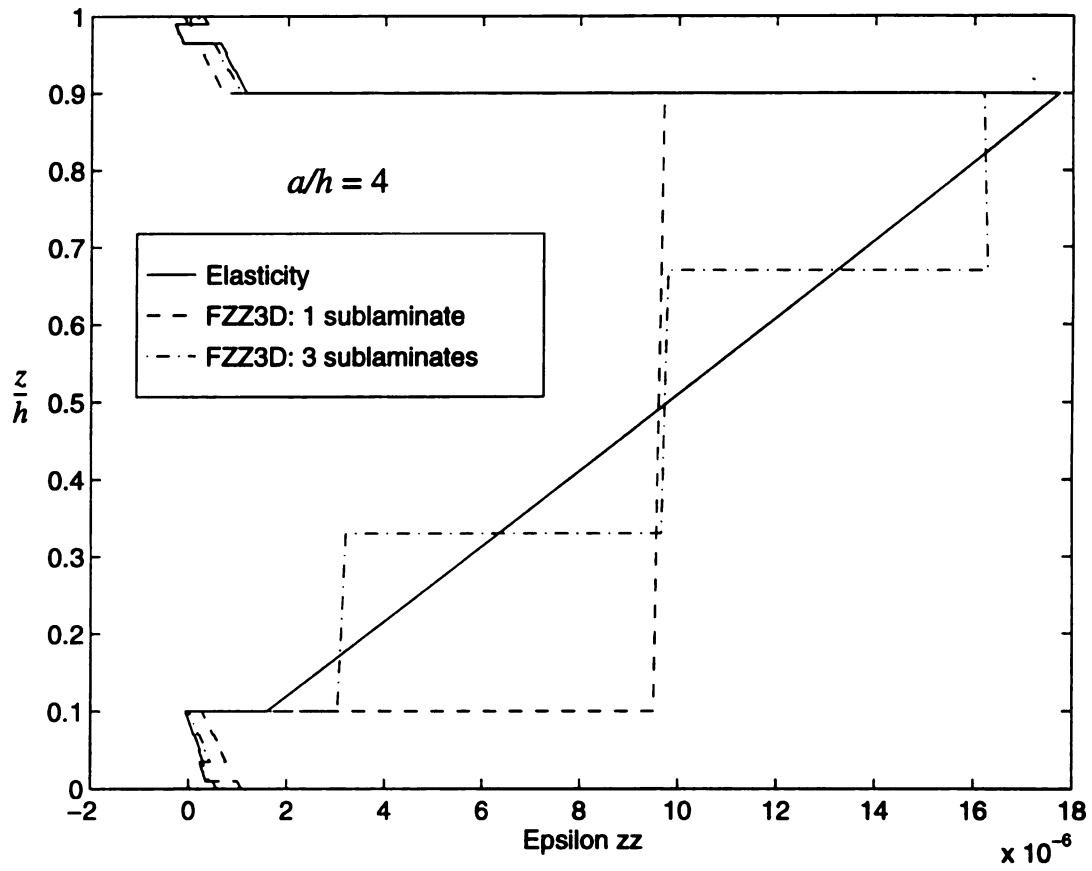
**Figure 30. Bending stress,  $\sigma_{xx}$ , in face sheets versus normalized thickness coordinate (Sandwich plate,  $a/h=4$ ).**



**Figure 31. In-plane shear stress in the face sheets,  $\sigma_{xy}$ , versus normalized thickness coordinate at  $x, y = 0$ , (CAV Panel,  $a/h=4$ ).**

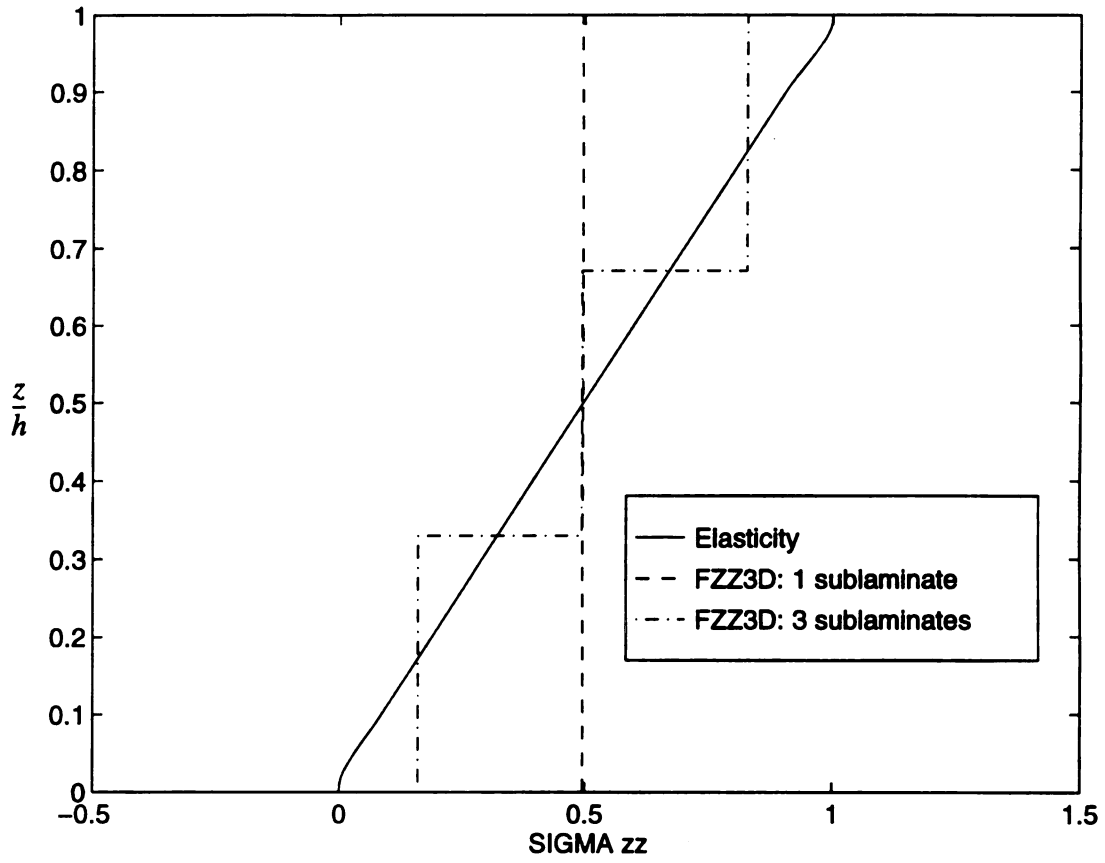


**Figure 32. Transverse shear stress,  $\sigma_{xz}$ , versus normalized thickness coordinate at  $x = 0$ ,  $y = 2/a$  (Sandwich Panel,  $a/h=4$ ).**



**Figure 33. Transverse normal strain,  $\epsilon_{zz}$ , versus normalized thickness coordinate at the center of the plate (Sandwich Panel,  $a/h=4$ ).**





**Figure 34. Transverse normal strain,  $\sigma_{zz}$ , versus normalized thickness coordinate at the center of the plate (Sandwich Panel,  $a/h=4$ ).**

# **CHAPTER IV**

## **PLATE ELEMENT**

### **BASED ON CUBIC ZIG-ZAG THEORY**

#### **4.1 Introduction**

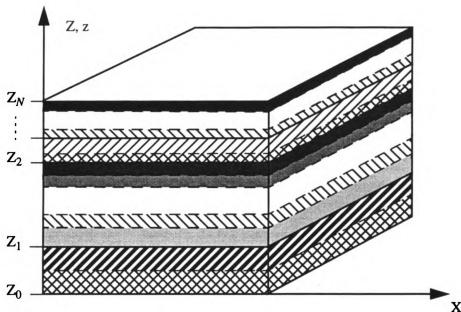
Composite materials are being considered for use in the front end structures of vehicles to help reduce overall vehicle mass and, thus, improve fuel efficiency. Acceptance of composite materials in structural members will depend on their ability to do crash energy management. Numerical simulations can greatly aid in the design of these critical structures and reduce the number of crash tests. When the laminated composites undergo crash, the fracture of composite materials results from four possible major reasons: (i) fiber fracture, (ii) matrix cracking, (iii) fiber/matrix interface debonding, and (iv) fiber buckling or kinking. These modes of failure don't occur simultaneously in all constituents but initiate as micro-cracks at the constituent level, and unite or propagate as external loading (for example, impact force, heat and moisture) is increased. As the micro-crack propagates parallel to the fiber direction through resin rich region, delamination occurs. The simplest way to predict delamination is to evaluate the interlaminar shear stress/strain states in each layer, neglecting the stress/strains in the constituent level. The discretization in the thickness direction gives a good prediction of the interlaminar shear strains, but makes the com-

putational cost expensive (especially for dynamic analysis). For this purpose, the development of more accurate theories and elements to predict interlaminar shear stress/strain is indispensable.

A new finite element, which is based on laminated plate theory with cubic zig-zag approximations, has been developed to model the relevant mechanics that occur in composite materials during crash events. In the plate theory, the in-plane displacement fields in a laminate are assumed to be piecewise cubic functions and vary in a zig-zag fashion through the thickness of the laminate. The zig-zag functions are obtained by satisfying the continuity of transverse shear stresses at layer interfaces. This in-plane displacement field assumption accounts for discrete layer effects without increasing the number of degrees of freedom as the number of layers is increased. The transverse normal strain predictions are improved by assuming a constant variation of transverse normal stress through the thickness in a laminate. The finite element is developed with the topology of an eight-noded brick. Each node has five engineering degrees of freedom, three translations and two rotations. Thus, this element can be conveniently implemented into general purpose finite element codes. Numerical performance of the current element is investigated by simulating a plate with a damaged layer.

## 4.2 Formulation of Plate Theory

A laminate is assumed to be composed of  $N$  perfectly bonded layers. Each layer in the laminate may have independent thickness and material properties. The global  $Z$ -axis is taken perpendicular to the inplane  $X$ ,  $Y$  coordinate axes and has its origin at the bottom of the laminate (see Figure 35).



**Figure 35. Schematic of a laminate and layer divisions in the cubic zig-zag laminated plate theory.**

To develop the element, three dimensional kinematic assumption were utilized, and the three dimensional constitutive equation in the  $k$ th layer can be written as:

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \sigma_{zz}^{(k)} \\ \sigma_{yz}^{(k)} \\ \sigma_{zx}^{(k)} \\ \sigma_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(k)} & C_{12}^{(k)} & C_{13}^{(k)} & 0 & 0 & C_{16}^{(k)} \\ C_{12}^{(k)} & C_{22}^{(k)} & C_{23}^{(k)} & 0 & 0 & C_{26}^{(k)} \\ C_{13}^{(k)} & C_{23}^{(k)} & C_{33}^{(k)} & 0 & 0 & C_{36}^{(k)} \\ 0 & 0 & 0 & C_{44}^{(k)} & C_{45}^{(k)} & 0 \\ 0 & 0 & 0 & C_{45}^{(k)} & C_{55}^{(k)} & 0 \\ C_{16}^{(k)} & C_{26}^{(k)} & C_{36}^{(k)} & 0 & 0 & C_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \epsilon_{zz}^{(k)} \\ \gamma_{yz}^{(k)} \\ \gamma_{zx}^{(k)} \\ \gamma_{xy}^{(k)} \end{Bmatrix} \quad (4.2.1)$$

For most practical laminated composites, the 13 coefficients of the material stiffness matrix are obtained from 9 independent material constants and a transformation law. The

stress - strain relationship is defined as:

$$\begin{aligned} \epsilon_{xx}^{(k)} &= \frac{\partial u_x^{(k)}}{\partial x}, & \epsilon_{yy}^{(k)} &= \frac{\partial u_y^{(k)}}{\partial y}, & \epsilon_{zz}^{(k)} &= \frac{\partial u_z^{(k)}}{\partial z} \\ \gamma_{yz}^{(k)} &= \frac{\partial u_z^{(k)}}{\partial y} + \frac{\partial u_y^{(k)}}{\partial z}, & \gamma_{xz}^{(k)} &= \frac{\partial u_z^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial z}, & \gamma_{xy}^{(k)} &= \frac{\partial u_y^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial y} \end{aligned} \quad (4.2.2)$$

The displacement fields are initially assumed in the following form:

$$\begin{aligned} u_x^{(k)}(x, y, z, t) &= u_b + z\psi_x + z^2\alpha_x + z^3\beta_x + \sum_{i=1}^{k-1} (z - z_i)\xi_i \\ u_y^{(k)}(x, y, z, t) &= v_b + z\psi_y + z^2\alpha_y + z^3\beta_y + \sum_{i=1}^{k-1} (z - z_i)\eta_i \\ u_z^{(k)}(x, y, z, t) &= w_b \left(1 - \frac{z}{h}\right) + w_t \left(\frac{z}{h}\right) \end{aligned} \quad (4.2.3)$$

where  $h$  is the thickness of the laminate,  $u_b$  and  $v_b$  are the axial displacements in  $x$  and  $y$  directions, respectively, at  $z = 0$ ,  $\psi_x$  and  $\psi_y$  are rotations of the normal at  $z = 0$ , and  $w_b$  and  $w_t$  are the transverse deflections of the bottom and top surfaces, respectively, of the laminate.  $\alpha_x, \alpha_y, \beta_x$  and  $\beta_y$  represent the high order rotations of the normal at  $z = 0$ , and account for warping of the normal, which occurs when the material properties in each layer are significantly different. Comparing Eq. (4.2.3) with Eq. (3.3.1), it can be observed that this plate theory is reduced to the first order zig-zag plate theory by setting these high order rotations to zero. It is assumed in Eq. (4.2.3) that a piecewise linear zig-zag function is superimposed on cubic displacement function, so that  $u_x^{(k)}$  and  $u_y^{(k)}$  vary in a piecewise cubic fashion through the thickness of a laminate, and  $u_z^{(k)}$  varies linearly through the thickness. The resulting transverse shear strains vary a quadratic fashion through thick-

ness, so interlaminar shear stresses can be predicted accurately. The parameters  $\xi_i$  and  $\eta_i$  of Eq. (4.2.3) are removed by enforcing the continuity of shear stress at each interface [33, 34]. At the  $k$ th interface in the laminate, the shear stress continuity condition can be given by:

$$\sigma_{yz}^{(k)} = \sigma_{yz}^{(k+1)} \Big|_{z=z_k} \quad \sigma_{zx}^{(k)} = \sigma_{zx}^{(k+1)} \Big|_{z=z_k} \quad (4.2.4)$$

The transverse shear stress - strain relationship can be obtained from the constitutive equation (4.2.1) and is defined as:

$$\begin{aligned} \sigma_{yz}^{(k)} &= C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)} \\ \sigma_{xz}^{(k)} &= C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)} \end{aligned} \quad (4.2.5)$$

By differentiating displacement fields with respect to  $z$ ,  $x$  or  $y$  properly, the transverse shear strains are obtained, and the relationship between the transverse shear strains in the  $k$ th layer and  $(k+1)$ th layer are derived as:

$$\begin{aligned} \gamma_{zx}^{(k+1)} &= \psi_x + 2z\alpha_x + 3z^2\beta_x + \sum_{i=1}^k \xi_i + \frac{\partial w}{\partial x} b \left(1 - \frac{z}{h}\right) + \frac{\partial w}{\partial x} \left(\frac{z}{h}\right) \\ &= \gamma_{zx}^{(k)} + \xi_k \\ \gamma_{yz}^{(k+1)} &= \psi_y + 2z\alpha_y + 3z^2\beta_y + \sum_{i=1}^k \eta_i + \frac{\partial w}{\partial y} b \left(1 - \frac{z}{h}\right) + \frac{\partial w}{\partial y} \left(\frac{z}{h}\right) \\ &= \gamma_{yz}^{(k)} + \eta_k \end{aligned} \quad (4.2.6)$$

The transverse shear stress continuity condition is rewritten as:

$$\begin{aligned}
(C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=z_k} &= \{ C_{44}^{(k+1)} (\gamma_{yz}^{(k)} + \eta_k) + C_{45}^{(k)} (\gamma_{xz}^{(k)} + \xi_k) \} \Big|_{z=z_k} \\
(C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=z_k} &= \{ C_{45}^{(k+1)} (\gamma_{yz}^{(k)} + \eta_k) + C_{55}^{(k)} (\gamma_{xz}^{(k)} + \xi_k) \} \Big|_{z=z_k}
\end{aligned} \tag{4.2.7}$$

The following equation is obtained by solving (4.2.7) for  $\eta_k$  and  $\xi_k$ :

$$\begin{aligned}
\eta_k &= \hat{a} \gamma_{yz}^{(k)} + \hat{b} \gamma_{xz}^{(k)} \\
\xi_k &= \hat{c} \gamma_{yz}^{(k)} + \hat{d} \gamma_{xz}^{(k)}
\end{aligned} \tag{4.2.8}$$

where

$$\begin{aligned}
\hat{a}_k &= \frac{1}{\Delta_{k+1}} (C_{55}^{(k+1)} C_{44}^{(k)} - C_{45}^{(k+1)} C_{45}^{(k)}) - 1 \\
\hat{b}_k &= \frac{1}{\Delta_{k+1}} (C_{55}^{(k+1)} C_{45}^{(k)} - C_{45}^{(k+1)} C_{55}^{(k)}) \\
\hat{c}_k &= \frac{1}{\Delta_{k+1}} (C_{44}^{(k+1)} C_{45}^{(k)} - C_{45}^{(k+1)} C_{44}^{(k)}) \\
\hat{d}_k &= \frac{1}{\Delta_{k+1}} (C_{44}^{(k+1)} C_{55}^{(k)} - C_{45}^{(k+1)} C_{45}^{(k)}) - 1 \\
\Delta_{k+1} &= C_{44}^{(k+1)} C_{55}^{(k)} - C_{45}^{(k+1)^2}
\end{aligned} \tag{4.2.9}$$

Substitution of Eq. (4.2.6) into Eq. (4.2.7) gives recursive equations about  $\xi_i$  and  $\eta_i$ , and

solving the recursive equations for  $\xi_i$  and  $\eta_i$  renders:

$$\begin{aligned}
\xi_i &= c_{1i} \left( \psi_y + \frac{\partial w_b}{\partial y} \right) + c_{2i} \alpha_y + c_{3i} \beta_y + c_{4i} \left( \frac{\partial w_t}{\partial y} - \frac{\partial w_b}{\partial y} \right) \\
&\quad + d_{1i} \left( \psi_x + \frac{\partial w_b}{\partial x} \right) + d_{2i} \alpha_x + d_{3i} \beta_x + d_{4i} \left( \frac{\partial w_t}{\partial x} - \frac{\partial w_b}{\partial x} \right) \\
\eta_i &= a_{1i} \left( \psi_y + \frac{\partial w_b}{\partial y} \right) + a_{2i} \alpha_y + a_{3i} \beta_y + a_{4i} \left( \frac{\partial w_t}{\partial y} - \frac{\partial w_b}{\partial y} \right) \\
&\quad + b_{1i} \left( \psi_x + \frac{\partial w_b}{\partial x} \right) + b_{2i} \alpha_x + b_{3i} \beta_x + b_{4i} \left( \frac{\partial w_t}{\partial x} - \frac{\partial w_b}{\partial x} \right)
\end{aligned} \tag{4.2.10}$$

where

$$\begin{aligned}
 a_{ik} &= \hat{a}_k \left( f_{ik} + \sum_{j=1}^{k-1} a_j \right) + \hat{b}_k \sum_{j=1}^{k-1} c_{ij} \\
 b_{ik} &= \hat{b}_k \left( f_{ik} + \sum_{j=1}^{k-1} d_j \right) + \hat{a}_k \sum_{j=1}^{k-1} b_{ij} \\
 c_{ik} &= \hat{c}_k \left( f_{ik} + \sum_{j=1}^{k-1} a_j \right) + \hat{d}_k \sum_{j=1}^{k-1} c_{ij} \\
 d_{ik} &= \hat{d}_k \left( f_{ik} + \sum_{j=1}^{k-1} d_j \right) + \hat{c}_k \sum_{j=1}^{k-1} b_{ij}, \quad (i = 1, 2, 3, 4) \\
 f_{ik} &= i \cdot (z_k)^{i-1}
 \end{aligned} \tag{4.2.11}$$

From Eqs. (4.2.9)-(4.2.11) it can be observed that  $\xi_i$  and  $\eta_i$  depend on the ratios of shear properties between adjacent layers and the shear deformation in the laminate. If the difference in shear stiffness quantities between two adjacent layers or the shear deformation in the laminate is small, then discrete-layer effects will also be small.

Convenient finite element models (i.e. regenerated elements) can be developed by writing the rotational variables  $\psi_x$  and  $\psi_y$  in the displacement field in terms of the translational variables  $u_t$  and  $v_t$ , the inplane translations at the top surface of the laminate [41,42]. Thus, in-plane displacement is described more conveniently by the translation at two points using surface quantities. Moreover, since the existence of the first derivatives of transverse deflections,

$$\frac{\partial w_b}{\partial x}, \frac{\partial w_t}{\partial x}, \frac{\partial w_b}{\partial y} \text{ and } \frac{\partial w_t}{\partial y} \tag{4.2.12}$$

in the displacement fields result in the presence of their second derivatives in the strain



energy functional,  $C^1$  continuity of  $w_b$  and  $w_t$  is required. In order to facilitate the development of more convenient finite element models, it is desirable to alleviate such a requirement by introducing new rotational degrees of freedom as follows:

$$\frac{\partial w_b}{\partial x} = -\theta_{yb}, \frac{\partial w_t}{\partial x} = -\theta_{yt}, \frac{\partial w_b}{\partial y} = \theta_{xb}, \frac{\partial w_t}{\partial y} = \theta_{xt} \quad (4.2.13)$$

These relationships will be imposed as constraints in the energy functional. Furthermore,  $\alpha_x$ ,  $\alpha_y$ ,  $\beta_x$  and  $\beta_y$  can be eliminated by using the free shear traction conditions at the top and bottom of the laminate as follows [15]:

$$\begin{aligned} \tau_{yz}^{(1)} \Big|_{z=0} &= (C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=0} = 0 \\ \tau_{xz}^{(1)} \Big|_{z=0} &= (C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=0} = 0 \\ \tau_{yz}^{(N)} \Big|_{z=h} &= (C_{44}^{(k)} \gamma_{yz}^{(k)} + C_{45}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=h} = 0 \\ \tau_{xz}^{(N)} \Big|_{z=h} &= (C_{45}^{(k)} \gamma_{yz}^{(k)} + C_{55}^{(k)} \gamma_{xz}^{(k)}) \Big|_{z=h} = 0 \end{aligned} \quad (4.2.14)$$

Substituting the modified displacement fields into Eq. (4.2.14), solving for  $\alpha_x$ ,  $\alpha_y$ ,  $\beta_x$  and  $\beta_y$ , we can attain final modified displacement fields. Here, we need to pay attention to the fact that since the free shear traction conditions were utilized, the discretization through thickness is impossible in finite element model. Therefore, the displacement fields can be written as follows:

$$u_x^{(k)} = \bar{u}_{\alpha\beta} \Phi_{\alpha\beta}^{(k)}, \quad u_y^{(k)} = \bar{u}_{\alpha\beta} \Psi_{\alpha\beta}^{(k)}, \quad u_z^{(k)} = w_\beta \Omega_\beta, \quad \alpha = 1 \dots 4; \beta = 1, 2 \quad (4.2.15)$$

where the index  $\beta$  is used to denote the top and bottom of the laminate with 1: bottom and 2: top, and  $\Phi_{\alpha\beta}^{(k)}$ ,  $\Psi_{\alpha\beta}^{(k)}$  and  $\Omega_\beta$  are shape functions in the thickness direction, which are

explained in Appendix B. Unless noted otherwise, summation on repeated indices is implied. The variables  $\bar{u}_{\alpha\beta}$  are:

$$\bar{u}_{1\beta} = u_{\beta}, \bar{u}_{2\beta} = v_{\beta}, \bar{u}_{3\beta} = \theta_{x\beta}, \bar{u}_{4\beta} = \theta_{y\beta} \quad (4.2.16)$$

The transverse normal strain, which is constant in the laminate due to the assumption of displacement for thickness direction, can be improved by instead assuming a constant transverse normal stress  $\bar{\sigma}_{zz}$  through-the-thickness of the laminate.  $\bar{\sigma}_{zz}$  can be determined using Reissner's mixed variational principle [81]. From Eq. (4.2.1), the transverse normal strain is obtained as:

$$\bar{\epsilon}_{zz}^{(k)} = -\frac{C_{13}^{(k)}}{C_{33}^{(k)}}\epsilon_{xx}^{(k)} - \frac{C_{23}^{(k)}}{C_{33}^{(k)}}\epsilon_{yy}^{(k)} - \frac{C_{36}^{(k)}}{C_{33}^{(k)}}\gamma_{xy}^{(k)} + \frac{1}{C_{33}^{(k)}}\bar{\sigma}_{zz} \quad (4.2.17)$$

$\bar{\sigma}_{zz}$  is then determined by analytically solving the following relation:

$$0 = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} (\bar{\epsilon}_{zz}^k - \epsilon_{zz}^k) dz \quad (4.2.18)$$

Substituting the strain - displacement relations and Eq. (4.2.17) into Eq. (4.2.18), the constant transverse normal stress  $\bar{\sigma}_{zz}$  is found to be:

$$\begin{aligned}
\bar{\sigma}_{zz} = & \frac{\partial u_b}{\partial x}(P_1 + S_6) + \frac{\partial u_t}{\partial x}(P_2 - S_6) + \frac{\partial v_b}{\partial x}(P_3 + S_7) + \frac{\partial v_t}{\partial x}(-P_3 + S_8) + \frac{\partial \theta^{yb}}{\partial x}(P_4 + S_9) \\
& + \frac{\partial \theta^{yt}}{\partial x}(P_4 + S_9) + \frac{\partial \theta^{xb}}{\partial x}(P_5 + S_{10}) + \frac{\partial \theta^{xt}}{\partial x}(P_5 + S_{10}) + \frac{\partial u_b}{\partial y}(Q_1 + S_1) + \frac{\partial u_t}{\partial y}(-Q_1 + S_2) \\
& + \frac{\partial v_b}{\partial y}(Q_2 + S_3) + \frac{\partial v_t}{\partial y}(Q_3 - S_3) + \frac{\partial \theta^{yb}}{\partial y}(Q_4 + S_4) + \frac{\partial \theta^{yt}}{\partial y}(Q_4 + S_4) + \frac{\partial \theta^{xb}}{\partial y}(Q_5 + S_5) \\
& + \frac{\partial \theta^{xt}}{\partial y}(Q_5 + S_5) + w_b(-T_1) + w_t(T_1)
\end{aligned} \tag{4.2.19}$$

where the coefficients are defined in the Appendix B. The newly defined transverse normal strain can now be written as:

$$\varepsilon_{zz}^{(k)} = \bar{u}_{\alpha\beta, x} \lambda_{x\alpha\beta}^{(k)} + \bar{u}_{\alpha\beta, y} \lambda_{y\alpha\beta}^{(k)} + w_\beta \chi_\beta \quad \alpha = 1 \dots 4; \beta = 1, 2 \tag{4.2.20}$$

where the functions  $\lambda_{x\alpha\beta}^{(k)}$ ,  $\lambda_{y\alpha\beta}^{(k)}$  and  $\chi_\beta$  are defined in the Appendix B.

The current plate theory has the following features:

- 1) A piecewise linear zig-zag function is superimposed on a cubic in-plane displacement function, thus,  $u_x^{(k)}$  and  $u_y^{(k)}$  vary in a piecewise cubic fashion through the thickness of a laminate, while  $u_z^{(k)}$  varies linearly through the thickness.
- 2) Discrete-layer effects are captured within a laminate by allowing a piecewise (layer-wise) continuous through-the-thickness variation of the in-plane displacements.
- 3) The transverse shear strains vary in a piecewise quadratic fashion through thickness, and interlaminar shear stresses can be predicted accurately.
- 4) Continuity of transverse shear stresses through the thickness of a laminate is satisfied.
- 5) By introducing rotational degrees of freedom, the  $C^0$  plate theory was developed.
- 6) Because of utilizing homogeneous shear traction boundary conditions, refinement through the thickness can not be accomplished.

6) Transverse normal strain distribution through the thickness of a laminate was improved by assuming constant transverse normal stress.

7) In the displacement fields, only five engineering d.o.f. (three translations + two rotations) are present.

The equations of motion, the essential and natural boundary conditions, as well as the displacement-based finite element model can be developed from Hamilton's principle. The energy functional is modified here to include the imposition of the constraints Eq. (4.2.13) via the penalty method. Hamilton's principle can be defined as follows:

$$\delta \int_{t_0}^{t_1} \{U - K - W + P\} dt = 0 \quad (4.2.21)$$

$U$  is the internal strain energy:

$$U = \sum_{k=1}^{N_m} \int_{V_k} \frac{1}{2} [\sigma_{xx}^{(k)} \epsilon_{xx}^{(k)} + \sigma_{yy}^{(k)} \epsilon_{yy}^{(k)} + \sigma_{zz}^{(k)} \epsilon_{zz}^{(k)} + \sigma_{yz}^{(k)} \gamma_{yz}^{(k)} + \sigma_{zx}^{(k)} \gamma_{zx}^{(k)} + \sigma_{xy}^{(k)} \gamma_{xy}^{(k)}] dV_k \quad (4.2.22)$$

where  $V_k$  is the volume of the  $k$ -th layer.  $K$  is the kinetic energy:

$$K = \sum_{k=1}^N \int_{V_k} \frac{1}{2} \rho^{(k)} [\dot{u}_x^{(k)^2} + \dot{u}_y^{(k)^2} + \dot{u}_z^{(k)^2}] dV_k \quad (4.2.23)$$

where  $\rho^{(k)}$  is the density of the  $k$ -th layer and a superposed dot indicates differentiation with respect to time.  $W$  is the work of external forces:

$$W = \int_{\Omega} w_{\alpha} P_{\alpha} d\Omega + \int_{\Gamma} t_i u_i d\Gamma \quad (4.2.24)$$

$$t_i = \sigma_{ij} n_j; \alpha = 1, 2; i = 1, 2$$

$P$  is the penalty constraint:

$$P = \frac{\gamma}{2} \int_{\Omega} \left[ \sum_{\beta=1}^2 \left\{ \left( u_{3\beta} - \frac{\partial w_{\beta}}{\partial y} \right)^2 + \left( u_{4\beta} + \frac{\partial w_{\beta}}{\partial x} \right)^2 \right\} \right] d\Omega \quad (4.2.25)$$

where  $\gamma$  is the penalty parameter. As  $\gamma$  is assigned successively larger values, the constraints of Eq. (4.2.13) are satisfied more exactly in the least squares sense.

