ANALYSIS OF THE LIGHT CURVES OF TEN ECLIPSING BINARY SYSTEMS

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ABSTRACT

ANALYSIS OF THE LIGHT CURVES OF
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By

Deanne Dorothy Proctor

Computer programs for the Fourier analysis, rectification, and solution of eclipsing binary light curves have been written. Both Kopal's method and the method of differential corrections have been generalized to include third light. The method of differential corrections has been further generalized to include orbital eccentricity directly.

Synthetic light curves were used to validate the computer programs, as well as to determine the effect of dispersion and number of observations on the ability to extract the desired parameters. Analysis of synthetic data indicated limb-darkening coefficients may be extracted from observations of sufficient accuracy and density. This conclusion was found to hold for partial as well as completely eclipsing systems. In addition, it has been found possible to extract values of third light. In some cases, however, correlation between parameters, combined with observations of insufficient quality or quantity, may prevent convergence.

The data from 10 eclipsing binary systems have been rectified and subsequently analyzed using differential corrections. The systems are CO Lacertae, CM Lacertae, RX Arietis, V338 Herculis, Y Leonis, RW Monocerotis, BR Cygni, BV 430, BV 412, and SW Lyncis.

It was often found necessary to solve the light curves for each combination of assumptions as to type of primary minimum (occultation

or transit) and possible presence of third light. Calculation and comparison of σ(est.) and σ(cal.), the estimated and calculated standard deviations, proved valuable in the determination of convergence. Equality of the standard deviation of the Fourier analysis and the standard deviation of the entire light curve, to within their probable errors, indicated the adequacy of the fit for each curve. For those light curves for which b was varied, choice of b, the exponent of the light in the weight, did not seem to cause significant change in the parameters obtained.

Convergence of the iterative procedure was obtained for the systems CO Lacertae, CM Lacertae, RX Arietis, and Y Leonis. Convergence for the V curves of BV 412 and BV 430 was also satisfactory. However, convergence of the B curves of these two systems was obtained only if the number of variables included in the differential corrections was limited to six. V338 Herculis and RW Monocerotis exhibited satisfactory convergence; however, the standard deviations of the Fourier analyses for these light curves were not in good agreement with the respective standard deviations obtained from the differential corrections analysis. The V light curves of BR Cygni and SW Lyncis exhibited satisfactory convergence. The B light curves did not. Further observation of V338 Herculis, RW Monocerotis, BR Cygni, and SW Lyncis is recommended.

Of the ten systems studied, two (CO Lacertae and BR Cygni) showed evidence of third light.

Limb-darkening coefficients resulting from the analyses are compared to the theoretical values. Results for limb-darkening coefficients in V show satisfactory agreement with theory, while limb-darkening coefficients in B show more scatter.

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By

Deanne Dorothy Proctor

A THESIS

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In memory of
my father

Charles Albert Blake

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I. INTRODUCTION

A. Historical Aspects

The term double star is generally applied to pairs of stars that appear single to the unaided eye, but are resolvable with the aid of a telescope. The discovery of the first double star, around 1650, was made by Riccioli (Aitken 1964, p. 1). Several dozen double stars were discovered in the following century and in 1767 it was first suggested that the phenomenon was due to something other than chance projection. John Michell (1767) suggested actual physical association of the members of some double stars. This was confirmed 36 years later by William Herschel. Herschel (1803) presented results of measurements and analyses of the relative position of the components of six double stars. He concluded that certain double stars are true binary systems, that is, systems of physically associated stars. At present we have calculations for the orbits of over 500 such systems. (The catalogue of Baize (1950) contains calculated orbits for 252 visual binaries.)

Stars exhibiting variation in brightness had been observed for hundreds of years when, in 1783, John Goodricke suggested an eclipsing nature for some of the variables. In that year Goodricke reported observing periodic minima in the light of Algol and suggested that the cause of the loss of light was the interposition of a body revolving around Algol. Proof of the binary nature of Algol came with spectroscopic investigations of radial velocities. Doppler in 1842 presented his formula for the shift in the wavelength of light as a function of relative velocity of the source and observer. Then, Vogel (1889) observed the periodic shifting of the radial velocity of Algol and noted that the times of conjunction obtained from the radial velocity curve coincided with the minima of light.

Estimates for the percentage of double or multiple stars in the vicinity of the solar system range from 30 to 50% of the total population (Kuiper 1935). No preferential orientation has been found for the inclination of the orbital planes for binary stars (Chang 1929, Finsen 1933, Huang and Wade 1966). Based on this assumption, the probability P of a system having an inclination between i_1 and i_2 is

$$P = \cos i_1 - \cos i_2 \tag{1.1}$$

(Binnendijk 1970). Eclipsing binaries should not be uncommon. As of 1968, over 20,000 variable stars had been catalogued. Of those stars, 4062 have been classified as eclipsing binaries (Kukarkin et. al. 1969). (The catalogue of Koch, Plavec, and Wood (1970) contains the results of the analysis of 216 eclipsing binary systems.)

From observations of the light of an eclipsing binary as a function of time it is possible to extract information regarding the physical and geometrical properties of the system. This is done by adopting a physically reasonable model and expressing the theoretical value of the light as a function of time in terms of the related parameters. The parameters of the model are then adjusted to obtain the best fit between the theoretical light curve and the observed light curve. The model is discussed in the next section.

B. Discussion of Model

The members of a binary system are distorted by tidal action and rotation. As a first approximation, each component may be regarded as a tri-axial ellipsoid (Jeans 1928, p. 225). The adopted model for the binary system consists of a pair of similar tri-axial ellipsoids. The errors introduced by the assumption of similarity will be discussed later in this section.

Let a_g, b_g, and c_g be the axes of the larger star and a_s, b_s and c_s be the axes of the smaller star, each expressed in terms of the separation of the components as unit of length. The axes a_g and a_s are taken along the line joining the centers of the components, the axes c_g and c_s are taken as the polar axes or axes of rotation, and the axes b_g and b_s are the remaining axes. The axes c_g and c_s are assumed parallel to the orbital angular momentum vector. Assuming the axes of rotation constant in magnitude and direction, their projections, as viewed by the observer, are constant. It has been shown by Russell (1945) that, for a particular form of the surface brightness, the light observed for a pair of similar tri-axial ellipsoids (axes a_g, b_g, c_g and a_s, b_s, c_s) with orbital inclination i is the same as would be observed for a pair of similar oblate spheroids (axes a_g, b_g, b_g and a_s, b_s, b_s) with inclination j, where

$$\tan^2 j = \frac{c_g^{\,2}}{b_g^{\,2}} \tan^2 i = \frac{c_s^{\,2}}{b_s^{\,2}} \tan^2 i \quad . \tag{1.2}$$

The form of the apparent surface brightness assumed is

$$J(\gamma) \propto \left(\frac{\cos\gamma}{H}\right)^n , \tag{1.3}$$

where $J(\gamma)$ is the apparent surface brightness, γ is the angle of foreshortening (the angle between the normal to the surface of the star and the line of sight), H is the perpendicular distance from the center of the star to the tangent plane of the point under consideration, and n is unrestricted. This form of apparent surface brightness is consistent with the theoretical form described later. Thus there is no loss of generality in replacing the model of similar tri-axial ellipsoids with the model of similar oblate spheroids of axes a_g, b_g, b_g and a_s, b_s, b_s and orbital inclination j, where j is given by equation (1.2).

The assumption of similarity (equal oblateness) of the equatorial forms is not an unreasonable first approximation. The dynamical theory of equilibrium gives

$$\frac{a_g - b_g}{\bar{r}_g} = \frac{3}{2}\frac{m_s}{m_g}\bar{r}_g^{\,3}(1+2K_g) , \tag{1.4}$$

where m_g is the mass of the star of larger radius, m_s is the mass of the star of smaller radius, and K_g is a function of the variation of density with radius that does not exceed 0.02 in any well-determined case (Russell and Merrill 1952, p. 40). The quantity $\bar{r}_g$ is defined by

$$\bar{r}_g^{\,3} = a_g b_g c_g . \tag{1.5}$$

Corresponding expressions hold for the smaller star.

Defining the oblateness of the equatorial shape as

$$\varepsilon = \frac{a-b}{a} \quad , \tag{1.6}$$

we have

$$\frac{\varepsilon_g}{\varepsilon_s} \simeq \left(\frac{m_s}{m_g}\right)^2 \left(\frac{\bar{r}_g}{\bar{r}_s}\right)^3 \quad . \tag{1.7}$$

For main-sequence stars the mass-radius relation (Russell and Moore 1940, p. 112) gives

$$r \propto m^{.7} \tag{1.8}$$

so

$$\frac{\varepsilon_g}{\varepsilon_s} \simeq \left(\frac{m_g}{m_s}\right)^{.1} \tag{1.9}$$

and the oblateness ratio is a weak function of the mass ratio.

The assumption of similarity of the polar flattening is perhaps less justified. Russell and Merrill (1952, p. 40) give

$$\frac{b_g - c_g}{\bar{r}_g} = \frac{m_g + m_s}{2m_g} \bar{r}_g^{\,3} \; (1+2K_g)\omega_g^{\,2} \quad , \tag{1.10}$$

where ω_g is the ratio of angular velocity of rotation to angular velocity of revolution for the larger component. A corresponding expression holds for the smaller star. Define the oblateness for the polar flattening as

$$\eta = \frac{b-c}{b} \quad . \tag{1.11}$$

Then, if the stars are taken to have synchronous rotation and revolution (Koch, Olson, and Yoss 1965 and Olson 1968)

$$\frac{\eta_g}{\eta_s} \simeq \left(\frac{m_g}{m_s}\right)^{1.1} \quad . \tag{1.12}$$

The assumption of similarity of shape is seen to be best for components of nearly equal mass. If the masses are reasonably similar and the radii are less than one third of the separation of centers of the components, the departure from spherical shape will not be more than a few percent.

Consider the errors resulting from the assumption of similarity of shape. From equation (1.2)

$$j = \tan^{-1}\left(\frac{c}{b}\tan i\right), \tag{1.13}$$

so

$$\frac{dj}{d\left(\frac{c}{b}\right)} = \frac{\tan i}{1 + \left(\frac{c}{b}\right)^2 \tan^2 i} \quad . \tag{1.14}$$

Thus

$$\Delta j < \frac{1}{\left(\frac{c}{b}\right)^2 \tan i}\,\Delta\left(\frac{c}{b}\right) \quad , \tag{1.15}$$

where Δj is the error in j resulting from an error of $\Delta\left(\frac{c}{b}\right)$ in $\left(\frac{c}{b}\right)$. Assuming the error in $\left(\frac{c}{b}\right)$ resulting from the assumption of similarity of shape is no greater than the difference of the values of $\left(\frac{c}{b}\right)$ for each star we have

$$\Delta\left(\frac{c}{b}\right) < \left|\frac{c_g}{b_g} - \frac{c_s}{b_s}\right| \tag{1.16}$$

or

$$\Delta\left(\frac{c}{b}\right) < |\eta_g - \eta_s| \ . \tag{1.17}$$

Thus for the error in the inclination

$$\Delta j < \frac{1}{\left(\frac{c}{b}\right)^2 \tan i} |\eta_g - \eta_s| \ . \tag{1.18}$$

Next, considering the error in b we have, from equation (1.6), that

$$b = a(1-\varepsilon) \quad , \tag{1.19}$$

so

$$\Delta b = a\Delta\varepsilon \quad , \tag{1.20}$$

where Δb is the error in b resulting from an error $\Delta\varepsilon$ in the equatorial oblateness. Again assuming that the error in the oblateness caused by assuming similarity of shape is less than the difference between the values of oblateness for each star, we have

$$\Delta b < a|\varepsilon_g - \varepsilon_s| \quad , \tag{1.21}$$

or

$$\Delta b < a\varepsilon \left| 1 - \frac{\varepsilon_g}{\varepsilon_s} \right| \ . \tag{1.22}$$

As an example consider a system with components $a_g = .25$, $a_s = .20$, and $i = 76°$. From equation (1.4) and equation (1.10) values of the oblateness for this system are $\varepsilon_g = \varepsilon_s = .017$, $\eta_g = .014$, $\eta_s = .010$. Then from equations (1.8), (1.19), and (1.22) we have $\Delta j < .06°$ and $\Delta b < .0002$. These errors in the inclination and radii are to be compared with the corresponding errors resulting from observational error in the light. For a synthetic light curve of approximately 800 points and a standard deviation in the light values of $\frac{1}{2}$%, the standard deviation in the inclination is approximately 0.1° and the standard deviation in the radii is approximately .0006 (Linnell and Proctor 1970a). Light curves discussed in Chapter V of this work typically have 400 observations and standard deviations in the light of $\frac{3}{4}$%. The errors in the inclination and radii due to observational errors are correspondingly greater for these curves. Typically the standard deviation in the inclination is 0.2° and the standard deviation in the radii is .004. Comparing these values with the values $\Delta j < .06°$ and $b < .0002$, we see that errors in assuming similarity of shape are less than the errors resulting from observational scatter in the light.

C. Review of Notation and Units

Based on the discussion of the previous section, we substitute for the similar tri-axial ellipsoids with inclination i, the mathematically equivalent prolate spheroids with inclination j. The latter form is called the Russell Model. Reflection from each star will also be included. Providing for the possibility of excess or uneclipsed third light, light curves based on this model may then be considered a function of fourteen parameters and the time. These parameters are as follows:

r_g - semi-major axis, larger star

r_s - semi-major axis, smaller star

j - inclination of plane of orbit of equivalent oblate spheroids

L_g - light of larger star

L_s - light of smaller star

L_3 - excess uneclipsed light, third light

x_g - limb-darkening coefficient, larger star

x_s - limb-darkening coefficient, smaller star

t_o - time of minimum projected distance of centers during primary minimum

e - eccentricity of orbit

ω - longitude of periastron

ε - oblateness of ellipsoids

S_c, S_h - parameters related to the light reflected from the cooler and hotter stars.

We note that for orbits of small eccentricity ($e \leq .02$) the change in the oblateness due to the variation in distance between the components is at most of the same order of magnitude as the error occuring due to the assumption of similarity of the components. We thus take the unit of length to be constant and equal to a_o, the semi-major axis of the orbit.

The unit of light intensity, U, is defined initially as

$$U = L_g + L_s + L_3 \quad . \tag{1.23}$$

It is customary to normalize the light such that U = 1, so

$$L_g{}^n + L_s{}^n = 1 \quad . \tag{1.24}$$

where

$$L_g^n = L_g/(U-L_3) \quad (1.25)$$

and

$$L_s^n = L_s/(U-L_3) \quad . \quad (1.26)$$

Third light may arise from an unresolved third companion, from gas streams, or from gas shells in the system. Alternatively, the source of excess light may not be physically associated with the binary system. A field star may be of such small angular displacement from the system that its light cannot be eliminated from the measurements. Koch (1970) gives a discussion of sources of third light.

Diagrams of the orbital parameters are given in Figure 1 and Figure 2. The angle Ω is the position angle of the nodal point between 0° and 180°. It cannot be determined from the light curve. Note that for the orbital parameters the convention used is that of spectroscopic notation. The primary is moving about the secondary and ω is measured from the ascending node, the node at which the star is moving away from the observer (Aitken 1964, p. 154). Observations do not give the sign of the inclination and therefore do not tell the quadrant. The angle θ is the phase angle measured in the plane of the orbit from the time of conjunction (primary minimum). The angle υ, called the true anomaly, is measured from periastron in the plane of the orbit and in the direction of motion of the primary. Thus we have the relation

$$\theta = \upsilon+\omega-90^{\circ} \quad . \quad (1.27)$$

The limb-darkening coefficients are parameters in an expression giving the distribution of brightness over the apparent projected

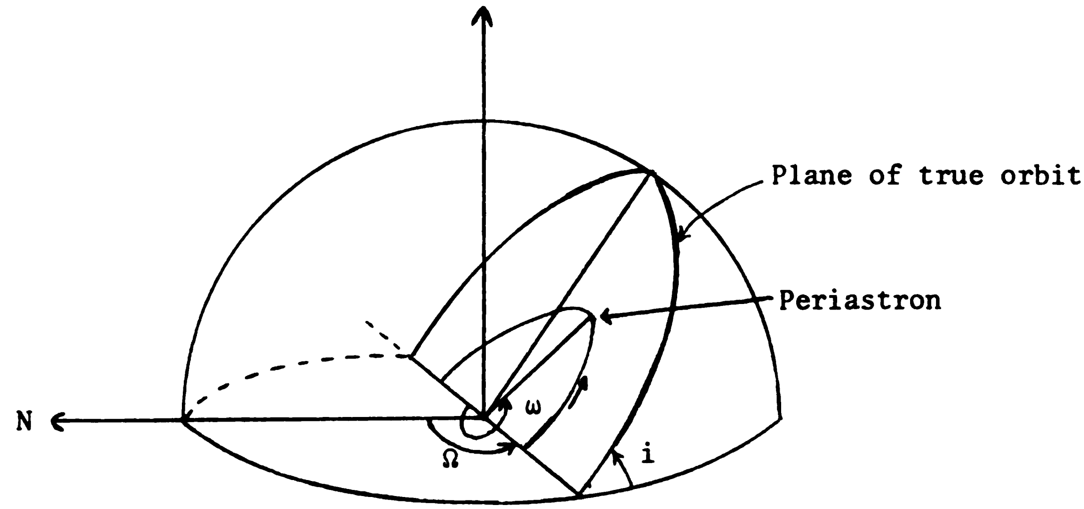

Figure 1. Orbital parameters of the true ellipse in space.

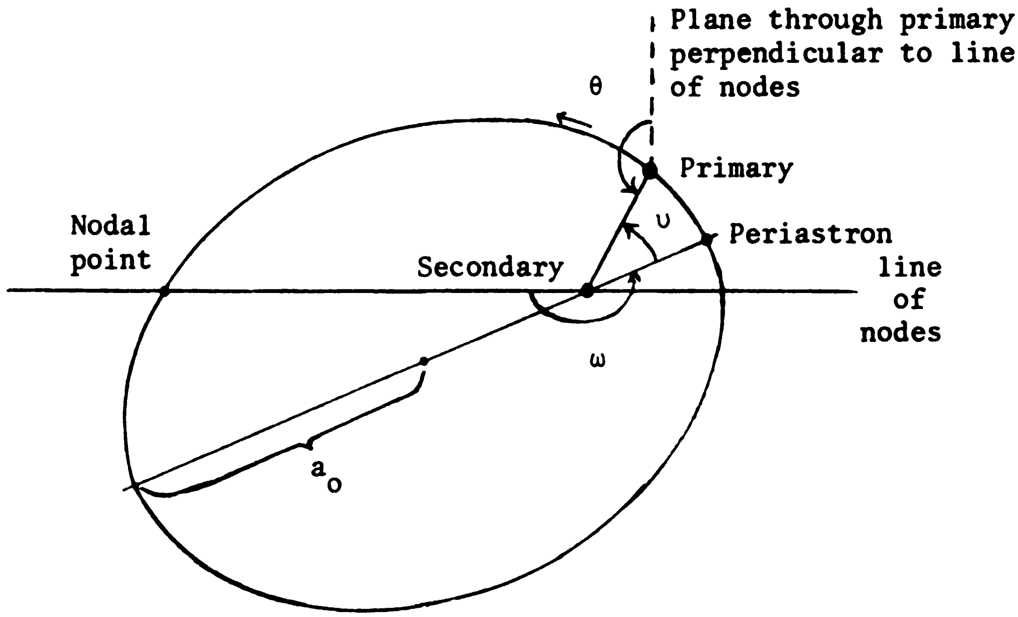

Figure 2. Orbital parameters for the true orbit.

stellar disks. This variation is due to both the finite optical depths of the atmospheres and the variation of temperature with depth in the atmosphere. Thus the apparent surface brightness depends on the angle of foreshortening. For a given wavelength the adopted form of the expression for apparent surface brightness $J(\gamma)$ is

$$J(\gamma) = J(0)(1 - x + x \cos \gamma) \quad , \tag{1.28}$$

where γ is the angle of foreshortening, $J(0)$ is the surface intensity at the center of the projected disk, and x is the limb-darkening coefficient. The values of x are restricted such that

$$-1 \leq x \leq 1 \quad . \tag{1.29}$$

The expression for apparent surface brightness is an approximation linear in the limb-darkening coefficient. Comparison of the first order theoretical values of x (Munch and Chandrasekhar 1949) with those produced by the third order theory of Kopal (1959 p. 160) indicate that over the wavelengths covered by the UBV system, the maximum discrepancy in the values of x is about 0.03. This error is comparable with the probable error in x resulting from observational dispersion in a light curve of 800 points and observational scatter of $\frac{1}{2}$% (Linnell and Proctor 1970a). However, for the observed light curves treated in Chapter V of this work the probable errors in x are typically three to four times as great. Until more accurate curves containing greater numbers of observations are available, use of equation (1.28) is an adequate approximation.

In general, x is expected to be a function of the wavelength of observation, atmospheric absorption coefficients, and the effective tex

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temperature of the star (Kopal 1959, p. 159). (A derivation of the limb-darkening law, equation (1.28), is given in Appendix A.) A completely theoretical determination of x involves knowledge of the chemical and physical processes occuring in the atmosphere. Model stellar atmospheres and their effective linear limb-darkening coefficients are discussed by Grygar (1965), Gingerich (1966), Margrave (1969), and Parsons (1971). These theoretical values of limb darkening can be compared with observed values.

D. Statement of Problem

The continual change in the size of the apparent projected area of the spheroidal stars as a function of phase angle results in a variation of the light received by an observer. The amount of light reflected from each star in the direction of the observer also changes as a function of phase angle. Schematic light curves showing the effects of oblateness and reflection are shown in Figure 5 and Figure 6. Estimates of the oblateness and ratio of reflected lights can be obtained from information resulting from Fourier analysis of the non-eclipse variation combined with knowledge of the spectral type of the primary and the ratio of the depths of eclipse. The oblateness and ratio of reflected lights are used to transform the curve from that of similar spheroids to that of certain equivalent spheres, in a manner to be described later. This transformation process is called rectification. The underlying reason for rectification is that this transformation eliminates the necessity to tabulate or calculate special functions for every value of oblateness and ratio of reflected light. Each light curve can be transformed to its equivalent Spherical Model light curve and functions for the Spherical Model can then be used. We also note that the Spherical Model is the limiting case of zero "interaction" between the components.

An initial value of t_o is generally available in the literature, along with P, the period of the orbit. Observations of minima over several years allow very accurate period determinations.

The geometric parameters, lights of each star, and limb-darkening coefficients remain to be determined. Various methods have been devised for the analysis of the Spherical Model light curve, the method commonly employed being the graphical one of Russell and Merrill (1952). The method of Russell and Merrill was initially designed to provide preliminary estimates of the parameters (Russell and Merrill 1952, p. 27). Subsequent modifications of the method can be applied to produce parameters of greater weight (Russell and Merrill 1952 p. 58). Certain specifically chosen points are taken from a free-hand curve drawn through the observations. Thus each observation is not, in general, given equal weight in the solution. However, for visual, photographic, and photometric observations with probable errors of a single observation commonly 4%, this method produces parameters that satisfactorily fit the light curves. Values of limb-darkening coefficients are assumed in this method of solution. For modern photoelectric light curves, probable errors of a single observation of ½% are not uncommon. It is likely that an analytic method of solution can extract more information from the data. In particular, limb-darkening coefficients and probable errors of the parameters are desired. Solution by computer is desirable to handle the large amounts of data used in the more rigorous methods.

E. Previous Work

Some of the earliest attempts at utilization of a computer included those of Hamid, Huffer and Kopal (1951) on RZ Cassiopeiae and Huffer and Collins (1962) on S Cancri and AR Cassiopeiae. Kopal's Second Method (1959) was used. A maximum of six parameters were determined. These were r_g, r_s, i, λ (depth of eclipse), x (limb-darkening coefficient of eclipsed star), and U (unit of light). In the Huffer and Collins analysis of S Cancri and AR Cassiopeiae corrections to the limb-darkening coefficient and its probable error were suspiciously small.

Jurkevich (1964) and West (1965) also programmed Kopal's Second Method. Neither included corrections to the depths of eclipse, unit of light, or limb-darkening coefficients, although West did examine variance as a function of successive values of limb-darkening.

Tabachnik (1969) programmed Kopal's First Method (1959, p. 319). Tabachnik minimizes a variance which is a function of x, the appropriate limb-darkening coefficient, and k, the ratio of radii. In principle the method allows for corrections to the depths of eclipse and unit of light, but, in the published results not all of the variables were included simultaneously.

Wilson (1969) presented an ingenious method for finding limb-darkening coefficients by enforcing a condition between the coefficients for the larger and smaller components at corresponding phases ($\theta, \theta+180°$). This method though is rather restricted. It requires completely eclipsing systems, small, well-known eccentricity and absence of third light. One of the eclipses must be represented piecewise by an analytic series. A computer is useful for the method.

Kitamura (1965) developed a procedure involving the Fourier transform of the light curve. It provides uniform treatment of partial, total, and annular eclipses. The method also determines whether the rectification process is satisfactory and whether the rectified light curve is acceptably represented by the eclipse effect alone. Kitamura ultimately resorts to the method of differential corrections for his final analysis.

F. Purpose

The purpose of this study is to extend and apply the equations involved in the differential corrections method and Kopal's Second Method. Both approaches are generalized to allow for the possibility of excess or uneclipsed third light. In the case of differential corrections, the effects of orbital eccentricity are explicitly included. Previously, light curves were analyzed for "fictitious" circular elements. These fictitious elements could be transformed to the true elements if values of the eccentricity and longitude of periastron were available. The results were good to second or third order in e, the orbital eccentricity (Kopal 1950, p. 106ff).

The requisite equations for rectification and analysis are described in Chapter II. Chapter III provides a brief description of the programs and Chapter IV describes the validation of the programs using synthetic light curves. Finally, Chapter V contains the results of analysis of published observations for 10 eclipsing binary systems. The systems considered are relatively well-separated systems, thus the relations discussed in Part B of this chapter should provide good first approximations.

II. METHOD

It is necessary to calculate the theoretical light intensity seen by a distant observer for a spheroidal star as a function of phase angle θ. The effects of limb darkening, gravity darkening, and reflection will first be considered individually in Section A. In Section B of this chapter the rectification equations are discussed. Sections A and B thus relate to the formal properties of the Russell Model. Details of the analysis procedure begin in Section C.

A. Theoretical Light Intensities

The form of the limb-darkening law of apparent surface brightness allows calculation of ${}^{x}\ell$, the light from a limb-darkened star, as a linear combination of ℓ^U, the light of a uniformly bright star (x=0.0), and ℓ^D, the light of a completely limb-darkened star (x=1.0). The apparent surface brightness $J(\gamma)$ at γ is

$$J(\gamma) = J(0)(1 - x + x \cos \gamma) \quad , \tag{2.1}$$

where γ is the angle between the normal to the surface of the star and the line of sight and $J(0)$ is the central surface brightness. Thus

$${}^{x}\ell = \int J(\gamma) \cos \gamma \, d\sigma \quad , \tag{2.2}$$

where $d\sigma$ is the surface element, facing the observer, at angle γ. So

$${}^{x}\ell = J(0)(1-x) \int \cos \gamma \, d\sigma + J(0)\, x \int \cos^2\gamma \, d\sigma \tag{2.3a}$$

$$= (1-x)\, \ell^U + x\, \ell^D \quad , \tag{2.3b}$$

where

$$\ell^U = J(0) \int \cos \gamma \, d\sigma \tag{2.4}$$

and

$$\ell^D = J(0) \int \cos^2\gamma \, d\sigma \quad . \tag{2.5}$$

The derivations below follow the presentation of Binnendijk (1960, p. 290ff), with minor changes in notation.

1. Oblate star, uniform brightness

For the uniformly bright star we may calculate ℓ^U by multiplying the surface brightness J(0) by the apparent projected area A_p of the star. Thus

$$\ell^U = J(0) \int \cos \gamma \, d\sigma \tag{2.6a}$$

$$= J(0) \, A_p \quad . \tag{2.6b}$$

The orbital geometry is given in Figure 3. (In Figure 3 the plane containing the angle (90-j) is perpendicular to the line of nodes.) The ellipsoid has axes a, b, and b, with major axis along the line of centers. The projected ellipse has axes d and b. Figure 4 is a view of the components in the plane of the orbit.

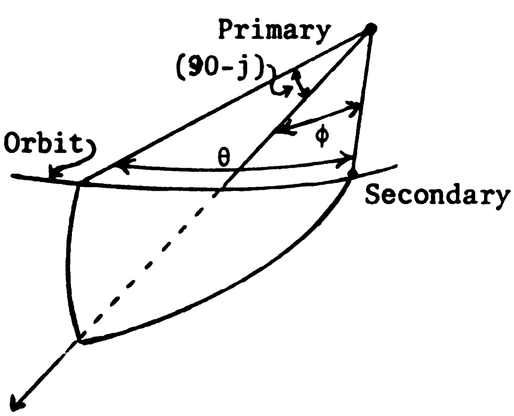

Figure 3. Orbital geometry.

