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CHANGING DISTRIBUTIONS IN THE STRONG SENSE AND THEIR COMPARATIVE STATICS: A RATIO APPROACH

By

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ABSTRACT

CHANGING DISTRIBUTIONS IN THE STRONG SENSE AND
THEIR COMPARATIVE STATICS: A RATIO APPROACH

By

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This dissertation analyzes the comparative static effects on the choice behavior of an economic agent who faces an uncertain situation when the given random situation changes into another. The decision maker's problem is to choose b to maximize $E[u(z(x, b))]$ where x is an exogenous random variable, and z is the payoff function which, as the argument of utility, depends on both the choice variable and the random variable. Given this general type of decision model, the existing comparative static results are developed by imposing restrictions on the following three components: (i) the set of changes in randomness; (ii) the set of decision makers; and (iii) the structure of the concerned decision model.

The purpose of this study is to develop general comparative static statements regarding various types of shifts in probability distribution function (pdf) or cumulative distribution function (CDF) of the given random variable. This dissertation concerns six special types of first-degree stochastic dominance shifts in chapters 3 and 4, four types of Rothschild-Stiglitz sense of increases in risk in chapters 5 and 6, and two types of second-degree stochastic dominance shifts in chapter 7. For each given type of change in the randomness, we develop conditions on the structure of the decision model and on the

risk preferences of the decision makers that are sufficient for making a general comparative static statement.

There are three special features in this dissertation. First, defining the subsets of changes in the randomness, the ratio between an initial and a final pdf's or CDF's is often restricted to be monotonic increasing or decreasing. Thus, we call this analysis a 'ratio approach' to comparative static problem. Second, the general comparative static statement developed for each of these subsets contains a relatively large class of decision makers, such as 'all individuals with non-decreasing utility functions,' 'all risk averse agents,' or 'all risk averse agents with non-negative third derivative of utility functions.' In this sense, all the changes in randomness examined in this study are referred to as 'CDF changes in the strong sense.'

Lastly, a special technique is used to generalize existing comparative static statements. It is often shown that: a CDF change that satisfies one type of CDF change can always be decomposed into a series of CDF changes that satisfy another type of CDF change. Then the comparative static statement made for the latter type of CDF change can also be applied for the former type of CDF change, without requiring any additional assumption. Using this technique, this study improves the robustness of the existing comparative static statements within the context of the ratio approach.

Dedicated to My Parents and Brother

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Chapter 1

INTRODUCTION

Uncertainty is an element of our environment and accordingly it affects every aspect of our life. Economists have recognized the pervasive influence of uncertainty on the choice behavior of economic agents. However, only after the pioneering work of von Neumann and Morgenstern (1947), who formalized the expected utility theory, has the analysis of risk and uncertainty been one of the main themes in the economics literature. During the last five decades, there has been much progress in the economic theory of uncertainty. Today the issue of uncertainty is often included in many fields of study such as firms' production, stock market, insurance, futures markets, public policy, international trade and finance.

In the study of choice under uncertainty, there are two main topics. One is to find selection rules governing the choice behavior of an individual who faces an uncertain situation. Every choice available to the individual gives a random outcome represented by a probability density function (pdf) or a cumulative distribution function (CDF). Then one is concerned with the question: which is the preferred probability distribution among those that are feasible? The stochastic dominance (SD) approach and the more traditional mean-variance analysis are two such selection rules. Each gives necessary and/or sufficient conditions on a pair of risky[1] prospects for one to be preferred to the other by all decision makers in some appropriate set.

[1] Throughout the paper, the terms 'uncertain,' 'random,' and 'risky' are used interchangingly.

The second main topic is to find predictable changes in choice behavior when a given random situation changes into another. In other words, one is often interested in finding necessary and/or sufficient conditions for determining the effect of a particular type of change in randomness on the optimal choice made by a decision maker. These general comparative static statements usually contain restrictions on the changes in pdf or CDF of the concerned random variable, the set of decision makers, and the structure of the given decision model. General SD orders play an important role in this comparative static analysis. Most research papers are concerned with particular types of CDF changes which are subsets of 'first-degree stochastic dominance'[2] (FSD) shifts, 'second-degree stochastic dominance'[3] (SSD) shifts, or 'mean-preserving spreads'[4] (MPS) which are 'increases in risk in the Rothschild-Stiglitz (R-S) sense.'

Many researchers have provided general comparative static statements dealing with the effects of various types of changes in the distribution of the random variable. Recent papers by Black and Bulkley (1989) and Landsberger and Meilijson (1990) have shown the importance of the likelihood ratio[5] between a pair of pdf's. Imposing monotonicity restrictions on the likelihood ratio, they define particular types of shifts from one pdf to another pdf. Black and Bulkley introduce the concept of a 'relatively strong increase in risk' (RSIR) which defines a subset of R-S increases in risk. Landsberger and Meilijson's analysis concerns a 'monotone likelihood ratio' (MLR) order which is widely used in the statistical literature. MLR shifts are a subset of general FSD shifts. In another recent paper, Eeckhoudt and Gollier (1995) give a definition of a 'monotone probability ratio' (MPR) order, imposing a monotonicity restriction on the probability ratio[6] between a pair of CDF's. The MPR order also specifies a subset of

[2] The concept of FSD is introduced by Quirk and Saposnik (1962) and Schneeweiss (1969).
[3] The concept of SSD is introduced by Hadar and Russell (1969, 1971) and Hanoch and Levy (1969).
[4] In order to give a formal definition of an increase in risk, Rothschild and Stiglitz (1970) introduce the concept of MPS.
[5] Given a pair of pdf's $f(x)$ and $g(x)$, the likelihood ratio means the ratio of one pdf to the other.
[6] Given a pair of CDF's $F(x)$ and $G(x)$, the probability ratio means the ratio of one CDF to the other.

FSD changes and is more general than the MLR order. The comparative static statements developed in these papers contain relatively weak restrictions on the set of decision makers. All individuals with non-decreasing utility functions are allowed in Landsberger and Meilijson's analysis, and all risk averse decision makers are considered in the other two cited papers. The likelihood ratio for pdf's and the probability ratio for CDF's play an important role in the analysis. In order to differentiate this particular work from the rest, we call their analysis a 'ratio approach' to the comparative statics problem.

In this dissertation, the analysis of comparative statics under uncertainty is continued. Our main purpose is to improve the robustness of the existing comparative static results within the context of the ratio approach. The results contained in the ratio approach literature are generalized by weakening various assumptions and considering more general subsets of CDF changes. Two such subsets of FSD shifts are specified by the concepts of a 'left-side monotone likelihood ratio' (L-MLR) order and a 'right-side monotone likelihood ratio' (R-MLR) order which are defined by imposing monotonicity restrictions on the likelihood ratio between a pair of pdf's. These are closely related to the subsets given by the MLR and the MPR orders, and the basic relationships among these subsets are as follows: The set of all CDF changes that satisfy the MLR order is a subset of the set of all CDF changes that satisfy the L-MLR order, which, in turn, is a subset of the set of all CDF changes that satisfy the MPR order. Also, the set that satisfies the MLR order is a subset of the set satisfying the R-MLR order.

Two more subsets of FSD shifts are specified. One is defined by the concept of an 'FSD improvement with respect to a point' (point-FSD). This subset is not related in a known way to the above four subsets of FSD shifts. The other is specified by the concept of an 'MPR with respect to a point' (point-MPR). Defined by weakening the conditions required for an MPR shift, the point-MPR order specifies a more general type of CDF change than an MPR shift. The sets of CDF changes defined by these two orders are determined by a given point. Let's denote, for a given point *k*, the orders as *k*-FSD and *k*-

MPR. Then these two orders are related to each other in such a way that a *k*-MPR shift can always be understood as the sum of two FSD shifts, one MPR and the other *k*-FSD. All these subsets of FSD shifts are used to improve the robustness of the existing comparative static results regarding the subsets of FSD shifts.

Regarding the case of R-S increases in risk, three subsets are considered when doing comparative static analysis. These sets are defined by introducing three special concepts of R-S increases in risk. A 'left-side relatively strong increase in risk' (L-RSIR) is defined by imposing a weaker condition than the one used by Black and Bulkley in defining an RSIR. An L-RSIR requires the monotonicity of the likelihood ratio only on the left-side of the given pair of pdf's. Monotonicity restrictions on the probability ratio between a pair of CDF's are used for obtaining the concepts of an 'extended strong increase in risk' (ESIR) and a 'left-side extended strong increase in risk' (L-ESIR). Each of the three concepts specifies a more general class of R-S increases in risk than does the RSIR order.[7] Basic relations among these subsets of R-S increases in risk are: The set of all CDF changes that satisfy the RSIR order is a subset of the set of all CDF changes that satisfy the ESIR or the L-RSIR order, which, in turn, are subsets of the set of all CDF changes that satisfy the L-ESIR order. For each of these subsets, we develop a general comparative static statement, and the case of ESIR shifts improves the robustness of the results in Black and Bulkley's analysis.

All the comparative static results obtained in this dissertation are associated with relatively large sets of decision makers; either the set of all individuals with non-decreasing utility functions, the set of all risk averse agents, or the set of risk averse agents with non-negative third derivative of their utility functions. These sets include utility functions representing quite plausible preferences, such as the ones exhibiting decreasing absolute risk aversion (DARA) generally accepted as a reasonable attitude

[7] We often interchangingly use the following terms: an 'order' between a pair of CDF's, a 'type' of CDF change, and a 'concept' of CDF change. Each means a definition of partial rankings (orderings) over probability distributions.

toward risk. These weak restrictions on the set of decision makers require a relatively strong restrictions on the changes in CDF for making general comparative static statements. In this sense, the changes in distributions examined in this study are all referred to as 'CDF changes in the strong sense.' Next the decision model used in the comparative static analysis is discussed.

An economic decision model involving randomness typically includes the following components: (i) random exogenous parameters; (ii) choice variables; (iii) an objective function; and (iv) a set of decision makers. In some decision models, more than one random and one choice variable are included, or utility functions depend on more[8] than one argument. For many of these decision models, it is generally difficult to make determinate comparative static statements. If the decision model includes multiple sources of randomness, every random variable affects the choice variables, and a change in the distribution of one random variable is generally accompanied by a simultaneous change in the distributions of the other random variables. Thus, even in the case of one choice variable, the presence of stochastic dependence among random variables makes it difficult to analyze the comparative statics for changing distribution of any one random variable. To avoid this problem, research papers often assume that the random variables are independent of one another, allowing one to change while the others are unchanged. This assumption allows the comparative static analysis for a change in any one random variable, using the standard SD orders as the restrictions on the change in the random variable. An example of this is Hadar and Seo's (1990) analysis of the portfolio model with more than one risky asset.

When there are several choice variables, more than one first-order condition defines the optimal solution, and this makes the comparative static analysis more difficult than for the case of one choice variable. Most papers studying the comparative statics

[8] That is, the objective function maps into R^n space where $n > 1$, rather than R^1.

assume that the first and the second-order conditions are satisfied to guarantee an interior unique solution. For the models with only one choice variable, the comparative static effect of a change in randomness is determined by the direction of the change in marginal expected utility of the choice variable (calculated at the initial optimal choice level). This is not true for the cases with more than one choice variable because of the interactions among the choice variables.

Comparative static analysis for models with more than one choice variable is usually possible by imposing additional restrictions on the concerned decision model. Examples of two choice variable decision models include Batra and Ullah (1974) who examine a competitive firm's input decisions with two inputs under output price uncertainty. Feder, Just and Schmitz (1977) deal with an international trade model, and Katz, Paroush and Kahana (1982) examine a price discriminating firm's optimal sales in two markets under price uncertainty in one of the two markets. Regarding a general form model with one random and two decision variables, Choi (1992) examines a special case of a corner solution.

When a utility function depends on two or more outcome values, that is, the outcome function is multidimensional, the standard risk aversion measures introduced by Pratt (1964) and Arrow (1971) are not able to be used when describing the risk preference of decision makers. In order to obtain general comparative static statements for these models, researchers need other definitions of risk aversion measures. Sandmo (1970) analyzes the saving behavior in the two-period income-capital model and uses his own definition of the 'temporal risk aversion' measure. Dardanoni (1988) treats a somewhat general form of two-outcome decision model and the obtained results depend on the 'proportional risk aversion' measure introduced by Menezes and Hanson (1970). There are other examples of imposing restrictions on the multidimensional utility function, such as Rothschild and Stiglitz (1971) and Mirman (1971) who analyze a two-period consumption-saving model, using an additively separable utility function.

The decision model used in this study includes only one random and one choice variable and utility depends only on one argument, that is, the objective function is single dimensional. Relatively standard notation for the general form of this one-argument decision model is:

$$\max_b E[u(z(x, b))] \tag{1.1}$$

where b is a scalar choice variable and x is an exogenous scalar random variable which interact to yield, z, the outcome which is the argument of utility. Many authors[9] have used this general decision model in their comparative static analysis. In this framework, problems involving multidimensionality are avoided because utility depends only on the value of a single dimensional outcome given by the intermediate function z.

Though the model (1.1) is a simple form, a variety of economic decision problems are of this type. Included are the often studied examples of a competitive firm facing random output price analyzed in Sandmo (1971), and the portfolio choice model with one risky and one riskless asset in Rothschild and Stiglitz (1971) and in Fishburn and Porter (1976), and the coinsurance problem in Dionne, Eeckhoudt and Gollier (1993). Sandmo considers a perfectly competitive firm choosing output level q to maximize expected utility of profits when the output price p is uncertain. In this model, the intermediate function z is understood as profits π given by $\pi = pq - c(q) - F$ where $c(q)$ is the variable cost function and F is the fixed cost. In the portfolio model, an individual is assumed to have initial wealth W_0 which is to be allocated between one riskless asset paying a zero rate of return and the risky asset paying a random rate of return e. In this case, the function z is understood as terminal wealth W given by $W = W_0[1 + be]$ where b is the proportion of W_0 invested in the risky asset.

[9] To name just a few, they are Feder (1977), Kraus (1979), Katz (1981), Meyer and Ormiston (1983, 1985, 1989), Black and Bulkley (1989).

Coinsurance is a contract that reimburses the policy holder a certain specified percentage for a loss occurred. In the standard coinsurance problem, the payoff function is given by the final wealth $W = W_0 - x + bx - b\lambda\mu$ where W_0 is the initial value of a given asset, x the random loss with mean μ, b the coinsurance rate, $b\lambda\mu$ the insurance premium, and $\lambda\mu$ the marginal cost of insurance rate. An individual chooses b to maximize expected utility from the final wealth. Typically the coinsurance rate b belongs to the interval [0, 1], the marginal cost of insurance rate $\lambda\mu$ is less than the maximum loss and larger than the minimum loss, and the maximum loss is usually less than initial wealth W_0. Other examples of models of this simple form include the problem of hiring workers in Feder (1977), the cooperative firm model in Paroush and Kahana (1980). By using the general form (1.1), each result developed in this dissertation is applicable to these specific models as well as others.

Regarding the decision model, there are several assumptions needed for conducting comparative static analysis. All decision makers are assumed to maximize expected utility. In other words, a decision maker always chooses that alternative which yields the highest mathematical expectation of a von Neumann-Morgenstern utility function $u(\cdot)$ which is defined over the set of all outcomes. The utility function u and the payoff function z are assumed to be continuous and thrice differentiable with respect to their arguments. In some results, it is enough for the functions to be differentiable only once or twice. Restrictions on these functions are often needed to guarantee an interior unique solution as well as to generate definite comparative static statements. To simplify the analysis, we follow the literature and focus only on the case where $z_x(x, b) \geq 0$. With the assumption $u'(z) \geq 0$, this implies that higher values of the random variable are preferred to lower values. The case where $z_x(x, b) \leq 0$ can be handled with an appropriate redefinition of the random variable.[10]

[10] The coinsurance problem cited above is of this case, and thus, if we redefine the random loss x as $x \underset{d}{=} -y$ where "$\underset{d}{=}$" means "has the same distribution as," the outcome function for y satisfies $z_y(y, b) \geq 0$.

The random variable x is assumed to have its initial and final distributions[11] characterized by CDF's $G(x)$ and $F(x)$, with their corresponding pdf's $g(x)$ and $f(x)$, respectively. Our study includes all the cases where the random variable x is continuous, discrete or mixed. While the results are usually proved for the continuous case (using integrals for summation), they can be adapted for the case where x is discrete or mixed. Let b_G and b_F denote the optimal choice[12] of the problem (1.1), under the CDF's G and F, respectively. We analyze the comparative static effects on the sign of $b_F - b_G$ for changes in CDF from G to F. The supports of x under both the distributions are assumed to be finite intervals. The notations for these finite supports are introduced as needed.

This dissertation is organized as follows. In the next chapter we will give a short review of the literature concerning the SD approach and the previous analysis concerning the comparative statics under uncertainty. The review of the SD selection rules is for the univariate case and the review of the comparative static analysis is limited to the decision models of the form (1.1). General comparative static statements are obtained by imposing restrictions on the following components: (i) the structure of the decision model; (ii) the set of decision makers; and (iii) the set of changes in distribution of the random variable. In an analysis, relatively strong restrictions on one component are usually associated with relatively weak restrictions on the other two components.

Previous work on this topic can be classified according to the restrictions placed on these components. Structural restrictions are sometimes imposed on payoff functions. Most papers concerning the decision model (1.1) adopt some restrictions such as $z_x \geq 0$, $z_{bx} \geq 0$ and $z_{bxx} \leq 0$. Since many economic problems already satisfy the restrictions, they are not very harmful for conducting comparative static analysis. As somewhat stronger restrictions, authors sometimes restrict the concerned payoffs to be linear in the choice

[11] To keep the notation for F to be stochastically dominant over G, the CDF's F and G for the case of an R-S increase in risk are regarded as an initial and a final distribution of x, respectively.
[12] Some parts of our study follows the usual assumption that the first and the second-order conditions are satisfied for defining an optimal solution of the problem (1.1), but the results concerning subsets of FSD shifts are established without requiring the second-order condition.

variable and/or the random variable. This restriction on the structure of the decision model limits the analysis to more specialized decision problems.

Given a decision model structure, some research imposes relatively strong restrictions on the changes in the pdf or CDF and relatively weak restrictions on the risk preference of decision makers, and other research treats the reverse case. The studies using relatively strong restrictions on the changes in distributions usually consider proper subsets of the set of FSD changes or R-S increases in risk.

According to the type of restrictions used for defining sets of changes in pdf or CDF, there are three approaches to the comparative static analysis. The deterministic transformation approach obtains a change in distribution by transforming an initial random variable into another random variable through a given deterministic function. The CDF approach directly imposes a restriction on the relation between the initial and the final CDF's G and F or pdf's g and f, often restricting $G - F$. Recently, interesting sets of changes are defined by imposing restrictions on the ratio of pdf's or the ratio of CDF's. In order to differentiate these changes from the others, the studies employing them are said to use the 'ratio approach.' The focus in this dissertation is on the ratio approach to comparative static analysis.

In chapters 3 and 4, we examine subsets of FSD shifts. For the ratio approach, they include the MLR shifts used in Landsberger and Meilijson (1990) and the MPR shifts in Eeckhoudt and Gollier (1995). We define four additional subsets of the set of general FSD shifts. They are specified by the L-MLR, the R-MLR, the point-FSD, and the point-MPR orders. Chapter 3 discusses the relationships among these subsets of FSD shifts. The restrictions used in defining these subsets are discussed and graphical examples are given to illuminate the basic relations among the subsets. For the basic relations, it is shown that the MLR order specifies a special case of the L-MLR or the R-MLR order, and that the L-MLR order defines a special case of the MPR order which, in turn, gives a special case of the point-MPR order.

The property of transitivity is also discussed with regard to these orders. Transitivity holds for both the MLR and the MPR orders, but not for the L-MLR or the R-MLR. Given a point *k*, both the *k*-FSD and the *k*-MPR orders also display transitivity. Whether a CDF order is transitive or not has important implications in doing comparative static analysis. If a given CDF order is sufficient for a general comparative static statement, then any shift that can be decomposed into a series of shifts satisfying the CDF order also has a predictable outcome. Thus, if the given order is not transitive, the admissible set of CDF changes for the comparative static result include more shifts than are included by the CDF order itself. The L-MLR order is generally non-transitive, but has an important relationship with the MPR order resulting from transitivity. We show that an MPR shift can always be decomposed into a series of L-MLR shifts. Another important relationship exists with regard to the point-MPR order. Given a point *k*, a *k*-MPR shift can always be decomposed into two shifts, one *k*-FSD and the other MPR. In chapter 4, these properties are used to prove some generalized comparative static statements.

Chapter 4 provides general comparative static statements for the subsets of FSD shifts presented in chapter 3. Each developed result gives a generalization of an existing comparative static result. The results in Landsberger and Meilijson (1990), and Eeckhoudt and Gollier (1995) are for decision models with linear payoffs. This chapter shows that the same results also hold for general non-linear decision models, without imposing additional assumptions. The extension of Eeckhoudt and Gollier's result is demonstrated using the L-MLR order and its relationship with the MPR order presented in chapter 3. The proposed comparative static statement for the set of MLR shifts applies to the set of all decision makers with non-decreasing utility functions, and the one for the set of MPR shifts applies to the set of all risk averse decision makers.

In addition, for each of the two comparative static results, we explore the trade-off between the structural restrictions on the decision model and the restrictions on the change in the distribution. When the concerned payoff (outcome) function is restricted to be linear in the choice variable *b* ($z_{bb} = 0$), even more general classes of changes in CDF yield determinate comparative static statements. Given a decision model satisfying the linearity restriction, the value of *x* satisfying $z_b = 0$ does not depend on the choice variable *b*. Let *c* be such the point of *x*. Then two generalized comparative static results are derived. One is that, in addition to the set of MLR shifts, L-MLR shifts satisfying the monotonicity restriction on the left-side of the point *c*, and R-MLR shifts satisfying the monotonicity restriction on the right-side of the point *c* are also sufficient for Landsberger and Meilijson's result. The other is that the set of *c*-FSD shifts and the set of *c*-MPR shifts are also sufficient for Eeckhoudt and Gollier's result. Each generalization implies that, for a fixed set of decision makers, the finding is extended to a more general class of CDF changes with the cost of adding the linearity restriction on the decision model.

Chapters 5 and 6 treat the cases of R-S increases in risk. In addition to the existing subsets of R-S increases in risk, we define three additional subsets of R-S increases in risk specified by the CDF orders: ESIR, L-RSIR and L-ESIR. These are more general than the RSIR order defined by Black and Bulkley (1989). The restrictions on the pdf ratio used in defining the RSIR and the L-RSIR orders are replaced by the restrictions on the CDF ratio when we define the ESIR and L-ESIR orders. In chapter 5, the relationships among these orders are examined and the basic relations are illustrated with some graphical examples. It is shown that the monotonicity restrictions on the CDF ratio are less severe than the ones on the pdf ratio. Thus, we have the basic relationships among the orders as: the RSIR order specifies a special case of the L-RSIR or the ESIR order, which, in turn, give a special case of the L-ESIR order.

In general, transitivity does not hold for these CDF orders. This implies that, given one of these orders, a CDF change that can be decomposed into two or more distinct shifts that are generated by the order may not satisfy the order. This guarantees that, if a general comparative static statement is made for the CDF order, the order is only sufficient but not necessary for the result. Though they are all non-transitive, there exist two important relationships between the RSIR and the ESIR orders, and between the L-RSIR and the L-ESIR orders. They are: (i) an ESIR shift can always be decomposed into a series of RSIR shifts; and (ii) an L-ESIR shift can always be decomposed into a series of L-RSIR shifts. Using these properties, chapter 6 provides two useful generalizations of the comparative static results given in the same chapter.

Chapter 6 gives general comparative static statements regarding the subsets of R-S increases in risk, presented in chapter 5. The comparative static result in Black and Bulkley (1989) is generalized to the set of ESIR shifts which is larger than the set of RSIR shifts. This generalization is, without any cost of additional assumption. The proposed comparative static statement is for the set of all risk averse decision makers (same as in Black and Bulkley) and the set of ESIR shifts.

When the concerned payoff is restricted to be linear in the random variable, another extension of Black and Bulkley's result is given by using a trade-off between the restrictions on the set of decision makers and the restrictions on the set of R-S increases in risk. With the given structural restriction, the proposed result contains, compared with Black and Bulkley's result, a larger set of changes in CDF and a smaller set of decision makers. That is, the set of risk averse agents with non-negative third derivative of utility functions and the set of L-RSIR shifts which includes all RSIR shifts are contained in the result. A further generalized result is provided in order to include a more general class of CDF changes. The set of L-ESIR shifts replaces the set of L-RSIR shifts in the result, and this generalization is directly comes from the relationship between the L-RSIR and the L-ESIR orders given in chapter 5.

Finally, chapter 7 gives a summary for the results developed in the previous chapters. Also some remarks regarding the comparative static results for subsets of second-degree stochastic dominance (SSD) shifts are given in section 7.2. We show that the results obtained in previous chapters can be applied for cases of SSD shifts. In general, the SSD order defines a more general type of changes in distribution of a random variable than does the FSD order or the concept of an R-S increase in risk. According to Hadar and Seo (1990), any SSD change can be decomposed into an FSD shift and an R-S decrease in risk (mean-preserving contract (MPC)). This can be applied for defining subsets of the set of SSD shifts which are combinations of FSD shifts and R-S decreases in risk, which satisfy the CDF orders defined in chapters 3 and 5.

Two subsets of SSD shifts are defined: 'extended strong SSD' (ESSSD) shifts and 'left-side extended strong SSD' (L-ESSSD) shifts. An ESSSD shift can always be decomposed into an MPR and an ESIR, and an L-ESSSD shift can always be understood as the sum of an MPR and an L-ESIR. These properties allow one to make general comparative static statements for these subsets, by combining the results obtained in chapters 4 and 6.

Chapter 2

LITERATURE REVIEW

There are two main topics in the analysis of choice under uncertainty. One is to find selection rules governing the choice behavior under uncertainty, and the other is to analyze the comparative static effects on the choice made by an economic agent when a given random situation changes into another. This chapter reviews a selected portion of the literature on these two topics. Section 2.1 presents the well-known standard stochastic dominance (SD) selection rules. Section 2.2 gives a brief review of the literature concerning the comparative static analysis. Finally, section 2.3 contains some remarks specific to this study. For expositional ease, let x be a random variable characterized by an initial CDF $G(x)$ and a final CDF $F(x)$, and if not stated otherwise, the supports of these CDF's are assumed to be in the finite interval [0, 1].

2.1 General Stochastic Dominance Selection Rules

When a decision maker is confronted with alternative uncertain prospects, he wants to choose to maximize expected utility. SD selection rules provide a partial ranking of these alternatives. Each selection rule gives a necessary and/or sufficient condition on a pair of risky prospects (a pair of distribution functions) for one to be preferred to the other by all individuals with non-decreasing and/or concave utility functions. Restricting ourselves to the case of univariate distributions, we give the definitions of the standard SD orders and then explain several SD selection rules.

2.1.1 Basic Definitions of Stochastic Dominance Orders

We review three[13] types of SD orders for pairs of CDF's. The first-degree stochastic dominance (FSD) introduced by Quirk and Saposnik (1962) and Schneeweiss (1969), and the second-degree stochastic dominance (SSD) introduced by Hadar and Russell (1969) and Hanoch and Levy (1969) are defined as follows:

Definition 2.1. *F*(*x*) is said to stochastically dominate *G*(*x*) in the first-degree (denoted by *F* FSD *G*) if and only if

$$G(x) - F(x) \geq 0, \text{ for all } x \in [0,\ 1].$$

Definition 2.2. *F*(*x*) is said to stochastically dominate *G*(*x*) in the second-degree (denoted by *F* SSD *G*) if and only if

$$\int_0^s [G(x) - F(x)]dx \geq 0, \text{ for all } s \in [0,\ 1].$$

Each definition gives a partial ordering on a set of probability distributions. It is well known and obvious from these definitions that if *F* FSD *G* then *F* SSD *G*. This implies that the set of CDF's that can be ordered by the concept of FSD is included in the set that can be ordered by the concept of SSD. These stochastic dominance orderings have the property of transitivity. That is, considering CDF's F_1, F_2 and F_3, if F_1 FSD (SSD) F_2 and F_2 FSD (SSD) F_3, then F_1 FSD (SSD) F_3.

[13] There are more stochastic dominance orderings, which are not concerned in this paper, such as the traditional mean-variance ordering by Markowitz (1952, 1959) and Tobin (1965), the third and higher degree stochastic dominance by Whitmore (1970, 1989), and SSD with respect to a function by Meyer (1977a, 1977b).

Finally, Rothschild and Stiglitz (R-S) (1970) provide a general definition of an 'increase in risk.' According to R-S, the following three conditions are equivalent and are used to define '*G* is riskier than *F*.' The conditions are:

1. A random variable y can be obtained from another random variable x by an addition of random noise ε, where y and x are the random variables defined by CDF's G and F, respectively, that is,

$$y \underset{d}{=} x + \varepsilon,$$

   where $E(\varepsilon|x) = 0$ and "$\underset{d}{=}$" means "has the same distribution as."

2. All risk averse individuals prefer F to G, that is, for every concave utility function $u(\cdot)$,

$$\int_0^1 u(x)dF(x) \geq \int_0^1 u(x)dG(x).$$

3. G can be obtained from F by a sequence of mean-preserving spreads (MPS).[14]

The condition 3 can be stated with the well-known 'integral conditions,' being the restrictions imposed on the difference $G - F$. Because the three conditions are equivalent, an increase in risk is formally defined by the integral conditions as:

Definition 2.3. $G(x)$ is said to be riskier than $F(x)$ in the Rothschild-Stiglitz sense (denoted by G MPS F) if and only if

(a) $\int_0^1 [G(x) - F(x)]dx = 0$

(b) $\int_0^s [G(x) - F(x)]dx \geq 0$, for all $s \in [0, 1]$.

[14] Using an alternative definition of a mean-preserving spread, Machina and Pratt (1997) give a more explicit result that, for an arbitrary pair of distributions F and G satisfying the integral conditions given in Definition 2.3, there always exists a sequence of MPS's by which F converge directly to G.

The condition (a) implies that CDF's F and G have the same mean, and the condition (b) is the one used in defining F SSD G. Thus an R-S increase in risk is a special case of SSD, requiring in addition, equal means. It too gives a transitive partial ordering on a set of probability distributions. That is, considering CDF's F_1, F_2 and F_3, if F_1 MPS F_2 and F_2 MPS F_3, then F_1 MPS F_3. For almost thirty years now, this definition has replaced variance or standard deviation, as the basic concept for studying comparative risk.

2.1.2 Stochastic Dominance Selection Rules

Three selections rules which are used in this dissertation are presented here. To generate these rules, each of the above stochastic dominance relations is associated with a particular set of utility functions. These sets are defined as:

Definition 2.4. Assuming that a utility function $u(\cdot)$ is continuous, bounded and twice differentiable for the interval [0, 1], define the following three sets as:

(a) $U_1 = \{u(\cdot)|\ u' \geq 0\}$

(b) $U_2 = \{u(\cdot)|\ u' \geq 0 \text{ and } u'' \leq 0\}$

(c) $U_3 = \{u(\cdot)|\ u'' \leq 0\}$.

Denote, for utility function u, EU_F and EU_G as the expected utility computed with respect to CDF's F and G, respectively. Then selection rules for the cases of FSD and SSD are expressed as:

Theorem 2.1. $EU_F \geq EU_G$ for every u in U_1 if and only if F FSD G.

Theorem 2.2. $EU_F \geq EU_G$ for every u in U_2 if and only if F SSD G.

The proof of FSD rule is given by Quirk and Saposnik (1962), Schneeweiss (1969), Hadar and Russell (1969) and Hanoch and Levy (1969), and the proof of SSD rule is given by the last two papers.

The last selection rule given in this sub-section is the one provided by Rothschild and Stiglitz (1970). From the three equivalent conditions shown in sub-section 2.2.1, the selection rule is expressed as:

Theorem 2.3. $EU_F \geq EU_G$ for every u in U_3 if and only if G MPS F.

Concavity is the only restriction used in defining the set U_3. This general property plays an important role in their companion paper, Rothschild and Stiglitz (1971), studying the comparative static effects of an R-S increase in risk.

2.2 Comparative Static Analysis

The question of interest for comparative statics under uncertainty is to find necessary and/or sufficient conditions for determining the direction of change in the choice variable when a change in the random variable occurs. Many economic models are used in addressing this question. Some are of a specific form and others are of a general form. There are models including two or more random variables, choice variables and/or outcome values.[15] However, since our concern is restricted to the general one-argument decision model (1.1) introduced in chapter 1, the literature review is restricted to studies concerning decision models of this form.

[15] See the dissertation by Choi (1992), for the comparative static analysis concerning decision models with multiple outcomes.

2.2.1 An Overview

Regarding the decision models of the form (1.1), the standard monotonicity or concavity properties of utility functions and the standard SD shifts are not sufficient for making general comparative static statements. That is, for an arbitrary FSD, MPS or SSD shift in distribution, one can not say that every decision maker with a non-decreasing and/or concave utility function changes his choice in the same direction. This is more formally stated by Meyer and Ormiston (1983) who show that any FSD or any SSD shift in distribution causes every risk-averse individual to adjust his choice level to the same direction if and only if the optimal choice under certainty is independent of the value of the random exogenous variable. Obviously, this is not an interesting case. Following this line, all the existing comparative static results have used some restrictions on the following three components:

(i) the structure of a given decision model

(ii) the set of decision makers concerned

(iii) the set of changes in distribution of the concerned random variable.

Imposing 'relatively weak restrictions' on one of the three components means that more cases or a relatively larger class of the component are included in an analysis, and vice versa. Generally speaking, when relatively strong restrictions are imposed on one component, the derived comparative static results are usually associated with relatively weak restrictions on the other components.

As the structural restrictions imposed on the general decision model (1.1), $z_x \geq 0$, $z_{hx} \geq 0$ and $z_{hxx} \leq 0$ are often assumed by many authors. Because many economic problems already satisfy the assumptions, they are not very harmful for the comparative static analysis. Some authors add the assumptions $z_{hh} = 0$ and/or $z_{xx} = 0$, that is, the concerned payoffs are restricted to be linear in the choice variable and/or the random variable. Two of the most often studied examples are the standard portfolio problem in which the final wealth is linear in both the choice and the random variable, and a competitive firm model under price uncertainty in which the profit function is linear in the random variable. Other examples are found in Dionne, Eeckhoudt and Gollier (1993), and Eeckhoudt and Gollier (1995). As an extreme case, Meyer and Ormiston (1983) use such a payoff that the optimal choice under certainty does not depend on the value of the random variable. Thus the restriction on the first component limits the analysis to special economic problems.

Given a decision model, the comparative static question is to find a set of changes in CDF or pdf which is sufficient for signing the effect on the choice variable selected by an arbitrary decision maker in a specified set. In a general comparative static statement, a relatively large (small) class of shifts in CDF is usually followed by relatively strong (weak) restrictions on the risk preference of decision makers. The trade-offs between the restrictions on these two components are often examined.

Examples using relatively strong restrictions on the risk preferences of agents include Rothschild and Stiglitz (1971), Hadar and Russell (1978) and Hadar and Seo (1990). They examine the general classes of FSD, MPS and/or SSD changes. Concerning the standard portfolio choice problem, they found that, for an FSD shift in the distribution of the return on a risky asset, relative risk aversion to be less than one is sufficient to cause investors to demand more of the risky asset. For an MPS shift in the random return on a risky asset, the conditions that absolute risk aversion is decreasing and relative risk aversion is increasing and smaller than one are sufficient for investors to decrease their demand for the risky asset. In these studies, large classes of changes in distribution are considered but relatively strong restrictions are imposed on the attitude toward risk of the concerned decision makers.

Many authors have used relatively strong restrictions on the third component, the changes in randomness. They have found sufficient conditions on the changes in distribution which cause every individual in a fairly large set to adjust his choice in the same direction. Subsets of FSD shifts are examined in Sandmo (1971), Landsberger and Meilijson (1990), Ormiston (1992), Ormiston and Schlee (1993), and Eeckhoudt and Gollier (1995), among others. Various subsets of MPS shifts or R-S increases in risk are analyzed in Sandmo (1971), Ishii (1977), Kraus (1979), Eeckhoudt and Hansen (1980), Katz (1981), Meyer and Ormiston (1985, 1989), Black and Bulkley (1989), Dionne, Eeckhoudt and Gollier (1993), and more.[16]

According to the type of restrictions used in defining various sets of changes in CDF or pdf, these studies can be divided into three approaches. Deterministic transformation approach defines a set of changes in distribution by mapping every possible initial outcome value into a new value through a given deterministic function. CDF approach uses restrictions on the initial and the final distribution functions G and F, often imposing restrictions on the difference $G - F$. Other interesting sets are defined by imposing restrictions on the likelihood ratio between the two pdf's or the probability ratio between the two CDF's. These studies are referred to as the 'ratio approach.' The following sub-sections present each approach in more detail.

