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THE CENTRALITY OF A TEACHER'S PROFESSIONAL TRANSFORMATION IN THE DEVELOPMENT OF MATHEMATICAL POWER: A CASE STUDY OF ONE HIGH SCHOOL MATHEMATICS TEACHER

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Janice Simonson Gormas

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By

Janice Simonson Gormas

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ABSTRACT

THE CENTRALITY OF A TEACHER'S PROFESSIONAL TRANSFORMATION IN THE DEVELOPMENT OF MATHEMATICAL POWER: A CASE STUDY OF ONE HIGH SCHOOL MATHEMATICS TEACHER

By

Janice Simonson Gormas

This dissertation is the result of a five year study of a high school mathematics teacher engrossed in change; changes that added up to a professional transformation that positioned him to provide opportunities for his students to develop mathematical power, as defined by the National Council of Teachers of Mathematics. The data analyzed, including informal conversations, formal interviews, field observations, the teacher's writings and student work, ended up revealing epistemological shifts and philosophical shifts, informing and informed by a radically altered instructional practice.

The body of the dissertation (Chapters 2-4) describes the components of the transformation, including a changed view of the goals of learning mathematics, a changed philosophy of mathematics and a changed view of the nature of teaching mathematics. These changes and the resulting mathematical power are illustrated in three vignettes (representing three progressive stages of the teacher's thinking and instruction) that appear in the Introduction.

As the teacher expands his goals for student learning to include a close look at the "why's" of mathematics, he ends up inviting them to explore, make conjectures and to use logical reasoning and mathematical conversations to support or refute conjectures and

work toward a consensus. These signs of mathematical power were made possible by changing his view of school mathematics to include the analysis of patterns and number systems rather than merely developing isolated symbolic manipulation skills. Eventually, he comes to recognize mathematics as a human construction, developed in a historical and cultural context -- ideas that he transfers to his students mathematical activities, allowing them the opportunity to socially construct mathematics. He encourages them to communicate their mathematical ideas, connect their mathematical conceptions and develop an understanding that can transfer to different situations.

The changes this teacher experienced within himself and incorporated in his instructional practice were heart-felt and deeply rooted. The transformation happened largely as a response to his own frustrations around his students' inability to make connections. Other factors that contributed to this transformation include a school atmosphere that promoted change as the school worked closely with a local university that provided resources including an innovative curriculum, technological tools and mathematics educators interested in discussing change and innovation. The real story here is the transformation the teacher experienced, completely dismantling his views of mathematics, teaching and learning. He then reconstructed his practice based on a new set of beliefs that reinforced one another and were well thought-out. This transformation positioned him to provide opportunities for all of his students to gain mathematical power. We can learn from this experience that professional transformation of this nature can and did happen to a mathematics teacher and is closely related to developing mathematical power, in both the teacher experiencing the transformation and in the students.

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Apart from the grace of my Lord Jesus Christ, the love of God, and the fellowship of the Holy Spirit this study would never have come to fruition. For that I am very thankful.

I would like to extend a sincere thanks to the teacher featured in this work for recognizing the need for change in the teaching of high school mathematics and for being completely willing and extremely helpful in getting this version of his story told. It is his dedication to the profession that sets the stage for such a dynamic transformation and for that I am both grateful and encouraged.

The research related to this study took place in a working Professional Development School, an arrangement introduced by the Holmes Group (1990). The relationship the university had with this high school served to open the door for many conversations with the teacher and for the data collection. This connection advanced an atmosphere at the school that promoted inquiry into practice and at the university that allowed practice to inform inquiry. I am very grateful for the support and opportunities afforded me in conducting this research made possible by the PDS arrangement

I would like to thank the members of my dissertation committee, Perry Lanier, Chris Clark, Steve Weiland, Bill Rosenthal and Dan Chazan for sharing their insight, asking hard questions, making thoughtful contributions and engaging with me throughout this endeavor.