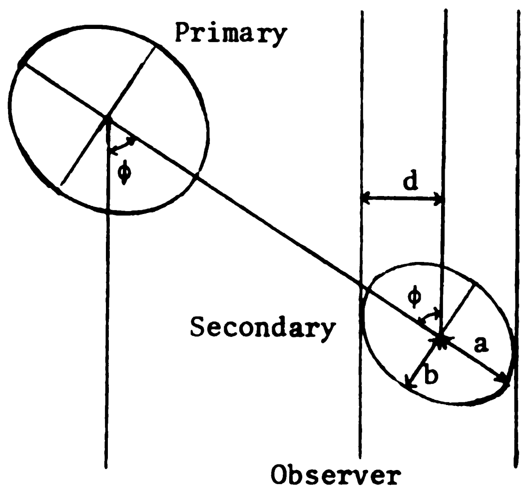

Figure 4. Projection of components.

The value of d is found by requiring that the equation for the intersection of the line of sight and the ellipse have a single root. This gives

$$d^2 = a^2 \sin^2\phi + b^2 \cos^2\phi \tag{2.7a}$$

$$= a^2 \left(1 - e_e{}^2 \cos^2\phi\right) \quad . \tag{2.7b}$$

Here e_e is the eccentricity of the ellipsoid. From the cosine rule of spherical trigonometry

$$\cos \phi = \sin j \cos \theta \quad . \tag{2.8}$$

Thus we have

$$A_p = \pi db \tag{2.9a}$$

$$= \pi ab\left(1 - e_e{}^2 \sin^2 j \cos^2\theta\right)^{1/2} \tag{2.9b}$$

$$\simeq \pi ab\left(1 - \tfrac{1}{2}e_e{}^2 \sin^2 j \cos^2\theta\right) \tag{2.9c}$$

to first order in $e_e{}^2$. For small e_e we have for the oblateness ε

$$\varepsilon = (a - b)/a = 1 - \left(1 - e_e{}^2\right)^{1/2} \tag{2.10a}$$

$$\varepsilon \simeq \tfrac{1}{2} e_e{}^2 \quad . \tag{2.10b}$$

Substituting equation (2.9c) in equation (2.6b) and using equation (2.10b) we have

$$\ell^U \simeq J(0)\pi ab\left(1-\varepsilon \sin^2 j \cos^2\theta\right) \tag{2.11a}$$

$$\simeq \ell^U(90)\left(1-\varepsilon \sin^2 j \cos^2\theta\right) \quad . \tag{2.11b}$$

where $\ell^U(90)$, the light at quadrature, is

$$\ell^U(90) = J(0)\pi ab \quad . \tag{2.12}$$

2. Oblate star, complete darkening

Evaluation of the integral in equation (2.5) to first order in ε results in

$$\ell^D \simeq \frac{2}{3}\left(1 + \frac{1}{5}\varepsilon\right)\pi abJ(0)\left(1 - \frac{8}{5}\varepsilon \sin^2 j \cos^2\theta\right) \tag{2.13a}$$

$$\simeq \ell^D(90)\left(1 - \frac{8}{5}\varepsilon \sin^2 j \cos^2\theta\right) \tag{2.13b}$$

where

$$\ell^D(90) = \frac{2}{3}\left(1 + \frac{1}{5}\varepsilon\right)\pi abJ(0) \tag{2.14}$$

(Binnendijk 1960, p. 301).

3. Oblate star, intermediate darkening

For intermediate limb darkening of an oblate star

$$^x\ell = (1 - x)\ell^U + x\ell^D \tag{2.15a}$$

$$\simeq (1 - x)\ell^U(90)\left(1 - \varepsilon \sin^2 j \cos^2\theta\right) + x\ell^D(90)\left(1 - \frac{8}{5}\varepsilon \sin^2 j \cos^2\theta\right) \quad , \tag{2.15b}$$

so

$$\frac{^x\ell}{^x\ell(90)} = 1 - f(x)\ \varepsilon \sin^2 j \cos^2\theta \quad , \tag{2.16}$$

where

$$f(x) = \frac{(1-x)\ell^U(90) + \frac{8}{5} x\ell^D(90)}{(1-x)\ell^U(90) + x\ell^D(90)} \tag{2.17}$$

and

$$^x\ell(90) = (1 - x)\ell^U(90) + x\ell^D(90) \quad . \tag{2.18}$$

With equation (2.12) and equation (2.14)

$$\frac{\ell^D(90)}{\ell^U(90)} = \frac{2}{3}\left(1 + \frac{1}{5}\varepsilon\right) , \tag{2.19}$$

and to first order in ε

$$\frac{x_\ell}{x_{\ell(90)}} = 1 - \frac{15+x}{15-5x}\,\varepsilon\,\sin^2 j\,\cos^2\theta \tag{2.20}$$

This variation is shown schematically in Figure 5.

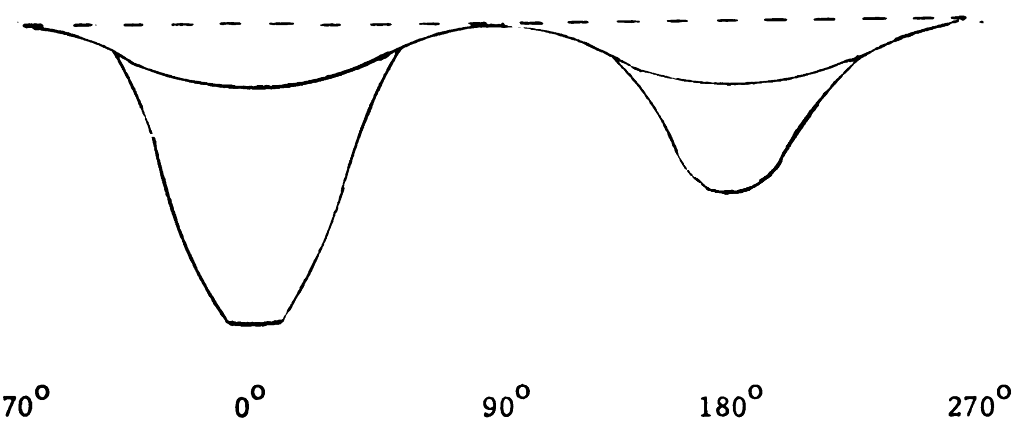

Figure 5.
Schematic light curve with influence of oblateness.

4. Oblate star, gravity darkening

Von Zeipel (1924) first demonstrated the proportionality of the emergent flux and local gravity at a point on the surface of a star. This relation has also been derived by Chandrasekhar (1933). Let H be the intensity of the total radiation emergent normally from the atmosphere and let g be the local surface gravity. Then

$$\frac{H}{H_o} = \frac{g}{g_o} = 1 - \left(1 - \frac{g}{g_o}\right) \quad , \tag{2.21}$$

where g_o is the mean surface gravity and H_o is the corresponding intensity. Proof of Equation (2.21) along with the assumptions involved is given in Appendix B. Kopal (1959, p. 172) has shown that, assuming stars radiate like black bodies, the surface brightness at wavelength λ will be

$$\frac{H_\lambda}{H_o} = 1 - y \left(1 - \frac{g}{g_o}\right) \quad , \tag{2.22}$$

where y, the gravity darkening coefficient, is a function of wavelength and effective temperature. Integration of equation (2.22) over the surface of the star gives the light variation associated with local gravity,

$$\frac{y_\ell}{y_{\ell(90)}} = 1 - (1 + y)\ \varepsilon \sin^2 j \cos^2 \theta \quad . \tag{2.23}$$

Again $y_{\ell(90)}$ is the light at quadrature.

5. Reflection

Let L_h and L_c be the intrinsic luminosities of the hotter and cooler stars respectively. Let 2 S_h be the total amount of light reflected from the hotter star and let 2 S_c be the total amount of light reflected from the cooler star. If stars reflected light like mirrors the light outside of eclipse would be

$$\ell = L_h + S_h(1 + \cos\phi) + L_c + S_c(1 - \cos\phi) \quad , \tag{2.24}$$

where ϕ is the angle between the line of sight and the line of centers of the stars in space. The difference in sign for the cosine terms is a result of the difference in phase of π between the components. Thus we may write

$$\ell = L_h + 2.5\ S_h\ f(\phi) + L_c + 2.5\ S_c\ f(\phi+\pi) \quad , \qquad (2.25)$$

where for "mirror-like" stars

$$f(\phi) = \frac{2}{5}\,(1 + \cos\phi) \quad . \qquad (2.26)$$

The normalization of the phase function, equation (2.26), is chosen so that the form for "mirror-like" stars may be compared to the form generally adopted.

Rigorous calculation of $f(\phi)$ for more physically realistic models of reflection is extremely complicated. The form generally adopted is

$$f(\phi) = 0.2 + 0.4\cos\phi + 0.2\cos^2\phi \quad , \qquad (2.27)$$

where

$$\cos\phi = \sin j\,\cos\theta \quad . \qquad (2.28)$$

(See Russell and Merrill 1952, p. 44.) Then

$$\ell = (L_c + L_h) + \frac{1}{2}\,(S_c + S_h) - (S_c - S_h)\cos\phi + \frac{1}{2}\,(S_c + S_h)\cos^2\phi \quad . \qquad (2.29)$$

This variation is shown schematically in Figure 6.

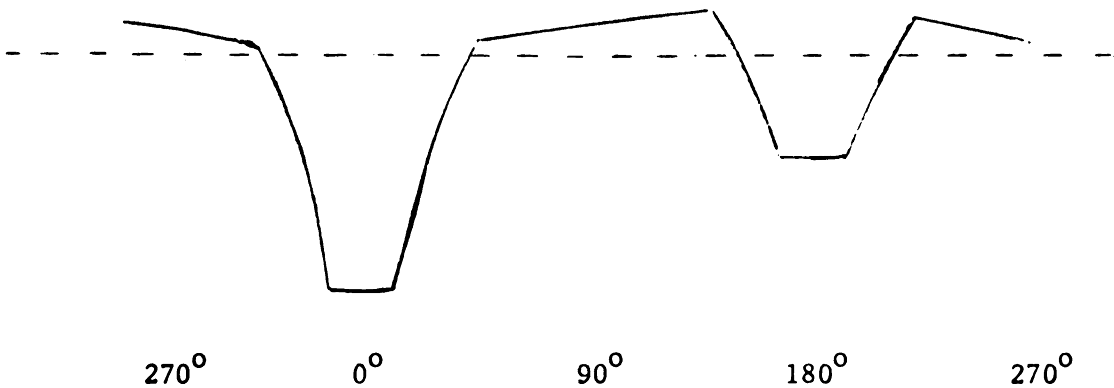

Figure 6. Schematic light curve with influence of reflection.

6. Combined effects

The combined effects of reflection, oblateness, limb-darkening and gravity darkening produce for the light received from both stars outside eclipse,

$$\ell = \ell(90)\left(1 - N\,\varepsilon\,\sin^2 j\,\cos^2\theta\right) + \frac{1}{2}\,(S_c + S_h)$$
$$+ (S_c - S_h)\,\sin j\,\cos\theta + \frac{1}{2}\,(S_c + S_h)\,\sin^2 j\,\cos^2\theta \quad , \tag{2.30}$$

where

$$N = \frac{15 + x}{15 - 5x}\,(1 + y) \quad , \tag{2.31}$$

and $\ell(90)$ is the sum of two terms of the form of equation (2.18), one for each star. It has been assumed for the purpose of rectification that the limb-darkening and gravity darkening are the same for both stars.

B. Rectification

The result desired from rectification is the elimination of the variation in the light curve due to reflection and oblateness in such a manner as to retain the physical meaning associated with the parameters of the model.

If ℓ(obs.) is the observed light value at phase angle θ and ℓ is the corresponding theoretical light value, then we may write

$$\ell(\text{obs.}) = \ell + \delta\ell \tag{2.32}$$

where $\delta\ell$ is the associated observational error. For points outside eclipse ℓ is given by equation (2.30), so that to first order in ε

$$\begin{aligned}\ell(\text{obs.}) = \ell(90)\left(1 - N\,\varepsilon \sin^2 j \cos^2\theta\right) &+ \frac{1}{2}(S_c + S_h) \\ &- (S_c - S_h)\sin j \cos\theta \\ &+ \frac{1}{2}(S_c + S_h)\sin^2 j \cos^2\theta + \delta\ell \quad .\end{aligned} \tag{2.33}$$

1. Rectification for reflection

From equation (2.29) and equation (2.33) it is seen that the light from the cooler and hotter stars may be symmetrized by the addition of the quantities of light $\Delta\ell_c$ and $\Delta\ell_h$, where

$$\Delta\ell_c = \frac{1}{2}S_c + S_c \sin j \cos\theta + \frac{1}{2}S_c \sin^2 j \cos^2\theta \tag{2.34a}$$

and

$$\Delta\ell_h = \frac{1}{2}S_h - S_h \sin j \cos\theta + \frac{1}{2}S_h \sin^2 j \cos^2\theta \quad . \tag{2.34b}$$

Thus if ℓ(obs.) is the observed light at phase angle θ, then ℓ_{rp}, the observed light partially rectified for reflection, may be defined as

$$\ell_{rp} = \ell(\text{obs.}) + \frac{1}{2}(S_c+S_h) + (S_c-S_h)\sin j\cos\theta + \frac{1}{2}(S_c+S_h)\sin^2 j\cos^2\theta \quad . \tag{2.35}$$

In effect, this partial rectification adds sufficient luminosity to the outer faces to bring them to equality with the illuminated sides. It thus provides completely illuminated stars at all phase angles. While the above rectification is exact for the non-eclipse portion of the light curve, no sensible error occurs by continuing its application right through eclipse (Russell and Merrill 1952, p. 48). Note that ℓ_{rp} still varies as a function of phase angle. This is due to the "non-mirror like" quality of the reflection. We can eliminate this variation and complete the rectification for reflection by division as follows:

$$\ell_r = \frac{\ell_{rp}}{(L_g+L_s)+(S_c+S_h)+(S_c+S_h)\sin^2 j\cos^2\theta}$$

$$= \frac{\ell(\text{obs.})+\frac{1}{2}(S_c+S_h)+(S_c-S_h)\sin j\cos\theta+\frac{1}{2}(S_c+S_h)\sin^2 j\cos^2\theta}{(L_g+L_s)+(S_c+S_h)+(S_c+S_h)\sin^2 j\cos^2\theta} , \tag{2.36}$$

where we have used the substitution

$$\ell(90) = L_g + L_s \quad . \tag{2.37}$$

The denominator of the right-hand side of equation (2.36) has been determined by the combination of equation (2.33) and equation (2.35).

2. Rectification for oblateness

To eliminate the non-eclipse variation due to oblateness, the light at θ is divided by the appropriate value of $(1 - N\,\varepsilon \sin^2 j \cos^2\theta)$. The quantity $N\,\varepsilon \sin^2 j$ is known as the photometric ellipticity. Thus if $\ell_{c,r}$ and $\ell_{h,r}$ are the lights of the cooler and hotter stars rectified for reflection and we define

$$\ell_{c,rr} = \frac{\ell_{c,r}}{1 - N_c\varepsilon_c \sin^2 j \cos^2\theta} \ , \tag{2.38}$$

$$\ell_{h,rr} = \frac{\ell_{h,r}}{1 - N_h\varepsilon_h \sin^2 j \cos^2\theta} \ , \tag{2.39}$$

the total rectified light ℓ_{rr} is

$$\ell_{rr} = \ell_{c,rr} + \ell_{h,rr} \tag{2.40}$$

$$= \frac{\ell_{c,r}}{1 - N_c\varepsilon_c \sin^2 j \cos^2\theta} + \frac{\ell_{h,r}}{1 - N_h\varepsilon_h \sin^2 j \cos^2\theta} \tag{2.41}$$

$$= \frac{\ell_{c,r} + \ell_{h,r}}{1 - \overline{N\varepsilon} \sin^2 j \cos^2\theta} \tag{2.42}$$

$$= \frac{\ell_r}{1 - \overline{N\varepsilon} \sin^2 j \cos^2\theta} \ , \tag{2.43}$$

where

$$\overline{N\varepsilon} = \frac{\ell_{c,r}N_c\varepsilon_c + \ell_{h,r}N_h\varepsilon_h}{\ell_{c,r} + \ell_{h,r}} \ . \tag{2.44}$$

If it is assumed that $\varepsilon_c = \varepsilon_h = \varepsilon$ and $N_c = N_h = N$, then

$$\overline{N\varepsilon} = N\varepsilon \quad . \tag{2.45}$$

3. Light rectification formulas

Combining equation (2.36) and equation (2.43), we have for the light rectification formula

$$\ell_{rr} = \frac{\ell(\text{obs.}) + \frac{1}{2}(S_c+S_h) + (S_c-S_h)\sin j \cos\theta + \frac{1}{2}(S_c+S_h)\sin^2 j \cos^2\theta}{(L_g+L_s+S_c+S_h)(1-N\varepsilon \sin^2 j \cos^2\theta)\left[1 + \frac{(S_c+S_h)\sin^2 j \cos^2\theta}{L_g+L_s+S_c+S_h}\right]} \quad . \tag{2.46}$$

We note that, excluding observational error, if ℓ(obs.) is the light of spheroidal stars with reflection and gravity darkening, then the rectified light is that which would be observed for spherical stars with Russell Model parameters r_g, r_s, x_g, x_s, L_g, L_s, and j. The variation of the non-eclipse portion of the light curve has been eliminated. (See Russell 1946, 1948.)

To first order in small quantities (S_c, S_h, ε) the order of the rectification for reflection and oblateness is immaterial.

It is necessary to obtain an expression for the rectified light in terms of empirically determinable quantities. Define

$$D_0 = \frac{1}{2}(S_c + S_h) \quad , \tag{2.47}$$

$$D_1 = -(S_c - S_h)\sin j \quad , \tag{2.48}$$

$$D_2 = \frac{1}{2}(S_c + S_h)\sin^2 j \quad . \tag{2.49}$$

From the theoretical expression for the observed light outside eclipse, equation (2.30), using equations (2.47), (2.48), and (2.49) we have the relation

$$\ell = \ell(90)\left(1 - N\varepsilon \sin^2 j \cos^2\theta\right) + D_0 + D_1 \cos\theta + D_2 \cos^2\theta \quad . \tag{2.50}$$

A Fourier analysis of the non-eclipse variation produces

$$\ell = A_0 + A_1 \cos\theta + A_2 \cos 2\theta \tag{2.51a}$$

$$= A_0' + A_1 \cos\theta + A_2' \cos^2\theta \quad , \tag{2.51b}$$

where

$$A_0' = A_0 - A_2 \quad , \tag{2.52}$$

$$A_2' = 2\,A_2 \quad . \tag{2.53}$$

We then have

$$D_1 = A_1 \quad , \tag{2.54}$$

$$D_0 = \frac{1}{2}(S_c + S_h) = -\frac{1}{2}\left(\frac{S_c+S_h}{S_c-S_h}\right)\frac{A_1}{\sin j} \quad , \tag{2.55}$$

$$D_2 = D_0 \sin^2 j \quad . \tag{2.56}$$

Thus we can empirically determine D_0, D_1, and D_2 if we know S_c/S_h. The procedure for estimating S_c/S_h is discussed in Appendix C.

We have that

$$\frac{1}{2}(S_c+S_h) + (S_c-S_h)\sin j \cos\theta + \frac{1}{2}(S_c+S_h)\sin^2 j \cos^2\theta$$

$$= D_0 - D_1 \cos\theta + D_2 \cos^2\theta \tag{2.57}$$

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$$(S_c+S_h+L_g+L_s)(1 - N\varepsilon \sin^2 j \cos^2\theta)\left(1 + \frac{(S_c+S_h)\sin^2 j \cos^2\theta}{L_g+L_s+S_c+S_h}\right)$$

$$= \ell(90)(1 - N\varepsilon \sin^2 j \cos^2\theta) + 2\, D_0 + 2\, D_2 \cos^2\theta \qquad (2.58a)$$

$$= (A_0' + D_0) + (A_2' + D_2) \cos^2\theta \quad , \qquad (2.58b)$$

where we have neglected quantities of second order in S_c, S_h, and ε, used equations (2.50) and (2.51b), and equation (2.37). Thus, substituting from equations (2.57) and (2.58b), equation (2.46) becomes

$$\ell_{rr} = \frac{\ell(\text{obs.}) + D_0 - D_1 \cos\theta + D_2 \cos^2\theta}{(A_0' + D_0) + (A_2' + D_2) \cos^2\theta} \qquad (2.59)$$

We have in equation (2.59) a formula for the rectified light in terms of empirically determinable quantities.

4. Phase rectification

It is possible to express the geometrical dependence of the theoretical value of light as a function of two dimensionless variables (Kopal 1946, p. 24ff). These variables are normally taken to be the ratio of radii k, where

$$k = \frac{r_s}{r_g} \quad , \qquad (2.60)$$

and the geometrical depth of eclipse p, where

$$p = \frac{\delta - d_g}{d_s} \quad . \qquad (2.61)$$

In the above formula d_g and d_s are the apparent projected axes along the projected line of centers and δ is the projected distance of centers. From Figure 7 and Figure 8 it can be seen that

$$\delta^2 = \delta_1{}^2 + \delta_2{}^2 \tag{2.62}$$

$$= R^2(\sin^2\theta \sin^2 j + \cos^2 j) \quad . \tag{2.63}$$

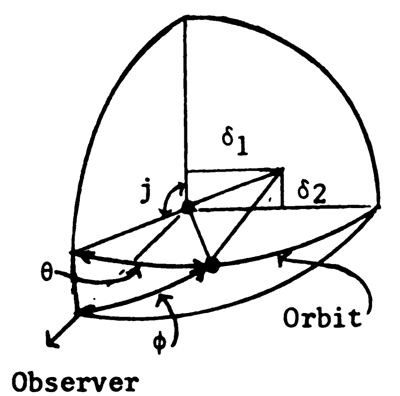


Figure 7. Orbital geometry for projected distance of centers.

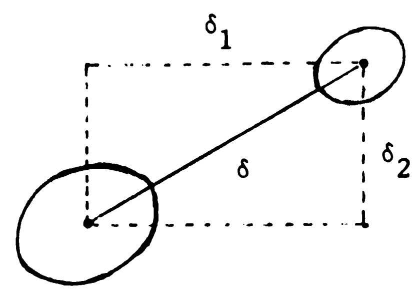


Figure 8. Projection of components against the sky.

Here R is the separation of the centers of the components

$$R = \frac{a_o(1 - e^2)}{1 + e \cos(\theta - \omega + 90)} \tag{2.64}$$

and a_o is the semi-major axis of the components ($a_o = 1$).

External contact of the apparent projected ellipsoids occurs at $p = +1$; internal contact occurs at $p = -1$. An eclipse is called total or complete if the minimum value (p_{min}) of the geometrical depth during eclipse satisfies

$$p_{min} \leq -1 \quad . \tag{2.65}$$

An eclipse is called partial if

$$-1 \leq p_{min} \leq 1 \quad . \tag{2.66}$$

For

$$1 < p_{min} \tag{2.67}$$

no eclipse occurs.

We may fit the rectified light with the equation

$$\ell_{rr}(\theta) = \ell(k,p(\theta)) \quad , \tag{2.68}$$

where $\ell(k,p(\theta))$ is the theoretical light for spherical stars with Russell Model parameters r_g, r_s, L_g, L_s, x_g, x_s, and j at phase angle θ and geometrical depth

$$p(\theta) = \frac{\delta - d_g}{d_s} \quad . \tag{2.69}$$

The radii project in the same ratio so that

$$k = \frac{r_s}{r_g} = \frac{d_s}{d_g} \quad . \tag{2.70}$$

Using equation (2.63) and equation (2.70)

$$p(\theta) = \frac{R\sqrt{\dfrac{\sin^2\theta \, \sin^2 j + \cos^2 j}{1 - z\cos^2\theta}}}{r_s} - \frac{1}{k} \quad , \tag{2.71}$$

where

$$z = 2\,\varepsilon \sin^2 j = e_e{}^2 \sin^2 j \quad . \tag{2.72}$$

However, with the substitutions

$$\sin^2\theta_r = \frac{\sin^2\theta}{1 - z\cos^2\theta} \quad , \tag{2.73}$$

$$\sin^2 i_r = \frac{\sin^2 j - z}{1 - z} \quad , \tag{2.74}$$

$$\cos^2 i_r = \frac{\cos^2 j}{1 - z} \quad , \tag{2.75}$$

we may write the geometrical depth as

$$p(\theta_r) = \frac{R\sqrt{\sin^2\theta_r \sin^2 i_r + \cos^2 i_r} - r_g}{r_s} \quad . \tag{2.76}$$

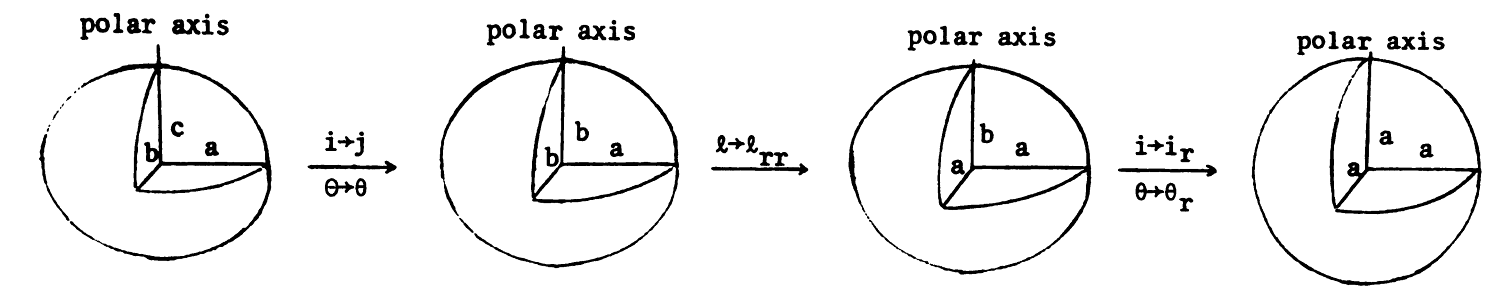
We note that this is the value of p that would be obtained for a pair of spherical stars with paremeters r_g, r_s, L_g, L_s, x_g, x_s, and i_r at θ_r. Thus we observe that as an alternative to equation (2.68) we may fit the rectified light with the equation

$$\ell_{rr}(\theta_r) = \ell(k,p(\theta_r)) \tag{2.77}$$

where $\ell(k,p(\theta_r))$ is the theoretical light for spherical stars with Spherical Model parameters r_g, r_s, L_g, L_s, x_g, x_s, and i_r at phase angle θ_r. This transformation procedure will be followed.

A summary of the transformations from the tri-axial model to the spherical model is given in Figure 9.

Figure 9. Summary of transformation from tri-axial ellipsoid with reflection, gravity darkening and limb darkening to sphere with limb darkening.



tri-axial ellipsoid
limb darkening x
inclination i
gravity darkening
reflection

prolate spheroid
limb darkening x
inclination j
gravity darkening
reflection

oblate spheroid
limb darkening x
inclination j
gravity darkening
Variation due to equatorial oblateness, reflection and gravity darkening removed (note gravity darkening still present).

spherical star
limb darkening x
inclination i_r
no gravity darkening

Transformation

$$\tan^2 j = \frac{c^2}{b^2}\tan^2 i$$

Transformations

$$\cos^2 i_r = \frac{\cos^2 j}{1 - z}$$

$$\sin^2 i_r = \frac{\sin^2 j - z}{1 - z}$$

$$z = 2\,\varepsilon\,\sin^2 j$$

$$\sin^2\theta_r = \frac{\sin^2\theta}{1 - z\cos^2\theta}$$

$$\cos^2\theta_r = \frac{1 - z}{1 - z\cos^2\theta}\,\cos\theta$$

Figure 9.