2.2.2 Deterministic Transformation Approach

According to Meyer and Ormiston (1989), a single-valued function $t(\cdot)$, what they call a 'deterministic transformation function,' can be used for defining a type of change in a random variable. That is, given an initial random variable x, a transformed random variable y is defined by $y = t(x)$. The transformation function is assumed to be non-decreasing, continuous and piecewise differentiable. The non-decreasing assumption is related to the following two properties. First, combined with the monotonicity property of utility functions, it ensures that the transformation does not reverse the preference ordering over the set of all possible outcomes of the original random variable. Second, letting the random variables x and y be characterized by CDF's G and F, respectively, the

[16] More examples are found in Coes (1977), Feder (1977), Paroush and Kahana (1980), Gollier (1995).

non-decreasing assumption implies that $G(x) = F(t(x))$ for all possible outcomes of x. This non-decreasing assumption is necessary to make general statements about the effects of some deterministic transformations of a random variable on the expected utility.

Using the transformation method, Meyer and Ormiston provide a fourth characterization of an R-S increase in risk as follow:

Theorem 2.4. The random variable given by a transformation $t(x)$ represents an R-S increase in risk from an initial random variable x described by a CDF $G(x)$ if and only if the function $k(x) \equiv t(x) - x$ satisfies

(a) $\int_0^1 k(x)dG(x) = 0$

(b) $\int_0^s k(x)dG(x) \leq 0$, for all $s \in [0, 1]$.

By the condition (a), the mean of the random variable is preserved. The condition (b) guarantees that the initial random variable dominate the transformed random variable in the second-degree. This theorem is proved by showing that the given transformation reduces the expected utility for all risk averse decision makers.

Similarly, FSD and SSD shifts can also be characterized in terms of transformations. According to Meyer (1989), we have:

Theorem 2.5. The random variable given by a transformation $t(x)$ dominates an initial random variable x described by a CDF $G(x)$ in the first-degree if and only if $k(x) \equiv t(x) - x \geq 0$, for all $x \in [0, 1]$.

The given condition implies that the transformed CDF lies to the right side of the initial CDF G. Thus, as we saw in section 2.1, the transformation gives a greater expected utility for all individuals with non-decreasing utility functions.

Theorem 2.6. The random variable given by a transformation $t(x)$ dominates an initial random variable x described by a CDF $G(x)$ in the second-degree if and only if the function $k(x) \equiv t(x) - x$ satisfies $\int_0^s k(x)dG(x) \geq 0$, for all $s \in [0, 1]$.

Only the condition (b) in Theorem 2.4 is used and the transformation gives a higher expected utility for all risk averse economic agents.

Imposing further restrictions on the transformation function produces interesting sets of changes in distribution within the context of comparative static analysis. A well-known example is a linear transformation analyzed in Sandmo (1971) and Ishii (1977). In their papers, the transformation function used is a linear form as:

$$y = \gamma(x - \bar{x}) + \theta + \bar{x}$$

where γ is a multiplicative shift parameter, θ is an additive shift parameter and $\bar{x}$ is the mean of the random variable x. An increase in θ (from $\gamma = 1$ and $\theta = 0$) increases the mean of the random variable without changing its shape or dispersion of the initial distribution. Since it gives a parallel shift of the original CDF $G(x)$ to the right, it defines a type of FSD shift.

Sandmo, studying the behavior of a competitive firm under price uncertainty, provides a general comparative static statement that, facing an increase in θ in the random price, the firm exhibiting decreasing absolute risk aversion increases its level of output. Sandmo also examines an increase in the parameter γ (from $\gamma = 1$ and $\theta = 0$). Preserving the mean, this change causes a 'stretching out' of the initial distribution about its mean. This transformation defines a type of R-S increases of risk. Treating the same problem in Sandmo, Ishii gives a proof for the comparative static result that, facing an increase in γ in the random price, the firm exhibiting decreasing absolute risk aversion decreases its level of output.

The comparative static results concerning the linear transformations are generalized by introducing 'simple deterministic transformations.' Meyer and Ormiston (1989) introduce a 'simple increase in risk' as a type of R-S increases in risk, and Ormiston (1992) defines a 'simple FSD transformation' as a class of FSD shifts. These are given as:

Definition 2.5. The random variable given by a transformation $t(x)$ represents a simple R-S decrease in risk from an initial random variable x described by a CDF $G(x)$ if the function $k(x) \equiv t(x) - x$ satisfies

(a) $\int_0^1 k(x)dG(x) = 0$

(b) $\int_0^s k(x)dG(x) \geq 0$, for all $s \in [0,\ 1]$

(c) $k'(x) \leq 0$.

Definition 2.6. The random variable given by a transformation $t(x)$ represents a simple FSD shift from an initial random variable x described by a CDF $G(x)$ if $k(x) \equiv t(x) - x \geq 0$ and $k'(x) \leq 0$, for all $x \in [0,\ 1]$.

These simple transformations generalize the above linear transformations by not requiring the shifting or the stretching to be of uniform intensity. Using the general decision model (1.1), each of the simple transformations allows one to make general comparative static statement. Assume that $u'(z) \geq 0$, $u''(z) \leq 0$ and $z_{bb} < 0$. Meyer and Ormiston provide the following theorem:

Theorem 2.7. Facing a simple R-S increase in risk in the random variable x, a decision maker will decrease the optimal value of b, if

(a) $u(z)$ displays decreasing absolute risk aversion

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{bx} \geq 0$ and $z_{bxx} \leq 0$.

Similarly, Ormiston[17] gives a general comparative static statement as:

Theorem 2.8. Facing a simple FSD shift in the random variable *x*, a decision maker will increase the optimal value of *b*, if

(a) $u(z)$ displays decreasing absolute risk aversion

(b) $z_x \geq 0$, $z_{xx} \leq 0$ and $z_{bx} \geq 0$.

With these simple transformations, the comparative static results by Sandmo and Ishii are generalized in two senses. First the same results are obtained for more general classes of transformations, with no additional restrictions required. Second, the decision model concerned is replaced by the general decision model which includes the competitive firm model as a special case.

2.2.3 CDF Approach

CDF approach to comparative static analysis uses restrictions on the initial and the final CDF's *G* and *F* to define sets of changes in distribution of the random variable. Sandmo (1971) considers a special type of R-S increases in risk, namely, an 'introduction of risk' or a 'global increase in risk' which implies an R-S increase in risk from an initial non-random situation. He found that the optimal output level of production of a risk-averse competitive firm is smaller when the price is uncertain, compared to the production level when the price is known with certainty.

Sandmo's negative effect of an introduction of risk is generalized by Kraus (1979) and Katz(1981). Kraus presents the same result using the general one-argument decision

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[17] Ormiston (1992) also defines a simple SSD transformation and provided its comparative static theorem. However, since an SSD shift can be considered as combinations of an FSD and an R-S decrease in risk or mean-preserving contract (MPC), the result is not presented here.

model (1.1), that is, he derives a sufficient condition for signing the effect of a global change in risk on the optimal level of b selected by an arbitrary risk averse decision maker. The condition includes restrictions on the payoff function $z(x, b)$ as follows:

Theorem 2.9. Assume that $u' > 0$, $u'' < 0$, $z_x > 0$ and $z_{bb} < 0$, then all risk averse decision makers, facing a global increase in risk in the random variable x, will decrease the optimal value of b if $z_{bx} \geq 0$ and $z_{bxx} \leq 0$.

The condition in Theorem 2.9 is somewhat more generalized by Katz who provides a shorter proof of the result.

Though the above results apply to a fairly large set of decision makers (all risk averse agents), the restriction that risk change is a global change in risk is quite severe. A global increase in risk is only a small subset of the set of all R-S increases in risk. This limits significantly the situations to which the result can be applied. Eeckhoudt and Hansen (1981) analyze the comparative static effect for a less restrictive set of R-S increases in risk. They introduce the concept of a 'min-max truncation' or a 'mean-preserving truncation.' Let the supports of CDF's G and F be the finite intervals, $[x_1, x_4]$ and $[x_2, x_3]$, respectively, where $x_1 \leq x_2 \leq x_3 \leq x_4$. This is a slight modification of the original notation, using the points x_2 and x_3 as the minimum and the maximum points where the truncation occurs. A mean-preserving truncation is defined as:

Definition 2.7. $F(x)$ represents a mean-preserving truncation from $G(x)$ if their difference $G(x) - F(x)$ satisfies

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]dx = 0$

(b) $F(x) = 0$ for all $x \in [x_1, x_2)$, $F(x) = 1$ for all $x \in (x_3, x_4]$, and $G(x) = F(x)$ for all $x \in [x_2, x_3]$.

Since the conditions (a) and (b) imply that $\int_{x_1}^{s}[G(x)-F(x)]dx \geq 0$ for all $s \in [x_1, x_4]$, a mean-preserving truncation is an R-S decrease in risk and it includes a global decrease in risk as a special case. It specifies a transfer of probability mass such that both the truncated left-tail and right-tail probability mass are moved into the minimum and the maximum end points, respectively. Eeckhoudt and Hansen show that, using Sandmo's competitive firm model, the optimal level of production of a risk averse firm increases when a mean-preserving truncation occurs in the random price. Thus they improve the robustness of Sandmo's result because the same comparative static result is obtained for a more general class of R-S increases in risk without additional assumptions required.

Meyer and Ormiston (1985) provide a further generalization of the results in Eeckhoudt and Hansen. They introduce a 'strong increase in risk' (SIR) as a concept of an R-S increase in risk. The condition (b) in Definition 2.7 is replaced by a much less restrictive form as:

Definition 2.8. $G(x)$ represents a strong increase in risk from $F(x)$ (denoted by F SIR G) if their difference $G(x)-F(x)$ satisfies[18]

(a) $\int_{x_1}^{x_4}[G(x)-F(x)]dx = 0$

(b) $G(x)-F(x)$ is non-increasing in $x \in [x_2, x_3]$.

An SIR specifies a transfer of probability mass such that some probability mass from the interval $[x_2, x_3]$ is moved into both the tail intervals $[x_1, x_2]$ and $[x_3, x_4]$. Thus it defines a more general class of changes in distribution than the one by a mean-preserving

[18] The condition, used in the original paper, that $\int_{x_1}^{s}[G(x)-F(x)]dx \geq 0$ for all $s \in [x_1, x_4]$ is omitted here because it is implied by the conditions (a) and (b) in Definition 2.8.

truncation. Using the general decision model (1.1), Meyer and Ormiston's comparative static result improves the robustness of the previous results. Without any cost of additional assumptions, the results in Kraus and Katz are obtained for a more general class of R-S increases in risk, and the result in Eeckhoudt and Hansen is extended to the general decision model and to a more general class of changes in distribution.

All the subsets, presented in this sub-section, are defined by imposing restrictions on the difference between the initial and the final CDF. They are all subsets of R-S increases in risk. There are more studies giving further extensions of the above results. In the next sub-section, we review the ratio approach which treats more sets of changes in distribution, including subsets of FSD shifts as well as subsets of R-S increases in risk.

2.2.4 Ratio Approach

Recent papers such as Black and Bulkley (1989), Landsberger and Meilijson (1990), Dionne, Eeckhoudt and Gollier (1993) and Eeckhoudt and Gollier (1995) have shown the importance of the likelihood ratio between a pair of pdf's and the probability ratio between a pair of CDF's in defining sets of changes in pdf or CDF. As mentioned earlier, our comparative static analysis follows this ratio approach and thus these previous papers are crucial to our comparative static analysis. In this subsection, we give a brief review of the ratio approach and more detailed review is given in later chapters.

Landsberger and Meilijson introduce a monotone likelihood ratio (MLR) shift which is defined by imposing a monotonicity restriction on the likelihood ratio between a pair of pdf's. The set of MLR shifts is a subset of general FSD shifts. Using the standard portfolio choice problem, Landsberger and Meilijson provide a general comparative static statement that, when an MLR shift in the random return on a risky asset occurs, all investors with non-decreasing utility functions increase their demand for the risky asset.

Another recent paper by Eeckhoudt and Gollier gives a definition of a monotone probability ratio (MPR) shift by imposing a monotonicity restriction on the probability ratio between a pair of CDF's. Being also a subset of general FSD shifts, the set of MPR shifts is larger than the set of MLR shifts. Assuming that the concerned payoff is linear in both the choice and the random variables, they show that an MPR shift in the random variable causes all risk averse individuals to increase their optimal choice level. Compared to Landsberger and Meilijson's analysis, this implies a trade-off between the restrictions used. That is, Eeckhoudt and Gollier provide a larger set of admissible changes in distribution for the comparative static result with the cost of a smaller set of decision makers. Since these two papers play an important role in our comparative static analysis for the case of FSD shifts, more detailed discussions are given in chapter 4.

Black and Bulkley introduce a relatively strong increase in risk (RSIR) which is defined by imposing a monotonicity restriction on the likelihood ratio between a pair of pdf's. As a subset of R-S increases in risk, the set of RSIR shifts is larger than the set of SIR shifts given in Definition 2.8. Black and Bulkley show that, based on the general decision model (1.1), an RSIR shift reduces the level of optimal choice selected by an arbitrary risk averse decision maker. Accordingly, this is a generalization of the results in Meyer and Ormiston (1985). Without the cost of additional assumptions, the same result is obtained for a more general class of changes in distribution. However the comparative static results in these two papers require a common restriction that payoffs should be strictly concave in the choice variable, that is, $z_{bb} < 0$. This restriction excludes the decision models in which payoffs are linear in the choice variable $(z_{bb} = 0)$ such as the standard portfolio problem.

In general, when $z_{bb} = 0$, the second-order condition for the decision problem (1.1) is not guaranteed and thus the cases of an unbounded and a corner solutions should be examined for making general comparative static statements. Dionne, Eeckhoudt and Gollier raise this issue and show that, when the concerned payoff is linear in the choice variable b, the results in Black and Bulkley are also obtainable for a 'relatively weak increase in risk' which is a more general concept than an RSIR. They examined the cases of an unbounded and a corner solutions in the proof of their comparative static result. More detailed discussions about these papers are given in chapter 6 which considers the comparative static analysis for the cases of R-S increases in risk.

2.3 Concluding Remarks

Reviewing the comparative static analysis gives a general notion that, when relatively strong restrictions are imposed on one of the component of the decision model, the derived comparative static results are usually associated with relatively weak restrictions on the other components. The comparative static statements developed in the context of the ratio approach contain relatively weak restrictions on the set of decision makers such as all risk averse agents or all individuals with non-decreasing utility functions. Our study follows the ratio approach in order to generate interesting comparative static statements. That is, most sets of changes in randomness examined in this study are defined by imposing monotonicity restrictions on the likelihood ratio between a pair of pdf's or the probability ratio between a pair of CDF's. The purpose of this dissertation is to generalize and extend the existing comparative static results presented in sub-section 2.2.3. The generalization includes the cases where the same result is derived for a more general class of changes in distribution or for more general decision problems. Some trade-offs between restrictions used in a general comparative static statement are explored. In this study, we also examine the changes in the admissible set of shifts in distribution when the linearity (with respect to the choice and/or the random variable) restriction is added to the general decision model (1.1).

Chapter 3

RELATIONSHIPS AMONG SUBSETS OF FSD SHIFTS

This chapter gives the definitions for six special types of FSD shifts. Five definitions result from imposing restrictions on the likelihood or the probability ratio between an initial and a final random variables. That is, either the ratio of the pdf's, g and f, or the corresponding CDF's, G and F, is restricted. The other one is given by imposing restrictions on the pdf difference $g-f$ and the CDF difference $G-F$. Each definition characterizes a class of CDF changes for which a general comparative static statement is made in chapter 4. Before doing the comparative static analysis, this chapter is used to present some important relationships among those six types of FSD changes.

This and the next chapter consider only subsets of FSD shifts, that is, the final distribution F always dominates the initial distribution G in the sense of first degree (denoted by F FSD G). Both the CDF's G and F are assumed to have their points of increase in bounded intervals. For notational convenience, we assume that the support of $G(x)$ is a finite interval $[x_1, x_3]$ and the support of $F(x)$ is another finite interval $[x_2, x_4]$ where $x_1 \le x_2$ and $x_3 \le x_4$. Defining the supports in this way, we also assume that: for the continuous case, $G(x) > 0$ for all $x \in (x_1, \infty)$ and $F(x) > 0$ for all $x \in (x_2, \infty)$; and for the discrete case, $G(x) > 0$ for all $x \in [x_1, \infty)$ and $F(x) > 0$ for all $x \in [x_2, \infty)$.[19]

[19] Defining sets of CDF changes, the ratio of f to g or the ratio of F to G is often restricted to be non-decreasing in the overlapped interval $[x_2, x_3]$. Then this assumption allows both the ratios to be strictly positive for the interval $[x_2, x_3]$ in a discrete case, and for the interval $(x_2, x_3]$ in a continuous case, which are often used to simplify the proofs of the results developed in chapters 3 and 4.

3.1 Important Subsets of FSD Shifts

In this section, we give the definitions for six special types of FSD changes. They include two previously introduced ones: a monotone likelihood ratio (MLR) order from Landsberger and Meilijson (1990) and a monotone probability ratio (MPR) order from Eeckhoudt and Gollier (1995); and four newly defined ones: a left-sided monotone likelihood ratio (L-MLR) order, a right-side monotone likelihood ratio (R-MLR) order, a point-FSD shift, and a point-MPR shift. In sub-section 3.1.1, basic definitions of these concepts are presented. Graphical examples in sub-section 3.1.2 are given to help the reader understand the basic relations among the six orders.

3.1.1 Basic Definitions of Subsets of FSD Shifts

In order to obtain a general comparative static statement in the portfolio problem, Landsberger and Meilijson (1990) use the concept of an MLR which was previously defined in the statistical literature. The MLR order specifies a type of FSD shift in a distribution function, satisfying a monotonicity restriction on the ratio between a pair of pdf's f and g.[20]

Definition 3.1. $F(x)$ represents a monotone likelihood ratio FSD shift from $G(x)$ (denoted by F MLR G) if there exists a non-decreasing function $h:[x_2, x_3] \rightarrow [0, \infty)$ such that $f(x) = h(x)g(x)$ for all $x \in [x_2, x_3]$.

According to the condition in Definition 3.1, $g(x) \geq f(x)$ when $h \leq 1$ and $g(x) \leq f(x)$ when $h \geq 1$, and the function h non-decreasing implies that the pdf's f and g cross only

[20] Following our notation about the supports of CDF's F and G, Definition 3.1 is slightly modified from the original one in Landsberger and Meilijson's paper.

once. Thus it is easy to confirm that an MLR order is an FSD shift.[21] Let k be the point at which the crossing occurs, and we know that $k \in [x_2, x_3]$. Then an MLR shift specifies a probability transformation such that some probability mass on the points to the left of the point k is transferred to the points to right of the point k, requiring the ratio of the final pdf f to the initial pdf g to be non-decreasing.

The MLR order can be applied for cases of discrete or mixed random variables as well. In general, the condition that the function h is non-decreasing in $x \in [x_2, x_3]$ requires that the ratio of f to g is non-increasing for the interval. As we noted in footnote 19, if the ratio h is non-decreasing then h is positive for all $x \in (x_2, x_3]$ regardless of discrete or continuous case. Thus the MLR condition also requires that, for the interval (x_2, x_3),[22] if $g(x) = 0$ then $f(x) = 0$, and if $f(x) = 0$ then $g(x) = 0$. This implies a restriction that, for the interval (x_2, x_3), the outcome values which were possible under the initial distribution must be also possible under the final distribution, and the outcomes which were previously impossible must have zero probability of occurring under the final distribution.

According to Landsberger and Meilijson (1990), and Ormiston and Schlee (1993) who also use the MLR order in their comparative static analysis, there are special examples of shifts that already possess the MLR order. First, we do not exclude the case where $x_3 \leq x_2$, that is, all possible outcome values under the final CDF F is larger than those under the initial CDF G. For any shift of this type, it satisfies that F MLR G. Second, consider a random variable which has only two outcomes. A transfer of probability from the lower outcome to the higher one is an MLR shift, while an increasing in the magnitude of either one or both outcomes (still the previously higher outcome being larger than the new lower outcome) is not an MLR change. Third, if the

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[21] The MLR condition implies that $G(x) \geq F(x)$ for all $x \in [x_1, x_4]$, and this is the condition for an FSD shift (see Definition 2.1).

[22] Both the end points are dropped from the interval $[x_2, x_3]$, because it is possible that $h(x_2) = 0$ or $h(x_3) = \infty$ for continuous case.

random variable is continuous, then the MLR order gives complete orderings over the set of all the uniform densities with the same left (right) endpoint. In this instance, a pdf with a higher right (left) endpoint is MLR dominant over a pdf with a lower right (left) endpoint.

Lastly, if an initial pdf is log-concave, adding a positive constant to the random variable leads to an MLR change. To show this, assume that the final random variable y is obtained from the initial random variable x as $y \underset{d}{=} x + c$, where c is a positive constant. With the given pdf of x, $g(\cdot)$, the pdf of y, $f(\cdot)$ is given by $f(t) = g(t-c)$ for all $t \in R$. Then the condition that $g(t-c)/g(t)$ is non-decreasing in t is equivalent to

$$\frac{g'(t-c)}{g(t-c)} \geq \frac{g'(t)}{g(t)}$$

and this is guaranteed by the condition that $g'(x)/g(x) = d\ln g(x)/dx$ is non-increasing in x or equivalently that $\ln g(x)$ is weakly concave. Examples possessing this property include normal, uniform, and exponential distributions.[23]

A second specialized FSD change is defined in the analysis of Eeckhoudt and Gollier (1995). They impose a monotonicity restriction on the ratio between the two CDF's F and G, and refer to the change as a monotone probability ratio (MPR) change. Its definition follows:[24]

Definition 3.2. $F(x)$ represents a monotone probability ratio FSD shift from $G(x)$ (denoted by F MPR G) if there exists a non-decreasing function $H:[x_2, x_3] \to [0, 1]$ such that $F(x) = H(x)G(x)$ for all $x \in [x_2, x_3]$.

[23] See Karlin and Rubin (1956) for more examples.

[24] With the same reason as in footnote 20, Definition 3.2 is a slight modification of Definition 2 in Eeckhoudt and Gollier.

The fact that $H \in [0, 1]$ guarantees $G(x) \geq F(x)$ for all $x \in [x_1, x_4]$, and thus an MPR shift is an FSD shift. Compared with an MLR shift, an MPR shift imposes less stringent restrictions on the change in the random variable. The feature accruing to the MPR order is that it does not restrict the number of times of crossing between the pdf's *f* and *g*. This comes from replacing the monotonicity restriction on the likelihood ratio *h* with the monotonicity restriction on the probability ratio *H*. Thus, the assertion that the MLR order implies the MPR order can be made by showing that the condition that the likelihood ratio *h* is non-decreasing implies the condition that the probability ratio *H* is non-decreasing. While the proof for this statement is formally given in the next section, here we provide some cases showing that an MPR shift is not an MLR one.

One example satisfying the MPR but not the MLR condition is found in a shift in a two-outcome random variable. Contrary to the MLR case, both the cases of a transfer of probability from the lower outcome to the higher one and an increase in the magnitude of lower outcome are MPR changes. Another notable thing is that, for the interval (x_2, x_3), while *F* MLR *G* requires that if $g(x) = 0$ then $f(x) = 0$, and if $f(x) = 0$ then $g(x) = 0$, the condition *F* MPR *G* does not require that $f(x) = 0$ when $g(x) = 0$. Thus it allows that $f(x^0) > 0$ when $g(x^0) = 0$, for some $x^0 \in (x_2, x_3)$. This is a useful extension for discrete cases where the final pdf *f* often has some points with positive probability which were impossible under the initial pdf *g*. An example of this case is given in Figure 3.9 in sub-section 3.1.2.

As Eeckhoudt and Gollier note, there is a class of CDF's in which adding a positive constant to the random variable leads to an MPR shift. Similar to the MLR case, assume that the final random variable *y* is obtained from the initial random variable *x* as $y \underset{d}{=} x + c$ where *c* is a positive constant, and that the initial CDF *G* is log-concave. Then the final CDF *F*, given by $F(t) = G(t - c)$ for all $t \in R$, is an MPR shift from *G*. This is because the condition that $G(t - c) / G(t)$ is non-decreasing in *t* is equivalent to

$$\frac{G'(t-c)}{G(t-c)} \geq \frac{G'(t)}{G(t)}$$

and this is guaranteed by the condition that $G'(x)/G(x) = d\ln G(x)/dx$ is non-increasing in x or equivalently that $\ln G(x)$ is weakly concave. Bagnoli and Bergstrom (1991) show that the log-concavity of the CDF is less demanding than the log-concavity of the pdf. Thus, examples possessing this property include the normal, uniform, and exponential distributions which satisfy the log-concavity of the pdf. They also show that, if the pdf is non-increasing, adding a positive constant to the random variable is an MPR shift but does not lead to an MLR shift.

In addition to the MLR and the MPR orders, there are other interesting types of changes in distribution. An MLR shift from g to f restricts the two pdf's to cross only once, and requires the ratio of f to g to be non-decreasing for both side of the crossing point. Relaxing some of these restrictions, we introduce two more special types of FSD shifts. One, called a left-side monotone likelihood ratio (L-MLR) order, is obtained from relaxing the monotonicity requirement for points to the right-side of the crossing point. The other, called a right-side monotone likelihood ratio (R-MLR) order, drops the monotonicity requirement to the left-side of the crossing point. Each of these relaxed restrictions specifies a class of FSD shifts including all MLR shifts. Thus all the examples of shifts given in the discussion of the MLR case are also examples of the L-MLR or the R-MLR shifts. Formally, these changes are defined as:

Definition 3.3. $F(x)$ represents a left-side monotone likelihood ratio FSD shift from $G(x)$ (denoted by F L-MLR G) if there exist a point $k \in [x_2, x_3]$ and a non-decreasing function $h: [x_2, k) \to [0, 1]$ such that $f(x) = h(x)g(x)$ for all $x \in [x_2, k)$ and $g(x) \leq f(x)$ for all $x \in [k, x_3]$.[25]

[25] Only one crossing at a point $k \in [x_2, x_3]$ implies that $f(x) \leq g(x)$ for all $x \in [x_2, k)$ and $f(x) \geq g(x)$ for all $x \in (k, x_3]$. At the point k, it is usual that $f(k) = g(k)$, but if the pdf g or f is discontinuous at k, both the cases of $f(k) < g(k)$ and $f(k) > g(k)$ are possible, and thus Definition 3.3 includes the case where

Definition 3.4. $F(x)$ represents a right-side monotone likelihood ratio FSD shift from $G(x)$ (denoted by F R-MLR G) if there exist a point $k \in [x_2, x_3]$ and a non-decreasing function $h: (k, x_3] \rightarrow [1, \infty)$ such that $g(x) \geq f(x)$ for all $x \in [x_2, k]$ $f(x) = h(x)g(x)$ for all $x \in (k, x_3]$.

Both the L-MLR and the R-MLR conditions require that the pdf's f and g cross only once at the point k and that $g(x) \geq f(x)$ for all points to the left-side of k and $g(x) \leq f(x)$ for all points to the right-side of k. This implies that each of the two orders defines an FSD shift. Since an L-MLR requires the condition of monotone likelihood ratio only for the left-side of the point k, and an R-MLR requires the monotonicity only for the right-side of the point k, both the orders are more general than the MLR order. An L-MLR shift specifies a probability transformation such that a decreasing proportion of probability mass of the left-side of the point k is transferred to the right-side of the point k. An R-MLR shift gives a transformation that some probability mass taken from the left-side of k is transferred to the right-side of k, keeping the ratio of the final pdf to the initial pdf non-decreasing only for the right-side of k.

In order to give special examples of an L-MLR and an R-MLR shift, consider an initial random variable which has only two outcomes. If the lower (higher) outcome becomes larger than before, then it is the case of an L-MLR shift where $k = x_2$ (an R-MLR shift where $k = x_3$). Generally, given an initial pdf $g(x)$ with its support $[x_1, x_3]$, let's consider a conditional pdf g^l, given by $g^l(t) = g(t)/G(T)$ with its support $[x_1, T]$ where $T \leq x_3$. Then an MLR shift in g^l gives an L-MLR shift in the initial $g(x)$. If we define a conditional pdf $g^r(t) = g(t)/[1-G(T)]$ with its support $[T, x_3]$ where $x_1 \leq T$, an MLR shift in g^r gives an R-MLR shift in the initial $g(x)$.

$f(x) \leq g(x)$ for all $x \in [x_2, k]$ and $f(x) \geq g(x)$ for all $x \in (k, x_3]$. This discussion is applied for all the other definitions given in this section.

With the restrictions used, it is obvious that an MLR shift is also an L-MLR or an R-MLR shift, but not vice versa. Similarly, the restrictions for the MLR condition that, for the interval (x_2, x_3), if $g(x)=0$ then $f(x)=0$, and if $f(x)=0$ then $g(x)=0$, is somewhat relaxed. The condition *F* L-MLR *G* allows that $f(x^0)>0$ when $g(x^0)=0$, for some $x^0 \in [k, x_3)$, and the condition *F* R-MLR *G* allows that $g(x^0)>0$ when $f(x^0)=0$, for some $x^0 \in (x_2, k]$. Another basic relationship exists between the L-MLR and the MPR orders, although there is no direct relationship[26] between the R-MLR and the MPR orders. In section 3.2, we show that the monotonicity restriction on the likelihood ratio between a pair of pdf's for the MLR and the L-MLR shift implies the monotone probability ratio between the corresponding CDF's. Thus the MLR order implies the L-MLR order, which in turn implies the MPR order.

In what follows, two more types of FSD shifts are defined. As a special feature, these two CDF orders are specified by a given point and thus each set of CDF changes satisfying the orders is determined by the given point. One is called an 'FSD improvement with respect to a point' (point-FSD). This is defined by imposing restrictions on the CDF difference $G-F$ and the pdf difference $g-f$, and the restrictions depend on a given point. A point-FSD shift does not use any monotonicity restriction on the likelihood or the probability ratio between a pair of distributions. This is defined as:

Definition 3.5. Given a point $k \in [x_2, x_3]$, *F*(*x*) represents an FSD shift with respect to the point *k* from *G*(*x*) (denoted by *F k*-FSD *G*) if $G(x) \geq F(x)$ for all $x \in [x_1, x_4]$, and $g(x) \leq f(x)$ for all $x \in [k, x_3]$.

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[26] While *F* R-MLR *G* allows that $g(x^0)>0$ when $f(x^0)=0$, for some $x^0 \in (x_2, k]$, which is not allowed in *F* MPR *G*, an MPR shift allows multiple crossings between the two pdf's, which is not allowed in an R-MLR shift.

Since the condition includes the general FSD condition, a k-FSD shift is also an FSD shift. In addition to the condition F FSD G, a k-FSD change requires only the restriction that, given the point k, $G(x) - F(x)$ is non-increasing or $g(x) \leq f(x)$ for the interval $[k, x_3]$. For the left-side of the point k, the only restriction required is F FSD G. Thus, as the given point k becomes large, the restriction on the point-FSD becomes less severe. That is, for the given points k_1 and k_2 where $k_1 \leq k_2$, all k_1-FSD shifts are also k_2-FSD shifts, but not vice versa. For any FSD change from g to f, there exists a point $k \in [x_2, x_3]$ satisfying the restriction that $g(x) \leq f(x)$ for all $x \in [k, x_4]$. This implies that any FSD change can be classified as a k-FSD shift, for an appropriately chosen k.

Compared with the above four definitions of FSD shifts, a k-FSD does not restrict the number of times of crossing between the pdf's f and g, nor is there any monotonicity restriction on the likelihood or the probability ratio between the two distributions. While the set of CDF changes defined by the point-FSD depends on a given point k, it generally includes shifts that do not satisfy any of the four types of FSD shifts defined above.

Finally, another CDF order which depends on a given point is called an 'MPR with respect to a point' (point-MPR). This is obtained from weakening the conditions required for an MPR shift, and the degree of the weakened restriction depends on a given point. A point-MPR requires the monotonicity restriction on the CDF ratio only for the points to the right-side of the given point, and the change is defined as:

Definition 3.6. Given a point $k \in [x_2, x_3]$, $F(x)$ represents a monotone probability ratio FSD shift with respect to the point k from $G(x)$ (denoted by F k-MPR G) if, with the ratio function H given by H:$[x_2, x_3] \rightarrow [0, 1]$ such that $F(x) = H(x)G(x)$, the function H is non-decreasing in $x \in [k, x_3]$ and $H(x) \leq H(k)$ for all $x \in [x_2, k]$.

The MPR condition is relaxed for the interval $[x_2, k]$ in which a k-MPR does not require the CDF ratio function H to be non-decreasing. Similar to the k-FSD case, as the given point k becomes large, the restriction on the point-FSD becomes less severe. That is, for the given points k_1 and k_2 where $k_1 \le k_2$, all k_1-MPR shifts are included in the set of k_2-MPR shifts. As a special case, the set of MPR shifts is equal to the set of k-MPR shifts where $k = x_2$. A point-MPR is related to a point-FSD shift in such a way that, given a point k, a k-MPR shift can always be decomposed into two shifts, one MPR and the other k-FSD. This is formally given in section 3.2.

3.1.2 Graphical Examples

This sub-section provides graphical examples of the six concepts of FSD shifts defined above, and shows the basic relationships among them. In order to contrast the differences among those concepts, we consider the following nine examples of FSD shifts, seven continuous cases and two discrete cases. Examining each example, we can visualize the basic relationships discussed in section 3.1.1.

(1) Example 3.1: F MLR G in Figure 3.1. As Figure 3.1 shows, the pdf's g and f cross only once at a point k. The likelihood ratio function h given in Definition 3.1 can be defined by $h = f / g$ for the interval $[x_2, x_3]$. Since the function h is non-decreasing in the interval, the shift from G to F satisfies the MLR condition. To show that the ratio f / g is non-decreasing is equivalent to show that $f' \cdot g \ge f \cdot g'$. When $g \ne 0$, this is equivalent to the following condition:

$$\begin{cases} f' / g' \ge f / g, \text{ when } g' > 0 \\ f' \ge 0, \text{ when } g' = 0 \\ f' / g' \le f / g, \text{ when } g' < 0. \end{cases}$$

Graphically, this implies that, when $g \ge f$ and $g' > 0$ ($g' < 0$), the tangent lines l_g and l_f in Figure 3.1 do not meet to the left (must meet to the right) direction in the

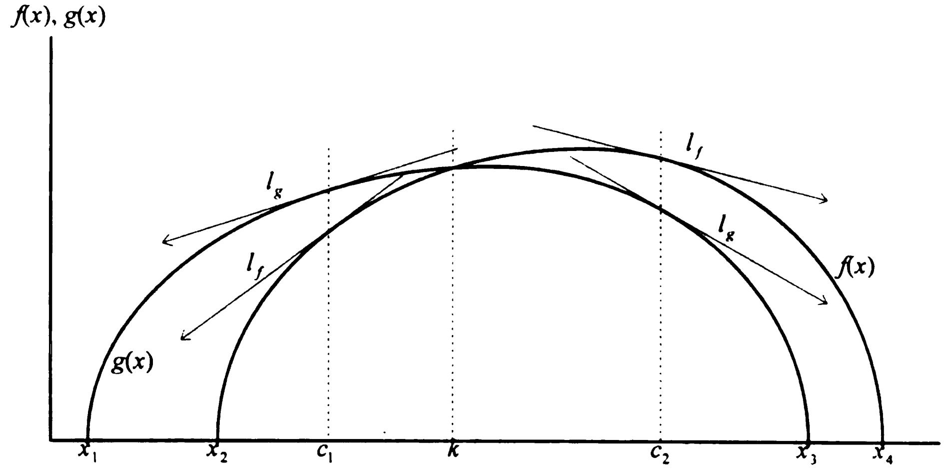

Figure 3.1. Example 3.1: *F* MLR *G*.

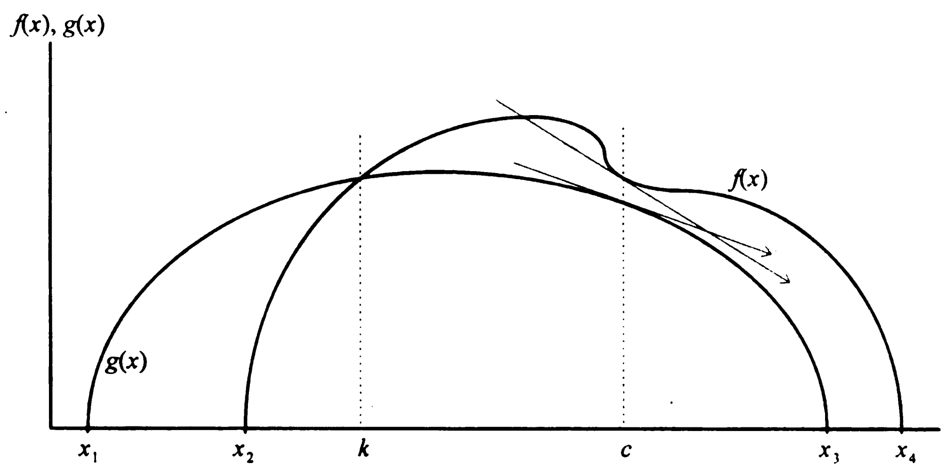

Figure 3.2. Example 3.2: *F* L-MLR *G*.