Substituting Eqs. (4.2.2), (4.2.15), (4.2.20), (4.2.22), (4.2.23), (4.2.24), and (4.2.25) into Eq. (4.2.21), taking the variation of Eq. (4.2.21), and integrating by parts, Eq. (4.2.21) can be rewritten as:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega} \left( \left\{ \delta \bar{u}_{\alpha\beta} - \frac{\partial N_{x\alpha\beta}}{\partial x} - \frac{\partial N_{y\alpha\beta}}{\partial y} - \frac{\partial N_{z\alpha\beta}}{\partial z} - \frac{\partial Q_{z\alpha\beta}}{\partial y} + Q_{y\alpha\beta} + Q_{x\alpha\beta} - \frac{\partial R_{xy\alpha\beta}}{\partial y} - \frac{\partial R_{yx\alpha\beta}}{\partial x} \right. \right. \\ & \quad \left. \left. - \gamma \left( \bar{u}_{\alpha\beta} - \frac{\partial w_{\beta}}{\partial y} \right) \delta_{3\alpha} - \gamma \left( \bar{u}_{\alpha\beta} + \frac{\partial w_{\beta}}{\partial x} \right) \delta_{4\alpha} + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^u + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^v \right\} \right. \\ & \quad \left. + \delta w_{\beta} \left\{ R_{z\beta} - \frac{\partial R_{y\beta}}{\partial y} - \frac{\partial R_{x\beta}}{\partial x} - P_{\beta} - \gamma \left( \frac{\partial \bar{u}_{3\beta}}{\partial y} - \frac{\partial^2 w_{\beta}}{\partial y^2} \right) + \gamma \left( \frac{\partial \bar{u}_{4\beta}}{\partial x} + \frac{\partial^2 w_{\beta}}{\partial x^2} \right) + \frac{\partial^2 w_{\beta}}{\partial t^2} I_{\beta p}^w \right\} \right) d\Omega \\ & \quad + \oint_s \left( \delta \bar{u}_{\alpha\beta} \left\{ N_{x\alpha\beta} n_x + N_{y\alpha\beta} n_y + N_{z\alpha\beta} n_z + Q_{z\alpha\beta} n_y + R_{xy\alpha\beta} n_y + R_{yx\alpha\beta} n_x \right\} \right. \\ & \quad \left. + \delta w_{\beta} \left\{ R_{y\beta} n_y + R_{x\beta} n_x + \gamma n_y \left( \bar{u}_{3\beta} - \frac{\partial w_{\beta}}{\partial y} \right) - \gamma n_x \left( \bar{u}_{4\beta} + \frac{\partial w_{\beta}}{\partial x} \right) \right\} \right) ds \\ & \quad + \int \sum_{s=k=1}^N \int_{z_{k-1}}^{z_k} \left( \sigma_{1j}^{(k)} n_j \delta \bar{u}_{\alpha\beta} \Phi_{\alpha\beta}^{(k)} + \sigma_{2j}^{(k)} n_j \delta \bar{u}_{\alpha\beta} \Psi_{\alpha\beta}^{(k)} \right) dz ds \Bigg] dt = 0 \\ & \quad (\alpha, r = 1 \dots 4; \beta, p = 1, 2) \end{aligned} \quad (4.2.26)$$

where  $N_{x\alpha\beta}$ ,  $N_{y\alpha\beta}$ ,  $N_{z\alpha\beta}$ ,  $Q_{z\alpha\beta}$ ,  $Q_{y\alpha\beta}$ ,  $Q_{x\alpha\beta}$ ,  $R_{xy\alpha\beta}$ ,  $R_{yx\alpha\beta}$ ,  $R_{z\beta}$ ,  $R_{y\beta}$ ,  $R_{x\beta}$  are the stress resultants shown in the Appendix B,  $n_x$ ,  $n_y$  are the directional cosines,

$I_{\alpha\beta rp}^u, I_{\alpha\beta rp}^v, I_{\beta p}^w$  are the plate inertias defined in the Appendix B, and  $\delta_{\alpha\beta}$  is the Kronecker delta. Considering that variations of the generalized displacement degrees of freedom are independent for  $t_0 < t < t_1$  and zero for  $t = t_0, t_1$ , the equations of motion are derived as:

$$\begin{aligned} \delta \bar{u}_{\alpha\beta}: \quad & \frac{\partial N_{x\alpha\beta}}{\partial x} + \frac{\partial N_{y\alpha\beta}}{\partial y} + \frac{\partial N_{z\alpha\beta}}{\partial x} + \frac{\partial Q_{z\alpha\beta}}{\partial y} - Q_{y\alpha\beta} - Q_{x\alpha\beta} + \frac{\partial R_{xy\alpha\beta}}{\partial y} + \frac{\partial R_{yx\alpha\beta}}{\partial x} \\ & + \gamma \left( \bar{u}_{\alpha\beta} - \frac{\partial w_{\beta}}{\partial y} \right) \delta_{3l} + \gamma \left( \bar{u}_{\alpha\beta} + \frac{\partial w_{\beta}}{\partial x} \right) \delta_{4l} = \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^u + \frac{\partial^2 \bar{u}_{rp}}{\partial t^2} I_{\alpha\beta rp}^v \quad (4.2.27) \\ \delta w_{\beta}: \quad & \frac{\partial R_{y\beta}}{\partial y} + \frac{\partial R_{x\beta}}{\partial x} - R_{z\beta} + P_{\beta} + \gamma \left( \frac{\partial \bar{u}_{3\beta}}{\partial y} - \frac{\partial^2 w_{\beta}}{\partial y^2} \right) - \gamma \left( \frac{\partial \bar{u}_{4\beta}}{\partial x} + \frac{\partial^2 w_{\beta}}{\partial x^2} \right) = \frac{\partial^2 w_p}{\partial t^2} I_{\beta p}^w \end{aligned}$$

The corresponding boundary conditions are:

Essential B.C

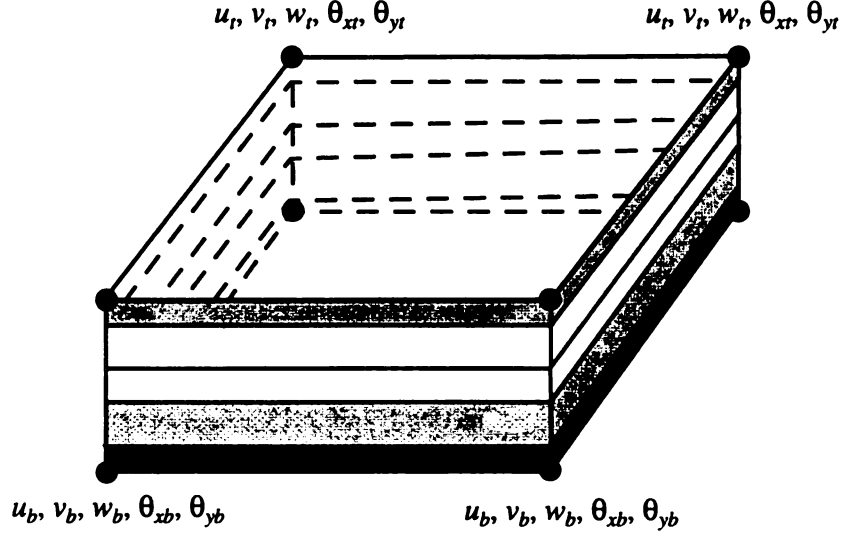
Natural B.C.

$$\begin{aligned} \bar{u}_{\alpha\beta} \quad & (N_{x\alpha\beta} n_x + N_{y\alpha\beta} n_y + N_{z\alpha\beta} n_x + Q_{z\alpha\beta} n_y + R_{xy\alpha\beta} n_y + R_{yx\alpha\beta} n_x) \\ & + \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \left( \sigma_{1j}^{(k)} n_j \Phi_{\alpha\beta}^{(k)} + \sigma_{2j}^{(k)} n_j \Psi_{\alpha\beta}^{(k)} \right) dz \quad (4.2.28) \\ w_{\beta} \quad & R_{y\beta} n_y + R_{x\beta} n_x + \gamma n_y \left( \bar{u}_{3\beta} - \frac{\partial w_{\beta}}{\partial y} \right) - \gamma n_x \left( \bar{u}_{4\beta} + \frac{\partial w_{\beta}}{\partial x} \right) \end{aligned}$$

### 4.3 Finite Element Formulation

Because the degrees of freedom represent quantities at the surfaces of the laminate, to take a major advantage of the present formulation, the finite element model was cast in the form of an eight-noded element with five degrees of freedom per node (see Figure 36). From Eq. (4.2.3), it can be seen that the primary variables are functions of  $x$ ,  $y$  and  $t$  only,

thus we need to consider only two dimensional shape functions for the finite element model.



**Figure 36. Element topology and nodal degrees of freedom.**

Inplane displacement and rotation degrees of freedom are approximated by the bilinear Lagrange interpolation functions:

$$\begin{aligned} \bar{u}_{1\beta} &= \sum_{i=1}^4 u_{\beta i} P_i & \bar{u}_{2\beta} &= \sum_{i=1}^4 v_{\beta i} P_i \\ \bar{u}_{3\beta} &= \sum_{i=1}^4 \theta_{x\beta i} P_i & \bar{u}_{4\beta} &= \sum_{i=1}^4 \theta_{y\beta i} P_i \end{aligned} \quad (4.3.1)$$

where

$$P_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta) \quad (4.3.2)$$

and  $\xi, \eta$  are the element natural (local) coordinates. For *transverse deflection* degrees of freedom, an interdependent interpolation concept similar to that developed by Tessler and Hughes [75] is utilized. The interdependent interpolations functions are:

$$w_\beta = \sum_{i=1}^4 (P_i w_{\beta i} + N_{xi} \theta_{x\beta i} + N_{yi} \theta_{y\beta i}) \quad (4.3.3)$$

where  $N_{xi}$  and  $N_{yi}$  are:

$$\begin{aligned} N_{x1} &= -\frac{1}{8}(y_{12}N_5 - y_{41}N_8), & N_{y1} &= \frac{1}{8}(x_{12}N_5 - x_{41}N_8) \\ N_{x2} &= -\frac{1}{8}(y_{23}N_6 - y_{12}N_5), & N_{y2} &= \frac{1}{8}(x_{23}N_6 - x_{12}N_5) \\ N_{x3} &= -\frac{1}{8}(y_{34}N_7 - y_{23}N_6), & N_{y3} &= \frac{1}{8}(x_{34}N_7 - x_{23}N_6) \\ N_{x4} &= -\frac{1}{8}(y_{41}N_8 - y_{34}N_7), & N_{y4} &= \frac{1}{8}(x_{41}N_8 - x_{34}N_7) \end{aligned} \quad x_{ij} = x_i - x_j; \quad y_{ij} = y_i - y_j \quad (4.3.4)$$

where  $N_i$  are the quadratic serendipity shape functions.

The developed element locks in the thin regime even though the interdependent interpolation scheme was utilized. This shear locking phenomenon is mainly attributed to inconsistent terms in the transverse shear strain fields with respect to the in-plane coordinates,  $x$  and  $y$  [73,74]. The inconsistent terms do not vanish in the thin regime of the element, which causes locking. The inconsistencies in the thickness coordinate,  $z$ , also exist but can be neglected relative to the inconsistencies in in-plane interpolation. Therefore, we need to ensure only the transverse shear strain fields to be field consistent with respect to the inplane coordinates in order to prevent shear locking. This is performed using the approach presented in [73,74]. Through the similar procedure as in the previous chapter,



the consistent shear strain fields in covariant frame are obtained as follows:

$$\begin{aligned}\gamma_{xz}(\xi, \eta) = & [P_i \bar{u}_{\xi\alpha\beta i} \Phi_{\xi\alpha\beta, z}^{(k)} + (P_{i,\xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi\beta i}) \Omega_\beta] \bar{j}_{11} \\ & + [P_i \bar{u}_{\eta\alpha\beta i} \Phi_{\eta\alpha\beta, z}^{(k)} + (P_{i,\eta} w_{\beta i} + N_{\eta i, \eta} \theta_{\eta\beta i}) \Omega_\beta] \bar{j}_{12}\end{aligned}\quad (4.3.5)$$

where  $i = 1, \dots, 4$ ;  $\alpha = 1, 2$ ;  $\beta = 1, 2$

$$\begin{aligned}\gamma_{yz}(\xi, \eta) = & [P_i \bar{u}_{\xi\alpha\beta i} \Psi_{\xi\alpha\beta, z}^{(k)} + (P_{i,\xi} w_{\beta i} + N_{\xi i, \xi} \theta_{\xi\beta i}) \Omega_\beta] \bar{j}_{21} \\ & + [P_i \bar{u}_{\eta\alpha\beta i} \Psi_{\eta\alpha\beta, z}^{(k)} + (P_{i,\eta} w_{\beta i} + N_{\eta i, \eta} \theta_{\eta\beta i}) \Omega_\beta] \bar{j}_{22}\end{aligned}\quad (4.3.6)$$

where  $i = 1, \dots, 4$ ;  $\alpha = 1, 2$ ;  $\beta = 1, 2$

The consistent constraint fields are recast into the following form:

$$\begin{aligned}C_{x\beta}(\xi, \eta) = & [P_{i,\xi} w_{\beta i} + (N_{\eta i, \xi} + P_i) \theta_{\xi\beta i}] \bar{j}_{11} \\ & + [P_{i,\eta} w_{\beta i} + (N_{\xi i, \eta} - P_i) \theta_{\eta\beta i}] \bar{j}_{12}\end{aligned}\quad (4.3.7)$$

$$\begin{aligned}C_{y\beta}(\xi, \eta) = & [P_{i,\xi} w_{\beta i} + (N_{\eta i, \xi} + P_i) \theta_{\xi\beta i}] \bar{j}_{21} \\ & + [P_{i,\eta} w_{\beta i} + (N_{\xi i, \eta} - P_i) \theta_{\eta\beta i}] \bar{j}_{21}\end{aligned}\quad (4.3.8)$$

To make sure that an element is free of locking, still the derived consistent shear strain field and consistent constraints are required to satisfy the edge-consistency on the inter-element edges. The edge consistency is fulfilled by mapping back the covariant-based shear strain fields and the covariant-based penalty constraint fields to Cartesian coordinate at element nodal coordinates. The edge consistent shear strains are:

$$\begin{aligned}
\bar{\gamma}_{xz}(\xi, \eta) = & \sum_{i=1}^4 \left[ u_{\beta i} \left\{ (j_{11})_i P_i \Phi_{\xi 1 \beta}^{(k)} z \bar{j}_{11} + (j_{21})_i P_i \Phi_{\eta 1 \beta}^{(k)} z \bar{j}_{12} \right\} \right. \\
& + v_{\beta i} \left\{ (j_{12})_i P_i \Phi_{\xi 1 \beta}^{(k)} z \bar{j}_{11} + (j_{22})_i P_i \Phi_{\eta 1 \beta}^{(k)} z \bar{j}_{12} \right\} + w_{\beta i} (P_{i, \xi} \bar{j}_{11} + P_{i, \eta} \bar{j}_{12}) \Omega_{\beta} \\
& + \theta_{x \beta i} \left\{ (j_{22})_i (P_i \Phi_{\xi 2 \beta}^{(k)} z \bar{j}_{11} + N_{\xi i, \eta} \Omega_{\beta} \bar{j}_{12}) - (j_{12})_i (P_i \Phi_{\eta 2 \beta}^{(k)} z \bar{j}_{12} + N_{\eta i, \xi} \Omega_{\beta} \bar{j}_{11}) \right\} \\
& \left. + \theta_{y \beta i} \left\{ (j_{11})_i (P_i \Phi_{\eta 2 \beta}^{(k)} z \bar{j}_{12} + N_{\eta i, \xi} \Omega_{\beta} \bar{j}_{11}) - (j_{21})_i (P_i \Phi_{\xi 2 \beta}^{(k)} z \bar{j}_{11} + N_{\xi i, \eta} \Omega_{\beta} \bar{j}_{12}) \right\} \right] \quad (4.3.9)
\end{aligned}$$

$$\begin{aligned}
\bar{\gamma}_{yz}(\xi, \eta) = & \sum_{i=1}^4 \left[ u_{\beta i} \left\{ (j_{11})_i P_i \Psi_{\xi 1 \beta}^{(k)} z \bar{j}_{21} + (j_{21})_i P_i \Psi_{\eta 1 \beta}^{(k)} z \bar{j}_{22} \right\} \right. \\
& + v_{\beta i} \left\{ (j_{12})_i P_i \Psi_{\xi 1 \beta}^{(k)} z \bar{j}_{21} + (j_{22})_i P_i \Psi_{\eta 1 \beta}^{(k)} z \bar{j}_{22} \right\} + w_{\beta i} (P_{i, \xi} \bar{j}_{21} + P_{i, \eta} \bar{j}_{22}) \Omega_{\beta} \\
& + \theta_{x \beta i} \left\{ (j_{22})_i (P_i \Psi_{\xi 2 \beta}^{(k)} z \bar{j}_{21} + N_{\xi i, \eta} \Omega_{\beta} \bar{j}_{22}) - (j_{12})_i (P_i \Psi_{\eta 2 \beta}^{(k)} z \bar{j}_{22} + N_{\eta i, \xi} \Omega_{\beta} \bar{j}_{21}) \right\} \\
& \left. + \theta_{y \beta i} \left\{ (j_{11})_i (P_i \Psi_{\eta 2 \beta}^{(k)} z \bar{j}_{22} + N_{\eta i, \xi} \Omega_{\beta} \bar{j}_{21}) - (j_{21})_i (P_i \Psi_{\xi 2 \beta}^{(k)} z \bar{j}_{21} + N_{\xi i, \eta} \Omega_{\beta} \bar{j}_{22}) \right\} \right] \quad (4.3.10)
\end{aligned}$$

where  $j_{ij}$  and  $\bar{j}_{ij}$  are defined in Eqs. (3.5.32) and (3.5.33).

The edge consistent constraints are:

$$\begin{aligned}
\bar{C}_{x\beta}(\xi, \eta) = & \sum_{i=1}^4 [\theta_{x\beta i} \{ -(j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{11} + (j_{22})_i (N_{\xi i, \eta} - P_i) \bar{j}_{12} \} \\
& + \theta_{y\beta i} \{ (j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{11} - (j_{21})_i (N_{\xi i, \eta} - P_i) \bar{j}_{12} \} \\
& + w_{\beta i} (P_{i, \xi} \bar{j}_{11} + P_{i, \eta} \bar{j}_{12})] \quad \beta = 1, 2 \quad (4.3.11)
\end{aligned}$$

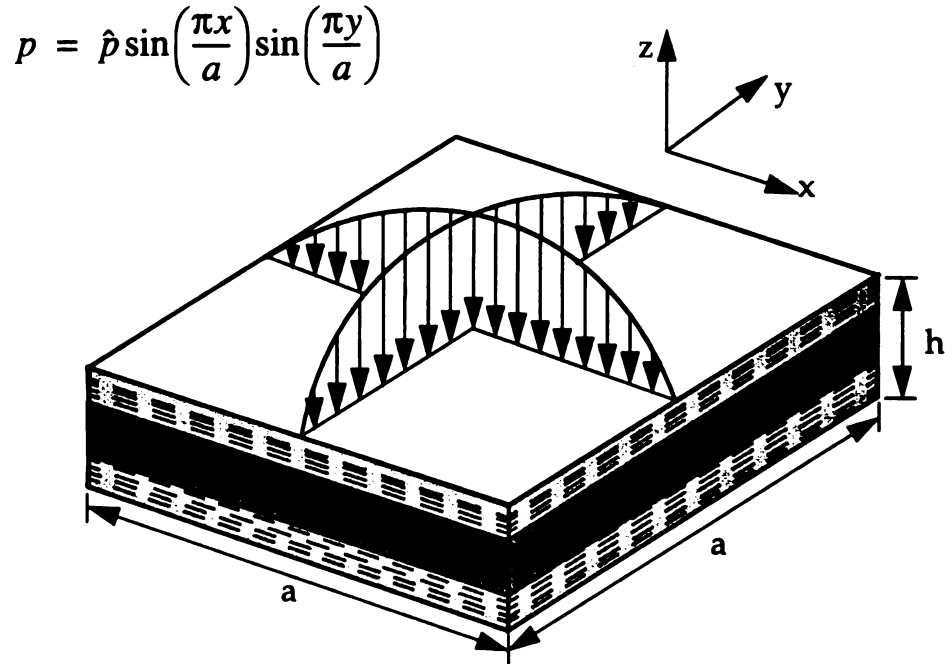
$$\begin{aligned}
\bar{C}_{y\beta}(\xi, \eta) = \sum_{i=1}^4 & [\theta_{x\beta i} \{-(j_{12})_i (N_{\eta i, \xi} + P_i) \bar{j}_{21} + (j_{22})_i (N_{\xi i, \eta} - P_i) \bar{j}_{22}\} \\
& + \theta_{y\beta i} \{(j_{11})_i (N_{\eta i, \xi} + P_i) \bar{j}_{21} - (j_{21})_i (N_{\xi i, \eta} - P_i) \bar{j}_{22}\} \\
& + w_{\beta i} \{P_{i, \xi} \bar{j}_{21} + P_{i, \eta} \bar{j}_{22}\}] \quad \beta = 1, 2
\end{aligned} \tag{4.3.12}$$

Since the adopted method ensures to alleviate locking phenomenon, therefore an exact number of integration points can be used to evaluate the shear strain energy without causing locking.

#### 4.4 Numerical Results

The response of simply-supported laminated plates with delamination or ply damage under a transverse double sinusoidal loading as shown in Figure 37 is presented. In order to investigate the ability of such a laminate model to capture the local kinematic variations associated with damage, a very thin layer with compliant material stiffnesses is placed between two adjacent layers. The material properties used in the analysis are listed in Table 6., and the lamination scheme is defined in Table 7.. The current plate results (CZZ3D) were compared with the elasticity solution of Burton and Noor [98]. For a given location in the plane of the plate, explicit expressions for the through-thickness variation of stresses are provided by the present model. In the plane of the plate, stresses were evaluated at the centroid of each element, where stress is super-convergent [73]. The bending stresses presented in the forthcoming discussion were evaluated at the center of the element that is nearest the center of the plate. The transverse deflections are obtained at the top surface of the center of the plate where load is applied. The in-plane displacements are

obtained at the edges of the plate where  $x = 0$  and  $y = a/2$  (for  $u_x$  displacement) or  $y = 0$  and  $x = a/2$  (for  $u_y$  displacement), and  $a$  is the length of each side of the plate. Due to both the geometric and boundary condition symmetries of the panel, only a quarter of the plate was modeled.



**Figure 37. Simply supported laminated composite plate subjected to sinusoidal loading.**

A uniform mesh of 8x8 elements was utilized in the plane of a quarter of the plate. A length-to-thickness aspect ratio of twenty was chosen in the present example in order to simulate the automotive-type fiber reinforced composite thin wall structures. The influence of aspect ratios on transverse deflection is shown in Figure 38. Eight different aspect ratios were considered to check a wide range of plate bending behavior ( $L/h = 4, 8, 10, 20, 30, 50, 100, 500$ , and  $1000$ ). The error in predicted deflection was about 3 percent for the case  $a/h = 10$ . In Figure 39, the variation of in-plane displacement,  $u_x$ , through the thick-

ness of the laminate is plotted at  $x = 0$  and  $y = a/2$ . The exact distribution of in-plane displacement is approximately linear within each layer, and globally piece-wise linear and of zig-zag pattern due to the difference in the shear moduli of adjacent layers. Moreover it can be observed that there is a big change in the distribution of in-plane displacement in the compliant layer. The bending stress,  $\sigma_{xx}$ , distribution through the thickness is illustrated in Figure 40. The stress was evaluated at the Gauss point nearest to the center of the plate. It can be observed that there is a sudden change of distribution of in-plane normal stress in each interface region, and the prediction of in-plane normal stress by the current model agrees well with the elasticity solution. The prediction of in-plane shear stress,  $\sigma_{xy}$ , is shown in Figure 41. It is shown that current model captures the in-plane stress distribution in interface region very accurately. The distribution of transverse shear stress,  $\sigma_{xz}$  through thickness is shown in Figure 42. This component of stress is quadratic through the thickness direction in each layer, and is under predicted compared with the elasticity solution. In Figure 43 the distribution of the transverse shear strain,  $\gamma_{xz}$ , is shown. It can be observed that the shear stress is large in the compliant layer, and the prediction of the current model is in good agreement with the elasticity solution. Thus, the developed element can capture delamination with a proper composite interface failure criterion when the delamination occurs at an interface. Figure 44 shows the distribution of transverse shear stress,  $\sigma_{yz}$  through the thickness. It can be observed that  $\sigma_{yz}$  varies in quadratic fashion like  $\sigma_{xz}$ , however it is over-predicted compared with the elasticity solution. The transverse normal strain,  $\epsilon_{zz}$ , is predicted in Figure 45. The trend of distribution of transverse strain is predicted very well, but the strain in the compliant layer is overpredicted by approximately a factor of two.

## **4.5 Summary**

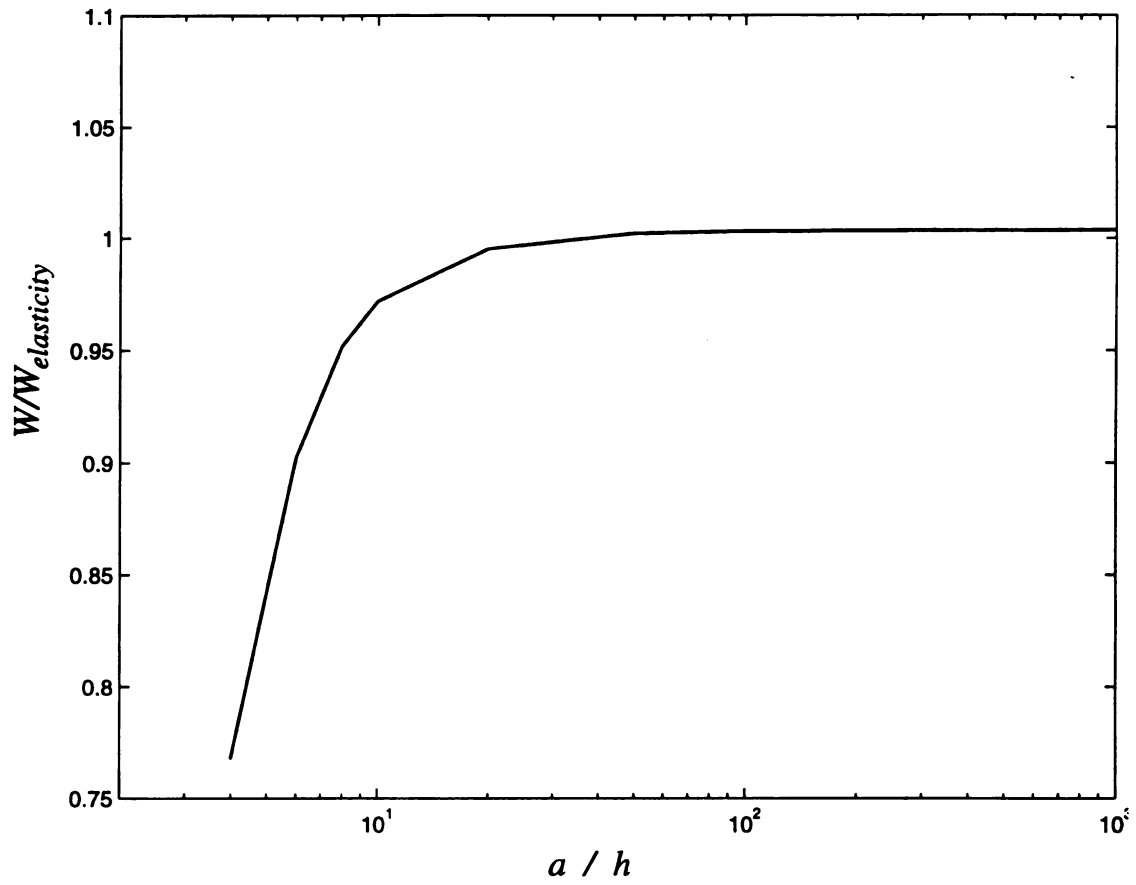
An improved cubic zig-zag theory and an associated finite element model are presented. The in-plane displacement fields in a laminate are assumed to be piecewise cubic functions and vary in a zig-zag fashion through the thickness of the laminate. This assumption accounts for discrete layer effects without increasing the number of degrees of freedom as the number of layers is increased. The finite element model is “regenerated” in the form of an eight-noded three dimensional element with five degrees of freedom (three translations and two rotations) per node. The finite element is free of shear locking, robust, and accurate. Thus, it is suitable for implementation into commercial finite element codes.