C. Fourier Analysis

As a preliminary to rectification, Fourier analysis of the non-eclipse variation is necessary. Data for light curves are commonly given in magnitude differences as a function of time, such that

$$\Delta m = m - m_c = -2.5 \log(\ell(\text{obs.})/\ell_c) \quad , \tag{2.78}$$

where m_c and ℓ_c are the magnitude and light of the comparison star and m and ℓ(obs.) are the magnitude and light of the eclipsing system (which may include excess light from an uneclipsed third source). Thus

$$\ell(\text{obs.})/\ell_c = 10^{-0.4(m-m_c)} = e^{-(m-m_c)/1.08573620} \quad . \tag{2.79}$$

The method of least squares is used to determine the Fourier coefficients in the equation

$$\frac{\ell(\text{obs.})}{\ell_c} = \frac{\ell_v+\ell_3}{\ell_c} = \frac{\alpha_0}{\ell_c} + \frac{A_1}{\ell_c}\cos\theta + \frac{A_2}{\ell_c}\cos 2\theta + \cdots\cdot$$
$$+ \frac{B_1}{\ell_c}\sin\theta + \frac{B_2}{\ell_c}\sin 2\theta + \cdots\cdot \quad . \tag{2.80}$$

Here ℓ_v is the light of the variable, ℓ_3 is the excess light and θ is the phase angle from conjunction at primary minimum. The Fourier analysis is normally carried to terms of order 2θ. The occurence of the odd harmonics will be discussed in Section D of this chapter. For non-eccentric orbits

$$\theta = \frac{2\pi}{P}(t - t_o) \quad , \tag{2.81}$$

where P is the period of the orbit and t is the time of the observation. For eccentric orbits Kepler's equation is solved. (See, for example, Kopal (1946, p. 94ff).)

Taking ℓ_c as unity and assuming that harmonic terms arise only from the variable we have

$$\ell_v = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \cdots\cdot \\ + B_1 \sin\theta + B_2 \sin 2\theta + \cdots\cdot \quad . \qquad (2.82)$$

Thus,

$$A_0 = \alpha_0 - \ell_3 \quad . \qquad (2.83)$$

Merrill (1970) states that conventional least squares analysis, carried through terms of 2θ, gives no indication as to the presence or absence of higher order harmonics in the data. He also demonstrates that failure to include the cos 3θ term can vitiate the resulting estimates of the reflection effect. On the other hand, inclusion of terms in 3θ and higher may diminish the weights of all the coefficients (Russell and Merrill 1952, p. 53). For these reasons, at least two least squares Fourier analyses were carried out on the non-eclipse variation of each light curve studied. The first analysis was the conventional series carried to terms of order 2θ. The second analysis included higher order terms (normally cos 3θ and sin 3θ). From an examination of the resulting residuals and standard deviations it could be determined whether or not the inclusion of higher order terms resulted in a significant deviation from a normal distribution.

Choice of phase ranges for the non-eclipse variation will be discussed in Chapter V.

D. Rectification Procedure

The rectification formula used to transform the light curve to the equivalent spherical model light curve was adopted from Jurkevich (1964) and Binnendijk (1960),

$$\ell_{rr}(\theta) = \frac{\ell(\theta)-\sum_{n=1}^{ns} B_n \sin n\theta - \sum_{n=3}^{nc} A_n \cos n\theta + D_0 - A_1 \cos\theta + D_2 \cos^2\theta}{(A_0' + D_0) + (A_2' + D_2)\cos^2\theta} \tag{2.84}$$

where ns is the number of significant sine terms and nc is the number of significant cosine terms from equation (2.82). The resultant $\ell_{rr}(\theta)$ is the observed intensity corrected for asymmetry, higher order cosine terms, reflection, and ellipticity. The higher order cosine terms and the sine terms as yet have no generally accepted theoretical justification. Their presence in the rectification formula represents an empirical correction. The various constants in the rectification formula are calculated from the Fourier coefficients, obtained from the outside-eclipse variation, and an estimate of the ratio of the reflected lights S_c/S_h. As discussed in Section B of this chapter

$$A_0' = A_0 - A_2 \quad , \tag{2.85}$$

$$A_2' = 2\,A_2 \quad , \tag{2.86}$$

$$D_0 = \frac{1}{2}(S_c + S_h) = -\frac{1}{2}\frac{(S_c + S_h)}{(S_c - S_h)}\frac{A_1}{\sin j} \quad , \tag{2.87}$$

$$D_1 = A_1 = -(S_c - S_h)\sin j \quad , \tag{2.88}$$

$$D_2 = \frac{1}{2}(S_c + S_h)\sin^2 j = D_0 \sin^2 j \quad . \tag{2.89}$$

We thus require an estimate for S_c/S_h.

For bolometric observations we may make the approximation

$$\frac{S_c}{S_h} = \frac{I_h}{I_c} = \frac{\text{Depth of Primary}}{\text{Depth of Secondary}} = \frac{1 - \ell_r(0)}{1 - \ell_r(\pi)} , \tag{2.90}$$

where I_h/I_c is the ratio of surface luminosities (Binnendijk 1960, p. 313). For observations taken at an effective wavelength λ it is necessary to include the effect of luminous efficiency when calculating S_c/S_h. Russell and Merrill (1952) provide a method of graphically evaluating S_c/S_h. For computer reduction, the required equations are given by Jurkevich (1964). The discussion presented by Jurkevich contains errors of a typographical nature. For this reason and for completeness the development is reproduced in Appendix C.

As discussed in Section B.4 of this chapter, it is necessary to rectify the phase angles to complete the transformation from the ellipsoidal model to the spherical model light curve. From Jurkevich (1964, p. 139), correcting the typographical error

$$\sin\theta_r = \frac{\sin\theta}{\sqrt{1 - z\cos^2\theta}} \tag{2.91}$$

and

$$\cos\theta_r = \sqrt{\frac{1 - z}{1 - z\cos^2\theta}}\,\cos\theta , \tag{2.92}$$

where

$$z = \frac{2\,\dfrac{D_2 - A_2'}{A_0' - D_0}}{\dfrac{15+x}{15-5x}(1+y)} . \tag{2.93}$$

Here the limb-darkening coefficient x and the gravity darkening coefficient y have been taken as the same for both stars. For the initial rectification, theoretical values for the brighter component may be used.

For computational purposes equations (2.91) and (2.92) may be combined to give

$$\theta_r = 2\tan^{-1}\left(\frac{\sin\theta_r}{1+\cos\theta_r}\right) . \tag{2.94}$$

In this process the inclination has been transformed as well, so that

$$\cos i_r = \frac{\cos j}{\sqrt{1-z}} , \tag{2.95}$$

$$\sin i_r = \sqrt{\frac{\sin^2 j - z}{1-z}} . \tag{2.96}$$

The (ℓ_{rr}, θ_r) data may now be analyzed according to the Spherical Model.

E. Effect of Third Light on Rectification

It is necessary to consider the effect of third light in the rectification of the light curve. Let

$$R(\theta) = -\sum_{n=1}^{ns} B_n \sin n\theta - \sum_{n=3}^{nc} A_n \cos n\theta + D_0 - A_1 \cos\theta + D_2 \cos^2\theta \tag{2.97}$$

and

$$E(\theta) = (A_0 - A_2 + D_0) + (A_2' + D_2)\cos^2\theta . \tag{2.98}$$

Then

$$\ell_{rr} = \ell_{rr}(\theta) = \frac{\ell_v + R(\theta)}{E(\theta)} . \tag{2.99}$$

And we have

$$\frac{\ell_{rr} + \dfrac{\ell_3}{E(\theta)}}{1 + \dfrac{\ell_3}{E(\theta)}} = \frac{\ell_v + \ell_3 + R(\theta)}{E(\theta) + \ell_3} = \frac{\ell + R(\theta)}{E(\theta) + \ell_3} . \tag{2.100}$$

Define

$$\ell_3^* = \frac{\ell_3}{E(\theta)} \tag{2.101}$$

and

$$\ell_c^* = 1 + \frac{\ell_3}{E(\theta)} . \tag{2.102}$$

Then

$$\ell^*_{rr} = \frac{\ell_{rr} + \ell_3^*}{\ell_c^*}$$

$$= \frac{\ell - \sum_{n=1}^{ns} B_n \sin n\theta - \sum_{n=3}^{nc} A_n \cos n\theta + D_0 - A_1 \cos\theta + D_2 \cos^2\theta}{(\alpha_0 - A_2 + D_0) + (A_2' + D_2)\cos^2\theta} . \tag{2.103}$$

Thus we see that use of the rectification formula, equation (2.84) produces rectified light where, if there is excess light, the rectified excess light ℓ_3^* and the rectified scale factor ℓ_c^* are slightly variable. For light curves of the "Algol" type this variation is in general less than 1% during eclipse. Thus, for a first approximation, ℓ^*_{rr} may be analyzed as though ℓ_3^* and ℓ_c^* were constant, say by the method of iterative differential corrections. From this analysis we obtain an estimate of ℓ_3^* and we may solve for ℓ_3

$$\ell_3^* = \frac{\ell_3}{(\alpha_0 - \ell_3 - A_2 + D_0) + (A_2' + D_2) \cos^2\theta} \;, \tag{2.104}$$

$$\ell_3 = \frac{\ell_3^*}{(1 + \ell_3^*)} \left[(\alpha_0 - A_2' + D_2) + (A_2 + D_2) \cos^2\theta \right] \;. \tag{2.105}$$

With this estimate of ℓ_3 we may use equation (2.83) for A_0 and eliminate the excess light during rectification.

From equation (2.93) we have

$$\varepsilon = \frac{\dfrac{D_2 - A_2'}{A_0' - D_0}}{N \sin^2 j} \tag{2.106}$$

and it can be seen that failure to exclude the excess light results in an under-estimation of the oblateness. Note, however, that for the systems discussed in Chapter V, that unless the excess light is a major fraction of the light of the system, failure to take it into account results in an error in the oblateness of approximately the same order as that caused by observational error in the Fourier coefficients.

F. Differential Corrections

Wyse (1939), Irwin (1947), and Kopal (1959, p. 367ff) have developed the initial equations necessary to determine the differential corrections to the initial parameters that describe the eclipsing binary system in the Spherical Model. The extended equations are described below.

At this point it is customary to drop the subscripts on ℓ_{rr}, i_r, and θ_r. This practice will be followed, keeping in mind that the quantities discussed are, in fact, the rectified values.

We adopt the terminology of Kopal (1959, p. 307). The deeper minimum will be called the primary and the shallower minimum will be called the secondary. The eclipse of the smaller star by the larger will be referred to as an occultation eclipse and the eclipse of the larger star by the smaller will be referred to as a transit.

The eclipsing binary light curve is to be fitted to the equation

$$\ell_o = \ell_c \tag{2.107a}$$

$$= U - {}^{x}f(k,p)L \tag{2.107b}$$

where ℓ_o is the observed light value, ℓ_c is the calculated or theoretical light value, U is the unit of light, ${}^{x}f$ is the fractional light loss appropriate to the type of eclipse (occultation or transit), and L is the total light of the eclipsed star. Further discussion of the function ${}^{x}f$ is given in Appendix D. Each observed point provides an equation of condition of the form

$$\ell_o = U - {}^{x}f(r_g', r_s', \cos^2 i', x', e', \omega', t_o', t)L' , \tag{2.108}$$

where r_g', r_s', $\cos^2 i'$, x'(limb-darkening coefficient of eclipsed star), e'(orbital eccentricity), ω'(longitude of periastron), t_o' (time of primary minimum), L', and U' are the true parameters. Assuming an initial approximate set of parameters r_g, r_s, $\cos^2 i$, x, e, ω, L, and U (U=1), we have, expanding equation (2.108) to first order in the differential corrections to the parameters

$$\ell_o = 1 - {}^x fL + \left\{\Delta U - {}^x f \Delta L - L\left(\frac{\partial\, {}^x f}{\partial r_g} \Delta r_g + \frac{\partial\, {}^x f}{\partial r_s}\ r_s + \frac{\partial\, {}^x f}{\partial \cos^2 i}\ \cos^2 i \right.\right.$$

$$\left.\left. + \frac{\partial\, {}^x f}{\partial x} \Delta x + \frac{\partial\, {}^x f}{\partial e} \Delta e + \frac{\partial\, {}^x f}{\partial \omega} \Delta\omega + \frac{\partial\, {}^x f}{\partial t_o} \Delta t_o\right)\right\} , \qquad (2.109)$$

where $\frac{\partial\, {}^x f}{\partial x} \Delta x$ is $\frac{\partial\, {}^x f^{oc}}{\partial x_s}\ x_s$ or $\frac{\partial\, {}^x f^{tr}}{\partial x_g} \Delta x_g$ as appropriate. Define ℓ_c calculated with the current estimate of the true parameters (the initial estimate on the first iteration) as

$$\ell_c = 1 - {}^x f(r_g, r_s, \cos^2 i, x, e, \omega, t_o, t)L \qquad (2.110)$$

and define

$$\Delta\ell(o\text{-}c) = \Delta U - {}^x f\Delta L - L\left(\frac{\partial\, {}^x f}{\partial r_g} \Delta r_g + \frac{\partial\, {}^x f}{\partial r_s} \Delta r_s + \frac{\partial\, {}^x f}{\partial \cos^2 i} \Delta\cos^2 i \right.$$

$$\left. + \frac{\partial\, {}^x f}{\partial x} \Delta x + \frac{\partial\, {}^x f}{\partial e} \Delta e + \frac{\partial\, {}^x f}{\partial \omega} \Delta\omega + \frac{\partial\, {}^x f}{\partial t_o}\ t_o\right) . \qquad (2.111)$$

The equation of condition, for a given iteration, is then

$$\ell_o - \ell_c = \Delta\ell(o\text{-}c) \quad . \qquad (2.112)$$

Note $\ell_o - \ell_c$ is the light residual with ℓ_c calculated from current parameters and $\Delta\ell(o\text{-}c)$ is an estimator of $\ell_o - \ell_c$. Thus, equation (2.112) is an attempt to account for the residuals in terms of changes in the current system parameters.

Writing $\Delta\ell$(o-c) explicitly for the various types of points we have

(a) for points outside eclipse

$$\Delta\ell(o-c) = \Delta U \quad , \tag{2.113}$$

(b) for points in transit eclipse

$$\Delta\ell(o-c) = \Delta U - x_f^{tr}\Delta L_g - L_g\left[\frac{\partial x_f^{tr}}{\partial r_g}\Delta r_g + \frac{\partial x_f^{tr}}{\partial r_s}\Delta r_s + \frac{\partial x_f^{tr}}{\partial \cos^2 i}\Delta\cos^2 i\right.$$

$$\left. + \frac{\partial x_f^{tr}}{\partial x_g}\Delta x_g + \frac{\partial x_f^{tr}}{\partial e}\Delta e + \frac{\partial x_f^{tr}}{\partial \omega}\Delta\omega + \frac{\partial x_f^{tr}}{\partial t_o}\Delta t_o\right] . \tag{2.114}$$

and (c) for points of occultation eclipse

$$\Delta\ell(o-c) = \Delta U - x_f^{oc}\Delta L_s - L_s\left[\frac{\partial x_f^{oc}}{\partial r_g}\Delta r_g + \frac{\partial x_f^{oc}}{\partial r_s}\Delta r_s + \frac{\partial x_f^{oc}}{\partial \cos^2 i}\Delta\cos^2 i\right.$$

$$\left. + \frac{\partial x_f^{oc}}{\partial x_s}\Delta x_s + \frac{\partial x_f^{oc}}{\partial e}\Delta e + \frac{\partial x_f^{oc}}{\partial \omega}\Delta\omega + \frac{\partial x_f^{oc}}{\partial t_o}\Delta t_o\right] . \tag{2.115}$$

We have the further condition that L_g, L_s, L_3, and U are related by

$$L_g + L_s + L_3 = U \quad . \tag{2.116}$$

Thus

$$\Delta L_s = \Delta U - \Delta L_g - \Delta L_3 \quad , \tag{2.117}$$

where L_3 is the possible excess light. We may now write equation (2.115) in the form

$$\Delta\ell(o\text{-}c) = (1 - {}^{x}f^{oc})\Delta U + {}^{x}f^{oc}\Delta L_g + {}^{x}f^{oc}\Delta L_3$$

$$- L_s\left(\frac{\partial\, {}^{x}f^{oc}}{\partial r_g}\Delta r_g + \frac{\partial\, {}^{x}f^{oc}}{\partial r_s}\Delta r_s + \frac{\partial\, {}^{x}f^{oc}}{\partial \cos^2 i}\Delta\cos^2 i\right.$$

$$\left. + \frac{\partial\, {}^{x}f^{oc}}{\partial x_s}\Delta x_s + \frac{\partial\, {}^{x}f^{oc}}{\partial e}\Delta e + \frac{\partial\, {}^{x}f^{oc}}{\partial \omega}\Delta\omega + \frac{\partial\, {}^{x}f^{oc}}{\partial t_o}\Delta t_o\right) . \qquad (2.118)$$

We have used $\cos^2 i$ as a parameter rather than i, following the recommendation of Irwin (1947).

The evaluation of the various partial derivatives of ${}^{x}f^{oc}$ and ${}^{x}f^{tr}$ is discussed in Appendix E.

The kth equation of condition is weighted according to

$$\sqrt{w_k} = \frac{\sqrt{w_{kI}}}{\ell_k^{\,b}} , \qquad (2.119)$$

where w_{kI} is the observational weight of the kth point and b = 0, $\frac{1}{2}$, or 1 according to the scale on which random errors are assumed constant. (Linnell and Proctor 1970b). Given the apparent magnitude of the system and the aperture of the telescope used for the observations, Young's Table IV (Young 1967, p. 794) may be used to estimate the most appropriate value for b.

Let S be the weighted sum of squares of residuals of the equation of condition, equation (2.112). Then

$$S = \sum_{k=1}^{N} w_k\left((\ell_o-\ell_c)_k - \Delta\ell(o\text{-}c)_k\right)^2 . \qquad (2.120)$$

Define

$$(\ell_o - \ell_c)_k = Y_k \qquad (2.121)$$

and

$$\Delta\ell(o-c)_k = \sum_{i=1}^{NP} \beta_{ik}\, C_i \quad . \tag{2.122}$$

Then

$$S = \sum_{k=1}^{N} w_k \left(Y_k - \sum_{i=1}^{NP} \beta_{ik}\, C_i \right)^2 \quad , \tag{2.123}$$

where N is the total number of observed points and i is summed over the differential corrections to be included, NP in all.

The various C_i and β_{ik} appear in Table 1. The dual use of i for the orbital inclination in $\cos^2 i$ and as a subscript should cause no confusion.

Application of the least squares criterion results in the matrix of normal equations

$$\underset{\sim}{A}\, \underset{\sim}{C} = \underset{\sim}{G} \quad , \tag{2.124}$$

where

$$A_{mj} = \sum_{k=1}^{N} w_k\, \beta_{mk}\, \beta_{jk} \tag{2.125}$$

and

$$G_m = \sum_{k=1}^{N} w_k\, \beta_{mk}\, Y_k \quad . \tag{2.126}$$

Then

$$\underset{\sim}{C} = \underset{\sim}{A}^{-1}\, \underset{\sim}{G} \tag{2.127}$$

produces the components of the $\underset{\sim}{C}$ vector which are the differential

Table 1. Differential Correction Terms.

i	C_i	Occultation β_{ik}	Transit β_{ik}	Outside Eclipse β_{ik}
1	Δr_g	$-L_s \frac{\partial x_f^{oc}}{\partial r_g}$	$-L_g \frac{\partial x_f^{tr}}{\partial r_g}$	0
2	Δr_s	$-L_s \frac{\partial x_f^{oc}}{\partial r_s}$	$-L_g \frac{\partial x_f^{tr}}{\partial r_s}$	0
3	$\Delta \cos^2 i$	$-L_s \frac{\partial x_f^{oc}}{\partial \cos^2 i}$	$-L_g \frac{\partial x_f^{tr}}{\partial \cos^2 i}$	0
4	ΔL_g	$+x_f^{oc}$	$-x_f^{tr}$	0
5	ΔU	$+1 - x_f^{oc}$	+1	+1
6	Δx_s	$-L_s \frac{\partial x_f^{oc}}{\partial x_s}$	0	0
7	Δx_g	0	$-L_g \frac{\partial x_f^{tr}}{\partial x_g}$	0
8	Δt_o	$-L_s \frac{\partial x_f^{oc}}{\partial t_o}$	$-L_g \frac{\partial x_f^{tr}}{\partial t_o}$	0
9	ΔL_3	$+x_f^{oc}$	0	0
10	Δe	$-L_s \frac{\partial x_f^{oc}}{\partial e}$	$-L_g \frac{\partial x_f^{tr}}{\partial e}$	0
11	$\Delta \omega$	$-L_s \frac{\partial x_f^{oc}}{\partial \omega}$	$-L_g \frac{\partial x_f^{tr}}{\partial \omega}$	0

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correction terms.

The covariance matrix SC is given by

$$(SC)_{ij} = \frac{S}{N - NP} \left(A^{-1}\right)_{ij} \tag{2.128}$$

(Ostle 1964, eq. 8.69 and 8.70). The simple correlation coefficient between the i^{th} and the j^{th} variables is defined by the covariance between the two variables divided by the product of their standard deviations. Thus the matrix of simple correlation coefficients S_{corr} is defined by the elements

$$(S_{corr})_{ij} = \frac{(SC)_{ij}}{\sqrt{(SC)_{ii}\ (SC)_{jj}}} \ . \tag{2.129}$$

The partial correlation coefficients are defined by

$$(P_{corr})_{ij} = \frac{(S^{-1}_{corr})_{ij}}{\sqrt{(S^{-1}_{corr})_{ii}\ (S^{-1}_{corr})_{jj}}} \ , \tag{2.130}$$

where (S^{-1}_{corr}) is the matrix inverse of S_{corr} (Smillie 1966, eq. 3.7.1). The values of the simple and partial correlation coefficients are limited to values in the range [-1,+1], with values near the end points indicating higher correlation.

We have for the standard deviation of the weighted light observations

$$\sigma(\text{est.}) = \sqrt{\frac{S}{N - NP}} \ , \tag{2.131}$$

where S may be calculated from the individual residuals (equation (2.123)) or, alternatively,

$$S = \sum_{k=1}^{N} Y_k^{\ 2} - \sum_{i=1}^{NP} C_i\ G_i \quad , \tag{2.132}$$

a form not requiring the calculation of the individual residuals.

We note that σ(est.) is an expression for the standard deviation of the observations from the spherical model light curve.that is first order in the differential corrections. Let

$$S_n = \sum_{k=1}^{N} w_k\ (\ell_k - \ell_{cn})^2 \quad , \tag{2.133}$$

where ℓ_{cn} is calculated with the incremented parameters ($r_g+\Delta r_g$, $r_s+\Delta r_s$, etc.). Then

$$\sigma(\text{cal.})^2 = \frac{S_n}{N - NP} \quad . \tag{2.134}$$

Equality of σ(est.) and σ(cal.) is a test of convergence, indicating $\Delta\ell$(o-c) does not contain systematic errors that can be accounted for by a significant change in the parameters. Equality of σ(est.) and σ(cal.) will not occur unless higher order terms in the expansion of ℓ_c are negligable compared to first order terms.

The probable errors of the parameters follow from the root of the appropriate covariance matrix element. For example, the probable error of Δr_g is

$$\text{P.E. } \Delta r_g = 0.6745\ (SC)_{1,1}^{\ \frac{1}{2}} \quad . \tag{2.135}$$

We have assumed that the uncertainty of the differential correction to a parameter is equal to the uncertainty of the respective parameter in the final iteration (Piotrowski 1948).

The method of differential corrections offers several advantages:

(1) Each observed point is given proper weight in the solution. This is not true with Russell's graphical solution.

(2) The same set of equations apply to partial as well as completely eclipsing systems.

(3). The effects of orbital eccentricity can be included directly.

G. KOPAL'S METHOD

Kopal (1959, p. 321ff) has developed an iterative method for the solution of eclipsing binary light curves that is suitable for adaption to a computer. The equation used in fitting the light curve follows from Kopal (1959, p. 332); but, applied to all parameters the equation becomes

$$\begin{aligned}\sqrt{w}(p^2-1)C_1+2\sqrt{w}(p+1)C_2+\sqrt{w}C_3+\frac{x_\alpha^{oc}}{\ell^b}C_4+\frac{(1-\alpha)}{\ell^b}C_5 \\ +\frac{(U-\lambda_o)}{\ell^b}\frac{\partial x_\alpha^{oc}}{\partial x_s}C_6+\frac{x_\alpha^{tr}}{\ell^b}C_7+\frac{(U-\lambda_t)}{\ell^b}\frac{\partial x_\alpha^{tr}}{\partial x_g}C_8 = \sqrt{w}\sin^2\theta \quad ,\end{aligned} \tag{2.136}$$

where the intrinsic weight of a given point is given by

$$\sqrt{w} = \frac{-(U-\lambda)\left(\frac{\partial\alpha}{\partial p}\right)}{2\ell^b C_2(1+kp)} \quad . \tag{2.137}$$

The α in the C_5 term is x_α^{oc} or x_α^{tr} as appropriate to the data point. Contributions to the C_4 and C_6 terms occur only for points in an occultation eclipse and contributions to the C_7 and C_8 terms occur only for transit eclipse. The variables λ_o and λ_t are the light values at internal tangency of the occultation and transit eclipse respectively. Choice of b (the weighting condition) depends on observational circumstances (Young 1967 and Linnell and Proctor 1970b). The regression equation, equation (2.136), must also be multiplied by the observational weight of the data point under consideration.

The C_i in equations (2.136) and (2.137) are related to the system parameters as follows:

$$C_1 = r_s^2 \csc^2 i \quad , \tag{2.138a}$$

$$C_2 = r_g r_s \csc^2 i \quad , \tag{2.138b}$$

$$C_3 = \sin^2\theta_{int.} \quad , \tag{2.138c}$$

$$C_4 = -\Delta\lambda_o \quad , \tag{2.138d}$$

$$C_5 = -\Delta U \quad , \tag{2.138e}$$

$$C_6 = \Delta x_s \quad , \tag{2.138f}$$

$$C_7 = -\Delta\lambda_t \quad , \tag{2.138g}$$

$$C_8 = \Delta x_g \quad , \tag{2.138h}$$

where $\theta_{int.}$ is the phase angle at internal tangency.

A least squares fit of equation (2.136) to the data produces the various C_i, which in turn can be solved for the system parameters. We note that the above equations apply only to completely eclipsing systems. A more detailed discussion of Kopal's method as used in computer solution of eclipsing binary light curves is given by Linnell and Proctor (1970a). In addition to the discussion by Linnell and Proctor, we note that

$$\lambda_o = U - L_s \quad , \tag{2.139}$$

$$\lambda_t = U - {}^{x}f^{tr}(k,-1)\ L_g \quad , \tag{2.140}$$

$$L_g + L_s + L_3 = U \quad , \tag{2.141}$$

(The function ${}^{x}f^{tr}$ is discussed in Appendix D.) Thus, with the values of λ_o, λ_t, and U obtained from the least squares solution we have

$$L_s = U - \lambda_o \quad , \tag{2.142}$$

$$L_g = (U-\lambda_t)/{}^{x}f^{tr}(k,-1) \quad , \tag{2.143}$$

and

$$L_3 = U - L_g - L_s \quad . \tag{2.144}$$

Several problems arise with the use of Kopal's method. The equations given above apply only to complete eclipses. Different equations must be used for partial eclipses. Also Kopal's method requires inversion of the α functions for the corresponding geometrical depths of eclipse at each point. This causes difficulty when the observed α values lie outside the theoretically permissible range. Further, eccentric orbits can't be handled directly. Finally, the normal equations used in Kopal's method do not rigorously satisfy the least squares condition in that the weights are not independent of the parameters, though they are treated as such in calculating the error sum of squares of the residuals.

Consistency of the results obtained by Kopal's method and the differential corrections method has been demonstrated for completely eclipsing systems (Linnell and Proctor 1971). However, because of the previously discussed limitations of Kopal's method, only the method of differential corrections was applied in the solutions of the systems discussed in Chapter V.

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III. DESCRIPTION OF COMPUTER PROGRAMS

A. Fourier Analysis Program

The program FOURIER calculates from one to ten Fourier coefficients for the non-eclipse portion of the light curve. The data points are fitted to an equation of the form

$$\ell(\text{obs.}) = \alpha_o + \sum_{i=1}^{NC} A_{n_i} \cos n_i\theta + \sum_{i=1}^{NS} B_{m_i} \sin m_i\theta \tag{3.1}$$

by the method of least squares. The n_i and m_i are the integers desired in the harmonic expansion, NC is the number of cosine terms, and NS is the number of sine terms. The various A_{n_i} and B_{m_i}, ℓ, and α_o are expressed in units of ℓ_c, the light of the comparison star. An abbreviated flow chart of the program FOURIER is given in Figure 10.

The program requires several control parameters to determine: (1) the form and order of the input data, (2) the number of data points, and (3) the number of Fourier analyses to be carried out for the current data set. Data may be in the form of phase or time units and light or magnitude units.

For each Fourier analysis to be carried out, the program requires a set of integers to determine which harmonic terms are to be included in the solution. A maximum of ten coefficients may be included without program modification. Phase limits of the non-eclipse portion of the light curve may be read in directly; alternatively, for circular orbits system parameters may be read in and phase limits calculated.