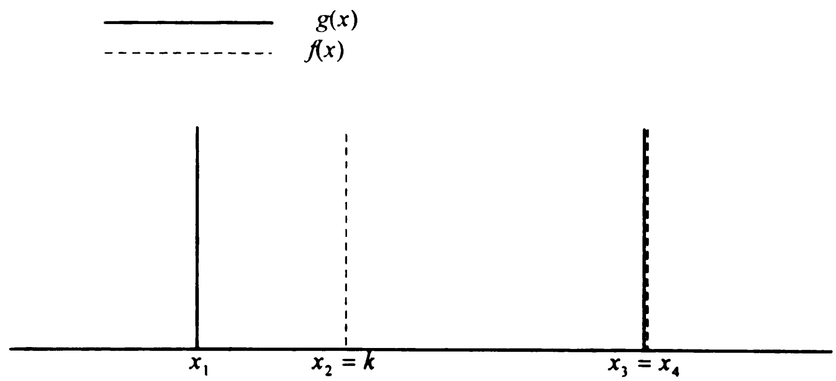

Figure 3.3. Example 3.3: *F* L-MLR *G* (a discrete case).

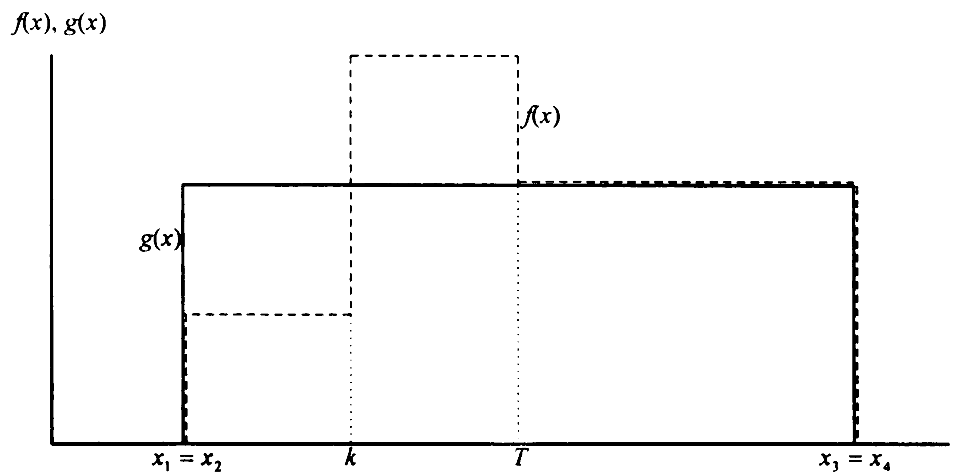

Figure 3.4. Example 3.4: *F* L-MLR *G*.

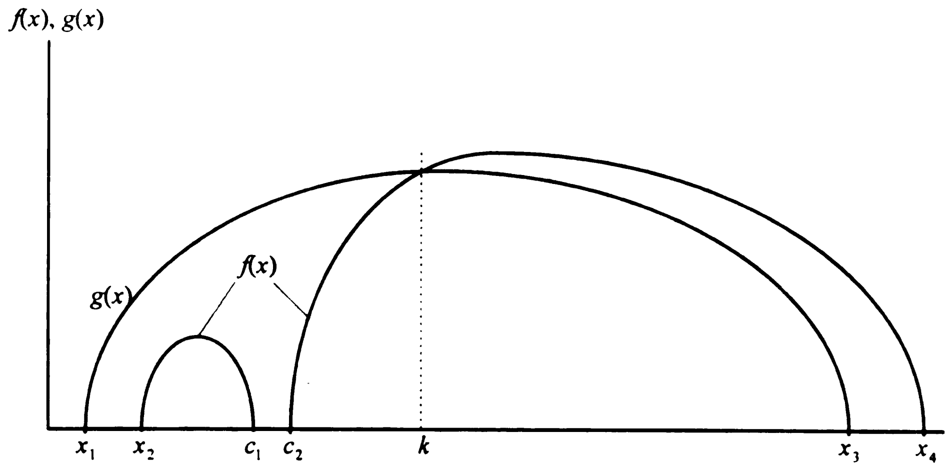

Figure 3.5. Example 3.5: *F* R-MLR *G*.

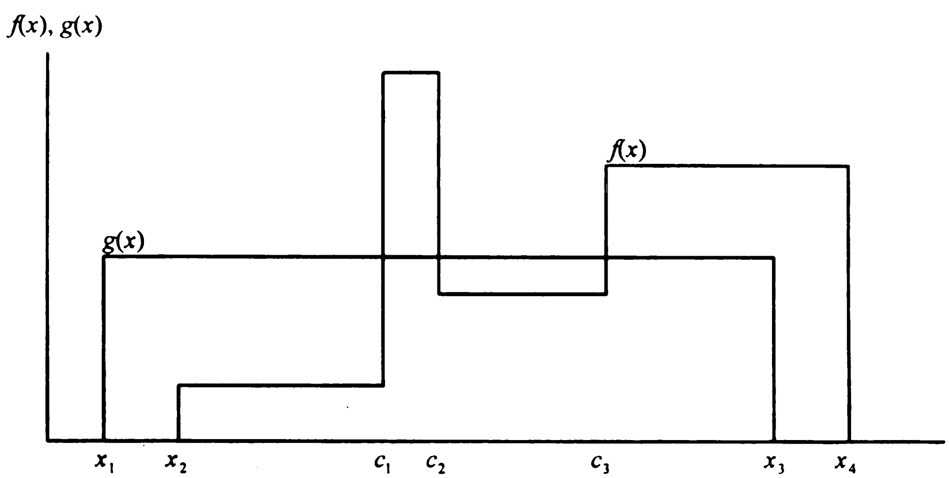

Figure 3.6. Example 3.6: *F* MPR *G*.

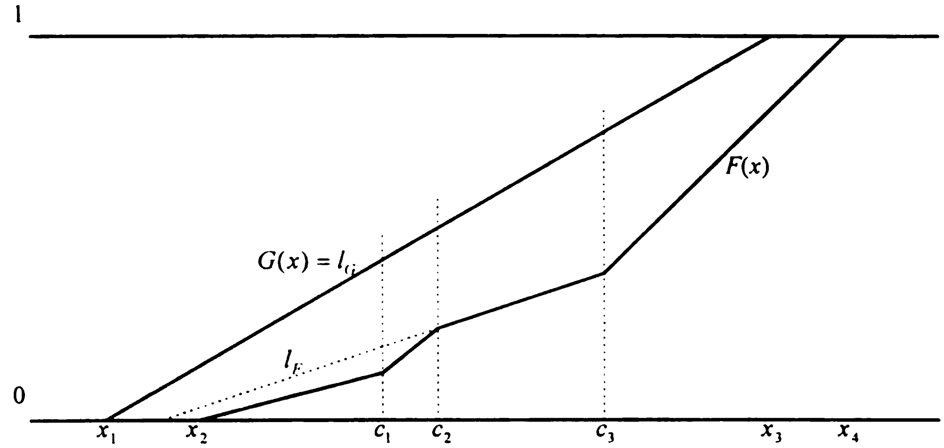

Figure 3.7. CDF representation of example 3.6.

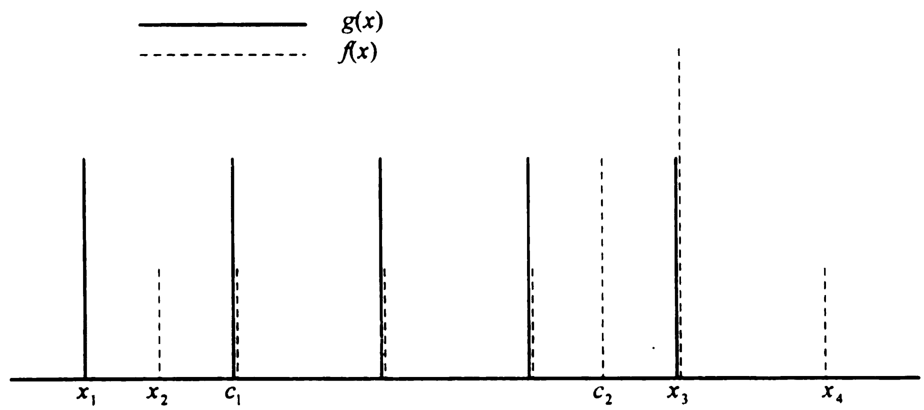

Figure 3.8. Example 3.7: F MPR G (a discrete case).

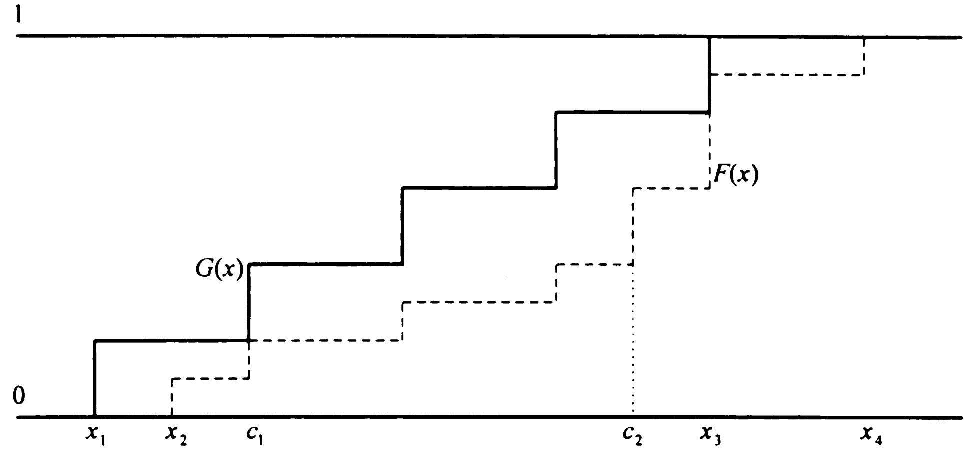

Figure 3.9. CDF representation of example 3.7.

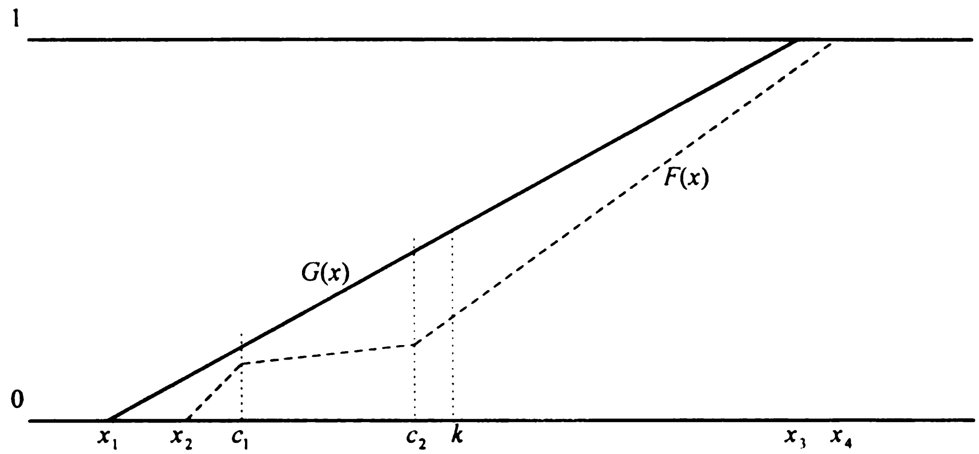

Figure 3.10. Example 3.8: F k-FSD G.

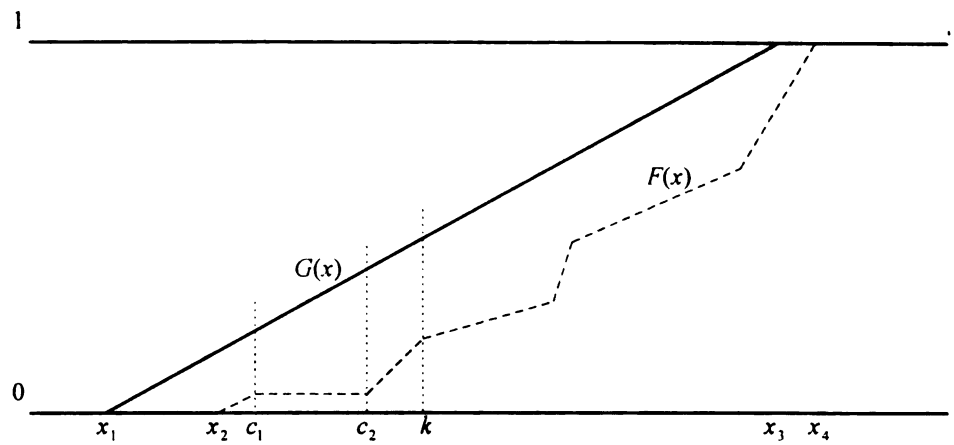

Figure 3.11. Example 3.9: F k-MPR G.

space above x axis, and when $g \le f$ and $g' < 0$ ($g' > 0$), the tangent lines l_g and l_f do not meet to the right (must meet to the left) direction in the space above x axis.

(2) Example 3.2: F L-MLR G in Figure 3.2. This is also an example of single crossing between the pdf's g and f. Let's define the likelihood ratio function h as $h = f / g$ for the interval $[x_2, k)$. Then the ratio is non-decreasing in the interval and thus the shift from G to F satisfies the L-MLR condition. The figure shows that the two tangent lines at point c ($g(c) \le f(c)$ and $g'(c) < 0$) meet in the right direction, which implies that, at the point c, the ratio f / g is decreasing. Thus the shift cannot be an MLR change.

(3) Example 3.3: F L-MLR G in Figure 3.3. Example 3.3 considers a shift in a two-outcome random variable. The initial random variable has only two outcomes, x_1 and x_3, with equal probability of occurring. Increasing the lower value x_1 to x_2 leads to an L-MLR shift. Let $x_2 = k$, then the domain of the non-decreasing ratio function h is empty and $f(x) \ge g(x)$ for all $x \in [k, x_3]$, which satisfy the L-MLR condition. The shift

cannot be an MLR shift because the ratio of *f* to *g* is $+\infty$ at x_2, and one at x_3, that is, the likelihood ratio function *h* should be decreasing at least one point in the interval $[x_2, x_3]$. In this case, if the higher outcome x_3 increases with the lower outcome unchanged then it is the case of an R-MLR shift.

(4) Example 3.4: *F* L-MLR *G* in Figure 3.4. This example shows that, given an initial pdf *g*(*x*) with its support $[x_1, x_3]$, an MLR type of shift occurred only on the left-side subinterval such as $[x_1, T]$ where $T \le x_3$ rives an L-MLR shift in the initial pdf *g*. Let's define a conditional pdf $g'(t) = g(t) / G(T)$ with its support $[x_1, T]$, then the shift shown in the figure can be understood as an MLR shift of the conditional pdf g'. Since the ratio *f* to *g* is constant and less than one for the interval $[x_2, k)$, and $f \ge g$ for all $x \in [k, x_3]$, it is obvious that *F* L-MLR *G*.

(5) Example 3.5: *F* R-MLR *G* in Figure 3.5. There is only one crossing between the pdf's *g* and *f*. Since the shift in this example satisfies that $f \le g$ for all $x \in [x_2, k)$, and that the ratio *f* to *g* is increasing and larger than one for the interval $[k, x_3]$, it is an R-MLR shift. The shift cannot be an MLR. As an evidence, the figure shows that, $f = 0$ and $g > 0$ for the interval $[c_1, c_2]$, which is not allowed for an MLR shift.

(6) Example 3.6: *F* MPR *G* in Figure 3.6 and 3.7. As Figure 3.6 shows, the pdf's *g* and *f* cross three times at the points c_1, c_2 and c_3, and thus the example cannot be an MLR, an L-MLR or an R-MLR shift. In Figure 3.7, we can see the CDF representation of example 3.6. The probability ratio function *H* given in Definition 3.2 can be defined by $H = F / G$ for the interval $[x_2, x_3]$. Since the function *H* is non-decreasing in the interval, the shift satisfies the MPR condition. In order to see this graphically, we can use the same logic used in the MLR example 3.1. *G* is always non-decreasing and greater than *F* in an FSD change. Thus, if the tangent lines $l_G (= G(x)$ in the given example) and

l_F for any point in the interval $[x_2, x_3]$ do not meet to the left direction in the space above *x* axis, the CDF ratio F / G is non-decreasing.

(7) Example 3.7: *F* MPR *G* in Figure 3.8 and 3.9. This is an example of a discrete case. To see that the shift is an MPR shift, consider the CDF representation of the shift given in Figure 3.9. Since the CDF ratio *F* to *G* is non-decreasing for the interval $[x_2, x_3]$, it is obvious that *F* MPR *G*. Figure 3.8 shows that the difference $g - f$ changes sign at three points x_2, c_1 and c_2. This implies that the pdf's *g* and *f* cross three times, and thus the shift cannot be an MLR, an L-MLR or an R-MLR change. Note that, at point c_2 which is in the interval (x_2, x_3), $f(c_2) > 0$ and $g(c_2) = 0$, which is not allowed for an MLR shift. Thus, this is an example showing that the MPR order is a useful extension for discrete cases where the final pdf *f* often has some points with positive probability which were impossible under the initial pdf *g*.

(8) Example 3.8: *F* *k*-FSD *G* in Figure 3.10. Given the point *k*, the fact that the difference $G - F$ is decreasing for the interval $[k, x_3]$ guarantees the shift to be a *k*-FSD shift. Since the ratio of *F* to *G* is decreasing for the interval $[c_1, c_2]$, this example cannot be an MPR shift. We noted that, as the given point *k* becomes large, the restriction on the point-FSD shift becomes less severe. This implies that, if the selected point is sufficiently large, all the above examples are also the examples of point-FSD shifts.

(9) Example 3.9: *F* *k*-MPR *G* in Figure 3.11. As the last example concerned in this sub-section, we present a point-MPR shift. Given the point *k*, the figure shows that the ratio F / G is non-decreasing for the interval $[k, x_3]$, and $F(x) / G(x) \le F(k) / G(k)$ for all $x \in [x_2, k]$. Thus example 3.9 satisfies the *k*-MPR conditions, but it cannot be an MPR shift because the CDF ratio is decreasing for the interval $[c_1, c_2]$. Note that the set of MPR shifts is a subset of the set of *k*-MPR shifts, for any given $k \in [x_2, x_3]$. The shift

also does not satisfy the k-FSD condition because the difference $G - F$ is increasing for some points in the interval $[k,\ x_3]$.

3.2 Some Properties among Subsets of FSD Shifts

This section points out several relationships among the concepts of FSD shifts, defined in section 3.1. They include important properties which play a key role for developing general comparative static statements in chapter 4. First, we discuss the basic relationships among the orders generated by these concepts.

Property 3.1. F MLR $G \Rightarrow F$ L-MLR $G \Rightarrow F$ MPR $G \Rightarrow F$ k-MPR G.

Proof. The claim that "if F MLR G then F L-MLR G" is obvious. It is because the restriction that the pdf ratio function $h(x)$ is non-decreasing in the interval $[x_2,\ x_3]$, which is required for an MLR shift, is relaxed for an L-MLR shift in such a way that, for the interval of x satisfying $h(x) \geq 1$, h need not be non-decreasing. The relationship that "if F MPR G then F k-MPR G" is also easily verified by the fact that the condition that the CDF ratio function $H(x)$ is non-decreasing for the interval $[x_2,\ x_3]$ implies $H(x) \leq H(k)$ for all $x \in [x_2,\ k]$.

Consider the second relationship "if F L-MLR G then F MPR G." Since $G > 0$ for all $x \in (x_1,\ \infty)$, we can define a function H for the interval $[x_2,\ x_3]$ as,[27]

$$H(x) = F(x)/G(x),\ \text{for all}\ x \in [x_2,\ x_3].$$

[27] If $x_1 = x_2$ and x is continuous, then $G(x_2) = 0$ and the proof is also true for this case with the function H defined as,

$$H = \begin{cases} 0, & \text{when}\ x = x_2 \\ F(x)/G(x), & \text{when}\ x \in (x_2,\ x_3]. \end{cases}$$

To prove the relationship, since $H \geq 0$, it is sufficient to show that the condition F L-MLR G implies that H is non-decreasing and less than or equal to one for the interval. It is clear that $H \leq 1$ for all $x \in [x_2, x_3]$ because an L-MLR shift in which the pdf's g and f have only a single crossing is an FSD shift. For the interval, the condition F / G non-decreasing is equivalent to:

$$f(x)G(x) - g(x)F(x) \geq 0, \text{ for all } x \in [x_2, x_3]. \qquad (3.1)$$

With the condition F L-MLR G, divide the interval $[x_2, x_3]$ into two sub-intervals, one such that $g \geq f$ and the other such that $g \leq f$. When $g \leq f$, the condition (3.1) is satisfied because $G \geq F$. When $g \geq f$, the L-MLR condition implies that

(i) if $g(x) = 0$, then $f(x) = 0$

(ii) if $g(x) \neq 0$, then $h(x) = f(x) / g(x)$.

Thus, for all the values of x such that $g = 0$, the condition (3.1) is satisfied. For case (ii), the condition (3.1) can be written as,

$$h(x)G(x) - F(x) \geq 0, \text{ for all } x \in [x_2, x_3] \text{ such that } g(x) \neq 0. \qquad (3.2)$$

Since $h(x_2)G(x_2) - F(x_2) \geq 0$ and the condition h non-decreasing in x implies that the LHS of (3.2) is non-decreasing, i.e.,

$$\partial[h(x)G(x) - F(x)] / \partial x = h'G + hg - f = h'G \geq 0,$$

the condition (3.2) is satisfied. Hence the condition (3.1) is satisfied for all x such that $g \geq f$. This completes the proof. Q.E.D.

Now we have formal relationships among the subsets of FSD changes. Let's define the following five subsets of FSD shifts, each representing a set of pairs of CDF's (F, G) such as:

$$\Omega_{ML} = \{(F,\ G) |\ F \text{ MLR } G\}$$
$$\Omega_{L-ML} = \{(F,\ G) |\ F \text{ L-MLR } G\}$$
$$\Omega_{R-ML} = \{(F,\ G) |\ F \text{ R-MLR } G\}$$
$$\Omega_{MP} = \{(F,\ G) |\ F \text{ MPR } G\}$$
$$\Omega_{k-MP} = \{(F,\ G) |\ F\ k\text{-MPR } G\}.$$

Then Property 3.1 implies,

$$\Omega_{ML} \subset \Omega_{L-ML} \subset \Omega_{MP} \subset \Omega_{k-MP},$$

and, since an R-MLR shift is less demanding than an MLR shift,

$$\Omega_{ML} \subset \Omega_{R-ML}.$$

Next we show that the four CDF orders, MLR, MPR, *k*-FSD, and *k*-MPR have the property of transitivity.

Property 3.2. If F_3 MLR F_2 and F_2 MLR F_1, then F_3 MLR F_1.

Proof. Let the support of F_i (and its corresponding pdf f_i) be a finite interval $[x_l^i,\ x_h^i]$, $i = 1,\ 2,\ 3$, where $x_l^1 \le x_l^2 \le x_l^3$ and $x_h^1 \le x_h^2 \le x_h^3$. Then the given two MLR shifts imply that there exist non-decreasing functions $h_1 : [x_l^2,\ x_h^1] \to [0,\ \infty)$ and $h_2 : [x_l^3,\ x_h^2] \to [0,\ \infty)$ such that $f_2 = h_1 f_1$ for $x \in [x_l^2,\ x_h^1]$ and $f_3 = h_2 f_2$ for $x \in [x_l^3,\ x_h^2]$, respectively. This implies that, for the interval $[x_l^3,\ x_h^1]$, $f_2 = h_1 f_1$ and $f_3 = h_2 f_2$, and thus $f_3 = h_1 h_2 f_1$. Since $h_1 \cdot h_2$ is also non-decreasing and non-negative for the interval $[x_l^3,\ x_h^1]$, and by Definition 3.1, F_3 MLR F_1. Q.E.D.

Property 3.3. If F_3 MPR F_2 and F_2 MPR F_1, then F_3 MPR F_1.

Proof. With the same supports defined in the proof of Property 3.2, the given two MPR shifts imply that there exist non-decreasing functions $H_1:[x_l^2, x_h^1] \to [0, 1]$ and $H_2:[x_l^3, x_h^2] \to [0, 1]$ such that $F_2 = H_1 F_1$ for $x \in [x_l^2, x_h^1]$ and $F_3 = H_2 F_2$ for $x \in [x_l^3, x_h^2]$, respectively. This implies that, for the interval $[x_l^3, x_h^1]$, $F_2 = H_1 F_1$ and $F_3 = H_2 F_2$, and thus $F_3 = H_1 H_2 F_1$. Since $H_1 \cdot H_2$ is also non-decreasing and between zero and one for all $x \in [x_l^3, x_h^1]$, and by Definition 3.2, F_3 MPR F_1. Q.E.D.

Property 3.4. Given a point *k*, if F_3 *k*-FSD F_2 and F_2 *k*-FSD F_1 then F_3 *k*-FSD F_1.

Proof. With the same supports given in the proof of Property 3.2, the given two *k*-FSD shifts imply that $F_1(x) - F_2(x)$ is non-negative for all $x \in R$ and non-increasing for all $x \in [k, x_h^1]$, and that $F_2(x) - F_3(x)$ is non-negative for all $x \in R$ and non-increasing for all $x \in [k, x_h^2]$. This implies that $F_1(x) - F_3(x)$ which is the sum of $F_1(x) - F_2(x)$ and $F_2(x) - F_3(x)$ is non-negative for all $x \in R$ and non-increasing for all $x \in [k, x_h^1]$. Thus, by Definition 3.5, F_3 *k*-FSD F_1. Q.E.D.

Property 3.5. Given a point *k*, if F_3 *k*-MPR F_2 and F_2 *k*-MPR F_1 then F_3 *k*-MPR F_1.

Proof. With the same notations used in the proof of Property 3.3, the given two *k*-MPR shifts imply that the ratio function $H_1:[x_l^2, x_h^1] \to [0, 1]$ is non-decreasing for all $x \in [k, x_h^1]$, and $H_1(x) \le H_1(k)$ for all $x \in [x_l^2, k]$, and that the function $H_2:[x_l^3, x_h^2] \to [0, 1]$ is non-decreasing for all $x \in [k, x_h^2]$ and $H_2(x) \le H_2(k)$ for all $x \in [x_l^3, k]$. With $F_3 = H_1 H_2 F_1$ for the interval $[x_l^3, x_h^1]$, this implies that $H_1 \cdot H_2$ is also non-decreasing and between zero and one for all $x \in [k, x_h^1]$, and $H_1(x) H_2(x) \le H_2(k) H_2(k)$ for all $x \in [x_l^3, k]$. Thus, by Definition 3.6, F_3 *k*-MPR F_1. Q.E.D.

From Property 3.2, 3.3, 3.4 and 3.5, we have a general statement that a series of MLR (MPR, *k*-FSD, *k*-MPR) shifts results in an MLR (MPR, *k*-FSD, *k*-MPR) shift. That is, a series of $n+1$ $(n \geq 1)$ number of MLR shifts such that F MLR G_n MLR G_{n-1} MLR $\cdots$ G_1 MLR G, implies that F MLR G, and the same is applied for the other three CDF orders. This can be stated in another way that any shift that can be decomposed into a series of MLR (MPR, *k*-FSD, *k*-MPR) shifts is also an MLR (MPR, *k*-FSD, *k*-MPR) shift. Thus, any shift that does not satisfy the MLR (MPR, *k*-FSD, *k*-MPR) order cannot be decomposed into a series of MLR (MPR, *k*-FSD, *k*-MPR) shifts.

Transitivity has important implications with regard to comparative static analysis under uncertainty. As we noted in chapter 2, one strategy to make a general comparative static statement is to find an admissible set of changes in CDF. Consider a CDF order which defines a particular type (set) of changes in CDF, and make assumptions so that the comparative static effect of any CDF change generated by the order is determinable for a given decision model considering all decision makers in a given set. Then the admissible set of changes in CDF for the comparative static result can be extended to include every shift that can be decomposed into a series of shifts which are all satisfying the given CDF order. If the given order has the property of transitivity, then the extended set is just the same as the set specified by the initial order and thus there is no gain in the comparative static result. The above four CDF orders, MLR, MPR, *k*-FSD, and *k*-MPR are of this type because they are transitive.

If the order is not transitive, the extended set of changes in CDF includes more changes that do not satisfy the given order. This implies that the order is only a sufficient but not necessary condition for the comparative static result and implies a possibility of extension giving an efficiency gain in the result. As a special case, consider that an arbitrary change in CDF generated by a CDF order can be decomposed into a series of shifts which are all satisfying another CDF order. If the latter order is sufficient for a general comparative static result, then all the shifts satisfying the former one are also admissible for the result.

Transitivity does not hold for the L-MLR order. It is easy to find an example that the sum of two single crossing L-MLR changes, from f_1 to f_2 and from f_2 to f_3, can involve multiple crossings between f_1 and f_3. In general, the relationships F_3 L-MLR F_2 and F_2 L-MLR F_1 do not necessarily mean F_3 L-MLR F_1. In this case, since an L-MLR shift is also an MPR shift, Property 3.3 implies F_3 MPR F_1. In other words, a CDF change that can be decomposed into a series of L-MLR shifts is not always an L-MLR shift but is an MPR shift. Considering the reverse case, there exists an important relationship between the L-MLR order and the MPR order. The next property shows that an MPR shift can always be decomposed into a series of L-MLR shifts.

Property 3.6. Any MPR shift can be decomposed into a series of L-MLR shifts, that is, if *F* MPR *G*, then there exists a series of CDF's $G_1, \cdots, G_n$ such that *F* L-MLR G_n L-MLR G_{n-1} L-MLR $\cdots$ L-MLR G_1 L-MLR *G*.

Proof. Given an arbitrary pair of pdf's *f* and *g* satisfying the condition *F* MPR *G*, let's define a function *h*: $R \rightarrow [0, \infty)$ as

$$h(x) = \begin{cases} f(x)/g(x), \text{ when } g(x) \neq 0 \\ \text{the defined last value of } f(x)/g(x), \text{ when } g(x) = f(x) = 0 \\ \infty, \text{ when } g(x) = 0 \text{ and } f(x) \neq 0. \end{cases}$$

Under the MPR condition, the function *h* is generally not monotone and thus there are points at which *h* changes from non-increasing to non-decreasing. Let's denote all such points as c_i, $i = 1, 2, \cdots, n$ where $n \geq 0$[28] and $c_n < c_{n-1} < \cdots < c_1$, and for each c_i define a CDF G_i and its corresponding pdf g_i as

[28] When $n = 0$, it just implies that the MPR shift is an MLR shift.

$$G_i(x) = \begin{cases} \lambda_i G(x), \text{ when } x < c_i \\ F(x), \text{ when } x \geq c_i \end{cases} \tag{3.3}$$

and

$$g_i(x) = \begin{cases} \lambda_i g(x), \text{ when } x < c_i \\ f(x), \text{ when } x \geq c_i \end{cases} \tag{3.4}$$

where $\lambda_i = F(c_i) / G(c_i)$, respectively. If we set $G = G_0$ and $F = G_{n+1}$ with $\lambda_0 = 1$, $\lambda_{n+1} = 0$, $c_0 = x_4$ and $c_{n+1} = x_2$, then from (3.4) we see that, for every $i = 1, 2, \cdots, n+1$, g_i is generally related to g_{i-1} as

$$g_i(x) = \begin{cases} (\lambda_i / \lambda_{i-1}) g_{i-1}(x), \text{ when } x < c_i \\ (h / \lambda_{i-1}) g_{i-1}(x)(= f(x)), \text{ when } c_i \leq x < c_{i-1} \\ g_{i-1}(x)(= f(x)), \text{ when } x \geq c_{i-1}. \end{cases} \tag{3.5}$$

With the given condition F MPR G, $\lambda_i / \lambda_{i-1} < 1$. From (3.3), the fact that $G_i(c_i) = F(c_i)$ and $G_i(x) \leq F(x)$ for all $x \in [x_1, c_i)$ implies that, at every point c_i, g_i meets f from the below. This implies $\lambda_i \leq h(c_i)$. For the interval $[c_i, c_{i-1})$, it is known that the function h is initially non-decreasing and changes to non-increasing through the end of the interval. Accordingly, (3.5) implies that there exist a point $k_i \in [c_i, c_{i-1})$ [29] and a non-decreasing function h_i:$[x_1, k_i) \rightarrow [0, 1]$ such that $g_i(x) = h_i(x) g_{i-1}(x)$ for all $x \in [x_1, k_i)$ and $g_i(x) \geq g_{i-1}(x)$ for all $x \in [k_i, x_4]$, where $h_i(x)$ is given as,

$$h_i(x) = \begin{cases} \lambda_i / \lambda_{i-1}, \text{ for } x \in [x_1, c_i) \\ h(x) / \lambda_{i-1}, \text{ for } x \in [c_i, k_i). \end{cases}$$

Hence, by Definition 3.3, G_i L-MLR G_{i-1} for all $i = 1, 2, \cdots, n+1$. Q.E.D.

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[29] If $g_i(c_i) \geq g_{i-1}(c_i)$, then c_i is the point k_i and from (3.3),

$$g_i(x) = (\lambda_i / \lambda_{i-1}) g_{i-1}(x), \text{ for } x \in [x_1, c_i)$$
$$g_i(x) \geq g_{i-1}(x), \text{ for } x \in [c_i, x_4]$$

which imply G_i L-MLR G_{i-1}. This case, as will be shown in Figure 3.12, leads to also G_{i+1} L-MLR G_{i-1}, and more generally, if $g_{i+j}(c_{i+j}) \geq g_{i-1}(c_{i+j})$ for every $j = 0, 1, \cdots, m$, then G_{i+m+1} L-MLR G_{i-1}.

To illustrate the result in Property 3.6, consider an example of an FSD shift given in Figure 3.12. As the figure shows, since the pdf's g and f cross seven times, the shift is neither an MLR nor an L-MLR shift. Following the method given in the proof of Property 3.6, four intermediate pdf's g_i's, for $i = 1, 2, 3, 4$, are defined. The shift from g to f is divided into five shifts, from g to g_1, $\cdots$, and from g_4 to f. Applying the conditions in Definition 3.3, it is easy to know that each shift is an L-MLR shift. Thus the example shows that a shift from g to f, satisfying the MPR[30] condition, can be decomposed into five L-MLR shifts such that:

$$F \text{ L-MLR } G_4 \text{ L-MLR } G_3 \text{ L-MLR } G_2 \text{ L-MLR } G_1 \text{ L-MLR } G.$$

As we noted, since $g_3(c_3) \geq g_2(c_3)$ and $g_4(c_4) \geq g_2(c_4)$, the shift can be decomposed into only three L-MLR shifts such that: F L-MLR G_2 L-MLR G_1 L-MLR G.

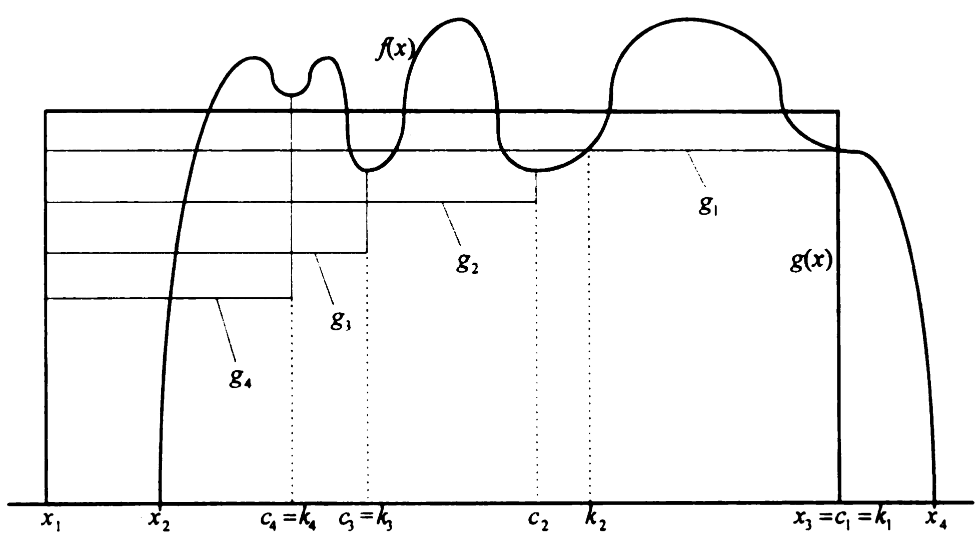

Figure 3.12. F MPR G that is the sum of five L-MLR shifts.

[30] Another implication from the example in Figure 3.11 is that the shift from g to f is guaranteed to be an MPR shift, without examining its CDF representation. It is because the five L-MLR shifts in the figure are also understood as the five MPR shifts (by Property 3.1) which are transitive (by Property 3.3).

Property 3.6 shows the case that a shift satisfying one CDF order can be decomposed into shifts satisfying another CDF order. In the next, as the last property examined in this section, we show another case that a shift satisfying one CDF order can be decomposed into shifts satisfying two other CDF orders. This interesting relationship exists among the orders, MPR, *k*-FSD and *k*-MPR. While they all have the property of transitivity, the following property shows that a *k*-MPR shift can always be understood as the sum of two shifts, one *k*-FSD and the other MPR.

Property 3.7. Any *k*-MPR shift can be decomposed into two FSD shifts one *k*-FSD and the other MPR, that is, if *F* *k*-MPR *G*, then there exists an intermediate CDF G_1 such that *F* *k*-FSD G_1 MPR *G*.