I owe a great deal of gratitude to Dan Chazan, who served as a mentor in many ways throughout this study. Dan's gentle prodding, wise advice, thoughtful questions and on-going encouragement helped me immensely. I would also like to thank Dan, the director of this dissertation, for an awesome experience in distance learning. As we communicated and argued across continents over the past six months, in spite of the pain, my learning grew by leaps and bounds with each rewrite. Thank you for your patience and careful scrutinizing of ideas, connections and for holding me accountable to a higher standard than I felt comfortable. I am looking forward to further collaboration around the ideas contained in this manuscript.

My gratitude to (Mr) Bill Rosenthal extends in many directions. First, thank you for taking the time to listen to my ideas, confront inconsistencies and for incorporating some of these raw ideas into the courses we taught together. Second, I am very grateful to you for teaching me about teaching and learning through, not only your words and your endless references, but also your example as you modeled the teaching I began to envision as ideal -- focusing on teaching students rather than course material. Third, thank you for the hours of editing and patient help in preparing this manuscript and the ideas contained within it.

I would like to thank my dear friend and colleague, Whitney Johnson, for listening and sharing her wisdom at opportune times, pushing me forward and encouraging me throughout this task.

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Introduction

> ***Mathematical power*** *includes the ability to explore, conjecture, and reason logically; to solve non-routine problems; to communicate about and through mathematics; and to connect ideas within mathematics and other intellectual activity.* ***Mathematical power*** *also involves the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power.* (National Council of Teachers of Mathematics (NCTM), 1991, p. 1)

This dissertation is a case study of a teacher I will call Marv Smith, a high school mathematics teacher, whose has made broad and significant changes in his thinking and the enactment of his instruction. The resulting professional transformation include changed views of what it means to learn mathematics, a changed philosophy of mathematics, a changed perception of what it means to teach mathematics, new experiences and confidence in his own mathematical understandings and, consequently, a transformed teaching practice. This confidence and transformed practice has positioned this teacher to work toward creating opportunities for his students to gain mathematical power.

I am using the words, ***professional transformation*** to indicate such dramatic changes that something similar to a metamorphosis occurred in this teacher's perceptions and actions within his profession. His goals for his students' learning has changed from memorizing procedures to making sense of patterns and relationships; his view of the nature of mathematics has changed from truth contained in a textbook to ideas constructed, and under continual construction, by human beings; while his classroom changed from presentation, practice and evaluation to a forum in which students are exploring, making conjectures, and discussing their findings, using mathematical reasoning to argue and develop their ideas. The assessment has changed from chapter tests with single right answers and procedures to a continual assessment that includes a great deal of writing, in which students explain their thinking, ask questions and investigate their ideas. The changes have left little, if any, signs of the teacher and classroom before the transformation.

Few people experience the power of personal self-confidence in the area of mathematics. In informal conversations, unlike with reading and writing, it is perfectly acceptable in our society to admit mathematical illiteracy and a strong aversion to the subject. As a teacher educator these issues give me special concern. Recently, while teaching a class of thirty-five elementary teacher candidates in a mathematics methods course, I asked them each to share a time when they learned something mathematical or anything from the field of mathematics -- anytime in any area of their lives. Much to my surprise, only three students shared such an experience; the entire rest of the class shared horror stories related to their experiences in school mathematics. These were college seniors soon to graduate from one of the top education schools in the country and then become certified schoolteachers, to teach children, among other subjects, mathematics.

If college graduates are leaving school feeling a lack of mathematical power, what are the implications for the rest of society? This is a serious question that plagues the mathematics education community. Is mathematics an elitist subject to which only certain people should receive access? The *National Council of Teachers of Mathematics Curriculum and Evaluation Standards* (1989), developed and compiled by mathematics educators at all levels, gives much the opposite message. In this document one of the major tenets is that curriculum goals for K-12 mathematics are for ***all*** students. At the secondary level, the *Standards* discusses core content goals to which ***all*** students, regardless if they are college-intending, should be exposed. According to the NCTM (1989), "The opportunity for all students to experience these components of mathematical training is at the heart of our vision of a quality mathematics program. ... [T]hey will gain mathematical power. ...[F]or ***each*** individual, mathematical power involves the development of personal self confidence" (p. 5, emphasis in the original).