For each point the phase angle θ is calculated. For circular orbits

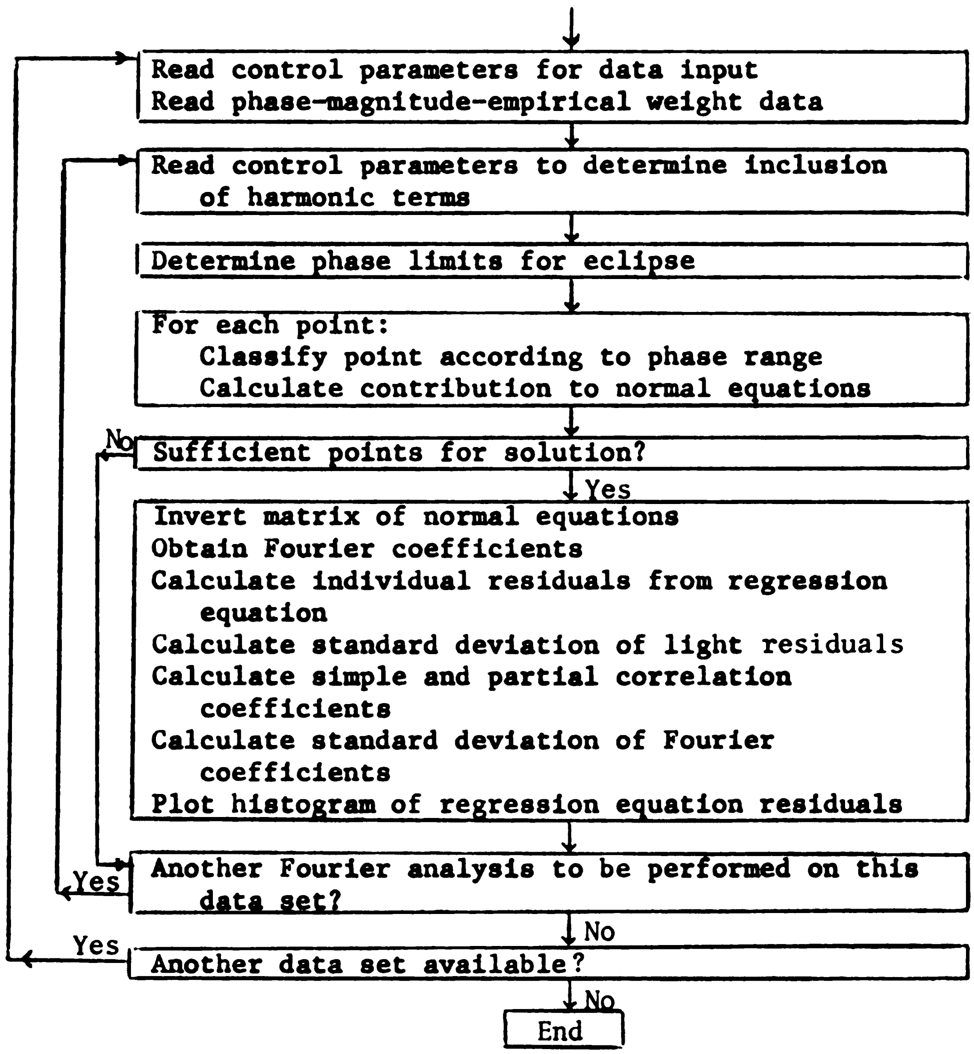
$$\theta = \frac{2\pi}{P}(t - t_o) \tag{3.2}$$

where P is the period and t_o is the time of minimum projected distance of centers. For eccentric orbits θ is calculated from Kepler's equation (equation (E-9)). The point is classified according to its phase value. If the point is in the non-eclipse portion of the light curve, its contribution to the normal equation is calculated following the standard formulas for the method of least squares. (See for example Ostle 1963, equation 8.59.) If the point is outside the non-eclipse portion of the light curve, it is omitted from the calculation. After each point has been processed a check is made to determine if there are sufficient points for solution. (The number of points must be greater than the number of coefficients being determined.) If there are insufficient points, solution for the present set of coefficients is terminated; otherwise, solution continues with the inversion of the matrix form of the normal equations and calculation of the Fourier coefficients. The matrices of simple and partial correlation coefficients are calculated. Standard deviations of the light residuals and individual Fourier coefficients are calculated. The program also calculates the Fourier coefficients and standard deviations normalized to α_0, the constant in the Fourier expansion. Individual residuals are calculated and plotted in a histogram. The histogram for a normal distribution with the same standard deviation is superimposed for comparison. The Kolmogorov-Smirnov goodness of fit test (Ostle 1963, p. 471) is applied to determine if the normal distribution satisfactorily fits the residuals.

Calculation of different sets of Fourier coefficients for the data is carried out as desired.

The entire process is repeated for each set of data points.

Figure 10. Flow Chart of FOURIER



B. Rectification Program

The program RRECK transforms observational data to the equivalent spherical model data. An abbreviated flow chart of the program is given in Figure 11.

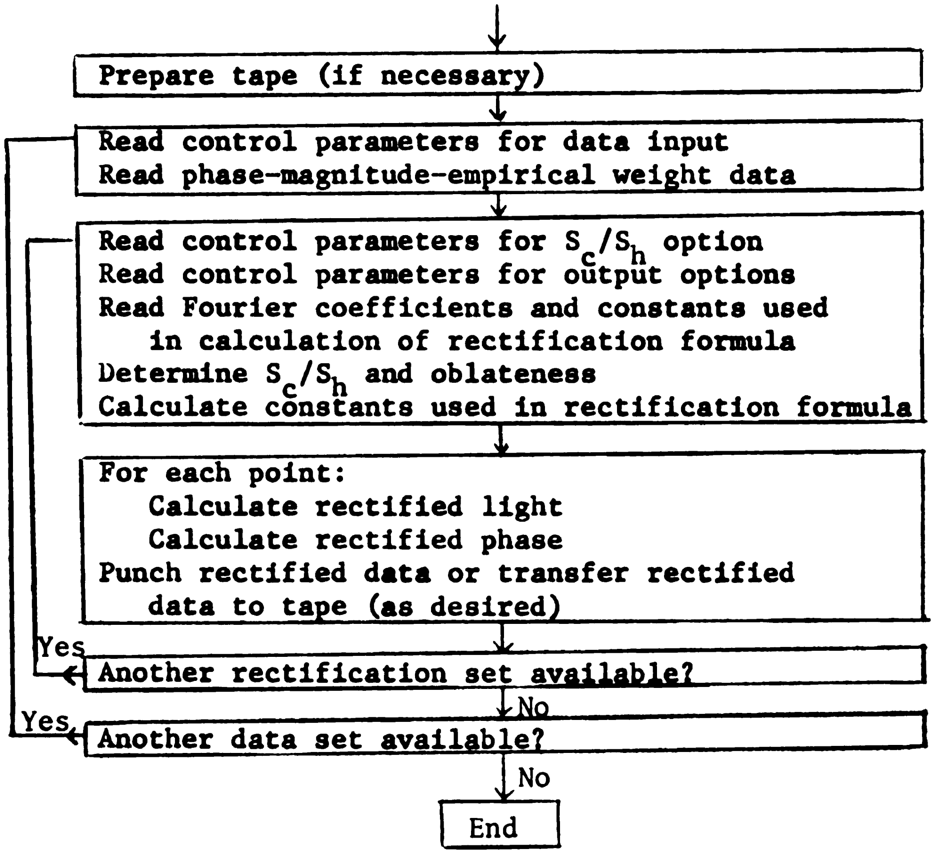
The program requires values of control parameters that determine: (1) the form and order of input data, (2) number of rectifications for the current data set, and (3) the number of data points in the data set. The data may be in the form of phase or time units and light or magnitude units. Conversion of the data to phase and light units is carried out as necessary.

The rectification formula is determined using the input values of Fourier coefficients, limb-darkening and gravity-darkening coefficients, angle of inclination and color temperature of the primary. Input control parameters allow three options for determining the ratio of reflected lights: (1) using an input value for the ratio of reflected lights, (2) using input depths of eclipse to calculate the ratio of reflected lights, or (3) using luminous efficiency calculations to find the ratio of reflected lights.

Rectified values of the light and phase may be output on cards or magnetic tape.

After each independent rectification of the data set has been performed, calculations continue on succeeding data sets.

Figure 11. Flow Chart of RRECK



C. Differential Corrections Program

The program DIFCORT produces from one to eleven differential corrections to spherical model parameters. An abbreviated flow chart of the program is given in Figure 12.

The program requires values of control parameters that determine: (1) the form and order of the input data, (2) the number of initial parameter sets for which differential corrections are to be found, and (3) the number of data points in the data set. The data may be in the form of phase or time units and light or magnitude units. Conversion of the data to phase and light units is carried out as necessary. The data may be on cards or magnetic tape. The program requires initial values for the spherical model parameters r_g, r_s, L_g, L_s, i, e, ω, t_o, and RF, where P is the period, and

$$RF = 2.5 \log U \tag{3.3a}$$

$$= 1.0857362 \ln U \quad . \tag{3.3b}$$

RF is the reference magnitude corresponding to the unit of light. Also required is a set of integers to indicate which differential corrections are to be included in the solution. (A maximum of ten differential corrections may be included simultaneously.) Control parameters to determine the maximum number of iterations and the type of solution (occulation eclipse, transit eclipse, or both) are also required.

Using the current values of the spherical model parameters, the minimum value of the geometrical depth for each eclipse is calculated along with ℓ_{min} the corresponding value of the light. Thus

$$\ell_{min}^{oc} = 1 - {}^{x}f^{oc}(k, p_{min}^{oc})\ L_s \tag{3.4}$$

and

$$\ell_{min}^{tr} = 1 - {}^{x}f^{tr}(k, p_{min}^{tr})\ L_g \quad , \tag{3.5}$$

where p_{min}^{oc} and p_{min}^{tr} are the minimum values of geometrical depth for occulation and transit eclipse respectively. The primary (deeper) minimum is then associated with the type of eclipse having the smallest value of minimum light. The ranges of partial and total phase of each eclipse are calculated. Each data point is then classified according to its phase range as being: (1) outside eclipse, (2) in partial phase of occulation, (3) in total phase of occulation, (4) in partial phase of transit, or (5) in total phase of transit. Partial derivatives required for the regression equation, (2.112), are calculated for the point. The value of ℓ_c is calculated using current spherical model parameters. The point's contribution to the normal equations, equation (2.124), is included.

At this point in the calculation a check is made to determine if there are sufficient points to obtain a solution. If there are sufficient points, calculation continues with the matrix inversion of the normal equations. Otherwise, solution of the present set of parameters is terminated.

The Gauss-Jordan method (Smillie 1966, p. 134) is used for solving the normal equations. As a check the program calculates the values of $\underset{\sim}{A}\underset{\sim}{A}^{-1}-\underset{\sim}{I}$, where $\underset{\sim}{I}$ is the unit matrix. Each matrix element should equal zero, within rounding errors.

Individual residuals of the regression equation are computed and used to calculate the standard deviation, σ(est.), of the observed points from the calculated values. The standard deviation is also calculated using equation (2.132), a form not requiring calculation of individual residuals. A histogram of the residuals is plotted. The Kolmogorov-Smirnov test of goodness of fit is applied to check the residuals for conformity with a normal distribution with standard deviation σ(est.).

The simple and partial correlation coefficients are calculated along with probable errors of the parameters. The current values of the spherical model parameters are then incremented by the differential corrections. The values of limb-darkening coefficients, luminosities, radii, and eccentricity are restricted as follows:

$$-1 \leq x \leq 1 \quad , \tag{3.6}$$

$$0 \leq L \leq 1 \quad , \tag{3.7}$$

$$0 \leq r \leq 1 \quad , \tag{3.8}$$

$$r_g \leq r_s \quad , \tag{3.9}$$

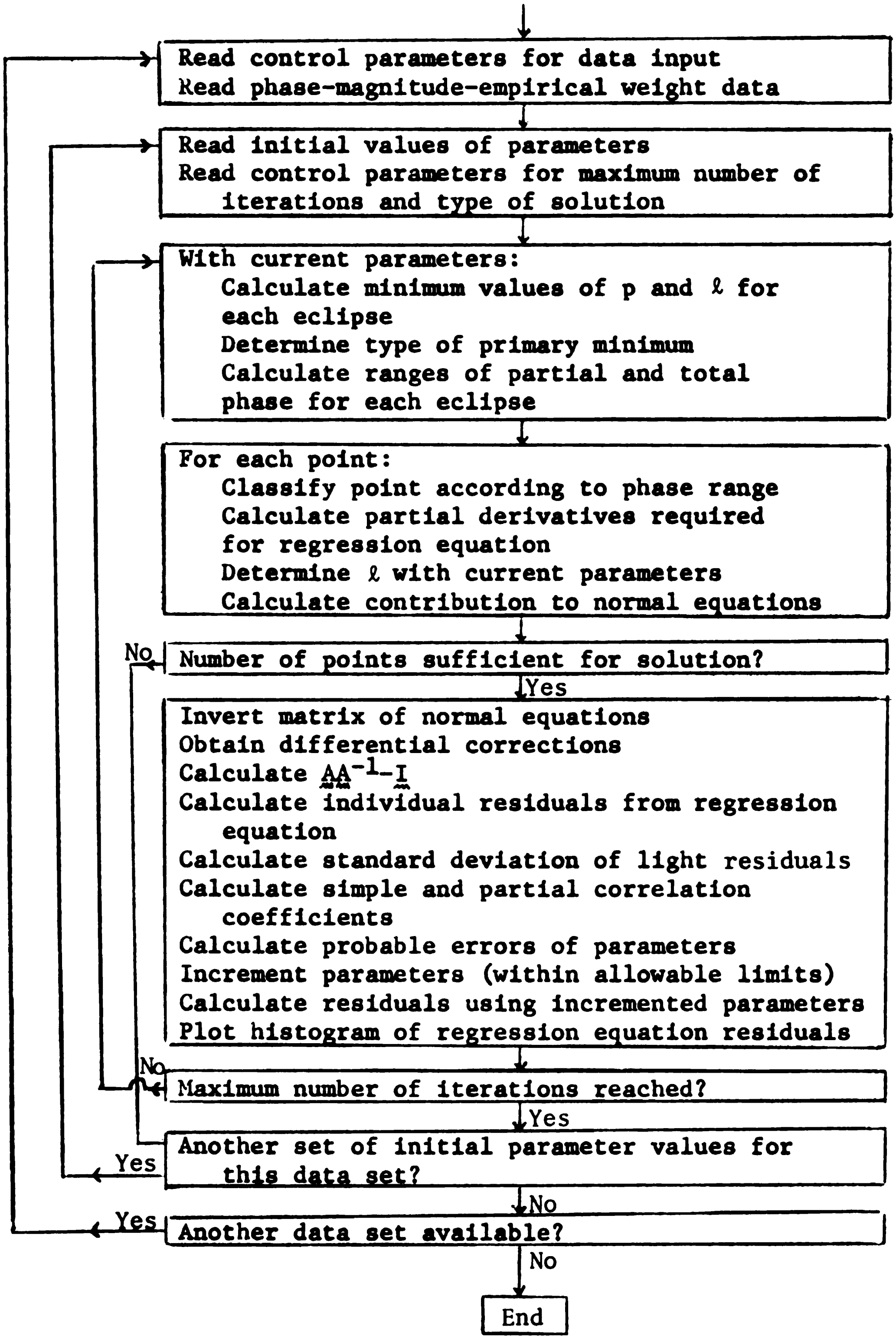
$$0 \leq e \leq 0.999 \quad . \tag{3.10}$$

Values of x less than zero are included to allow for the possibility of limb brightening. The theoretical values of light calculated with the incremented parameters are then used to calculate the standard deviation of the observed values. It is customary to normalize the light curve such that the non-eclipse portion is unity. Thus, the light values are normalized by the replacement

$$\frac{\ell}{1+\Delta U} \rightarrow \ell \quad . \tag{3.11}$$

Calculation of differential corrections for the incremented parameters is repeated for the maximum allowed number of iterations. Iteration on succeeding sets of initial parameter values is then carried out. The entire procedure is repeated for each data set.

Figure 12. Flow chart of DIFCORT



D. Kopal's Method Program

The Kopal's Method program, called CFIT, is restricted to completely eclipsing systems and spherical orbits. The data are assumed rectified. The program produces the values of the parameters C_1, C_2, and C_3, which are functions that may be solved for r_g, r_s, and $\cos^2 i$. The program also allows for inclusion of differential corrections to U, λ_o, λ_t, x_g, and x_s, where λ_o and λ_t are the values of light at internal tangency ($p = -1$) of the occulation and transit eclipse respectively. From the equations

$$\lambda_o = U - L_s \quad , \tag{3.13}$$

$$\lambda_t = U - {}^{x}f^{tr}(k,-1)\, L_g \quad , \tag{3.14}$$

$$L_g + L_s + L_3 = U \quad , \tag{3.15}$$

where ${}^{x}f^{tr}(k,p)$ is defined in Appendix D, we see that we may solve for L_s, L_g, and L_3 as follows

$$L_s = U - \lambda_o \quad , \tag{3.16}$$

$$L_g = (U - \lambda_t)/{}^{x}f^{tr}(k,-1) \quad , \tag{3.17}$$

$$L_3 = U - L_g - L_s \quad . \tag{3.18}$$

Further description of the program is given by Linnell and Proctor (1970a, p. 1043).

E. Program Accuracy

The programs FOURIER, RRECK, DIFCORT, and CFIT are written in the CDC 3600 FORTRAN language. The precision of the CDC 3600 in single precision is approximately 10 decimal digits.

The programing of the direct eclipse functions (${}^{x}\alpha^{oc}(k,p)$ and ${}^{x}\alpha^{tr}(k,p)$) used is described by Linnell (1965a,b; 1966a,b,c). The stated programming objective for these functions was to obtain a fractional error of 10^{-6}. This was obtained for most values of k and p. The maximum fractional error was given as 10^{-5}. In most regions of the k-p plane for which eclipsing systems have meaning the absolute errors are less than 10^{-6}.

The matrix inversion in the programs FOURIER, DIFCORT, and CFIT was carried out in double precision with corresponding word length accuracy of approximately 25 decimal digits. The Gauss-Jordan method (Smillie 1966) is used for carrying out the matrix inversion. Further, to insure minimum rounding and truncation error, the matrix inversion routine chooses as pivot element, at each stage of the matrix inversion, the element largest in absolute value in the rows and columns not containing previous pivot elements. As a check the program calculates the matrix $\underset{\sim}{A}\underset{\sim}{A}^{-1}-\underset{\sim}{I}$, where $\underset{\sim}{I}$ is the unit matrix. In no case has an element of this matrix been found to be larger than 10^{-19}. Typically the elements of this matrix are several orders of magnitude smaller. Further discussion of accuracy is given in Linnell and Proctor (1970a).

Final validation of the programs rests in the solution of synthetic light curves with known parameters. Discussion of such solutions is given in the following chapter.

Complete program listings are on file in the Astronomy Department, Michigan State University.

IV. SOLUTION OF SYNTHETIC LIGHT CURVES

This chapter contains the results of the application of the method of differential corrections to synthetic light curves. The synthetic light curves are constructed on the Spherical Model and include random errors with normal distribution and assignable dispersion.

Synthetic light curves are useful in the validation of the programs. Convergence on known parameters provides the most convincing test of program reliability. Synthetic light curves may also be used to evaluate the effect of the dispersion and number of observations on ability to extract the desired parameters. Synthetic light curves may be based on parameters obtained from the results of actual light curve analysis. Subsequent solution of these curves may provide further confidence in the results; alternatively, the solution may indicate the need for more observations of greater accuracy.

Synthetic light curves with zero dispersion were used to validate DIFCORT. Resulting light residuals were on the order of 10^{-6}.

Table 2 gives the results of analysis of a synthetic light curve similar to the light curve of the system S Cancri. The dispersion is comparable to that obtainable under optimum observational conditions. Primary minimum is a deep occulation eclipse, while secondary minimum is very shallow. Thus x_s can be reliably determined, but the uncertainty in x_g is rather large. Satisfactory convergence on the parameters is demonstrated.

Table 3 shows the results of a test for separability of r_s and x_s. Irwin (1947) has shown that for certain values of parameters the ratio of the coefficients of x_s and r_s is nearly constant. There is the possibility that the correlation is so great as to prevent separation. To test for this possibility, a synthetic light curve was constructed with 800 total points of intrinsic dispersion $\sigma = .005$. The parameters closely accord with the parameters adopted by Irwin for his example. Convergence was not as good as for the previous example; however, resulting parameter values were at most 2½ standard deviations from the true values.

As an example of a system with third light, a synthetic light curve corresponding to BR Cygni was constructed. The curve had a total of 430 points, 130 in occulation and 100 in transit. Convergence occured as shown in Table 4. Solution of actual data for BR Cygni is discussed in Chapter V.

An illustration of the complications that occur due to correlation of the system parameters is given using synthetic BV 412 data. For the σ=0.0 data with initial estimates of the parameters given in Table 5, convergence to the true parameters occurs in three iterations. The light curve with $\sigma = .0074$ was solved twice, once allowing L_3 to vary (Table 6) and once holding L_3 to its true value (Table 7). Convergence was obtained only by holding L_3 constant. It was noted that when L_3 was allowed to vary, the absolute magnitude of the correlation coeffi-cients between L_3 and r_g, L_g, and $\cos^2 i$ were greater than 0.99. Also ote that this is a partially eclipsing system. It has often been ssumed that it is not possible to determine limb-darkening coefficients or such systems (Wilson 1968). We conclude from these results that e determinability is based more on the accuracy of the light curve and e density of observations, and less on the geometrical depth of lipse.

Extensive tests on eccentric systems have not yet been carried out. However, the results of iterative solutions of a synthetic curve with zero dispersion are given in Tables 8, 9, and 10. Initial parameters were the same in all three cases. In the first solution only the differential corrections for i, e, ω, and t_o were calculated. Satisfactory convergence occured in three iterations. When all parameters except L_3 were included, there was no indication of convergence after three iterations. This is probably due to correlation of the variables combined with sufficiently large error in the initial estimates of the parameters. Higher order differential corrections then become significant. (Compare σ(est.) and σ(cal.) for the first iteration in Tables 8 and 9.) The absolute values of the simple correlation coefficients of L_g with r_s and $\cos^2 i$ are large (greater than 0.97). Since for total eclipse an estimate of L_g can be obtained from the light during total phase of occultation, a third solution, omitting differential corrections to L_g was carried out. Satisfactory convergence occurs. The results are given in Table 10.

Table 2. Iterative Solution of Synthetic S Cancri (b = 1)

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	p_o	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.44444	86.000	.18000	.08000	.500	.500	.04000	.96000	.00120	-1.378		
1	.58023	86.344	.17829	.10345	1.000	1.000	.03438	.96562	.00306	-1.107	4.4532	5.4307
2	.38340	84.423	.19206	.07363	-1.000	.331	.03218	.96782	.00043	-1.288	9.9707	0.5809
3	.50407	85.466	.18391	.09270	1.000	1.000	.03388	.96612	.00134	-1.131	2.6553	4.2528
4	.38240	83.914	.19648	.07513	.485	.417	.03216	.96784	.00024	-1.204	6.6127	0.4859
5	.45068	84.534	.19095	.08606	.773	.931	.03288	.96712	.00114	-1.112	2.5692	1.8289
6	.37965	83.589	.19927	.07565	.191	.418	.03217	.96783	.00043	-1.158	3.7761	0.4838
7	.40785	83.820	.19702	.08035	1.000	.580	.03237	.96763	.00062	-1.112	0.6449	0.7233
8	.38468	83.452	.20042	.07710	.628	.404	.03217	.96783	.00043	-1.120	0.6293	0.4858
9	.38381	83.416	.20076	.07705	.787	.392	.03218	.96782	.00038	-1.117	0.4855	0.4856
10	.38363	83.410	.20078	.07703	.801	.391	.03218	.96782	.00037	-1.117	0.4855	0.4855
p.e.	±.00081	± 0.046	±.00043	±.00033	±.386	±.012	±.00001		±.00017			
True	.38271	83.390	.20091	.07689	.400	.400	.03220		.00000		0.4868	

The synthetic S Cancri light curve contains 800 points. Each minimum contains 300 points.

Table 3. **Iterative Solution of Synthetic Light Curve.** (b = 0)
Test for limb darkening and radius correction.

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	p_o	σ(est.) (10^{-2})
Initial	.79167	89.500	.24000	.19000	.790	.750	.76000	.24000	.00200	-1.217	
1	.80060	90.000	.24761	.19823	.910	1.039	.76007	.23993	.00033	-1.205	0.493
2	.80032	90.000	.24891	.19921	.967	1.057	.76144	.23856	.00023	-1.249	0.489
3	.80050	90.000	.24923	.19951	.980	1.073	.76121	.23879	.00001	-1.249	0.488
4	.80057	90.000	.24931	.19959	.984	1.077	.76121	.23879	.00011	-1.249	0.488
5	.80058	90.000	.24932	.19960	.985	1.078	.76120	.23880	.00011	-1.249	0.488
p.e.	±.00019		±.00034	±.00022	±.005	±1.023	±.00046		±.00029		
True	.80000	89.500	.25000	.20000	1.000	1.000	.76000				

The synthetic light curve used to test for limb darkening and radius correction contains 800 points. Each minimum contains 350 points.

Table 4. Iterative Solution of Synthetic BR Cygni (b = ½)

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	P_o	$\sigma(b=0)$ (10^{-2})	$\sigma(cal.)$ (10^{-2})	$\sigma(est.)$ (10^{-2})
Initial	.8158	88.00	.3800	.3100	.400	.400	.1700	.5920	.2380	.0000	-1.11			
1	.8570	90.00	.3524	.3020	.533	.709	.1915	.5929	.2165	.0009	-1.17	1.271	1.36166	1.36626
2	.8723	90.00	.3529	.3078	.423	.872	.1881	.5928	.2194	.0013	-1.15	1.269	1.35727	1.35654
3	.8566	88.56	.3535	.3028	.366	.786	.1943	.5932	.2124	.0012	-1.08	1.269	1.35698	1.35691
4	.8547	88.48	.3536	.3022	.337	.770	.1962	.5949	.2111	.0012	-1.08	1.270	1.35840	1.35843
5	.8534	88.43	.3536	.3018	.332	.763	.1964	.5913	.2123	.0012	-1.08	1.270	1.36103	1.35690
6	.8531	88.42	.3537	.3017	.329	.762	.1965	.5936	.2099	.0012	-1.08	1.269	1.35690	1.35690
p.e.	±.0258	± 1.56	±.0017	±.0106	±.221	±.181	±.0148	±.0148	±.0150	±.0007				
True	.7960	86.14	.3557	.2831	.400	.560	.2148	.5947	.1905	.0000	-1.02		1.33	

The synthetic BR Cygni light curve contains 430 points, 130 in occultation and 100 in transit.

Table 5. Iterative Solution of Snythetic BV 412 (σ = 0.0)

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-4})	σ(cal.) (10^{-4})	σ(est.) (10^{-4})
Initial	.5096	77.93	.4180	.2130	.600	.600	.9470	.0530	.0000	.0000	-.981			
1	.5251	77.76	.4061	.2133	.380	.775	.8978	.0636	.0386	.0000	-.910	2.584	2.767	0.567
2	.5166	77.18	.4111	.2124	.415	.777	.9315	.0643	.0042	.0000	-.891	0.469	0.526	0.021
3	.5161	77.12	.4114	.2123	.417	.780	.9356	.0643	.0001	.0000	-.888	0.022	0.022	0.005
True	.5160	77.12	.4114	.2123	.417	.780	.9357	.0643	.0000	.0000	-.888		.0000	

The synthetic BV 412 (σ=0.0) light curve contains 730 points, 110 in occultation and 240 in transit.

Table 6. Iterative Solution of Synthetic BV 412 ($\sigma = 0.0074$, $b = \frac{1}{2}$)

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	$\sigma(b=0)$ (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.5096	77.93	.4180	.2130	.600	.600	.9470	.0530	.0000	.0000	-.981			
1	.4769	75.20	.4329	.2065	.666	(1.880)	(1.1425)	.1493	-.1493	.0000	-.860	2.492	2.587	.7364
2	.4749	74.02	.4399	.2089	.754	(1.252)	(1.1547)	.2172	-.2173	.0000	-.788	4.062	4.210	.7365
3	.4918	75.22	.4309	.2119	.705	(1.067)	(1.0630)	.1213	-.1213	.0000	-.829	1.713	1.769	.7363
p.e.	±.0159	± 4.13	±.0280	±.0069	±.188	±.259	±.2952		±.3055					
True	.5160	77.12	.4114	.2123	.417	.780	.9357	.0643	.0000	.0000	-.888		.7381	

Table 7. Iterative Solution of Synthetic BV 412 ($\sigma = 0.0074$, $b = \frac{1}{2}$) excluding L_3

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	$\sigma(b=0)$ (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.5096	77.93	.4180	.2130	.600	.600	.9470	.0530	.0000	.0000	-.981			
1	.5127	76.83	.4175	.2140	.561	(1.814)	.9402	.0599		.0000	-.886	.7184	.7370	.7363
2	.5144	76.63	.4189	.2155	.606	(1.278)	.9401	.0599		.0000	-.871	.7184	.7370	.7364
3	.5141	76.62	.4193	.2155	.613	(1.287)	.9402	.0598		.0000	-.872	.7185	.7371	.7364
p.e.	±.0025	± 0.44	±.0041	±.0031	±.075	±.275	±.0019			±.0003				
True	.5160	77.12	.4114	.2123	.417	.780	.9357	.0643	.0000	.0000	-.888		.7381	

The synthetic BV 412 light curve (σ=0.0074, b=½) contains 730 points, 110 in occultation and 240 in transit.