Proof. With a shift from *G* to *F* satisfying the condition *F* *k*-MPR *G* where $k \in [x_2, x_3]$, define an intermediate CDF G_1 as

$$G_1(x) = \begin{cases} \lambda_k G(x), \text{ when } x < k \\ F(x), \text{ when } x \geq k \end{cases} \tag{3.6}$$

where $\lambda_k = F(k)/G(k)$, respectively. Then we know that G_1 MPR *G* because the ratio of *G* to G_1 is non-decreasing for all $x \in [x_1, x_4]$, and that *F* *k*-FSD G_1 because the *k*-MPR condition implies that $F(x)/G(x) \leq \lambda_k$ for all $x \in [x_2, k]$ which, in turn, implies $G_1(x) \geq F(x)$ for all $x \in [x_2, k]$. Q.E.D.

A graphical example is given to illustrate the result in Property 3.7. The shift from *G* to *F* shown in Figure 3.13 satisfies the condition *F* *k*-MPR *G*, but does not satisfy the MPR or *k*-FSD condition given in Definition 3.2 and 3.5, respectively. The figure shows that the intermediate CDF G_1, given by using (3.6), satisfies the relationship that *F* *k*-FSD G_1 MPR *G*.

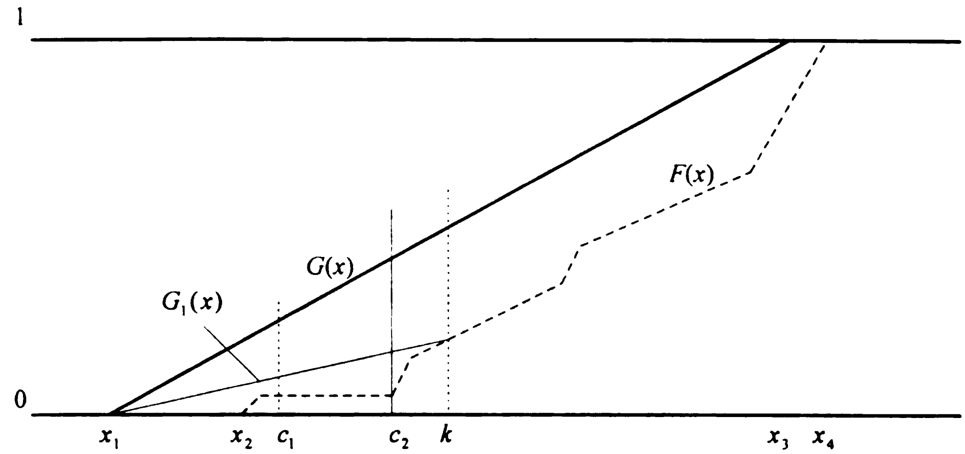

Figure 3.13. *F k*-MPR *G* that is the sum of a *k*-FSD and an MPR.

The last two properties can be considered as the examples showing that a CDF order can be decomposed into shifts that can be generated by other one or more CDF orders. These properties have important consequences with regard to providing general comparative static statements. By Property 3.6, the comparative static statement for the set of MPR shifts can be obtained from the one made for the L-MLR order. By Property 3.7, the comparative static statement for the set of *k*-MPR shifts can be obtained from the ones made for the MPR and the *k*-FSD orders. These are shown in chapter 4.

Chapter 4

COMPARATIVE STATIC RESULTS WITH SUBSETS OF FSD SHIFTS

This chapter provides general comparative static statements regarding the classes of FSD shifts discussed in chapter 3. Both the concepts of an MLR shift and an MPR shift are considered in Landsberger and Meilijson (1990) and Eeckhoudt and Gollier (1995), respectively. However their comparative static results are restricted to the special case of the general framework (1.1) such that the concerned payoff function is linear in both the choice and the random variables. Their results are generalized in this chapter by showing that the same comparative static results hold for the general decision model without any additional assumption required. The concept of an L-MLR shift and the discussion given in chapter 3 are useful for the generalization of Eeckhoudt and Gollier's result. We also discuss trade-offs between the structural restrictions on the given decision model and the restrictions on the set of changes in the pdf or CDF. In particular, when the concerned payoff function is restricted to be linear in the choice variable, it is possible to make an even more general comparative static statement. Four CDF orders, L-MLR, R-MLR, *k*-FSD, and *k*-MPR are used in these trade-offs.

Section 4.1 presents preliminary discussion about these two key papers and some related work, and the main results in this chapter are provided in section 4.2. In this chapter, let b_G and b_F be the optimal choices under the CDF's G and F, respectively, then we examine the sign of $b_F - b_G$ when a shift from an initial CDF G to a final CDF F occurs.

4.1 Literature Review and Preliminary Discussions

Meyer and Ormiston (1983) show that an arbitrary FSD improvement causes every risk averse agent to increase his level of choice variable if and only if its optimal level under uncertainty is independent of the value of the random exogenous variable. However, it is not an interesting problem because the restriction used on the structure of the concerned decision model is too severe. Using the simple case of a one-risky and one-safe asset portfolio problem, Fishburn and Porter (1976) also show that an FSD improvement in the return of the risky asset does not necessarily induce a risk averse investor to increase his demand for the risky asset. These papers generally imply that, only for subsets of the set of general FSD shifts, interesting comparative static statements regarding the choice made by an arbitrary risk averse decision maker can be made. Two recent papers by Landsberger and Meilijson (1990) and Eeckhoudt and Gollier (1995) introduce two special types of FSD shifts, an MLR shift and an MPR shift, respectively. These were described in chapter 3. This section provides discussion concerning comparative static results for these types of shifts, which are about to be generalized in the next section.

4.1.1 Monotone Likelihood Ratio FSD Shifts

Landsberger and Meilijson made a general comparative static statement for the set of MLR shifts. The decision model used in their analysis is the standard portfolio choice problem. In that problem, an individual is assumed to have one unit (normalized) of wealth which is to be allocated between one safe asset paying a zero rate of return and the other risky asset paying a random rate of return e. Then the individual wants to maximize his expected utility of final wealth W given by:

$$W = 1 + be \tag{4.1}$$

where b is the fraction of the unit wealth allocated to the risky asset. Landsberger and Meilijson's finding for this problem is that an MLR shift in the distribution of the random return on the risky asset induces all investors with non-decreasing utility functions to increase their demand for the risky asset. This result is very strong in the sense that the comparative static statement is for a very large and general class of decision makers.

A point concerning their analysis is worthy of mention. Most other comparative static analysis assumes that the first and the second-order conditions for the maximum problem (1.1) are satisfied to guarantee an interior bounded solution. Along with the continuity and the differentiability assumptions, these conditions are then used to prove comparative static results under uncertainty. That is, to prove that $b_F - b_G \geq 0$ for a particular type of change from G to F, or correspondingly from g to f, it is sufficient to show that,

$$Q(b_G) = \int_{x_1}^{x_4} u'[z(x,\ b_G)]z_b(x,\ b_G)[g(x) - f(x)]dx \leq 0. \tag{4.2}$$

As Dionne, Eeckhoudt and Gollier (1993) note, this approach can be quite restrictive and may exclude some interesting cases such as a corner or an unbounded solution. In particular, when decision makers are risk-neutral and the payoff function $z(x,\ b)$ is linear in the choice variable, the sufficient second-order condition is not met and the expression (4.2) is not as useful for the comparative static analysis. Landsberger and Meilijson's analysis relies on a very weak optimality condition of the decision problem. Their analysis requires only the condition that the optimal solution b_G is determinable for an initial CDF G, including an unbounded or a corner solution.[31]

The method used in the proof of their result is interesting because it includes the cases of a corner and an unbounded solution along with the interior solution case. Let the expected utility in the above portfolio problem be expressed as a function of b, the

[31] Furthermore it does not require the continuity and the differentiability assumptions on the utility functions and the payoff function W in (4.1).

fraction invested in the risky asset as: for an initial and a final CDF's G and F of the random return e,

$$EU_G(b) = E_G[u(1+be)]$$

and

$$EU_F(b) = E_F[u(1+be)]$$

respectively. To prove that $b_G \leq b_F$, it is sufficient to show that for any pair b_1 and b_2,

$$\text{if } b_1 \leq b_2 \text{ and } EU_G(b_1) \leq EU_G(b_2), \text{ then } EU_F(b_1) \leq EU_F(b_2). \tag{4.3}$$

This is because (4.3) implies that $EU_F(b) \leq EU_F(b_G)$ for every $b \leq b_G$ which in turn implies that the maximum of $EU_F(b)$ exists at a value of b larger than b_G and thus $b_G \leq b_F$. Landsberger and Meilijson show that an MLR shift from G to F satisfies the condition (4.3) for the portfolio problem.

However Landsberger and Meilijson's comparative static analysis is restricted to a specific problem, i.e., the standard portfolio choice model. This is a special case of the general framework (1.1) as the final wealth W in (4.1) is linear in both the choice and the random variables. Subsequently, Ormiston and Schlee (1993) provide a comparative static result based on the general decision model of the form:

$$\max_b E[u(x, b)] \tag{4.4}$$

where a utility function, u, depends on both the choice variable b and the random exogenous variable x. Since in this general framework the utility function may depend on two or more outcome values, the general framework (1.1) is a special case of (4.4). Treating the same subset of FSD changes, they examine a trade-off between assumptions on risk preferences and ordinal preferences under certainty. They show that an MLR shift induces all the individuals whose choices under certainty are non-decreasing in the exogenous variable to increase their choice level.

The restriction imposed on ordinal preferences under certainty implies, in the problems of the form (1.1), the set of decision makers with non-decreasing utility functions. In this sense Ormiston and Schlee's comparative static result is more general than Landsberger and Meilijson's result. However, the assumptions required for defining a unique solution of the problem (4.4) is quite restrictive. They also assume that the first and the second-order conditions are satisfied, and thus only the case of an interior solution is examined. When the decision models are of the form (1.1) and have linear payoffs such as the portfolio problem, the cases where a corner or an unbounded solution prevails and the risk neutral decision maker are excluded from the analysis.

So far, we have discussed the comparative static analysis for the MLR order. In section 4.2, we generalize Landsberger and Meilijson's result by showing that the same result is obtainable for the general decision model (1.1) for interior or non-interior solution cases – an efficiency gain compared to the analysis of Ormiston and Schlee. Furthermore, we provide another important generalization. Since the MLR order is sufficient (but not necessary) for Landsberger and Meilijson's result, we show that the same result is true for a more general class of FSD shifts than the MLR order.

4.1.2 Monotone Probability Ratio FSD Shifts

Eeckhoudt and Gollier's paper is an extension of the Landsberger and Meilijson contribution. They examine a trade-off between the restrictions on the set of decision makers and the set of changes in distribution. While the MLR order concerns the ratio between density functions, the MPR order is defined using the ratio between cumulative distribution functions. As we have seen in section 3.2, the MLR condition is a stronger restriction than the MPR condition. Eeckhoudt and Gollier show that an MPR shift in the distribution of the concerned random variable increases the choice level selected by an arbitrary risk-averse decision maker. Thus, they consider a more general class of changes in CDF, but their comparative static result is for a smaller set of decision makers.

The decision model used in their paper is restricted to the special case of (1.1), requiring the payoff to be linear in both the choice and the random variables. Their maximization is of the form:

$$\max_{b}\ E[u(z_0 + bx)] \tag{4.5}$$

where z_0 is some given scalar. Examples of such linear payoffs are the standard portfolio problem, a competitive firm with constant marginal costs facing uncertain output price, and the coinsurance problem. Assuming the program (4.5) is concave in the choice variable b, Eeckhoudt and Gollier prove their result by showing that the condition F MPR G implies (4.2). Thus, the first and the second-order conditions are assumed to be satisfied for the optimal solution of the problem (4.5). They consider only the interior solution case. For the proof, they used an important property (Lemma 3 in their paper) that:

$$\text{if } F \text{ MPR } G, \text{ then } E_F[x\colon x \le t] \ge E_G[x\colon x \le t] \text{ for all } t \in [x_1, x_4] \tag{4.6}$$

where $E_F[x|\ x \le t]$ and $E_G[x|\ x \le t]$ denote the conditional $(x \le t)$ expectation of the distributions F and G, respectively. Gollier (1995) shows that a modification of the latter condition in (4.6) has an important implication in the analysis of comparative statics for decision models with linear payoffs. He provide a necessary and sufficient condition on a shift in distribution, for signing its effect on the optimal choice made by an arbitrary risk averse decision maker. The condition is as follows:

$$\begin{gathered}\text{There exists a scalar } \gamma \in R \text{ such that, for all } t \in [x_1, x_4],\\ \gamma \int_{x_1}^{t} xf(x)dx = \gamma F(t)E_F(x;\ x \le t) \ge G(t)E_G(x;\ x \le t) = \int_{x_1}^{t} xg(x)dx.\end{gathered} \tag{4.7}$$

It means that, regarding the decision model (4.5), $b_F - b_G \geq 0$ if and only if a shift from G to F satisfies the condition (4.7). The latter condition in (4.6) is a special case where $\gamma = 1$ which is named as a linear stochastic dominance of factor 1 (LSD_1), and (4.6) means that if F MPR G then F LSD_1 G.

According to Eeckhoudt and Gollier, there are advantages of using the MPR order over the MLR order. Consider a shift from a random variable x (defined by a CDF G) to another random variable y (defined by a CDF F) such that $y = x - s$ where s is a positive constant. In the portfolio problem this can be viewed as a per-unit tax s on the returns of risky assets. With the presence of an income effect, the effect of such a shift on the optimal investment is ambiguous. However, as Ormiston and Schlee (1993) note, if the original density function g is log-concave, then the shift satisfies that G MLR F and thus, by Landsberger and Meilijson's result, imposing a per-unit tax reduces investment in the risky asset for all investors with non-decreasing utility functions. Similarly, Eeckhoudt and Gollier show that if the original distribution function G is log-concave, then the shift satisfies that G MPR F and thus imposing the per-unit tax reduces investment in the risky asset for all risk averse investors. It is shown, by Bagnoli and Bergstrom (1991), that the log-concavity restriction on a CDF is less stringent than the log-concavity of the corresponding pdf and all non-decreasing density functions are included in the cases of the log-concave CDF.

Another advantage of using the MPR order is related to the study by Eeckhoudt and Hansen (1980) who analyze the effect of a minimum and a maximum price regulations on the output level of a competitive firm. A minimum price regulation gives a change from an initial price distribution G to another F such that

$$F(x) = 0 \text{ for all } x < x_{\min} \text{ and } G(x) = F(x) \text{ for all } x \geq x_{\min} \tag{4.8}$$

and a maximum price regulation gives a change such that

$$F(x)=1 \text{ for all } x \geq x_{\max} \text{ and } G(x)=F(x) \text{ for all } x < x_{\max}, \tag{4.9}$$

where $x_{\min}$ and $x_{\max}$ denote the minimum and the maximum price given each price regulation, respectively. Eeckhoudt and Hansen found that, while the minimum price regulation increases the output level of a risk-averse competitive firm, the effect of the maximum price regulation on the output level of the firm is ambiguous. This lack of symmetry between the two regulations can be explained by Eeckhoudt and Gollier's result. The minimum price regulation (4.8) implies F MPR G but the maximum price regulation (4.9) does not imply G MPR F. However this explanation is restricted to the firms with constant marginal costs because Eeckhoudt and Gollier examine only the linear payoffs as shown in (4.5).

Lastly Eeckhoudt and Gollier suggest a possible extension. Regarding the decision model (4.5), the MPR order is only sufficient (but not necessary) to obtain their comparative static result. They provide a special type of changes in distribution which is not an MPR shift but gives the same comparative static result. Let's rewrite the payoff function shown in the decision model (4.5) as,

$$z(x,\ b)=z_0+b(x-c) \tag{4.10}$$

where z_0 and c are exogenous parameters. This linear form is used in Dionne, Eeckhoudt and Gollier (1993). The point c is the value of x satisfying $z_b(x,\ b)=0$, and it is understood as the rate of return on a safe asset in the portfolio problem, the constant marginal cost in the competitive firm model under price uncertainty, or the marginal cost of insurance rate in the coinsurance problem. Eeckhoudt and Gollier gives a type of change in CDF satisfying,

$$G(x)\geq F(x) \text{ for all } x \in [x_1,\ c] \text{ and } G(x)=F(x) \text{ for all } x \in [c,\ x_4], \tag{4.11}$$

and they informally say that the type of shift satisfies the result $b_F \geq b_G$.

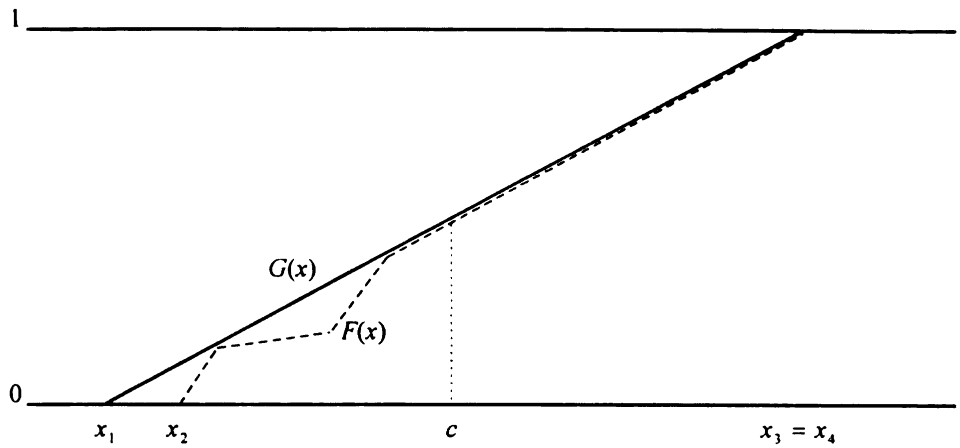

Figure 4.1. A special case of c-FSD shift.

Figure 4.1 gives an example of a shift satisfying the condition (4.11). It is easy to see that the example is neither an MPR shift nor an MLR shift. According to Definition 3.5, given a point c, the shift in the figure is a special case of c-FSD shift. A c-FSD shift requires the restriction that $G(x) - F(x)$ is non-increasing in the interval $[c, x_1]$, which is weaker than the restriction, used in the condition (4.11), that $G(x) = F(x)$ for all $x \in [c, x_4]$. Eeckhoudt and Gollier's notion is formalized in the next section, using the more general types of changes in CDF, c-FSD shifts and c-MPR shifts.

In section 4.2, the comparative static result in Eeckhoudt and Gollier's analysis is also generalized in two ways. One is obtained from generalizing their result to the general one-argument decision model (1.1). Proving this generalization, we follow the same technique used in Landsberger and Meilijson's analysis. Thus imposing a weak restriction on the optimality of the problem, our result improves the efficiency of Eeckhoudt and Gollier's result. The other generalization is given when the payoff is restricted to be linear in the choice variable b. This shows that the set of admissible changes in CDF is extended from the set of MPR shifts to the set including both the point-FSD shifts and the point-MPR shifts, given an appropriately chosen point.

4.2 Comparative Static Analysis: Generalized Results

Using the general one-argument decision model (1.1), this section provides general comparative static statements concerning various subsets of FSD shifts. First we examine the MLR shifts.

Theorem 4.1. For all decision makers with non-decreasing utility functions, $b_F \geq b_G$ if

(a) F MLR G

(b) $z_{bx} \geq 0$.

Proof. With the CDF's F and G given, each expected utility can be expressed as a function of the choice variable b,

$$EU_F(b) = \int_{x_2}^{x_4} u[z(x,b)]f(x)dx$$

and

$$EU_G(b) = \int_{x_1}^{x_3} u[z(x,b)]g(x)dx,$$

respectively. To prove the theorem it suffices[32] to show that, for a pair of points b_1 and b_2,

$$\text{if } b_1 \leq b_2 \text{ and } EU_G(b_1) \leq EU_G(b_2), \text{ then } EU_F(b_1) \leq EU_F(b_2). \tag{4.12}$$

This is because (4.12) implies that $\Delta_F = EU_F(b_G) - EU_F(b) \geq 0$ for every $b \leq b_G$ which in turn implies $b_F \geq b_G$.[33] Thus, assuming that $\Delta_G = EU_G(b_2) - EU_G(b_1) \geq 0$ where $b_1 < b_2$, we show that the following is non-negative,

[32] We follow the technique used in Landsberger and Meilijson (1990).

$$\Delta_F = \int_{x_2}^{x_4} A(x)f(x)dx \tag{4.13}$$

where $A(x) = u[z(x, b_2)] - u[z(x, b_1)]$. The difference $z(x, b_2) - z(x, b_1)$ is non-decreasing in x, since $z_{bx} \geq 0$ by assumption. The assumption $\Delta_G = \int_{x_1}^{x_3} A(x)g(x)dx \geq 0$ excludes the case where $z(x, b_2) - z(x, b_1) \leq 0$ for all $x \in [x_2, x_3]$ because it contradicts the assumption, that is, with the assumption $u' \geq 0$, the case implies $A \leq 0$ for all $x \in [x_1, x_3]$ and thus $\Delta_G \leq 0$. If $z(x, b_2) - z(x, b_1) \geq 0$ for all $x \in [x_2, x_3]$, then the assumption $u' \geq 0$ implies that $A \geq 0$ for all $x \in [x_2, x_4]$ and thus $\Delta_F \geq 0$. Since these cases are true for any FSD shift from G with its support $[x_1, x_3]$ to F with its support $[x_2, x_4]$ where $x_1 \leq x_2$ and $x_3 \leq x_4$,[34] they will not be considered any more in later analysis.

Now consider the case that, with b_1, b_2 and the payoff function z given, there exists a point $x^*(b_1, b_2, z) \in [x_2, x_3]$ such that the difference $z(x, b_2) - z(x, b_1)$ is non-positive for all $x \leq x^*$ and non-negative for all $x \geq x^*$. This implies that $A \leq 0$ for all $x \leq x^*$ and $A \geq 0$ for all $x \geq x^*$.[35] According to Definition 3.1, the condition $h: [x_2, x_3] \to [0, \infty)$ non-decreasing implies that there exists a point $k \in [x_2, x_3]$ such that $0 \leq h \leq 1$ for all $x \in [x_2, k)$ and $1 \leq h < \infty$ for all $x \in [k, x_3]$, or $0 \leq h \leq 1$ for all

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[33] If there is no such pair of points b_1 and b_2 satisfying $b_1 \leq b_2$ and $EU_G(b_1) \leq EU_G(b_2)$, this is the case of corner solution and the optimal choice is the lowest among the feasible set. In this case, $b_F \geq b_G$ for any shift from G to F.

[34] We do not exclude the case $x_3 \leq x_2$ where all the possible outcome values of x under F are higher than under G. For any shift of this type, it is easy to see that $\Delta_F \geq 0$.

[35] This proof will be useful for the later comparative static result. However a shorter proof for the case $x^*(b_1, b_2, z) \in [x_2, x_3]$ is suggested by Jack Meyer: According to Definition 3.1, if F MLR G then there exists a non-decreasing function $h: [x_2, x_3] \to [0, \infty)$ such that $f(x) = h(x)g(x)$ for all $x \in [x_2, x_3]$. Without loss of generality, let $h(x) = 0$ for the interval $x \in [x_1, x_2)$. Since $f(x) = 0$ for all $x \in [x_1, x_2)$ and $A \geq 0$ for all $x \geq x_3$, (4.13) can be rewritten as,

$$\Delta_F = \int_{x_1}^{x_4} A(x)f(x)dx \geq \int_{x_1}^{x_3} A(x)f(x)dx = \int_{x_1}^{x_3} A(x)h(x)g(x)dx.$$

Since $A(x)$ changes its sign from negative to positive at $x = x^*$ and $h(x)$ is non-decreasing,

$$\int_{x_1}^{x_3} A(x)h(x)g(x)dx \geq h(x^*) \int_{x_1}^{x_3} A(x)g(x)dx = h(x^*)\Delta_G \geq 0.$$

Thus, with the assumption $\Delta_G \geq 0$, we have $\Delta_F \geq 0$. Q.E.D.

$x \in [x_2, k]$ and $1 \le h < \infty$ for all $x \in (k, x_3]$.[36] At the point k, both cases $h(k) \ge 1$ and $h(k) \le 1$ are possible. Since the proof for each case follows the same procedure, only the case $h(k) \ge 1$ is considered in the below. Without loss of generality, the non-decreasing function h can be written as,

$$h(x) = \begin{cases} 1 - \delta(x), & \text{for } x \in [x_1, k) \\ 1 + \eta(x), & \text{for } x \in [k, x_3] \end{cases}$$

where $\delta = 1$ for $x \in [x_1, x_2)$, $0 \le \delta \le 1$ for $x \in [x_2, k)$ and $\eta \ge 0$.

Case (i): when $x^* \le k$. Since $f(x) = 0$ for $x \in [x_1, x_2)$, (4.13) can be rewritten as,

$$\Delta_F = \int_{x_1}^{x_4} A(x) f(x) dx$$

$$= \int_{x_1}^{k} A(x)[1 - \delta(x)] g(x) dx + \int_{k}^{x_3} A(x)[1 + \eta(x)] g(x) dx + \int_{x_3}^{x_4} A(x) f(x) dx. \quad (4.14)$$

Rearranging (4.14),

$$\Delta_F = \Delta_G + \int_{x_1}^{k} A(x)[-\delta(x)] g(x) dx + \int_{k}^{x_3} A(x) \eta(x) g(x) dx + \int_{x_3}^{x_4} A(x) f(x) dx.$$

Since $A(x)$ change its sign from negative to positive at $x = x^*$ and $\delta(x)$ is non-increasing,

$$\Delta_F \ge \Delta_G - \delta(x^*) \int_{x_1}^{k} A(x) g(x) dx + \int_{k}^{x_3} A(x) \eta(x) g(x) dx + \int_{x_3}^{x_4} A(x) f(x) dx.$$

Since $\Delta_G \ge 0$ by assumption, $0 \le \delta(x^*) \le 1$ and $\Delta_G \ge \int_{x_1}^{k} A(x) g(x) dx$, we have $\Delta_F \ge 0$.

Case (ii): when $k \le x^*$. For the first term in the RHS of (4.14), since $A(x) \le 0$ and $1 - \delta(x) \le 1$ for all $x \in [x_1, k)$,

---

[36] Note that, as two special cases, if $k = x_2$ or $k = x_3$, then the interval of x such that $0 \le h \le 1$ or $1 \le h < \infty$ is null, respectively, and we can follow the same procedure for the proof of the cases.

$$\int_{x_1}^{k} A(x)[1-\delta(x)]g(x)dx \geq [1+\eta(x^*)]\int_{x_1}^{k} A(x)g(x)dx$$

and for the second term, since $A(x)$ change its sign from negative to positive at $x = x^*$ and $\eta(x)$ is non-decreasing,

$$\int_{k}^{x_3} A(x)[1+\eta(x)]g(x)dx \geq [1+\eta(x^*)]\int_{k}^{x_3} A(x)g(x)dx.$$

Thus from (4.14), we have

$$\Delta_F \geq [1+\eta(x^*)]\int_{x_1}^{x_3} A(x)g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx = [1+\eta(x^*)]\Delta_G + \int_{x_3}^{x_4} A(x)f(x)dx$$

and hence the assumption $\Delta_G \geq 0$ implies that $\Delta_F \geq 0$. Q.E.D.

Theorem 4.1 is a direct extension of Landsberger and Meilijson's study, showing that the same comparative static result holds for decision models with non-linear payoff functions. Proving our result, we follow the same technique used in their study. Thus it improves the robustness of their result without any additional cost of assumptions.

In addition if we closely examine the proof of Theorem 4.1, we see some notable things. For the result in Theorem 4.1, if $x^* \leq k$ then only the condition L-MLR which is weaker than an MLR is sufficient, and if $x^* \geq k$ then only the condition R-MLR which is also weaker than an MLR is sufficient. These findings allow us to make another comparative static statement. In particular, if the concerned payoff is restricted to be linear in the choice variable, a further generalized result is possible.

Theorem 4.2. For all decision makers with non-decreasing utility functions, $b_F \geq b_G$ if

(a) $z(x, b)$ is linear in b

(b) F L-MLR G with $k \in [c, x_3]$ or F R-MLR G with $k \in [x_2, c]$, where c is the value of x satisfying $z_b(x) = 0$ and k is the point of crossing between the pdf's g and f

(c) $z_{bx} \geq 0$.

Proof. Since the payoff function z is restricted to be linear in the choice variable b, the point $x^*(b_1, b_2, z)$ defined in the proof of Theorem 4.1 is independent of the choice values b_1 and b_2. This implies that x^* is the value of x satisfying $z_b(x) = 0$ and thus $x^* = c$. From the proof of Theorem 4.1, it is known that: (i) if $k \geq x^*$ (the point of single crossing between the pdf's f and g is larger than the value of c), then only the condition F L-MLR G is sufficient for the result $b_F \geq b_G$, and (ii) if $k \leq x^*$ (the point of single crossing is smaller than the value of c), then only the condition F R-MLR G is sufficient for the result $b_F \geq b_G$. Q.E.D.

The trade-off used between Theorem 4.1 and 4.2 is that a larger class of changes in CDF is allowable at the cost of the linearity assumption on the payoff function. If the payoff function is linear in the choice variable, then, according as the point of crossing between the pair of pdf's is smaller or larger than the point c, the restriction of monotone likelihood ratio on either one of the left- or the right-side (of the point of crossing) is not necessary for the result. Consider the linear payoff function given in (4.10), in which $x^* = c$. The point c can be, as we noted in sub-section 4.1.2, understood as the rate of return on a safe asset in a portfolio problem, the constant marginal cost in a firm theory, or the marginal cost of insurance rate in an insurance model. Two examples of FSD shifts are shown in Figure 4.2 and 4.3, and both are satisfying the condition (b) in Theorem 4.2. The result in Theorem 4.2 implies that, for both the shifts in the figures, $b_F \geq b_G$ for all the individuals with non-decreasing utility functions. Therefore the MLR condition for the comparative static result in Landsberger and Meilijson's study is unduly

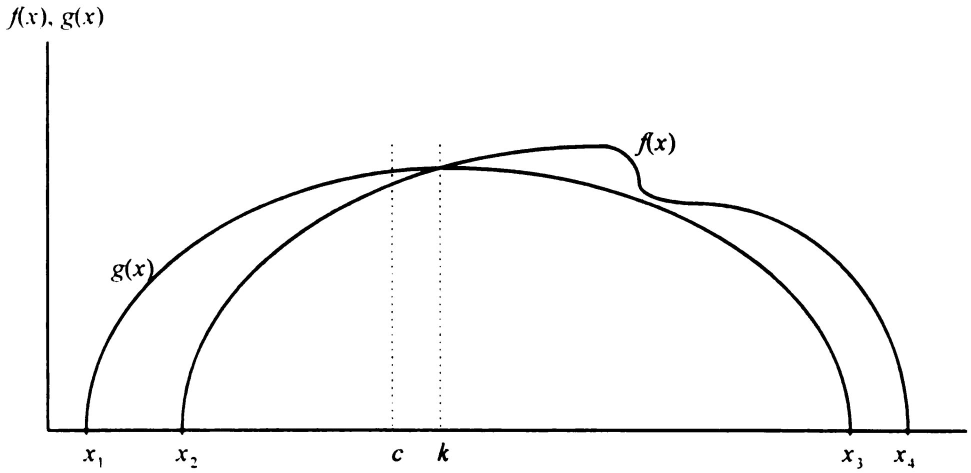

Figure 4.2. F L-MLR G with $k \in [c, x_3]$.

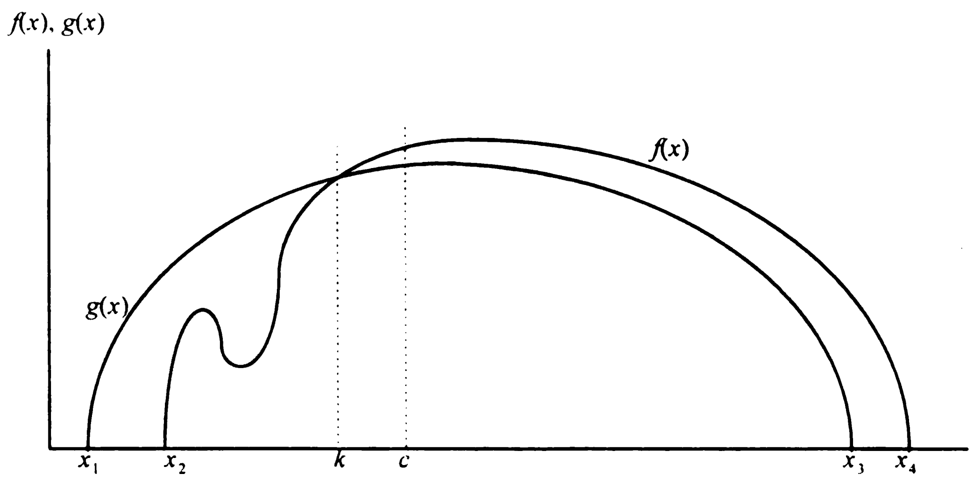

Figure 4.3. F R-MLR G with $k \in [x_2, c]$.

restrictive for the case of linear payoff and Theorem 4.2 improves the robustness of their result without additional cost of assumptions.

From this point forward, we replace the set of non-decreasing utility functions with the set of non-decreasing and concave utility functions; that is, the concerned decision makers are assumed to be risk-averse. In what follows, we show that adding this restriction on the risk preferences of decision maker allows the expansion of the addmissible changes in CDF for desirable comparative static statements. The next theorem concerns the set of L-MLR shifts which is larger than the set of MLR shifts and smaller than the set of MPR shifts.

Theorem 4.3. For all risk averse decision makers, $b_F \geq b_G$ if

(a) F L-MLR G

(b) $z_x \geq 0$ and $z_{bx} \geq 0$.

Proof. We follow the technique and notation used in the proof of Theorem 4.1, and as we noted, we consider only the case where $x_2 \leq x^* \leq x_3$. According to Definition 3.3, the condition F L-MLR G implies that there exists a point $k \in [x_2, x_3]$ such that $f(x) \leq g(x)$ for all $x \in [x_1, k)$ and $f(x) \geq g(x)$ for all $x \in [k, x_4]$. Since the proof for the case that $f(k) \leq g(k)$ and $f(x) \leq g(x)$ for all $x \in [x_1, k]$ and $f(x) \geq g(x)$ for all $x \in (k, x_4]$ follows the same procedure, we omit the case. Hence, with the condition F L-MLR G, $f(x)$ can be expressed as,

$$f(x) = \begin{cases} [1-\delta(x)]g(x), \text{ for } x \in [x_1, k) \\ [1+\eta(x)]g(x), \text{ for } x \in [k, x_3) \\ f(x), \text{ for } x \in [x_3, x_4] \end{cases}$$

where δ is non-increasing in x, $\delta = 1$ for $x \in [x_1,\ x_2)$ and $0 \le \delta \le 1$ for $x \in [x_2,\ k)$, and η is a non-negative function. Then, for the case where $x^* \le k$, the proof is exactly same as in the proof of Theorem 4.1. Consider the case where $k \le x^*$. Let's rewrite (4.13) as,

$$\Delta_F = \int_{x_1}^{x^*} A(x) f(x) dx + \int_{x^*}^{x_3} A(x) f(x) dx + \int_{x_3}^{x_4} A(x) f(x) dx. \tag{4.15}$$

Integrating the first term in RHS of (4.15) by parts, and by adding and subtracting,

$$\begin{aligned} \Delta_F &= A(x)F(x)\Big|_{x_1}^{x^*} - \int_{x_1}^{x^*} B(x)F(x)dx + \int_{x^*}^{x_3} A(x)f(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx \\ &= A(x)[F(x) - G(x)]\Big|_{x_1}^{x^*} + A(x)G(x)\Big|_{x_1}^{x^*} - \int_{x_1}^{x^*} B(x)[F(x) - G(x)]dx - \int_{x_1}^{x^*} B(x)G(x)dx \\ &\quad + \int_{x^*}^{x_3} A(x)[f(x) - g(x)]dx + \int_{x^*}^{x_3} A(x)g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx \end{aligned}$$

where $B = dA/dx = u'(z(x,\ b_2))z_x(x,\ b_2) - u'(z(x,\ b_1))z_x(x,\ b_1)$. By rearranging the above,

$$\begin{aligned} \Delta_F &= \Delta_G + A(x)[F(x) - G(x)]\Big|_{x_1}^{x^*} - \int_{x_1}^{x^*} B(x)[F(x) - G(x)]dx \\ &\quad + \int_{x^*}^{x_3} A(x)[f(x) - g(x)]dx + \int_{x_3}^{x_4} A(x)f(x)dx. \end{aligned} \tag{4.16}$$

Note that $z(x,\ b_2) \le z(x,\ b_1)$ when $x \le x^*$ and $z_x(x,\ b_2) \ge z_x(x,\ b_1)$ when $x \ge x_1$ by the assumption $z_{bx} \ge 0$. The assumptions $u' \ge 0$, $u'' \le 0$ and $z_x \ge 0$ imply that $B(x) \ge 0$ for $x \in [x_1,\ x^*]$. Since $A \le 0$ for all $x \le x^*$ and $A \ge 0$ for all $x \ge x^*$, and the L-MLR condition implies that $F(x) \le G(x)$ for all $x \in [x_1,\ x_4]$ and $f(x) \ge g(x)$ for all $x \in [k,\ x_3]$, the assumption $\Delta_G \ge 0$ implies that $\Delta_F \ge 0$. Q.E.D.