Implementing this Vision

The definition the *Standards* uses for mathematical power, quoted above, includes exploring and making conjectures. Exploring what? Making conjectures about what? Before coming to graduate school, I taught high school mathematics for nine years. It was very hard for me, as a mathematics teacher, to imagine how exploration and making conjectures fit with my vision of mathematics and teaching mathematics. My experience, as has been the experience of most secondary mathematics teachers, had been teaching students established mathematical methods to solve routine problems. How do you reconcile learning and practicing methods until you become proficient at them with exploring and making conjectures, regardless of the content or teaching method you choose? There seemed to be a mismatch between how I was used to teaching and the *Standard's* vision.

Without a change in my perceptions of doing and teaching school mathematics, the proposals presented in the *Standards* did not seem applicable to my classroom. This transformation is related to the foundational issues of what ***is*** mathematics and what it means to ***learn*** mathematics. A renowned mathematician captures this idea in a celebrated quote: "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, quoted in Ernest, 1991, p. xiii). The role of the content of school mathematics, students and the teacher in a mathematics classroom are then based on this agreed-upon foundation. The first two documents of the *NCTM Standards* (1989, 1991) lay out the goals for reforming school mathematics while only subtly challenging the traditional foundation that mathematics is absolute, an existing body of knowledge. Answers typically remain either right or wrong. Moreover, the *Standards* are not explicit about the professional and philosophical transformation in a teacher's experience and teaching inherent to the reform being proposed.

The *NCTM Curriculum and Evaluation Standards for School Mathematics* (1989) discuss changing the content of school mathematics, maintaining that our changed society and the advent of technology call for students to know different content and use different tools. How will this give students mathematical power? It is easy to imagine that this could be read to mean that students will become less mathematically powerful. The amount of mathematical knowledge out there is so large and growing so fast schools cannot keep up, so we will teach students less and make them dependent on technology. This sounds like moving from a thinking society to a society controlled by machines. Using the above-mentioned traditional definition of mathematics, programming the existing right methods into a calculator or a computer may give you speed and accuracy, but is that mathematical power? Adding and subtracting content does not, by itself, provide opportunities for students to explore, conjecture or reason logically or to gain mathematical self- confidence. I contend that changes in curriculum content, teaching methods and assessment alone will not empower students mathematically nor provide the motivation for developing reasoning skills or help students build their confidence in their own ability to make sense of mathematics. According to Nelson (1995), the changes for most teachers will involve a set of epistemological shifts.

> The epistemological position inherent in the NCTM Standards is a socio-constructivist one. Knowledge is considered to be the dynamic and conditional product of individuals working in intellectual communities, not a fixed body of immutable facts and procedures. For most teachers, developing a practice based on such a view of the nature of knowledge, learning, and teaching will not be accomplished merely by adding new techniques to their current repertoire. It will require a set of epistemological shifts -- changing their beliefs about the nature of knowledge and learning, deepening their knowledge of mathematics, and reinventing their classroom practice from within the new conceptual framework. (p. 2)

Professional transformation, as it is used in this work, includes the epistemological shifts mentioned above, with the addition of changing one's philosophy of the very nature of mathematics, and then reinventing their teaching practice in ways consistent with these changes. Marv's professional transformation illustrates the relationship between a teacher's understandings of the nature of knowledge and learning to the classroom experience, and the importance of having a coordinated view of learning, subject matter and teaching to fulfill the ideal presented in the *NCTM Standards*, -- in particular, equipping all students with mathematical power. Fostering a similar professional transformation of this type, rather than focusing professional development and teacher education on isolated aspects of the desired changes, seems to offer a solution to the different kind of teaching suggested in mathematics education literature (Ball, 1991; Lampert, 1989, 1992; *NCTM,* 1989, 1991).