Table 8. **Iterative Solution of Synthetic Eccentric Light Curve (e = .15, ω = 45.0°)** for i, e, ω, and t_o

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	e	ω	t_o	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.7500	86.00	.2000	.1500	.500	.500	.8000	.2000	.0000	.1300	40.00	.0005		
1		88.09								.1489	45.18	-.0002	8.285	1.102
2		88.59								.1500	44.98	.0000	0.348	0.170
3		88.60								.1500	45.00	.0000	0.003	0.000
True	.7500	88.60	.2000	.1500	.500	.500	.8000	.2000	.0000	.1500	45.00	.0000	0.000	0.000

The synthetic eccentric light curve (e=.15, ω=45.0°) contains 334 points. Each minimum contains 134 points.

Table 9. Iterative Solution of Synthetic Eccentric Light Curve ($e = .15, \omega = 45.0°$)

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	e	ω	t_o	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.7500	86.00	.2000	.1500	.500	.500	.8000	.2000	.0000	.1300	40.00	.0005		
1	.5212	90.00	.2159	.1125	.181	.025	.9125	.0871	-.0004	.1419	40.96	-.0002	8.84	1.116
2	.6153	90.00	.2126	.1308	1.000	-1.000	.8110	.1895	.0002	.0605	3.20	.0008	21.28	1.765
3	.6784	87.14	.2323	.1576	.744	-1.000	.8840	.0984	-.0191	.1037	248.17	-.0002	23.11	6.672
True	.7500	88.60	.2000	.1500	.500	.500	.8000	.2000	.0000	.1500	45.00	.0000	.00	

Table 10. Iterative Solution of Synthetic Eccentric Light Curve ($e = .15, \omega = 45.0°$) excluding L_g

Iteration	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	e	ω	t_o	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.7500	86.00	.2000	.1500	.500	.500	.8000	.2000	.0000	.1300	40.00	.0005		
1	.7437	86.85	.2076	.1544	.801	.921		.2000	.0000	.1520	46.94	-.0023	10.93	1.117
2	.7328	88.90	.2022	.1482	.640	.379		.2003	.0003	.1404	41.58	.0001	8.92	.460
3	.7496	88.52	.2002	.1501	.518	.514		.1997	.0000	.1496	45.17	-.0001	5.93	.310
4	.7496	88.61	.2001	.1500	.503	.509		.2000	.0000	.1501	45.03	.0000	0.23	.099
True	.7500	88.60	.2000	.1500	.500	.500		.2000	.0000	.1500	45.00	.0000	.00	

The synthetic eccentric light curve ($e=.15, \omega=45.0^{o}$) contains 334 points. Each minimum contains 134 points.

V. ANALYSIS OF PUBLISHED DATA

An attempt was made to look at each set of data listed in the catalog of Koch, Plavec and Wood (1970). From the more than two hundred systems listed, ten were chosen for further study. Selection, although somewhat subjective, was based on the following criteria:

(1) Well-separated systems

(2) Coverage of entire phase range

(3) Number and quality of observations

(4) Individual observations published

(5) Lack of obvious complications.

Before the analyses of the individual systems are discussed, it is necessary to consider the type of eclipse to be associated with the consecutive minima. If spectroscopic radial velocity curves are available, it is possible, in principle, to determine whether the primary minimum is an occultation or a transit.

Let L_1 be the luminosity and J_1 the mean surface brightness of the star of greater surface brightness (the star being eclipsed during primary minimum). Let L_2 and J_2 be the corresponding quantities for the star of lesser surface brightness. The star approaching the observer immediately before primary minimum is thus the star of luminosity L_2 and mean surface brightness J_2. For bolometric light we have

$$\frac{L_1}{L_2} = \frac{{r_1}^2 J_1}{{r_2}^2 J_2} \qquad (5.1)$$

where r_1 is the radius of the star of greater surface brightness and r_2 is the radius of the star of lesser surface brightness. Thus

$$\frac{r_1^2}{r_2^2} = \frac{L_1/L_2}{J_1/J2} \qquad (5.2)$$

where an estimate of L_1/L_2 is obtained from the spectroscopic observations and an estimate of J_1/J_2 is obtained from the depths of eclipse. In this way it can be determined if the star of greater surface brightness is the larger or smaller star and hence whether the primary minimum is an occultation or a transit.

Unfortunately, spectroscopic information is not always available; or if it is, the errors associated with the estimates of L_1/L_2 and J_1/J_2 may prevent positive determination of the type of eclipse. Thus it is not always possible to make an "a priori" judgement as to the type of eclipse. Both possibilities must then be considered.

The results of analysis of the light curves of 10 eclipsing binary systems are presented in the following sections. For each system a general discussion is presented and tabular data follow. In each table the source of the original photoelectric data is given, along with the spectral type of the primary. The spectral type of the secondary is given if available. The spectral type or range of spectral types of the secondary, as found by subsequent luminous efficiency calculations, is given in parentheses. The value of the period is followed by the adopted designation for the type of primary minimum.

The data in each table are divided into three sections.

Section A contains the results of the Fourier analysis. The phase ranges of the points included in the Fourier analysis are given in parentheses. For each light curve the results of two Fourier analyses are presented. Normally the first analysis for each color is the analysis carried to terms of order 2θ, while the second analysis is to terms of order 3θ. The standard deviation (normalized to α_0) of the resulting residuals is presented in the second column of Section A. This is followed by α_0, the constant of the Fourier expansion, and the remaining Fourier coefficients (normalized to α_0). (Note $\alpha_0 = A_0$ if it is assumed that there is no third light.) The Fourier coefficients are followed by NDF, the number of data points used in the Fourier analysis. The adopted values of the Fourier coefficients are followed by initial estimates of the ratio of surface brightnesses and the color temperatures of the primary components. The color temperature of the primary component is taken from Figure 15, using the known spectral type of the star. This procedure for estimation of the temperature follows the recommendation of Jurkevich (1964, p. 185). The temperature of the secondary and the ratio of reflected lights resulting from the luminous efficiency calculations is given next, followed by the subsequent value of ε, the oblateness of the equatorial cross section and N (given by the equation (2.31)).

Section B of each table gives the coefficients used in the rectification formula, equation (2.84). RF_0, if given, is the value of the reference magnitude initially subtracted from observed magnitude differences in order to normalize the non-eclipse portion of the light curve to unity.

Section C of each table contains the equivalent spherical model parameters. The value of b, the exponent of the light in the weights of the conditional equations, is given in parentheses. The parameters designated as "Initial" are those determined by the author publishing the original data. Often it was not clear whether the value of inclination given by the author was i_r, j, or i. The values of inclination are simply included in the tables as they were given in the original paper. In addition to the geometric parameters and luminosities obtained with the differential corrections method, the reference magnitude RF is given, where

$$RF = 2.5 \log U \tag{5.3}$$

The geometric depth of eclipse p_o is also given. For eccentric orbits the value of p_o from the primary minimum is used. The last three columns present the various standard deviations of the rectified data. The standard deviation $\sigma(b = 0)$ of the light values is given, followed by the standard deviation of the weighted light values σ(cal.) and its estimator σ(est.). (See equations (2.131) and (2.134).)

The number of observations used in the solution of the light curve is given beneath the tabular data. This information is followed by values of λ_o and λ_t. For partially eclipsing systems λ_o and λ_t are the calculated values of light for the occultation eclipse and the transit eclipse, respectively, at minimum geometrical depth. For completely eclipsing systems λ_o and λ_t are the calculated values of light for internal contact ($p = -1$) of the occultation and transit eclipse respectively. Also included are the ratio of the mean surface intensities J_g/J_s and the ratio of the central surface intensities $(J_g/J_s)_c$, where

$$\frac{J_g}{J_s} = \frac{r_s^2 L_g}{r_g^2 L_s} \tag{5.4}$$

and

$$\left(\frac{J_g}{J_s}\right)_C = \frac{3 - x_s}{3 - x_g} \frac{J_g}{J_s} \qquad (5.5)$$

(Kopal 1950, p. 53).

A. CO Lacertae

CO Lacertae is a tenth magnitude system exhibiting a small value of orbital eccentricity. The system is also notable for the short period of its apsidal motion. Semeniuk (1967), from an analysis of 27 times of minima, obtained $e = 0.027$ and $\omega = 65.4°$ for the epoch of her observations. Smak (1967), from his spectroscopic analysis of the system, classified the primary component as B8.5IV and the secondary component as B9.5V.

The recent photoelectric observations of Semeniuk (1967), have been chosen for analysis.

Semeniuk, in her analysis of the data, reflected the descending branches of minima onto the respective ascending branches and grouped the observations into normal points. These normal points were rectified for ellipticity only. She assumed values for the limb-darkening coefficients and made a preliminary analysis using the iterative method of Piotrowski (1948) and Kopal (1959). She reported lack of convergence for the primary minimum. Using the values obtained for the analysis of the secondary minimum, she made a single differential corrections solution for the geometric parameters and luminosities. The results of the individual B and V Semeniuk solutions are listed as the initial values of the parameters in Table 11C.

Results of analysis of individual observations, using the programs FOURIER, RRECK, and DIFCORT, are given in Table 11. The values of e and ω given by Semeniuk were used.

Inclusion of terms of order 3θ decreased the standard deviation of residuals in both the B and V light curves. Thus the corresponding coefficients were used in the subsequent rectification.

Five errors were found in the published phase values. The Julian dates of these observations, along with the corrected phase values, are listed in Appendix F.

Following rectification, a solution of the B curve was attempted assuming that the primary minimum was a transit and that there was no excess light. Convergence did not occur. The results of the sixth iteration are given in Table 11C under the designation "B1". Extreme divergence was exhibited in the seventh iteration. The B2 solution is discussed below.

Iterative analysis including excess light did converge. The parameters and their respective probable errors are given in Table 11C under the designation "Adopted". The V light curve also converged under the assumption of primary minimum a transit allowing excess light. Results are listed in the table. The geometric parameters from the B and V light curves are in good agreement. The difference in the standard deviation obtained from the Fourier analysis and the differential corrections analysis is comparable to the probable error of the standard deviation and can be accounted for by the error in the choice of b, the exponent of the light in the weights. To determine if the choice of b significantly affects resulting parameters, the V light curve was analyzed again with $b = 1$. All resulting parameters were less than ½ standard deviation from the values obtained with $b = ½$.

In the process of analysis, a solution of the B curve was carried out on the assumption that primary minimum was an occultation. Surprisingly, convergence was obtained in this case also. The resulting parameters are given in Table 11C under the designation "B2". It has commonly been assumed that iterations will converge only if the type of eclipse has been correctly identified (Kopal 1959, p. 334). Note the close correspondence of the geometrical parameters and luminosities of this solution with those of the solution assuming the primary minimum a transit. However, the resulting values of limb-darkening coefficients assuming primary minimum an occultation are not in good agreement with the theoretical values discussed later.

Unfortunately, the spectroscopic data of Smak were not sufficiently accurate to determine the type of eclipse; however, it is felt that the assumption of primary minimum a transit and presence of third light provides the best solution. The standard deviation is a few percent smaller for this case. In addition, limb-darkening coefficients resulting from this assumption are in good agreement with the theoretical values discussed later. The presence of apsidal motion lends weight to the existence of third light.

Table 11. Parameters of CO Lacertae.

Original data : I. Semeniuk, A.A. 17, 223 (1967)
Spectral type : B8.5IV, B9.5V (B9.5 - uncorrected for third light)
Period : 1.542 d.
Primary minimum: transit

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.08 - 0.43 \text{ and } 0.59 - 0.92 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
B	.751	1.0951	-.0023	-.0139		+.0017	-.0007		171						
p.e.		±.0007	±.0009	±.0011		±.0008	±.0007								
B	.745	1.0952	-.0043	-.0138	-.0026	+.0016	-.0008	+.0008	171	1.15	18000	16900	1.43	.0182	2.2
p.e.		±.0007	±.0014	±.0011	±.0013	±.0008	±.0007	±.0008							
V	.674	1.0802	+.0007	-.0125		+.0007	+.0009		166						
p.e.		±.0006	±.0009	±.0010		±.0007	±.0006								
V	.662	1.0803	+.0002	-.0126	-.0005	+.0005	+.0005	+.0019	166	1.17	16000	14800	1.57	.0112	2.2
p.e.		±.0006	±.0012	±.0010	±.0011	±.0007	±.0006	±.0007							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	+.0134	-.0047	+.0134	1.1237	-.0168
V	-.0005	+.0002	-.0004	1.0935	-.0276

Table 11 (cont'd.)

C. Equivalent Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
B Initial	.85	85.2	.251	.216	.4	.4	.601	.399						
B1	.8262	84.50	.2550	.2107	.596	-.055	.6046	.3954		-.0004	-0.77	.9317	.8390	.7923
p.e.	±.0078	± 0.35	±.0011	±.0029	±.045	±.119	±.0075			±.0004				
B2	.8251	87.71	.2465	.2034	-.667	-.177	.5577	.3991	.0432	-.0002	-0.64	.7387	.7869	.7869
p.e.	±.0029	± 0.20	±.0007	±.0013	±.080	±.101	±.0079		±.0084	±.0004				
B Adopted	.8427	87.86	.2558	.2155	.645	.340	.5228	.3564	.1207	.0004	-1.02	.7189	.7661	.7661
p.e.	±.0069	± 0.31	±.0008	±.0024	±.053	±.095	±.0129		±.0133	±.0004				
V Initial	.85	85.5	.255	.216	.4	.4	.608	.392						
V Adopted	.8410	87.56	.2535	.2132	.477	.534	.5591	.3597	.0812	-.0022	-0.99	.6918	.7401	.7401
p.e.	±.0063	± 0.34	±.0008	±.0022	±.057	±.078	±.0116		±.0119	±.0004				

The B1 solution is based on the assumption that primary minimum is a transit. It excludes third light.
The B2 solution is based on the assumption that primary minimum is an occultation.
The adopted solutions are based on the assumption that primary minimum is a transit and allow third light.

The B light curve contains 454 observations, 124 in primary minimum and 143 in secondary minimum.
For the adopted B solution λ_t = .5981, λ_o = .6436, J_g/J_s = 1.04 and $(J_g/J_s)_c$ = 1.18.

The V light curve contains 447 observations, 125 in primary minimum and 145 in secondary minimum.
For the adopted V solution λ_t = .5824, λ_o = .6409, J_g/J_s = 1.10 and $(J_g/J_s)_c$ = 1.08.

B. CM Lacertae

CM Lacertae, an eighth magnitude double-line spectroscopic binary, was observed by Alexander (1958) and later by Barnes, Hall, and Hardie (1968). The former investigator did not report individual observations. Thus, only the more recent photoelectric observations of Barnes, Hall, and Hardie were chosen for study. Spectroscopic observations were obtained by Sanford (1934) and re-examined by Popper (1967).

Alexander, in the analysis of his data, concluded that the primary minimum was an occultation. Barnes, Hall, and Hardie felt the alternative assumption was better for fitting their data. Consequently, the Barnes data were studied under both assumptions. The results are presented in Table 12C. For the U light curve convergence was obtained for both assumptions. However, the residuals assuming primary minimum an occultation have a significantly smaller standard deviation. It is interesting to note the correspondence of the values of the limb-darkening coefficients. For the two assumptions $x = .33$ and $.24$ for the brighter star and $x = .87$ and $.78$ for the cooler star. While the B and V light curves did not converge assuming primary minimum a transit, observe that there is agreement between the values of the radii of the hotter and cooler stars for each assumption. The V light curve was analyzed allowing inclusion of excess light. The resulting value of excess light was approximately two standard deviations from zero. Since the geometrical parameters of the V solution agreed well with the B and U solutions, it was not felt worthwhile to re-analyze the V data excluding the excess light.

During the analysis four observations in the U light curve, four observations in the B light curve and one observation in the V light curve were found to have residuals greater than three standard deviations from the calculated light curve. These points were omitted from subsequent iterations. They are listed in Appendix F.

The distinction between occultation and transit eclipse diminishes as the ratio of radii approaches unity. However, it is seen from this analysis that the assumption of primary minimum an occultation provides the more consistent results for CM Lacertae. Parameters based on this assumption have been adopted.

Table 12. Parameters of CM Lacertae.

Original data : R. C. Barnes, D. S. Hall, R. H. Hardie, P.A.S.P. 80, (1968)
Spectral type : A2V, A8V (A7 - A8)
Period : 1.605 d.
Primary minimum: occultation

A. Fourier Analysis ($\frac{\theta}{2\pi}$ = 0.06 - 0.44 and 0.56 - 0.94)

	σ/A_0 (10^{-2})	A_0 (10^{-3})	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	T_h (°K)	T_c (°K)	S_c/S_h	ε	N
U	.994	.42413	+.0009	-.0058		-.0002	-.0004		94	2.44	11000	8800	2.40	.0040	2.6
p.e.		±.00046	±.0016	±.0017		±.0014	±.0014								
U	1.002	.42413	+.0007	-.0056	+.0001	-.0003	-.0003	+.0010	94						
p.e.		±.00047	±.0018	±.0018	±.0018	±.0015	±.0014	±.0014							
B	.786	.44789	-.0016	-.0079		+.0007	-.0007		94						
p.e.		±.00038	±.0012	±.0013		±.0011	±.0011								
B	.736	.44803	-.0025	-.0081	-.0039	+.0011	-.0009	-.0021	94	2.70	15000	10600	5.93	.0069	2.6
p.e.		±.00036	±.0013	±.0013	±.0013	±.0011	±.0010	±.0010							
V	.638	.52252	+.0012	-.0072		-.0009	+.0013		94						
p.e.		±.00036	±.0010	±.0011		±.0009	±.0009								
V	.605	.52242	+.0025	-.0080	+.0018	-.0008	+.0010	-.0028	94	2.16	12500	9400	4.53	.0053	2.6
p.e.		±.00035	±.0011	±.0011	.0011	±.0009	±.0008	±.0009							

Table 12 (cont'd.)

B. Rectification Formula Coefficients

	D_0	D_1	D_2	A'_0+D_0	A'_2+D_2	RF_0
U	-.0008	+.0006	-.0008	1.0048	-.0120	-8.4313
B	+.0018	-.0026	+.0018	1.0099	-.0145	-8.3717
V	-.0020	+.0025	-.0020	1.0060	-.0179	-8.2050

Table 12 (cont'd)

C. *Equivalent* Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
U Initial	.80	87.9	.187		.6	.6		.233						
B Initial	.92	87.1	.180		.6	.6		.236						
V Initial	.89	87.7	.184		.6	.6		.268						
U1	.9616	87.25	.1793	.1724	.329	.865	.7486	.2514		-.0001	-.762	1.0043	1.1203	1.1203
p.e.	±.0076	± 0.14	±.0031	±.0016	±.095	±.148	±.0098			±.0008				
B1	1.0142	86.91	.1720	.1745	.512	.109	.7150	.2850		.0004				
p.e.	±.0164	± 0.08	±.0047	±.0019	±.110	±.202	±.0157			±.0005				
V1	1.0493	86.88	.1683	.1766	.198	.440	.6819	.3180		-.0001				
p.e.	±.0128	± 0.10	±.0045	±.0026	±.110	±.150	±.0183			±.0005				
V2	1.1098	87.87	.1689	.1874	.724	.928	.5796	.3012	.1189	-.0001				
p.e.	±.0010	± 0.33	±.0029	±.0031	±.183	±.182	±.0289		±.0336	±.0005				
U Adopted	.9958	87.05	.1755	.1748	.775	.240	.2672	.7328		.0000	-.710	.8580	.9583	.9583
p.e.	±.0116	± 0.12	±.0020	±.0040	±.157	±.115	±.0137			±.0007				
B Adopted	.9603	86.90	.1758	.1688	-.011	.435	.2959	.7041		.0003	-.721	.6626	.7149	.7149
p.e.	±.0185	± 0.07	±.0022	±.0054	±.260	±.136	±.0182			±.0005				
V Adopted	.9234	87.49	.1815	.1675	.528	.474	.3128	.6271	.0601	.0001	-.822	.6311	.6900	.6900
p.e.	±.0160	± 0.27	±.0022	±.0050	±.165	±.161	±.0185		±.0208	±.0005				

The U1, B1, V1, and V2 solutions are based on the assumption that primary minimum is a transit. The adopted solutions are based on the assumption that primary minimum is an occultation.

The U light curve contains 291 observations, 84 in primary minimum and 102 in secondary minimum. For the adopted U solution λ_o = .3926, λ_t = .7706, J_g/J_s = .361, and $(J_g/J_s)_c$ = .449.

The B light curve contains 291 observations, 83 in primary minimum and 102 in secondary minimum. For the adopted B solution λ_o = .3979, λ_t = .7728, J_g/J_s = .387, and $(J_g/J_s)_c$ = .330.

The V light curve contains 294 observations, 87 in primary minimum and 102 in secondary minimum. For the adopted V solution λ_o = .4216, λ_t = .7480, J_g/J_s = .425, and $(J_g/J_s)_c$ = .435.

C. RX Arietis

RX Arietis is a ninth magnitude eclipsing binary system. The solution obtained by McCluskey (1966) for his data indicated primary minimum was a transit. The light curves were analyzed using this assumption. Both the U and V light curves were solved permitting and then excluding excess light.

Two errors were found in the published phase values for RX Arietis. There were four observations with residuals greater than six standard deviations from the calculated light curve. It is felt that these are the result of typographical errors. These four points were omitted for the final iterations. Each of these points is listed in Appendix F.

The results of the iterative solutions are given in Table 13C. The solutions excluding excess light have been adopted. Agreement between B and V solutions for this assumption is good. Also, the V solution converged to a value of L_3 within its probable error of zero. It is felt that the L_3 convergence to a non-zero value in the B solution allowing third light is due to correlation (as in the case of the synthetic BV 412 light curve). As seen in the table, limb-darkening coefficients for the primary component can be determined with reasonable accuracy; but, since the secondary eclipse is very shallow, the probable error in the limb-darkening coefficient for the smaller star is quite large.

Table 13. Parameters of RX Arietis.

Original data : G. E. McCluskey, Jr., A.J. 71, 527 (1966)
Spectral type : F5 (>G0)
Period : 1.030 d.
Primary minimum: transit

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.1 - 0.4 \text{ and } 0.6 - 0.9 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
B	.582	.52167	-.0130	-.0332		-.0003	-.0004	242	15.1	7500	4700	2.83	.0367	2.2
p.e.		±.00031	±.0008	±.0009		±.0004	±.0005							
B	.582	.52171	-.0119	-.0331	+.0007			242						
p.e.		±.00031	±.0016	±.0009	±.0011									
V	.563	.52752	-.0127	-.0334		-.0003	-.0015	242	9.84	7500	4600	5.25	.0348	2.2
p.e.		±.00030	±.0007	±.0009		±.0004	±.0005							
V	.551	.52749	-.0183	-.0331	-.0046			242						
p.e.		±.00029	±.0015	±.0009	±.0010									

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	+.0072	-.0068	+.0070	.54621	-.0277
V	+.0050	-.0067	+.0049	.55016	-.0304

Table 13 (cont'd.)

C. Equivalent Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
B Initial	.57	79.9	.38	.22	.4	.4	.9808	.0192						
B1	.6529	82.98	.3519	.2297	-.082	-2.519	.7884	.0208	.1908	-.0002	-1.00	.5360	.5490	.5483
p.e.	±.0041	± 0.52	±.0059	±.0024	±.175	±1.925	±.0277		±.0276	±.0002				
B Adopted	.5851	79.77	.3745	.2191	.254	-3.381	.9785	.0215		.0000	-0.90	.5381	.5517	.5509
p.e.	±.0007	± 0.33	±.0037	±.0019	±.094	±2.116	±.0011			±.0002				
V Initial	.57	79.9	.38	.22	.4	.4	.9667	.0333						
V1	.5514	80.34	.3839	.2117	.519	-1.318	.9817	.0305	-.0122	-.0006	-1.02	.5319	.5476	.5476
p.e.	±.0030	± 0.86	±.0099	±.0043	±.127	±1.312	±.0754		±.0752	±.0003				
V Adopted	.5543	80.47	.3833	.2125	.522	-1.273	.9695	.0305		.0008	-1.02	.5314	.5471	.5471
p.e.	±.0033	± 0.13	±.0038	±.0008	±.082	±1.302	±.0007			±.0002				

The solutions are based on the assumption that primary minimum is a transit.

The B light curve contains 501 observations, 140 in primary minimum and 110 in secondary minimum. For the adopted B solution λ_o = .9793, λ_t = .6659, J_g/J_s = 15.6 and $(J_g/J_s)_c$ = 22.7.

The V light curve contains 498 observations, 137 in primary minimum and 112 in secondary minimum. For the adopted V solution λ_o = .9695, λ_t = .6803, J_g/J_s = 9.76 and $(J_g/J_s)_c$ = 15.8.

D. V338 Herculis

V338 Herculis, an eclipsing binary of approximately the tenth magnitude, has been observed independently by Vetesnik (1968) and Walter (1969). Both investigators classified the primary minimum as a transit. The more numerous observations of Vetesnik were chosen for study.

The analysis presented here is based on the assumption that primary minimum is a transit and that there is no third light. There is good agreement between the resulting geometrical parameters of the B and V solutions for this assumption. Iterative solution of the B curve allowing excess light converged to parameters that were within one standard deviation of those obtained for the B analysis excluding third light.

It is seen in Table 14C that while the limb-darkening coefficient of the larger star can be reasonably well determined, the secondary minimum is too shallow to permit reliable evaluation of the limb-darkening coefficient of the smaller star.

One error was found in the published phase values for the V light curve. The Julian date of this observation is $J.D._{Hel.}$ 2439648.4767 and the corrected phase value is 0.9683. There is also evidence of systematic error in the data. There are runs of constant sign in the residuals. Agreement between standard deviations of the Fourier analysis and the differential corrections analysis is poor. The B light curve residuals do not fit a normal distribution satisfactorily. A possible short period (about 0.01 day) small amplitude oscillation (0.015 magnitude) in the luminosity of one of the stars is suspected. This is most clearly indicated by the residuals of the V observations between J.D.Hel. 2439639.3975 and 2439639.4558. (There are no B observations covering this time period.)

Even though there is this evidence of systematic error, it is felt that the error is small enough that the set of geometrical parameters given in Table 14C provides good representation of the system.

Table 14. Parameters of V338 Herculis.

Original data : M. Vetesnik, B.A.C. 19, 135 (1968)
Spectral type : A9 (>G0)
Period : 1.306 d.
Primary minimum: transit

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.088 - 0.412 \text{ and } 0.588 - 0.912, \text{B}; \quad = 0.1 - 0.4 \text{ and } 0.6 - 0.9, \text{V} \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
B	.499	.99597	-.0062	-.0087		-.0013	+.0028		420						
p.e.		±.00033	±.0005	±.0005		±.0003	±.0003								
B	.497	.99599	-.0078	-.0086	-.0014	-.0014	+.0029	-.0002	420	19.8	10500	5500	9.49	.0085	2.6
p.e.		±.00033	±.0009	±.0005	±.0006	±.0003	±.0003	±.0004							
V	.757	.78104	-.0057	-.0024		-.0034	+.0002		392						
p.e.		±.00045	±.0008	±.0009		±.0004	±.0005								
V	.715	.78078	-.0144	-.0028	-.0067	-.0030	+.0005	+.0021	392	10.3	8500	4900	7.84	.0063	2.4
p.e.		±.00043	±.0016	±.0009	±.0011	±.0004	±.0005	±.0005							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	+.0048	-.0078	+.0048	1.0094	-.0124
V	+.0073	-.0113	+.0073	0.7902	+.0029

Table 14 (cont'd.)