**Table 6. . Material properties for damaged composite plate**

|            | material 1 | material 2 | material 3 |
|------------|------------|------------|------------|
| $E_{11}$   | 139.e9     | 4.30e9     | 4.30e6     |
| $E_{22}$   | 9.86e9     | 4.30e9     | 4.30e6     |
| $E_{33}$   | 9.86e9     | 4.30e9     | 4.30e6     |
| $\nu_{23}$ | 0.37       | 0.34       | 0.34       |
| $\nu_{13}$ | 0.30       | 0.34       | 0.34       |
| $\nu_{12}$ | 0.30       | 0.34       | 0.34       |
| $G_{23}$   | 3.25e9     | 1.60e9     | 1.60e6     |
| $G_{13}$   | 5.24e9     | 1.60e9     | 1.60e6     |
| $G_{12}$   | 5.24e9     | 1.60e9     | 1.60e6     |

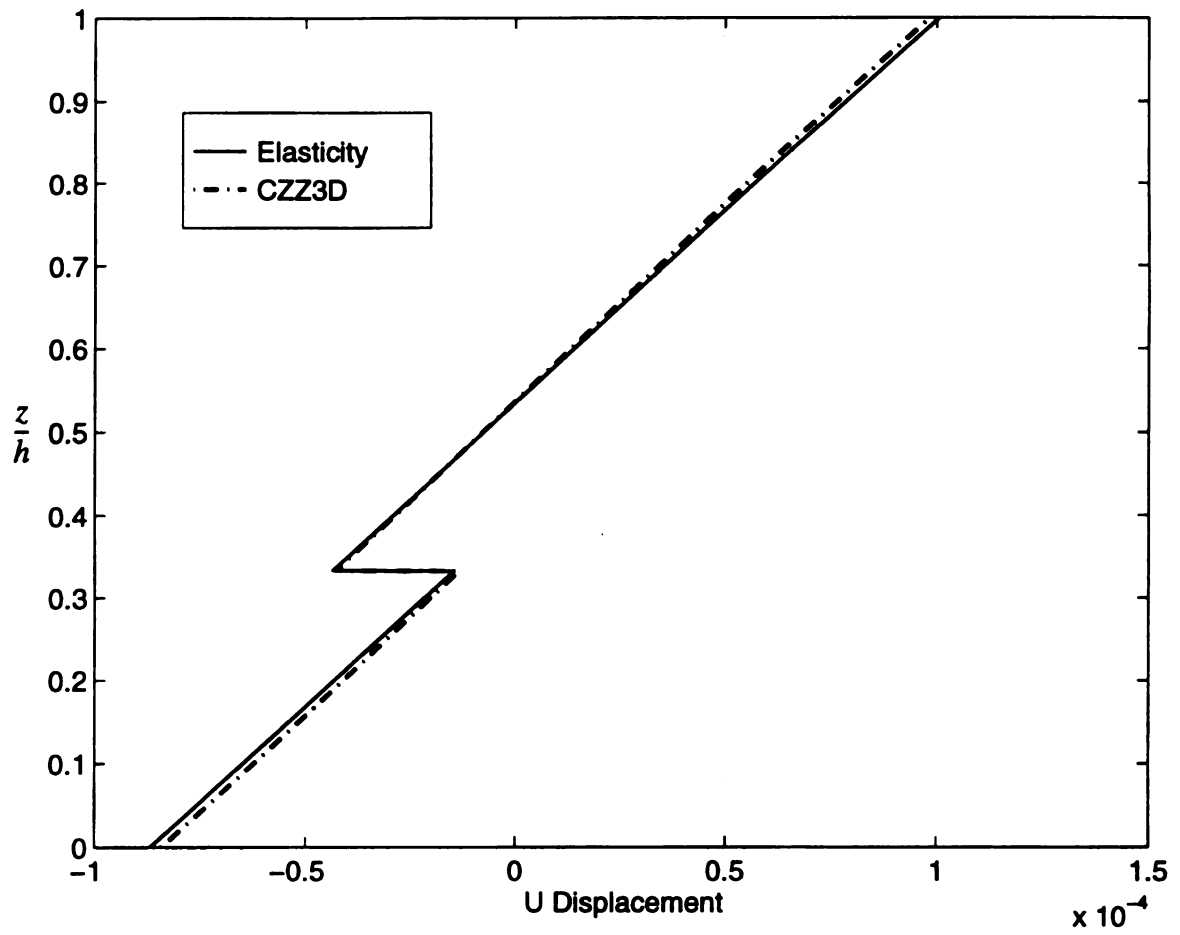
**Table 7. . Lamination Scheme for damaged composite plate**

| Layer No. | Material | Thickness | Angle |
|-----------|----------|-----------|-------|
| 1         | 1        | 0.16583   | 0.00  |
| 2         | 2        | 0.00100   | 0.00  |
| 3         | 1        | 0.16583   | 90.0  |
| 4         | 3        | 0.00100   | 0.00  |
| 5         | 1        | 0.16583   | 0.00  |
| 6         | 2        | 0.00100   | 0.00  |
| 7         | 1        | 0.16583   | 0.00  |
| 8         | 2        | 0.00100   | 0.00  |
| 9         | 1        | 0.16583   | 90.0  |
| 10        | 2        | 0.00100   | 0.00  |
| 11        | 1        | 0.16583   | 0.00  |

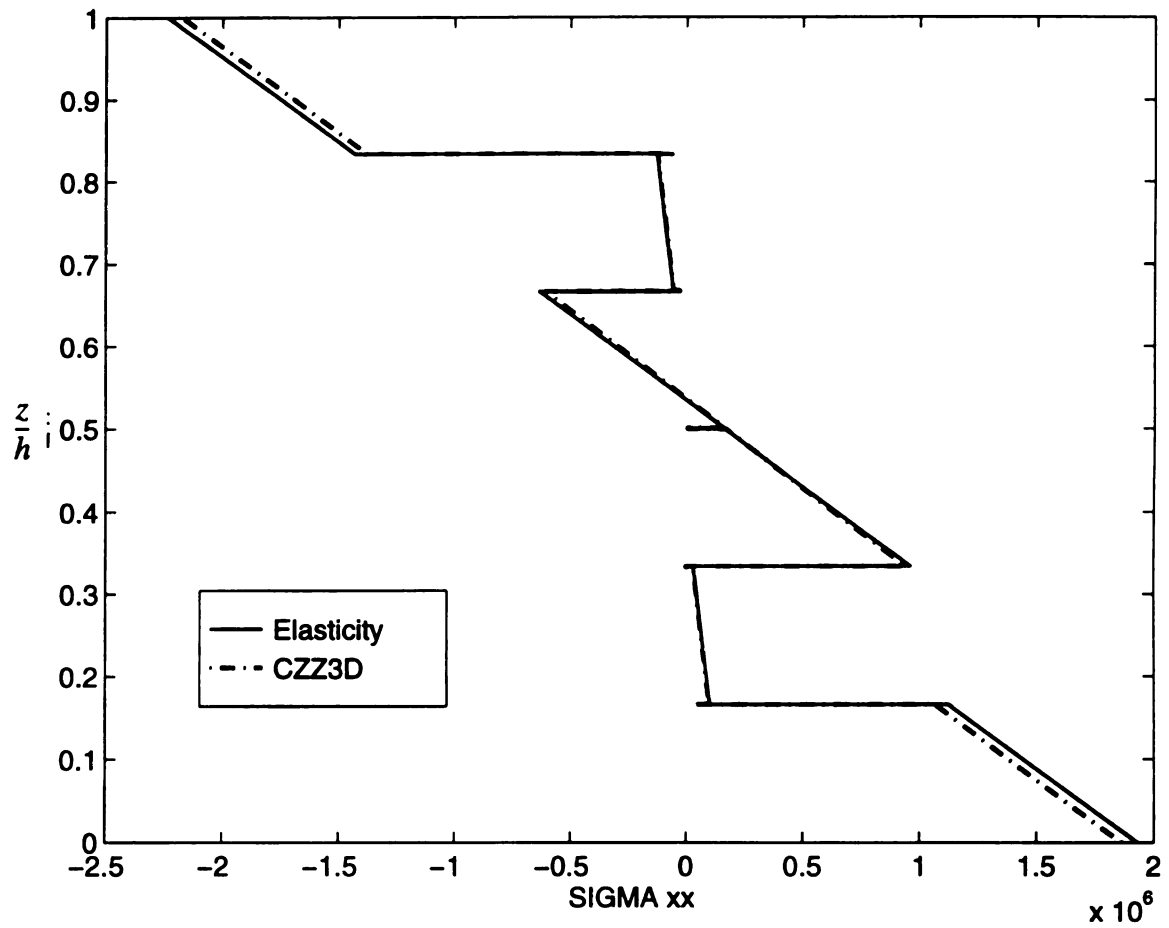


**Figure 38. Variation of normalized center deflection versus plate aspect ratio at the center of the plate.**

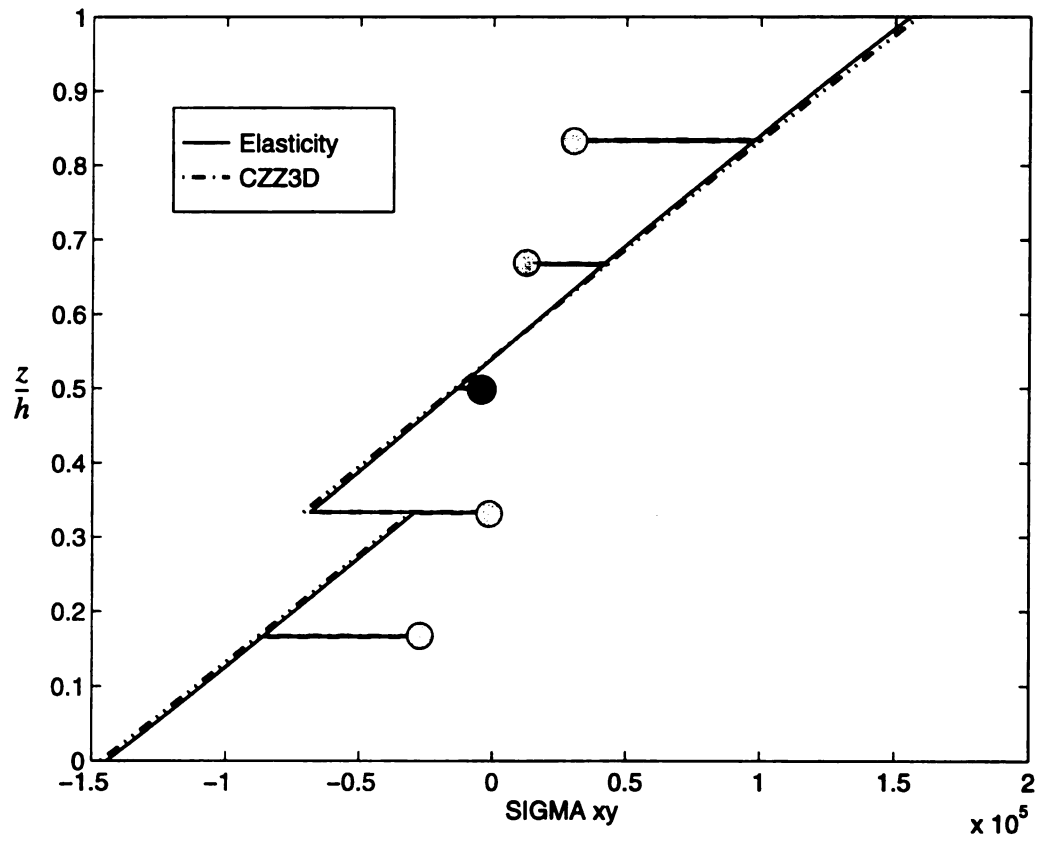




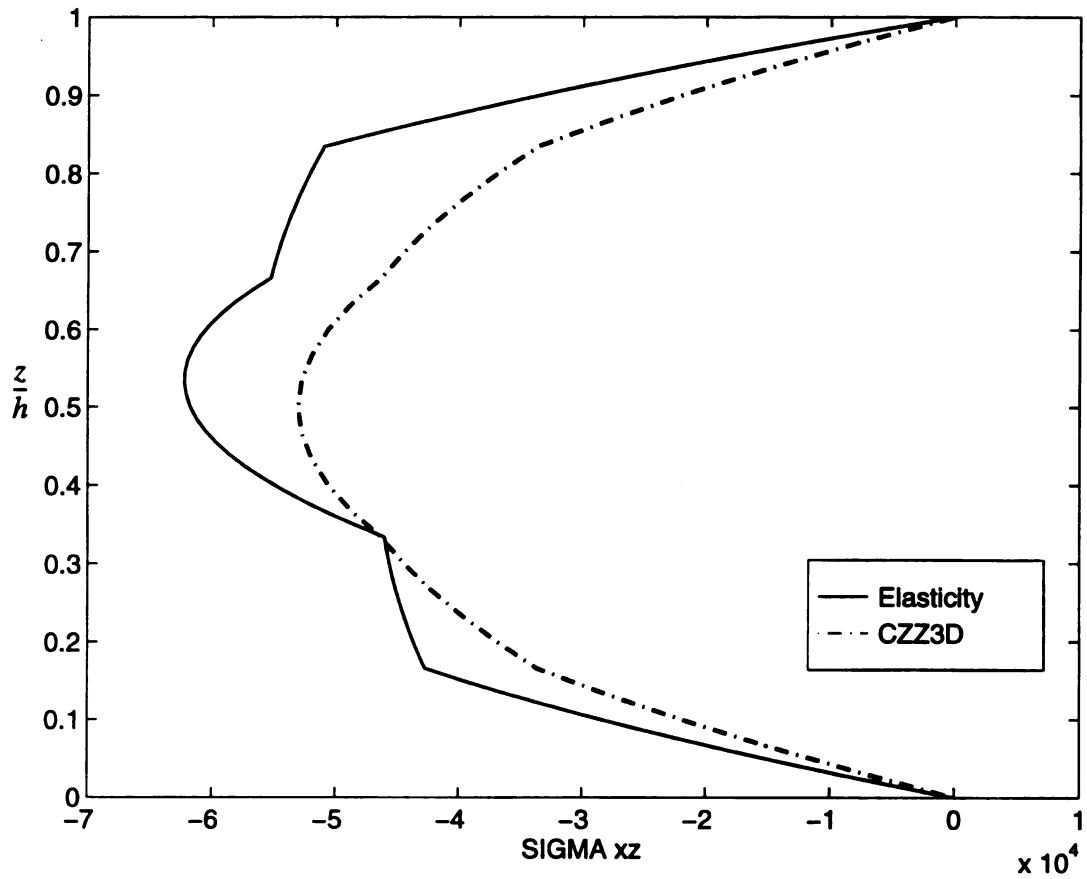
**Figure 39. In-plane displacement,  $u$ , versus normalized thickness coordinate ( $a/h=20$ )**



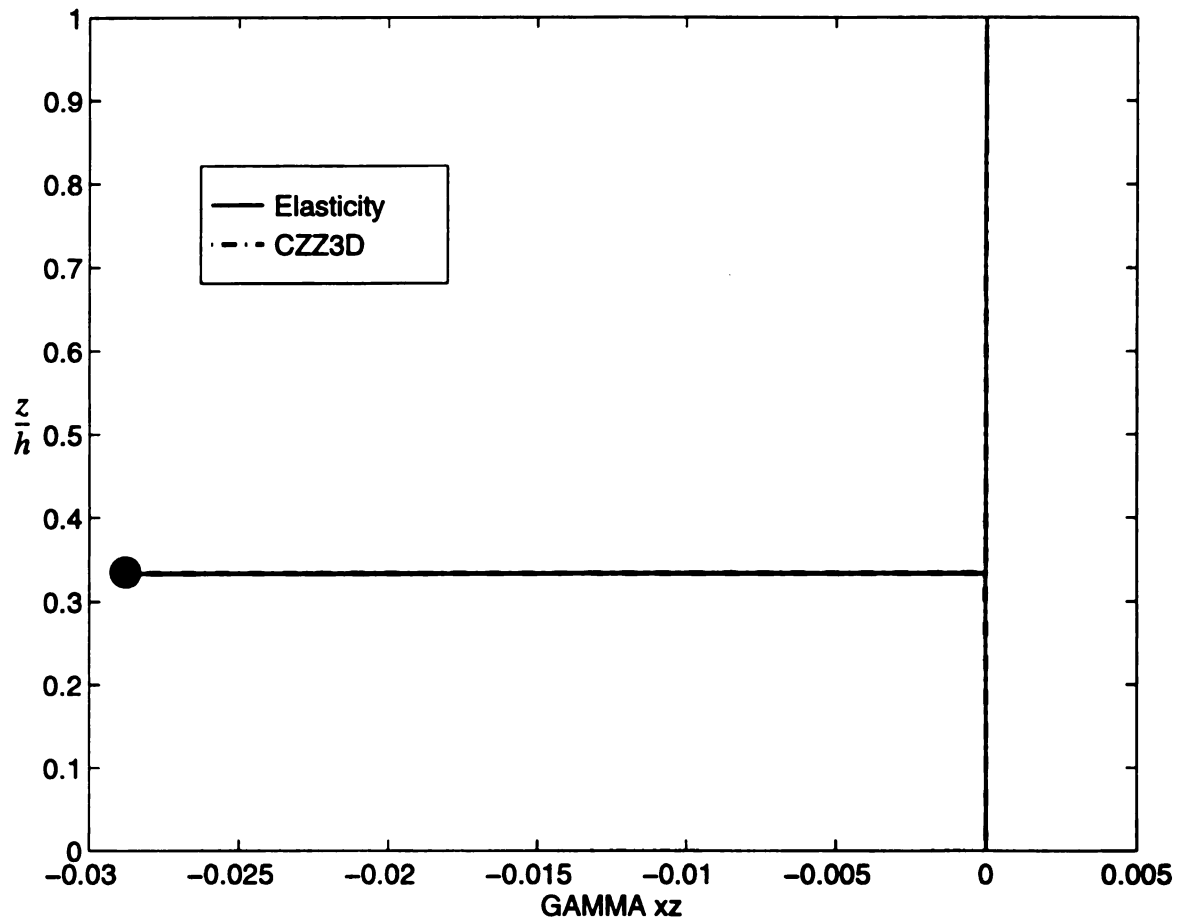
**Figure 40. Bending stress,  $\sigma_{xx}$ , versus normalized thickness coordinate at the center of the plate ( $a/h=20$ ).**



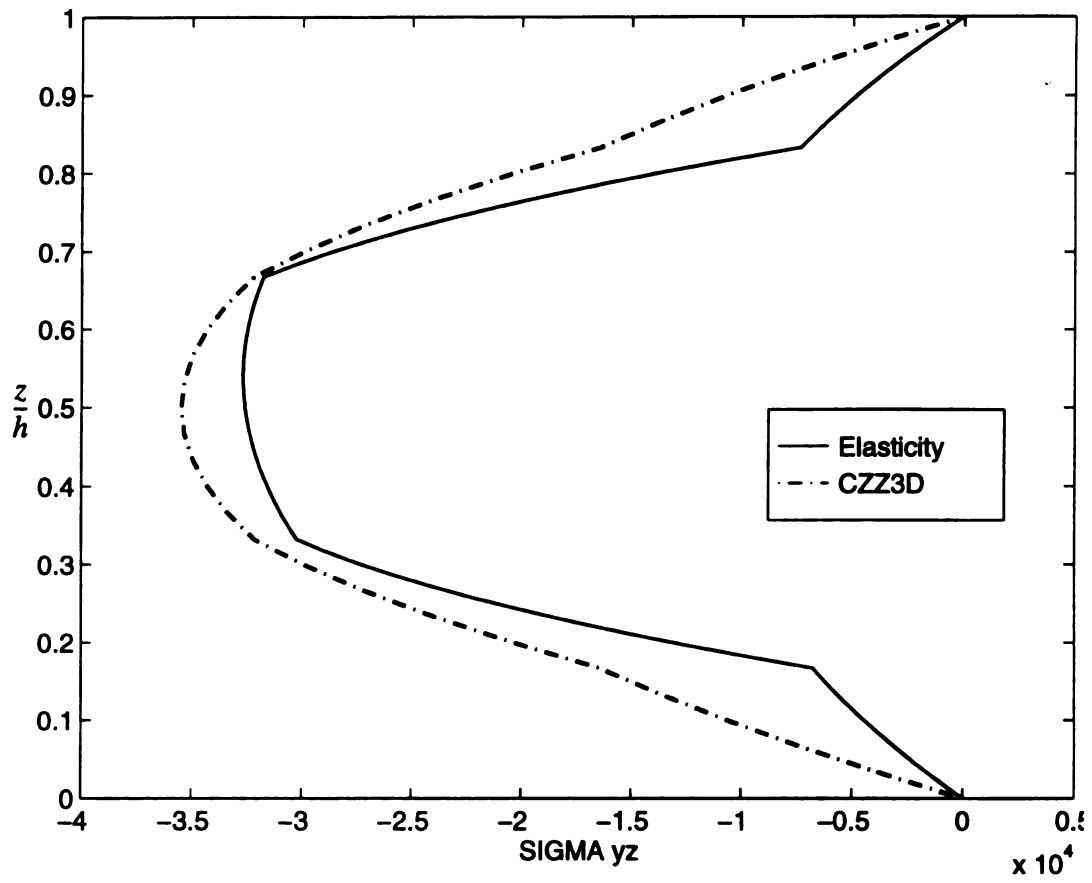
**Figure 41. In-plane Shear stress,  $\sigma_{xy}$ , versus normalized thickness coordinate at  $x, y = 0$ , ( $a/h=20$ ).**



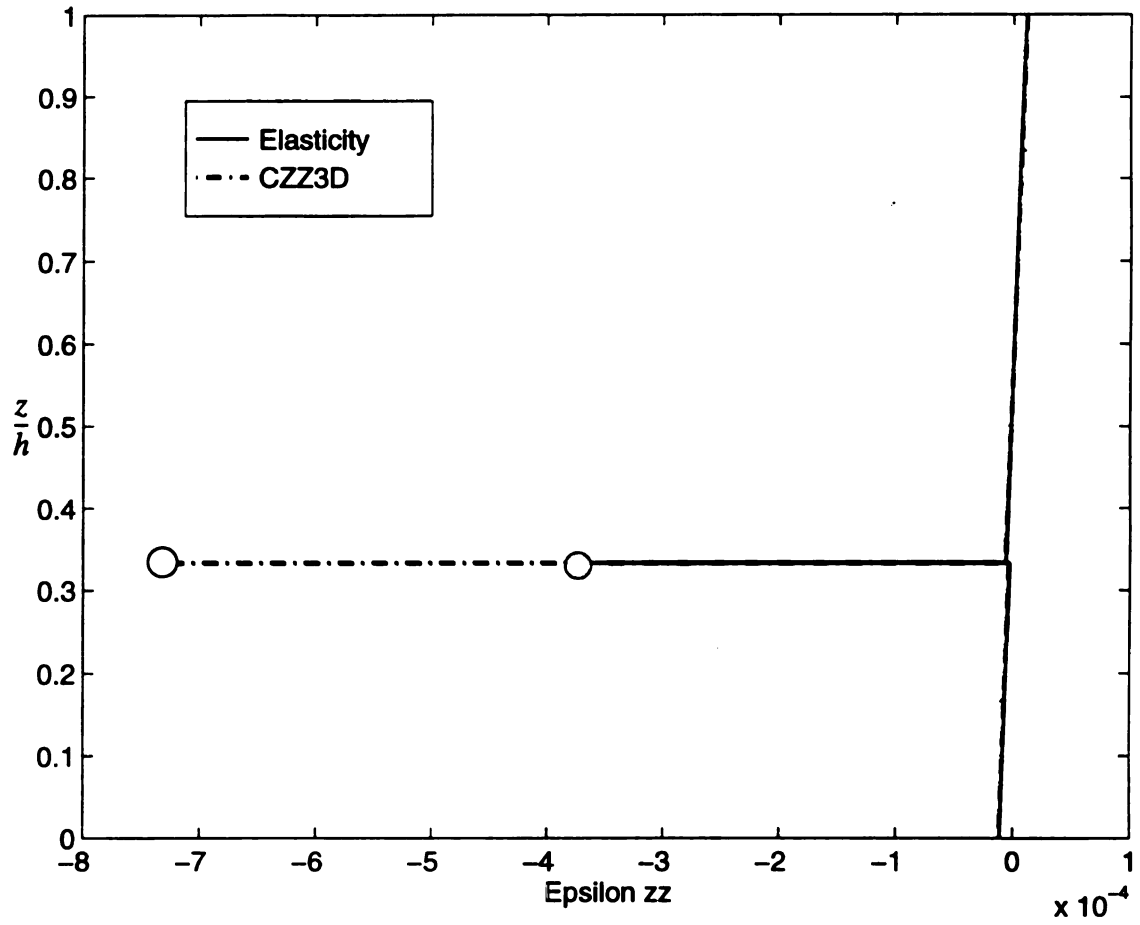
**Figure 42. Transverse shear stress,  $\sigma_{xz}$ , versus normalized thickness coordinate at  $x = 0$ ,  $y = 2/a$**



**Figure 43. Transverse shear strain,  $\gamma_{xz}$ , versus normalized thickness coordinate at  $x = 0$ ,  $y = 2/a$**



**Figure 44. Transverse shear stress,  $\sigma_{yz}$ , versus normalized thickness coordinate at  $x = 2/a, y = 0$**



**Figure 45. Transverse normal strain,  $\epsilon_{zz}$ , versus normalized thickness coordinate at the center of the plate ( $a/h=20$ ).**

## **CHAPTER V**

### **FREE VIBRATION ANALYSIS**

#### **5.1 Introduction**

In free vibration analysis, the mass matrix accounts for inertia and represents discretely the continuous distribution of mass in a structure. In this chapter, consistent mass matrices for the previously developed elements are derived, and the lumped mass matrices associated with the consistent mass matrices are obtained employing the HRZ lumping scheme. On the basis of the derived mass matrices, free vibration analysis of simply supported laminated composites will be accomplished. Since the derived lumped mass matrices render the decoupled dynamics governing equations in explicit dynamic analysis, they will be utilized for explicit dynamic analysis in the next chapter.

#### **5.2 Mass Matrices**

Inertial effects must be included when analyzing the response of a structure if it vibrates freely or if excitation frequencies are higher than  $1/3$  of the lowest natural frequency of the structure. In the finite element analysis of the dynamic response of a structure, the mass matrix accounts for inertia of a structure, and represents discretely the



continuous distribution of mass in a structure.

A consistent mass matrix can be obtained from the kinetic energy Eq. (3.4.3) using the displacement fields Eq. (3.3.14) (for cubic zig-zag plate element, Eqs. (4.2.23) and (4.2.15)) and the same shape functions are employed as those used to develop the stiffness matrix [55,56]. The consistent mass matrix can be written as:

$$[M] = \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}^{(k)}]^T [\tilde{N}^{(k)}] dz d\Omega \quad (5.2.1)$$

where  $N_m$  is the number of layers in the  $m$ th sublaminate or the laminate,  $\rho^{(k)}$  is the density of the  $k$ th layer, and  $[\tilde{N}^{(k)}]$  is the shape function matrix in  $x$ ,  $y$  and  $z$  coordinates.

The lumped mass matrix is derived on the basis of the consistent mass matrix. Among the lumping schemes that produce a diagonal mass matrix, the row-sum lumping, the optimal lumping [57] and proportional lumping (so called HRZ lumping) schemes [58] are well-known. However, since the current finite element model includes rotational d.o.f., the HRZ lumping scheme which can treat rotational d.o.f. reasonably is employed in this research. The basic idea of the HRZ lumping scheme is that only the diagonal terms of the consistent mass matrix are scaled in such a way that the total mass of the element is maintained. The procedure of the HRZ lumping scheme is as follows. First, we need to obtain the total mass of the element:

$$m = \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} d\Omega dz \quad (5.2.2)$$

Second, calculate only diagonal components ( $M_{ii}$ ) of the consistent mass matrix.

$$[M_{ii}] = \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_i^{(k)}]^T [\tilde{N}_i^{(k)}] dz d\Omega \quad (5.2.3)$$

Third, compute total masses,  $(s_u, s_v, s_w)$  in each translational direction  $(u, v, w)$  respectively:

$$\begin{aligned} s_u &= \sum_{j=5i-4}^{40} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \\ s_v &= \sum_{j=5i-3}^{40} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \\ s_w &= \sum_{j=5i-2}^{40} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \end{aligned} \quad (5.2.4)$$

where  $i = 1 \dots 8$

Fourth, scale all the diagonal components by multiplying them by the appropriate ratios.

For  $u$  d.o.f:

$$[M_{jj}] = \frac{m}{s_u} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \quad (5.2.5)$$

where  $j = 5i - 4, i = 1 \dots 8$

For  $v$  d.o.f:

$$[M_{jj}] = \frac{m}{s_v} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \quad (5.2.6)$$

where  $j = 5i - 3, i = 1 \dots 8$

For  $w$  d.o.f:

$$[M_{jj}] = \frac{m}{s_w} \sum_{k=1}^{N_m} \int_{\Omega} \int_{z_{k-1}}^{z_k} \rho^{(k)} [\tilde{N}_j^{(k)}]^T [\tilde{N}_j^{(k)}] dz d\Omega \quad (5.2.7)$$

where  $j = 5i - 2, i = 1 \dots 8$

For rotational d.o.f, attention should be paid to which  $s$  will be used. In this dissertation, the factor,  $s$ , for rotations was obtained by averaging three factors as follows:

$$s_{\theta x} = s_{\theta y} = \frac{(s_u + s_v + s_w)}{3} \quad (5.2.8)$$

Then, by similar procedures, the rotational d.o.f. components in the lumped mass matrix are obtained. The most important feature of this lumping scheme is that it always produces positive lumped masses without regard to the element type, because the diagonal terms in the consistent mass matrix must be positive by virtue of the positive-definiteness property. For low order elements, the accuracy of this lumping scheme in the prediction of natural frequencies often surpasses that of the consistent mass matrix.