Marv's Professional Transformation

The story that follows is a story of change -- changes so profound that I chose to describe them as a professional transformation, essentially a series of epistemological shifts similar to those described by Nelson. Evidence from this study supports both that a professional transformation occurred in this teacher's practice and that, as a result, his students were provided with opportunities to gain mathematical power as it is defined in the *NCTM Standards* (1989, 1991). Faced with the lack of successful performance demonstrated by many of his students, this teacher initiated a dynamic change in his teaching following a year of sharing teaching assignments with another teacher. Marv's teaching practice continued to be transformed in part through the partnership between his school -- a professional development school (Holmes Group, 1990) that I call Hawkins High School (HHS) -- and a nearby public university. Key components of this exposure included differing philosophies of mathematics, a forum where he could pursue his mathematical questions, the historical underpinnings of the mathematics he was teaching, the use of technological tools, and a variety of approaches to the development of curricula.

In referring to the changes in Marv's practice as a professional transformation, I will be describing a phenomena that includes many specific changes to his activity as a teacher, but is much more encompassing. The transformation I witnessed involved ***coordinated*** changes in his beliefs and actions, changes that are widespread, even all-encompassing. These changes have been sustained over a time period and, I believe, will continue to be sustained, because they have not been imposed, are not experimental, and are not dependent on any outside resources. Instead, they represent a transformed paradigm of what it means to learn and teach mathematics. Marv is intent on pursuing the ideas that make sense to him and bringing his practice into line with his changing philosophies around teaching, learning and mathematics. Because this transformation is so encompassing, although any one of the changes Marv has incorporated (e.g. innovative algebra and calculus curricula, use of technology, conversations in the classroom, using writing as an assessment tool, etc.) would make an interesting story, these changes in practice are not the focus of this dissertation. Instead, after analyzing Marv's developing transformed philosophies that have informed and changed his practice, it became apparent that the story demanding to be told is one of a professional transformation that addresses his changing philosophy of the goals of student learning in mathematics (Chapter Two), the content and activities of school mathematics, as it relates to the discipline of mathematics (Chapter Three), and the role a teacher plays in this new paradigm (Chapter Four). This professional transformation is illustrated and evidenced by the vignettes that follow, as they are analyzed in light of Marv's commentary around his questions, goals and changing views.

One of the most striking outcomes of this teacher's professional transformation was his mathematical empowerment and the opportunities he now provides for his students to gain mathematical power, according to the above definition. It is interesting that a high school mathematics teacher would be in a position to have only limited mathematical power. There seems to be a direct relationship between mathematical power and a person's vision of mathematics. One of the main features of Marv's new found mathematical power is related to pursuing **his** questions about mathematics and thus being in a position to explore and make conjectures, rather than only make sense of someone else's findings.

Contrasting Vignettes of Marv's Teaching

In the passages that follow I will present the reader with a progression of three different lessons Marv taught in order to provide an image of the changes he has made in his instruction. The first lesson occurred in the fall semester of 1993, the second one was taught in the spring of 1995 and the third in the spring of 1996. These lessons are intended to illustrate the changes in Marv's thinking and enactment of teaching that I have come to see as a transformation of his teaching practice. Further analysis of these vignettes will occur in Chapters Two, Three and Four.

On this particular day, in the fall of 1993, Marv was working with this introductory algebra class (made up of 10th - 12th graders) solving for unknown quantities in an equation, going over questions on homework problems (see below, from UCSMP, 1990a, p. 176).

8. Solve $\frac{1}{5} = \frac{1}{80}y$.

The class was following this algebra textbook section by section. Marv was being asked about the following question while going over homework solutions.