C. Equivalent Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
B Initial	.754	86.88	.300	.227	.6	.6	.965	.035					
B Final	.7664	83.69	.3117	.2389	1.018	-1.333	.9621	.0379	.0002	-.844	.658	.746	.742
p.e.	±.0144	± 0.26	±.0078	±.0015	± .115	±1.355	±.0022						
V Initial	.758	86.82	.300	.227	.5	.5	.945	.055					
V Final	.7485	83.86	.3190	.2387	.991	-5.258	.9269	.0730	-.0003	-.888	.778	.849	.826
p.e.	±.0086	± 0.30	±.0061	±.0018	± .081	± .722	±.0028		.0003				

The solutions are based on the assumption that primary minimum is a transit.

The B light curve contains 780 observations, 189 in primary minimum and 178 in secondary minimum. For the final B iteration λ_o = .9651, λ_t = .3928, J_g/J_s = 14.9, and $(J_g/J_s)_c$ = 29.8.

The V light curve contains 962 observations, 357 in primary minimum and 186 in secondary minimum. For the final V iteration λ_o = .9307, λ_t = .4245, J_g/J_s = 7.11, and $(J_g/J_s)_c$ = 14.1.

E. Y Leonis

Y Leonis is a single-line spectroscopic binary of approximately the tenth magnitude. Struve (1945) derived spectroscopic elements. The system is notable for its deep primary minimum. The photometric observations studied here are the broad band (3000A wide) infared (8000A) observations of Johnson (1960). The UBV observations of Johnson covered essentially only one primary minimum. It was felt that they were not sufficiently numerous to warrent analysis.

There was one error found in the published phase values. The Julian date of this observation along with the corrected phase value is given in Appendix F.

Preliminary elements obtained by Johnson are given as the initial parameter values in Table 15C.

The Johnson IR data were analyzed assuming primary minimum an occultation, both allowing and then excluding excess light. The results are given in Table 15C. The difference between the resulting parameters are negligible. Notice, however, exclusion of third light significantly reduces the probable errors of the parameters. The values of the standard deviation of the residuals from the Fourier analysis and from the differential corrections analysis are in very good agreement. However, there are relatively few observations contributing to the determination of the limb-darkening coefficients (48 observations in the occultation eclipse and 51 observations in the transit.). It is felt that re-observation of Y Leonis in UBV covering the entire phase range would be worthwhile.

Table 15. Parameters of Y Leonis

Original data : H. L. Johnson, Ap.J. 131, 127 (1960)
Spectral type : A3, (>G0)
Period : 1.686 d.
Primary minimum: occultation

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.1 - 0.4 \text{ and } 0.6 - 0.9 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ϵ	N
IR	.724	.13632	-.0216	-.0159		+.0057	+.0029		76	10.1	9500	4400	44.4	.0196	2.2
p.e.		±.00015	±.0016	±.0025		±.0020	±.0013								
IR	.735	.13629	-.0218	-.0151	-.0003	+.0063	+.0027	-.0005	76						
p.e.		±.00023	±.0026	±.0054	±.0030	±.0043	±.0017	±.0028							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
IR	+.00155	-.00295	+.00154	.14004	-.00280

Table 15 (cont'd.)

C. Equivalent Spherical Model Elements (b = 0)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
Initial	.8300	85.17	.2800	.2300	.400	.400								
With L_3	.7957	85.20	.2848	.2266	-.166	.368	.1503	.8478	.0020	.0009	-.887	.7216	.7216	.7216
p.e.	.0193	1.30	.0029	.0078	.305	.164	.0080		.0532	.0006				
No L_3	.7952	85.17	.2848	.2265	-.166	.363	.1508	.8492		.0019	-.886	.7200	.7200	.7196
p.e.	.0149	0.17	.0016	.0055	.300	.156	.0080			.0006				

The solutions are based on the assumption that primary minimum is an occultation.

The light curve contains 184 observations, 48 in primary minimum and 51 in secondary minimum. For the adopted solution $\lambda_o = .1830$, $\lambda_t = .9103$, $J_g/J_s = .112$, and $(J_g/J_s)_c = .0985$.

F. RW Monocerotis

RW Monocerotis, a ninth magnitude system, has been classified as a single-spectrum binary by Heard and Newton (1969). The system has been studied photometrically in two series of infrared observations by Brukalska, Rucinski, Smak, and Stepien (1969). From their preliminary analysis Brukalska, et. al., reported a negative limb-darkening coefficient for the secondary component.

As the Brukalska Series I observations did not cover the non-eclipse portion of the light curve, only the Series II observations are discussed here.

The Fourier analysis carried to terms of order 3θ has a significantly smaller standard deviation than the analysis carried to terms of order 2θ. The large sine terms are a preliminary indication of complications in the system. Although there are a large number of observations, a significant range of the non-eclipse portion of the light curve is not covered.

Analysis has been carried out on the assumption that primary minimum is an occultation. Initial analysis indicated asymmetry in the residuals and absence of third light. Thus differential corrections to t_o were calculated and differential corrections L_3 were excluded in succeeding iterations. Inclusion of t_o was accompanied by a significant reduction of the standard deviation. (It should be observed, however, that inclusion of sine terms in the rectification introduces systematic variation which may partially simulate a change in the reference time t_o.)

Contrary to the analysis of Brukalska, et. al., the resulting limb-darkening coefficients are in reasonable agreement with the theoretical values. However, the standard deviation of the residuals from the entire light curve is not in good agreement with the standard deviation from the Fourier analysis. As shown in Table 16C this cannot be accounted for by change in the choice of b. The results of iterations with three different values of b show little variation.

The observation on $J.D._{Hel.}$ 2439454.8463, apparently containing a typographical error, was omitted from the solution. Fourteen observations between phases 0.067 and 0.087 have systematically positive residuals between 2½ and 6½ standard deviations from the calculated curve. This phase range was covered on only one night during the photometric study. Thus the solution presented in Table 16C should be viewed with some reserve. Further observation of the system would be useful.

Table 16. Parameters of RW Monocerotis.

Original data : R. Brukalska, M. Rucinski, J. Smak, K. Stepien, A.A. 19, 257 (1969)
Spectral type : A0 (>G0)
Period : 1.906 d.
Primary minimum: occultation

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.09 - 0.41 \text{ and } 0.59 - 0.91 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
IR	.822	1.3364	-.0230	-.0336		+.0017	+.0027		308						
p.e.		±.0014	±.0018	±.0011		±.0013	±.0013								
IR	.736	1.3547	-.0050	-.0184	-.0003	+.0095	+.0156	+.0130	308	5.34	11000	6000	23.0	.0177	2.2
p.e.		±.0026	±.0027	±.0020	±.0012	±.0016	±.0020	±.0016							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
IR	+.0037	-.0067	+.0037	1.3833	-.0461

Table 16 (cont'd.)

C. Equivalent Spherical Model Elements

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	t_o	RF	p_o	$\sigma(b=0)$ (10^{-2})	$\sigma(cal.)$ (10^{-2})	$\sigma(est.)$ (10^{-2})
Initial	.72	90.	.2896	.2085	.4	.4	.273	.727						
b=1 Final	.7406	90.00	.3011	.2230	.430	.574	.2419	.7581	.00253	-.0010	-1.35	0.974	1.161	1.120
p.e.	±.0027		±.0004	±.0011	±.050	±.035	±.0004		±.00006	±.0004				
b=$\frac{1}{2}$ Final	.7386	90.00	.3018	.2229	.463	.529	.2415	.7585	.00292	-.0009	-1.35	0.959	1.030	1.018
p.e.	±.0026		±.0005	±.0012	±.047	±.044	±.0008		±.00008	±.0004				
b=0 Final	.7386	90.00	.3025	.2234	.493	.505	.2404	.7596	.00335	-.0008	-1.35	0.952	0.952	0.950
p.e.	±.0029		±.0007	±.0014	±.045	±.057	±.0014		±.00009	±.0004				

Solutions are based on the assumption that primary minimum is an occultation.

The light curve contains 880 observations, 183 in primary minimum and 374 in secondary minimum. For the solution with b = 0, $\lambda_o = .2404$. $\lambda_t = .8591$, $J_g/J_s = .173$, and $(J_g/J_s)_c = .172$.

G. BR Cygni

BR Cygni, a ninth magnitude system, has been observed by Wehinger (1968). Wehinger presented a solution for the primary minimum of the V curve only. The V curve exhibits uniform light for phase values within approximately 0.011 days of t_o. On the strength of this feature Wehinger assumed primary minimum was a complete occultation eclipse, even though the B curve did not indicate a similar characteristic. The B light curve shows night-to-night variation of about 0.03 magnitudes. This variation was particularly apparent in the phase ranges 0.1 to 0.2 and 0.4 to 0.5.

Solution of the B and V curves were attempted assuming primary minimum a transit and excluding third light. Apparent convergence on the parameters was obtained. Resulting parameters are given with the designation "B1" and "V1" in Table 17C. Note, however, the values of σ(est.) and σ(cal.) are significantly different in both B and V solutions. Solution of the V light curve assuming primary minimum a transit and allowing excess light resulted in large negative values of excess light and hence was not considered further.

Using Wehinger's results as initial parameter values, an iterative solution assuming primary minimum an occultation eclipse and excluding third light was attempted. The results of three iterations are given in Table 17C with the designation "V2". Note the negative values of the limb-darkening coefficients. In an attempt to find a more satisfactory fit, the solution was repeated using the same initial parameters, but in this case allowing third light. Convergence occured in three iterations. The resulting limb-darkening coefficient for the primary component is not unreasonable. Iterative solution of the B curve assuming primary minimum an occultation, both allowing and excluding third light, were divergent. A solution of the B curve assuming geometric parameters of the V3 solution, but excluding differential corrections to r_s was then attempted. Convergence occured. The results of this solution are given in Table 17C with the designation "B2". Further iteration excluding differential corrections to L_g were divergent.

Designation of the primary eclipse as an occultation eclipse seems to provide the most satisfactory results. It is felt that further observation, especially in B, will be needed to determine the system parameters with greater reliability.

Table 17. Parameters of BR Cygni

Original data : P. A. Wehinger, A.J. 73, 159 (1968)
Spectral type : B9 (F2 - F4)
Period : 1.333d.
Primary minimum: occultation (?)

A. Fourier Analysis ($\frac{\theta}{2\pi} = 0.11 - 0.39$ and $0.61 - 0.89$)

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ϵ	N
B	1.21	1.7739	-.0121	-.0217		+.0004	+.0000		196						
p.e.		±.0024	±.0016	±.0024		±.0011	±.0011								
B	1.21	1.7743	-.0092	-.0214	+.0020	+.0003	-.0000	-.0012	196	11.40	17500	8000	47.6	.0214	2.2
p.e.		±.0024	±.0042	±.0024	±.0028	±.0011	±.0011	±.0015							
V	1.14	1.1270	-.0079	-.0194		+.0043	-.0014		196						
p.e.		±.0014	±.0016	±.0023		±.0010	±.0010								
V	1.14	1.1271	-.0119	-.0191	-.0029	+.0043	-.0015	-.0011	196	3.13	15500	8100	31.4	.0201	2.2
p.e.		±.0015	±.0040	±.0023	±.0026	±.0010	±.0010	±.0014							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	+.0085	-.0163	+.0085	1.8207	-.0673
V	+.0072	-.0134	+.0071	1.1558	-.0359

Table 17 (cont'd.)

C. Equivalent Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	$\sigma(b=0)$ (10^{-2})	$\sigma(cal.)$ (10^{-2})	$\sigma(est.)$ (10^{-2})
B1	.7423	90.00	.3856	.2863	0.996	1.761	.9450	.0550		.0029	-1.35	1.511	1.764	1.469
p.e.	±.0167		±.0106	±.0014	±.132	±0.519	±.0026			±.0007				
B2	.7867	83.94	.3478	.2736	1.081	0.565	.1033	.6980	.1988	-.0002	-0.89	1.266	1.335	1.334
p.e.	±.0048	± 2.59	±.0021		±.262	±0.130	±.0103		±.0708	±.0006				
V Initial	.773	86.06	.360	.278	.4	.4	.393	.607						
V1	.7450	90.00	.3780	.2816	0.741	0.843	.8959	.1041		.0005	-1.34	1.363	1.511	1.381
p.e.	±.0072		±.0054	±.0013	±0.089	±0.289	±.0025			±.0007				
V2	.5325	79.50	.3828	.2039	-0.635	-1.760	.4057	.5943		-.0018	-0.98	1.315	1.403	1.361
p.e.	±.0029	± 0.19	±.0014	±.0019	±0.280	±0.199	±.0023							
V3	.7706	85.33	.3550	.2736	-0.518	0.586	.2028	.5927	.2044	-.0005	-1.00	1.237	1.314	1.314
p.e.	±.0222	± 0.72	±.0022	±.0096	±0.334	±0.186	±.0151		±.0155	±.0007				

The B1 and V1 solutions are based on the assumption that primary minimum is a transit.
The B2, V2, and V3 solutions are based on the assumption that primary minimum is an occultation.

The B light curve contains 427 observations, 122 in primary minimum and 90 in secondary minimum.
For the B2 solution λ_o = .3237, λ_t = .9296, J_g/J_s = .0916, and $(J_g/J_s)_c$ = .112.

The V light curve contains 431 observations, 130 in primary minimum and 94 in secondary minimum.
For the V3 solution λ_o = .4037, λ_t = .8860, J_g/J_s = .203, and $(J_g/J_s)_c$ = .139.

H. BV 430

BV 430 (RS Cha), a sixth magnitude system, has been observed independently by Chambliss (1967) and Schoffel and Mauder (1967). Since the latter's observations were not published, the Chambliss data were chosen for further study.

Chambliss based his solution on the assumption that the primary minimum was a transit. Using this assumption, the B and V light curves were analyzed both allowing and then excluding third light. When third light was included, convergence to large negative values of excess light occured in both the B and V light curves. Results of the third iterations of B and V excluding third light are presented in Table 18C, under the designation "B1" and "V1". Note the attempted "interchange" of the larger and smaller star, as indicated by values of $k > 1.0$.

Solutions assuming primary minimum at occultation and excluding third light were then attempted. The results are given in Table 18C under the designations "B2" and "V2". Observe the close correspondence between the V2 and the B1 and V1 geometrical parameters and also between the limb-darkening coefficients for the brighter and less bright components for the V1 and V2 solutions. With the ratio of radii close to unity it is extremely difficult to distinguish between occultation and transit eclipse. The results of the B2 solution were somewhat puzzling considering the correspondence of the other three solutions. Examination revealed correlation coefficient between r_s and L_g was -0.98. A solution assuming the V2 results for r_s and $\cos^2 i$ and excluding differential corrections to r_s and $\cos^2 i$ was attempted. The luminosities of this solution were used and differential corrections to the remaining parameters were calculated. The resulting values of the geometrical parameters were within ½ probable error of the input values. For the final three iterations only the differential corrections to r_s were excluded. The resulting parameters were virtually unchanged from the input values. These parameters are designated "B3" in Table 18C. Even though the procedure followed for the B3 solution is somewhat subjective, the resulting parameters and reduced standard deviation seem to justify the procedure. The B3 geometrical parameters are in good agreement with the V2 values.

Table 18. Parameters of BV 430.

Original data : C. R. Chambliss, A.J. 72, 518 (1967)
Spectral type : A5 (A7)
Period : 1.679 d.
Primary minimum: occultation

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.09 - 0.41 \text{ and } 0.59 - 0.91 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
B	1.26	.42555	-.0062	-.0272		+.0036	-.0016	290						
p.e.		±.00046	±.0015	±.0017		±.0009	±.0010							
B	1.22	.42517	-.0283	-.0082	-.0016	+.0032	-.0020	290	1.34	13000	11700	1.67	.0226	2.6
p.e.		±.00046	±.0017	±.0019	±.0025	±.0009	±.0010							
V	1.13	.55835	-.0033	-.0233		-.0009	-.0016	289						
p.e.		±.00055	±.0013	±.0015		±.0008	±.0009							
V	1.13	.55820	-.0062	-.0236	-.0024	-.0010	-.0017	289	1.18	10000	9500	1.33	.0266	2.6
p.e.		±.00056	±.0025	±.0016	±.0018	±.0008	±.0009							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	A'_0+D_0	A'_2+D_2
B	+.00135	-.00068	+.00134	.43854	-.02270
V	+.01212	-.00345	+.01201	.58349	-.01433

Table 18 (cont'd.)

C. Equivalent Spherical Model Elements (V1: b = 1/2; B1, B2, B3, V2: b = 1)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	RF	p_o	$\sigma(b=0)$ (10^{-2})	$\sigma(cal.)$ (10^{-2})	$\sigma(est.)$ (10^{-2})
B Initial	.79	84.7	.264	.209	.6	.6	.669	.331					
B1	1.0697	83.58	.2387	.2554	.656	.656	.5305	.4695	.0016				
p.e.	±.0093	± .29	±.0084	±.0067	±.124	±.159	±.0315		±.0006				
B2	.7560	86.01	.2689	.2033	.054	.131	.5753	.4247	.0030	-.981	1.343	1.480	1.480
p.e.	±.0043	± 0.30	±.0013	±.0021	±.089	±.139	±.0047		±.0006				
B3	.9715	83.45	.2520	.2448	.722	.726	.4502	.5498	.0058	-.563	1.297	1.429	1.429
p.e.	±.0107	± 0.23	±.0028		±.094	±.106	±.0061		±.0006				
V Initial	.79	84.7	.264	.209	.6	.6	.658	.342					
V1	1.0076	82.70	.2514	.2533	.748	.848	.5467	.4533	-.0003				
p.e.	±.0046	± 0.27	±.0085	±.0074	±.123	±.150	±.0327		±.0005				
V2	.9867	82.72	.2540	.2506	.794	.759	.4575	.5425	-.0007	-.507	1.111	1.226	1.226
p.e.	±.0077	± 0.28	±.0061	±.0079	±.143	±.113	±.0279		±.0005				

The B1 and V1 solutions are based on the assumption that primary minimum is a transit.
The B2, B3, and V2 solutions are based on the assumption that primary minimum is an occultation.

The B light curve contains 554 observations, 120 in primary minimum and 136 in secondary minimum. For the B3 solution λ_o = .5741, λ_t = .6693. J_g/J_s = .773, and $(J_g/J_s)_c$ = .771.

The V light curve contains 555 observations, 117 in primary minimum and 137 in secondary minimum. For the V2 solution λ_o = .5994, λ_t = .6691, J_g/J_s = .821, and $(J_g/J_s)_c$ = .834.

I. BV 412

BV 412, an eighth magnitude system, was observed spectroscopically by Mammano, Margoni, and Stagni and photoelectrically by Harris (1968). Harris states that the spectroscopic observations indicate primary eclipse is a transit. The Harris observations were analyzed under this assumption.

Two errors were found in the published phase values. The Julian dates of these observations are listed in Appendix F.

The V observations were first analyzed allowing third light. The iterations converged to a large negative value of excess light. It was assumed this was due to correlation between the parameters. The V data were subsequently re-analyzed excluding third light. Satisfactory convergence occured. The results of this analysis appear with the designation "Adopted" in Table 19C.

While the B curve iterative solution excluding third light was divergent, analysis allowing third light converged. The results of the convergent solution are given in Table 19C under the designation "B1". The geometrical parameters of the B1 and the adopted V solutions are not in good agreement. Examination of the correlation coefficients of the B1 solution showed a correlation coefficient between r_s and $\cos^2 i$ of 0.99. A procedure similar to that used for BV 430 was used in an attempt to find accordant results for the B and V light curves. Geometrical parameters of the V solution were used as initial values of an iterative solution of the B light curve. Differential corrections to r_s were excluded and it was assumed that there was no third light. Convergence was obtained in four iterations. The resulting parameters were used in an iterative solution excluding differential corrections to L_g. The parameters changed by less than one standard deviation. There is a discrepency between the standard deviation of the Fourier analysis and the differential corrections analysis for the B light curve. Residuals between $2\frac{1}{2}$ and 6 standard deviations from the calculated curve were found for eighteen B observations. Fifteen of these observations are in the non-eclipse portion of the light curve. This accounts for the standard deviation of the Fourier analysis being 10% greater than that obtained by the differential corrections analysis. Resultant B parameters, designated "Adopted" in Table 19C, show good agreement with the V solution.

Table 19. Parameters of BV 412

Original data : A. J. Harris, A.J. <u>73</u>, 164 (1968)
Spectral type : A0 (F2.5 - F4.5)
Period : 0.771 d.
Primary minimum: transit

A. Fourier Analysis $\left(\frac{\theta}{2\pi} = 0.1 - 0.4 \text{ and } 0.6 - 0.9 \right)$

	σ/A_0 (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	B_1/A_0	B_2/A_0	B_3/A_0	NDF	J_h/J_c	$T_h(^oK)$	$T_c(^oK)$	S_c/S_h	ε	N
B	.916	2.0726	-.0043	-.0490		+.0012	+.0018		378						
p.e.		±.0015	±.0009	±.0012		±.0006	±.0006								
B	.900	2.0734	-.0114	-.0485	-.0054	+.0012	+.0018	+.0005	378	10.83	16500	7800	38.2	.0398	2.6
p.e.		±.0015	±.0021	±.0012	±.0014	±.0006	±.0006	±.0007							
V	.773	1.3823	-.0057	-.0414		+.0020	+.0003		382						
p.e.		±.0009	±.0008	±.0010		±.0005	±.0005								
V	.756	1.3830	-.0109	-.0407	-.0039	+.0022	+.0001	+.0016	382	4.85	14500	8100	21.7	.0340	2.6
p.e.		±.0009	±.0017	±.0010	±.0012	±.0005	±.0005	±.0006							

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	+.0127	-.0236	+.0122	2.1867	-.1889
V	+.0084	-.0150	+.0080	1.4477	-.1045

Table 19 (cont'd.)

C. Equivalent Spherical Model Parameters (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	σ(b=0) (10^{-2})	σ(cal.) (10^{-2})	σ(est.) (10^{-2})
B Initial	.51	77.56	.418	.213	.6	.6	.971	.029						
B1	.6885	75.42	.3682	.2535	.071	-1.032	.7374	.0642	.1986	.0008	-.459	.8035	.8203	.8203
p.e.	±.1183	± 6.29	±.0588	±.0841	±.303	±1.682	±.1834		±.1471	±.0003				
B Adopted	.5284	75.73	.4109	.2171	.471	.509	.9539	.0461		.0011	-.757	.8040	.8209	.8209
p.e.	±.0039	± 0.48	±.0038	±.0036	±.090	± .565	±.0019			±.0003				
V Initial	.51	77.56	.418	.213	.6	.6	.947	.053						
V Adopted	.5159	77.12	.4114	.2123	.417	.780	.9357	.0643		.0002	-.888	.7386	.7595	.7595
±p.e.	±.0016	± 0.40	±.0038	±.0026	±.086	± .319	±.0020			±.0003				

Solutions are based on the assumption that primary eclipse is a transit.

The B light curve contains 731 observations, 245 in primary minimum and 103 in secondary minimum. For the adopted B solution λ_o = .9574, λ_t = .7513, J_g/J_s = 5.77, and $(J_g/J_s)_c$ = 5.68.

The V light curve contains 732 observations, 244 in primary minimum and 105 in secondary minimum. For the adopted V solution λ_o = .9368, λ_t = .7499, J_g/J_s = .387, and $(J_g/J_s)_c$ = 3.33.

J. SW Lyncis

SW Lyncis, a ninth magnitude system, has been observed by Gleim (1967) and Vetesnik (1968). Fourier analysis indicated the standard deviation of the Gleim data is approximately twice as large as the standard deviation of the Vetesnik data. Since the Gleim data are also less numerous, only the analysis of the Vetesnik data is discussed here.

Both Gleim and Vetesnik concluded that primary minimum of SW Lyncis is a transit eclipse. Analysis was based on this assumption. Vetesnik indicated that some of his data apparently contained systematic errors. He omitted certain observations from his analysis. Accordingly, fourteen V observations and thirteen B observations designated by Vetesnik were thus excluded from the differential corrections analysis. One V observation felt to contain a typographical error was also excluded. Julian dates of these observations are given in Appendix F.

The V light curve was first analyzed allowing third light. Convergence occured. The resulting parameters are designated as "V1" in Table 20C. Iterative solution of the V light curve excluding third light was not completely convergent. The parameters of the iteration having the smallest value of σ(cal.) are given in Table 20C with the designation "V2". An attempt was made to improve convergence by omitting differential corrections to L_g. Although the resulting standard deviation is a few percent larger than the V1 solution, convergence was satisfactory. The results of this solution are designated "V3" in Table 20C.

Iterative solution of the B light curve allowing third light was divergent. Iterative solution excluding third light did not exhibit satisfactory convergence. The parameters of the iteration having the smallest value of σ(cal.) are given in Table 20C with the designation "B1". The partial correlation coefficient between r_s and $\cos^2 i$ for this iteration is 0.98. The geometrical parameters of the V2 solution were used as initial parameter values in an iterative solution excluding differential corrections to $\cos^2 i$. A small decrease in the standard deviation of the residuals was obtained. The results are designated "B2" in Table 20C. Further iterations using the B2 solution and excluding differential corrections to r_s were divergent.

The B2 and V3 solutions seem to provide the most consistent results. However, these parameters should be regarded with considerable reserve until they can be supplemented with the results of more numerous observations of greater accuracy.

Table 20. Parameters of SW Lyncis

Original data : M. Vetesnik, B.A.C. 19, 110 (1968)
Spectral type : F2 (>G0)
Period : 0.644 d.
Primary minimum: transit

A. Fourier Analysis ($\frac{\theta}{2\pi}$ = 0.13 - 0.37 and 0.63 - 0.87)

	$/A_0$ (10^{-2})	A_0	A_1/A_0	A_2/A_0	A_3/A_0	A_4/A_0	B_1/A_0	B_2/A_0	B_3/A_0	B_4/A_0	J_h/J_c	T_h (°K)	T_c (°K)	S_c/S_h	ε	N
B	.856	.9926	+.0139	-.0566			-.0006	+.0068			39.9	8500	4300	5.39	.0372	2.6
p.e.		±.0013	±.0013	±.0018			±.0006	±.0008								
B	.850	.9871	+.0119	-.0658	-.0016	-.0047	-.0014	+.0034	-.0020	-.0035						
p.e.		±.0057	±.0050	±.0089	±.0026	±.0037	±.0008	±.0017	±.0011	±.0015						
V	.706	.9013	+.0040	-.0427			+.0006	+.0042			8.44	7500	4700	5.21	.0303	2.6
p.e.		±.0009	±.0010	±.0014			±.0005	±.0006								
V	.704	.8862	+.0070	-.0699	+.0013	-.0108	+.0005	+.0067	+.0001	+.0025						
p.e.		±.0043	±.0041	±.0076	±.0022	±.0031	±.0006	±.0013	±.0008	±.0013						

B: NDF=297; V: NDF=324

B. Rectification Formula Coefficients

	D_0	D_1	D_2	$A_0'+D_0$	$A_2'+D_2$
B	-.0100	+.0138	-.0100	1.0387	-.1224
V	-.0026	+.0036	-.0025	0.9371	-.0798

Table 20 (cont'd.)

C. Equivalent Spherical Model Elements (b = 1/2)

	k	i	r_g	r_s	x_g	x_s	L_g	L_s	L_3	RF	p_o	$\sigma(b=0)$ (10^{-2})	$\sigma(cal.)$ (10^{-2})	$\sigma(est.)$ (10^{-2})
B Initial	.620	72.6	.463	.287			.981	.019						
B1	.6970	79.81	.4215	.2938	.566	- 5.0	.9943	.0051		.0001	-0.83	.9858	1.062	1.015
p.e.	±.0270	± 0.37	±.0189	±.0017	±.187	± 8.6	±.0008			±.0004				
B2	.6212	83.67	.4490	.2789	.860	-10.8	.9943	.0056		.0004	-1.21	.9701	1.034	1.033
p.e.	±.0190		±.0149	±.0007	±.164	± 8.6	±.0009			±.0004				
V Initial	.62	72.6	.463	.287	.6	.6	.9480	.0520						
V1	.7248	90.0	.4240	.3073	.297	- 1.7	.7399	.0522	.2072	-.0029	-1.38	.7692	.8020	.8004
p.e.	±.0005		±.0031	±.0025	±.063	± .9	±.0155			±.0003				
V2	.6040	83.67	.4615	.2788	.746	- 4.0	.9500	.0500		-.0007	-1.26	.7892	.8316	.8165
p.e.	±.0058	± 0.33	±.0065	±.0013	±.069	± 1.0	±.0009			±.0003				
V3	.6198	82.72	.4553	.2822	.615	- 2.0	.9502	.0498		-.0008	-1.16	.7802	.8205	.8199
p.e.	±.0038	± 0.27	±.0044	±.0010	±.068	± 0.9				±.0003				

Solutions are based on the assumption that primary minimum is a transit.

The B light curve contains 631 observations, 132 in primary minimum and 194 is secondary minimum. For the B2 solution λ_o = .9944, λ_t = .5578, J_g/J_s = 68.6, and $(J_g/J_s)_c$ = 128.