### 5.3 Free Vibration Analysis

The natural frequency of a structure can be obtained by solving the dynamic equations without damping and external force as follows:

$$[M]\{\ddot{u}(x, y, z, t)\} + [K]\{u(x, y, z, t)\} = \{0\} \quad (5.3.1)$$

where  $[M]$  and  $[K]$  are global mass and stiffness matrices respectively. A solution to the above equation may be obtained by assuming a displacement vector as

$$\{u(x, y, z, t)\} = \{\bar{u}(x, y, z)\}e^{i\omega t} \quad (5.3.2)$$

where  $\{\bar{u}(x, y, z)\}$  is the modal vector and  $\omega$  is the natural frequency of the plate. The acceleration vector can be derived by differentiating Eq. (5.3.2) twice with respect to time and can be written as:

$$\{\ddot{u}(x, y, z, t)\} = -\omega^2 \{\bar{u}(x, y, z)\}e^{i\omega t} \quad (5.3.3)$$

Substitution of Eq. (5.3.2) and Eq. (5.3.3) into Eq. (5.3.1) leads to the eigenvalue problem:

$$([K] - \omega^2[M])\{\bar{u}\} = \{0\} \quad (5.3.4)$$

Eq. (5.3.4) is solved by using the standard eigenvalue solution schemes after boundary conditions are imposed. Thus, the modal vectors ( $\{\bar{u}\}$ ) and the natural frequencies ( $\omega$ ) are obtained.

## 5.4 Numerical Results

The accuracies of the mass matrix and lumping scheme were evaluated by simulating the response of a simply-supported laminated plate. The example is a sandwich panel which consists of a thick core between two sets of five face sheets. In this example, a uniform 16x16 mesh was used for a full plate, which is equivalent to an 8x8 mesh for a quarter plate. The reason why a full plate was used for free vibration analysis is that both the geometric and boundary condition symmetries of the panel do not imply symmetry of all vibration modes, so that all antisymmetric modes would be excluded by imposing symmetry. The material properties used in the analysis are listed in Table 8, and the lamination

scheme is defined in Table 9. The core occupies eighty percent of the thickness of the plate, while each set of face sheets contains five layers and occupies ten percent of the total thickness.

**Table 8. Material Properties for Sandwich Composite Plate**

|            | Material 1 | Material 2 | Material 3 | Core   |
|------------|------------|------------|------------|--------|
| $E_{11}$   | 1.0e6      | 3.26e7     | 25.e6      | 50.e3  |
| $E_{22}$   | 25.0e6     | 1.0e7      | 1.0e6      | 150.e3 |
| $E_{33}$   | 1.0e6      | 1.0e7      | 1.0e6      | 50.e3  |
| $G_{23}$   | 0.50e6     | 0.5e6      | 0.2e6      | 42.0e3 |
| $G_{13}$   | 0.20e6     | 8.21e6     | 0.5e6      | 21.7e3 |
| $G_{12}$   | 0.50e6     | 8.21e6     | 0.5e6      | 21.7e3 |
| $\nu_{23}$ | 0.25       | 0.25       | 0.25       | 0.15   |
| $\nu_{13}$ | 0.25       | 0.10       | 0.25       | 0.15   |
| $\nu_{12}$ | 0.01       | 0.10       | 0.25       | 0.01   |
| $\rho$     | 0.065      | 0.0542     | 0.0578     | 0.0047 |

The predictions of the lowest natural frequency by FZZ3D and CZZ3D are shown in Figures 46 and 47. The lowest natural frequencies were obtained at the various plate aspect ratios using both a consistent mass matrix and a lumped mass matrix, and normalized with the solution of Burton and Noor [98]. In Figure 46, it can be observed that the prediction of the lowest frequency by ad hoc lumping scheme is very poor when the aspect ratio is small. This comes from the fact that the rotary inertia terms are excluded in ad hoc lumping. Thus we can recognize that keeping rotary inertia terms in the lumped mass

matrix is very important when the plate is thick. The biggest error in the prediction of the lowest natural frequency is less than 1.6% for both the consistent mass matrix and the lumped mass matrix as seen in Figures 46 and 47, and it can be concluded that treatment of the total mass factor for rotational d.o.f was correct and that the HRZ lumping scheme is effective and accurate. In Table 10, the higher natural frequencies in the 16x16 mesh based on the consistent and the lumped mass matrices are listed. It can be observed that the lumped mass matrix yields a lower the highest natural frequency than the consistent mass matrix. Therefore, in explicit dynamic analysis, the lumped mass matrix not only enables the dynamic equations to be decoupled but also gives rise to larger time step (this fact will be discussed in the next chapter).

Figures 48, 49, 50, 51 and 52 show the first, second, third, fourth and fifth eigen mode shapes of transverse deflection d.o.f. of the simply supported sandwich plate at the aspect ratio of  $a/h = 4$ . From the figures we can see that all the eigen modes are linearly independent. As the plate considered above is an orthotropic plate, we observe different eigen values for the second and third eigen modes given in Figure 49 and Figure 50. But in the case of an isotropic plate, the mode shapes shown in Figure 53 and Figure 54 degenerate into a single mode with the same eigen value. This results from the orthotropic material properties of the sandwich plate.

## 5.5 Summary

The consistent mass matrices have been derived from the previously developed elements and the lumped mass matrices associated with the consistent mass matrices have been also derived. Free vibration analysis of simply supported sandwich panel has been

carried out. From the free vibration analysis, it has been shown that the lowest frequency predicted by the consistent and the lumped mass matrices are in good agreement with elasticity solution. It was observed that the predicted highest natural frequency by lumped mass matrix was lower than that by the consistent mass matrix. Also from the results it is seen that the eigen modes for orthotropic materials are distinctly different from conventional isotropic materials.

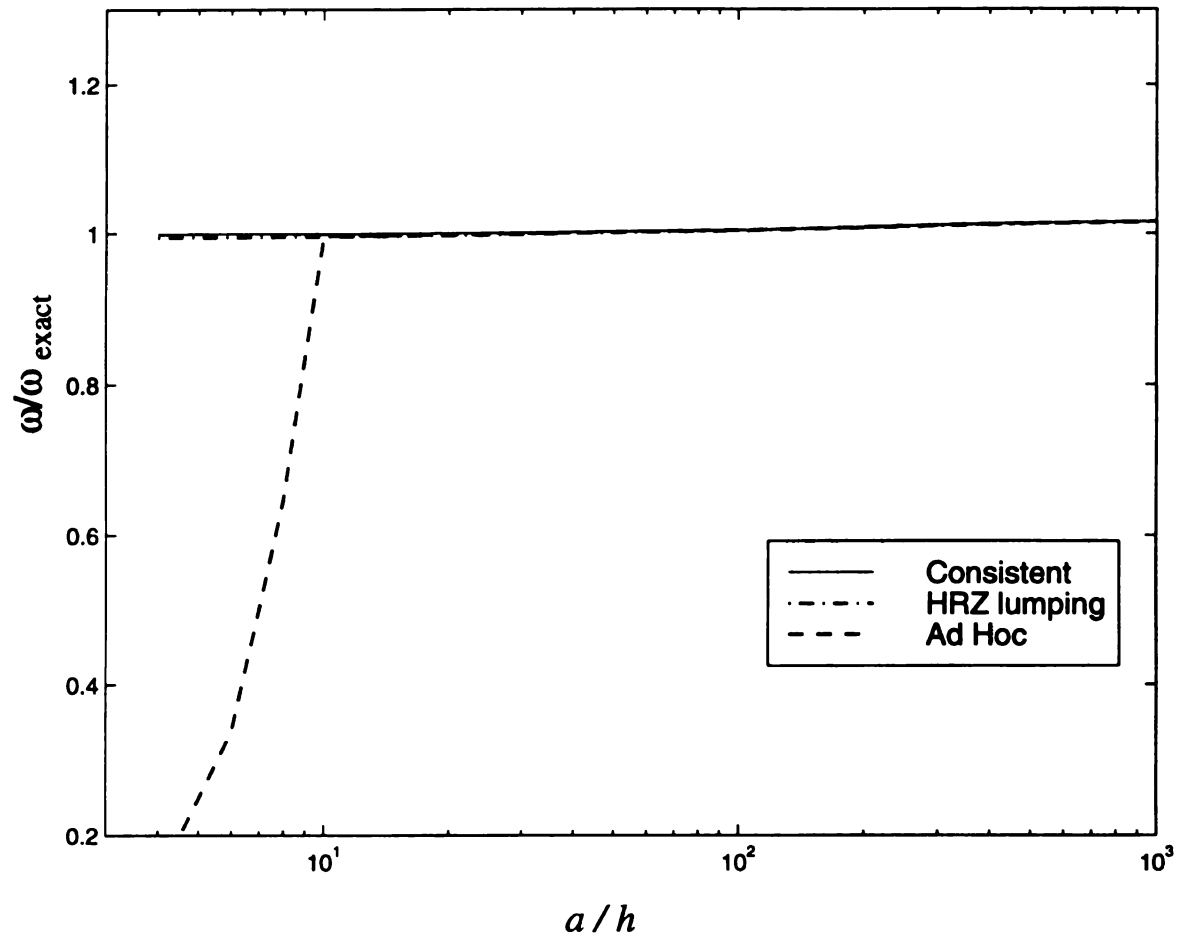
**Table 9. Lamination Scheme for Sandwich Panel**

| Layer No. | Material | Thickness |
|-----------|----------|-----------|
| 1         | 1        | 0.010     |
| 2         | 2        | 0.025     |
| 3         | 3        | 0.015     |
| 4         | 1        | 0.020     |
| 5         | 3        | 0.030     |
| 6         | Core     | 0.800     |
| 7         | 3        | 0.030     |
| 8         | 1        | 0.020     |
| 9         | 3        | 0.015     |
| 10        | 2        | 0.025     |
| 11        | 1        | 0.010     |

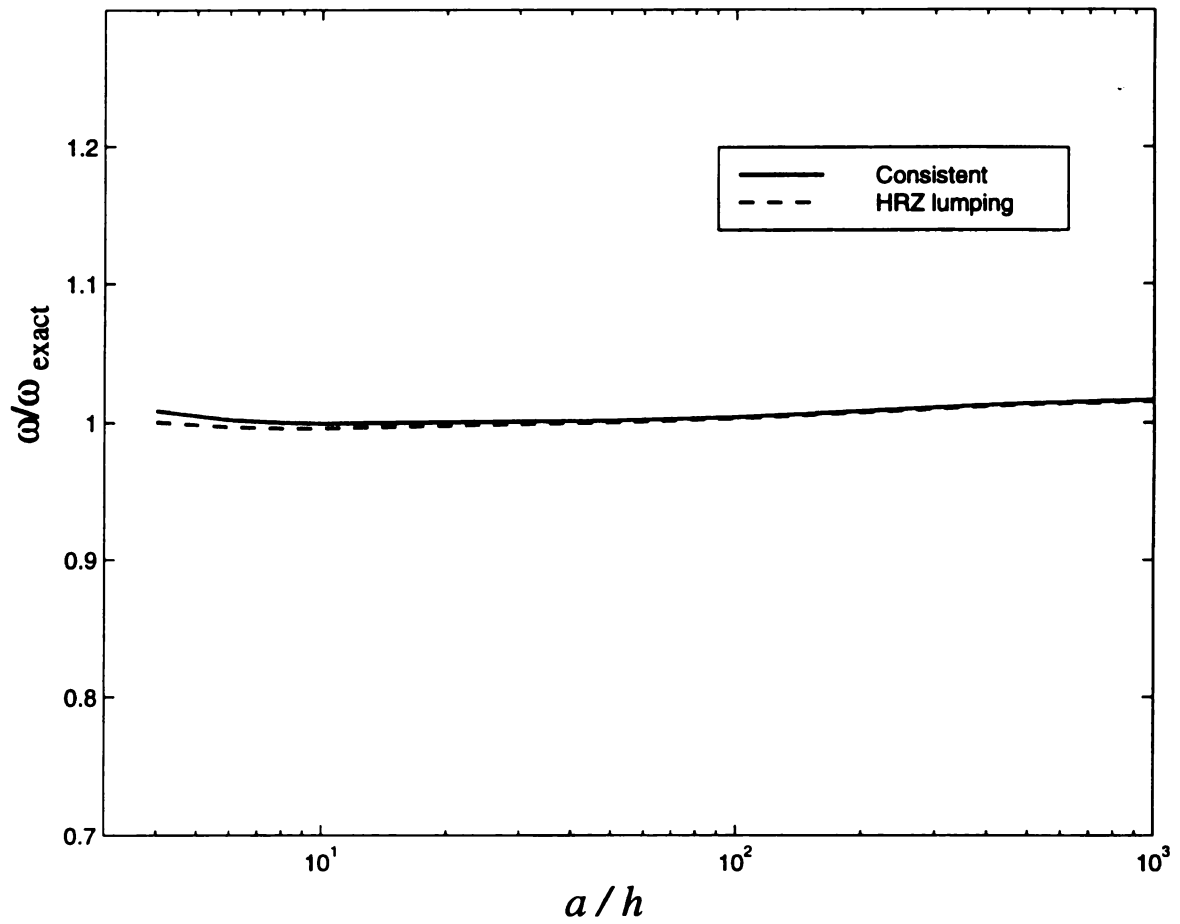
**Table 10. The higher natural frequencies in the mesh**

| Mode No. | consistent | HRZ lumping |
|----------|------------|-------------|
| 2478     | 0.1375E+08 | 0.3574E+07  |
| 2479     | 0.1379E+08 | 0.3581E+07  |
| 2480     | 0.1380E+08 | 0.3581E+07  |
| 2481     | 0.1380E+08 | 0.3583E+07  |
| 2482     | 0.1381E+08 | 0.3583E+07  |

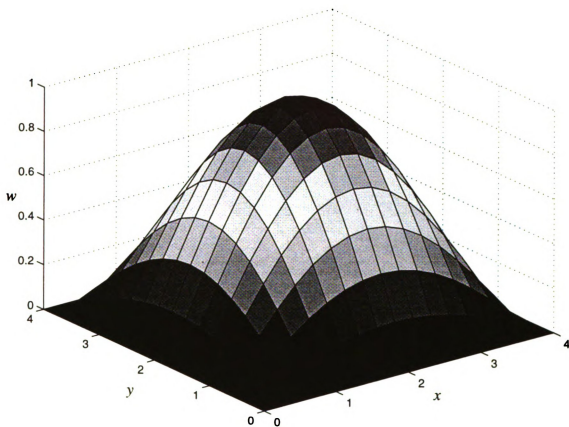




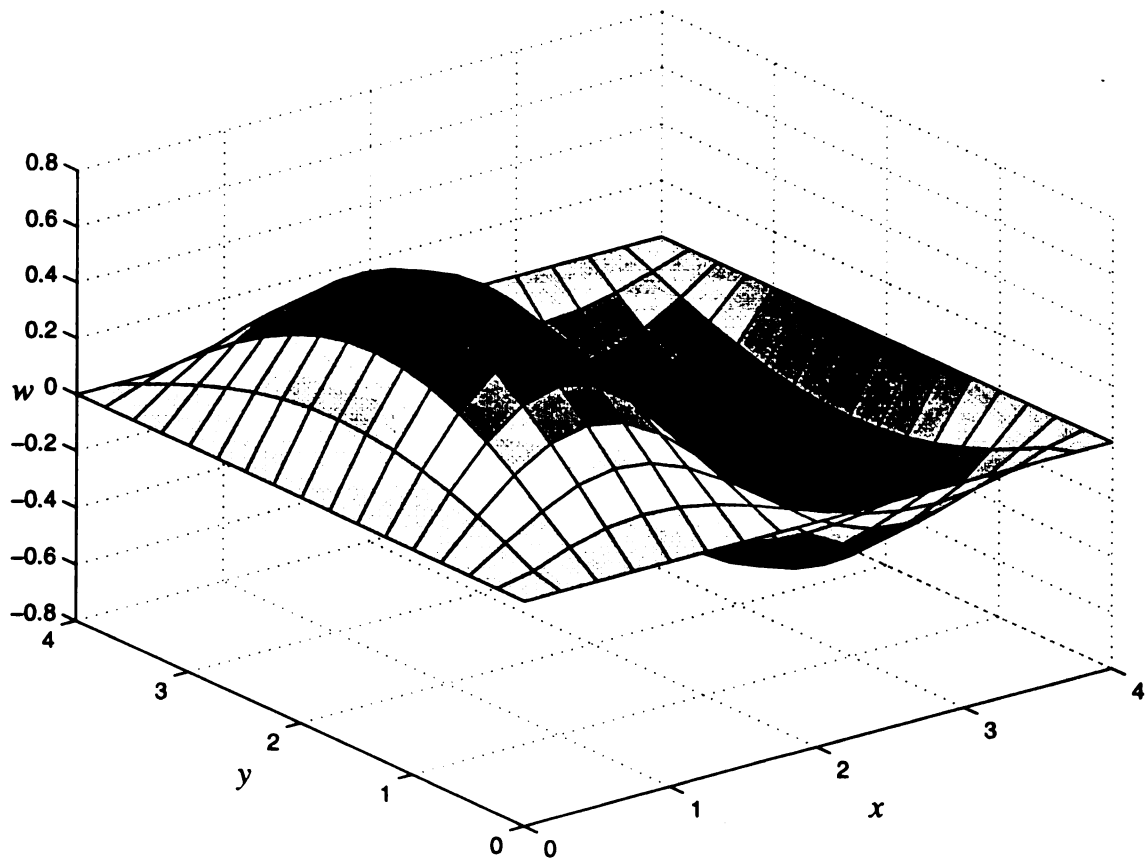
**Figure 46. Variation of normalized lowest natural frequency versus plate aspect ratio by consistent mass matrix and lumped mass matrix (FZZ3D).**



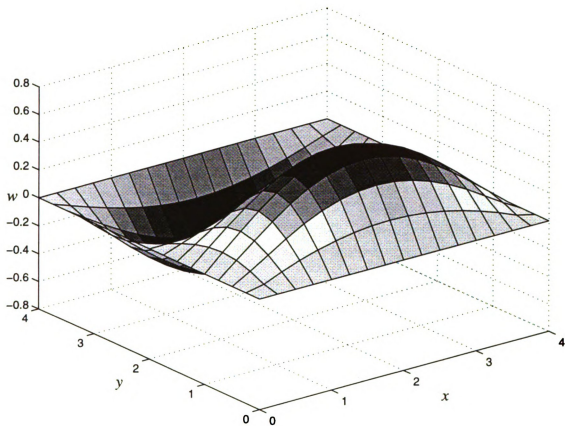
**Figure 47. Variation of normalized lowest natural frequency versus plate aspect ratio by consistent mass matrix and lumped mass matrix (CZZ3D).**



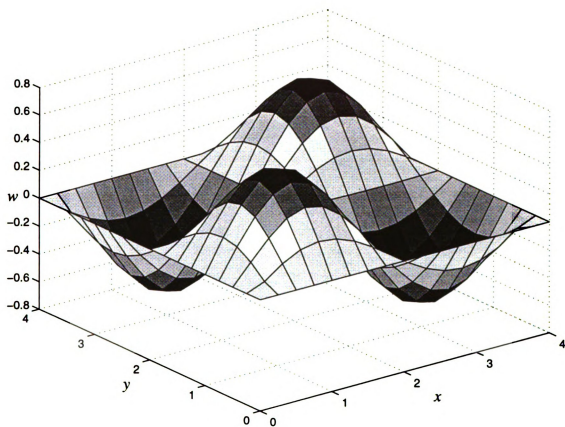
**Figure 48.** The first eigen mode of transverse deflection d.o.f of simply supported sandwich panel ( $a/h=4$ )



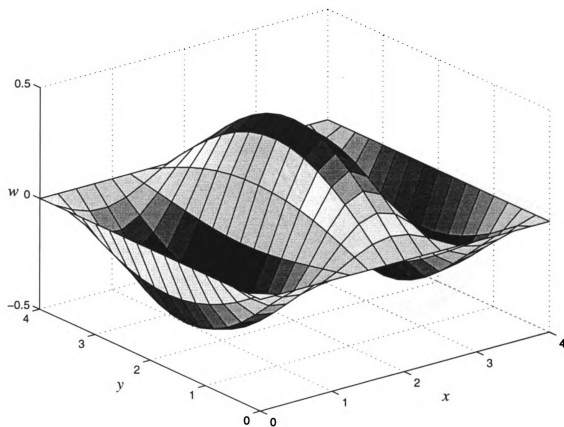
**Figure 49.** The second eigen mode of transverse deflection d.o.f of simply supported sandwich panel ( $a/h=4$ )



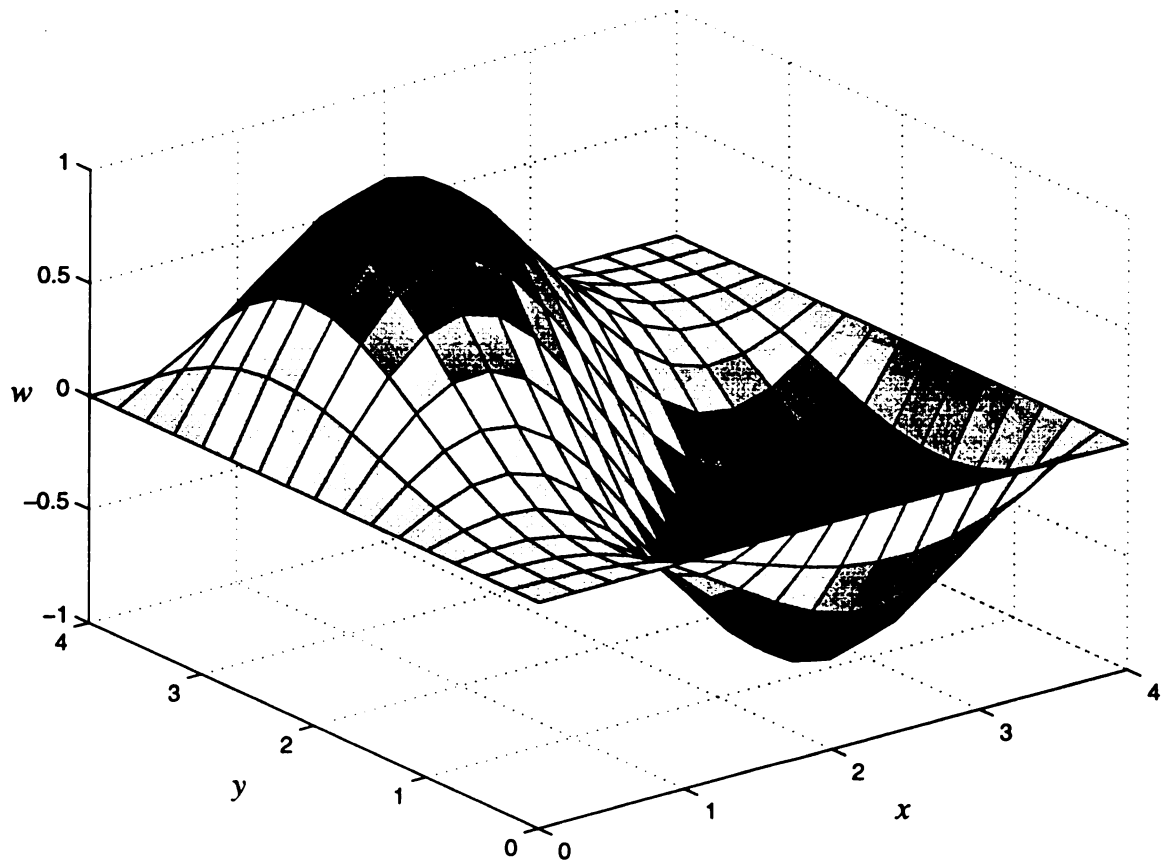
**Figure 50.** The third eigen mode of transverse deflection d.o.f of simply supported sandwich panel ( $a/h=4$ )



**Figure 51. The fourth eigen mode of transverse deflection d.o.f of simply supported sandwich panel ( $a/h=4$ )**

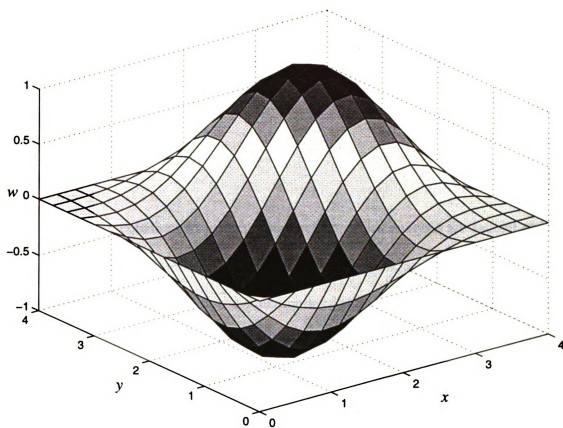


**Figure 52.** The fifth eigen mode of transverse deflection d.o.f of simply supported sandwich panel ( $a/h=4$ )



**Figure 53. The second eigen mode of transverse deflection d.o.f of simply supported isotropic plate ( $a/h=4$ )**





**Figure 54.** The third eigen mode of transverse deflection d.o.f of simply supported isotropic plate ( $a/h=4$ )

## **CHAPTER VI**

### **EXPLICIT DYNAMIC ANALYSIS**

#### **6.1 Introduction**

The governing equations in general structural dynamics analysis are second order or fourth order partial differential equations in the space domain, and hyperbolic in the time domain. In this case, the solution is a function defined both on the space domain and on the time domain. Methods of dynamic analysis have been developed to solve the dynamic equations on the time domain, and the popular methods are modal methods and direct integration methods. In modal method, the dynamic equations are uncoupled, and then each of the equations can be solved independently of the others [99,100]. In direct integration method the dynamic equations are discretized in time, and a sequence of simultaneous algebraic equations is obtained [99-101]. The direct integration methods can be divided into two types - explicit direct integration and implicit direct integration methods.

Since the developed elements are implemented into Argonne National Laboratory's in-house code, NEPTUNE [102-104], which utilizes explicit direct integration method, only the explicit direct integration method is studied in this chapter. In NEPTUNE the internal force vector is calculated from the developed elements at each time step.

## 6.2 Explicit Dynamic Analysis

In structural dynamic analysis, the major goal is to investigate how a structure moves with time under external excitation, that is the time-history of movement of the structure.

The dynamic equations for the undamped forced vibration of an elastic system undergoing small displacements can be derived by the principle of virtual work and rewritten as follows by describing displacement fields in terms of nodal displacement and shape functions.

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{R^{ext}(t)\} \quad (6.2.1)$$

where  $[M]$  is the global mass matrix,  $[K]$  is the global stiffness matrix,  $\{R^{ext}(t)\}$  is the external time varying force vector due to body forces, traction, concentrated forces:

$$\{R^{ext}\} = \int_{V_e} [N]^T \{F\} dV + \int_{S_e} [N]^T \{\Psi\} dS + \sum_{i=1}^n \{p\}_i \quad (6.2.2)$$

$\{F\}$ : body force vector,  $\{\Psi\}$ : traction vector,  $\{p\}$ : concentrated load vector

$\{u(t)\}$  and  $\{\ddot{u}(t)\}$  are global displacement and acceleration vectors respectively. The above dynamic equation is a system of coupled, second order and ordinary differential equations in time. Although displacement and acceleration vectors are discrete functions of space, they are still continuous functions of time. Methods of dynamic analysis have been developed to solve this equation, and direct integration methods are the popular methods to deal with a large problem. In the direct integration method, the finite element method is used in the space domain, and the finite difference method is applied in the time domain. Thus, the dynamic equations are discretized in time, and a sequence of simulta-

neous algebraic equations is obtained [99-101]. At a specific instant of time, the dynamic equation is:

$$[M]\{\ddot{u}(t)\}_n + [K]\{u(t)\}_n = \{R^{ext}(t)\}_n \quad (6.2.3)$$

where  $n$  represents time  $n\Delta t$  and  $\Delta t$  is the size of time step.

The explicit direct integration method utilizes the central difference method to solve a sequence of simultaneous algebraic equations. In the central difference method, velocity and acceleration at time  $(n\Delta t)$  is approximated as follows by expanding  $\{u(t)\}_{n+1}$  and  $\{u(t)\}_{n-1}$  in Taylor series about the time  $n\Delta t$ .

$$\begin{aligned} \{\dot{u}(t)\}_n &= \frac{1}{2\Delta t}(\{u(t)\}_{n+1} - \{u(t)\}_{n-1}) \\ \{\ddot{u}(t)\}_n &= \frac{1}{\Delta t^2}(\{u(t)\}_{n+1} - 2\{u(t)\}_n + \{u(t)\}_{n-1}) \end{aligned} \quad (6.2.4)$$

Substituting Eq. (6.2.4) into Eq. (6.2.3), the dynamic equation can be described as:

$$\frac{1}{\Delta t^2}[M]\{u(t)\}_{n+1} = \{R^{ext}(t)\}_n - [K]\{u(t)\}_n + \frac{1}{\Delta t^2}[M](2\{u(t)\}_n - \{u(t)\}_{n-1}) \quad (6.2.5)$$

In Eq. (6.2.5), the term,  $[K]\{u(t)\}_n$  is called the internal force vector and represents loads at nodes caused by straining of the material. The internal force vector can be redefined as:

$$\{R(t)^{int}\}_n = [K]\{u(t)\}_n = \int_{V_e} [B]^T [D] [B] dV \cdot \{u(t)\}_n = \int_{V_e} [B]^T \{\sigma\}_n dV \quad (6.2.6)$$

where  $[B]$  is the strain - displacement matrix,  $\{\sigma\}$  is the stress vector, and  $[D]$  is the

material property matrix. It can be seen in Eq. (6.2.6) that the internal force vector at each time step can be calculated by using the strain - displacement matrix and the stress vector, and the element stiffness matrix does not need to be calculated and assembled into a global stiffness matrix. Thus, the computing time can be decreased by reducing the number of matrix multiplications, and computer storage can be saved because element and global stiffness matrices need not be stored. In NEPTUNE the internal force vector is calculated for the developed element at each time step. Since the internal force vector needs to be calculated at each time step, the use of reduced integration quadrature can decrease computational cost, but in this case, hour glass control or a stability matrix is required to add artificial stiffness to the element to suppress the zero-energy modes [106,107]. Thus, the reduced integration quadrature will not be used.

If  $[M]$  is a diagonal matrix, then the Eq. (6.2.5) is a system of decoupled linear algebraic equations. By solving Eq. (6.2.5), the displacements at the end of the time step,  $n$  are obtained.