Compared with Theorem 4.1, the comparative static result in Theorem 4.3 includes a larger set of FSD changes and a smaller class of decision makers. Since we follow the same technique used in the proof of Theorem 4.1, the result in Theorem 4.3 is also established on the weak restriction on the optimality of the decision problem (1.1).

While the result itself is meaningful as a general comparative static statement, a further generalization is given in the next theorem. From Property 3.6 developed in chapter 3, it is known that an MPR shift can be decomposed into a series of L-MLR shifts. This allows a direct generalization of the comparative static result in Theorem 4.3, without any additional cost of assumptions.

Theorem 4.4. For all risk averse decision makers, $b_F \geq b_G$ if

(a) F MPR G

(b) $z_x \geq 0$ and $z_{bx} \geq 0$.

Proof. From Property 3.6 we know that an MPR shift can always be decomposed into a series of L-MLR shifts. That is, if F MPR G then there exists a series of CDF's $G_1, \cdots, G_n$ such that F L-MLR G_n L-MLR G_{n-1} L-MLR $\cdots$ L-MLR G_1 L-MLR G. With the result in Theorem 4.3, it completes the proof. Q.E.D.

Theorem 4.4 is an easy and important generalization of Theorem 4.3, and it is also an important generalization of the result in Eeckhoudt and Gollier's analysis. That is, the MPR condition is sufficient for the general decision model (1.1). Another efficiency gain is that our result is established on the weak restriction on the optimal solution of the decision problem. That is, the cases of a corner or an unbounded solution and the risk neutral decision makers are not excluded from our result. This completes the Eeckhoudt and Gollier's discussion about the lack of symmetry in the effects of the minimum and the maximum price regulations in Eeckhoudt and Hansen (1980).

Finally, we formalize the intuition given by Eeckhoudt and Gollier. As we discussed in sub-section 4.1.2, given a linear (in the choice variable b) payoff function, they informally say that it is possible to determine the effect of the type of changes in CDF defined in (4.11) on the optimal choice made by an arbitrary risk averse decision maker. It is also discussed that the shift is neither an MPR shift nor an MLR shift, but a special case of c-FSD shift. Hence it is meaningful to make a comparative static statement for the set of c-FSD shifts.

Adding a structural restriction that the concerned payoff is linear in the choice variable b, the extension of the admissible changes in CDF for the result in Theorem 4.4 is obtained from the point-MPR as well as the point-FSD shifts. Next theorem shows that, given a point c, the set of c-FSD shifts is first shown to be sufficient for a desirable comparative static statement and then the extension to the set of c-MPR shifts comes directly from Property 3.7.

Theorem 4.5. For all risk averse decision makers, $b_F \geq b_G$ if

(a) $z(x, b)$ is linear in b and $z_b(c, b) = 0$

(b) F c-FSD G or F c-MPR G

(c) $z_x \geq 0$ and $z_{bx} \geq 0$.

Proof. First consider the set of c-FSD shifts. We know that the linearity assumption implies that the point $x^*(b_1, b_2, z)$, defined in the proof of Theorem 4.1, is constant for all the pair of choice values b_1 and b_2. Now x^* is the value of x satisfying $z_b(x) = 0$ and thus $x^* = c$. The condition F c-FSD G implies that $G(x) - F(x) \geq 0$ for the interval $x \in [x_1, x^*]$ and $g(x) - f(x) \leq 0$ for the interval $x \in [x^*, x_3]$. This is the case that $k \leq x^*$ in the proof of Theorem 4.3, and from (4.16) we know that the condition is sufficient for $\Delta_F \geq 0$.

Now, given the proof for the set of c-FSD shifts, the proof for the set of c-MPR shift can be completed by Property 3.7 and Theorem 3.4. By Property 3.7, it is known that a c-MPR shift can always be decomposed into two FSD shifts, one c-FSD and the other MPR. Q.E.D.

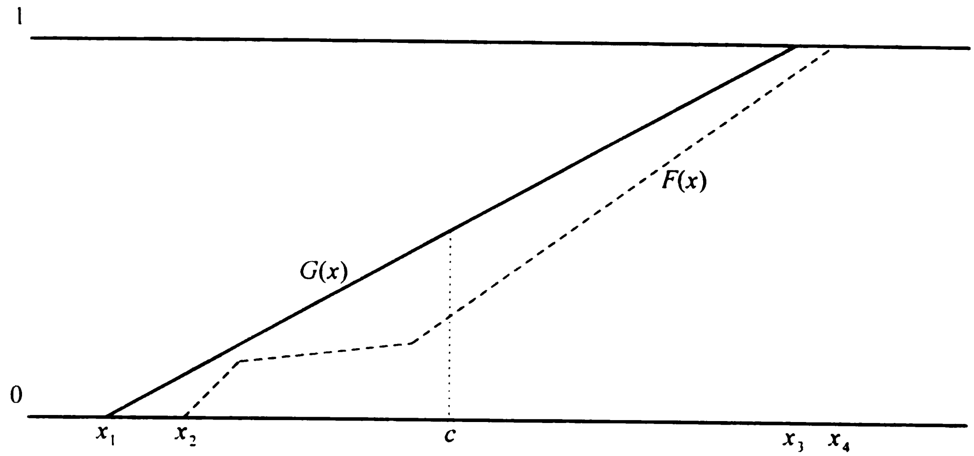

Figure 4.4. A *c*-FSD shift.

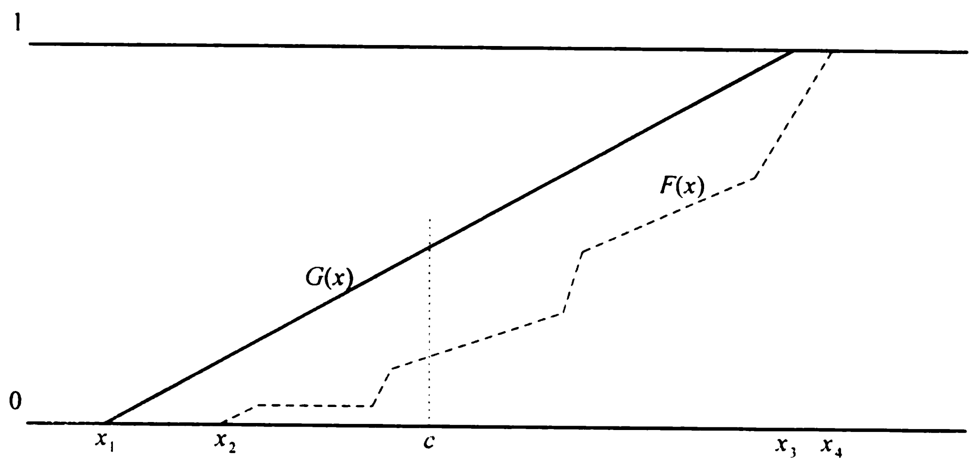

Figure 4.5. A *c*-MPR shift.

Adding the linearity restriction, the comparative static result given in Theorem 4.5 includes a more general set of changes in CDF than the set of MPR shifts contained in the result in Theorem 4.4. This is an example of a trade-off between the restrictions on the set of changes in CDF and the structure of the concerned decision model. This result is also a generalization of the comparative static result in Eeckhoudt and Gollier's analysis. Consider the two examples of FSD shifts shown in Figure 4.4 and 4.5. It is easy to see that each of the examples is neither an MLR shift nor an MPR shift. Given the point *c*, the shifts satisfy the conditions, *F* *c*-FSD *G* and *F* *c*-MPR *G*, respectively.
Now, regarding the decision models of the form (4.10), we know the result that, for both the shifts, $b_F \geq b_G$ for all risk-averse decision makers.

In this chapter, comparative static theorems are developed with regard to subsets of the set of general FSD shifts. All the results developed in this chapter are associated with relatively weak restrictions on the risk preference of decision makers, i.e., the set all risk averse agents or the set of all individuals with non-decreasing utility functions. In this sense, we refer all the subsets of CDF changes concerned in this chapter to "FSD shifts in the strong sense."

Chapter 5

RELATIONSHIPS AMONG SUBSETS OF R-S INCREASES IN RISK

This chapter gives the definitions for four special classes of R-S increases in risk. Each class is specified by adding additional restrictions to the 'integral conditions' which define the general concept of an R-S increase in risk (Definition 2.3). As in the case of defining subsets of FSD changes, monotonicity restrictions on the likelihood ratio (between a pair of pdf's) or on the probability ratio (between a fair of CDF's) are also important in defining subsets of R-S increases in risk. For each subset, a general comparative static statement is developed in chapter 6. Before doing this, this chapter examines several important relationships among these four types of R-S increases in risk.

In chapters 5 and 6, since only subsets of R-S increases in risk are considered, a final[37] distribution *G* is assumed to be always riskier than an initial distribution *F* in the R-S sense, that is, *G* is a mean-preserving spread from *F* (denoted by *G* MPS *F*). Assume that both the CDF's *F* and *G* have their points of increase in bounded intervals. For notational convenience, let the support of *F*(*x*) be a finite interval $[x_2, x_3]$ and the support of *G*(*x*) be another finite interval $[x_1, x_4]$ where $x_1 \le x_2 \le x_3 \le x_4$. Defining the supports in this way, we also assume that: for continuous case, $G(x) > 0$ for all $x \in (x_1, \infty)$, $1 - G(x) > 0$ for all $x \in (-\infty, x_4)$, $F(x) > 0$ for all $x \in (x_2, \infty)$, and $1 - F(x) > 0$ for all $x \in (-\infty, x_3)$; and for discrete case, $G(x) > 0$ for all $x \in [x_1, \infty)$,

[37] In order to keep the notation for *F* to be stochastically dominant over *G*, as in chapters 3 and 4, the CDF's *F* and *G* are regarded as an initial and a final distribution of the random variable *x*, respectively.

$1-G(x)>0$ for all $x \in(-\infty, x_4]$, $F(x)>0$ for all $x \in[x_2, \infty)$, and $1-F(x)>0$ for all $x \in(-\infty, x_3]$.[38]

5.1 Important Subsets of R-S Increases in Risk

This section gives the definitions for several subsets of R-S increases in risk. Including the concept of a 'relatively strong increase in risk' (RSIR) introduced by Black and Bulkley (1989), we add three concepts of R-S increases in risk. Being more general than an RSIR, they are a 'left-side relatively strong increase in risk' (L-RSIR), an 'extended strong increase in risk' (ESIR), and a 'left-side extended strong increase in risk' (L-ESIR). In sub-section 5.1.1, basic definitions of these four concepts are given. Graphical examples in sub-section 5.1.2 are presented to help the reader understand the basic relations among the concepts.

5.1.1 Basic Definitions of Subsets of R-S Increases in Risk

In order to make a desirable comparative static statement, Meyer and Ormiston (1985) introduce the concept of a 'strong increase in risk' (SIR). Defining a shift from *F*(*x*) to *G*(*x*), an SIR specifies a transfer of probability mass from points in the interval $[x_2, x_3]$ to points in the intervals $[x_1, x_2)$ and $(x_3, x_4]$. It is the case where the initially possible outcomes are less likely to occur, and the previously impossible outcomes which were outside the support of the initial CDF *F*(*x*) now have a non-zero probability mass. As we saw in chapter 2, the set of SIR shifts includes, as special cases, 'mean-preserving

[38] Similar to FSD cases, sets of R-S increases in risk used in this chapter are defined by imposing monotonicity restrictions on the ratio of *f* to *g* or the ratio of *F* to *G* in the overlapped interval $[x_2, x_3]$. Then this assumption allows both the ratios to be strictly positive in the interval $[x_2, x_3]$ for discrete case, and in the interval (x_2, x_3) for continuous case, which are often used to simplify the proofs of the results developed in chapters 5 and 6.

truncations' defined by Eeckhoudt and Hansen (1980) and 'global increases in risk' analyzed in Kraus (1979), Katz (1981). Recently, as a more general concept than an SIR, Black and Bulkley (1989) use an RSIR in their comparative static analysis. This is defined as:

Definition 5.1. *G*(*x*) represents a relatively strong increase in risk from *F*(*x*) (denoted by *G* RSIR *F*) if[39]

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]dx = 0$

(b) there exists a pair of points $k_1,\ k_2 \in [x_2,\ x_3]$ where $k_1 \leq k_2$, such that $f(x) \geq g(x)$ for all $x \in [k_1,\ k_2]$ and $f(x) \leq g(x)$ for all *x* in $[x_2,\ k_1)$ and $(k_2,\ x_3]$

(c) for all $x \in [x_2,\ k_1)$, there exists a non-decreasing function $h_1{:}[x_2,\ k_1) \rightarrow [0,\ 1]$ such that $f(x) = h_1(x)g(x)$

(d) for all $x \in (k_2,\ x_3]$, there exists a non-decreasing function $h_2{:}(k_2,\ x_3] \rightarrow [0,\ 1]$ such that $f(x) = h_2(x)g(x)$.[40]

Condition (a) implies the same mean between the two distributions. Condition (b) imposes the restriction that the two pdf's cross only twice and specifies a probability transformation from points within a centric interval $[k_1,\ k_2]$ to points lying outside the interval. These two conditions are sufficient for *G*(*x*) to represent an R-S increase in risk from *F*(*x*), satisfying the integral conditions in Definition 2.3. It is obvious that an SIR, given in Definition 2.8, is a special case of an RSIR where $k_1 = x_2$ and $k_2 = x_3$.

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[39] We defined an RSIR in somewhat different way from the original one. This is to include not only the continuous case but the discrete case where *g*(*x*) and/or *f*(*x*) is not always positive in the given supports.

[40] The two crossings at points $k_1, k_2 \in [x_2,\ x_3]$ imply that $f(x) \leq g(x)$ for all x in $[x_2,\ k_1)$ and $(k_2,\ x_3]$ and $f(x) \geq g(x)$ for all $x \in (k_1,\ k_2)$. At the points $k_i,\ i = 1,\ 2,$ it is usual that $f(k_i) = g(k_i)$, but if the pdf *g* or *f* is discontinuous at the points, it is possible that $f(k_i) < g(k_i)$ and $f(k_i) > g(k_i)$. Definition 5.1, thus, includes the cases where $f(x) \leq g(x)$ for all *x* in $[x_2,\ k_1]$ and/or $[k_2,\ x_3]$, and $f(x) \geq g(x)$ for all *x* in $[k_1,\ k_2)$ or $(k_1,\ k_2]$. This discussion is applied for all the other definitions given in this section.

Conditions (c) and (d) restrict the transformation from $f(x)$ to $g(x)$ to be such that, for both the intervals $[x_2, k_1)$ and $(k_2, x_3]$, the ratio of f to g is non-decreasing as x goes farther from the end points x_2 and x_3. In other words, considering an R-S decrease in risk from G to F, the condition G RSIR F implies that a non-increasing proportion of left-tail probability mass and a non-decreasing proportion of right-tail probability mass are transferred to the centric interval.

The RSIR order is closely related to the L-MLR and the R-MLR orders defined in section 3.1. Conditions (a) and (b) in Definition 5.1 imply that the CDF's F and G cross only once, and there exists a point $k \in [k_1, k_2]$ such that $F(x) \le G(x)$ when $x \le k$ and $F(x) \ge G(x)$ when $x \ge k$. Given an RSIR, it can be divided into two shifts: one for the left and the other for the right-side of the point k. In each side of k, there is only one crossing between the two pdf's, and for each end interval such that $f(x) \le g(x)$, the ratio between the pdf's should be monotone. Applying Definition 3.3 and 3.4, it is obvious that the shift for the left-side is understood as 'F dominant over G in the L-MLR sense,' and the shift for the right-side is understood, in the opposite direction, as 'G dominant over F in the R-MLR sense.' Thus an RSIR shift can be viewed as the sum of two shifts, keeping the mean same, an L-MLR dominated shift and an R-MLR dominant shift.

In the next definition of an R-S increase in risk, the restriction on the right-side of the point k is relaxed, and we name it a 'left-side relatively strong increase in risk' (L-RSIR).

Definition 5.2. $G(x)$ represents a left-side relatively strong increase in risk from $F(x)$ (denoted by G L-RSIR F) if

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]dx = 0$

(b) there exists a point $k \in [x_2, x_3]$ such that

$F(x) \le G(x)$ for all $x \in [x_2, k)$ and $F(x) \ge G(x)$ for all $x \in [k, x_3]$

(c) there exist a point $k' \in [x_2, k]$ and a non-decreasing function $h: [x_2, k') \rightarrow [0, 1]$ such that $f(x) = h(x)g(x)$ for all $x \in [x_2, k')$ and $g(x) \leq f(x)$ for all $x \in (k', k]$.

Condition (b) imposes the restriction that the two CDF's cross only once at a point *k*, and combined with condition (a), this obviously implies that an L-RSIR is an R-S increase in risk. Condition (c) implies that, for the left-side of the point *k*, an L-RSIR requires the same restriction used to define an RSIR. Now for the right-side of the point *k*, there is no restriction on the number of times of crossing between the pdf's *f* and *g*, nor is there any monotonicity restriction on the ratio between the pdf's. Only the required restriction on the right-side of *k* is that $F(x) \geq G(x)$. Therefore the set of RSIR shifts are a subset of the set of L-RSIR shifts.

Similar to the case of an RSIR shift, an L-RSIR shift can be divided into two shifts: one for the left and the other for the right-side of the point *k*. The shift for the left-side is understood as '*F* dominant over *G* in the L-MLR sense' and the shift for the right-side is understood, in the opposite direction, as '*G* dominant over *F* in the FSD sense.' Thus an L-RSIR shift can be viewed as the sum of two shifts, an L-MLR dominated shift and an FSD dominant shift.

To see a special case of L-RSIR shifts, consider a random variable which has only two outcome values. If the smaller outcome becomes riskier (mean-preserving) than before, then it is the case of an L-RSIR, while an increase in risk in the higher outcome does not lead to an L-RSIR. Generally, given an initial pdf $f(x)$ with its support $[x_2, x_3]$, let's consider a conditional pdf $f'(t) = f(t) / F(T)$ with its support $[x_2, T]$ where $T \leq x_3$. Then an RSIR shift in f' leads to an L-RSIR shift in the initial $f(x)$.

Compared with an RSIR, while an L-RSIR is defined by relaxing the restrictions imposed on the right-side of the point *k*, another less stringent type of R-S increase in risk is introduced and it is named as an 'extended strong increase in risk' (ESIR). Instead of using the monotonicity restrictions on the pdf ratio of *f* to *g*, an ESIR is defined by imposing monotonicity restrictions on the ratio of an initial CDF F to a final CDF G and on the ratio of $(1-F)$ to $(1-G)$ as:

Definition 5.3. $G(x)$ represents an extended strong increase in risk from $F(x)$ (denoted by G ESIR F) if

(a) $\int_{x_1}^{x_4}[G(x)-F(x)]dx=0$

(b) there exists a point $k\in[x_2,\ x_3]$ such that

$F(x)\le G(x)$ for all $x\in[x_1,\ k)$ and $F(x)\ge G(x)$ for all $x\in[k,\ x_4]$

(c) for all $x\in[x_2,\ k)$, there exists a non-decreasing function $H_1:[x_2,\ k)\to[0,\ 1]$ such that

$F(x)=H_1(x)G(x)$

(d) for all $x\in[k,\ x_3]$, there exists a non-increasing function $H_2:[k,\ x_3]\to[0,\ 1]$ such that $1-F(x)=H_2(x)[1-G(x)]$.

The same conditions (a) and (b) in Definition 5.2 are used and they guarantee that G represents an R-S increase in risk from F. An ESIR requires the restriction that the two CDF's cross only once at a point k, but does not restrict the number of times of crossing between the pdf's. There is no direct relationship between an ESIR and an L-RSIR. For the left-side of the point k, while an ESIR has no restriction on the number of times of crossing between the pdf's, an L-RSIR requires only one crossing between them. For the right-side of k, while an ESIR requires that the ratio of $(1-F)$ to $(1-G)$ should be monotone, an L-RSIR requires only the condition that $F(x)\ge G(x)$. Thus, for the interval $[x_2,\ k)$, an ESIR is less demanding than an L-RSIR, and vice versa for the interval $[k,\ x_3]$.

However an RSIR implies an ESIR. In section 5.2, we provide a formal proof for this assertion, showing that the monotonicity restrictions imposed on the ratio of f to g imply that the ratio of F to G in the interval $[x_1,\ k)$ and the ratio of $(1-F)$ to $(1-G)$ in the interval $[k,\ x_4]$ are monotone. The steps used in the proof is similar to the ones used in the proof of the assertion that an L-MLR implies an MPR (Property 3.1). This implies that, compared with an RSIR, an ESIR uses less stringent restrictions on the changes in CDF. It does not use the restrictions on both the number of times of crossing between the pair of pdf's f and g and the monotonicity of the likelihood ratio between the pdf's.

The ESIR order is closely related to the MPR order defined in section 3.1. Given a shift from F to G satisfying G ESIR F, it can be divided into two shifts: one for the left and the other for the right-side of the point k. Then, applying Definition 3.2, the shift for the left-side can be understood as 'F dominant over G in the MPR sense.' The shift for the right-side is a special FSD shift. If we define a new random variable y as $y = -x$ and the initial and the final pdf's of y, $\hat{f}(y)$ and $\hat{g}(y)$, are given as $f(x) = f(-y) = \hat{f}(y)$ and $g(x) = g(-y) = \hat{g}(y)$, then the shift for the right-side is understood as $\hat{F}$ MPR $\hat{G}$. Thus an ESIR shift can be viewed as the sum of two shifts, one MPR dominated shift and the other specialized FSD dominant shift.

As the last specialized concept of an R-S increase in risk presented in this section, the next one is obtained from further relaxing the restrictions imposed on either an ESIR or an L-RSIR. Now the condition (d) used in Definition 5.3 is dropped. Thus in the following definition, the right-side monotonicity restriction on the CDF ratio is relaxed and we name it a 'left-side extended strong increase in risk' (L-ESIR).

Definition 5.4. $G(x)$ represents a left-side extended strong increase in risk from $F(x)$ (denoted by G L-ESIR F) if

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]dx = 0$

(b) there exist a point $k \in [x_2,\ x_3]$ and a non-decreasing function H:$[x_2,\ k) \rightarrow [0,\ 1]$ such that $F(x) = H(x)G(x)$ for all $x \in [x_2,\ k)$ and $G(x) \leq F(x)$ for all $x \in [k,\ x_3]$.

An L-ESIR still has the restriction that the two CDF's cross only once at a point *k*, which is implied by condition (b). Combined with condition (a), this guarantees that an L-ESIR is an R-S increase in risk. Since an L-ESIR requires only the left-side monotonicity of the CDF ratio, it is more general concept than an ESIR. The L-ESIR order is also more general than the L-RSIR order because, with the same restrictions on the points to the right-side of the point *k*, an L-ESIR is less demanding than an L-RSIR for the points to the left-side of *k*. This can be proved by showing that the monotonicity of the pdf ratio implies the monotonicity of the CDF ratio, which is given in the next section. Similar to the above three cases of R-S increases in risk, an L-ESIR shift from *F* to *G* can also be understood as the sum of two shifts: one for the left-side of the point *k* is *F* MPR *G*, and the other for the right-side of the point *k* is, in the opposite direction, *G* FSD *F*.

As in the case of an L-RSIR, a special L-ESIR is as follows. Given an initial pdf $f(x)$ with its support $[x_2, x_3]$, let's consider a conditional pdf $f'(t) = f(t) / F(T)$ with its support $[x_2, T]$ where $T \le x_3$. Then any ESIR shift in f' results in an L-ESIR shift in the initial $f(x)$.

5.1.2 Graphical Examples

Each of the above four concepts of R-S increases in risk specifies a particular subset of the set of all R-S increases in risk. They all have a common restriction that the CDF's *F* and *G* cross only once, but each type of shifts is defined by imposing additional restrictions on the changes in distribution functions. In this sub-section, using graphical examples including seven continuous cases and two discrete cases, we show the basic relationships among those four types of R-S increases in risk discussed in sub-section 5.1.1. In what follows, we examine each example to see the differences among the restrictions used to define each type of R-S increase in risk. Thus every example is assumed to preserve the mean same before and after the given shift in pdf or CDF. First two examples are special cases of RSIR shifts: example 5.1 shows a mean-preserving

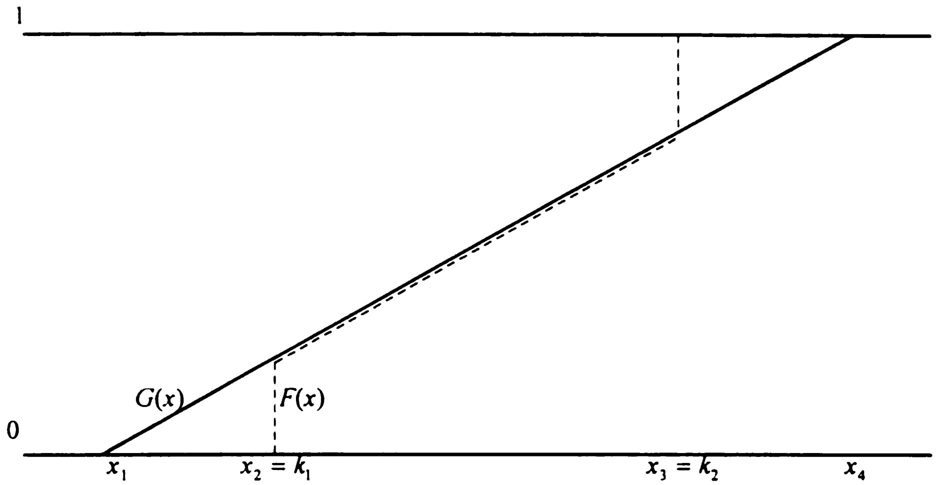

Figure 5.1. Example 5.1: Mean-preserving truncation from G to F.

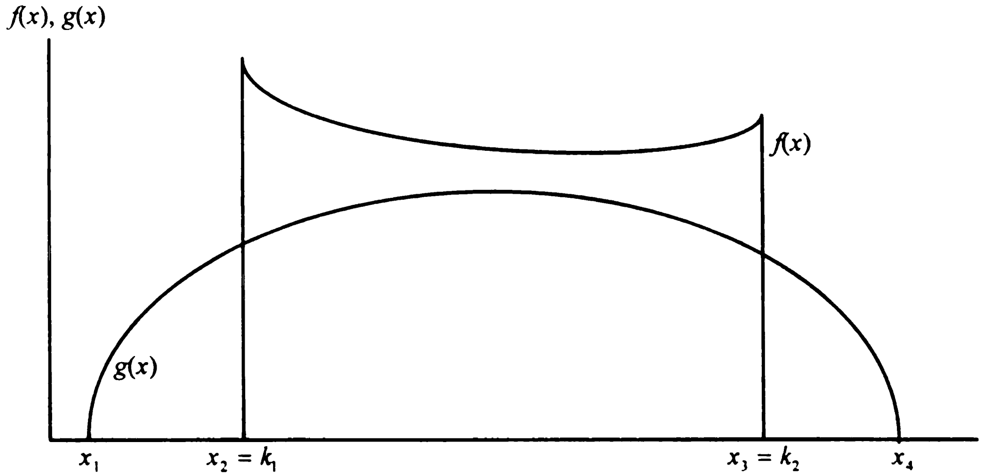

Figure 5.2. Example 5.2: G SIR F.

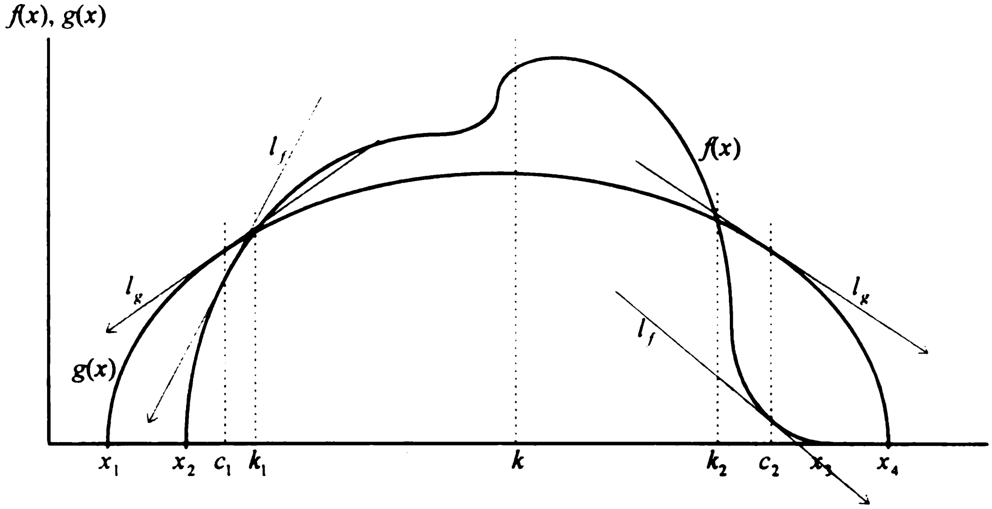

Figure 5.3. Example 5.3: *G* RSIR *F*.

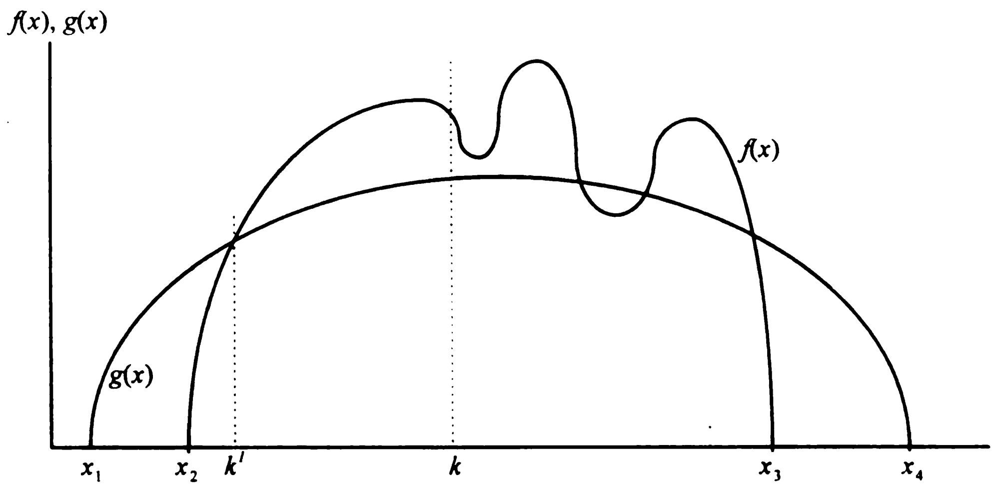

Figure 5.4. Example 5.4: *G* L-RSIR *F*.

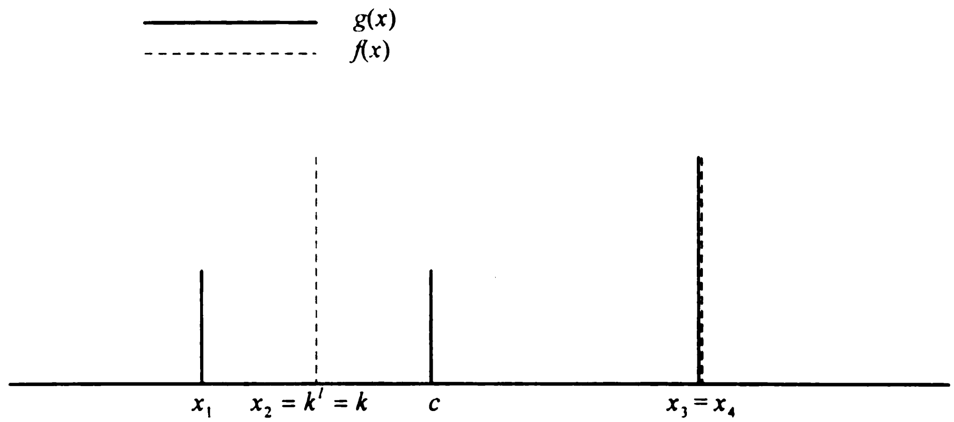

Figure 5.5. Example 5.5: G L-RSIR F (a discrete case).

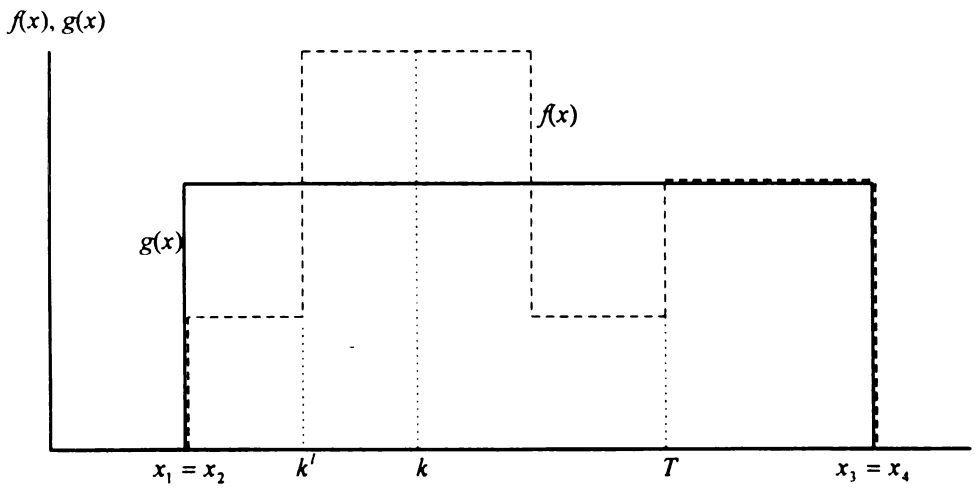

Figure 5.6. Example 5.6: G L-RSIR F.

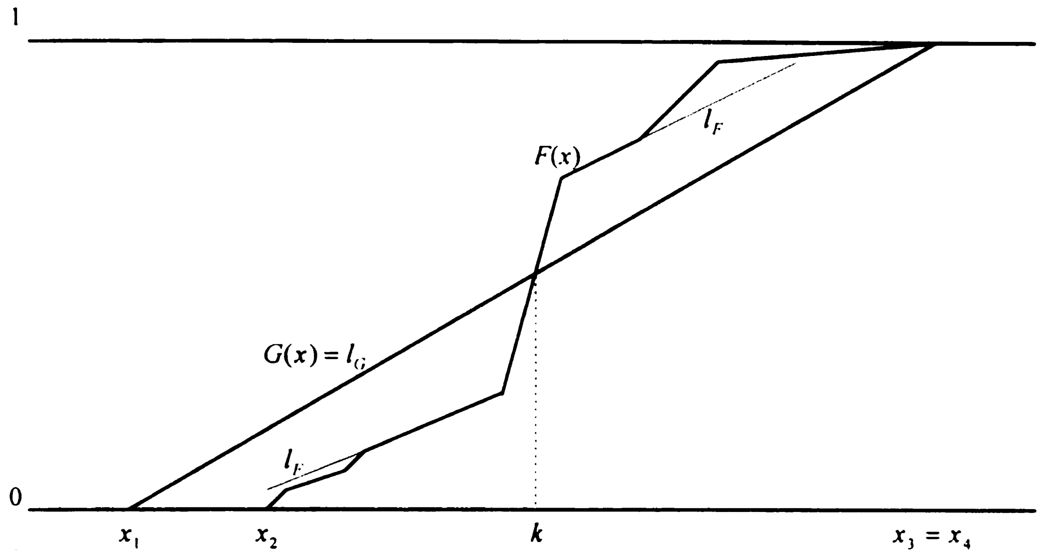

Figure 5.7. Example 5.7: G ESIR F.

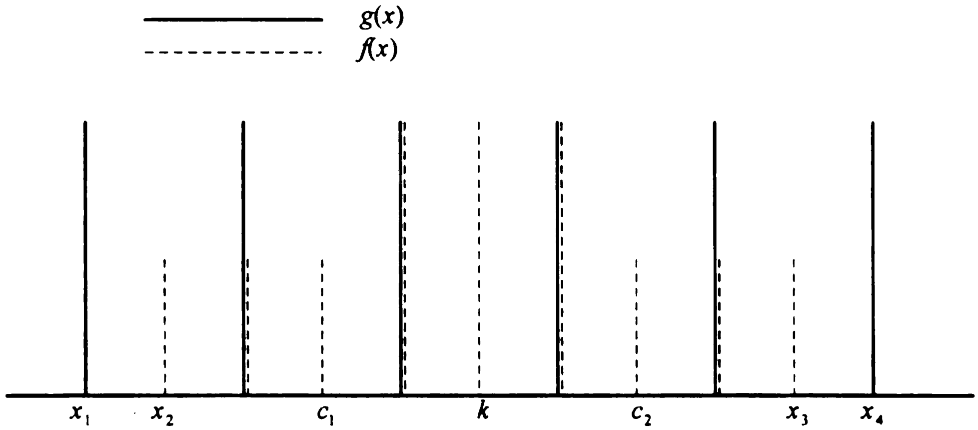

Figure 5.8. Example 5.8: G ESIR F (a discrete case).