The V light curve contains 632 observations, 132 in primary minimum and 170 in secondary minimum. For the V3 solution λ_o = .9502, λ_t = .5994, J_g/J_s = 7.33, and $(J_g/J_s)_c$ = 12.3.

VI. SUMMARY AND CONCLUSIONS

To summarize, we have discussed the transformation from the model of similar tri-axial ellipsoids to the spherical model for an eclipsing binary system. Kopal's method and the method of differential corrections were discussed. Both methods were generalized to include third light. The method of differential corrections was further generalized to include orbital eccentricity directly. Synthetic light curves were used to validate the computer programs, as well as to determine the effect of dispersion and number of observations on the ability to extract the desired parameters. Analysis of synthetic data indicated limb-darkening coefficients may be extracted from observations of sufficient accuracy and density. This conclusion was found to hold for partial as well as completely eclipsing systems. In addition, it has been found possible to extract values of third light. In some cases, however, correlation between parameters, combined with observations of insufficient quality or quantity, may prevent convergence.

The data from 10 eclipsing binary systems have been rectified and subsequently analyzed using differential corrections. The systems are CO Lacertae, CM Lacertae, RX Arietis, V338 Herculis, Y Leonis, RW Monocerotis, BR Cygni, BV 430, BV 412, and SW Lyncis.

It was often necessary to solve the light curves for each combination of assumptions as to type of primary minimum and possible presence of third light. Calculation and comparison of σ(est.) and σ(cal.), the estimated and calculated standard deviations, proved valuable in the determination of convergence. Equality of the standard deviations of the Fourier analysis and the standard deviation of the entire light curve indicated the adequacy of the fit. For those systems in which b was varied, choice of b, the exponent of the light in the weight, did not seem to cause significant change in the parameters obtained.

Excellent results were obtained for the systems CO Lacertae, CM Lacertae, RX Arietis, and Y Leonis. For these light curves convergence was obtained for the standard set of parameters (r_g, r_s, $\cos^2 i$, L_g, L_s, x_g, x_s, and, if necessary, L_3). For each of these light curves the standard deviation for the entire light curve was in good agreement with the standard deviation obtained from the Fourier analysis of the non-eclipse variation. In addition, for the systems with multi-color observations the resulting geometric parameters from the separate curves showed good agreement.

For the B light curves of BV 412 and BV 430 convergence was obtained only if the number of variables in the parameter set for a given iteration was limited to six. The V curves converged with a complete set of seven variables. Resulting geometric parameters and standard deviations showed satisfactory agreement.

The iterative analysis of V338 Herculis and RW Monocerotis exhibited satisfactory convergence for the entire set of seven parameters (again excluding e, ω, t_o, and possibly L_3). The agreement of the geometric parameters from the individual color curves is good for V338 Herculis. However, for each of the light curves of these two systems the standard deviations of the Fourier analysis is not in good agreement with the standard deviation obtained for the entire light curve. Further observation and analysis is indicated.

The V light curve solution of BR Cygni was satisfactory; however complete convergence was not obtained for the B light curve. Similar results were obtained for SW Lyncis. It is felt that the parameters for BR Cygni and SW Lyncis should be viewed with reserve until further observations are made.

With the exception of V338 Herculis, the dispersion of the observations was larger for the B light curve than for the V light curve of each system. It is interesting to note that for the systems where difficulty in the convergence of one curve occured, lack of convergence was in the B light curve.

Of the ten systems studied, two (CO Lacertae and BR Cygni) showed evidence of third light.

Values obtained for V and B limb-darkening coefficients and their probable errors are given in Figures 13 and 14. The theoretical results are given in the figures for comparison. The theoretical values for spectral types B0 through A0 are from Grygar (1965). The remainder of the theoretical values result from least squares fits of the model stellar-atmospheres limb darkening given by Gingerich (1966) and Margrave (1969) to the linear limb-darkening law, equation (1.28). Results for the limb-darkening coefficients in V show reasonable agreement with theory, while limb-darkening coefficients in B show somewhat more scatter.

In conclusion, it is suggested that greater numbers of high quality observations are needed to reduce the uncertainty of the limb-darkening coefficients. Of the 20 limb-darkening coefficients given in Figures 13 and 14, only four had more than 150 observations in the corresponding minimum. Light curves containing 300 observations per minimum should provide satisfactory determination of the corresponding limb-darkening coefficient.

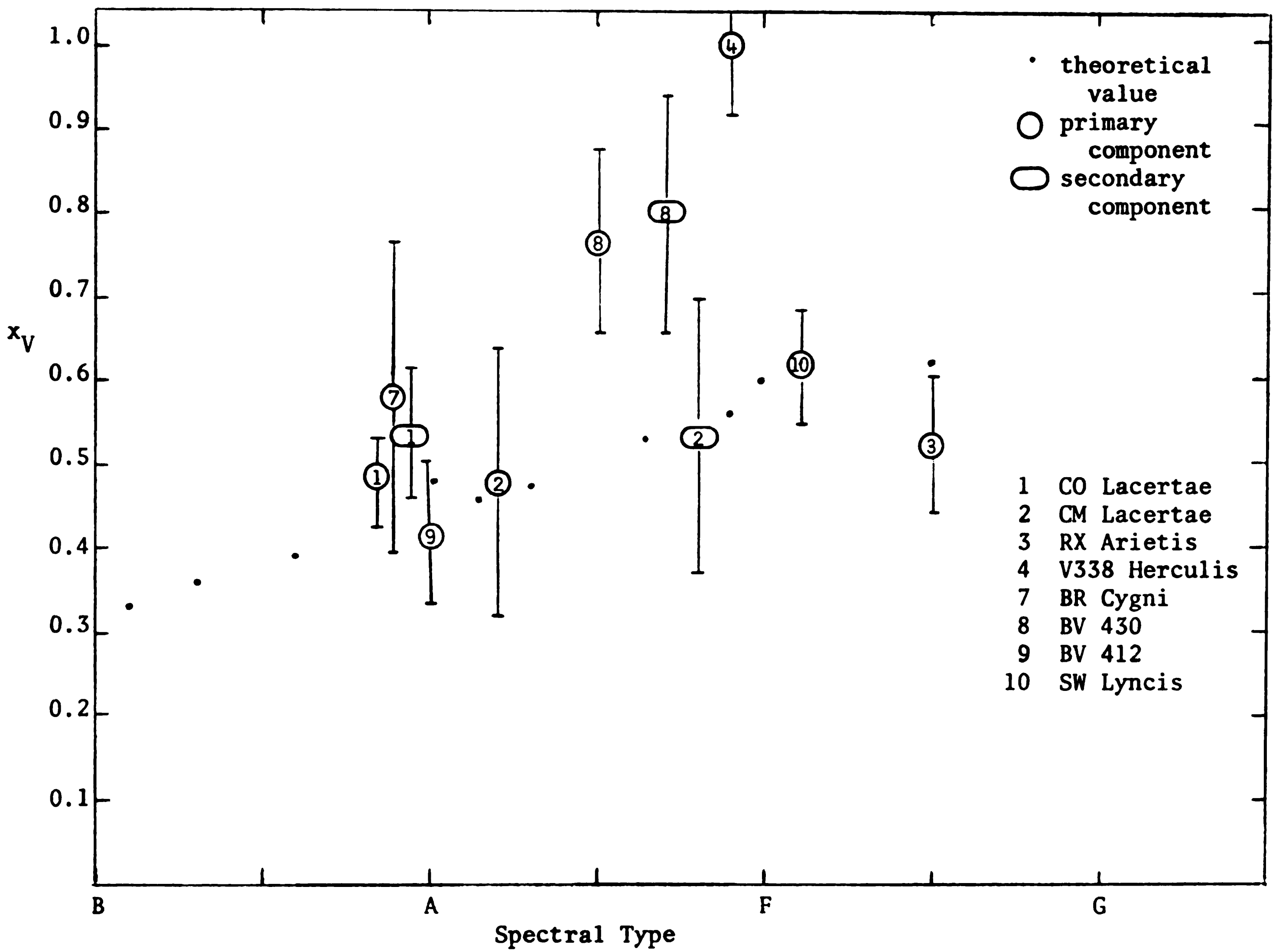


Figure 13. Limb-darkening coefficients in V.

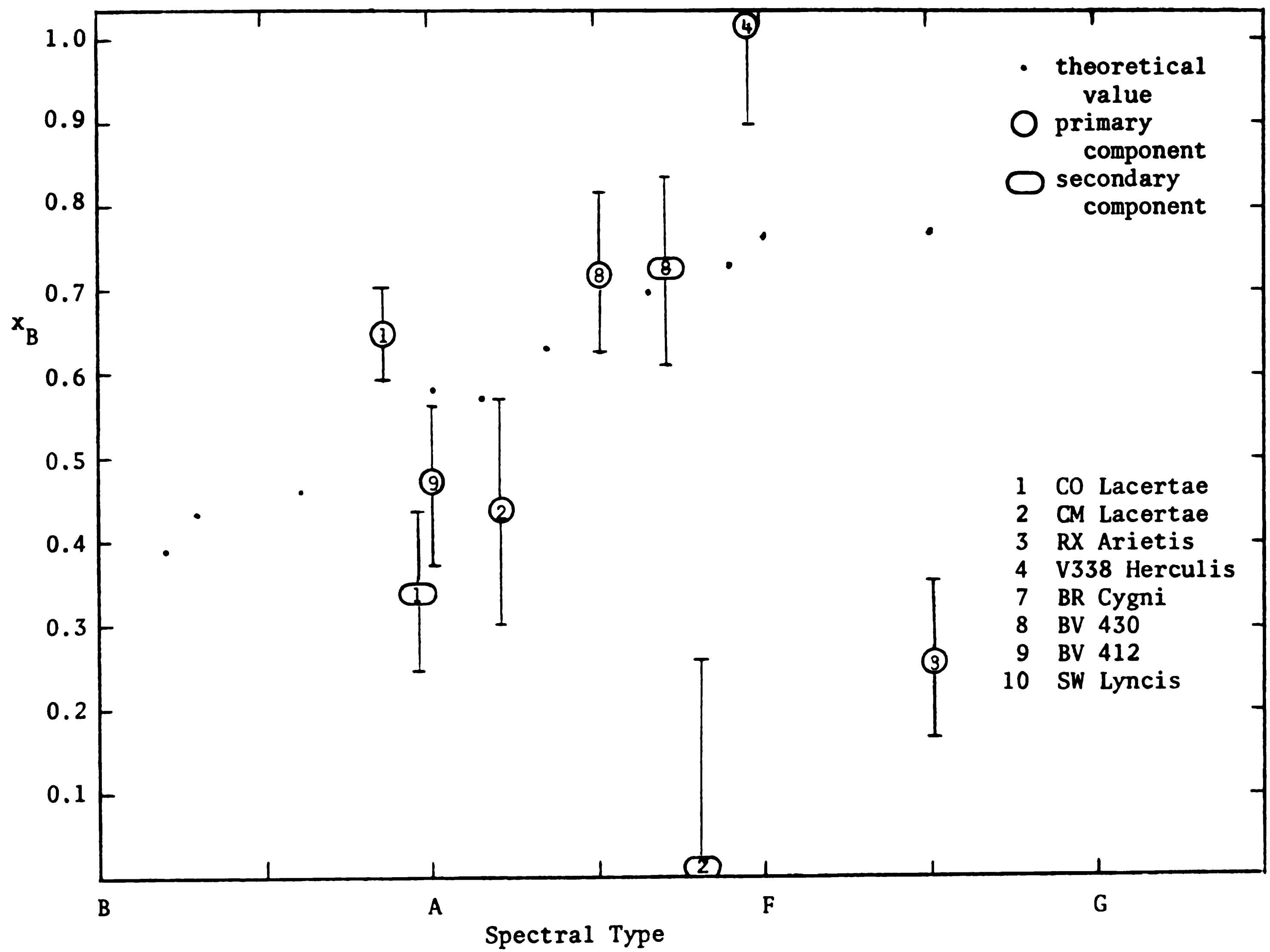


Figure 14. Limb-darkening coefficients in B.

APPENDICES

APPENDIX A. DISTRIBUTION OF APPARENT BRIGHTNESS ON THE STELLAR DISK

The discussion of limb darkening presented here follows closely the discussion given by Kopal (1959, p. 150ff). However we have used x for the limb-darkening coefficient rather than u.

Temperature variation in the semi-transparent stellar atmosphere results in an apparent surface brightness that is dependent on the angle of foreshortening. Radiation viewed normally originates, on the average, at greater stellar depth than that viewed tangentially.

Assuming the semi-transparent atmosphere represents such a small fraction of the total stellar radius that it can be regarded as plane-parallel layers, the equation of transfer of the radiation is

$$\cos \gamma \frac{dI}{dr} = \kappa\rho(B - I) \quad , \tag{A-1}$$

where I(r) is the intensity of the radiation at a distance r from the center of the star, B is the source function (emissivity), γ is the angle of foreshortening (angle between the radius vector and the line of sight), and κ is the coefficient of opacity and ρ is the density of the stellar material. (More complete discussion of the equation of transfer is given by Mihalis (1970, Ch. 1).)

Define the optical depth τ such that

$$d\tau = -\kappa\rho \, dr \quad ; \tag{A-2}$$

then

$$\mu \frac{dI}{d\tau} = I - B \quad , \tag{A-3}$$

where

$$\mu = \cos \gamma \quad . \tag{A-4}$$

Assuming that the energy sources in the atmosphere are neglibible, so that the atmospheric layers merely transmit radiation without gain or loss, the net flux of radiation is

$$F = 2 \int_0^{\pi} I \sin \gamma \cos \gamma \, d\sigma = 2 \int_{-1}^{1} I\mu \, d\mu \quad . \tag{A-5}$$

F is thus constant and independent of τ. Also under the assumption of negligible atmospheric energy generation, the source function (giving the radiation emitted at a point) will be

$$B(\tau) = \frac{1}{2} \int_0^{\pi} I \sin \gamma \, d\gamma = \frac{1}{2} \int_{-1}^{1} I \, d\mu \quad . \tag{A-6}$$

The source function consists of incident light from all directions.

Combining equations (A-3) and (A-6), we have

$$\mu \frac{dI}{d\tau} = I - \frac{1}{2} \int_{-1}^{1} I \, d\mu \quad . \tag{A-7}$$

Equation (A-7) is an integro-differential equation for the intensity $I(\tau,\mu)$ of radiation at any optical depth τ in an arbitrary direction γ. It describes the radiative transfer of energy which is absorbed and re-emitted (or isotropically scattered with unit albedo) in the plane-parallel atmosphere.

The two boundary conditions are that the net flux is constant and independent of τ and that no radiation is incident on the star, thus

$$I(0,\mu) = 0 \tag{A-8}$$

for

$$0 \geq \mu \geq -1 \quad . \tag{A-9}$$

Equation (A-7) has no known closed form solution. But, for the case of interest ($\tau = 0$) Wiener and Hopf (1931) have shown the solution to be

$$I(0,\mu) = \frac{\sqrt{3}}{4} \frac{F}{\sqrt{1+\mu}} \exp\left(\frac{1}{\pi} \int_0^{\pi/2} \frac{\theta \tan^{-1}(\mu \tan \theta)}{1 - \theta \cot \theta} d\theta \right), \quad \text{(A-10)}$$

where F is given by equation (A-5). Equation (A-10) may be expanded in a Taylor series around $\mu = 1$. For the linear approximation Milne (1921) has given the result

$$\frac{I(0,\mu)}{I(0,1)} = 1 - x + x \cos \gamma \quad , \quad \text{(A-11)}$$

where $x = 0.6$.

The above result is valid for bolometric observations. Assuming local thermodynamic equilibrium, Kopal (1959, p. 155ff) has shown

$$I_\lambda(0,\mu) = B_\lambda(T_e) \left(A_0 + A_1\mu + A_2\mu^2 + A_3\mu^3 + \cdots \right) , \quad \text{(A-12)}$$

where B_λ is the Planck function and the A_i are functions of τ_e(the optical depth at which the temperature equals the effective temperature), the Planck function and its derivatives evaluated at τ_e, and the ratio of the mean absorption coefficient to the frequency dependent absorption coefficient. (The A_i discussed here are not to be confused with the A_i used in the Fourier expansion of the non-eclipse variation of the light curve.)

In addition, Kopal (1959, p. 158) has shown that third-order theory may be approximated by the linear theory, so that in adopting the linear limb-darkening law

$$J(\gamma)/J(0) = (1 - x + x \cos \gamma) \tag{A-13}$$

the coefficient x is given by

$$x = \frac{32(A_1 + A_2) + 30\ A_3}{32(A_0 + A_1) + 28\ A_2 + 25\ A_3} \quad . \tag{A-14}$$

Explicit expressions for the A_i in terms of the parameters discussed above are given by Kopal (1959, equations (1-24)). Thus, in general, we expect the limb-darkening coefficients to be a function of effective wavelength and spectral type (effective temperature), as well as the absorption coefficient of the stellar atmosphere.

APPENDIX B. VON ZEIPEL'S THEOREM

H. von Zeipel (1924) proved the emergent radiation flux at the surface of a rotationally or tidally distorted star in radiative equilibrium is proportional to the local gravity. The version of the derivation of von Zeipel's theorum given here follows the derivations of Kopal (1959, p. 170ff) and Chandrasekhar (1933, p. 539ff).

If p_r is the radiation pressure, κ_ν the frequency dependent absorption coefficient and F_ν the energy flux, then, as has been shown by Mihalas (1970, p. 13ff), the variation of the radiation pressure with depth in the stellar atmosphere is

$$\frac{dp_r}{dz} = - \frac{\rho}{c} \int_0^\infty \kappa_\nu \, F_\nu \, d\nu \quad , \tag{B-1}$$

where c is the velocity of light, ρ is the density of stellar material and z, measured normal to the surface, increases outward in the atmosphere. Scattering has been excluded. Defining the mean absorption coefficient κ as

$$\kappa = \frac{1}{F} \int_0^\infty \kappa_\nu \, F_\nu \, d\nu \quad , \tag{B-2}$$

equation (B-1) becomes

$$\frac{dp_r}{dz} = - \frac{\kappa\rho}{c} F \quad , \tag{B-3}$$

or more generally

$$\vec{\nabla} p_r = - \frac{\kappa\rho}{c} \vec{F} \tag{B-4}$$

(Motz 1970, p. 101), where $\vec{F}$ is the total energy flux.

Assuming all energy is transported by radiation (the condition of radiative equilibrium), conservation of energy requires

$$\vec{\nabla} \cdot \vec{F} = \varepsilon\rho \quad , \tag{B-5}$$

where ε is the rate of energy liberation per unit mass. Substituting for $\vec{F}$ from equation (B-4), equation (B-5) becomes

$$\vec{\nabla} \cdot \left(\frac{1}{\kappa\rho} \vec{\nabla} p_r \right) = - \frac{\varepsilon\rho}{c} \quad . \tag{B-6}$$

Expressing p_r as a function of P, the total pressure (gas plus radiation), the left-hand side of equation (B-6) may be written as

$$\vec{\nabla} \cdot \left(\frac{1}{\kappa\rho} \vec{\nabla} p_r \right) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{1}{\kappa\rho} \frac{\partial p_r}{\partial x_i} \right) \tag{B-7}$$

$$= \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \frac{1}{\rho} \frac{\partial P}{\partial x_i} \right) \tag{B-8}$$

$$= \frac{d}{dP} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \right) \frac{1}{\rho} \left(\frac{dP}{dn} \right)^2 + \frac{1}{\kappa} \frac{dp_r}{dP} \vec{\nabla} \cdot \left(\frac{1}{\rho} \vec{\nabla} P \right) \quad , \tag{B-9}$$

where we have used

$$\left(\frac{dP}{dn} \right)^2 = \sum_{i=1}^{3} \left(\frac{\partial P}{\partial x_i} \right)^2 \tag{B-10}$$

and measured n normal to the surfaces of constant potential.

If the force on a particle arises from a potential V, then we may write the equivalent potential for motion with respect to axes that are rotating with constant angular velocity ω about a polar axis x_3 as

$$\Psi = V + \frac{1}{2} \omega^2 (x_1^{\,2} + x_2^{\,2}) \quad . \tag{B-11}$$

(See for example Danby (1962, p. 47).) Thus for the rotating star with gravitational potential V

$$\vec{\nabla} \cdot \vec{\nabla}\Psi = -\ 4\pi G\rho + 2\omega^2 \quad . \tag{B-12}$$

If hydrostatic equilibrium is assumed then

$$\vec{\nabla}P = \rho\ \vec{\nabla}\Psi \tag{B-13}$$

and equation (B-12) becomes

$$\vec{\nabla} \cdot \left(\frac{1}{\rho} \vec{\nabla}P \right) = -4\pi G\rho + 2\omega^2 \quad . \tag{B-14}$$

Combining equations (B-6), (B-9), and (B-14) we have

$$\frac{d}{dP} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \right) \frac{1}{\rho} \left(\frac{dP}{dn} \right)^2 = \frac{2}{\kappa} \frac{dp_r}{dP} \left(2\pi G\rho - \omega^2 \right) - \frac{\varepsilon\rho}{c} \quad . \tag{B-15}$$

Along an equipotential surface the right-hand side of equation (B-15) is constant (Kopal 1959, p. 170ff), thus

$$\frac{d}{dP} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \bigg|_{\text{surface}} \right) \frac{1}{\rho} \left(\frac{dP}{dn} \bigg|_{\text{surface}} \right)^2 = \text{constant} \quad . \tag{B-16}$$

If it is assumed the constant is non-zero, the equipotential surfaces must be equidistant. But, in a rotationally or tidally distorted star, this is not possible. Thus the constant is zero and for each equipotential surface

$$\frac{d}{dP} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \right) = 0 \quad , \tag{B-17}$$

since the pressure gradient is non-zero. Then

$$\frac{1}{\kappa} \frac{dp_r}{dP} = \text{constant} \quad . \tag{B-18}$$

Thus for the normal component of $\vec{F}$ we have

$$F_n = - \frac{c}{\kappa\rho} \frac{dp_r}{dn} = - \frac{c}{\kappa\rho} \frac{dp_r}{dP} \frac{dP}{dn} \quad , \tag{B-19}$$

and

$$F_n \propto \frac{1}{\rho} \frac{dP}{dn} \propto \frac{d\Psi}{dn} \quad . \tag{B-20}$$

Thus we expect that at the boundary of the star the intensity H of total radiation emerging normally from the atmosphere should vary as

$$\frac{H - H_o}{H_o} = \frac{g - g_o}{g_o} \quad , \tag{B-21}$$

where g and g_o are the local and mean surface gravities and H and H_o are the corresponding intensities. Further, Kopal has shown that assuming black body radiation the surface brightness H_λ at a particular wavelength λ may be expressed as

$$\frac{H_\lambda}{H_o} = 1 - y\left(1 - \frac{g}{g_o}\right) \quad , \tag{B-22}$$

where

$$y = \frac{1}{4} \left(\frac{T}{B} \frac{dB}{dT}\right)\Bigg|_{T_e} \quad . \tag{B-23}$$

T is the actual temperature of the atmosphere and T_e is the effective temperature. B is the Planck function.

The theory of stellar atmospheres thus indicates that the limb-darkening coefficient and the gravity-darkening coefficient are not independent. (See Kopal (1959, p. 159 and p. 172).) The adopted values of the gravity-darkening coefficient as a function of the limb-darkening coefficient are those suggested by Russell and Merrill (1952) and tabulated by Jurkevich (1964, p. 186). A tabulation of y as a function of x is given in Table 21. The values of N as calculated by equation (2.31) are also given in Table 21.

Table 21. Gravity Darkening as a Function of Limb Darkening.

x	y	N
0.4	0.08571	2.2
0.6	1.00000	2.6
0.8	1.2278	3.2
1.0	1.2500	3.6

APPENDIX C. LUMINOUS EFFICIENCY CALCULATIONS

Portions of the following treatment have been adopted from Jurkevich (1964, p. 140ff) and Linnell (1971).

Let $L_{h,b}$ and $L_{c,b}$ be the intrinsic bolometric luminosities of the hotter and cooler stars. Then

$$L_{h,b} = \sigma\, T_{h,b}^{\;4}\, 4\pi r_h^{\;2} \tag{C-1}$$

and

$$L_{c,b} = \sigma\, T_{c,b}^{\;4}\, 4\pi r_c^{\;2} \quad , \tag{C-2}$$

where r_h and r_c are the radii of the hotter and cooler stars expressed in physical units and $T_{h,b}$ and $T_{c,b}$ are the corresponding effective temperatures.

The total energy from the hotter star intercepted by the cooler star is

$$\Delta L_c^h = L_h \frac{r_c^{\;2}}{4a^2} \quad , \tag{C-3}$$

where a is the separation of the stars expressed in physical units and $L_h/4\pi r_h^{\;2}$ is the surface luminosity of the inner hemisphere and includes heating by the radiation of the cooler star. Similarly

$$\Delta L_h^c = L_c \frac{r_h^{\;2}}{4a^2} \quad . \tag{C-4}$$

In deriving equations (C-3) and (C-4) it has been assumed that all of the incident external radiation is absorbed and that the "heating" is uniform over the inner faces.

Thus for the inner hemisphere of the hotter star we have

$$\frac{1}{2} L_h = \frac{1}{2} L_{h,b} + \Delta L_h^c = \sigma \; T_h{}^4 \; 2\pi r_h{}^2 \quad , \tag{C-5}$$

where T_h is the effective temperature of the hotter face of the hotter component. Similarly, for the cooler component

$$\frac{1}{2} L_c = \frac{1}{2} L_{c,b} + \Delta L_c^h = \sigma \; T_c{}^4 \; 2\pi r_c{}^2 \quad . \tag{C-6}$$

Define the luminous efficiency, with σ the Stefan - Boltzmann constant, as

$$E(T) = \frac{J_T(\lambda)}{\sigma T^4} \quad , \tag{C-7}$$

where $J_T(\lambda)$ is the wavelength distribution of the emitted energy. Then

$$E_h\left(\frac{1}{2} L_{h,b} + \Delta L_h^c \right) = E_h \; \sigma T_h{}^4 \; 2\pi r_h{}^2 \tag{C-8}$$

$$= \quad J_{T_h}(\lambda) \; 2\pi r_h{}^2 \quad . \tag{C-9}$$

Define the increase in radiation at effective wavelength λ caused by the incident external radiation as $2S_h$. Then

$$2S_h = E_h(T_h)\left(\frac{1}{2} L_{h,b} + \Delta L_h^c \right) - \frac{1}{2} E_h(T_{h,b})L_{h,b} \tag{C-10}$$

$$= E_h(T_h)\Delta L_h^c + \frac{1}{2} \left(E_h(T_h) - E_h(T_{h,b})\right) L_{h,b} \quad . \tag{C-11}$$

If, as is customary, it is assumed that the change in effective temperature is small, then

$$2S_h = E_h(T_h)\Delta L_h^c \quad . \tag{C-12}$$

Similarly for S_c

$$2S_c = E_c(T_c)\Delta L_c^h \quad . \tag{C-13}$$

Thus we have for the ratio of reflected lights

$$\frac{S_c}{S_h} = \frac{E_c \Delta L_c^h}{E_h \Delta L_h^c} = \frac{E_c L_h r_c^2}{E_h L_c r_h^2} \quad . \tag{C-14}$$

With equations (C-5) and (C-6), equation (C-14) becomes

$$\frac{S_c}{S_h} = \frac{E_c^2}{E_h} \frac{T_h^4}{T_c^4} \quad . \tag{C-15}$$

Substituting for T^4 (see equation (C-7)), we have

$$\frac{S_c}{S_h} = \frac{E_c^2}{E_h^2} \frac{J_{T_h}(\lambda)}{J_{T_c}(\lambda)} \quad . \tag{C-16}$$

If it is assumed that the stars radiate like black bodies, Planck's law gives the emergent surface flux distribution as a function of wavelength,

$$J_T(\lambda) = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1} \quad . \tag{C-17}$$

Here c_1 and c_2 are Planck's first and second radiation constants, λ is the effective wavelength, and T is the absolute temperature. The total emergent surface flux is given by the Stefan-Boltzmann relation

$$J_b = \sigma T^4 \quad , \tag{C-18}$$

where σ is the Stephen-Boltzmann constant.