Since  $\{\sigma\}$  in Eq. (6.2.6) could be a nonlinear function of strain, Eq. (6.2.6) is valid for both linear and nonlinear material behavior. Therefore, the explicit method is well suited to treating material nonlinearity.

### 6.3 Critical Time Step

The central difference algorithm is relatively simple; however, it is only conditionally stable. To assure a stable solution, the computational time step must be equal to or less than the critical time step. Thus, the determination of the critical time step is a key to reducing computational time and maintaining a stable solution. Overly conservative esti-

mates will increase computing times and estimates that are too large will produce numerical instabilities.

### ***Stability Analysis of the Equations of Motion***

The method of assessing stability of the equations of motion based on direct integration methods can be roughly divided into two categories: spectral (Fourier) and energy stability analysis. In spectral stability analysis, a single equation of motion is derived by modally uncoupling the original global equations of motion, and then the effects of a time integration method on a single equation of motion are examined. While energy stability method utilizes the original global stiffness and mass matrices, and leads to the conditions that a norm of the solution at time,  $t = n\Delta t$  can be bounded a norm of the solution at time,  $t = 0$ . Since spectral stability analysis usually provides insight to interpret the results (stability conditions) and sometimes predicts more precise results, the spectral stability analysis will be discussed.

If the equations of motion based on direct integration methods are stable without any external loading, they will also be stable with external loading. Thus, we need to consider only the homogeneous form of the equations of motion in stability analysis. The modally uncoupled, undamped, and homogeneous equations of motion are given by:

$$\ddot{Z}_i + \omega_i^2 Z_i = 0 \quad (6.3.1)$$

where  $Z_i$  and  $\omega_i$  are the modal amplitude and natural frequency of mode  $i$ , respectively. The subscript  $i$  will be omitted for brevity hereafter. Applying the central difference method Eq. (6.2.4) to Eq. (6.3.1), a central difference approximation of Eq. (6.3.1) at time

$t = n\Delta t$  can be written as:

$$Z_{n+1} + (\omega^2 \Delta t^2 - 2)Z_n + Z_{n-1} = 0 \quad (6.3.2)$$

The exact solution of a central difference approximation, Eq. (6.3.2), consists of a linear combination of two parts as follows:

$$Z_n = C_1 \lambda_1^n + C_2 \lambda_2^n \quad \text{when } \lambda_1 \neq \lambda_2 \quad (6.3.3)$$

$$Z_n = C_1 \lambda_1^n + n\Delta t C_2 \lambda_2^n \quad \text{when } \lambda_1 = \lambda_2 \quad (6.3.4)$$

where  $C_1$  and  $C_2$  are constants and determined using initial conditions, and

$$\lambda_i = e^{\mu_i \Delta t} \quad i = 1, 2 \quad (6.3.5)$$

in which  $\mu_1$  and  $\mu_2$  are generally complex.  $Z_n$  of Eq. (6.3.3) will decay or remain steady only if  $\lambda_1 \neq \lambda_2$ ,  $|\lambda_1| \leq 1$  and  $|\lambda_2| \leq 1$ . Therefore, the criterion of time step size,  $\Delta t$ , providing a stable computation can be derived by assuming that  $\lambda_1 \neq \lambda_2$  and selecting initial conditions such that  $C_1$  or  $C_2$  is zero.  $Z_n$  of Eq. (6.3.4) becomes:

$$Z_n = C \lambda^n \quad \text{where } \lambda^n = e^{\mu n \Delta t} \quad (6.3.6)$$

Substituting Eq. (6.3.6) into Eq. (6.3.2) and dividing by  $C \lambda^{n-1}$ , the following characteristic equation can be obtained:

$$\lambda^2 + (\omega^2 \Delta t^2 - 2)\lambda + 1 = 0 \quad (6.3.7)$$

The solutions of Eq. (6.3.7) are:

$$\lambda_{1,2} = \frac{2 - \omega^2 \Delta t^2 \pm \omega \Delta t \sqrt{\omega^2 \Delta t^2 - 4}}{2} \quad (6.3.8)$$

Since the characteristic equation of Eq. (6.3.7) is quadratic, we can recognize that the multiplication of the solutions of the quadratic equation,  $\lambda_1 \lambda_2$  is 1. If the radicand of Eq. (6.3.8) is positive, that is,  $\sqrt{\omega^2 \Delta t^2 - 4} > 0$ , then both  $\lambda_1$  and  $\lambda_2$  are real and  $\lambda_1 \lambda_2 = 1$  implies that the absolute value of one  $\lambda$  is less than unity while that of the other is greater than 1. Thus instability results because one  $\lambda$  does not satisfy  $|\lambda| \leq 1$ . If the radicand is negative, that is,  $\sqrt{\omega^2 \Delta t^2 - 4} < 0$ , then  $\lambda_1$  and  $\lambda_2$  are complex conjugates and each is of unit modulus by virtue of  $\lambda_1 \lambda_2 = 1$ . Thus the dynamic equation is always stable. If  $\sqrt{\omega^2 \Delta t^2 - 4} = 0$ ,  $\lambda_1 = \lambda_2 = -1$  is obtained. This violates  $\lambda_1 \neq \lambda_2$ , which is required for stability. The stability condition of the equations of motion associated with central difference method is such that:

$$\Delta t < \frac{2}{\omega} \quad (6.3.9)$$

However the equation of motion of Eq. (6.3.1) is only one of many uncoupled equations. Thus we need to evaluate Eq. (6.3.1) for every possible mode and select the most conservative critical time step as follows:

$$\Delta t < \frac{2}{\omega_{max}} \quad (6.3.10)$$

where  $\omega_{max}$  is the maximum natural frequency. From Eq. (6.3.10) it can be observed that



the size of the time step depends on the highest frequency of the mesh. Moreover, it is known and mathematically proven that the highest frequency of the mesh is bounded by the maximum frequency of a single, unsupported element. For structural and continuum elements a bound on the critical time step can normally be obtained from the highest eigenvalue of the unconstrained elements. Simple formulas based on element eigenvalues have been derived for some elements [111,112], however, isotropic elastic material properties were assumed. The upper bound of the highest frequency of an element can be obtained by Gerschgorin's theorem [113].

### ***Gerschgorin's Theorem***

The approach followed here to obtain an estimate for the critical time step is to apply Gerschgorin's theorem to the unassembled element stiffness matrix of each element in the mesh. The time step,  $\Delta t_{il}^e$ , for the  $i$ th degree-of-freedom of node  $I$  of element  $e$  can be expressed as [114]:

$$\Delta t_{il}^e \leq \frac{2(m_{ilil})^{0.5}}{\left( K_{ilil} + \sum_{\substack{i \neq j \\ I \neq J}} |K_{iljJ}| \right)^{0.5}} \quad (6.3.11)$$

where  $m_{ilil}$  is the element nodal mass of node  $I$  in the  $i$ th direction,  $K_{ilil}$  is the diagonal stiffness, and  $K_{iljJ}$  is an off-diagonal stiffness. The maximum time step for the mesh is determined from:

$$\Delta t_{crit} = \min(\Delta t_{il}^e) \quad (6.3.12)$$

In order to decrease computational cost, it is desirable to decrease the highest frequency of a single element. It is known that a lumped mass matrix provides for larger time steps. Moreover, a lumped mass matrix provides uncoupled dynamic equations and generally more accurate results than a consistent mass matrix.

## 6.4 Amplitude and Period Errors in Central Difference Approximation

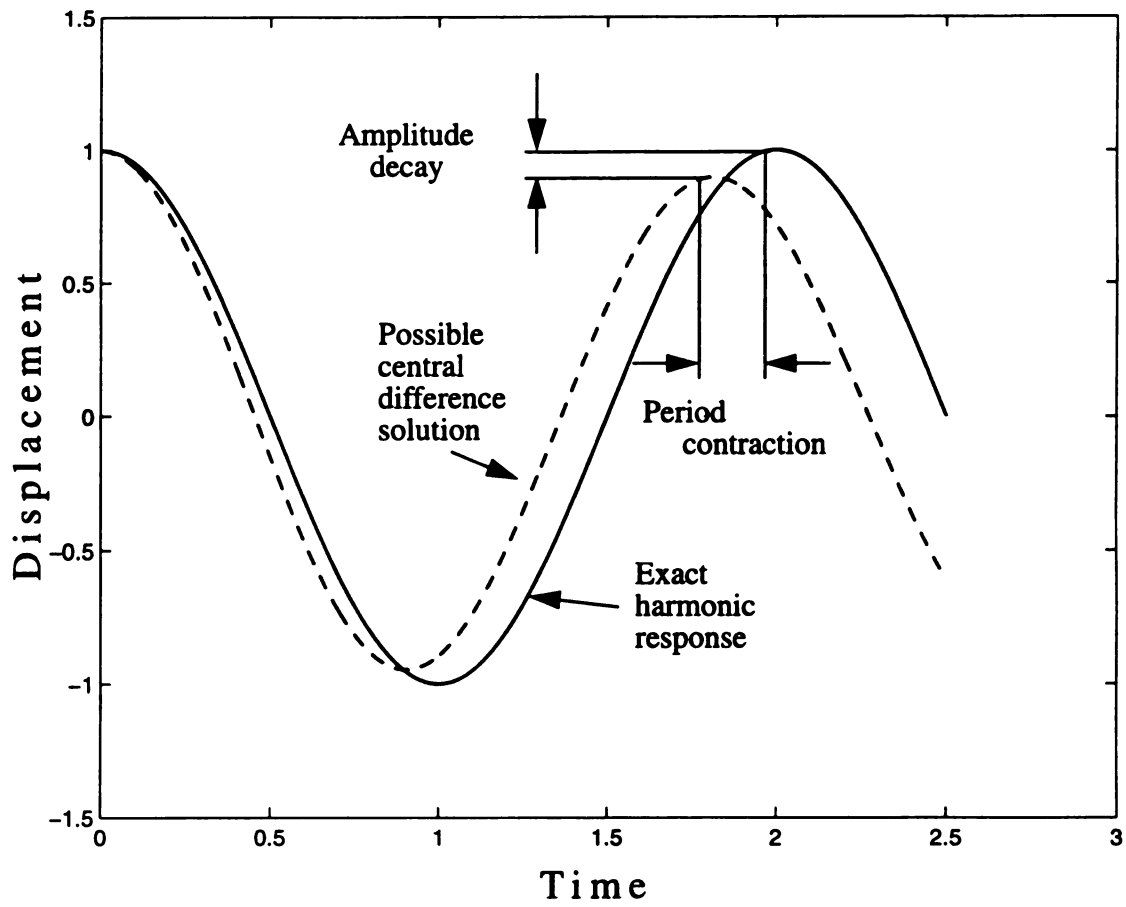
In order to analyze the errors in using the central-difference method, the modally uncoupled, undamped, and homogeneous equation of motion, Eq. (6.3.1), is utilized. The exact solution of Eq. (6.3.1) is given by:

$$Z_n^{exact} = \tilde{C}_1(\cos \omega t + i \sin \omega t) + \tilde{C}_2(\cos \omega t - i \sin \omega t) \quad (6.4.1)$$

where  $\tilde{C}_1$  and  $\tilde{C}_2$  are determined from initial conditions, and  $i = \sqrt{-1}$ . However, the approximate solution of Eq. (6.3.4) obtained by the central difference method may have amplitude error and period error. Amplitude error can be either amplitude increase or amplitude decay. The amplitude increase can be interpreted as instability, and amplitude decay can be viewed as artificial damping or viscosity. Period error can be either period elongation or period contraction [99]. These errors are illustrated in Figure 55. Since  $\lambda_1$  and  $\lambda_2$  of Eq. (6.3.5) are complex conjugates,  $\mu_1$  and  $\mu_2$  are also conjugate and can be written as:

$$\mu_1 = a + ib \quad \mu_2 = a - ib \quad (6.4.2)$$

Thus,



**Figure 55. Amplitude and period errors in central difference solution**

$$\begin{aligned}
\lambda_1^n &= e^{\mu_1 n \Delta t} = e^{an \Delta t} e^{ibn \Delta t} = e^{an \Delta t} (\cos bn \Delta t + i \sin bn \Delta t) \\
\lambda_2^n &= e^{\mu_2 n \Delta t} = e^{an \Delta t} e^{-ibn \Delta t} = e^{an \Delta t} (\cos bn \Delta t - i \sin bn \Delta t)
\end{aligned}
\tag{6.4.3}$$

Substitution of Eq. (6.4.3) into the exact solution of a central difference approximation, Eq. (6.3.3) gives:

$$\begin{aligned}
Z_n &= C_1 \lambda_1^n + C_2 \lambda_2^n \\
&= C_1 e^{an \Delta t} (\cos bn \Delta t + i \sin bn \Delta t) + C_2 e^{an \Delta t} (\cos bn \Delta t - i \sin bn \Delta t)
\end{aligned}
\tag{6.4.4}$$

When the exact solution of the central difference approximation, Eq. (6.4.4) is compared with the exact analytical solution of Eq. (6.4.1), the possible errors can be distinguished. If  $a \neq 0$ , amplitude error occurs, and if  $b \neq \omega$ , period error result.

### ***Period Error***

Period error,  $P$ , is defined as:

$$P = \frac{2\pi/\omega}{2\pi/b} = \frac{\omega}{b} \tag{6.4.5}$$

where  $2\pi/\omega$  and  $2\pi/b$  are the period of the actual system and the period of the system created by the central difference algorithm. In order to characterize the period error, we need to determine  $b$  of Eq. (6.4.5).  $b$  is obtained by using Eq. (6.4.3) with  $n = 1$  in the following procedure:

$$\frac{\text{Im}(\lambda)}{\text{Re}(\lambda)} = \frac{\sin(b \Delta t)}{\cos(b \Delta t)} = \tan(b \Delta t) \tag{6.4.6}$$

Using Eq. (6.3.8) (remember that the radicand in Eq. (6.3.8) is complex), the following

equation is derived:

$$\frac{\text{Im}(\lambda)}{\text{Re}(\lambda)} = \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \quad (6.4.7)$$

Equating Eq. (6.4.6) and Eq. (6.4.7),  $b$  is found to be:

$$b = \frac{1}{\Delta t} \text{atan} \left( \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \right) \quad (6.4.8)$$

Substitution of Eq. (6.4.8) into Eq. (6.4.5) yields the period error of the central difference method as:

$$P = (\omega \Delta t) / \text{atan} \left( \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \right) \quad (6.4.9)$$

The possible period error can be one of the followings:

$$\begin{array}{ll} P > 1 & \text{period elongation} \\ P = 1 & \text{no period error} \\ P < 1 & \text{period contraction} \end{array} \quad (6.4.10)$$

From Eq. (6.4.9), we can see that the period error is subject to the value of  $\Delta t$ . Furthermore, the frequency,  $\omega$  is the natural frequency for mode  $i$  among many possible modes that a multi-d.o.f structure can have. Therefore the appropriate selection of the time step,  $\Delta t$ , can prevent period error only for one mode. In addition, even though  $\Delta t$  is chosen to be small enough to maintain the stable condition of the equations of motion, the period error occurs. Hence it can be concluded that the period error always happens in the central

difference method. However it is observed that period error is partially compensated when a lumped mass matrix is used with the central difference method.

### ***Amplitude Error***

The following amplitude errors are possible:

$$\begin{aligned}
 a > 1 & \quad \text{amplitude growth (instability)} \\
 a = 1 & \quad \text{no amplitude error} \\
 a < 1 & \quad \text{amplitude decay (artificial damping)}
 \end{aligned} \tag{6.4.11}$$

Amplitude error in the central difference method can be checked as in the following procedure by obtaining the expression for  $a$ . The sum of  $\lambda_1$  and  $\lambda_2$  using Eq. (6.4.3) with  $n = 1$  is expressed as:

$$\begin{aligned}
 \lambda_1 + \lambda_2 &= e^{a\Delta t} (\cos b\Delta t + i \sin b\Delta t) + e^{a\Delta t} (\cos b\Delta t - i \sin b\Delta t) \\
 &= 2e^{a\Delta t} \cos b\Delta t
 \end{aligned} \tag{6.4.12}$$

While the sum of  $\lambda_1$  and  $\lambda_2$  using Eq. (6.3.8) is derived as:

$$\begin{aligned}
 \lambda_1 + \lambda_2 &= \frac{2 - \omega^2 \Delta t^2 + \omega \Delta t \sqrt{\omega^2 \Delta t^2 - 4}}{2} + \frac{2 - \omega^2 \Delta t^2 - \omega \Delta t \sqrt{\omega^2 \Delta t^2 - 4}}{2} \\
 &= 2 - \omega^2 \Delta t^2
 \end{aligned} \tag{6.4.13}$$

Equating Eq. (6.4.12) to Eq. (6.4.13) provides:

$$2e^{a\Delta t} \cos b\Delta t = 2 - \omega^2 \Delta t^2 \tag{6.4.14}$$

Substitution of the expression for  $b$  of Eq. (6.4.8) into Eq. (6.4.14) leads to the following expression:

$$a\Delta t = \ln \left\{ \frac{1 - (\omega^2 \Delta t^2)/2}{\cos \left[ \operatorname{atan} \left( \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \right) \right]} \right\} \quad (6.4.15)$$

If Eq. (6.4.15) is evaluated for  $0 \leq \omega \Delta t < 2$ , it is shown that  $a\Delta t = 0$ . Hence we can conclude that the central difference method has no amplitude error. However, we need to pay attention to the fact that this does not imply that the amplitude predicted by the central difference method agrees with the exact one. When the time step is very close to the critical time step, the radicand of Eq. (6.3.8) approaches to zero. Thus,  $\lambda_1$  and  $\lambda_2$  are almost equal and the expressions in the parentheses of Eq. (6.4.4) are almost linear combinations of one another. In this case,  $C_1$  and  $C_2$  of Eq. (6.4.4) may be very different from  $\tilde{C}_1$  and  $\tilde{C}_2$  of Eq. (6.4.1), and hence the beating phenomenon occurs. Thus, zero amplitude error in central difference method implies that the numerical solution provides a mean value of amplitude that does not grow or decay in comparison with the exact solution.

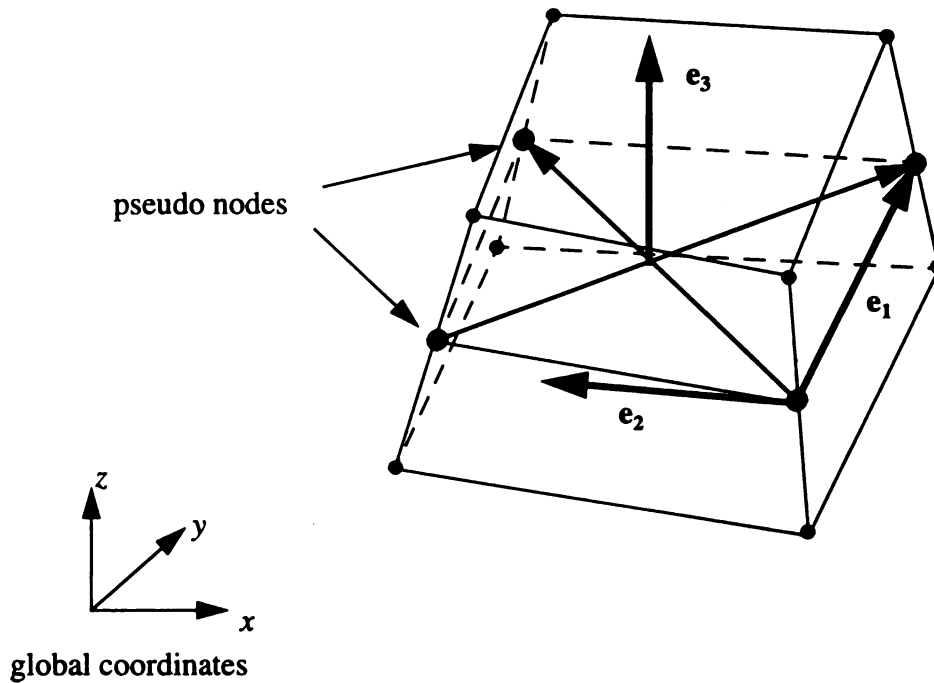
## 6.5 Geometrical Nonlinearities - Corotation Approach

The use of the present element in crashworthiness simulations requires the capability to handle geometrical nonlinearities. Geometrical nonlinearities can be accounted for using a co-rotation approach [108-111], wherein a local coordinate system is attached to the element and moves in space with the element. For this eight-noded hex-plate element, a corotational coordinate system is embedded in the mid-plane of the element and translates and rotates with the element. The coordinate system is developed as follows. A set of mid-

plane pseudo-node coordinates,  $\bar{X}_i$ ,  $\bar{Y}_i$  and  $\bar{Z}_i$ , is defined between each pair of corner nodes as follows:

$$\begin{aligned}\bar{X}_i &= \frac{1}{2}(x_i + x_{i+4}) \\ \bar{Y}_i &= \frac{1}{2}(y_i + y_{i+4}) \\ \bar{Z}_i &= \frac{1}{2}(z_i + z_{i+4})\end{aligned} \quad i = 1 \dots 4 \quad (6.5.1)$$

The orthonormal basis vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are computed from the pseudo-nodes. The  $\mathbf{e}_1$  basis vector is taken to be along the line connecting pseudo-nodes 1 and 2. The  $\mathbf{e}_3$  basis is normal to the cross product of the diagonal vectors between pseudo-nodes 1 and 3 and pseudo-nodes 2 and 4. The remaining basis vector,  $\mathbf{e}_2$ , is obtained from the cross product of  $\mathbf{e}_3$  and  $\mathbf{e}_1$ .



**Figure 56. The corotational coordinate system**



## 6.6 Numerical Results

The numerical convergence and accuracy of the transient behavior of the previously developed element are investigated through several cases. In order to prove the validity of the present formulation, the results obtained are compared with those available in the literature. In the examples, it is assumed that the plates remain elastic.

### ***Example 1. Simply Supported 0° Orthotropic Plate***

A simply supported 0° orthotropic plate subjected to a suddenly applied uniformly distributed load of 0.1 MPa is considered. The dimensions of the plate are 0.25 m x 0.25 m with aspect ratio 5. A 4 x 4 mesh is used in a quarter plate model due to the biaxial symmetry of the problem. The material properties of material 1 in Table 11 are used. The loading and boundary conditions and the geometry are shown in Figure 57. From the Gershgorin's theorem, Eq. (6.3.11) the critical time step,  $\Delta t = 1.2337\text{e-}7$  sec. was obtained. However, the larger time step  $\Delta t = 8.542\text{e-}7$  sec. derived by the trial and error method was used in the analysis. The deflection at the center of the plate is obtained and compared with Kant *et al.*'s solutions using nonlinear first and high order shear deformation theories [115], and Swaddiwudhipong *et al.*'s nonlinear results [116]. It is shown in Figure 58 that the amplitude and the period response at the center of the plate are in good agreement with Kant *et al.* and Swaddiwudhipong *et al.*'s predictions. The small deviation in the amplitude and period of the response may come from the fact that the plate is very thick ( $a/h = 5$ ) so that the transverse normal effect, which only the current element has the ability to predict, is significant. The top and bottom deflections are different, and a through-thickness (barrelling) mode of vibration can be observed in Figure 58.

### ***Example 2. Simply Supported Two-Layered Angle-Ply Plate***

The behavior of a simply supported two-layered angle-ply ( $-45^\circ/45^\circ$ ) plate is investigated in this example. The thickness of each layer is 0.025 m. The material properties, the loading and the geometry are the same as in the example 1. However, the biaxial symmetry can not be utilized even when the plate geometry and boundary condition are symmetric, because the bending is coupled with stretching in an angle ply. Thus, the full plate is modeled with  $8 \times 8$  mesh. The purpose of this example is to demonstrate the effect of lamination scheme on the transient response of a composite plate. The transient response of center deflection of plate is shown in Figure 59, and it indicates good agreement with Kant *et al.* and Swaddiwudhipong *et al.*'s prediction. If Figure 59 is compared with Figure 58, it is observed that the difference in the lamination scheme causes a big difference in both the amplitude and period distribution in the dynamic response of a laminated plate.

### ***Example 3. The Effect of Material Orthotropy on the Transient Center Deflection***

#### ***Analysis***

In order to examine the effect of material orthotropy on the transient center deflection analysis, the following case was analyzed. The loading, boundary conditions and geometry used are as shown in Figure 57. Due to the biaxial symmetry of the problem, a  $4 \times 4$  mesh is used in the quarter plate. The Materials 2 and 3 in Table 11 is used for orthotropic plate and isotropic plates, respectively. It is shown in Figure 60 that even though only  $E_{11}$ 's are different among the material properties, the effect of orthotropy results in decreasing both amplitude and period of center deflection [117].

***Example 4. An Impulsively Loaded Cylindrical Sandwich Panel***

The panel is clamped at both ends along its length and simply supported at both ends along the circumferential direction as shown in Figure 58. The dynamic response of the sandwich panel was obtained in the elastic range when the impact load is applied on the panel at the initial velocity of 5650 in/sec over a 3.08 in x 10.205 in area. The material properties and the lamination scheme of the panel are shown in Table 8 and Table 9, respectively. A mesh of 16x8 elements was utilized in the panel. Using Eq. (6.3.11)  $\Delta t = 2.92459\text{E-}07$  was obtained as the critical time step. However, the larger time step  $\Delta t = 2.60\text{E-}06$  was derived by the trial and error method.

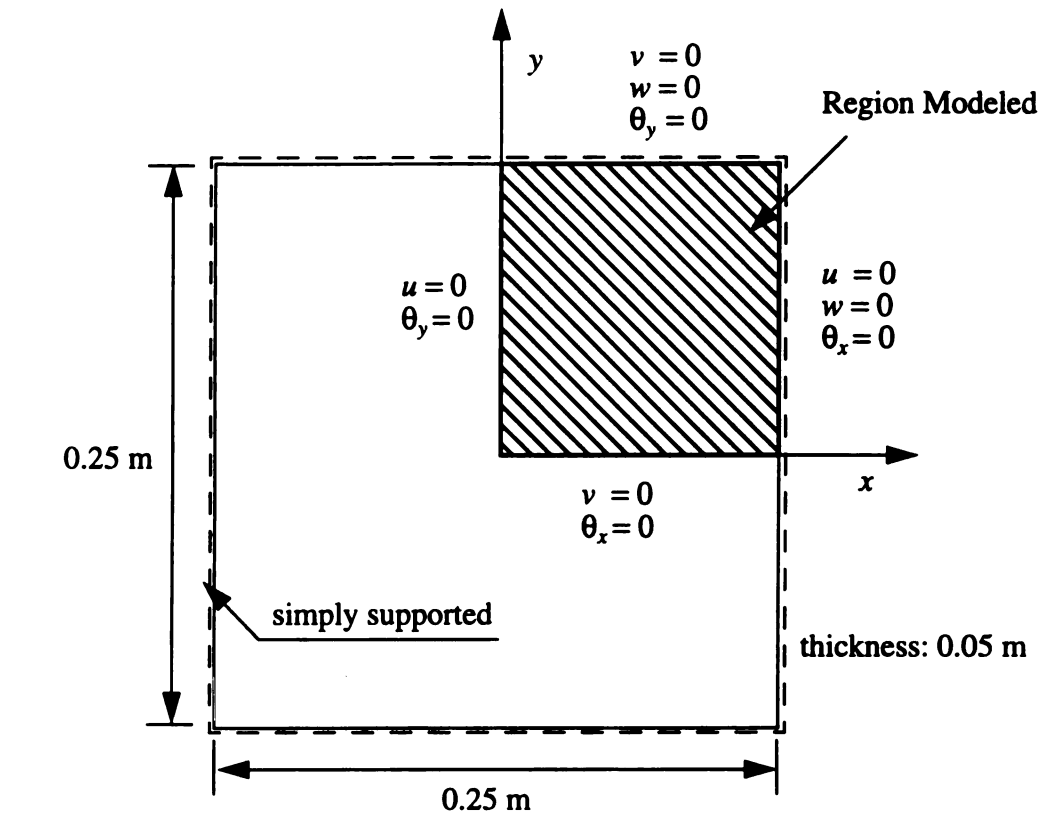
In Figure 61, the time history of displacement in the y direction of the panel subjected to impulsive loading is represented at the middle of the outer surfaces of the panel and at node 13 ( $x = -1.31134\text{E-}07$ ,  $y = 3$ ,  $z = -9.42$ ). By comparison of examples 1, 2 and 3, it is shown that the response of the sandwich panel is irregular in the periodicity of amplitude and period, primarily due to geometric nonlinearity.