Marv wrote $1/5 = 1/80\ y$ on the board
M: "What is the coefficient of the unknown?"
Silence
M: "What is the unknown?"
Several Students: "y"
M: What is the number it is being multiplied by?
S1: "One over eighty"
M: What should you do to find "y"?
S1: "Multiply by eighty?"
M: "Yes, remember you always multiply by the reciprocal of the coefficient of the unknown. Multiply what?"
A Chorus of Students: "Both sides"
M: What do you get?
S2: "Eighty times one-fifth"
M: "How do you do that?"
Silence
M: "How can we make eighty look like a fraction? Remember that if want to somehow combine a whole number and a fraction, turn the whole number into a fraction."
S1: "Eighty over one"
M: "Then what?"
S1: "You do eighty times one and one times five."
M: "What do you get on the other side?"
Several students: "y"
M: "What happened to the one over eighty?"
S2: "It cancels."
M: "I think what you mean is that when you multiply by eighty, you get eighty over eighty, which is?"
A Chorus of students: "One"

Marv was very careful to repeat important points, rules and skills that the students should remember to be able to successfully complete the problems on their own. In order for the students to successfully solve for the unknown in these types of problems, they needed to remember the process. Thus Marv repeated the rule to multiply the unknown by the reciprocal of the coefficient (see the following explanation from the textbook [UCSMP, 1990a, p. 174]) -- after he was sure everyone knew what number represented the coefficient.

To solve $ax = b$ for x (when a is not zero) multiply both side of the equation by the reciprocal of a.

In the term ax, a is called the coefficient of x. So to solve $ax =$ multiply both sides by the reciprocal of the coefficient of x.

It was obvious that he had done this quite well in the past because, for example, when he asked leading questions, such as "Multiply what?," and the class responded in unity, in this case, "Both sides." Students had been taught from a previous section in the book that equivalent expressions are still equivalent if both sides are multiplied by the same number. It was an impressive performance and it seemed that Marv's careful attention to details and slow deliberate presentation of ideas gave his students a calming peace during his presentation. However, once he gave the assignment, only one or two of the students worked on it (Field Notes, 9/18/93).

The purpose of the teaching Marv exhibited appears to me to have been to help the students become proficient at working this type of problem, a mathematical skill of sorts. Students were to learn a process presented by the textbook and teacher, practice it and apply it with the goal of consistently obtaining the correct answer. Marv paid special attention to places students often made mistakes that might cause them problems in the future, such as "canceling" rather than noticing that dividing a number by itself is equivalent to one.

Two years later, in the fall of 1995, I was again observing Marv teach. The configuration of the students' desks has been changed from straight rows facing the board to a semicircle facing the center of the room. Marv handed out a worksheet developed by HHS mathematics teachers entitled "Number Recipes #1", see below).

Number Recipes #1

Below are two rules:

1. Take the input, multiply it by -3 and then add 2.
2. Take the input, multiply it by 5 and then take away 4.

For each rule above:

(a) Make a table of ten inputs and outputs.

(b) Find any inputs which will make the output be 0.

(c) Find inputs which will make the output negative.

(d) Find two input which will make the output not a whole number.

(e) Write down any observations you have about the rule.

According to Marv, the goal of this worksheet includes giving the students the opportunity to think about repeated calculation procedures, allowing the procedure time to take on a life rather than imposing $-3x + 2$ as an abstract object right away. In Marv's view, taking time to think about this procedure gives purpose to the symbols, whereas if you go to the abstract too quickly, many students are discouraged and alienated. Over the course of the year, they will treat a function as an abstract object, but they are not read for that yet. Part (b) on this worksheet, according to Marv, is intended to "plant a seed that there are times that you are interested in a specific output." Marv saw that this particular problem, having a non-integer answer, serves to probe students' thinking to see how they make sense of our number system, which was particularly important since these students were supposedly "behind" in mathematics. The students that made up this class were upper level high school (10th -12th grade) taking introductory algebra. This question offers an opportunity for interaction and learning, forcing their sense-making to come to the forefront in a context where they are not directly answering a question about the number system. In this case, according to Marv, students' articulating what kinds of things fall between integers gives purpose to those kinds of numbers, while in the past, many have not seen a purpose to, or even cared about numbers between integers.