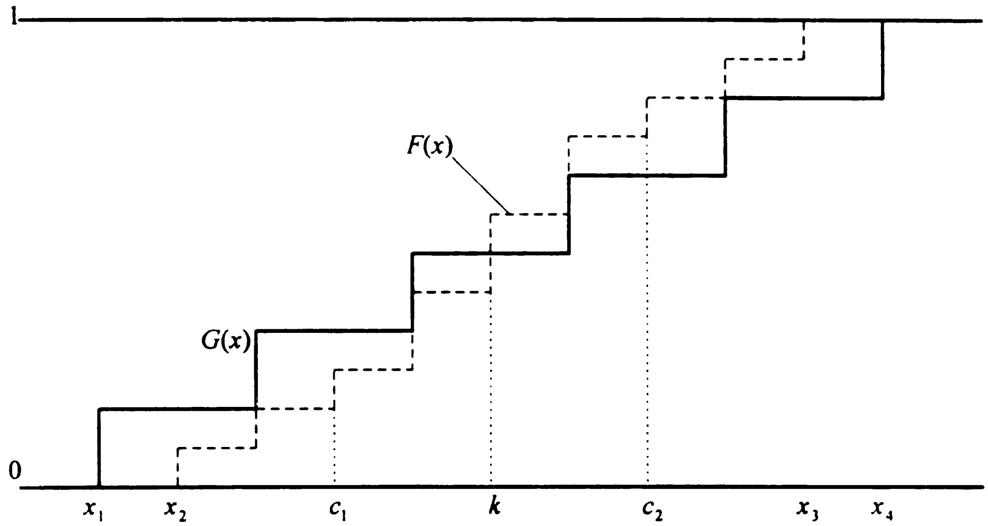

Figure 5.9. CDF representation of example 5.8.

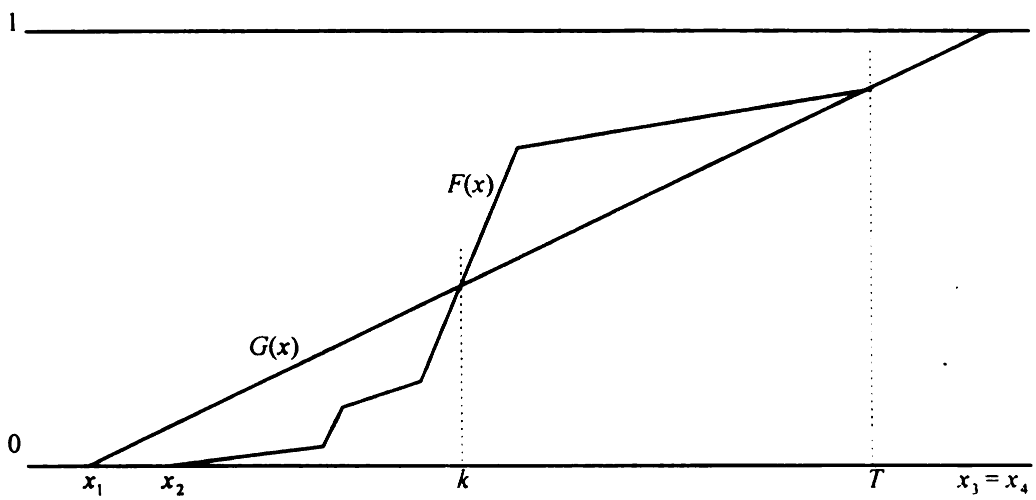

Figure 5.10. Example 5.9: G L-ESIR F.

truncation, and example 5.2 gives a case of a strong increase in risk (SIR).

(1) Example 5.1: Mean-preserving truncation from G to F in Figure 5.1. This type of R-S increase in risk is defined by imposing somewhat severe restrictions on the changes in CDF. It specifies a probability transformation such that truncated both tail probability mass is transferred to each end point. As Figure 5.1 shows, all the probability of occurring for points less than x_2 (larger than x_3) is moved to the point x_2 (x_3). This transformation is a special case of an RSIR. Since $x_2 = k_1$ and $x_3 = k_2$, the intervals $[x_2, k_1)$ and $(k_2, x_3]$ shown in Definition 5.1 are empty, thus the mean-preserving truncation satisfies the RSIR condition.

(2) Example 5.2: G SIR F in Figure 5.2. The restriction that $F(x) = G(x)$ for the interval $[x_2, x_3]$, which is required for a mean-preserving truncation, is relaxed. Instead, an SIR restricts the CDF difference $G - F$ to be non-increasing for the interval $[x_2, x_3]$. Equivalently, the restriction means that $f \geq g$ for all $x \in [x_2, x_3]$, as shown in Figure 5.2. It specifies a probability transformation from points in the interval $[x_2, x_3]$ to points outside the interval. An SIR is also an RSIR, specifying a special case that $x_2 = k_1$ and $x_3 = k_2$. Note that both shifts given in examples 3.1 and 3.2 satisfy the condition that the pdf's f and g cross only twice (at points k_1 and k_2), which is required for an RSIR.

(3) Example 5.3: G RSIR F in Figure 5.3. This example shows that the pdf's f and g cross only at points k_1 and k_2. Let's define the likelihood ratios h_1 and h_2 as $h_1 = f / g$ for the interval $[x_2, k_1)$, and $h_2 = f / g$ for the interval $(k_2, x_3]$. Then, for each given interval, the ratio function h_1 is non-decreasing and h_2 is non-increasing, respectively. Thus example 5.3 satisfies the RSIR conditions given in Definition 5.1. To see this monotonicity in the graphic example, we can follow the logic used in the discussion of the MLR example given in Figure 3.1. Since $g \geq f$ for all $x \in [x_2, k_1)$, the

ratio f/g is non-decreasing in the interval when $g' > 0$ ($g' < 0$) and the tangent lines l_g and l_f in Figure 5.3 do not meet to the left (meet to the right) direction in the space above *x* axis. Similarly, since $g \geq f$ for all $x \in (k_2, x_3]$, the ratio f/g is non-increasing in the interval if $g' < 0$ ($g' > 0$) and the tangent lines l_g and l_f do not meet to the right (meet to the left) direction in the space above *x* axis.

(4) Example 5.4: *G* L-RSIR *F* in Figure 5.4. In this example, the point *k* is selected such that $G(k) = F(k)$, and it is assumed that $G \geq F$ for all $x \in [x_2, k)$, and $G \leq F$ for all $x \in [k, x_3]$. For the left-side of the point *k*, the pdf's *f* and *g* cross only once at the point k'. Define the likelihood ratio *h* as $h = f/g$ for the interval $[x_2, k')$. Since the ratio is non-decreasing in the interval and *f* is larger than *g* for all $x \in (k', k]$, the shift in the example satisfies the L-RSIR condition in Definition 5.2. The pdf's *f* and *g* cross four times, and thus the shift cannot be an RSIR change.

(5) Example 5.5: *G* L-RSIR *F* in Figure 5.5. This is a special case of an L-RSIR in which the initial random variable has only two outcomes, x_2 and x_3, with equal probability of occurring. In this example, an R-S increase in risk occurs only in the smaller outcome x_2. Since it is such a case that $x_2 = k' = k$, the non-decreasing ratio function *h* shown in Definition 5.2 has, as its domain, an empty interval and $F(x) \geq G(x)$ for all $x \in [k, x_3]$. Thus the shift satisfies the L-RSIR condition. However the example does not satisfy the RSIR condition. Since, for the interval including the points *c* and x_3 satisfying $f \leq g$, the ratio of *f* to *g* is zero at *c* and one at x_3, the ratio likelihood function h_2 should be increasing at least one point in the interval $[c, x_3]$. The shift cannot be an ESIR either. For the interval $[k, x_3]$, the CDF ratio $(1-F)/(1-G)$ should be increasing at least one point in the interval, which contradicts the ESIR condition in Definition 5.3.

In this instance, if an increase in risk occurs in the higher outcome x_3 with the lower outcome unchanged, then it is the case where $k' = k = x_3$. This does not satisfy the

L-RSIR condition because the pdf ratio function h should be decreasing at least one point in the interval $[x_2, k')$. The shift is not an L-ESIR either because the CDF ratio H defined by $H = F / G$ should be decreasing at least one point in the interval $[x_2, k)$. Consider the case where R-S increases in risk occur in both the outcome values. Then it is the case where the CDF's F and G cross three times, and thus it does not satisfy any of the four types of R-S increases in risk defined in this section.

(6) Example 5.6: G L-RSIR F in Figure 5.6. As another special case of L-RSIR, example 5.6 shows that, given an initial pdf $f(x)$ with its support $[x_2, x_3]$, an RSIR type of change occurred only on a left-side sub-interval $[x_1, T]$ where $T \le x_3$ results in an L-RSIR shift in the initial pdf f. In Figure 5.6, let's define a conditional pdf $f'(t) = f(t) / F(T)$ with its support $[x_2, T]$, then the shift shown in the figure can be understood as an RSIR change from the pdf f'. The shift satisfies that $F \le G$ for all $x \in [x_2, k)$ and $F \ge G$ for all $x \in [k, x_3]$. Since the ratio f to g is constant and less than one for the interval $[x_2, k')$, and $f \ge g$ for all $x \in [k', k]$, it is obvious that F L-RSIR G.

(7) Example 5.7: G ESIR F in Figure 5.7. While the shift is represented by CDF change, it is easy to see that the pdf's f and g cross more than two times in the interval $[x_2, k]$. This implies that the shift is neither an RSIR nor an L-RSIR. The CDF's F and G cross only once at the point k. The probability ratio functions H_1 and H_2 in Definition 5.3 can be defined by $H_1 = F / G$ for the interval $[x_2, k)$, and by $H_2 = (1 - F) / (1 - G)$ for the interval $[k, x_3]$, and the figure shows that H_1 is non-decreasing and H_2 is non-increasing for each corresponding interval, respectively. This guarantees that G ESIR F. The way to see the monotonicity follows the discussion about the example of an MPR shift given in Figure 3.7. If, for any point in the interval $[x_2, k)$ $([k, x_3])$, the tangent lines l_F and $l_G (= G(x)$ in example 5.7) do not meet to the left (right) direction in the space between zero and one, the CDF ratio H_1 (H_2) is non-decreasing (non-increasing).

(8) Example 5.8: *G* ESIR *F* in Figure 5.8 and 5.9. This is an example of an ESIR, for a discrete random variable case. To see that *G* ESIR *F*, consider the CDF representation of the shift given in Figure 5.9. Since there is only one crossing between the CDF's and the monotonicity restrictions on the two CDF ratios are satisfied, it is obvious that *G* ESIR *F*. Figure 5.8 shows that the difference $g - f$ changes sign more than two times for the left-side of the point *k*. This implies that the shift cannot be an RSIR or an L-RSIR change.

(9) Example 5.9: *G* L-ESIR *F* in Figure 5.10. As the last example of single-crossing between the CDF's *F* and *G*, example 5.9 gives an L-ESIR change. The shift satisfies that $F \le G$ for all $x \in [x_2, k)$ and $F \ge G$ for all $x \in [k, x_3]$. For the left-side of the point *k*, the CDF ratio F / G is non-decreasing, which guarantees that *G* L-ESIR *F*. The example is also a special case of an L-ESIR change. Figure 5.10 shows that $G = F$ for all $x \in [T, x_3]$. Similar to the example 5.6, let's define a conditional pdf $f'(t) = f(t) / F(T)$ with its support $[x_2, T]$, then the shift shown in Figure 5.10 can be understood as an ESIR change from the pdf f'. Thus an ESIR type of change occurred only on a left-side sub-interval $[x_1, T]$ where $T \le x_3$ results in an L-ESIR change in the original pdf.

5.2 Some Properties among Subsets of R-S Increases in Risk

This section examines several relationships among the four concepts of R-S increases in risk, defined in section 5.1. First, two properties clarify the basic relationships among the orders generated by these concepts. One relationship is that the RSIR order implies the ESIR order which, in turn, implies the L-ESIR order, and the other is that the RSIR order implies the L-RSIR order which, in turn, implies the L-ESIR order. We provide formal proofs for these relationships.

In addition, there are important relationships between an RSIR and an ESIR, and between an L-RSIR and an L-ESIR. In chapter 4, we have shown that the property that an MPR shift can be decomposed into a series of L-MLR shifts gives an important generalization of a comparative static result, extending admissible set of CDF changes from the L-MLR shifts to the MPR shifts. Similarly, this section presents two assertions that: (i) "an ESIR shift can be decomposed into a series of RSIR shifts," and (ii) "an L-ESIR shift can be decomposed into a series of L-RSIR shifts." They are the key relationships that allow us to make general comparative static statements in chapter 6. Let's begin with the statement that the ESIR order lies between the RSIR and the L-ESIR orders.

Property 5.1. *G* RSIR *F* $\Rightarrow$ *G* ESIR *F* $\Rightarrow$ *G* L-ESIR *F*.

Proof. It is obvious that "if *G* ESIR *F* then *G* L-ESIR *F*" because only the conditions (a), (b) and (c) in Definition 5.3 are sufficient for the L-ESIR order. The proof for the claim that "if *G* RSIR *F* then *G* ESIR *F*" follows the same steps used in the proof of Property 3.1. The condition (a) in Definition 5.1 and 5.3 is just the same mean condition. The condition (b) in Definition 5.1 implies that there exists a point $k \in [k_1, k_2]$ such that $F(x) \le G(x)$ for all $x \in [x_1, k)$, and $F(x) \ge G(x)$ for all $x \in [k, x_4]$, which is the condition (b) in Definition 5.3. Since $G(x) > 0$ for all $x \in (x_2, \infty)$ and $1 - G(x) > 0$ for all $x \in (-\infty, x_3)$, we can define a pair of functions H_1 for the interval $[x_2, k)$ and H_2 for the interval $[k, x_3]$ as,[41]

[41] If x is continuous and $x_1 = x_2$ or $x_3 = x_4$, then $G(x_2) = 0$ or $G(x_3) = 0$, and the proof is also true for this case with the function defined as,

$$H_1(x) = \begin{cases} 0, & \text{when } x = x_2 \\ F(x)/G(x), & \text{for } x \in (x_2,\ k) \end{cases} \text{ or } H_2(x) = \begin{cases} F(x)/G(x), & \text{for } x \in [k,\ x_3) \\ 0, & \text{when } x = x_3. \end{cases}$$

$$H_1(x) = F(x)/G(x), \text{ for } x \in [x_2, k)$$

and

$$H_2(x) = [1 - F(x)]/[1 - G(x)], \text{ for } x \in [k, x_3]$$

respectively. Given G RSIR F, since $H_1 \in [0, 1]$ and $H_2 \in [0, 1]$, it is sufficient to show that H_1 is non-decreasing in $x \in [x_2, k)$ and H_2 is non-increasing in $x \in [k, x_3]$. These monotonicity conditions are equivalent to,

$$f(x)G(x) - g(x)F(x) \geq 0, \text{ for all } x \in [x_2, k) \tag{5.1}$$

and

$$f(x)[1 - G(x)] - g(x)[1 - F(x)] \geq 0, \text{ for all } x \in [k, x_3], \tag{5.2}$$

respectively. For the interval $[k_1, k)$, the condition (5.1) is satisfied because G RSIR F implies $g(x) \leq f(x)$ and $G(x) \geq F(x)$, and for the interval $[k, k_2]$, the condition (5.2) is satisfied because G RSIR F implies $g(x) \leq f(x)$ and $1 - G(x) \geq 1 - F(x)$. For the intervals $[x_2, k_1)$ and $(k_2, x_3]$, the RSIR condition implies that

(i) if $g(x) = 0$, then $f(x) = 0$

(ii) if $g(x) \neq 0$, then $\begin{cases} h_1(x) = f(x)/g(x), \text{ for } x \in [x_2, k_1) \\ h_2(x) = f(x)/g(x), \text{ for } x \in (k_2, x_3]. \end{cases}$

Thus, for all values of x such that $g(x) = 0$, the conditions (5.1) and (5.2) are satisfied. For case (ii), the condition (5.1) and (5.2) can be written as,

$$h_1(x)G(x) - F(x) \geq 0, \text{ for all } x \in [x_2, k_1) \text{ such that } g(x) \neq 0 \tag{5.3}$$

and

$$h_2(x)[1-G(x)]-[1-F(x)] \geq 0, \text{ for all } x \in (k_2,\ x_3] \text{ such that } g(x) \neq 0, \qquad (5.4)$$

respectively. Since $h_1(x_2)G(x_2) - F(x_2) \geq 0$ and the condition h_1 non-decreasing implies that the LHS of (5.3) is non-decreasing, i.e.,

$$\partial[h_1(x)G(x) - F(x)] / \partial x = h_1'G + h_1 g - f = h_1'G \geq 0,$$

the condition (5.3) is satisfied. Since $h_2(x_3)[1-G(x_3)]-[1-F(x_3)] \geq 0$ and the condition h_2 non-increasing implies that the LHS of (5.4) is non-increasing, i.e.,

$$\partial\{h_2(x)[1-G(x)]-[1-F(x)]\} / \partial x = h_2'[1-G]-h_2 g + f = h_2'[1-G] \geq 0,$$

the condition (5.4) is satisfied. Hence the conditions (5.1) and (5.2) are satisfied for all $x \in [x_2,\ k_1)$ and $x \in (k_2,\ x_3]$, respectively. This completes the proof. Q.E.D.

Next property gives the relationship that the RSIR order implies the L-RSIR order and the L-RSIR order implies the L-ESIR order.

Property 5.2. *G* RSIR *F* $\Rightarrow$ *G* L-RSIR *F* $\Rightarrow$ *G* L-ESIR *F*.

Proof. According to Definition 5.1, 5.2 and 5.4, three types of shifts *G* RSIR *F*, *G* L-RSIR *F* and *G* L-ESIR *F* require a common restriction that the CDF's *G* and *F* cross only once. Given such a single-crossing R-S increase in risk from *F* to *G*, let *k* be the point at which the crossing occurs. Then the first relationship that "if *G* RSIR *F* then *G* L-RSIR *F*" is assured by comparing the restriction used in Definition 5.1 and 5.2. For the interval $[x_2,\ k]$, both *G* RSIR *F* and *G* L-RSIR *F* require the same condition, but for the interval $(k,\ x_3]$, the L-RSIR order requires only that $G(x) \leq F(x)$ which is less stringent than the conditions required for the RSIR order.

Consider the second relationship that "if G L-RSIR F then G L-ESIR F." For the interval $(k,\ x_3]$, both the orders L-RSIR and L-ESIR require a common restriction that $G(x) \le F(x)$. For the interval $[x_2,\ k]$, the proof follows the same method used in the proof of Property 5.1. Q.E.D.

Let's define four subsets of R-S increases in risk, each being composed of pairs of CDF's (F, G), such as:

$$\Omega_{RS} = \{(F,\ G) |\ G \ \text{RSIR}\ F\}$$
$$\Omega_{L-RS} = \{(F,\ G) |\ G \ \text{L-RSIR}\ F\}$$
$$\Omega_{ES} = \{(F,\ G) |\ G \ \text{ESIR}\ F\}$$
$$\Omega_{L-ES} = \{(F,\ G) |\ G \ \text{L-ESIR}\ F\}.$$

Now we have, from Property 5.1 and 5.2, the following two formal relationships among the subsets of R-S increases in risk:

$$(1)\ \Omega_{RS} \subset \Omega_{ES} \subset \Omega_{L-ES}$$
$$(2)\ \Omega_{RS} \subset \Omega_{L-RS} \subset \Omega_{L-ES}.$$

There is a notable thing before going into the next property. While both the MLR and the MPR orders, discussed in section 3.2, have the property of transitivity, the RSIR and the ESIR orders do not have this property. That is, the relationships G_1 ESIR G_2 and G_2 ESIR G_3 do not necessarily imply that G_1 ESIR G_3, and the same is applied for the RSIR order. This implies that an R-S increase in risk that can be decomposed into a series of RSIR or ESIR shifts is not always an RSIR or an ESIR shift. In Figure 5.11, we give an example of this case. It shows that G ESIR (RSIR) G_1 and G_1 ESIR F, but it is not true that G ESIR F because the CDF's F and G cross three times.

In general, a series of RSIR (ESIR) shifts does not necessarily imply an RSIR (ESIR) shift. However, as a special case, if a series of ESIR shifts have a single common

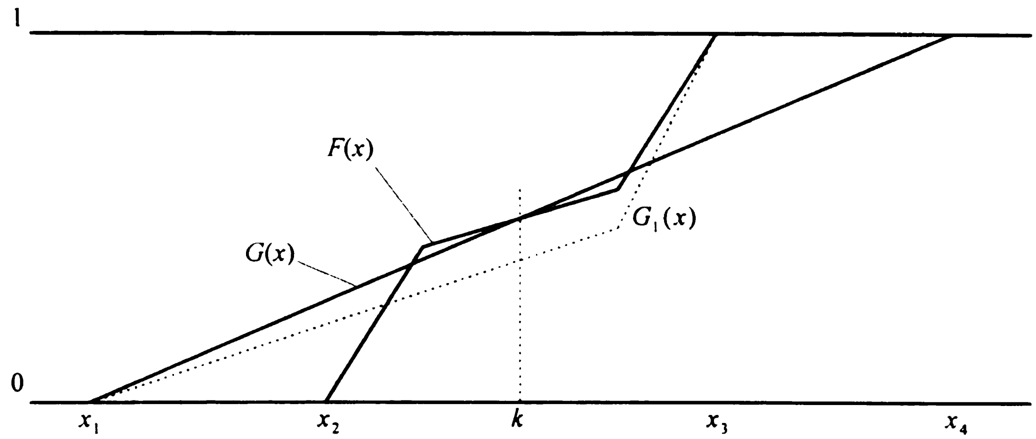

Figure 5.11. G ESIR G_1 ESIR F but not G ESIR F.

crossing between every pair of CDF's, the summation of all the shifts results in an ESIR. That is, if a series of shifts G ESIR G_1 ESIR $\cdots$ G_{n-1} ESIR G_n ESIR F, where all CDF's G, F, and G_i's for all $i = 1, \cdots, n$ have a single common crossing, the shift from F to G is an ESIR. The reason is as follows. Let k be the single point of crossing, then the series of ESIR shifts implies that $G \geq G_1 \geq \cdots \geq G_n \geq F$ for all points to the left-side of k, and $G \leq G_1 \leq \cdots \leq G_n \leq F$ for all points to the right-side of k. Define a new random variable $y = -x$ and let $f(x) = f(-y) = \hat{f}(y)$, $g(x) = g(-y) = \hat{g}(y)$ and $g_i(x) = g_i(-y) = \hat{g}_i(y)$ for all $i = 1, \cdots, n$. Then, as we saw in section 5.1, every ESIR shift can be understood as the sum of two specialized FSD shifts. The series of shifts are expressed as: for the left-side of the point k, F MPR G_n MPR $\cdots$ G_1 MPR G, and for the right-side of k, $\hat{F}$ MPR $\hat{G}_n$ MPR $\cdots$ $\hat{G}_1$ MPR $\hat{G}$. Since the MPR order is transitive (Property 3.3), the shift from F to G can be understood as the sum of two FSD shifts such that F MPR G for the left-side of k, and $\hat{F}$ MPR $\hat{G}$ for the right-side of k. These two shifts sum up to imply a shift satisfying G ESIR F.

A special family of CDF's holds the transitivity of the ESIR order. If the initial and final CDF's F and G are restricted to be symmetric[42] from the mean, every ESIR shift from F to G has a common crossing at the point of mean and thus the ESIR order is transitive. The family of symmetric distributions include normal, triangular and uniform distributions.

While the RSIR and the ESIR orders, in general, do not display transitivity, the non-transitivity gives important implications with regard to comparative static analysis under uncertainty. As we have discussed in section 3.2, if a given CDF order is sufficient for making a general comparative static statement and the order is non-transitive, the admissible set of changes in CDF includes every shift that can be decomposed into a series of shifts generated by the given order. The CDF change given in Figure 5.11 is an example of this case. That is, the comparative static result made for the set of ESIR changes is also obtained for the shift given in the figure.

The above discussion leads to another implication of the non-transitivity with regard to the comparative static analysis. Assume that, given two CDF orders, an arbitrary change in CDF generated by one CDF order can always be decomposed into a series of shifts that satisfy the other CDF order. Then, if the latter order gives a sufficient condition for a general comparative static result, then all the shifts that satisfy the former order are also admissible for the result. Therefore, examining the presence of this kind of relationship between CDF orders is meaningful for making a general comparative static statement. In what follows, two such relationships are presented: one between the RSIR and the ESIR orders, and the other between the L-RSIR and the L-ESIR orders which are also non-transitive. They are: (i) "an ESIR shift can always be decomposed into a series of RSIR shifts," and (ii) "an L-ESIR shift can always be decomposed into a series of L-RSIR shifts."

[42] A CDF F or a corresponding pdf f, with its mean $\bar{x}$, is said to be 'symmetric from the mean' if, for any number c, $F(\bar{x}-c)=1-F(\bar{x}+c)$ or $f(\bar{x}-c)=f(\bar{x}+c)$.

Property 5.3. Any ESIR shift can be decomposed into a series of RSIR shifts, that is, if G ESIR F, then there exists a series of CDF's $G_1,\cdots,G_n$ such that G RSIR G_1 RSIR $\cdots$ G_{n-1} RSIR G_n RSIR F.

Proof. Given an arbitrary pair of pdf's f and g satisfying G ESIR F, let's define a function $h\colon R \to [0, \infty)$ as

$$h(x) = \begin{cases} f(x)/g(x), \text{ when } g(x) \neq 0 \\ \text{the defined last value of } f(x)/g(x), \text{ when } g(x) = f(x) = 0 \\ \infty, \text{ when } g(x) = 0 \text{ and } f(x) \neq 0. \end{cases}$$

Note that, according to Definition 5.3, there is a point $k \in [x_2, x_3]$ such that $F(x) \leq G(x)$ for all $x \in [x_1, k)$ and $F(x) \geq G(x)$ for all $x \in [k, x_4]$, and there is no restriction on the number of times of crossing between the pdf's f and g. This implies that the function h is generally not monotone and thus there are points at which h changes from non-increasing to non-decreasing. Assume that there are $n(=n_1+n_2)$ number of such points and denote them as: for the left-side of the point k, $c_{i_1}^L$ for $i_1 = 1, 2, \cdots, n_1$ where $n_1 \geq 0$ and $c_{n_1}^L < c_{n_1-1}^L < \cdots < c_1^L$; and for the right-side of k, $c_{i_2}^R$ for $i_2 = 1, 2, \cdots, n_2$ where $n_2 \geq 0$[43] and $c_1^R < c_2^R < \cdots < c_{n_2}^R$. For every $c_{i_1}^L$ and $c_{i_2}^R$, define CDF's $G_{i_1}^L(x)$ and $G_{i_2}^R(x)$ as,

$$G_{i_1}^L(x) = \begin{cases} \lambda_{i_1}^L G(x), \text{ when } x < c_{i_1}^L \\ F(x), \text{ when } c_{i_1}^L \leq x \leq d_{i_1}^R \\ 1 - \lambda_{i_1}^R[1 - G(x)], \text{ when } x > d_{i_1}^R \end{cases} \tag{5.5}$$

and

[43] When $n_1 = n_2 = 0$, it is the case where the ESIR shift is also an RSIR shift.

$$G_{i_2}^R(x) = \begin{cases} \lambda_{i_2}^L G(x), \text{ when } x < d_{i_2}^L \\ F(x), \text{ when } d_{i_2}^L \le x \le c_{i_2}^R \\ 1-\lambda_{i_2}^R[1-G(x)], \text{ when } x > c_{i_2}^R, \end{cases} \tag{5.6}$$

where $\lambda_{i_1}^L = F(c_{i_1}^L)/G(c_{i_1}^L)$, $\lambda_{i_1}^R = [1-F(d_{i_1}^R)]/[1-G(d_{i_1}^R)]$, $\lambda_{i_2}^L = F(d_{i_2}^L)/G(d_{i_2}^L)$ and $\lambda_{i_2}^R = [1-F(c_{i_2}^R)]/[1-G(c_{i_2}^R)]$, and $d_{i_1}^R(\ge k)$ and $d_{i_2}^L(\le k)$ are the selected points such that $G_{i_1}^L(x)$ and $G_{i_2}^R(x)$ have the same mean as *F* or *G*, respectively. Since *G* ESIR *F* implies that the order of *n* values of $\lambda_{i_1}^L$ and $\lambda_{i_2}^L$ is same as the order of the corresponding *n* values of $\lambda_{i_1}^R$ and $\lambda_{i_2}^R$, the order of *n* points $c_{i_1}^L$ and $d_{i_2}^L$ (left-side of *k*) is same as the reversed order of the corresponding *n* points $c_{i_1}^R$ and $d_{i_2}^R$ (right-side of *k*). Let (s_i^L, s_i^R) for $i = 1, 2, \cdots, n$, be *n* pairs of points where $s_n^L < s_{n-1}^L < \cdots < s_1^L$ are the ordered *n* values of $c_{i_1}^L$ and $d_{i_2}^L$, and $s_1^R < s_2^R < \cdots < s_n^R$ are the ordered *n* values of $c_{i_2}^R$ and $d_{i_1}^R$, respectively. Then the defined *n* CDF's can be rewritten, for each *i*-th pair of points (s_i^L, s_i^R), as

$$G_i(x) = \begin{cases} \lambda_i^L G(x), \text{ when } x < s_i^L \\ F(x), \text{ when } s_i^L \le x \le s_i^R \\ 1-\lambda_i^R[1-G(x)], \text{ when } x > s_i^R \end{cases} \tag{5.7}$$

where $\lambda_i^L = F(s_i^L)/G(s_i^L)$ and $\lambda_i^R = [1-F(s_i^R)]/[1-G(s_i^R)]$, and its corresponding pdf is

$$g_i(x) = \begin{cases} \lambda_i^L g(x), \text{ when } x < s_i^L \\ f(x), \text{ when } s_i^L \le x \le s_i^R \\ \lambda_i^R g(x), \text{ when } x > s_i^R. \end{cases} \tag{5.8}$$

Now all CDF's *F*, *G* and $G_1, \cdots G_n$ have the same mean and have a common crossing at the point *k*. This implies that, if we let $G = G_0$ and $F = G_{n+1}$, then for all $i = 1, 2, \cdots, n+1$, $G_i(x) \le G_{i-1}(x)$ for all $x \in [x_1, k)$ and $G_i(x) \ge G_{i-1}(x)$ for all $x \in [k, x_4]$, and we have an order that G_0 MPS G_1 MPS $\cdots$ G_n MPS G_{n+1}. In addition, set $\lambda_0^L = \lambda_0^R = 1$, $\lambda_{n+1}^L = \lambda_{n+1}^R = 0$, $s_0^L = s_0^R = k$, $s_{n+1}^L = x_2$ and $s_{n+1}^R = x_3$, then from (5.8) we see that, for every $i = 1, 2, \cdots, n+1$, g_i is generally related to g_{i-1} as

$$g_i(x) = \begin{cases} (\lambda_i^L / \lambda_{i-1}^L) g_{i-1}(x), \text{ when } x < s_i^L \\ (h / \lambda_{i-1}^L) g_{i-1}(x) (= f(x)), \text{ when } s_i^L \le x < s_{i-1}^L \\ g_{i-1}(x)(= f(x)), \text{ when } s_{i-1}^L \le x \le s_{i-1}^R \\ (h / \lambda_{i-1}^R) g_{i-1}(x)(= f(x)), \text{ when } s_{i-1}^R < x \le s_i^R \\ (\lambda_i^R / \lambda_{i-1}^R) g_{i-1}(x), \text{ when } x > s_i^R. \end{cases} \tag{5.9}$$

By the condition G ESIR F, $\lambda_i^L / \lambda_{i-1}^L < 1$ and $\lambda_i^R / \lambda_{i-1}^R < 1$, for every $i = 1, 2, \cdots, n+1$. From (5.7), the fact that $G_i(s_i^L) = F(s_i^L)$ and $G_i(x) \le F(x)$ for all $x \in [x_1, s_i^L)$ implies that at every point s_i^L, g_i meets f from the below, and the fact that $G_i(s_i^R) = F(s_i^R)$ and $G_i(x) \le F(x)$ for all $x \in (s_i^R, x_4]$ implies that at every point s_i^R, g_i meets f from the above. Thus $\lambda_i^L \le h(s_i^L)$ and $\lambda_i^R \le h(s_i^R)$. Now for each of the intervals $[s_i^L, s_{i-1}^L)$ and $(s_{i-1}^R, s_i^R]$, there is no point at which the function h changes from non-increasing to non-decreasing, and (5.7) and (5.9) imply that $g_i(x) \le g_{i-1}(x)$ for all $x \in [x_1, s_i^L)$ and $(s_i^R, x_4]$, and $G_i(s_{i-1}^L) = G_{i-1}(s_{i-1}^L)$ and $G_i(s_{i-1}^R) = G_{i-1}(s_{i-1}^R)$. Accordingly, there exists a pair of points $k_i^L \in [s_i^L, s_{i-1}^L)$ and $k_i^R \in (s_{i-1}^R, s_i^R]$[44] such that $g_i(x) \ge g_{i-1}(x)$ for all $x \in [k_i^L, k_i^R]$, and there exist a non-decreasing function $h_i^L : [x_1, k_i^L) \to [0, 1]$ and a non-increasing function $h_i^R : (k_i^R, x_4] \to [0, 1]$ such that $g_i(x) = h_i^L(x) g_{i-1}(x)$ for all $x \in [x_1, k_i^L)$ and $g_i(x) = h_i^R(x) g_{i-1}(x)$ for all $x \in (k_i^R, x_4]$ where

[44] If s_i^L is one of $c_{i_1}^L$'s and $g_i(s_i^L) \ge g_{i-1}(s_i^L)$, or s_i^R is one of $c_{i_2}^R$'s and $g_i(s_i^R) \ge g_{i-1}(s_i^R)$, then $s_i^L = k_i^L$ or $s_i^R = k_i^R$ and from (5.7),

$$g_i(x) = (\lambda_i^L / \lambda_{i-1}^L) g_{i-1}(x), \text{ for } x \in [x_1, s_i^L)$$
$$g_i(x) \ge g_{i-1}(x), \text{ for } x \in [s_i^L, s_i^R]$$
$$g_i(x) = (\lambda_i^R / \lambda_{i-1}^R) g_{i-1}(x), \text{ for } x \in (s_i^R, x_4]$$

which imply G_{i-1} RSIR G_i. This is also the case where G_{i-1} RSIR G_{i+1}, and more generally for every $j = 0, 1, \cdots, m$, if s_{i+j}^L is one of $c_{i_1}^L$'s and $g_{i+j}(s_{i+j}^L) \ge g_{i-1}(s_{i+j}^L)$, or s_{i+j}^R is one of $c_{i_1}^R$'s and $g_{i+j}(s_{i+j}^R) \ge g_{i-1}(s_{i+j}^R)$, then G_{i-1} RSIR G_{i+m+1}.

$$h_i^L(x) = \begin{cases} \lambda_i^L / \lambda_{i-1}^L, & \text{for } x \in [x_1, s_i^L) \\ h(x) / \lambda_{i-1}^L, & \text{for } x \in [s_i^L, k_i^L) \end{cases}$$

and

$$h_i^R(x) = \begin{cases} h(x) / \lambda_{i-1}^R, & \text{for } x \in (k_i^R, s_i^R] \\ \lambda_i^R / \lambda_{i-1}^R, & \text{for } x \in (s_i^R, x_4]. \end{cases}$$

Hence, by Definition 5.1, G_{i-1} RSIR G_i for all $i = 1, 2, \cdots, n+1$. Q.E.D.

Property 5.4. Any L-ESIR shift can be decomposed into a series of L-RSIR shifts, that is, if *F* L-ESIR *G*, then there exists a series of CDF's $G_1, \cdots, G_n$ such that *G* L-RSIR G_1 L-RSIR $\cdots$ G_{n-1} L-RSIR G_n L-RSIR *F*.

Proof. For a given CDF change from *F* to *G*, if it satisfies the condition *G* L-ESIR *F*, then there is only one crossing between the two CDF's. As we noted, an L-ESIR shift can be understood as the sum of two FSD shifts: an MPR dominated shift for the left-side of the crossing point, and an FSD dominant shift for the right-side of the point. This implies that the proof of the assertion that "any MPR shift can be decomposed into a series of L-MLR shifts" (Property 3.6) can be used for the proof of Property of 5.4.

To illustrate the result in Property 5.3, consider an example of an R-S increase in risk given in Figure 5.12. Since the pdf representation (while it does not provided) for the shift shows more than two crossing between the pdf's *f* and *g*, it cannot be an RSIR shift. Following the method given in the proof of Property 5.3, three intermediate CDF's G_i's, for $i = 1, 2, 3$, are defined. The shift from *F* to *G* is divided into four shifts, from *F* to G_3, $\cdots$, and from G_1 to *G*. By closely examining the shifts, we know that each shift satisfies the RSIR condition in Definition 5.1. Thus the example shows that a shift from

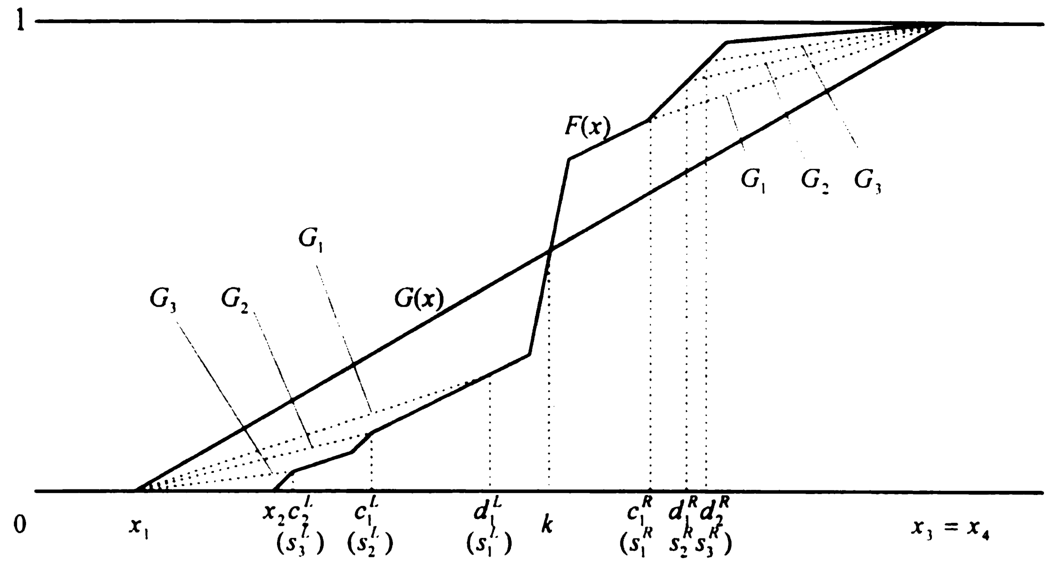

Figure 5.12. G ESIR F that is the sum of four RSIR shifts.