## **6.7 Summary**

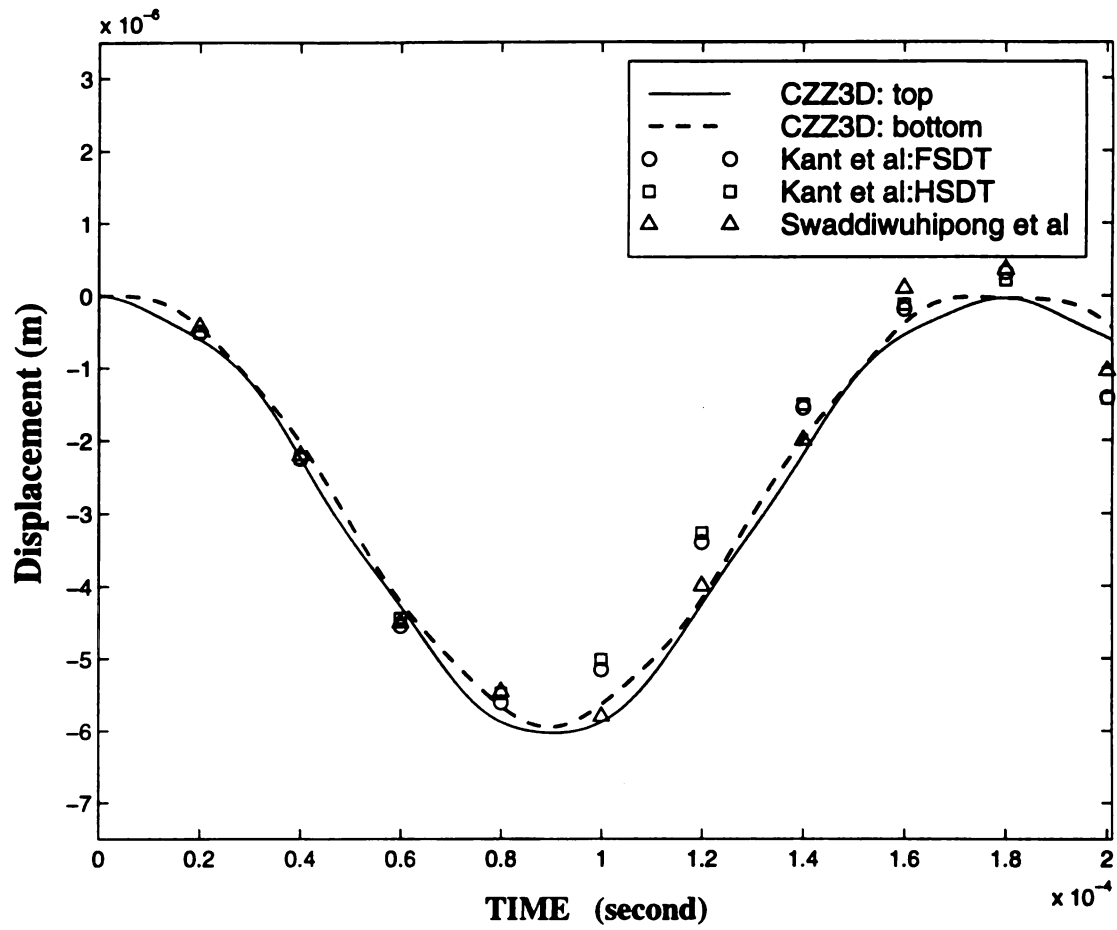
For explicit structural dynamics analysis, the elements are implemented into Argonne National Laboratory's in-house code, NEPTUNE wherein the internal force vector at each time step is calculated for the developed element. Gerschgorin's theorem predicts the lower bound of critical time step of an element, and the larger time step can be obtained by try and error method based on the prediction of Gerschgorin's theorem. Numerical results demonstrate that the current models are accurate, efficient, and robust for dynamic analysis of laminated plates.

**Table 11. Material properties**

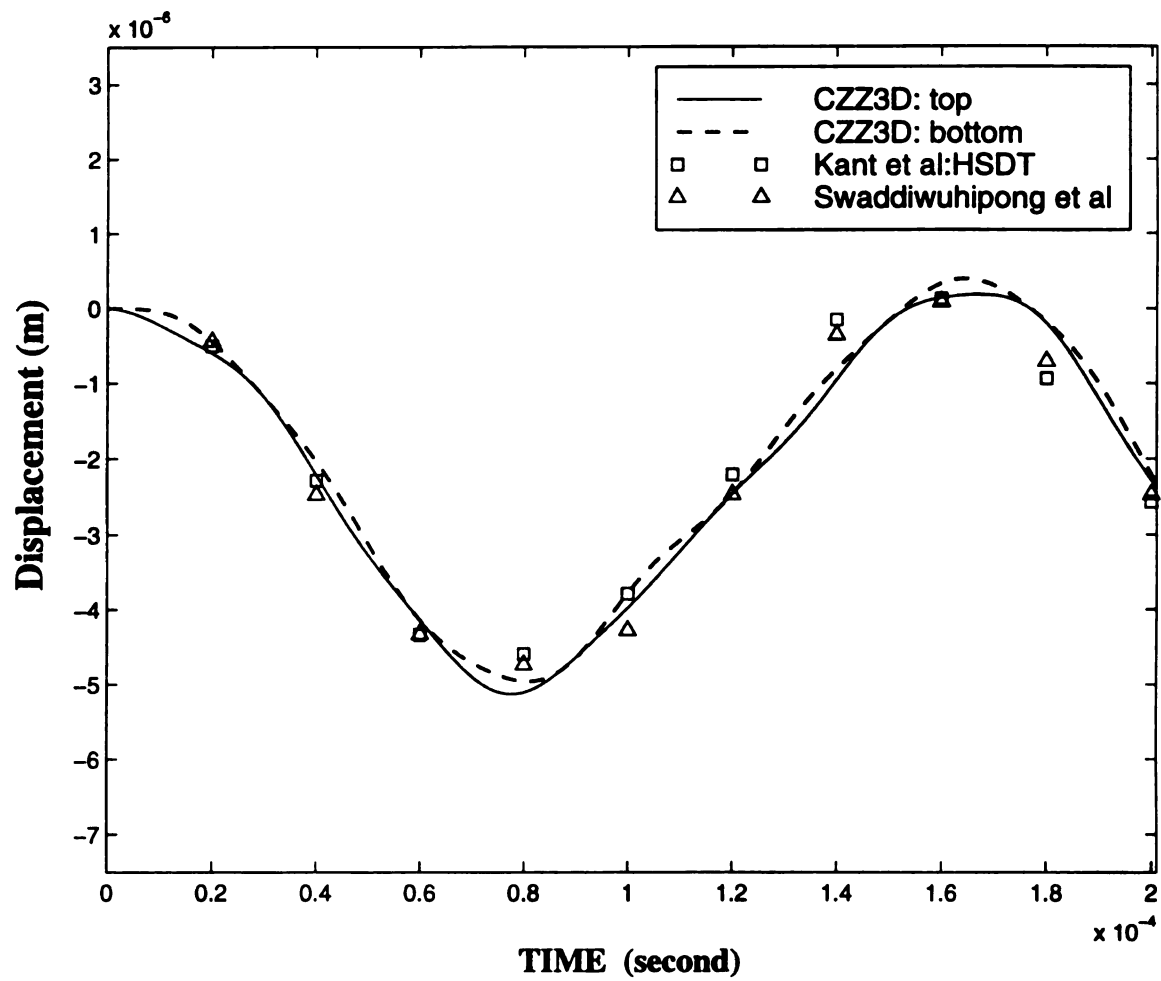
|            | Material 1 | Material 2 | Material 3 |
|------------|------------|------------|------------|
| $E_{11}$   | 5.25e11    | 5.25e11    | 21e9       |
| $E_{22}$   | 21e9       | 21e9       | 21e9       |
| $E_{33}$   | 21e9       | 21e9       | 21e9       |
| $\nu_{12}$ | 0.25       | 0.25       | 0.25       |
| $\nu_{23}$ | 0.25       | 0.25       | 0.25       |
| $\nu_{13}$ | 0.25       | 0.25       | 0.25       |
| $G_{12}$   | 1.05e10    | 8.4e9      | 8.4e9      |
| $G_{23}$   | 1.05e10    | 8.4e9      | 8.4e9      |
| $G_{13}$   | 1.05e10    | 8.4e9      | 8.4e9      |
| $\rho$     | 0.8e3      | 0.8e3      | 0.8e3      |



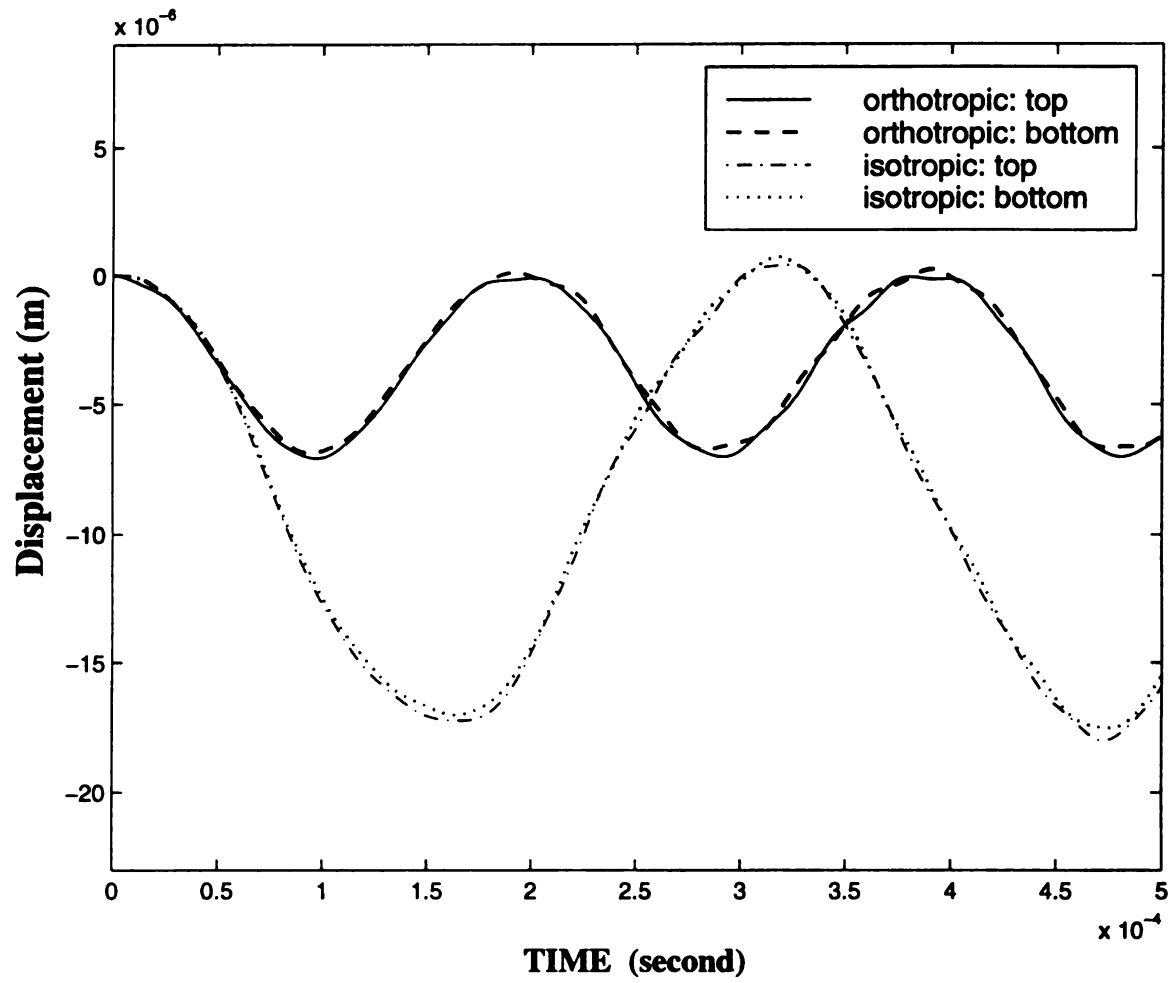
**Figure 57. The loading and boundary conditions and the geometry of plate**



**Figure 58. The time history of Center deflection of orthotropic square plate under suddenly applied pressure loading.**

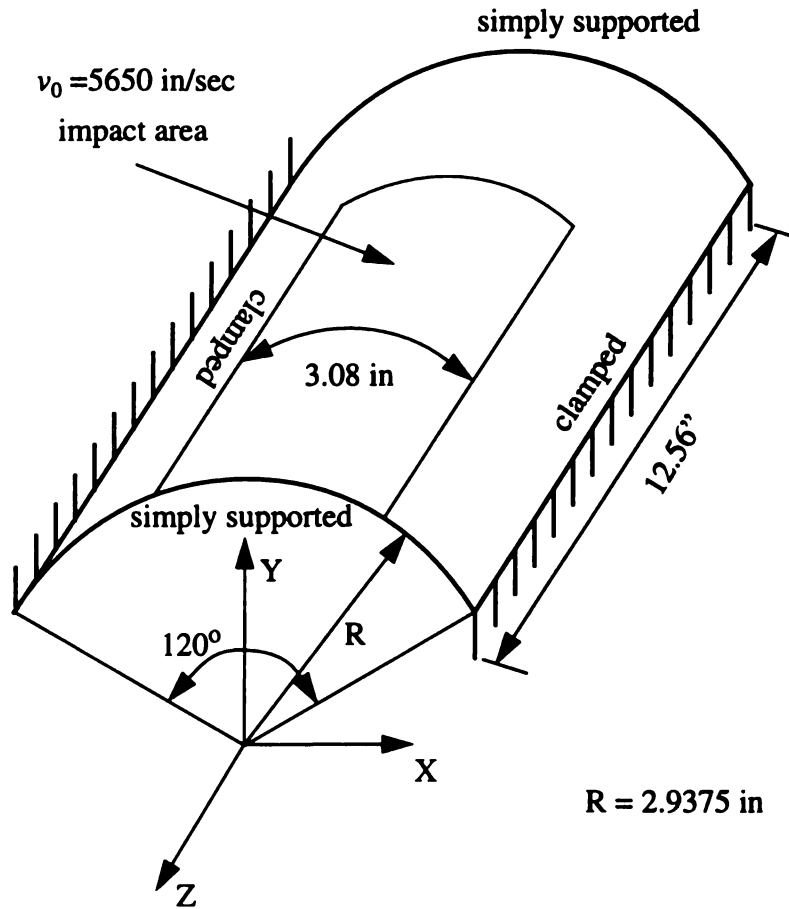


**Figure 59.**The transient response at the center of two-layered angle-ply ( $-45^\circ/45^\circ$ ) plate under suddenly applied pressure loading.

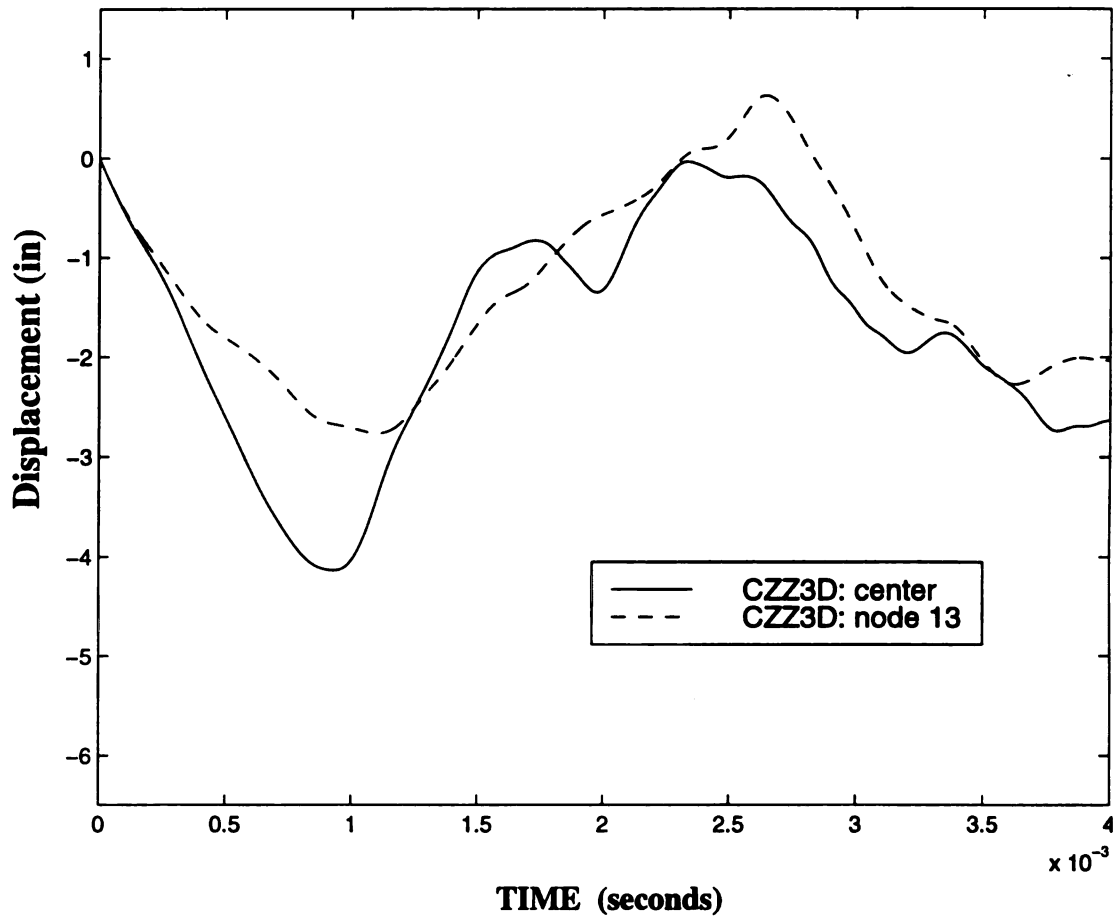


**Figure 60. The effect of material orthotropy on the transient center deflection analysis**





**Figure 61. An Impulsively Loaded Cylindrical Sandwich Panel**



**Figure 62. The time history of displacement of the sandwich cylindrical panel at the center of outer surface and at node 13.**

## **CHAPTER VII**

### **CONCLUSIONS**

#### **7.1 Summary**

The primary goals of the current research to develop the first order and the third order zig-zag  $C^0$  laminated plate theories and associated finite element models are accomplished. On the basis of the developed elements, static, free vibration, and explicit dynamic analysis for sandwich and laminated composite panels were performed. The first order zig-zag model allows the representation of a laminate as an assemblage of sublaminae in order to refine the model through the thickness, when needed. In the third order zig-zag model, discretization through the thickness of a laminate is not allowed. However, since in-plane displacement fields are assumed to vary in a piecewise cubic fashion through the thickness of a laminate, accurate analysis of composite behavior can be achieved without any discretization through the thickness. In particular, the associated element is very efficient to analyze and predict the crack (delamination) propagation as time marches in explicit dynamic analysis of composite plates under impact loads. In these plate theories, all variables are required to be  $C^0$  continuous, and the total number of degrees of freedom is independent of the number layer in a laminate. The first order zig-zag theory has some common features of other approaches of laminated plate theories,

namely the discrete layerwise and the zig-zag plate theories. The cubic zig-zag plate theory has some common characteristics of conventional cubic zig-zag theories. However, both the first order and the cubic order zig-zag theories can be distinguished from the already existing theories in that the variables are expressed in terms of surface quantities in order to facilitate the development of convenient finite element models. In each of these models, the transverse normal strain is improved by assuming a constant normal stress.

## 7.2 Conclusions

With regard to the use of the proposed laminated plate theories and associated finite element approximations for static and explicit dynamic structural analysis of sandwich and laminated composite panels, the following conclusions can be made:

- Zig-zag discrete layer-wise plate theories wherein local effects are accounted for very well are developed on the basis of 3-D kinematics.
- The continuity conditions of transverse shear stresses are explicitly satisfied in the derivation of displacement fields, and hence transverse shear stresses are continuous through the thickness.
- The transverse shear stresses in the first order zig-zag theory are constant within a sub-laminate.
- The number of degrees of freedom of the finite element model associated with the plate theory are independent of the number of layers in the composites.
- The developed finite elements are “regenerated” elements in the form of an eight-noded three dimensional element, and they have only engineering degrees of freedom (3 translations and 2 rotations) to facilitate implementation into commercial finite ele-

ment codes.

- The constant transverse normal strain is improved by assuming constant transverse normal stress instead, and the Poisson stiffening effect in the associated finite element is thus alleviated.
- The element stiffness coefficients are integrated exactly, yet the element exhibits no shear locking. This is achieved by using the consistent transverse shear strain fields as well as the edge-consistency of the tangential shear strain on any common inter-element edge.
- To be used in structural dynamics problems, the consistent mass matrices are developed. On the basis of the consistent mass matrices, the lumped mass matrices according to the HRZ lumping scheme are developed.
- The predicted highest natural frequency for the given mesh by employing a lumped mass matrix is lower than that by employing a consistent mass matrix.
- When the plate is thick, rotary inertia terms are found to play a very important role in the prediction of natural frequencies. Thus, the rotary inertia terms should not be neglected in the lumping scheme.
- Gershgorin's theorem predicts a conservative critical time step for explicit structural dynamic analysis.
- In structural dynamics problems, the internal force vector is calculated for the developed element.
- Numerical results demonstrate that the current models are accurate, efficient, and robust for analysis of a wide variety of thick or thin laminated plates in static and explicit dynamic analysis.

### 7.3 Recommendation

There are many needs to improve the proposed plate theories and associated finite elements in order to obtain better prediction of static and dynamic responses of laminated plates and sandwich panels.

- The developed elements have the capability to simulate shell analysis by allowing the in-plane integration points of Gauss quadrature to be varied in each layer. The shell geometry can not be described exactly by this method, however, reasonable results can be obtained when the curvature radius is not too small. Thus, by introducing transformation matrix to transform stiffness matrix and force vector in local coordinate to global coordinate, shell analysis can be performed.
- In the developed plate theories, it is assumed that the thickness of a laminate is constant. To account for the change of thickness of a laminate due to ply drop-off etc., it is desirable to add the capability to model non-uniform thickness through in-plane directions.
- For computationally efficient explicit dynamic analysis, an element with a larger critical time step needs to be developed. This can be accomplished by adding rigid motion in the transverse direction between the top and bottom nodes.
- In order to predict the location and time of delamination commencement due to dynamic loading, it is desirable to obtain the time history of stress in the composite plate.

## **APPENDICES**

## **APPENDIX A**



## APPENDIX A

### • *Shape Functions:*

The rotational variables  $\psi_x$  and  $\psi_y$  are eliminated in favor of the translational variables  $u_i$  and  $v_i$ , the inplane translations at the top surface of the sublaminate. In that process the following variables are obtained.

$$\begin{aligned} A &= h + \sum_{i=1}^{N_m} (h - z_i) d_i & B &= \sum_{i=1}^{N_m} (h - z_i) c_i \\ C &= \sum_{i=1}^{N_m} (h - z_i) b_i & D &= h + \sum_{i=1}^{N_m} (h - z_i) a_i & \bar{\Delta} &= A \cdot D - B \cdot C \end{aligned} \quad \text{A.(1)}$$

where  $h$  is the thickness of  $m$ th sublaminate, and  $a_i, b_i, c_i, d_i$  are defined in Eq. (3.3.11).

On the basis of the variable of Eq. A.(1), new variables are introduced as:

$$\begin{aligned} \alpha_1 &= \frac{D}{\bar{\Delta}}, & \alpha_2 &= -\frac{B}{\bar{\Delta}}, & \alpha_3 &= \frac{\{B \cdot C - D \cdot (A - h)\}}{2\bar{\Delta}}, & \alpha_4 &= \frac{-B \cdot h}{2} \\ \beta_1 &= -\frac{C}{\bar{\Delta}}, & \beta_2 &= \frac{A}{\bar{\Delta}}, & \beta_3 &= -\frac{C \cdot h}{2\bar{\Delta}}, & \beta_4 &= \frac{\{B \cdot C - A \cdot (D - h)\}}{2\bar{\Delta}} \end{aligned} \quad \text{A.(2)}$$

$$\begin{aligned} \bar{A}_1 &= \alpha_1 + \beta_1 \sum_{i=1}^{k-1} c_i + \alpha_1 \sum_{i=1}^{k-1} d_i & \bar{A}_2 &= \beta_1 \sum_{i=1}^{k-1} c_i z_i + \alpha_1 \sum_{i=1}^{k-1} d_i z_i \\ \bar{B}_1 &= \alpha_2 + \beta_2 \sum_{i=1}^{k-1} c_i + \alpha_2 \sum_{i=1}^{k-1} d_i & \bar{B}_2 &= \beta_2 \sum_{i=1}^{k-1} c_i z_i + \alpha_2 \sum_{i=1}^{k-1} d_i z_i \\ \bar{C}_1 &= \alpha_3 + \beta_3 \sum_{i=1}^{k-1} c_i + \left(\alpha_3 + \frac{1}{2}\right) \sum_{i=1}^{k-1} d_i & \bar{C}_2 &= \beta_3 \sum_{i=1}^{k-1} c_i z_i + \left(\alpha_3 + \frac{1}{2}\right) \sum_{i=1}^{k-1} d_i z_i \\ \bar{D}_1 &= \alpha_4 + \left(\beta_4 + \frac{1}{2}\right) \sum_{i=1}^{k-1} c_i + \alpha_4 \sum_{i=1}^{k-1} d_i & \bar{D}_2 &= \left(\beta_4 + \frac{1}{2}\right) \sum_{i=1}^{k-1} c_i z_i + \alpha_4 \sum_{i=1}^{k-1} d_i z_i \\ \hat{A}_1 &= \beta_1 + \beta_1 \sum_{i=1}^{k-1} a_i + \alpha_1 \sum_{i=1}^{k-1} b_i & \hat{A}_2 &= \beta_1 \sum_{i=1}^{k-1} a_i z_i + \alpha_1 \sum_{i=1}^{k-1} b_i z_i \end{aligned} \quad \text{A.(3)}$$

$$\begin{aligned}
\hat{B}_1 &= \beta_2 + \beta_2 \sum_{i=1}^{k-1} a_i + \alpha_2 \sum_{i=1}^{k-1} b_i & \hat{B}_2 &= \beta_2 \sum_{i=1}^{k-1} a_i z_i + \alpha_2 \sum_{i=1}^{k-1} b_i z_i \\
\hat{C}_1 &= \beta_3 + \beta_3 \sum_{i=1}^{k-1} a_i + \left(\alpha_3 + \frac{1}{2}\right) \sum_{i=1}^{k-1} b_i & \hat{C}_2 &= \beta_3 \sum_{i=1}^{k-1} a_i z_i + \left(\alpha_3 + \frac{1}{2}\right) \sum_{i=1}^{k-1} b_i z_i \\
\hat{D}_1 &= \beta_4 + \left(\beta_4 + \frac{1}{2}\right) \sum_{i=1}^{k-1} a_i + \alpha_4 \sum_{i=1}^{k-1} b_i & \hat{D}_2 &= \left(\beta_4 + \frac{1}{2}\right) \sum_{i=1}^{k-1} a_i z_i + \alpha_4 \sum_{i=1}^{k-1} b_i z_i
\end{aligned}$$

The shape functions for  $u_x^{(k)}$  are:

$$\begin{aligned}
\Phi_{11}^{(k)} &= -\bar{A}_1 z + 1 + \bar{A}_2, & \Phi_{12}^{(k)} &= \bar{A}_1 z - \bar{A}_2, & \Phi_{21}^{(k)} &= -\bar{B}_1 z + \bar{B}_2, & \Phi_{22}^{(k)} &= \bar{B}_1 z - \bar{B}_2 \\
\Phi_{31}^{(k)} &= \Phi_{32}^{(k)} = \bar{D}_1 z - \bar{D}_2, & \Phi_{41}^{(k)} &= \Phi_{42}^{(k)} = -\bar{C}_1 z + \bar{C}_2
\end{aligned} \quad \text{A.(4)}$$

The shape functions for  $u_y^{(k)}$  are:

$$\begin{aligned}
\Psi_{11}^{(k)} &= -\hat{A}_1 z + \hat{A}_2, & \Psi_{12}^{(k)} &= \hat{A}_1 z - \hat{A}_2, & \Psi_{21}^{(k)} &= -\hat{B}_1 z + \hat{B}_2 + 1, & \Psi_{22}^{(k)} &= \hat{B}_1 z - \hat{B}_2 \\
\Psi_{31}^{(k)} &= \Psi_{32}^{(k)} = \hat{D}_1 z - \hat{D}_2, & \Psi_{41}^{(k)} &= \Psi_{42}^{(k)} = -\hat{C}_1 z + \hat{C}_2
\end{aligned} \quad \text{A.(5)}$$

The shape functions for  $u_z^{(k)}$  are:

$$\Omega_1 = 1 - \frac{z}{h}, \quad \Omega_2 = \frac{z}{h} \quad \text{A.(6)}$$

The shape functions associated with  $\epsilon_{zz}$  based on a constant transverse normal stress assumption are:

$$\begin{aligned}
\lambda_{x11}^{(k)} &= \bar{E}_1 z + \bar{E}_2, & \lambda_{x12}^{(k)} &= \bar{F}_1 z + \bar{F}_2, & \lambda_{x21}^{(k)} &= \bar{G}_1 z + \bar{G}_2, & \lambda_{x22}^{(k)} &= \bar{H}_1 z + \bar{H}_2 \\
\lambda_{x31}^{(k)} &= \lambda_{x32}^{(k)} = \bar{I}_1 z + \bar{I}_2, & \lambda_{x41}^{(k)} &= \lambda_{x42}^{(k)} = \bar{J}_1 z + \bar{J}_2 \\
\lambda_{y11}^{(k)} &= \hat{E}_1 z + \hat{E}_2, & \lambda_{y12}^{(k)} &= \hat{F}_1 z + \hat{F}_2, & \lambda_{y21}^{(k)} &= \hat{G}_1 z + \hat{G}_2, & \lambda_{y22}^{(k)} &= \hat{H}_1 z + \hat{H}_2 \\
\lambda_{y31}^{(k)} &= \lambda_{y32}^{(k)} = \hat{I}_1 z + \hat{I}_2, & \lambda_{y41}^{(k)} &= \lambda_{y42}^{(k)} = \hat{J}_1 z + \hat{J}_2 \\
\chi_1 &= -K_1, & \chi_2 &= K_1
\end{aligned} \quad \text{A.(7)}$$

where coefficients associated with the shape functions are:

$$\begin{aligned}
\bar{E}_1 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{A}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{A}_1, & \bar{E}_2 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} (\bar{A}_2 + 1) - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{A}_2 + \frac{P_1 + S_6}{C_{33}^{(k)}} \\
\bar{F}_1 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{A}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{A}_1, & \bar{F}_2 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{A}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{A}_2 + \frac{P_2 - S_6}{C_{33}^{(k)}} \\
\bar{G}_1 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{B}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{B}_1, & \bar{G}_2 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{B}_2 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} (\hat{B}_2 + 1) + \frac{P_3 + S_7}{C_{33}^{(k)}} \\
\bar{H}_1 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{B}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{B}_1, & \bar{H}_2 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{B}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{B}_2 + \frac{-P_3 + S_8}{C_{33}^{(k)}} \\
\bar{I}_1 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{C}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{C}_1, & \bar{I}_2 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{C}_2 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{C}_2 + \frac{P_4 + S_9}{C_{33}^{(k)}} \\
\bar{J}_1 &= -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{D}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{D}_1, & \bar{J}_2 &= \frac{C_{13}^{(k)}}{C_{33}^{(k)}} \bar{D}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \hat{D}_2 + \frac{P_5 + S_{10}}{C_{33}^{(k)}} \\
\hat{E}_1 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{A}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{A}_1, & \hat{E}_2 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{A}_2 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} (\bar{A}_2 + 1) + \frac{Q_1 + S_1}{C_{33}^{(k)}} \\
\hat{F}_1 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{A}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{A}_1, & \hat{F}_2 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{A}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{A}_2 + \frac{-Q_1 + S_2}{C_{33}^{(k)}} \\
\hat{G}_1 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{B}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{B}_1, & \hat{G}_2 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} (\hat{B}_2 + 1) - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{B}_2 + \frac{Q_2 + S_3}{C_{33}^{(k)}} \\
\hat{H}_1 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{B}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{B}_1, & \hat{H}_2 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{B}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{B}_2 + \frac{Q_3 - S_3}{C_{33}^{(k)}} \\
\hat{I}_1 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{C}_1 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{C}_1, & \hat{I}_2 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{C}_2 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{C}_2 + \frac{Q_4 + S_4}{C_{33}^{(k)}} \\
\hat{J}_1 &= -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{D}_1 - \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{D}_1, & \hat{J}_2 &= \frac{C_{23}^{(k)}}{C_{33}^{(k)}} \hat{D}_2 + \frac{C_{36}^{(k)}}{C_{33}^{(k)}} \bar{D}_2 + \frac{Q_5 + S_5}{C_{33}^{(k)}} \\
K_1 &= \frac{T_1}{C_{33}^{(k)}}
\end{aligned} \tag{A.8}$$