Marv invited the students to work together to complete the worksheet, after which they would discuss their work with the entire class. Some students were working in small groups, others with a partner and a couple by themselves. The students had been discussing number recipes -- doing operations on a variety of inputs, resulting in different outputs.

S1: How do you want us to make the tables?
M: In whatever way makes sense to you.
S2: Why don't you just tell us the answer?
M: Because I think you guys are smart enough to think about it.
S2: Can we make up our own values for the table?
S3: But we need the values that will give us zero.
M: Will (S2's) suggestion accomplish that for you?
S2: Can you just tell us what is right?
S4: There is no one answer.
S2: Can we make up our own values?
M: That will work.

Some of the students started to make the tables by hand, but soon realized it would be quicker to use a graphing calculator. Part (b), searching for inputs that would make the output zero, seemed to be a stopping point for all of the students. As they reached that point, the students seemed to diverge in different directions. It turned out that the exact x-intercept would never appear on the calculator, since it occurred when x was equal to 2/3, the calculator only showing a part of this repeating decimal.

S5: Try fractions. (Marv has her repeat her statement, but the others don't pay special attention.)
S6: We tried one and it doesn't work. We tried zero and it doesn't work. we tried negative zero and it doesn't work.
M: Do you remember what outputs you got?
S6: No. (They began looking at the output values.)
M: What else could you try?
S7: Negative two?
M: Yes, and I suggest that you expand your table to show the values so you can see if there is a pattern.

Students continue working and discussing a possible value for the x-intercept.

M: Is zero negative or positive?
S8: It is like a divider, isn't it? One way is positive and the way is negative.

A different group has noticed something they want to share with Marv.

S9: Everything below .6 give you a positive number and .7 and above give you a negative.
M.: Where do you think you should go looking for the number that give you zero?
S9: I don't know.
M: Keep thinking about it -- inputs below .6 give you a positive number and .7 and below give you a negative. If we were thinking about money, and everything was in cents and I gave you three quarters. How could you write that amount in decimals?
S9: .75
M: Think about the significance of that as it relates to this problem. Think about it for a while and if you don't get anywhere, go on and come back. But don't give up on it -- this is a very important idea.

Some students became very frustrated. Others began describing what was happening (identifying the interval of the input values inside of which they felt yielded an output of zero). Still others were busily narrowing the interval and at least one put the table aside and began to work on solving the problem using symbolic manipulation. The room was busy with activity, students walking across the room to talk with other students, others comparing notes with what they had tried. Some were very confident that their answer of ".667" was close enough; others strongly disagreeing. Class is convened as a whole group.

Two students put their table on the board. They used the values -1 to -10.
M: Take a look at your table and the table on the board. What do you notice?
S1: It goes up by three.
S3: None of the outputs are negative.
M: Question (b) asks for inputs which will make the output zero.
S10: We couldn't find any.
M: Any come up with any strategies to find it? Anyone get close?
S8: .1
M: .1 as an input or output?
S8: Output.

M: Anyone get closer?
S11: I got .2, but I don't know if that is any closer.
M: What was your input?
S11: .6
M: Anyone get any closer?
S9: I got .002.
M: What did you use as an input?
S9: .6666
M: Why did you try that?
S9: Because if I used .67 it went too far.
S5: Try another 6.
S9: you get .0000002.
M: What happens when you fill up the screen?
S3: Seven zero's
M: Anyone get any closer?
S6: -.0001
M: What input did you use?
S6: .66667
M: Why did you think there was not a number that would do this?
S10: We tried numbers.
M: What did you do?
S10: Started with one and worked my way up.
M: What happened?
S10: I don't know.
M: Everyone look at your tables and notice what happens when you go further up the number system.
S1: They go up by three every time.
M: If I were looking for zero, should I keep putting in larger numbers?
S10: No.
M: Should I look at large negative numbers?
S6: No, look in-between zero and one.
M: (S9), can you tell us what you told me about .6 and .7?
S9: Everything .7 and above is negative and everything .6 and below is positive.
M: (Looks at the clock) Has anyone every seen .6666666 when you were doing calculations?
Several students: Yes
M: Does anyone know a fraction that has that approximate value?
S5: 2/3
The bell rings.
M: (Stopping their movement) *Remember*, the point of mathematics is not punching a number in a calculator or finding the right answer, but the thinking that goes on around it. You did some very important mathematical thinking today. (Field Notes, 9/18/97)