G to F, satisfying G ESIR F, can be decomposed into four RSIR shifts such that:

$$G \text{ RSIR } G_1 \text{ RSIR } G_2 \text{ RSIR } G_3 \text{ RSIR } F.$$

These last two properties have important implications with regard to providing general comparative static statements. By Property 5.3, the comparative static statement made for the set of RSIR changes can be generalized to the set of ESIR changes. By Property 5.4, the comparative static statement made for the set of L-RSIR shifts can be generalized to the set of L-ESIR shifts. These are shown in chapter 6.

Chapter 6

COMPARATIVE STATIC RESULTS WITH SUBSETS OF R-S INCREASES IN RISK

This chapter provides general comparative static statements regarding the subsets of R-S increases in risk presented in chapter 5. The admissible set of R-S increases in risk for the comparative static result in Black and Bulkley (1995) is extended to the set of ESIR shifts which is larger than the set of RSIR shifts. This generalization is directly obtained from the relationship that an ESIR shift can always be decomposed into a series of RSIR shifts (Property 5.3), without any additional cost of assumptions. Another determinate comparative static statement is made for the set of L-RSIR shifts. Restricting the concerned payoff function to be linear in the random variable, we show that the effect of an L-RSIR shift is determinable for all risk averse decision makers with non-negative third derivative of utility functions. Another relationship established in chapter 5 gives a further generalization of this result. Using the relationship that an L-ESIR shift can always be decomposed into a series of L-RSIR shifts, the admissible set of R-S increases in risk is extended to the set of L-ESIR shifts without imposing additional assumptions.

Section 6.1 gives some preliminary discussions about the results in Black and Bulkley (1989), and Dionne, Eeckhoudt and Gollier (1993). Given the general decision model (1.1), section 6.2 examines the properties regarding the marginal utility with respect to the choice variable, measured at the optimal level. These properties are required for developing the comparative static results in section 6.3.

6.1 Literature Review and Preliminary Discussions

Rothschild and Stiglitz (1971) have developed a sufficient condition for signing the effect of an arbitrary R-S increase in risk on the optimal choice made by an economic agent. Given the general one-argument decision problem (1.1), assume that the solution of the problem satisfies the first and the second-order conditions. Then the R-S condition implies that the optimal value of the choice variable b does not increase (decrease) for any R-S increase in risk if $u'(z)z_b$ is concave (convex) in x, or the following expression is signed as non-positive (non-negative):

$$u' \cdot z_{bxx} + 2u'' \cdot z_x \cdot z_{bx} + z_b(u'' \cdot z_{xx} + u''' \cdot z_x^2). \tag{6.1}$$

However, as Kraus (1979) noted, the R-S sufficient condition cannot generally be satisfied for the general decision problem (1.1). This is because the first-order condition that $E[u'(z)z_b(x,\ b)] = 0$ at the optimal level of b implies that z_b must change its sign as x varies and thus the sign of the last term in (6.1) is generally ambiguous. In addition, since (6.1) includes the third derivative of the utility function, it is also obvious that the effect of an arbitrary R-S increase in risk in the random variable on the choice variable made by a risk averse agent is generally not determinable.

Instead, many authors have examined subsets of R-S increases in risk when analyzing the choice behavior of an arbitrary risk averse decision maker. Examples include the cases of a global increase in risk analyzed in Kraus (1979) and Katz (1981), a mean-preserving truncation in Eeckhoudt and Hansen (1981), and a strong increase in risk (SIR) in Meyer and Ormiston (1985). Being a subset of R-S increases in risk, SIR shifts include all reverse cases of mean-preserving truncations which in turn include all global increases in risk. As we saw in chapter 2, Meyer and Ormiston generalize the comparative static results developed in the other cited papers. Using the general decision model (1.1), they show that an SIR in the random variable x decreases the choice variable b made by an arbitrary risk averse decision maker.

A further generalization is given by Black and Bulkley (1989) who introduce the concept of a relatively strong increase in risk (RSIR). As shown in Definition 5.1, RSIR shifts include, as a special case, the set of SIR shifts. They prove that the same desirable comparative static result holds for the set of RSIR shifts.

Theorem 6.1. For all risk averse decision makers, $b_F \geq b_G$ if
(a) G RSIR[45] F
(b) $z_x \geq 0$, $z_{bx} \geq 0$, $z_{bb} < 0$ and $z_{bxx} \leq 0$.

Accordingly, the admissible set of changes in risk for the comparative static result is improved as: global changes in risk → mean-preserving truncations → strong increases in risk → relatively strong increases in risk.

All these results contain the same set of decision makers (all risk averse individuals) and the same restrictions on the structure of the concerned decision model, as the ones in condition (b) in Theorem 6.1. The structural restrictions on the payoff function such as, $z_x \geq 0$, $z_{bx} \geq 0$ and $z_{bxx} \leq 0$, are often assumed by many authors and they are not very harmful for the comparative static analysis because many economic problems already satisfy these restrictions. However, another common restriction that the concerned payoff should be strictly concave in the choice variable $(z_{bb} < 0)$ is quite restrictive. This excludes the decision models in which payoffs are linear in the choice variable $(z_{bb} = 0)$ such as the standard portfolio problem, the model of a competitive firm

[45] An RSIR defined in Definition 5.1 is slightly modified from the original one. Similarly, the proof of Theorem 6.1 needs a modification of the original proof, substituting the pdf ratio function h for the ratio g/f.

under price uncertainty with a constant marginal costs, and the coinsurance demand model.

Many authors have assumed that, doing their comparative static analysis, the first and the second-order conditions are satisfied to guarantee an interior unique solution. Let $f(x)$ be a given pdf of the random variable x, then the first-order condition which determines the optimal value of b is,

$$\int_{x_2}^{x_1} u'[z(x,\ b)]z_b(x,\ b)f(x)dx = 0. \tag{6.2}$$

The solution satisfying (6.2) is guaranteed to be a global optimum by the second-order condition that,

$$\int_{x_2}^{x_1} \{u''[z(x,\ b)]z_b^2 + u'[z(x,b)]z_{bb}\}f(x)dx < 0. \tag{6.3}$$

The condition (6.3) is saitisfied[46] with the restriction $z_{bb} < 0$ when risk aversion $(u'' \le 0)$, or with the restriction $z_{bb} \le 0$ when strict risk aversion $(u'' < 0)$. Assume that an initial optimal solution satisfies both the first and second-order conditions. Then, to prove the result that $b_F \ge b_G$ for a specified change in pdf from f to g, it is sufficient to show that

$$Q(b_F) = \int_{x_1}^{x_2} u'[z(x,\ b_F)]z_b(x,\ b_F)[f(x) - g(x)]dx \ge 0. \tag{6.4}$$

This is an often used technique to prove comparative static theorems. However, if the initial solution does not satisfy the first or the second-order condition, the method using (6.4) is generally not enough for proving a comparative static result.

When $z_{bb} = 0$, since z_b does not depend on the choice variable b, there are cases where an interior solution does not exist. If a decision maker is risk neutral, his choice depends only on the expectation of z_b and there is no interior unique solution. In this case, both the first and the second-order conditions are not satisfied. Given the assumption $z_{bx} \ge 0$, if $E(z_b) > (<)0$ then $b_F = +\infty\ (-\infty)$, and if $E(z_b) = 0$ then the

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[46] The non-decreasing utility function is implicitly assumed to be increasing at least some $x \in [x_2,\ x_1]$.

solution is indeterminate. For a strictly risk averse individual, $E(z_b) \leq 0$ implies $b_F = -\infty$, in which the solution satisfying (6.2) does not exist. Another problem caused by the restriction $z_{bb} = 0$ is that one cannot examine the effect of a change from no risk to risky situation because there is no interior unique solution under certainty. Thus, considering the decision models with payoffs being linear in the choice variable, the case of an unbounded or a corner solution should be examined for making a general comparative static statement.

Dionne, Eeckhoudt and Gollier (1993) raise this issue and show that, when the concerned payoff is linear in the choice variable *b*, Black and Bulkley's comparative static result can be extended to include more shifts[47] that cannot be generated by the RSIR order. They consider a linear decision model of the form,[48]

$$z(x,\ b) = z_0 + b(x - c) \tag{6.5}$$

where z_0 and *c* are exogenous parameters. This is the case where the payoff function is linear in both the choice and the random variable. The constant z_0 can be interpreted as an initial endowment in the portfolio problem and the coinsurance problem, or a fixed cost or an initial wealth in the competitive firm model, and *c* represents either a sure interest rate, a marginal cost of insurance, or a marginal cost of production. Restricting the choice variable to be non-negative, this linear model satisfies the above structural restrictions as, $z_x = b \geq 0$, $z_{bx} = 1$ and $z_{bxx} = z_{xx} = z_{bb} = 0$.

In order to make a general comparative static statement, Dionne et. al. examine the cases of an unbounded and a corner solution for the decision problem (6.5). Given an initial pdf *f*(*x*), every possible optimal choice of a risk averse decision maker is summarized in Table 6.1. When $E(x) > c$, b_F is strictly positive because the marginal expected utility, $E[u'(z)(x - c)]$, is strictly positive at $b = 0$, and when $E(x) < c$, $b_F = 0$

[47] They introduce the concept of a 'relatively weak increase in risk' which is not related to our definitions of R-S increases in risk. Thus this dissertation do not reproduce the definition of the concept.
[48] This linear model is already shown in (4.10).

Table 6.1. Different solutions when $b \geq 0$.

Cases	$u''(z) < 0$	$u''(z) = 0$
i) $c \leq x_2 < E(x) < x_3$	unbounded solution $b_F = +\infty$	unbounded solution $b_F = +\infty$
ii) $x_2 < c < E(x) < x_3$	interior or unbounded solution $0 < b_F \leq +\infty$	unbounded solution $b_F = +\infty$
iii) $x_2 < c = E(x) < x_3$	corner solution $b_F = 0$	undetermined $b_F = 0$
iv) $x_2 < E(x) < c \leq x_3$	corner solution $b_F = 0$	corner solution $b_F = 0$
v) $x_2 < E(x) < x_3 < c$	corner solution $b_F = 0$	corner solution $b_F = 0$
vi)* $c < x_2 = x_3$	unbounded solution $b_F = +\infty$	unbounded solution $b_F = +\infty$
vii) $x_2 = c = x_3$	corner solution $b_F = 0$	undetermined $b_F = 0$
viii) $x_2 = x_3 < c$	corner solution $b_F = 0$	corner solution $b_F = 0$

*** In Dionne et. al., only the first five cases are considered, but in order to examine the no risk to risk case, as Meyer and Ormiston (1985) note, we add the case of initially certain.**

because $E[u'(z)(x-c)] \le u'[z(c,\ b)][E(x)-c] < 0$ for all positive b. Consider the case, $E(x) = c$. If $u'' < 0$, then $b_F = 0$ because $E[u'(z)(x-c)] < u'[z(c,\ b)][E(x)-c] = 0$ for all positive b, and if $u'' = 0$, then the solution is indeterminate because, for all possible b, $E[u'(z)(x-c)] = u'(z)[E(x)-c)] = 0$. It is easy to see that when a corner or an unbounded solution prevails, any R-S increase in risk will not increase the optimal level of choice variable. First consider the risk neutral case. The solution depends only on the expectation of the random variable x. If $E(x) > (<)\ c$ then $b_F = +\infty$ (0), and if $E(x) = c$ then the solution is indeterminate. Since an R-S increase in risk does not affect the mean, $E(x)$, it does not change the solution.[49] Consider the case of strict risk aversion. When $b_F = +\infty$, the solution cannot increase any more, and when $b_F = 0$ due to $E(x) \le c$, the solution will not change because $E(x)$ remains unchanged after an R-S increase in risk.

Thus, for both the cases of a corner or an unbounded solution, we obtain the desired comparative static results that any R-S increase in risk does not increase the initial optimal solution. Now, in order to make a general comparative static statement for a particular type of R-S increase in risk, only the case of an interior solution is left to be considered. Since an interior solution is possible only when $u''(z) < 0$, the solution satisfies the second-order condition and the solution satisfying the first-order condition is unique. This allows Dionne et. al. to use the sufficient condition (6.4) to prove their comparative static result. With the above discussion, the result in Black and Bulkley's analysis can be applied for decision models with linear payoffs. That is, if the concerned decision model with the structural restriction $z_{bb} = 0$ follows the form (6.5) in which the payoff is also linear in the random variable, Theorem (6.1) can be generalized as follows.

Theorem 6.1′. For all risk averse decision makers, $b_F \ge b_G$ if

(a) G RSIR F

[49] This assertion is not valid for the case where the solution is indeterminate, but even in this case a risk neutral individual has no reason to change his original choice after the change in risk.

(b) $z_x \geq 0$, $z_{bx} \geq 0$, $z_{bb} \leq 0$, $z_{bxx} \leq 0$, and $z_{xx} = 0$ when $z_{bb} = 0$.

The types of changes in CDF such as, a global increase in risk, a mean-preserving truncation, an SIR, or an RSIR, have a common restriction that the pdf's f and g cross only twice[50] and the comparative static results developed for these shifts contain the same set of decision makers (all risk averse individuals). In this chapter, we extend the set of admissible R-S increases in risk allowing the desirable comparative static properties. First, the results in Theorem 6.1′ is generalized for the set of ESIR shifts in which the pdf's f and g are not restricted to cross only twice. Another extension is given by using a trade-off between the restrictions on the set of decision makers and the set of changes in CDF. If a risk averse individual is restricted to have a utility function with its non-negative third derivative, a desirable comparative static statement is obtained for the set of L-RSIR and this result is further generalized to the set of L-ESIR shifts.

Before going into the main body of this chapter, the next section examines several properties regarding the marginal utility with respect to the choice variable b, $u'(z)z_b$. They are used to develop a comparative static result in section 6.3.

6.2 Properties of the Function $u'(z)z_b$

Given a CDF $F(x)$, this section investigates several properties regarding the function $u'[z(x, b_F)]z_b(x, b_F)$ which is the marginal utility of b, measured at the optimal level b_F. The properties are derived from the structural restrictions on the payoff function such as, $z_x \geq 0$, $z_{bx} \geq 0$ and/or $z_{bxx} \leq 0$, and the assumptions on utility function such as $u' \geq 0$, $u'' \leq 0$, and/or $u''' \geq 0$. Some of the properties have already been used to prove the comparative static results obtained in the literature. Others are required for the

[50] For the case of a global change in risk, the two crossings occur at the point of mean.

results in section 6.3. In this section, assume that the support of the CDF F and its corresponding pdf f lie in a bounded interval [0, 1], and let b_F be the interior bounded solution satisfying the first and the second-order conditions for the maximum problem (1.1). The condition $z_{bx} \geq 0$ implies that $z_b(x, b_F)$ has a single crossing of the x axis, changing its sign from negative to positive, and if we let x^* be the value of x satisfying $z_b(x, b_F) = 0$ then $x^* \in [0, 1]$. The first two properties are easily derived and often encountered in the previous comparative static analysis.

Property 6.1. Assume that $u' \geq 0$ and $z_{bx} \geq 0$, then for all $y \in [0, 1]$, (i) $\int_0^y u'[z(x, b_F)]z_b(x, b_F)f(x)dx \leq 0$ and (ii) $\int_y^1 u'[z(x, b_F)]z_b(x, b_F)f(x)dx \geq 0$.

Proof. For a given CDF $F(x)$, the first-order condition for the problem (1.1) implies that

$$\int_0^1 u'(z)z_b(x, b_F)f(x)dx = 0,$$

and thus the assumptions $u' \geq 0$ and $z_{bx} \geq 0$ complete the proof. Q.E.D.

Property 6.2. Assume that $u' \geq 0$ and $z_{bx} \geq 0$, then (i) $u'(z)z_b \leq 0$ for all $x \in [0, x^*]$, and (ii) $u'(z)z_b \geq 0$ for all $x \in [x^*, 1]$.

Proof. Since the assumption $z_{bx} \geq 0$ implies that $z_b \leq 0$ when $x \leq x^*$, and $z_b \geq 0$ when $x \geq x^*$, the assumption $u' \geq 0$ completes the proof. Q.E.D.

Property 6.3. Assume that $u' \geq 0$, $u'' \leq 0$, $z_x \geq 0$, $z_{bx} \geq 0$ and $z_{bxx} \leq 0$, then (i) for the interval $[0, x^*]$, $u'(z)z_b$ is non-positive, non-decreasing in x and its derivative with respect to x is minimized at $x = x^*$; and (ii) for the interval $[x^*, 1]$, $u'(z)z_b$ is non-negative and its derivative with respect to x is maximized at $x = x^*$.

Proof. By Property 6.2, $u'(z)z_b \leq 0$ when $x \leq x^*$, and $u'(z)z_b \geq 0$ when $x \geq x^*$. The derivative of $u'(z)z_b$ with respect to x is

$$u''(z)z_x z_b + u'(z)z_{bx}. \tag{6.6}$$

(i) When $x \leq x^*$: The given assumptions imply that both terms in (6.6) are non-negative and thus $u'(z)z_b$ is non-decreasing in x. Also we know that the first term is zero at $x = x^*$ and the second term is non-increasing in x. This implies that (6.6) has its minimum value at $x = x^*$ within the interval $[0,\ x^*]$.

(ii) When $x \geq x^*$: The given assumptions imply that the first term in (6.6) is always non-positive and the second term is non-increasing in x. Thus (6.6) has its maximum at $x = x^*$ within the interval $[x^*,\ 1]$. Q.E.D.

A portion of the assertion in Property 6.3 can be restated as,

$$\left.\frac{\partial u'(z)z_b}{\partial x}\right|_{x \leq x^*} \geq \left.\frac{\partial u'(z)z_b}{\partial x}\right|_{x = x^*} \geq \left.\frac{\partial u'(z)z_b}{\partial x}\right|_{x \geq x^*}.$$

Property 6.4. Assume that $u' \geq 0$, $u'' \leq 0$, $z_x \geq 0$, $z_{bx} \geq 0$ and $z_{xx} = 0$, then

(i) $\dfrac{\partial u'(z)z_b}{\partial x} \geq \dfrac{u'(z)z_b}{x - x^*}$, for all $x \in [0,\ x^*)$

(ii) $\dfrac{\partial u'(z)z_b}{\partial x} \leq \dfrac{u'(z)z_b}{x - x^*}$, for all $x \in (x^*,\ 1]$.

Proof. Since we assume that z is linear in the random variable x, that is $z_{xx} = 0$, the payoff function can be expressed as

$$z(x,\ b) = r_1(b)x + r_2(b)$$

where r_1 and r_2 are arbitrary functions of b, satisfying $z_x = r_1(b) \geq 0$ and $z_{bx} = r_1'(b) \geq 0$.

With the linearity, we have $x^* = -r_2'(b_F)/r_1'(b_F)$.[51] This implies that, for all $x \neq x^*$,

$$\frac{z_h}{x-x^*} = \frac{r_1'(b_F)x + r_2'(b_F)}{x-x^*} = \frac{r_1'(b_F)x + r_2'(b_F)}{x + [r_2'(b_F)/r_1'(b_F)]} = r_1'(b_F) = z_{hx}, \tag{6.7}$$

and thus,

$$\frac{\partial u'(z)z_h}{\partial x} = u''(z)z_x z_h + u'(z)z_{hx} = u''(z)z_x z_h + u'(z)\frac{z_h}{x-x^*}. \tag{6.8}$$

The fact that $u''(z)z_x z_h$ is non-negative for the interval $[0, x^*)$ and non-positive for the interval $(x^*, 1]$ completes the proof. Q.E.D.

The term $\frac{u'(z)z_h}{x-x^*}\left(= \frac{u'[z(x, b_F)]z_h(x, b_F) - u'[z(x^*, b_F)]z_h(x^*, b_F)}{x-x^*}\right)$ is the average change rate of the marginal utility $u'(z)z_h$ from any x to the point x^*. Property 6.4 is same as the assertion that the average change rate is non-increasing in x, because the restriction $z_{xx} = 0$ implies that (6.7) and thus,

$$\partial\left(\frac{u'(z)z_h}{x-x^*}\right)\Big/\partial x = \partial u'(z)z_{hx} / \partial x \le 0, \text{ for all } x \in [0,\ 1] \text{ and } x \neq x^*.$$

Property 6.5. Assume that $u' \ge 0$, $u'' \le 0$, $z_x \ge 0$, $z_{hx} \ge 0$ and $z_{xx} = 0$, then (i) if $u''' \ge 0$, $u'(z)z_h$ is non-positive, non-decreasing and concave in $x \in [0, x^*]$; and (ii) if $u''' \le 0$, $u'(z)z_h$ is non-negative and concave in $x \in [x^*, 1]$.

Proof. Form Property 6.2 and 6.3, we know that $u'(z)z_h$ is non-positive and non-decreasing in $x \in [0, x^*]$, and non-negative for $x \in [x^*, 1]$. Since z is assumed to be linear in x, differentiating (6.6) with respect to x gives

$$u'''(z)z_x^2 z_h + 2u''(z)z_x z_{hx}. \tag{6.9}$$

[51] Since $z_h(x, b_F) = r_1'(b_F)x + r_2'(b_F)$ is assumed to have a single crossing at $x^* \in [0, 1]$, $r_1'(b_F) \neq 0$.

Hence with the given assumptions, (i) for the interval $[0,\ x^*]$, if $u''' \geq 0$ then (6.9) is non-positive and thus $u'(z)z_b$ is concave; and for the interval $[x^*,\ 1]$, if $u''' \leq 0$ then (6.9) is non-positive and thus $u'(z)z_b$ is concave. Q.E.D.

6.3 Comparative Static Analysis

Using the general one-argument decision model (1.1), this section provides general comparative static statements concerning the three types of R-S increases in risk, an ESIR, an L-RSIR, and an L-ESIR, presented in chapter 5. Again, let the supports of CDF's *G* and *F* (assuming that *G* is riskier than *F* in the R-S sense) be the finite intervals $[x_1,\ x_4]$ and $[x_2,\ x_3]$, where $x_1 \leq x_2 \leq x_3 \leq x_4$, respectively. First, consider the set of ESIR shifts defined in Definition 5.3. This class of R-S increases in risk includes, as a special case, the set of RSIR shifts (Property 5.1). Since every ESIR change can be decomposed into a sequence of RSIR shifts (Property 5.3), the result in Theorem 6.1′ is generalized as:

Theorem 6.2. For all risk averse decision makers, $b_F \geq b_G$ if

(a) *G* ESIR *F*

(b) $z_x \geq 0,\ z_{bx} \geq 0,\ z_{bb} \leq 0,\ z_{bxx} \leq 0,$ and $z_{xx} = 0$ when $z_{bb} = 0$.

Proof. By Property 5.3, if *G* ESIR *F* then there exists a series of CDF's $G_1, \cdots, G_n$ such that *G* RSIR G_1 RSIR $\cdots$ G_n RSIR *F*. Combined with the result in Theorem 6.1′, it completes the proof. Q.E.D.

Theorem 6.2 improves the robustness of Black and Bulkley's results without any cost of additional assumptions. Defining an ESIR, the restriction that the pdf's *f* and *g* cross only twice is dropped. Instead, it requires that the pair of CDF's F and G cross only once and that, for the left side of the crossing point, the ratio of F to G should be non-decreasing and, for the right side of the point, the ratio of $(1-F)$ to $(1-G)$ should be non-increasing.

One notable thing is that, as we saw in section 5.2, the ESIR order does not have the property of transitivity. Consider the example of an R-S increase in risk given in Figure 5.11, which shows that G ESIR G_1 and G_1 ESIR F, but not G ESIR F because the CDF's F and G cross three times. However, by Theorem 6.2, we know that $b_G \leq b_{G_1} \leq b_F$. In general, it is obvious that an R-S increase in risk that can be decomposed into a series of ESIR shifts allows one to make the desired comparative static statement provided in Theorem 6.2. This implies that the ESIR order is only a sufficient (but not necessary) condition for obtaining the result in Theorem 6.2. Allowing multiple crossing between the initial and the final CDF's F and G, it is an open question to find the set of admissible R-S increases in risk yielding the comparative static result.

In order to make another general comparative static statement, we investigate a trade-off between the restrictions on the set of decision makers and the set of changes in the random variable. In what follows, the concerned payoff is restricted to be linear in the random variable. We show that, when $z_{xx} = 0$, one can further extend the class of admissible R-S increases in risk with the cost of an additional restriction on the set of decision makers. In the next comparative static result, the set of L-RSIR shifts which includes all RSIR shifts is admissible by adding the restriction that risk averse decision makers have positive third derivative of utility functions. When $z_{bb} = 0$, we have shown in section 6.1 that, for the cases of a corner or an unbounded solution, an arbitrary R-S increase in risk will not increase the choice variable b. In the following theorem, we consider only the case of an interior solution, and thus, to prove that $b_F \geq b_G$ for a specified change in pdf from f to g, it is sufficient to show that

$$Q(b_F) = \int_{x_1}^{x_4} u'[z(x, b_F)]z_b(x, b_F)[f(x) - g(x)]dx \geq 0. \tag{6.4}$$

Theorem 6.3. For all decision makers with utility functions u's such that $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$, $b_F \geq b_G$ if

(a) G L-RSIR F

(b) $z_x \geq 0$, $z_{bx} \geq 0$, $z_{bb} \leq 0$ and $z_{xx} = 0$.

Proof. Let x^* be the value of x satisfying $z_b(x, b_F) = 0$, then x^* exists in the interval $[x_2, x_3]$. With the points k' and k used in Definition 5.2, where $x_2 \leq k' \leq k \leq x_3$, we consider the following three cases:

Case (i): when $k \leq x^* \leq x_3$. Integrating by parts, $Q(b_F)$ in (6.4) can be written as

$$Q(b_F) = \int_{x_1}^{x_4} [u''(z)z_x z_b + u'(z)z_{bx}](G - F)dx$$

With the given assumptions, we know that for the interval $[x_1, x^*]$, $u''(z)z_x z_b + u'(z)z_{bx}$ is non-increasing in x by Property 6.5, and for the interval $[x^*, x_4]$, it has its maximum at $x = x^*$ by Property 6.3. Since $k \leq x^*$, this implies

$$[u''(z)z_x z_b + u'(z)z_{bx}]\Big|_{x \leq k} \geq [u''(z)z_x z_b + u'(z)z_{bx}]\Big|_{x=k} \geq [u''(z)z_x z_b + u'(z)z_{bx}]\Big|_{x \geq k},$$

and by the condition G L-RSIR F which implies $G - F \geq 0$ for all $x \in [x_1, k]$, and $G - F \leq 0$ for all $x \in [k, x_4]$, we have the following inequality,

$$Q(b_F) \geq [u''(z)z_x z_b + u'(z)z_{bx}]\Big|_{x=k} \int_{x_1}^{x_4} (G - F)dx = 0.$$

Case (ii): when $k' \leq x^* \leq k$. Let's rewrite $Q(b_F)$ in (6.4) as

$$Q(b_F) = \int_{x_1}^{x^*} u'(z)z_b(f - g)dx + \int_{x^*}^{k} u'(z)z_b(f - g)dx + \int_{k}^{x_4} u'(z)z_b(f - g)dx.$$

With the given assumptions and by the condition G L-RSIR F,

$$Q(b_F) \geq u'[z(k',b_F)]\int_{x_1}^{x^*} z_b(f-g)dx + u'[z(k,b_F)]\int_{x^*}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx.$$

Adding and subtracting $u'[z(k,\ b_F)]\int_{x_1}^{x^*} z_b(f-g)dx$ in the RHS gives,

$$Q(b_F) \geq \{u'[z(k',\ b_F)] - u'[z(k,\ b_F)]\}\int_{x_1}^{x^*} z_b(f-g)dx$$

$$+ u'[z(k,\ b_F)]\int_{x_1}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx. \qquad (6.10)$$

Integrating by parts,

$$\int_{x_1}^{x^*} z_b(f-g)dx = \int_{x_1}^{x^*} z_{bx}(G-F)dt = z_{bx}\int_{x_1}^{x^*}(G-F)dt \geq 0$$

because z_{bx} does not depend on x and non-negative, and $\int_{x_1}^{x}(G-F)dt \geq 0$ for all $x \in [x_1,\ x_4]$. Thus, by the assumptions $u'' \leq 0$ and $z_x \geq 0$, the first term in (6.10) is non-negative. Integrating the second term in (6.10) by parts and using the assumption $z_{xx} = 0$, we have

$$u'[z(k,b_F)]\int_{x_1}^{k} z_{bx}(G-F)dx = u'[z(k,b_F)]z_{bx}\int_{x_1}^{k}(G-F)dx. \qquad (6.11)$$

Also using integrating by parts, the third term in (6.10) is equal to

$$\int_{k}^{x_4}[u''(z)z_x z_b + u'(z)z_{bx}](G-F)dx.$$

From Property 6.4, it is known that

$$\frac{\partial u'(z)z_b}{\partial x} \leq \frac{u'(z)z_b}{x-x^*}, \text{ for all } x \in (x^*,\ \infty)$$

and since $u'(z)z_b/(x-x^*)$ is non-increasing for all $x \in (x^*,\ \infty)$, we have

$$\int_{k}^{x_4}[u''(z)z_x z_b + u'(z)z_{bx}](G-F)dx \geq \frac{u'[z(k,b_F)]z_\alpha(k,b_F)}{k-x^*}\int_{k}^{x_4}(G-F)dx. \qquad (6.12)$$

From (6.7), it is shown that the linearity assumption $z_{xx} = 0$ implies, for all $x \neq x^*$,

$$\frac{z_b}{x-x^*} = z_{bx}. \qquad (6.13)$$

Hence, from (6.11), (6.12) and (6.13),

$$u'[z(k,b_F)]\int_{x_1}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx \geq u'[z(k,b_F)]z_{bx}\int_{x_1}^{x_4}(G-F)dx = 0. \quad (6.14)$$

Therefore we have the desired result that

$$Q(b_F) \geq \{u'[z(x_2,b_F)] - u'[z(k,b_F)]\}\int_{x_1}^{x^*} z_b(f-g)dx \geq 0.$$

Case (iii): when $x_2 \leq x^* \leq k'$. Consider the sign of the expression $\int_{x_1}^{k'} z_b(f-g)dx$.

First, assume that $\int_{x_1}^{k'} z_b(f-g)dx \geq 0$. Rewriting $Q(b_F)$ in (6.4) as

$$Q(b_F) = \int_{x_1}^{k'} u'(z)z_b(f-g)dx + \int_{k'}^{k} u'(z)z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx.$$

Using the given assumptions and the L-RSIR condition, we have

$$Q(b_F) \geq u'[z(x^*,b_F)]\int_{x_1}^{k'} z_b(f-g)dx + u'[z(k,b_F)]\int_{k'}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx.$$

Adding and subtracting $u'[z(k,\ b_F)]\int_{x_1}^{k'} z_b(f-g)dx$ in the RHS gives,

$$Q(b_F) \geq \{u'[z(x^*,\ b_F)] - u'[z(k,\ b_F)]\}\int_{x_1}^{k'} z_b(f-g)dx$$

$$+ u'(z(k,\ b_F))\int_{x_1}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx. \quad (6.15)$$

Thus (6.14) and the assumption $\int_{x_1}^{k'} z_b(f-g)dx \geq 0$ imply $Q(b_F) \geq 0$. Now assuming that $\int_{x_1}^{k'} z_b(f-g)dx \leq 0$, let's rewrite $Q(b_F)$ in (6.4) as

$$Q(b_F) \geq \int_{x_1}^{k'} u'(z)z_b(f-g)dx + u'[z(k,b_F)]\int_{k'}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx.$$

Adding and subtracting $u'[z(k,\ b_F)]\int_{x_1}^{k'} z_b(f-g)dx$ in the RHS gives,

$$Q(b_F) \geq \int_{x_1}^{k'} u'(z)z_b(f-g)dx - u'[z(k,b_F)]\int_{x_1}^{k'} z_b(f-g)dx$$

$$+ u'(z(k,\ b_F))\int_{x_1}^{k} z_b(f-g)dx + \int_{k}^{x_4} u'(z)z_b(f-g)dx.$$

From (6.14) and the assumption $\int_{x_1}^{k'} z_h(f-g)dx \le 0$,

$$Q(b_F) \ge \int_{x_1}^{k'} u'(z)z_h(f-g)dx = -\int_{x_1}^{x_2} u'(z)z_h g dx + \int_{x_2}^{k'} u'(z)z_h(f-g)dx. \qquad (6.16)$$

The first term in the RHS of (6.16) is non-negative, and since, according to Definition 5.2, the function h is non-decreasing and strictly positive for all $x \in [x_2, k']$, the second term can be written as

$$\int_{x_2}^{k'} u'(z)z_h(f-g)dx = \int_{x_2}^{k'} u'(z)z_h(1-\frac{1}{h})fdx \ge [1-\frac{1}{h(x^*)}]\int_{x_2}^{k'} u'(z)z_h f dx.$$

Since $h(x^*) \le 1$ and $\int_{x_1}^{k'} u'(z)z_h g dx \le 0$ by Property 6.1, $Q(b_F) \ge 0$. Q.E.D.

Adding the structural restriction $z_{xx} = 0$ is not very harmful for comparative static analysis because all the economic decision problems introduced in chapter 1 follow the restriction. Compared with the comparative static result in Black and Bulkley's analysis, the result in Theorem 6.3 is an example of a trade-off between the restrictions on the set of decision makers and the set of admissible set of changes in distribution. Our result extends the class of admissible R-S increases in risk with the cost of adding additional restriction on the risk preferences of decision makers, such as $u''' \ge 0$. Following the economic literature about the analysis under uncertainty, this set of decision makers includes utility functions representing quite plausible preferences, such as those exhibiting decreasing absolute risk aversion (DARA) which is a widely accepted property of a risk averse decision maker.

As the last result developed in this section, the admissible R-S increases in risk for the comparative static result in Theorem 6.3 is extended to the set of L-ESIR shifts which is larger than the set of L-RSIR shifts (Property 5.2). Property 5.4 gives an easy and direct generalization of the result in Theorem 6.3.

Theorem 6.4. For all decision makers with utility functions u's such that $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$, $b_F \geq b_G$ if

(a) G L-ESIR F

(b) $z_x \geq 0$, $z_{hx} \geq 0$, $z_{hh} \leq 0$ and $z_{xx} = 0$.

Proof. It is clear by Theorem 6.3 and Property 5.4.

Accordingly, Theorem 6.4 improves the robustness of the result in Theorem 6.3, and the efficiency gain is obtained from extending the set of admissible R-S increases in risk yielding the result, without any cost of additional assumptions. With the restriction of linearity in the payoff function, the result also shows a trade-off between the restrictions on the set of decision makers and the set of admissible set of changes in distribution. Compared with the result in Theorem 6.2, it is known that the result in Theorem 6.4 contains a larger set of changes in distribution and a smaller set of decision makers. This is another example that supports the general statement made in chapter 1 that, when relatively strong restrictions are imposed on one of the components composing a decision model, the derived comparative static results are usually associated with relatively weak restrictions on the other components.

In this chapter, comparative static theorems for subsets of the set of R-S increases in risk are developed. Following the ratio approach, all the subsets are defined by imposing restrictions on the ratio between a pair of pdf's or between a pair of CDF's. Those developed comparative static results are associated with relatively weak restrictions on the risk preferences of decision makers, i.e., the set of all risk averse agents or the set of risk averse agents with non-negative third derivative of utility function. In this sense, we refer all the subsets CDF changes concerned in this chapter to "R-S increases in risk in the strong sense."

Chapter 7

SUMMARY AND SOME REMARKS FOR COMPARATIVE STATICS WITH SUBSETS OF SSD SHIFTS

In this dissertation, the analysis of comparative statics under uncertainty is continued. Using the general one-argument decision model of the form (1.1), general comparative static statements are made for the subsets of FSD changes presented in chapter 3, and for the subsets of R-S increases in risk given in chapter 5. As a special feature in defining these subsets of changes in distribution, the ratio between an initial and a final pdf's or CDF's is often restricted to be monotonic increasing or decreasing. This chapter shows that the results obtained in previous chapters can be applied for cases of second-degree stochastic dominance (SSD) shifts. Following the ratio approach, special subsets of general SSD changes are defined and general comparative static statements are made for these subsets. We start this chapter by giving a summary for the results developed in the previous chapters.