The coefficients defined Eq. (3.3.21) are given as follows:

$$\begin{aligned}
 P_1 &= \left\{ \sum_{k=1}^N R_{31}^{(k)} [-\bar{A}_1 \kappa_2 + \kappa_1 (1 + \bar{A}_2)] \right\} / \Delta & P_2 &= \left\{ \sum_{k=1}^N R_{31}^{(k)} [\bar{A}_1 \kappa_2 - \kappa_1 \bar{A}_2] \right\} / \Delta \\
 P_3 &= \left\{ \sum_{k=1}^N R_{31}^{(k)} [-\bar{B}_1 \kappa_2 + \kappa_1 \bar{B}_2] \right\} / \Delta & P_4 &= \left\{ \sum_{k=1}^N R_{31}^{(k)} [-\bar{C}_1 \kappa_2 + \kappa_1 \bar{C}_2] \right\} / \Delta \\
 P_5 &= \left\{ \sum_{k=1}^N R_{31}^{(k)} [\bar{D}_1 \kappa_2 - \kappa_1 \bar{D}_2] \right\} / \Delta & Q_1 &= \left\{ \sum_{k=1}^N R_{32}^{(k)} [-\hat{A}_1 \kappa_2 + \kappa_1 \hat{A}_2] \right\} / \Delta \\
 Q_2 &= \left\{ \sum_{k=1}^N R_{32}^{(k)} [-\hat{B}_1 \kappa_2 + \kappa_1 (\hat{B}_2 + 1)] \right\} / \Delta & Q_3 &= \left\{ \sum_{k=1}^N R_{32}^{(k)} [\hat{B}_1 \kappa_2 - \kappa_1 \hat{B}_2] \right\} / \Delta \\
 Q_4 &= \left\{ \sum_{k=1}^N R_{32}^{(k)} [-\hat{C}_1 \kappa_2 + \kappa_1 \hat{C}_2] \right\} / \Delta & Q_5 &= \left\{ \sum_{k=1}^N R_{32}^{(k)} [\hat{D}_1 \kappa_2 - \kappa_1 \hat{D}_2] \right\} / \Delta \\
 S_1 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\bar{A}_1 \kappa_2 + \kappa_1 (1 + \bar{A}_2)] \right\} / \Delta & S_2 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [\bar{A}_1 \kappa_2 - \kappa_1 \bar{A}_2] \right\} / \Delta \\
 S_3 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\bar{B}_1 \kappa_2 + \kappa_1 \bar{B}_2] \right\} / \Delta & S_4 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\bar{C}_1 \kappa_2 + \kappa_1 \bar{C}_2] \right\} / \Delta \text{A. (9)} \\
 S_5 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [\bar{D}_1 \kappa_2 - \kappa_1 \bar{D}_2] \right\} / \Delta & S_6 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\hat{A}_1 \kappa_2 + \kappa_1 \hat{A}_2] \right\} / \Delta \\
 S_7 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\hat{B}_1 \kappa_2 + \kappa_1 (\hat{B}_2 + 1)] \right\} / \Delta & S_8 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [\hat{B}_1 \kappa_2 - \kappa_1 \hat{B}_2] \right\} / \Delta \\
 S_9 &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [-\hat{C}_1 \kappa_2 + \kappa_1 \hat{C}_2] \right\} / \Delta & S_{10} &= \left\{ \sum_{k=1}^N R_{34}^{(k)} [\hat{D}_1 \kappa_2 - \kappa_1 \hat{D}_2] \right\} / \Delta \\
 T_1 &= - \left\{ \sum_{k=1}^N R_{33}^{(k)} \kappa_1 / h \right\} / \Delta
 \end{aligned}$$

$$\text{where } R_{31}^{(k)} = -\frac{C_{13}^{(k)}}{C_{33}^{(k)}} \quad R_{32}^{(k)} = -\frac{C_{23}^{(k)}}{C_{33}^{(k)}} \quad R_{33}^{(k)} = \frac{1}{C_{33}^{(k)}} \quad R_{34}^{(k)} = -\frac{C_{36}^{(k)}}{C_{33}^{(k)}}$$

$$\kappa_2 = \frac{(z_k^2 - z_{k-1}^2)}{2} \quad \kappa_1 = (z_k - z_{k-1}) \quad \Delta = - \sum_{k=1}^N R_{33}^{(k)} \kappa_1$$

• **Plate Inertias:**

$$\begin{aligned}
 I_{\alpha\beta rp}^u &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \rho^{(k)} \Phi_{\alpha\beta}^{(k)} \Phi_{rp}^{(k)} dz, & I_{\alpha\beta rp}^v &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \rho^{(k)} \Psi_{\alpha\beta}^{(k)} \Psi_{rp}^{(k)} dz \\
 I_{p\beta}^w &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \rho^{(k)} \Omega_{\beta} \Omega_p dz & \alpha, r &= 1 \dots 4; \quad \beta, p = 1, 2
 \end{aligned} \tag{7.3.1}$$

• **Shape Functions in the Covariant Frame.**

The shape functions defined in Eqs. A.(4), A.(5) are expressed in covariant frame through the transformation matrix Eqs.(3.5.38) and (3.5.44) as follows:

$$\begin{aligned}
 \Phi_{\xi 1\beta} &= \Phi_{1\beta}^{(k)} + \frac{j_{11}}{j_{12}} \Phi_{2\beta}^{(k)}, & \Phi_{\xi 2\beta} &= \frac{j_{22}}{j_{11}} \Phi_{3\beta}^{(k)} - \frac{j_{12}}{j_{11}} \Phi_{4\beta}^{(k)} \\
 \Phi_{\eta 1\beta} &= \Phi_{1\beta}^{(k)} + \frac{j_{22}}{j_{12}} \Phi_{2\beta}^{(k)}, & \Phi_{\eta 2\beta} &= -\frac{j_{21}}{j_{12}} \Phi_{3\beta}^{(k)} - \frac{j_{11}}{j_{12}} \Phi_{4\beta}^{(k)}
 \end{aligned} \tag{A.10}$$

$$\begin{aligned}
 \Psi_{\xi 1\beta} &= \frac{j_{11}}{j_{21}} \Psi_{1\beta}^{(k)} + \Psi_{2\beta}^{(k)}, & \Psi_{\xi 2\beta} &= \frac{j_{22}}{j_{21}} \Psi_{3\beta}^{(k)} - \frac{j_{12}}{j_{21}} \Psi_{4\beta}^{(k)} \\
 \Psi_{\eta 1\beta} &= \frac{j_{12}}{j_{22}} \Psi_{1\beta}^{(k)} + \Psi_{2\beta}^{(k)}, & \Psi_{\eta 2\beta} &= -\frac{j_{21}}{j_{22}} \Psi_{3\beta}^{(k)} + \frac{j_{11}}{j_{22}} \Psi_{4\beta}^{(k)}
 \end{aligned} \tag{A.11}$$

• **Stress Resultants:**

$$\begin{aligned}
 N_{x\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} \Phi_{\alpha\beta}^{(k)} dz, & N_{y\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{yy}^{(k)} \Psi_{\alpha\beta}^{(k)} dz \\
 N_{z\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \lambda_{x\alpha\beta}^{(k)} dz, & Q_{z\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \lambda_{y\alpha\beta}^{(k)} dz \\
 R_{z\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \chi_{\beta} dz, & Q_{y\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{yz}^{(k)} \Psi_{\alpha\beta, z}^{(k)} dz \\
 R_{y\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{yz}^{(k)} \bar{\Omega}_{\beta} dz, & Q_{x\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{xz}^{(k)} \Phi_{\alpha\beta, z}^{(k)} dz \\
 R_{x\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{xz}^{(k)} \bar{\Omega}_{\beta} dz, & R_{xy\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{xy}^{(k)} \Phi_{\alpha\beta}^{(k)} dz, & R_{yx\alpha\beta} &= \sum_{k=1}^{N_m} \int_{z_{k-1}}^{z_k} \tau_{xy}^{(k)} \Psi_{\alpha\beta}^{(k)} dz
 \end{aligned} \tag{A.12}$$

## **APPENDIX B**

## APPENDIX B

• **Shape Functions:**

The rotational variables  $\psi_x$  and  $\psi_y$  are eliminated in favor of the translational variables  $u_t$  and  $v_t$ , the inplane translations at the top surface of the laminate. In that process the following variables are obtained.

$$\begin{aligned}
 A_1 &= \sum_{i=1}^N (h-z_i)d_{1i}, \quad B_1 = \sum_{i=1}^N (h-z_i)c_{1i}, \quad C_1 = \sum_{i=1}^N (h-z_i)b_{1i}, \quad D_1 = \sum_{i=1}^N (h-z_i)a_{1i} \\
 A_2 &= \sum_{i=1}^N (h-z_i)d_{2i}, \quad B_2 = \sum_{i=1}^N (h-z_i)c_{2i}, \quad C_2 = \sum_{i=1}^N (h-z_i)b_{2i}, \quad D_2 = \sum_{i=1}^N (h-z_i)a_{2i} \\
 A_3 &= \sum_{i=1}^N (h-z_i)d_{3i}, \quad B_3 = \sum_{i=1}^N (h-z_i)c_{3i}, \quad C_3 = \sum_{i=1}^N (h-z_i)b_{3i}, \quad D_3 = \sum_{i=1}^N (h-z_i)a_{3i} \\
 A_4 &= \sum_{i=1}^N (h-z_i)d_{4i}, \quad B_4 = \sum_{i=1}^N (h-z_i)c_{4i}, \quad C_4 = \sum_{i=1}^N (h-z_i)b_{4i}, \quad D_4 = \sum_{i=1}^N (h-z_i)a_{4i} \\
 \tilde{A} &= A_1 + h, \quad \tilde{D} = D_1 + h, \quad \bar{\Delta} = \tilde{A} \cdot \tilde{D} - B_1 \cdot C_1
 \end{aligned} \tag{A.13}$$

where  $h$  is the thickness of the laminate, and  $a_{ij}, b_{ij}, c_{ij}, d_{ij}, i = 1 \dots 4$  are defined in Eq. (4.2.11).

$$\begin{aligned}
 \alpha_1 &= \frac{\tilde{D}}{\bar{\Delta}}, \quad \alpha_2 = -\frac{B_1}{\bar{\Delta}}, \quad \alpha_3 = \frac{B_1 C_2 - \tilde{D}(h^2 - A_2)}{\bar{\Delta}}, \quad \alpha_4 = \frac{B_1(h^2 + D_2) - \tilde{D}B_2}{\bar{\Delta}} \\
 \alpha_5 &= \frac{B_1 C_3 - \tilde{D}(h^3 + A_3)}{\bar{\Delta}}, \quad \alpha_6 = \frac{B_1(h^3 + D_3) - \tilde{D}B_3}{\bar{\Delta}}, \quad \alpha_7 = \frac{\tilde{D}(A_4 - A_1) + B_1(C_1 - C_4)}{\bar{\Delta}} \\
 \alpha_8 &= \frac{\tilde{D}(B_4 - B_1) + B_1(D_1 - D_4)}{\bar{\Delta}}, \quad \alpha_9 = \frac{B_1 C_4 - \tilde{D}A_4}{\bar{\Delta}}, \quad \alpha_{10} = \frac{B_1 D_4 - \tilde{D}B_4}{\bar{\Delta}} \tag{A.14} \\
 \beta_1 &= -\frac{C_1}{\bar{\Delta}}, \quad \beta_2 = \frac{\tilde{A}}{\bar{\Delta}}, \quad \beta_3 = \frac{C_1(h^2 + A_2) - \tilde{A}C_2}{\bar{\Delta}}, \quad \beta_4 = \frac{C_1 B_2 - \tilde{A}(h^2 + D_2)}{\bar{\Delta}} \\
 \beta_5 &= \frac{C_1(h^3 + A_3) - \tilde{A}C_3}{\bar{\Delta}}, \quad \beta_6 = \frac{B_3 C_1 - \tilde{A}(h^3 + D_3)}{\bar{\Delta}}, \quad \beta_7 = \frac{\tilde{A}(C_4 - C_1) + C_1(A_1 - A_4)}{\bar{\Delta}}
 \end{aligned}$$



$$\beta_8 = \frac{\tilde{A}(D_4 - D_1) + C_1(B_1 - B_4)}{\bar{\Delta}}, \quad \beta_9 = \frac{C_1 A_4 - \tilde{A} C_4}{\bar{\Delta}}, \quad \beta_{10} = \frac{C_1 B_4 - \tilde{A} D_4}{\bar{\Delta}}$$

$$\begin{aligned} \bar{A}_1 &= \alpha_1 + \beta_1 \sum_{i=1}^{k-1} c_{1i} + \alpha_1 \sum_{i=1}^{k-1} d_{1i}, & \bar{A}_2 &= \beta_1 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_1 \sum_{i=1}^{k-1} d_{1i} z_i \\ \bar{B}_1 &= \alpha_2 + \beta_2 \sum_{i=1}^{k-1} c_{1i} + \alpha_2 \sum_{i=1}^{k-1} d_{1i}, & \bar{B}_2 &= \beta_2 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_2 \sum_{i=1}^{k-1} d_{1i} z_i \\ \bar{C}_1 &= \alpha_8 + (\beta_8 + 1) \sum_{i=1}^{k-1} c_{1i} + \alpha_8 \sum_{i=1}^{k-1} d_{1i} - \sum_{i=1}^{k-1} c_{4i} \\ \bar{C}_2 &= -(\beta_8 + 1) \sum_{i=1}^{k-1} c_{1i} z_i - \alpha_8 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} c_{4i} z_i \\ \bar{D}_1 &= -\alpha_7 - \beta_7 \sum_{i=1}^{k-1} c_{1i} - (1 + \alpha_7) \sum_{i=1}^{k-1} d_{1i} + \sum_{i=1}^{k-1} d_{4i} \\ \bar{D}_2 &= \beta_7 \sum_{i=1}^{k-1} c_{1i} z_i + (1 + \alpha_7) \sum_{i=1}^{k-1} d_{1i} z_i - \sum_{i=1}^{k-1} d_{4i} z_i \\ \bar{E}_1 &= \alpha_3 + \beta_3 \sum_{i=1}^{k-1} c_{1i} + \alpha_3 \sum_{i=1}^{k-1} d_{1i} + \sum_{i=1}^{k-1} d_{2i}, & \bar{E}_2 &= \beta_3 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_3 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} d_{2i} z_i \\ \bar{F}_1 &= \alpha_4 + \beta_4 \sum_{i=1}^{k-1} c_{1i} + \alpha_4 \sum_{i=1}^{k-1} d_{1i} + \sum_{i=1}^{k-1} c_{2i}, & \bar{F}_2 &= \beta_4 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_4 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} c_{2i} z_i \\ \bar{G}_1 &= \alpha_5 + \beta_5 \sum_{i=1}^{k-1} c_{1i} + \alpha_5 \sum_{i=1}^{k-1} d_{1i} + \sum_{i=1}^{k-1} d_{3i}, & \bar{G}_2 &= \beta_5 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_5 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} d_{3i} z_i \\ \bar{H}_1 &= \alpha_6 + \beta_6 \sum_{i=1}^{k-1} c_{1i} + \alpha_6 \sum_{i=1}^{k-1} d_{1i} + \sum_{i=1}^{k-1} c_{3i}, & \bar{H}_2 &= \beta_6 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_6 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} c_{3i} z_i \\ \bar{I}_1 &= \alpha_{10} \left( 1 + \sum_{i=1}^{k-1} d_{3i} \right) + \beta_{10} \sum_{i=1}^{k-1} c_{1i} + \sum_{i=1}^{k-1} c_{4i}, & \bar{I}_2 &= -\beta_{10} \sum_{i=1}^{k-1} c_{1i} z_i - \alpha_{10} \sum_{i=1}^{k-1} d_{1i} z_i - \sum_{i=1}^{k-1} c_{4i} z_i \\ \bar{J}_1 &= -\alpha_9 \left( 1 + \sum_{i=1}^{k-1} d_{1i} \right) - \beta_9 \sum_{i=1}^{k-1} c_{1i} - \sum_{i=1}^{k-1} d_{4i}, & \bar{J}_2 &= \beta_9 \sum_{i=1}^{k-1} c_{1i} z_i + \alpha_9 \sum_{i=1}^{k-1} d_{1i} z_i + \sum_{i=1}^{k-1} d_{4i} z_i \end{aligned} \quad \text{A. (15)}$$

$$\begin{aligned}
\hat{A}_1 &= \beta_1 + \beta_1 \sum_{i=1}^{k-1} a_{1i} + \alpha_1 \sum_{i=1}^{k-1} b_{1i} & \hat{A}_2 &= \beta_1 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_1 \sum_{i=1}^{k-1} b_{1i} z_i \\
\hat{B}_1 &= \beta_2 + \beta_2 \sum_{i=1}^{k-1} a_{1i} + \alpha_2 \sum_{i=1}^{k-1} b_{1i} & \hat{B}_2 &= \beta_2 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_2 \sum_{i=1}^{k-1} b_{1i} z_i \\
\hat{C}_1 &= \beta_8 + (\beta_8 + 1) \sum_{i=1}^{k-1} a_{1i} + \alpha_8 \sum_{i=1}^{k-1} b_{1i} - \sum_{i=1}^{k-1} a_{4i} \\
\hat{C}_2 &= -(\beta_8 + 1) \sum_{i=1}^{k-1} a_{1i} z_i - \alpha_8 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} a_{4i} z_i \\
\hat{D}_1 &= -\beta_7 - \beta_7 \sum_{i=1}^{k-1} a_{1i} - (1 + \alpha_7) \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} b_{4i} \\
\hat{D}_2 &= \beta_7 \sum_{i=1}^{k-1} a_{1i} z_i + (1 + \alpha_7) \sum_{i=1}^{k-1} b_{1i} z_i - \sum_{i=1}^{k-1} b_{4i} z_i \\
\hat{E}_1 &= \beta_3 + \beta_3 \sum_{i=1}^{k-1} a_{1i} + \alpha_3 \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} b_{2i}, & \hat{E}_2 &= \beta_3 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_3 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} b_{2i} z_i \\
\hat{F}_1 &= \beta_4 + \beta_4 \sum_{i=1}^{k-1} a_{1i} + \alpha_4 \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} a_{2i}, & \hat{F}_2 &= \beta_4 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_4 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} a_{2i} z_i \\
\hat{G}_1 &= \beta_5 + \beta_5 \sum_{i=1}^{k-1} a_{1i} + \alpha_5 \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} b_{3i}, & \hat{G}_2 &= \beta_5 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_5 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} b_{3i} z_i \\
\hat{H}_1 &= \beta_6 + \beta_6 \sum_{i=1}^{k-1} a_{1i} + \alpha_6 \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} a_{3i}, & \hat{H}_2 &= \beta_6 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_6 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} a_{3i} z_i \\
\hat{I}_1 &= \beta_{10} \left( 1 + \sum_{i=1}^{k-1} a_{1i} \right) + \beta_{10} \sum_{i=1}^{k-1} b_{1i} + \sum_{i=1}^{k-1} a_{4i}, & \hat{I}_2 &= -\beta_{10} \sum_{i=1}^{k-1} a_{1i} z_i - \alpha_{10} \sum_{i=1}^{k-1} b_{1i} z_i - \sum_{i=1}^{k-1} a_{4i} z_i \\
\hat{J}_1 &= -\beta_9 \left( 1 + \sum_{i=1}^{k-1} a_{1i} \right) - \alpha_9 \sum_{i=1}^{k-1} b_{1i} - \sum_{i=1}^{k-1} b_{4i}, & \hat{J}_2 &= \beta_9 \sum_{i=1}^{k-1} a_{1i} z_i + \alpha_9 \sum_{i=1}^{k-1} b_{1i} z_i + \sum_{i=1}^{k-1} b_{4i} z_i
\end{aligned}$$

The following variables are obtained when Eq. (4.2.14) is evaluated.

$$\begin{aligned}
&ub1, ut1, vb1, vt1, \theta xb1, \theta xt1, \theta yb1, \theta yt1, ub2, ut2, vb2, vt2, \theta xb2, \theta xt2, \theta yb2, \theta yt2, \\
&ub3, ut3, vb3, vt3, \theta xb3, \theta xt3, \theta yb3, \theta yt3, ub4, ut4, vb4, vt4, \theta xb4, \theta xt4, \theta yb4, \theta yt4
\end{aligned}$$

The shape functions for  $u_x^{(k)}$  are:

$$\begin{aligned}
\Phi_{11}^{(k)} &= ub2 \cdot z^3 + ub1 \cdot z^2 + (-\bar{A}_1 + ub1 \cdot \bar{E}_1 + ub3 \cdot \bar{F}_1 + ub2 \cdot \bar{G}_1 + ub4 \cdot \bar{H}_1) \cdot z \\
&\quad + 1 + \bar{A}_2 - ub1 \cdot \bar{E}_2 - ub3 \cdot \bar{F}_2 - ub2 \cdot \bar{G}_2 - ub4 \cdot \bar{H}_2 \\
\Phi_{12}^{(k)} &= ut2 \cdot z^3 + ut1 \cdot z^2 + (\bar{A}_1 + ut1 \cdot \bar{E}_1 + ut3 \cdot \bar{F}_1 + ut2 \cdot \bar{G}_1 + ut4 \cdot \bar{H}_1) \cdot z \\
&\quad - \bar{A}_2 - ut1 \cdot \bar{E}_2 - ut3 \cdot \bar{F}_2 - ut2 \cdot \bar{G}_2 - ut4 \cdot \bar{H}_2 \\
\Phi_{21}^{(k)} &= vb2 \cdot z^3 + vb1 \cdot z^2 + (-\bar{B}_1 + vb1 \cdot \bar{E}_1 + vb3 \cdot \bar{F}_1 + vb2 \cdot \bar{G}_1 + vb4 \cdot \bar{H}_1) \cdot z \\
&\quad + \bar{B}_2 - vb1 \cdot \bar{E}_2 - vb3 \cdot \bar{F}_2 - vb2 \cdot \bar{G}_2 - vb4 \cdot \bar{H}_2 \\
\Phi_{22}^{(k)} &= vt2 \cdot z^3 + vt1 \cdot z^2 + (\bar{B}_1 + vt1 \cdot \bar{E}_1 + vt3 \cdot \bar{F}_1 + vt2 \cdot \bar{G}_1 + vt4 \cdot \bar{H}_1) \cdot z \\
&\quad - \bar{B}_2 - vt1 \cdot \bar{E}_2 - vt3 \cdot \bar{F}_2 - vt2 \cdot \bar{G}_2 - vt4 \cdot \bar{H}_2 \\
\Phi_{31}^{(k)} &= \theta xb2 \cdot z^3 + \theta xb1 \cdot z^2 + (\bar{C}_1 + \theta xb1 \cdot \bar{E}_1 + \theta xb3 \cdot \bar{F}_1 + \theta xb2 \cdot \bar{G}_1 + \theta xb4 \cdot \bar{H}_1)z \\
&\quad + \bar{C}_2 - \theta xb1 \cdot \bar{E}_2 - \theta xb3 \cdot \bar{F}_2 - \theta xb2 \cdot \bar{G}_2 - \theta xb4 \cdot \bar{H}_2 \\
\Phi_{32}^{(k)} &= \theta xt2 \cdot z^3 + \theta xt1 \cdot z^2 + (\bar{I}_1 + \theta xt1 \cdot \bar{E}_1 + \theta xt3 \cdot \bar{F}_1 + \theta xt2 \cdot \bar{G}_1 + \theta xt4 \cdot \bar{H}_1)z \\
&\quad + \bar{I}_2 - \theta xt1 \cdot \bar{E}_2 - \theta xt3 \cdot \bar{F}_2 - \theta xt2 \cdot \bar{G}_2 - \theta xt4 \cdot \bar{H}_2 \\
\Phi_{41}^{(k)} &= \theta yb2 \cdot z^3 + \theta yb1 \cdot z^2 + (\bar{D}_1 + \theta yb1 \cdot \bar{E}_1 + \theta yb3 \cdot \bar{F}_1 + \theta yb2 \cdot \bar{G}_1 + \theta yb4 \cdot \bar{H}_1)z \\
&\quad + \bar{D}_2 - \theta yb1 \cdot \bar{E}_2 - \theta yb3 \cdot \bar{F}_2 - \theta yb2 \cdot \bar{G}_2 - \theta yb4 \cdot \bar{H}_2 \\
\Phi_{42}^{(k)} &= \theta yt2 \cdot z^3 + (\theta yt1) \cdot z^2 + (\bar{J}_1 + \theta yt1 \cdot \bar{E}_1 + \theta yt3 \cdot \bar{F}_1 + \theta yt2 \cdot \bar{G}_1 + \theta yt4 \cdot \bar{H}_1)z \\
&\quad + \bar{J}_2 - \theta yt1 \cdot \bar{E}_2 - \theta yt3 \cdot \bar{F}_2 - \theta yt2 \cdot \bar{G}_2 - \theta yt4 \cdot \bar{H}_2
\end{aligned} \tag{A.16}$$

The shape functions for  $u_y^{(k)}$  are:

$$\begin{aligned}
\Psi_{11}^{(k)} &= ub4 \cdot z^3 + ub3 \cdot z^2 + (-\hat{A}_1 + ub1 \cdot \hat{E}_1 + ub3 \cdot \hat{F}_1 + ub2 \cdot \hat{G}_1 + ub4 \cdot \hat{H}_1) \cdot z \\
&\quad + \hat{A}_2 - ub1 \cdot \hat{E}_2 - ub3 \cdot \hat{F}_2 - ub2 \cdot \hat{G}_2 - ub4 \cdot \hat{H}_2 \\
\Psi_{12}^{(k)} &= ut4 \cdot z^3 + ut3 \cdot z^2 + (\hat{A}_1 + ut1 \cdot \hat{E}_1 + ut3 \cdot \hat{F}_1 + ut2 \cdot \hat{G}_1 + ut4 \cdot \hat{H}_1) \cdot z \\
&\quad - \hat{A}_2 - ut1 \cdot \hat{E}_2 - ut3 \cdot \hat{F}_2 - ut2 \cdot \hat{G}_2 - ut4 \cdot \hat{H}_2 \\
\Psi_{21}^{(k)} &= vb4 \cdot z^3 + vb3 \cdot z^2 + (-\hat{B}_1 + vb1 \cdot \hat{E}_1 + vb3 \cdot \hat{F}_1 + vb2 \cdot \hat{G}_1 + vb4 \cdot \hat{H}_1) \cdot z \\
&\quad + 1 + \hat{B}_2 - vb1 \cdot \hat{E}_2 - vb3 \cdot \hat{F}_2 - vb2 \cdot \hat{G}_2 - vb4 \cdot \hat{H}_2 \\
\Psi_{22}^{(k)} &= vt4 \cdot z^3 + vt3 \cdot z^2 + (\hat{B}_1 + vt1 \cdot \hat{E}_1 + vt3 \cdot \hat{F}_1 + vt2 \cdot \hat{G}_1 + vt4 \cdot \hat{H}_1) \cdot z \\
&\quad - \hat{B}_2 - vt1 \cdot \hat{E}_2 - vt3 \cdot \hat{F}_2 - vt2 \cdot \hat{G}_2 - vt4 \cdot \hat{H}_2 \\
\Psi_{31}^{(k)} &= \theta xb4 \cdot z^3 + (\theta xb3) \cdot z^2 + (\hat{C}_1 + \theta xb1 \cdot \hat{E}_1 + \theta xb3 \cdot \hat{F}_1 + \theta xb2 \cdot \hat{G}_1 + \theta xb4 \cdot \hat{H}_1) \\
&\quad z + \hat{C}_2 - \theta xb1 \cdot \hat{E}_2 - \theta xb3 \cdot \hat{F}_2 - \theta xb2 \cdot \hat{G}_2 - \theta xb4 \cdot \hat{H}_2 \\
\Psi_{32}^{(k)} &= \theta xt4 \cdot z^3 + (\theta xt3) \cdot z^2 + (\hat{I}_1 + \theta xt1 \cdot \hat{E}_1 + \theta xt3 \cdot \hat{F}_1 + \theta xt2 \cdot \hat{G}_1 + \theta xt4 \cdot \hat{H}_1)z \\
&\quad + \hat{I}_2 - \theta xt1 \cdot \hat{E}_2 - \theta xt3 \cdot \hat{F}_2 - \theta xt2 \cdot \hat{G}_2 - \theta xt4 \cdot \hat{H}_2 \\
\Psi_{41}^{(k)} &= \theta yb4 \cdot z^3 + \theta yb3 \cdot z^2 + (\hat{D}_1 + \theta yb1 \cdot \hat{E}_1 + \theta yb3 \cdot \hat{F}_1 + \theta yb2 \cdot \hat{G}_1 + \theta yb4 \cdot \hat{H}_1)z \\
&\quad + \hat{D}_2 - \theta yb1 \cdot \hat{E}_2 - \theta yb3 \cdot \hat{F}_2 - \theta yb2 \cdot \hat{G}_2 - \theta yb4 \cdot \hat{H}_2 \\
\Psi_{42}^{(k)} &= \theta yt4 \cdot z^3 + \theta yt3 \cdot z^2 + (\hat{J}_1 + \theta yt1 \cdot \hat{E}_1 + \theta yt3 \cdot \hat{F}_1 + \theta yt2 \cdot \hat{G}_1 + \theta yt4 \cdot \hat{H}_1)z \\
&\quad + \hat{J}_2 - \theta yt1 \cdot \hat{E}_2 - \theta yt3 \cdot \hat{F}_2 - \theta yt2 \cdot \hat{G}_2 - \theta yt4 \cdot \hat{H}_2
\end{aligned} \tag{A.17}$$