Let

$$x = \frac{c_2}{\lambda T} \tag{C-19}$$

and define $J_T'(\lambda)$ as

$$J_T'(\lambda) = \frac{J_T(\lambda)}{c_1/c_2^5} = \frac{T^5 x^5}{e^x - 1} \quad . \tag{C-20}$$

The luminous efficiency is

$$E(x) = J_T(\lambda)/J_b = \frac{c_1}{\sigma c_2^4 \lambda} \left(\frac{x^4}{e^x - 1} \right) \quad . \tag{C-21}$$

Define $E'(x)$ as

$$E'(x) = \frac{E(x)}{c_1/(\sigma c_2^4 \lambda)} = \frac{x^4}{e^x - 1} \quad . \tag{C-22}$$

The maximum value of this function is

$$E'_{max} = 4.7798404 \quad , \tag{C-23}$$

which occurs at x_{max}, where

$$x_{max} = 3.9206904 \tag{C-24}$$

(Jurkevich 1964, p. 143). The values of T_{max} and $J'_{T,max}$ then follow as

$$T_{max} = c_2/(\lambda x_{max}) \tag{C-25}$$

and

$$J'_{T,max} = \frac{(T_{max}\, x_{max})^5}{(e^{x_{max}} - 1)} \quad . \tag{C-26}$$

Note that λ is fixed for a given set of observations and this discussion relates the maximum value of E to an effective temperature.

For the known spectral type a value of T_h is obtained from a plot of color temperature versus spectral type, as in Figure 15. The data for this plot is from Harris (1963).

We then have

$$x_h = c_2/(\lambda T_h) \tag{C-27}$$

and

$$J'_{T_h}(\lambda) = \frac{(T_h x_h)^5}{(e^{x_h}-1)} \quad . \tag{C-28}$$

The values of $J'_{T_h}(\lambda)$ and E'_h normalized to the values at x_{max} are

$$J^n_{T_h}(\lambda) = J'_{T_h}(\lambda)/J'_{max}(\lambda) \tag{C-29}$$

and

$$E^n_h = E'_h / E'_{max} \quad , \tag{C-30}$$

where

$$E'_h = \frac{x_h^4}{(e^{x_h}-1)} \quad . \tag{C-31}$$

With this normalization the range of E^n_h is [0,1].

The spectral type of the cooler star cannot always be determined. An alternate method of determining T_c is thus necessary. The ratio of the mean surface brightnesses of the smaller and larger component is given by

$$\frac{J_s}{J_g} = Y(k,p_o)\ \frac{1-\ell(p=p_o)_{occultation}}{1-\ell(p=p_o)_{transit}} \quad , \tag{C-32a}$$

where p_o is the geometrical depth of maximum eclipse and is defined as as -1 for total eclipses (Kopal 1959, p. 338 and p. 348). Approximating $Y(k,p_o)$ as unity (Kopal 1959, p. 343)

$$\frac{J_s}{J_g} = \frac{1-\ell(p_o)_{occultation}}{1-\ell(p_o)_{transit}} \quad . \tag{C-32b}$$

Associating the deeper eclipse with the eclipse of the hotter star and defining a as the ratio of surface luminosities of the hotter and cooler components

$$\underline{a} \equiv \frac{J'_{T_h}(\lambda)}{J'_{T_c}(\lambda)} \simeq \frac{1-\ell(p_o)_{primary\ minimum}}{1-\ell(p_o)_{secondary\ minimum}} \quad . \tag{C-33}$$

Thus

$$J'_{T_c}(\lambda) = \frac{J'_{T_h}(\lambda)}{\underline{a}} = \frac{1}{\underline{a}} \frac{(T_h x_h)^5}{(e^{x_h}-1)} \quad , \tag{C-34}$$

$$\frac{(T_c x_c)^5}{(e^{x_c}-1)} = \frac{1}{\underline{a}} \frac{(T_h x_h)^5}{(e^{x_h}-1)} \quad . \tag{C-35}$$

Solving equation (C-35) for x_c, we have

$$x_c = \ln\left(\underline{a}(e^{x_h}-1) + 1 \right) \tag{C-36}$$

and thus

$$T_c = c_2/(\lambda x_c) \quad . \tag{C-37}$$

Continuing with the calculation for S_c/S_h, we have

$$J'_{T_c}(\lambda) = \frac{(T_c x_c)^5}{(e^{x_c}-1)} \quad , \tag{C-38}$$

with its normalized value

$$J^n_{T_c}(\lambda) = J'_{T_c}(\lambda)/J'_{max}(\lambda) \quad , \tag{C-39}$$

and

$$E_c' = \frac{x_c^{\ 4}}{(e^{x_c}-1)} \quad , \tag{C-40}$$

with its normalized value

$$E_c^n = E_c'/E_{max}' \quad . \tag{C-41}$$

Finally, combining equation (C-29) and (C-39) with equation (C-16) we have

$$\frac{S_c}{S_h} = \frac{J_{T_h}^{\ n}(\lambda)/(E_h^n)^2}{J_{T_c}^{\ n}(\lambda)/(E_c^n)^2} \quad . \tag{C-42}$$

In summary, we have assumed:

(1) Stars radiate like black bodies.

(2) All incident radiation is absorbed and re-emitted at the effective temperature of the absorbing star.

(3) The luminous efficiency is not significantly changed by the external incident radiation.

(4) The heating is uniform over the inner face of the star.

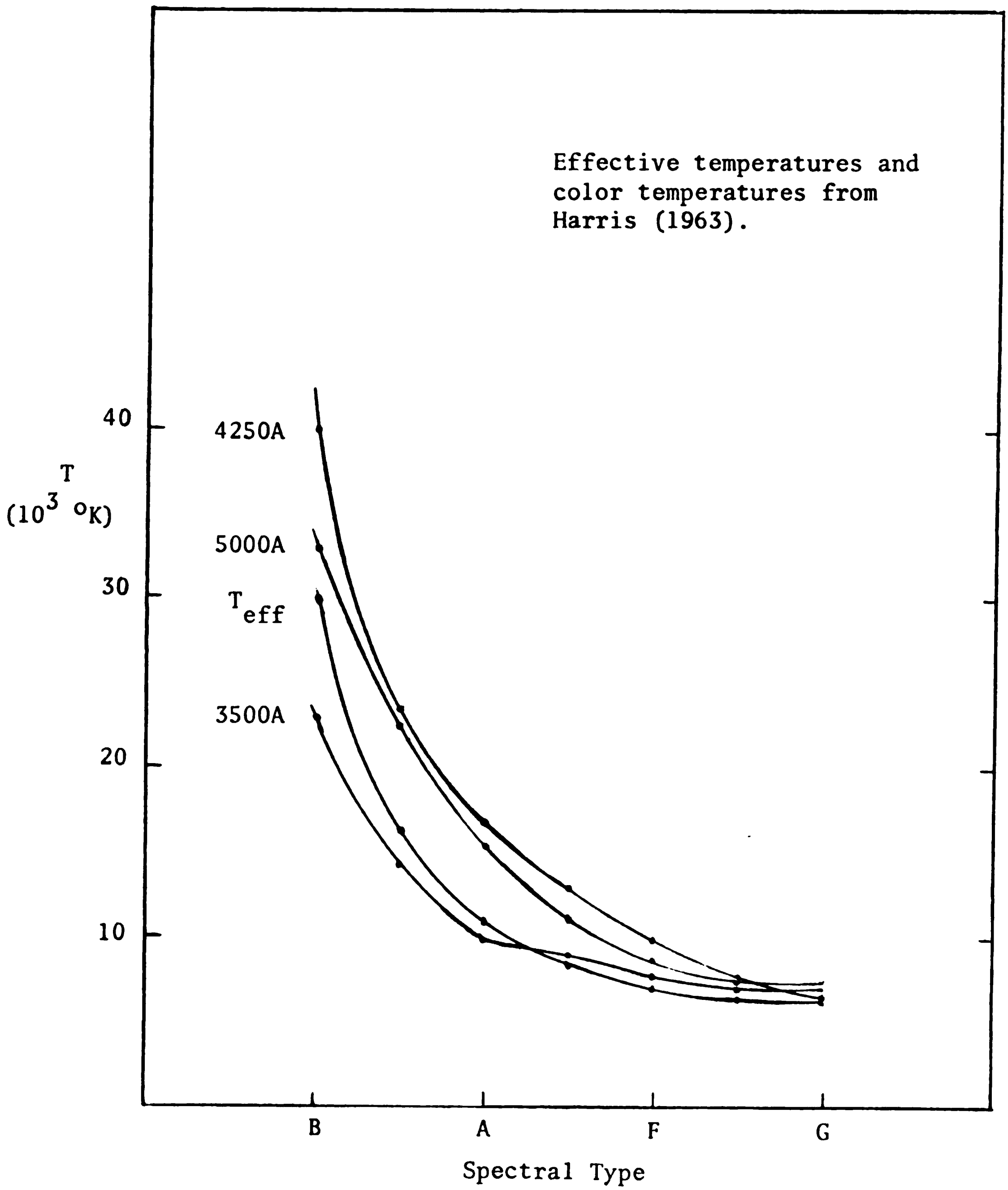


Figure 15. Effective Temperature and Color Temperature.

APPENDIX D. THE FUNCTIONS ${}^{x}f^{oc}$ AND ${}^{x}f^{tr}$

As was mentioned in Section A3 of Chapter II, it is convenient to express the geometrical dependence of the theoretical value of the light of an eclipsing binary as a function of the ratio of radii k and the geometrical depth of eclipse p. Thus

$$\ell = U - {}^{x}f(k,p)L \quad , \tag{D-1}$$

where ℓ is the theoretical value of the light of the eclipsing binary, U is the unit of light, L is the total light of the eclipsed star (with limb-darkening coefficient x), and ${}^{x}f(k,p)$ is the fraction of the light L lost by eclipse at geometrical depth p.

The fractional loss of light for occultation is expressed as

$${}^{x}f^{oc}(k,p) = {}^{x}\alpha^{oc}(k,p) \quad , \tag{D-2}$$

while the fractional loss of light for transit is

$${}^{x}f^{tr}(k,p) = {}^{x}\tau(k)\ {}^{x}\alpha^{tr}(k,p) \quad , \tag{D-3}$$

where

$${}^{x}\tau(k) = \frac{3(1 - x)}{3 - x}\, k^2 + \frac{2x}{3 - x}\, {}^{10}\tau(k) \tag{D-4}$$

with

$${}^{10}\tau(k) = \frac{2}{\pi}\left(\sin^{-1}\sqrt{k} + \frac{1}{3}\,(4k - 3)(2k + 1)\sqrt{k(1 - k)} \right) \quad . \tag{D-5}$$

The α function is the fractional amount of light lost at geometrical depth p normalized to the fractional amount of light lost at internal tangency (p = -1) for the respective eclipse. In turn, the α functions may be expressed in terms of the α functions for uniform and completely limb-darkened stars, ${}^{0}\alpha$ and ${}^{10}\alpha$, respectively. Thus

$$x_{\alpha}^{oc} = \frac{3(1 - x)}{3 - x}\,{}^{0}\alpha + \frac{2x}{3 - x}\,{}^{10}\alpha^{oc} \quad , \tag{D-6}$$

$$x_{\alpha}^{tr} = (1 - u^{tr})\alpha^{0} + u^{tr}\,\alpha^{tr} \quad , \tag{D-7}$$

where

$$u^{tr} = \frac{x\,\phi}{1 - x + x\phi} \tag{D-8}$$

and

$$\phi = \frac{2}{3k^2}\,{}^{10}\tau(k) \quad . \tag{D-9}$$

These relations are discussed by Irwin (1947) and Kopal (1950, p. 34ff). Merrill (1950) gives the generating equations for the various α functions. The special forms used in the computer routines are given by Linnell (1965a,b; 1966a,b,c).

APPENDIX E. PARTIAL DERIVATIVES OF ${}^{x}f^{oc}$ AND ${}^{x}f^{tr}$

The generating expressions for $\frac{\partial^{x}f}{\partial r_g}$, $\frac{\partial^{x}f}{\partial r_s}$, $\frac{\partial^{x}f}{\partial \delta}$, and $\frac{\partial^{x}f}{\partial x}$, where ${}^{x}f$ is ${}^{x}f^{oc}$ or ${}^{x}f^{tr}$, have been given by Kopal (1946, p. 78ff) and Irwin (1947, p. 386). The function ${}^{x}f$ is a homogeneous function of δ/r_g and δ/r_s of order zero (Kopal 1950, p. 88). The dependence of ${}^{x}f$ on $\cos^2 i$, e, ω, and t_o is through δ. Previous methods for including the effects of orbital eccentricity on eclipsing binary light curves employed "fictitious" elements and were correct to second or third order in the orbital eccentricity (Kopal 1950, p. 106). A more direct approach is possible.

We have

$$\frac{\partial^{x}f}{\partial \cos^2 i} = \frac{\partial^{x}f}{\partial \delta}\frac{\partial \delta}{\partial \cos^2 i} , \tag{E-1}$$

$$\frac{\partial^{x}f}{\partial e} = \frac{\partial^{x}f}{\partial \delta}\frac{\partial \delta}{\partial e} , \tag{E-2}$$

$$\frac{\partial^{x}f}{\partial \omega} = \frac{\partial^{x}f}{\partial \delta}\frac{\partial \delta}{\partial \omega} , \tag{E-3}$$

$$\frac{\partial^{x}f}{\partial t_o} = \frac{\partial^{x}f}{\partial \delta}\frac{\partial \delta}{\partial t_o} , \tag{E-4}$$

where

$$\delta = R(\sin^2\theta \sin^2 i + \cos^2 i)^{\frac{1}{2}} . \tag{E-5}$$

Here

$$R = \frac{a(1 - e^2)}{1 + e \cos \upsilon} \tag{E-6}$$

where a is the semi-major axis of the orbit (taken as unity), υ is the true anomaly measured from periastron, and θ is the phase angle measured from minimum δ (primary minimum). The phase angle θ is defined by

$$\theta = \upsilon + \omega - 90^{o} , \tag{E-7}$$

with the true anomaly υ given by

$$\tan \frac{\upsilon}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \tag{E-8}$$

(Binnendijk (1960, p. 101)). The eccentric anomaly E needed for equation (E-8) results from the solution of Kepler's equation

$$M = E - e \sin E , \tag{E-9}$$

where M is the mean anomaly with respect to periastron

$$M = \frac{2\pi}{P} (t - t_{pp}) . \tag{E-10}$$

(Binnendijk (1960, p. 102) discusses Kepler's equation in greater detail.) Here t_{pp} is the time of periastron passage. We wish to express M is terms of t_o, the time of minimum δ at primary eclipse, since t_o is more easily estimated from the light curve. Let the subscript "o" refer to a quantity evaluated at minimum δ, δ_o. Then the mean anomaly at minimum δ is

$$M_o = \frac{2\pi}{P} (t_o - t_{pp}) . \tag{E-11}$$

Substituting t_{pp} from equation (E-11) into equation (E-10) we have

$$M = \frac{2\pi}{P} (t - t_o) + M_o . \tag{E-12}$$

We can obtain the value of M_o from the geometrical and orbital parameters by integration of Kepler's second law. From Kopal (1946, p. 94) for the time interval $(t_2 - t_1)$, we have

$$\frac{2\pi}{P}(t_2 - t_1) = \frac{1}{a^2\sqrt{1 - e^2}} \int_{\upsilon_1}^{\upsilon_2} R^2 \, d\upsilon \tag{E-13a}$$

$$= (1 - e^2)^{3/2} \int_{\upsilon_1}^{\upsilon_2} \frac{1}{(1 + e \cos \upsilon)^2} \, d\upsilon \quad , \tag{E-13b}$$

where the indefinite integral is evaluated as

$$\int \frac{d\upsilon}{(1 + e \cos \upsilon)^2} = \frac{1}{1-e^2} \left(\frac{2}{\sqrt{1-e^2}} \tan^{-1}\left(\frac{1-e}{1+e} \tan \frac{\upsilon}{2} \right) - \frac{e \sin \upsilon}{1 + e \cos \upsilon} \right) \tag{E-14}$$

and υ_1 and υ_2 are the true anomalies at t_1 and t_2 respectively. Thus applying (E-13) and (E-14), we have for M_o

$$M_o = \frac{2\pi}{P}(t_o - t_{pp}) \tag{E-15}$$

$$= 2 \tan^{-1} \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\upsilon_o}{2} \right) - \frac{e\sqrt{1-e^2} \sin \upsilon_o}{1 + e \cos \upsilon_o} \quad , \tag{E-16}$$

where

$$\upsilon_o = \theta_o - \omega + 90^o \; . \tag{E-17}$$

The required value of θ_o is derived from the evaluation of the minimum geometrical depth of eclipse δ_o. The expression for δ is

$$\delta = \frac{a(1 - e^2)(\sin^2\theta \sin^2 i + \cos^2 i)^{1/2}}{1 + e \cos(\theta - \omega + 90^o)} \quad . \tag{E-18}$$

The requirement for minimum δ is given by $\frac{\partial \delta}{\partial \theta} = 0$. From Kopal (1950, p. 106) this expression is

$$\left(1-e\ \sin(\theta_o-\omega)\right)\sin^2 i\ \sin 2\theta_o + 2e\ \cos(\theta_o-\omega)\ \left(1-\cos^2\theta_o\ \sin^2 i\right) = 0 \quad . \tag{E-19}$$

This can be solved numerically for θ_o. Thus we have

$$\theta_o = \theta_o(\cos^2 i, e, \omega) \quad . \tag{E-20}$$

With the evaluation of θ_o the equations necessary for the evaluation of M_o are complete.

The evaluation of the derivatives of δ with respect to $\cos^2 i$, e, ω, and t_o is required. This proceeds as follows:

We have

$$\delta = R\ (\sin^2\theta\ \sin^2 i + \cos^2 i)^{1/2} \tag{E-21}$$

and

$$R = \frac{a(1-e^2)}{1 + e\ \cos\upsilon} \quad . \tag{E-22}$$

Thus

$$\frac{\partial\delta}{\partial t_o} = \frac{\partial R}{\partial t_o}\frac{\delta}{R} + \frac{R^2}{\delta}\sin^2 i\ \sin\theta\ \cos\theta\ \frac{\partial\theta}{\partial t_o} \quad , \tag{E-23}$$

$$\frac{\partial\delta}{\partial\cos^2 i} = \frac{\partial R}{\partial\cos^2 i}\frac{\delta}{R} + \frac{R^2}{\delta}\left\{\sin^2 i\ \sin\theta\ \cos\theta\ \frac{\partial\theta}{\partial\cos^2 i} + \frac{\cos^2\theta}{2}\right\} \quad , \tag{E-24}$$

$$\frac{\partial\delta}{\partial e} = \frac{\partial R}{\partial e}\frac{\delta}{R} + \frac{R^2}{\delta}\sin^2 i\ \sin\theta\ \cos\theta\ \frac{\partial\theta}{\partial e} \quad , \tag{E-25}$$

$$\frac{\partial\delta}{\partial\omega} = \frac{\partial R}{\partial\omega}\frac{\delta}{R} + \frac{R^2}{\delta}\sin^2 i\ \sin\theta\ \cos\theta\ \frac{\partial\theta}{\partial\omega} \quad , \tag{E-26}$$

where from (E-7)

$$\frac{\partial \theta}{\partial t_o} = \frac{\partial \upsilon}{\partial t_o} \quad , \tag{E-27}$$

$$\frac{\partial \theta}{\partial \cos^2 i} = \frac{\partial \upsilon}{\partial \cos^2 i} \quad , \tag{E-28}$$

$$\frac{\partial \theta}{\partial e} = \frac{\partial \upsilon}{\partial e} \quad , \tag{E-29}$$

$$\frac{\partial \theta}{\partial \omega} = 1 + \frac{\partial \upsilon}{\partial \omega} \quad , \tag{E-30}$$

and

$$\frac{\partial R}{\partial t_o} = \frac{R\ e \sin \upsilon}{1 + e \cos \upsilon} \frac{\partial \upsilon}{\partial t_o} \quad , \tag{E-31}$$

$$\frac{\partial R}{\partial \cos^2 i} = \frac{R\ e \sin \upsilon}{1 + e \cos \upsilon} \frac{\partial \upsilon}{\partial \cos^2 i} \quad , \tag{E-32}$$

$$\frac{\partial R}{\partial e} = \left(R\ e \sin \upsilon \frac{\partial \upsilon}{\partial e} - R \cos \upsilon - 2\ a\ e \right) / \left(1 + e \cos \upsilon\right) \quad , \tag{E-33}$$

$$\frac{\partial R}{\partial \omega} = \frac{R\ e \sin \upsilon}{1 + e \cos \upsilon} \frac{\partial \upsilon}{\partial \omega} \quad . \tag{E-34}$$

Solving equation (E-8) for υ we have

$$\upsilon = 2 \tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right) \quad . \tag{E-35}$$

The partial derivatives of υ with respect to t_o, $\cos^2 i$, e, and ω are thus

$$\frac{\partial \upsilon}{\partial t_o} = \sqrt{\frac{1+e}{1-e}} \cos^2\left(\frac{\upsilon}{2}\right) \left(1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon}{2}\right)\right) \frac{\partial E}{\partial t_o} \quad , \tag{E-36}$$

$$\frac{\partial \upsilon}{\partial \cos^2 i} = \sqrt{\frac{1+e}{1-e}} \cos^2\left(\frac{\upsilon}{2}\right) \left(1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon}{2}\right)\right) \frac{\partial E}{\partial \cos^2 i} \quad , \tag{E-37}$$

$$\frac{\partial \upsilon}{\partial e} = \sqrt{\frac{1+e}{1-e}} \cos^2\left(\frac{\upsilon}{2}\right) \left[1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon}{2}\right)\right] \frac{\partial E}{\partial e}$$

$$+ 2 \cos^2\left(\frac{\upsilon}{2}\right) \tan\left(\frac{\upsilon}{2}\right) \left[\frac{1}{1-e^2}\right] \quad , \tag{E-38}$$

$$\frac{\partial \upsilon}{\partial \omega} = \sqrt{\frac{1+e}{1-e}} \cos^2\left(\frac{\upsilon}{2}\right) \left[1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon}{2}\right)\right] \frac{\partial E}{\partial \omega} \quad . \tag{E-39}$$

From equation (E-9) and equation (E-12) we have

$$\frac{2\pi}{P} (t - t_o) + M_o = E - e \sin E \quad . \tag{E-40}$$

Implicit differentiation of this equation gives

$$\frac{\partial E}{\partial t_o} = - \frac{a}{R} \frac{2\pi}{P} \quad , \tag{E-41}$$

$$\frac{\partial E}{\partial \cos^2 i} = \frac{a}{R} \frac{\partial M_o}{\partial \cos^2 i} \quad , \tag{E-42}$$

$$\frac{\partial E}{\partial e} = \frac{a}{R} \frac{\partial M_o}{\partial e} + \frac{\sin \upsilon}{\sqrt{1-e^2}} \quad , \tag{E-43}$$

$$\frac{\partial E}{\partial \omega} = \frac{a}{R} \frac{\partial M_o}{\partial \omega} \quad , \tag{E-44}$$

where we have used

$$\frac{\partial M_o}{\partial t_o} = 0 \tag{E-45}$$

and

$$R = a(1 - e \cos E) \quad . \tag{E-46}$$

(See Bennendijk (1960, p. 101) for a proof of the latter equation.)

Repeating equation (E-16) we write the expression for M_o in terms of e and υ_o

$$M_o = 2 \tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\upsilon_o}{2} \right) - \frac{e\sqrt{1-e^2} \sin \upsilon_o}{1 + e \cos \upsilon_o} \quad . \tag{E-47}$$

Thus the derivatives of M_o are

$$\frac{\partial M_o}{\partial \cos^2 i} = \frac{\partial M_o}{\partial \upsilon_o} \frac{\partial \upsilon_o}{\partial \cos^2 i} \quad , \tag{E-48}$$

$$\frac{\partial M_o}{\partial e} = \frac{\partial M_o}{\partial \upsilon_o} \frac{\partial \upsilon_o}{\partial e} - \frac{\tan \frac{\upsilon_o}{2}}{1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon_o}{2}\right)} \sqrt{\frac{1+e}{1-e}} \frac{2}{(1+e)^2}$$

$$- \frac{\sin \upsilon_o}{1 + e \cos \upsilon_o} \left(\frac{\sqrt{1-e^2}}{1 + e \cos \upsilon_o} - \frac{e^2}{\sqrt{1-e^2}} \right) , \tag{E-49}$$

$$\frac{\partial M_o}{\partial \omega} = \frac{\partial M_o}{\partial \upsilon_o} \frac{\partial \upsilon_o}{\partial \omega} \quad , \tag{E-50}$$

where

$$\frac{\partial M_o}{\partial \upsilon_o} = \frac{2}{1 + \frac{1-e}{1+e} \tan^2\left(\frac{\upsilon_o}{2}\right)} \sqrt{\frac{1-e}{1+e}} \frac{1}{1 + e \cos \upsilon_o}$$

$$- \frac{e\sqrt{1-e^2}}{1 + e \cos \upsilon_o} \left(\cos \upsilon_o + \frac{e \sin^2 \upsilon_o}{1 + e \cos \upsilon_o} \right) . \tag{E-51}$$

With

$$\upsilon_o = \theta_o - \omega + 90^o \quad , \tag{E-52}$$

we have

$$\frac{\partial \upsilon_o}{\partial \cos^2 i} = \frac{\partial \theta_o}{\partial \cos^2 i} \quad , \tag{E-53}$$

$$\frac{\partial \upsilon_o}{\partial e} = \frac{\partial \theta_o}{\partial e} \quad , \tag{E-54}$$

$$\frac{\partial \upsilon_o}{\partial \omega} = -1 + \frac{\partial \theta_o}{\partial \omega} \quad . \tag{E-55}$$

Since θ_o is the root of equation (E-19), we may differentiate equation (E-19) implicitly to obtain the derivatives of θ_o with respect to $\cos^2 i$, e, and ω. With

$$\begin{aligned} D = {} & 2(1 - e \sin(\theta_o - \omega))\sin^2 i \cos 2\theta_o \\ & + e \cos(\theta_o - \omega)(\sin^2 i \sin 2\theta_o) \\ & - 2e \sin(\theta_o - \omega)(1 - \cos^2\theta_o \sin^2 i) \quad , \end{aligned} \tag{E-56}$$

we have

$$\frac{\partial \theta_o}{\partial \cos^2 i} = \left((1 - e \sin(\theta_o - \omega)) \sin 2\theta_o - 2e \cos(\theta_o - \omega) \cos^2\theta_o\right) / D \, , \tag{E-57}$$

$$\frac{\partial \theta_o}{\partial e} = \left(- 2 \cos(\theta_o - \omega)(1 - \cos^2\theta_o \sin^2 i) + \sin(\theta_o - \omega) \sin^2 i \sin 2\theta_o\right) / D, \tag{E-58}$$

$$\frac{\partial \theta_o}{\partial \omega} = \left(- e \cos(\theta_o - \omega) \sin^2 i \sin^2\theta - 2e \sin(\theta_o - \omega)(1 - \cos^2\theta_o \sin^2 i\right) / D. \tag{E-59}$$

The equations necessary for the evaluation of the partial derivatives of x_f^{oc} and x_f^{tr} are now complete.

If it is desired to calculate the corrections to e and ω in the form $\Delta(e \sin \omega)$ and $\Delta(e \cos \omega)$ the following transformations may be applied

$$\frac{\partial f}{\partial (e \sin \omega)} = \sin \omega \frac{\partial f}{\partial e} + \frac{\cos \omega}{e} \frac{\partial f}{\partial \omega} \quad , \tag{E-60}$$

$$\frac{\partial f}{\partial (e \cos \omega)} = \cos \omega \frac{\partial f}{\partial e} - \frac{\sin \omega}{e} \frac{\partial f}{\partial \omega} \quad . \tag{E-61}$$

APPENDIX F. ERRORS IN THE PUBLISHED DATA

This appendix contains a list of points that were found to contain errors in their published phase values. These points are listed with corrected phase values. Also included in the table are points that were omitted from subsequent iterative solutions because their residuals were greater than 3 standard deviations from the calculated light curve. These points are listed without corresponding phase values. It is felt that most of these errors are typographical in nature.

Table 22. Errors in published data.

System	Color	$J.D._{Hel.}$	Corrected Phase Values
CO Lac	B	2439033.5594	.5141
		2439034.3798	.0461
		2439060.4350	.9409
	V	2438990.4732	.5760
		2439029.6320	.9675
CM Lac	U	2434595.694	
		2434643.821	
		2437201.7242	
		2437201.7465	
	B	2434595.694	
		2434606.872	
		2434643.753	
		2434643.813	
	V	2434595.694	
RX Ari	B	2437984.6910	.0301
	V	2437637.7173	.0400
		2437639.6936	
		2438315.7749	
		2438398.6477	
		2438398.6513	
V338 Her	V	2439648.4767	.9683
Y Leo	IR	2436631.7214	.0104
RW Mon	IR	2439454.8463	

Table 22 (cont'd)

System	Color	J.D.$_{\text{Hel.}}$	Corrected Phase Values
BV 412	B	2439036.86266	.86950
	V	2439094.72037	.87819
SW Lyn	V	2439598.3325	
14 observations beginning with		2439615.3140	
	B		
13 observations beginning with		2439598.3201	

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