7.1 Summary

Within the context of ratio approach, most subsets of CDF changes concerned in this dissertation are defined by imposing restrictions on the ratio between an initial and a final pdf's or CDF's. They include six subsets of FSD shifts and four subsets of R-S increases in risk. The general comparative static statement derived for each of these subsets contains a relatively large class of decision makers, such as 'all individuals with non-decreasing utility functions,' 'all risk averse agents,' or 'all risk averse agents with non-negative third derivative of utility functions.' In sub-section 7.1.1, the results provided in chapters 3 and 5 are summarized and the comparative static results developed in chapters 4 and 6 are summarized in sub-section 7.1.2.

7.1.1 Relationships among Subsets of CDF Changes

First, consider the six types of FSD changes defined in section 3.1. Each definition specifies a particular type of shift from a CDF $G(x)$ to another $F(x)$ or, correspondingly, from a pdf $g(x)$ to another $f(x)$ such as:

- F MLR G (Definition 3.1): the ratio of f to g is monotonic increasing.
- F L-MLR G (Definition 3.3): decreasing proportion of the probability mass on points to the left of a given point is transferred to points to the right of that point.
- F R-MLR G (Definition 3.4): probability mass on points to the left of a given point is transferred to points to the right of that point, requiring the ratio of f to g to be monotonic increasing for the right-side of the point.
- F MPR G (Definition 3.2): the ratio of F to G is monotonic increasing.
- F k-FSD G (Definition 3.5): given a point k, $f(x) \geq g(x)$ for all $x \geq k$.
- F k-MPR G (Definition 3.6): given a point k, the ratio of F to G is monotonic increasing for $x \geq k$, and the ratio of F to G is maximized at k for $x \leq k$.

Section 3.2 examines several relationships and properties among these subsets. As shown in Figure 7.1, basic relationships among the subsets are:

- MLR $\Rightarrow$ L-MLR $\Rightarrow$ MPR $\Rightarrow$ c-MPR $\Rightarrow$ FSD (Property 3.1).

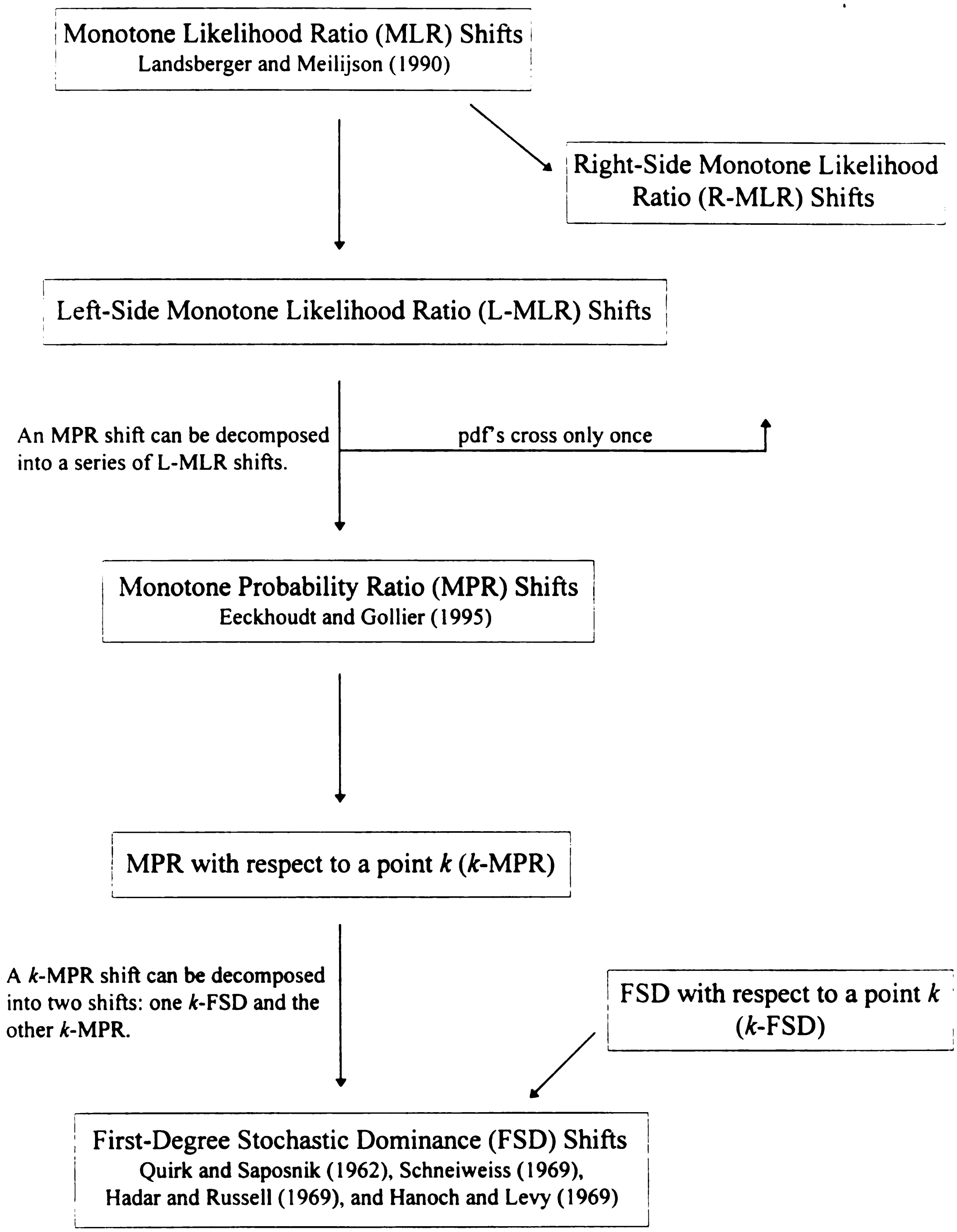

Figure 7.1. Relationships among subsets of FSD shifts.

· MLR $\Rightarrow$ R-MLR $\Rightarrow$ FSD.

· k-FSD $\Rightarrow$ FSD.

Regarding the property of transitivity, it is shown that each of the four CDF orders, MLR, MPR, k-FSD, and k-MPR are transitive:

· The MLR order is transitive (Property 3.2).

· The MPR order is transitive (Property 3.3).

· The k-FSD order is transitive (Property 3.4).

· The k-MPR order is transitive (Property 3.5).

Transitivity does not hold for the L-MLR order. Instead, since an L-MLR shift is also an MPR shift, a series of L-MLR shifts leads to an MPR shift. In addition to this, an important relationship exists between these two CDF orders.

· An MPR shift can always be decomposed into a series of L-MLR shifts (Property 3.6).

This allows the comparative statics for the set of L-MLR shifts to be also admissible for the set of MPR shifts.

As the last property concerned in section 3.2, a relationship exists between the k-MPR and k-FSD orders.

· A k-MPR shift can always be decomposed into an MPR and a k-FSD shifts (Property 3.7).

Given the general comparative static theorems for the set of MPR and for the set of k-FSD shifts, the last two properties extend the allowable set of changes in CDF without any additional assumption.

Next, consider the four types of R-S increases in risk defined in section 5.1. Each definition specifies a particular type of shift from a CDF $F(x)$ to another $G(x)$ or, correspondingly, from a pdf $f(x)$ to another $g(x)$ such as:

· G RSIR F (Definition 5.1): a probability mass from a centric interval is moved to both tails, requiring the ratio of f to g to be monotonic increasing for the left-tail interval and monotonic decreasing for the right-tail interval.

· G L-RSIR F (Definition 5.2): requires only the left-side monotonicity restriction on the pdf ratio in an RSIR.

· G ESIR F (Definition 5.3): when $G \geq F$, the ratio of F to G is monotonic increasing, and when $G \leq F$, the ratio of $1-F$ to $1-G$ is monotonic decreasing.

· G L-ESIR F (Definition 5.4): requires only the left-side monotonicity restriction on the CDF ratio in an ESIR.

Section 5.2 gives formal proofs for the basic relationships among these subsets, and they are given as:

· RSIR $\Rightarrow$ ESIR $\Rightarrow$ L-ESIR $\Rightarrow$ MPS (Property 5.1).

· RSIR $\Rightarrow$ L-RSIR $\Rightarrow$ L-ESIR $\Rightarrow$ MPS (Property 5.2).

Figure 7.2 shows these relationships including the concepts of a 'global increase in risk,' a 'mean-preserving truncation,' and a 'strong increase in risk.'

In general, the above four CDF orders do not have the property of transitivity. However there are important relationships among the orders and section 5.2 also provides the following two properties which play the key role for developing general comparative static statements in chapter 6.

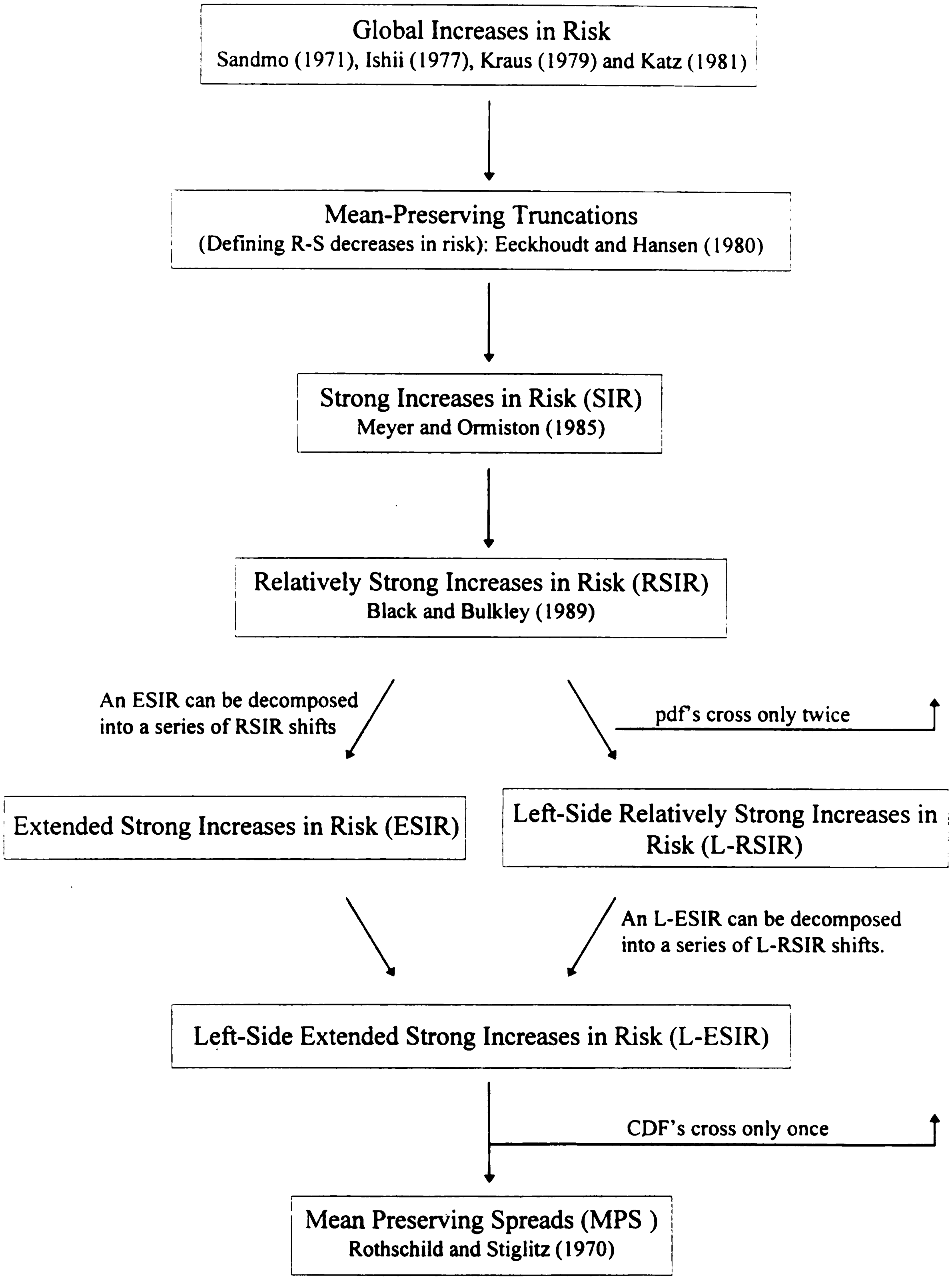

Figure 7.2. Relationships among subsets of R-S increases in risk.

· An ESIR can always be decomposed into a series of RSIR (Property 5.6).

· An L-ESIR can always be decomposed into a series of L-RSIR (Property 5.7).

With these properties, the general comparative static statement made for the set of RSIR (L-RSIR) shifts is generalized into the set of all ESIR (L-ESIR) shifts, without further assumptions required.

7.1.2 Comparative Static Analysis

Regarding the subsets of FSD shifts, the comparative static results developed in chapter 4 are summarized in Table 7.1. Each of the results contains restrictions on the following three components: (1) the set of changes in randomness, (2) the set of decision makers, and (3) the structure of the concerned decision model. It is obvious from the table that, as the restrictions on the type of CDF change become less demanding, the required restrictions on the risk preferences of decision makers and the structure of the concerned decision model are more demanding.

· Result (i) in Theorem 4.1 contains all MLR shifts, all the individuals with non-decreasing utility functions, and can be applied for decision models with linear or non-linear payoff functions – a generalization of the comparative static result in Landsberger and Meilijson (1990).

· Result (ii) in Theorem 4.2 contains, compared with the result (i), the same set of decision makers and a larger set of changes in CDF, and adds a structural restriction that the payoff is linear in the choice variable b – a trade-off between the restrictions on the components (1) and (3).

· Result (iii) in Theorem 4.3 contains, compared with the result (i), a larger set of changes in CDF and a smaller set of decision makers, and can be

Table 7.1. Comparative static results regarding subsets of FSD shifts.

	Types of shifts between CDF's F and G	Sufficient conditions for $b_G \le b_F$
i)	F MLR G	$u' \ge 0,\ z_{bx} \ge 0$
ii)	F L-MLR G with $k \in [c,\ x_3]$ F R-MLR G with $k \in [x_2,\ c]$	$u' \ge 0,\ z_{bx} \ge 0$ z is linear in b and $z_b(c,\ b) = 0$
iii)	F L-MLR G	$u' \ge 0,\ u'' \le 0,$ $z_x \ge 0,\ z_{bx} \ge 0$
iv)	F MPR G	$u' \ge 0,\ u'' \le 0,$ $z_x \ge 0,\ z_{bx} \ge 0$
v)	F c-FSD G F c-MPR G	$u' \ge 0,\ u'' \le 0,\ z_x \ge 0,\ z_{bx} \ge 0,$ z is linear in b and $z_b(c,\ b) = 0$

applied for decision models with linear or non-linear payoff functions – a trade-off between the restrictions on the components (1) and (2).

- Result (iv) in Theorem 4.4 contains, compared with the result (iii), a larger set of changes in CDF and the same restrictions on the other components – a generalization of the result (iii), which is directly obtained from Property 3.6.

- Result (v) in Theorem 4.5 contains, compared with the result (iv), the same set of decision makers and a larger set of changes in CDF, and adds a structural restriction that the concerned payoff is linear in the choice variable – a trade-off between the restrictions on the components (1) and (3), which is obtained from Property 3.7.

The comparative static results developed in chapter 6 are summarized in Table 7.2. These results concern the subsets of R-S increases in risk, and as in the cases of subsets of FSD shifts, each of the results contains restrictions on the three components (1), (2), and (3). The first four[52] comparative static results are from the literature and the last three are provided in chapter 6. As the table shows, all the results (i) - (v) contain the same restrictions on the components (2) and (3). Only the set of admissible changes in CDF is generalized in the order: global increases in risk → mean-preserving truncations → strong increases in risk → relatively strong increases in risk → extended strong increases in risk. The restrictions contained in the comparative static results (iv) - (vii) are compared as:

· Result (iv) in Theorem 6.1′ contains all RSIR changes, all risk-averse agents, and can be applied for decision models with linear or non-linear payoff functions – a generalization of the results (i) - (iii).

· Result (v) in Theorem 6.2 contains, compared with the result (iv), a larger set of changes in CDF and the same restrictions on the other components – a generalization of the result (iv), which is directly obtained from Property 5.3.

· Result (vi) in Theorem 6.3 contains, compared with the result (iv), a larger set of changes in CDF and a smaller set of decision makers, and adds a structural restriction that the concerned payoff is linear in the random variable – a trade-off between the restrictions on the component (1), and the components (2) and (3).

[52] The previous papers producing the results (i) - (iv) are: Sandmo (1971), Ishii (1977), Kraus (1979), and Katz (1981) for the result (i); Eeckhoudt and Hansen (1980) for the result (ii); Meyer and Ormiston (1985) for the result (iii); and Black and Bulkley (1989).

Table 7.2. Comparative static results regarding subsets of R-S increases in risk.

	Types of shifts between CDF's F and G	Sufficient conditions for $b_G \leq b_F$
i)	Global changes in risk from a certainty (F) to a risky G	$u' \geq 0,\ u'' \leq 0,$ $z_x \geq 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0,$* $z_{hxx} \leq 0$
ii)	Mean-Preserving Truncations from G to F	$u' \geq 0,\ u'' \leq 0,$ $z_x \geq 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0,\ z_{hxx} \leq 0$
iii)	G SIR F	$u' \geq 0,\ u'' \leq 0,$ $z_x \geq 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0,\ z_{hxx} \leq 0$
iv)	G RSIR F	$u' \geq 0,\ u'' \leq 0,$ $z_x \geq 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0,\ z_{hxx} \leq 0$
v)	G ESIR F	$u' \geq 0,\ u'' \leq 0,$ $z_x \geq 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0,\ z_{hxx} \leq 0$
vi)	G L-RSIR F	$u' \geq 0,\ u'' \leq 0,\ u''' \geq 0$ $z_x \geq 0,\ z_{xx} = 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0$
vii)	G L-ESIR F	$u' \geq 0,\ u'' \leq 0,\ u''' \geq 0$ $z_x \geq 0,\ z_{xx} = 0,\ z_{hx} \geq 0,\ z_{hh} \leq 0$

* In Table 7.2, when the concerned payoff is linear in the choice variable, the decision model is assumed to follow the form (6.5), that is, $z_{xx} = 0$ when $z_{hh} = 0$. Decision models of the linear form can be included in the first four comparative static results with the analysis given by Dionne, Eeckhoudt and Gollier (1993) – see section 6.1.

· Result (vii) in Theorem 6.4 contains, compared with the result (vi), a larger set of changes in CDF and the same restrictions on the other components – a generalization of the result (vi), which is directly obtained from Property 5.4.

7.2 Remarks for Comparative Statics Regarding Subsets of SSD Shifts

The second-degree stochastic dominance (SSD) order specifies a more general type of changes in distribution of a random variable than the FSD order or the concept of an R-S increase in risk defines. In section 2.1, we indicated that an FSD shift is also an SSD shift, and that an R-S increase in risk is a special case of an SSD. It is generally known that an SSD shift can be decomposed into a combination of an FSD shift and an R-S decrease in risk (mean-preserving contract (MPC)). This is formally stated by Hadar and Seo (1990) who prove the statement that "a pair of CDF's F and G satisfies F SSD G if and only if there exists an intermediate CDF F^0 such that F MPC F^0 and F^0 FSD G. In other words, any SSD change can be decomposed into an FSD shift and an MPC shift. Using the results just summarized for subsets of general FSD and MPC changes, this can be used for defining subsets of the set of SSD shifts which are admissible for making general comparative static statements. Two examples follow.

Considering SSD shifts, this section assumes that the final distribution F dominates the initial distribution G in the sense of second-degree (denoted by F SSD G). Both the CDF's G and F are assumed to have their points of increase in bounded intervals. Let the support of $G(x)$ be a finite interval $[x_1, x_4]$ and the support of $F(x)$ be another finite interval $[x_2, x_3]$ where $x_1 \le x_2$. We drop the restriction that $x_3 \le x_4$ which is required in section 5.1. Similar to the case of an R-S increase in risk, we assume that: for continuous case, $G(x) > 0$ for all $x \in (x_1, \infty)$, $1 - G(x) > 0$ for all $x \in (-\infty, x_4)$,

$F(x) > 0$ for all $x \in (x_2, \infty)$, and $1 - F(x) > 0$ for all $x \in (-\infty, x_3)$; and for discrete case, $G(x) > 0$ for all $x \in [x_1, \infty)$, $1 - G(x) > 0$ for all $x \in (-\infty, x_4]$, $F(x) > 0$ for all $x \in [x_2, \infty)$, and $1 - F(x) > 0$ for all $x \in (-\infty, x_3]$.

In what follows, first we define subsets of SSD shifts which can be understood as the combinations of FSD shifts and R-S increases in risk, that are generated by the CDF orders defined in sections 3.1 and 5.1, respectively. Then the comparative static statements for these subsets are derived in sub-section 7.2.2.

7.2.1 Subsets of SSD Shifts

In this sub-section two concepts of SSD shifts are introduced. Relaxing the equal means condition, the conditions used in Definition 5.3 (for an ESIR) and Definition 5.4 (for an L-ESIR) are modified to give the following two types of SSD shifts.

Definition 7.1. $F(x)$ represents an extended strong SSD shift from $G(x)$ (denoted by F ESSSD G) if

(a) $\int_{x_1}^{y} [G(x) - F(x)]dx \geq 0$ for all $y \in [x_1, \infty)$

(b) there exists a point $k \in [x_2, x_3]$ such that

$F(x) \leq G(x)$ for all $x \in [x_2, k)$ and $F(x) \geq G(x)$ for all $x \in [k, x_3]$

(c) for all $x \in [x_2, k)$, there exists a non-decreasing function $H_1 : [x_2, k) \to [0, 1]$ such that

$F(x) = H_1(x)G(x)$

(d) for all $x \in [k, x_3]$, there exists a non-increasing function $H_2 : [k, x_3] \to [0, 1]$ such that

$1 - F(x) = H_2(x)[1 - G(x)]$.

Definition 7.2. $F(x)$ represents a left-side extended strong SSD shift from $G(x)$ (denoted by F L-ESSSD G) if

(a) $\int_{x_1}^{y} [G(x) - F(x)]dx \geq 0$ for all $y \in [x_1, \infty)$

(b) there exist a point $k \in [x_2, x_3]$ and a non-decreasing function $H:[x_2, k) \to [0, 1]$ such that $F(x) = H(x)G(x)$ for all $x \in [x_2, k)$ and $G(x) \leq F(x)$ for all $x \in [k, x_3]$.

Condition (a) in both definitions is just the SSD condition shown in Definition 2.2. The other conditions in Definition 7.1 and 7.2 are same as the ones in Definition 5.3 and 5.4, respectively.

As a special case, if the CDF's *F* and *G* have a same mean, then the relationship *F* ESSSD *G* just implies *G* ESIR *F*, and *F* L-ESSSD *G* means *G* L-ESIR *F*. In addition, both the concepts include, as a special case, an MPR shift. Consider the case where $x_4 < x_3$. Condition (b) in both definitions implies that the point *k* must be equal to x_3, to satisfy the condition that $G(x) \leq F(x)$ for all $x \in [k, x_3]$. In this case, both an ESSSD and an L-ESSSD are reduced to MPR shifts.

The properties associated with the ESIR and the L-ESIR orders can be adopted for these two SSD orders. When $x_3 \leq x_4$, the CDF's *F* and *G* cross only once at the point $k \in [x_2, x_3]$, and an ESSSD (L-ESSSD) shift can be understood as the combination of an MPR dominant shift for the left-side of the point *k* and another dominated FSD shift for the right-side of the point. More importantly, there are properties which allow us to provide general comparative static statements for the two subsets of SSD shifts, which are given in the next sub-section. The properties are: (i) an ESSSD shift can always be decomposed into an MPR shift and an ESIR; and (ii) an L-ESSSD shift can always be decomposed into an MPR shift and an L-ESIR.

Property 7.1. If *F* ESSSD *G*, there always exists a CDF G_1 such that *F* MPR G_1, and *G* ESIR G_1.

Proof. If *F* ESSSD *G* and $x_4 < x_3$, then *F* and *G* meets at the point x_3 and this implies that an ESSSD shift is just an MPR shift. In this case, the claim in Property 7.1 is satisfied by letting $G_1 = G$. For the case where $x_3 \le x_4$, let's define an intermediate CDF $G_1(x)$ such that,

$$G_1(x) = \begin{cases} \lambda G(x), \text{ when } x < s \\ F(x), \text{ when } x \ge s \end{cases} \tag{7.1}$$

where $\lambda = F(s)/G(s)$ and s is a selected point in the interval $[x_2, k]$ such that G_1 has a same mean as G. Then, with the conditions in Definition 7.1, it is easy to see that F MPR G_1, and G ESIR G_1. Q.E.D.

Property 7.2. If F L-ESSSD G, there always exists a CDF G_1 such that F MPR G_1, and G L-ESIR G_1.

Proof. Similar to the case of ESSSD in Property 7.1, if F L-ESSSD G and $x_4 < x_3$, then this is just an MPR shift and the claim in Property 7.2 is satisfied by letting $G_1 = G$. When $x_3 \le x_4$, define an intermediate CDF $G_1(x)$ as in (7.1). Then, with the conditions in Definition 7.2, it is easy to see that F MPR G_1, and G L-ESIR G_1. Q.E.D.

Two graphical examples are given to illustrate the results in these two properties. The shift from a CDF G to F shown in Figure 7.3 satisfies F ESSSD G, and the one shown in Figure 7.4 satisfies F L-ESSSD G. The figures also show that there exists an intermediate G_1 such that F MPR G_1 in both examples and G ESIR G_1 in Figure 7.3 and G L-ESIR G_1 in Figure 7.4.

7.2.2 Comparative Statics for Subsets of SSD Shifts

For each of the subsets of ESSSD shifts and L-ESSSD shifts, the general comparative static statements are obtained from the properties given in sub-section 7.2.1. Since a shift from G to F satisfying F ESSSD G can always be decomposed into two

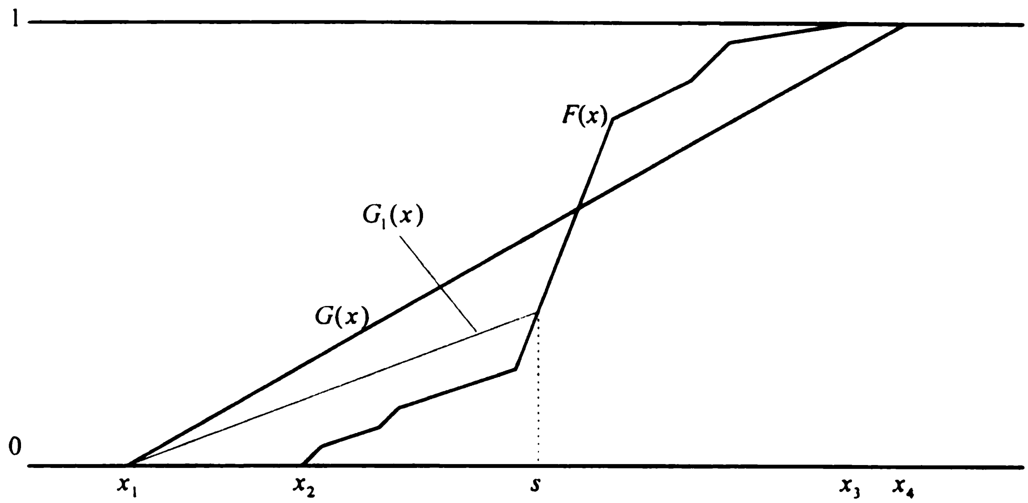

Figure 7.3. An ESSSD shift that is the sum of an MPR and an ESIR.

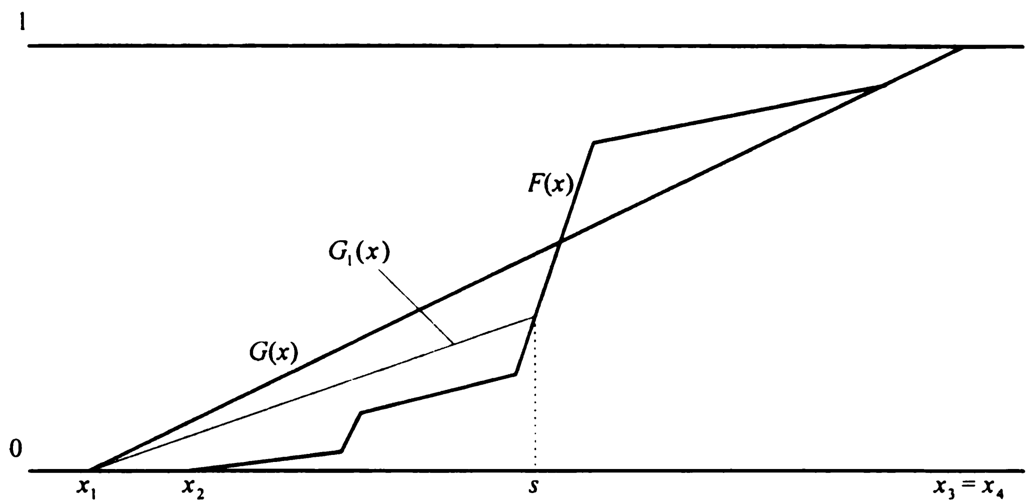

Figure 7.4. An L-ESSSD shift that is the sum of an MPR and an L-ESIR.

dominant shifts, one MPR and the other ESIR,[53] a general comparative static statement regarding an ESSSD shift can be derived from combining the two results in Theorem 4.4 and 6.2.

Theorem 7.1. For all risk averse decision makers, $b_F \geq b_G$ if

(a) F ESSSD G

(b) $z_x \geq 0$, $z_{bx} \geq 0$, $z_{bb} \leq 0$, $z_{bxx} \leq 0$, and $z_{xx} = 0$ when $z_{bb} = 0$.

Proof. Given Property 7.1, the comparative static results in Theorem 4.4 and 6.2 complete the proof. Q.E.D.

Similarly, for the subset of L-ESSSD shifts, we combine the results in Theorem 4.4 and 6.4. That is, adding the restriction that the concerned payoff is linear in the random variable, a general comparative static statement regarding an L-ESSSD shift can be derived from Property 7.2.

Theorem 7.2. For all decision makers with utility functions u's such that $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$, $b_F \geq b_G$ if

(a) F L-ESSSD G

(b) $z_x \geq 0$, $z_{bx} \geq 0$, $z_{bb} \leq 0$ and $z_{xx} = 0$.

Proof. Given Property 7.2, the comparative static results in Theorem 4.4 and 6.4 complete the proof. Q.E.D.

We finish this section by adding some remarks on the comparative static analysis regarding the subsets of SSD shifts. Transitivity does not hold for both the ESSSD and the L-ESSSD order. That is, the relationship F_3 ESSSD (L-ESSSD) F_2 ESSSD (L-

[53] Here an ESIR dominant shift means an R-S decrease in risk.

ESSSD) F_1 does not necessarily implies F_3 ESSSD (L-ESSSD) F_1. This implies that the admissible set of changes in CDF can be extended to other shifts that are not generated by the given two CDF orders. Figure 7.5 gives an example of this case, showing that the shift from G to F, while it does not satisfy F ESSSD G, allow the desirable comparative static result in Theorem 7.1 because the shift can be decomposed into two shifts such that F ESSSD G_1 ESSSD G.

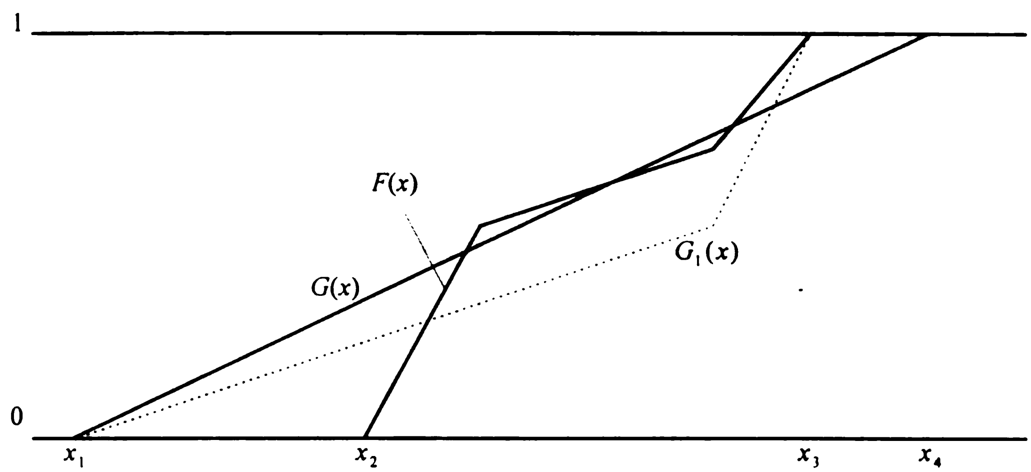

Figure 7.5. A shift showing that F ESSSD G_1 ESSSD G but not F ESSSD G.

Another extension for the set of admissible SSD shifts is possible when we restrict the concerned payoff to be linear in the choice variable. In section 4.2, we showed that, regarding a decision model of the form (4.10) in which a point c satisfying $z_b(c, b) = 0$ exists, the desirable comparative static property holds for the combined set of all c-FSD shifts and c-MPR shifts (Theorem 4.5). This can be shown by the fact that, given a point c, a c-MPR shift can always be decomposed into an MPR shift and a c-FSD shift (Property 3.7). In the same way, if the concerned decision model follows the form (4.10), the result in Theorem 7.1 (7.2) also holds for any shift that can be decomposed into two shifts, one ESSSD (L-ESSSD) and the other c-FSD. Graphical examples showing these

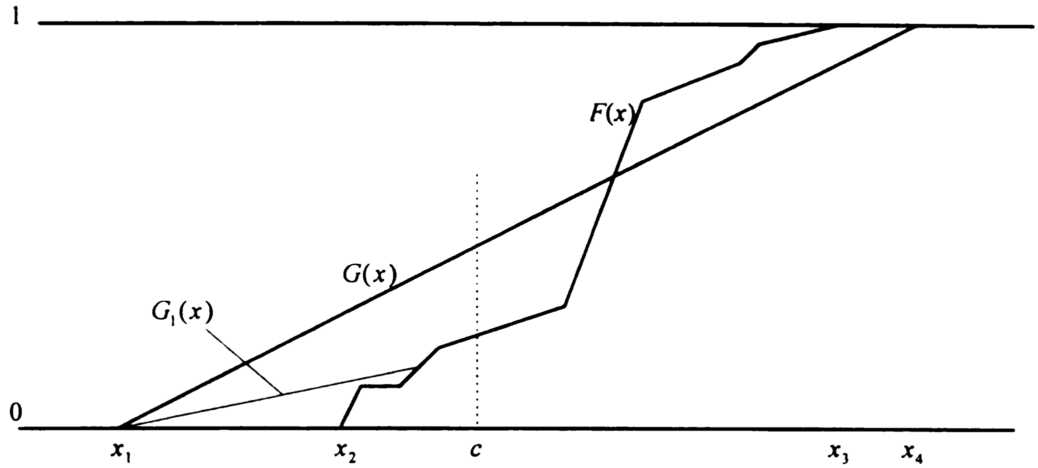

Figure 7.6. A shift that is the sum of an ESSSD and a *c*-FSD.

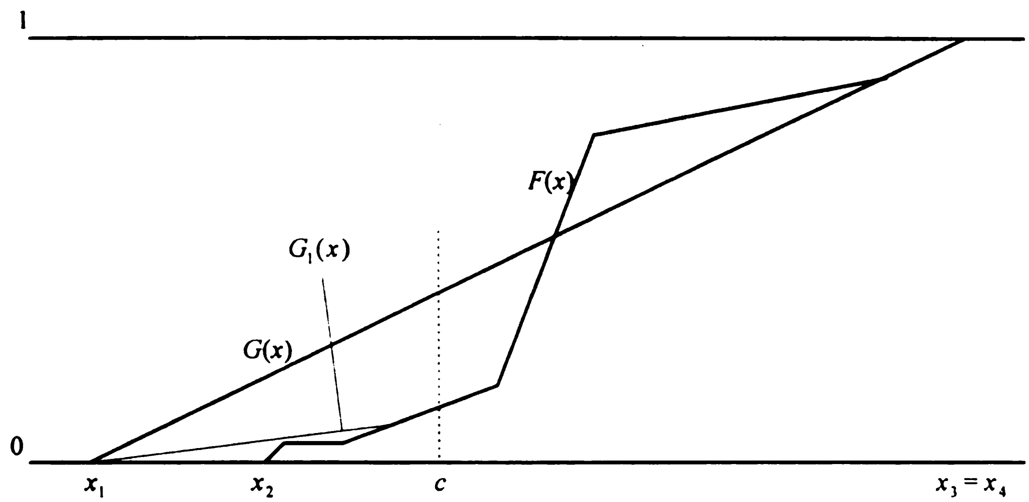

Figure 7.7. A shift that is the sum of an L-ESSSD and a *c*-FSD.

cases are given in Figure 7.6 and 7.7. While both the examples do not satisfy any CDF order concerned in this dissertation, each shift can be admissible for the comparative static results in Theorem 7.1 and 7.2, respectively. The former gives an example of a shift that can be decomposed into two shifts, one *c*-FSD and the other ESSSD, and the latter shows an example that can be decomposed into two shifts, one *c*-FSD and the other L-ESSSD.

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