The shape functions for  $u_z^{(k)}$  are:

$$\Omega_1 = 1 - \frac{z}{h}, \quad \Omega_2 = \frac{z}{h} \tag{A.18}$$

The shape functions associated with  $\epsilon_{zz}$  based on a constant transverse normal stress assumption are:

$$\begin{aligned}
\lambda_{x11}^{(k)} &= R_{31}^{(k)} \Phi_{11}^{(k)} + R_{34}^{(k)} \Psi_{11}^{(k)} + R_{33}^{(k)} (P_1 + S_9) \\
\lambda_{x12}^{(k)} &= R_{31}^{(k)} \Phi_{12}^{(k)} + R_{34}^{(k)} \Psi_{12}^{(k)} + R_{33}^{(k)} (P_2 + S_{10}) \\
\lambda_{x21}^{(k)} &= R_{31}^{(k)} \Phi_{21}^{(k)} + R_{34}^{(k)} \Psi_{21}^{(k)} + R_{33}^{(k)} (P_3 + S_{11}) \\
\lambda_{x22}^{(k)} &= R_{31}^{(k)} \Phi_{22}^{(k)} + R_{34}^{(k)} \Psi_{22}^{(k)} + R_{33}^{(k)} (P_4 + S_{12}) \\
\lambda_{x31}^{(k)} &= R_{31}^{(k)} \Phi_{31}^{(k)} + R_{34}^{(k)} \Psi_{31}^{(k)} + R_{33}^{(k)} (P_5 + S_{13}) \\
\lambda_{x32}^{(k)} &= R_{31}^{(k)} \Phi_{32}^{(k)} + R_{34}^{(k)} \Psi_{32}^{(k)} + R_{33}^{(k)} (P_6 + S_{14}) \\
\lambda_{x41}^{(k)} &= R_{31}^{(k)} \Phi_{41}^{(k)} + R_{34}^{(k)} \Psi_{41}^{(k)} + R_{33}^{(k)} (P_7 + S_{15}) \\
\lambda_{x42}^{(k)} &= R_{31}^{(k)} \Phi_{42}^{(k)} + R_{34}^{(k)} \Psi_{42}^{(k)} + R_{33}^{(k)} (P_8 + S_{16}) \\
\lambda_{y11}^{(k)} &= R_{32}^{(k)} \Psi_b^{(k)} + R_{34}^{(k)} \Phi_b^{(k)} + R_{33}^{(k)} (Q_1 + S_1) \\
\lambda_{y12}^{(k)} &= R_{32}^{(k)} \Psi_t^{(k)} + R_{34}^{(k)} \Phi_t^{(k)} + R_{33}^{(k)} (Q_2 + S_2) \\
\lambda_{y21}^{(k)} &= R_{32}^{(k)} \Psi_b^{(k)} + R_{34}^{(k)} \Phi_b^{(k)} + R_{33}^{(k)} (Q_3 + S_3) \\
\lambda_{y22}^{(k)} &= R_{32}^{(k)} \Psi_t^{(k)} + R_{34}^{(k)} \Phi_t^{(k)} + R_{33}^{(k)} (Q_4 + S_4) \\
\lambda_{y31}^{(k)} &= R_{32}^{(k)} \Psi_b^{(k)} + R_{34}^{(k)} \Phi_b^{(k)} + R_{33}^{(k)} (Q_5 + S_5) \\
\lambda_{y32}^{(k)} &= R_{32}^{(k)} \Psi_t^{(k)} + R_{34}^{(k)} \Phi_t^{(k)} + R_{33}^{(k)} (Q_6 + S_6) \\
\lambda_{y41}^{(k)} &= R_{32}^{(k)} \Psi_b^{(k)} + R_{34}^{(k)} \Phi_b^{(k)} + R_{33}^{(k)} (Q_7 + S_7) \\
\lambda_{y42}^{(k)} &= R_{32}^{(k)} \Psi_t^{(k)} + R_{34}^{(k)} \Phi_t^{(k)} + R_{33}^{(k)} (Q_8 + S_8) \\
K_1 &= R_{33}^{(k)} (T_1)
\end{aligned} \tag{A.19}$$

where coefficients associated with the shape functions are:

$$\begin{aligned}
P_1 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ ub2 \cdot thk4 + ub1 \cdot \kappa_3 + (-\bar{A}_1 + ub1 \cdot \bar{E}_1 + ub3 \cdot \bar{F}_1 + ub2 \cdot \bar{G}_1 \\
&\quad + ub4 \cdot \bar{H}_1) \kappa_2 + (1 + \bar{A}_2 - ub1 \cdot \bar{E}_2 - ub3 \cdot \bar{F}_2 - ub2 \cdot \bar{G}_2 - ub4 \cdot \bar{H}_2) \kappa_1 \} \\
P_2 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ ut2 \cdot thk4 + ut1 \cdot \kappa_3 + (\bar{A}_1 + ut1 \cdot \bar{E}_1 + ut3 \cdot \bar{F}_1 + ut2 \cdot \bar{G}_1 \\
&\quad + ut4 \cdot \bar{H}_1) \kappa_2 + (-\bar{A}_2 - ut1 \cdot \bar{E}_2 - ut3 \cdot \bar{F}_2 - ut2 \cdot \bar{G}_2 - ut4 \cdot \bar{H}_2) \kappa_1 \}
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
P_3 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ vb2 \cdot \kappa_4 + vb1 \cdot \kappa_3 + (-\bar{B}_1 + vb1 \cdot \bar{E}_1 + vb3 \cdot \bar{F}_1 + vb2 \cdot \bar{G}_1 \\
&\quad + vb4 \cdot \bar{H}_1) \kappa_2 + (\bar{B}_2 - vb1 \cdot \bar{E}_2 - vb3 \cdot \bar{F}_2 - vb2 \cdot \bar{G}_2 - vb4 \cdot \bar{H}_2) \kappa_1 \} \\
P_4 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ vt2 \cdot \kappa_4 + vt1 \cdot \kappa_3 + (\bar{B}_1 + vt1 \cdot \bar{E}_1 + vt3 \cdot \bar{F}_1 + vt2 \cdot \bar{G}_1 \\
&\quad + vt4 \cdot \bar{H}_1) \kappa_2 + (-\bar{B}_2 - vt1 \cdot \bar{E}_2 - vt3 \cdot \bar{F}_2 - vt2 \cdot \bar{G}_2 - vt4 \cdot \bar{H}_2) \kappa_1 \} \\
P_5 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ \theta xb2 \cdot \kappa_4 + \theta xb1 \cdot \kappa_3 + (\bar{C}_1 + \theta xb1 \cdot \bar{E}_1 + \theta xb3 \cdot \bar{F}_1 + \theta xb2 \cdot \bar{G}_1 \\
&\quad + \theta xb4 \cdot \bar{H}_1) \kappa_2 + (\bar{C}_2 - \theta xb1 \cdot \bar{E}_2 - \theta xb3 \cdot \bar{F}_2 - \theta xb2 \cdot \bar{G}_2 - \theta xb4 \cdot \bar{H}_2) \kappa_1 \} \\
P_6 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ \theta xt2 \cdot \kappa_4 + \theta xt1 \cdot \kappa_3 + (\bar{I}_1 + \theta xt1 \cdot \bar{E}_1 + \theta xt3 \cdot \bar{F}_1 + \theta xt2 \cdot \bar{G}_1 \\
&\quad + \theta xt4 \cdot \bar{H}_1) \kappa_2 + (\bar{I}_2 - \theta xt1 \cdot \bar{E}_2 - \theta xt3 \cdot \bar{F}_2 - \theta xt2 \cdot \bar{G}_2 - \theta xt4 \cdot \bar{H}_2) \kappa_1 \} \\
P_7 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ \theta yb2 \cdot \kappa_4 + \theta yb1 \cdot \kappa_3 + (\bar{D}_1 + \theta yb1 \cdot \bar{E}_1 + \theta yb3 \cdot \bar{F}_1 + \theta yb2 \cdot \bar{G}_1 \\
&\quad + \theta yb4 \cdot \bar{H}_1) \kappa_2 + (\bar{D}_2 - \theta yb1 \cdot \bar{E}_2 - \theta yb3 \cdot \bar{F}_2 - \theta yb2 \cdot \bar{G}_2 - \theta yb4 \cdot \bar{H}_2) \kappa_1 \} \\
P_8 &= -T_1 \sum_{k=1}^N R_{31}^{(k)} \{ \theta yt2 \cdot \kappa_4 + \theta yt1 \cdot \kappa_3 + (\bar{J}_1 + \theta yt1 \cdot \bar{E}_1 + \theta yt3 \cdot \bar{F}_1 + \theta yt2 \cdot \bar{G}_1 \\
&\quad + \theta yt4 \cdot \bar{H}_1) \kappa_2 + (\bar{J}_2 - \theta yt1 \cdot \bar{E}_2 - \theta yt3 \cdot \bar{F}_2 - \theta yt2 \cdot \bar{G}_2 - \theta yt4 \cdot \bar{H}_2) \kappa_1 \} \\
Q_1 &= -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ ub4 \cdot \kappa_4 + ub3 \cdot \kappa_3 + (-\hat{A}_1 + ub1 \cdot \hat{E}_1 + ub3 \cdot \hat{F}_1 + ub2 \cdot \hat{G}_1 \\
&\quad + ub4 \cdot \hat{H}_1) \kappa_2 + (\hat{A}_2 - ub1 \cdot \hat{E}_2 - ub3 \cdot \hat{F}_2 - ub2 \cdot \hat{G}_2 - ub4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_2 &= -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ ut4 \cdot \kappa_4 + ut3 \cdot \kappa_3 + (\hat{B}_1 + ut1 \cdot \hat{E}_1 + ut3 \cdot \hat{F}_1 + ut2 \cdot \hat{G}_1 \\
&\quad + ut4 \cdot \hat{H}_1) \kappa_2 + (\hat{B}_2 - ut1 \cdot \hat{E}_2 - ut3 \cdot \hat{F}_2 - ut2 \cdot \hat{G}_2 - ut4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_3 &= -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ vb4 \cdot \kappa_4 + vb3 \cdot \kappa_3 + (-\hat{B}_1 + vb1 \cdot \hat{E}_1 + vb3 \cdot \hat{F}_1 + vb2 \cdot \hat{G}_1 \\
&\quad + vb4 \cdot \hat{H}_1) \kappa_2 + (1 + \hat{B}_2 - vb1 \cdot \hat{E}_2 - vb3 \cdot \hat{F}_2 - vb2 \cdot \hat{G}_2 - vb4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_4 &= -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ vt4 \cdot \kappa_4 + vt3 \cdot \kappa_3 + (\hat{B}_1 + vt1 \cdot \hat{E}_1 + vt3 \cdot \hat{F}_1 + vt2 \cdot \hat{G}_1 \\
&\quad + vt4 \cdot \hat{H}_1) \kappa_2 + (\hat{B}_2 - vt1 \cdot \hat{E}_2 - vt3 \cdot \hat{F}_2 - vt2 \cdot \hat{G}_2 - vt4 \cdot \hat{H}_2) \kappa_1 \}
\end{aligned}$$

$$\begin{aligned}
& + \nu t 4 \cdot \hat{H}_1) \cdot \kappa_2 + (-\hat{B}_2 - \nu t 1 \cdot \hat{E}_2 - \nu t 3 \cdot \hat{F}_2 - \nu t 2 \cdot \hat{G}_2 - \nu t 4 \cdot \hat{H}_2) \cdot \kappa_1 \} \\
Q_5 = & -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ \theta x b 4 \cdot \kappa_4 + \theta x b 3 \cdot \kappa_3 + (\hat{C}_1 + \theta x b 1 \cdot \hat{E}_1 + \theta x b 3 \cdot \hat{F}_1 + \theta x b 2 \cdot \hat{G}_1 \\
& + \theta x b 4 \cdot \hat{H}_1) \kappa_2 + (\hat{C}_2 - \theta x b 1 \cdot \hat{E}_2 - \theta x b 3 \cdot \hat{F}_2 - \theta x b 2 \cdot \hat{G}_2 - \theta x b 4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_6 = & -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ \theta x t 4 \cdot \kappa_4 + \theta x t 3 \cdot \kappa_3 + (\hat{I}_1 + \theta x t 1 \cdot \hat{E}_1 + \theta x t 3 \cdot \hat{F}_1 + \theta x t 2 \cdot \hat{G}_1 \\
& + \theta x t 4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{I}_2 - \theta x t 1 \cdot \hat{E}_2 - \theta x t 3 \cdot \hat{F}_2 - \theta x t 2 \cdot \hat{G}_2 - \theta x t 4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_7 = & -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ \theta y b 4 \cdot \kappa_4 + \theta y b 3 \cdot \kappa_3 + (\hat{D}_1 + \theta y b 1 \cdot \hat{E}_1 + \theta y b 3 \cdot \hat{F}_1 + \theta y b 2 \cdot \hat{G}_1 \\
& + \theta y b 4 \cdot \hat{H}_1) \kappa_2 + (\hat{D}_2 - \theta y b 1 \cdot \hat{E}_2 - \theta y b 3 \cdot \hat{F}_2 - \theta y b 2 \cdot \hat{G}_2 - \theta y b 4 \cdot \hat{H}_2) \kappa_1 \} \\
Q_8 = & -T_1 \sum_{k=1}^N R_{32}^{(k)} \{ \theta y t 4 \cdot \kappa_4 + \theta y t 3 \cdot \kappa_3 + (\hat{J}_1 + \theta y t 1 \cdot \hat{E}_1 + \theta y t 3 \cdot \hat{F}_1 + \theta y t 2 \cdot \hat{G}_1 \\
& + \theta y t 4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{J}_2 - \theta y t 1 \cdot \hat{E}_2 - \theta y t 3 \cdot \hat{F}_2 - \theta y t 2 \cdot \hat{G}_2 - \theta y t 4 \cdot \hat{H}_2) \kappa_1 \} \\
S_1 = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ u b 2 \cdot \kappa_4 + u b 1 \cdot \kappa_3 + (-\bar{A}_1 + u b 1 \cdot \bar{E}_1 + u b 3 \cdot \bar{F}_1 + u b 2 \cdot \bar{G}_1 \\
& + u b 4 \cdot \bar{H}_1) \cdot \kappa_2 + (1 + \bar{A}_2 - u b 1 \cdot \bar{E}_2 - u b 3 \cdot \bar{F}_2 - u b 2 \cdot \bar{G}_2 - u b 4 \cdot \bar{H}_2) \kappa_1 \} \\
S_2 = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ u t 2 \cdot \kappa_4 + u t 1 \cdot \kappa_3 + (\bar{A}_1 + u t 1 \cdot \bar{E}_1 + u t 3 \cdot \bar{F}_1 + u t 2 \cdot \bar{G}_1 \\
& + u t 4 \cdot \bar{H}_1) \cdot \kappa_2 + (-\bar{A}_2 - u t 1 \cdot \bar{E}_2 - u t 3 \cdot \bar{F}_2 - u t 2 \cdot \bar{G}_2 - u t 4 \cdot \bar{H}_2) \kappa_1 \} \\
S_3 = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \nu b 2 \cdot \kappa_4 + \nu b 1 \cdot \kappa_3 + (-\bar{B}_1 + \nu b 1 \cdot \bar{E}_1 + \nu b 3 \cdot \bar{F}_1 + \nu b 2 \cdot \bar{G}_1 \\
& + \nu b 4 \cdot \bar{H}_1) \cdot \kappa_2 + (\bar{B}_2 - \nu b 1 \cdot \bar{E}_2 - \nu b 3 \cdot \bar{F}_2 - \nu b 2 \cdot \bar{G}_2 - \nu b 4 \cdot \bar{H}_2) \kappa_1 \} \\
S_4 = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \nu t 2 \cdot \kappa_4 + \nu t 1 \cdot \kappa_3 + (\bar{B}_1 + \nu t 1 \cdot \bar{E}_1 + \nu t 3 \cdot \bar{F}_1 + \nu t 2 \cdot \bar{G}_1 \\
& + \nu t 4 \cdot \bar{H}_1) \cdot \kappa_2 + (-\bar{B}_2 - \nu t 1 \cdot \bar{E}_2 - \nu t 3 \cdot \bar{F}_2 - \nu t 2 \cdot \bar{G}_2 - \nu t 4 \cdot \bar{H}_2) \kappa_1 \} \\
S_5 = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta x b 2 \cdot \kappa_4 + \theta x b 1 \cdot \kappa_3 + (\bar{C}_1 + \theta x b 1 \cdot \bar{E}_1 + \theta x b 3 \cdot \bar{F}_1 + \theta x b 2 \cdot \bar{G}_1 \\
& + \theta x b 4 \cdot \bar{H}_1) \kappa_2 + (\bar{C}_2 - \theta x b 1 \cdot \bar{E}_2 - \theta x b 3 \cdot \bar{F}_2 - \theta x b 2 \cdot \bar{G}_2 - \theta x b 4 \cdot \bar{H}_2) \kappa_1 \}
\end{aligned}$$

$$\begin{aligned}
S_6 &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{xt2} \cdot \kappa_4 + \theta_{xt1} \cdot \kappa_3 + (\bar{I}_1 + \theta_{xt1} \cdot \bar{E}_1 + \theta_{xt3} \cdot \bar{F}_1 + \theta_{xt2} \cdot \bar{G}_1 \\
&\quad + \theta_{xt4} \cdot \bar{H}_1) \cdot \kappa_2 + (\bar{I}_2 - \theta_{xt1} \cdot \bar{E}_2 - \theta_{xt3} \cdot \bar{F}_2 - \theta_{xt2} \cdot \bar{G}_2 - \theta_{xt4} \cdot \bar{H}_2) \kappa_1 \} \\
S_7 &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{yb2} \cdot \kappa_4 + \theta_{yb1} \cdot \kappa_3 + (\bar{D}_1 + \theta_{yb1} \cdot \bar{E}_1 + \theta_{yb3} \cdot \bar{F}_1 + \theta_{yb2} \cdot \bar{G}_1 \\
&\quad + \theta_{yb4} \cdot \bar{H}_1) \kappa_2 + (\bar{D}_2 - \theta_{yb1} \cdot \bar{E}_2 - \theta_{yb3} \cdot \bar{F}_2 - \theta_{yb2} \cdot \bar{G}_2 - \theta_{yb4} \cdot \bar{H}_2) \kappa_1 \} \\
S_8 &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{yt2} \cdot \kappa_4 + \theta_{yt1} \cdot \kappa_3 + (\bar{J}_1 + \theta_{yt1} \cdot \bar{E}_1 + \theta_{yt3} \cdot \bar{F}_1 + \theta_{yt2} \cdot \bar{G}_1 \\
&\quad + \theta_{yt4} \cdot \bar{H}_1) \cdot \kappa_2 + (\bar{J}_2 - \theta_{yt1} \cdot \bar{E}_2 - \theta_{yt3} \cdot \bar{F}_2 - \theta_{yt2} \cdot \bar{G}_2 - \theta_{yt4} \cdot \bar{H}_2) \kappa_1 \} \\
S_9 &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ ub4 \cdot \kappa_4 + ub3 \cdot \kappa_3 + (-\hat{A}_1 + ub1 \cdot \hat{E}_1 + ub3 \cdot \hat{F}_1 + ub2 \cdot \hat{G}_1 \\
&\quad + ub4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{A}_2 - ub1 \cdot \hat{E}_2 - ub3 \cdot \hat{F}_2 - ub2 \cdot \hat{G}_2 - ub4 \cdot \hat{H}_2) \kappa_1 \} \\
S_{10} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ ut4 \cdot \kappa_4 + ut3 \cdot \kappa_3 + (\hat{B}_1 + ut1 \cdot \hat{E}_1 + ut3 \cdot \hat{F}_1 + ut2 \cdot \hat{G}_1 \\
&\quad + ut4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{B}_2 - ut1 \cdot \hat{E}_2 - ut3 \cdot \hat{F}_2 - ut2 \cdot \hat{G}_2 - ut4 \cdot \hat{H}_2) \kappa_1 \} \\
S_{11} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ vb4 \cdot \kappa_4 + vb3 \cdot \kappa_3 + (-\hat{B}_1 + vb1 \cdot \hat{E}_1 + vb3 \cdot \hat{F}_1 + vb2 \cdot \hat{G}_1 \\
&\quad + vb4 \cdot \hat{H}_1) \cdot \kappa_2 + (1 + \hat{B}_2 - vb1 \cdot \hat{E}_2 - vb3 \cdot \hat{F}_2 - vb2 \cdot \hat{G}_2 - vb4 \cdot \hat{H}_2) \kappa_1 \} \\
S_{12} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ vt4 \cdot \kappa_4 + vt3 \cdot \kappa_3 + (\hat{B}_1 + vt1 \cdot \hat{E}_1 + vt3 \cdot \hat{F}_1 + vt2 \cdot \hat{G}_1 \\
&\quad + vt4 \cdot \hat{H}_1) \cdot \kappa_2 + (-\hat{B}_2 - vt1 \cdot \hat{E}_2 - vt3 \cdot \hat{F}_2 - vt2 \cdot \hat{G}_2 - vt4 \cdot \hat{H}_2) \kappa_1 \} \\
S_{13} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{xb4} \cdot \kappa_4 + \theta_{xb3} \cdot \kappa_3 + (\hat{C}_1 + \theta_{xb1} \cdot \hat{E}_1 + \theta_{xb3} \cdot \hat{F}_1 + \theta_{xb2} \cdot \hat{G}_1 \\
&\quad + \theta_{xb4} \cdot \hat{H}_1) \kappa_2 + (\hat{C}_2 - \theta_{xb1} \cdot \hat{E}_2 - \theta_{xb3} \cdot \hat{F}_2 - \theta_{xb2} \cdot \hat{G}_2 - \theta_{xb4} \cdot \hat{H}_2) \kappa_1 \} \\
S_{14} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{xt4} \cdot \kappa_4 + \theta_{xt3} \cdot \kappa_3 + (\hat{I}_1 + \theta_{xt1} \cdot \hat{E}_1 + \theta_{xt3} \cdot \hat{F}_1 + \theta_{xt2} \cdot \hat{G}_1 \\
&\quad + \theta_{xt4} \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{I}_2 - \theta_{xt1} \cdot \hat{E}_2 - \theta_{xt3} \cdot \hat{F}_2 - \theta_{xt2} \cdot \hat{G}_2 - \theta_{xt4} \cdot \hat{H}_2) \kappa_1 \} \\
S_{15} &= -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta_{yb4} \cdot \kappa_4 + \theta_{yb3} \cdot \kappa_3 + (\hat{D}_1 + \theta_{yb1} \cdot \hat{E}_1 + \theta_{yb3} \cdot \hat{F}_1 + \theta_{yb2} \cdot \hat{G}_1 \\
&\quad + \theta_{yb4} \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{D}_2 - \theta_{yb1} \cdot \hat{E}_2 - \theta_{yb3} \cdot \hat{F}_2 - \theta_{yb2} \cdot \hat{G}_2 - \theta_{yb4} \cdot \hat{H}_2) \kappa_1 \}
\end{aligned}$$



$$\begin{aligned}
& + \theta y b 4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{D}_2 - \theta y b 1 \cdot \hat{E}_2 - \theta y b 3 \cdot \hat{F}_2 - \theta y b 2 \cdot \hat{G}_2 - \theta y b 4 \cdot \hat{H}_2) \kappa_1 \} \\
S_{16} = & -T_1 \sum_{k=1}^N R_{34}^{(k)} \{ \theta y t 4 \cdot \kappa_4 + \theta y t 3 \cdot \kappa_3 + (\hat{J}_1 + \theta y t 1 \cdot \hat{E}_1 + \theta y t 3 \cdot \hat{F}_1 + \theta y t 2 \cdot \hat{G}_1 \\
& + \theta y t 4 \cdot \hat{H}_1) \cdot \kappa_2 + (\hat{J}_2 - \theta y t 1 \cdot \hat{E}_2 - \theta y t 3 \cdot \hat{F}_2 - \theta y t 2 \cdot \hat{G}_2 - \theta y t 4 \cdot \hat{H}_2) \kappa_1 \} \\
T_1 = & \frac{1}{N \sum_{k=1}^N R_{33}^{(k)} \cdot t h k 1}, \quad \kappa_4 = \frac{1}{4}(z_k^4 - z_{k-1}^4), \quad \kappa_3 = \frac{1}{3}(z_k^3 - z_{k-1}^3), \quad \kappa_2 = \frac{1}{2}(z_k^2 - z_{k-1}^2) \\
\kappa_1 = & (z_k - z_{k-1}), \quad R_{31}^{(k)} = -\frac{C_{31}^{(k)}}{C_{33}^{(k)}}, \quad R_{32}^{(k)} = -\frac{C_{32}^{(k)}}{C_{33}^{(k)}}, \quad R_{33}^{(k)} = \frac{1}{C_{33}^{(k)}}, \quad R_{34}^{(k)} = -\frac{C_{36}^{(k)}}{C_{33}^{(k)}}
\end{aligned}$$

Plate Inertias:

$$\begin{aligned}
I_{\alpha\beta rp}^u &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \rho^{(k)} \Phi_{\alpha\beta}^{(k)} \Phi_{rp}^{(k)} dz, & I_{\alpha\beta rp}^v &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \rho^{(k)} \Psi_{\alpha\beta}^{(k)} \Psi_{rp}^{(k)} dz \\
I_{p\beta}^w &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \rho^{(k)} \Omega_{\beta} \Omega_p dz & \alpha, r &= 1 \dots 4; \quad \beta, p = 1, 2
\end{aligned} \tag{A.21}$$

Stress Resultants:

$$\begin{aligned}
N_{x\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} \Phi_{\alpha\beta}^{(k)} dz, & N_{y\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \sigma_{yy}^{(k)} \Psi_{\alpha\beta}^{(k)} dz \\
N_{z\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \lambda_{x\alpha\beta}^{(k)} dz, & Q_{z\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \lambda_{y\alpha\beta}^{(k)} dz \\
R_{z\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \sigma_{zz}^{(k)} \chi_{\beta} dz, & Q_{y\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{yz}^{(k)} \Psi_{\alpha\beta, z}^{(k)} dz \\
R_{y\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{yz}^{(k)} \Omega_{\beta} dz, & Q_{x\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{xz}^{(k)} \Phi_{\alpha\beta, z}^{(k)} dz \\
R_{x\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{xz}^{(k)} \Omega_{\beta} dz, & R_{xy\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{xy}^{(k)} \Phi_{\alpha\beta}^{(k)} dz, & R_{yx\alpha\beta} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \tau_{xy}^{(k)} \Psi_{\alpha\beta}^{(k)} dz
\end{aligned} \tag{A.22}$$

• ***Shape Functions in the Covariant Frame.***

The shape functions defined in Eqs. A.(16), A.(17) are expressed in covariant frame through the transformation matrix Eqs.(3.5.38) and (3.5.44) as follows:

$$\begin{aligned}\Phi_{\xi 1\beta} &= \Phi_{1\beta}^{(k)} + \frac{j_{11}}{j_{12}}\Phi_{2\beta}^{(k)}, & \Phi_{\xi 2\beta} &= \frac{j_{22}}{j_{11}}\Phi_{3\beta}^{(k)} - \frac{j_{12}}{j_{11}}\Phi_{4\beta}^{(k)} \\ \Phi_{\eta 1\beta} &= \Phi_{1\beta}^{(k)} + \frac{j_{22}}{j_{12}}\Phi_{2\beta}^{(k)}, & \Phi_{\eta 2\beta} &= -\frac{j_{21}}{j_{12}}\Phi_{3\beta}^{(k)} - \frac{j_{11}}{j_{12}}\Phi_{4\beta}^{(k)}\end{aligned}\tag{A.23}$$

$$\begin{aligned}\Psi_{\xi 1\beta} &= \frac{j_{11}}{j_{21}}\Psi_{1\beta}^{(k)} + \Psi_{2\beta}^{(k)}, & \Psi_{\xi 2\beta} &= \frac{j_{22}}{j_{21}}\Psi_{3\beta}^{(k)} - \frac{j_{12}}{j_{21}}\Psi_{4\beta}^{(k)} \\ \Psi_{\eta 1\beta} &= \frac{j_{12}}{j_{22}}\Psi_{1\beta}^{(k)} + \Psi_{2\beta}^{(k)}, & \Psi_{\eta 2\beta} &= -\frac{j_{21}}{j_{22}}\Psi_{3\beta}^{(k)} + \frac{j_{11}}{j_{22}}\Psi_{4\beta}^{(k)}\end{aligned}\tag{A.24}$$

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