The small group and whole class discussion ended up including many fundamental mathematical ideas including rational numbers, number sense, estimations, tabular values of linear functions, and algebraic manipulations of rules. (Marv later told me that S5 had told him, "My grandmother said if you don't know what to do in math, just undo what you did -- so I subtracted 2 and divided by -3 and got 2/3.")

After class, Marv talked about being pleased with students' thinking and engagement. He repeated what various students had shared in private and public conversations, demonstrating a new interest in listening to his students sense-making, and he was beginning to think about the situations he might pose for the next day to challenge, guide or encourage their investigations. During the class the students commented more than once that Mr. Smith never answered their questions, only asked them more questions, but I noticed that they did not hesitate to pursue his questions.

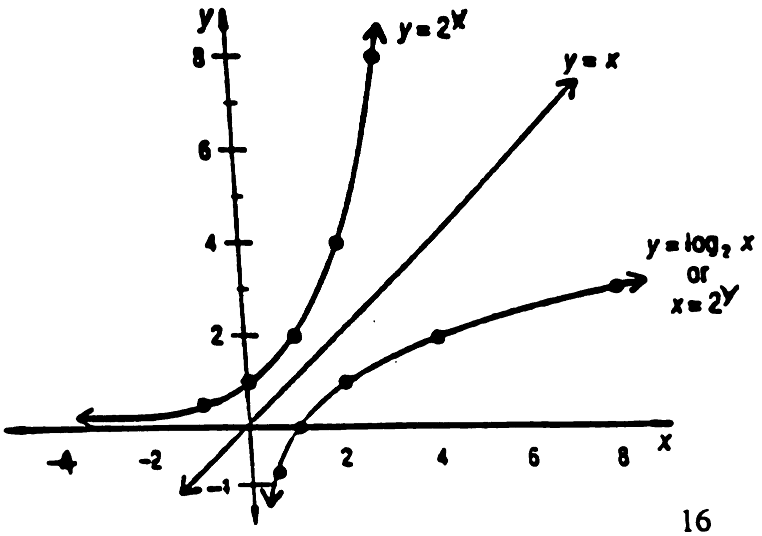
The changes in the two classes described above take on many dimensions, from the set up of the room to the interactions with the students. Marv is no longer describing a certain process for students to memorize with the purpose of using it to find one correct answer. Although the class was in search of the x-intercept (the value of x when the output is zero), they were not following a predetermined path to find that value. The paths taken were determined by the students and the students interacted with each other's ideas and suggestions. The students were actually making the decisions and following up on their ideas. Marv's goal for the students, in addition to finding the x-intercept, was for them to be thinking about number patterns, and he seemed interested in all aspects of their mathematical musings. As they discussed their findings in their small groups they were forced to articulate their thinking and defend their decisions -- a form of mathematical conversation and reasoning that is much more difficult in a class that is learning a predetermined process with only the goal of finding the one correct answer. Marv's interaction with the students was giving them space, the opportunity to think and encouragement to pursue their ideas, as he listened closely to what they were thinking. He was, however, directly involved in the whole-class discussion with most comments going directly to and through him.

The next vignette comes from an Algebra 2 class in the spring of 1996. The students were investigating logarithms. While studying the graphs of logarithmic functions, they noticed that the range was all real numbers and wondered why the domain was restricted to positive real numbers. Marv decided that one way to approach this question was to look at the definition in the textbook, analyze what was being said in it, then critique the definition by looking at alternative restrictions. The textbook definition and graph appear below:

Definition:

Let $b > 0$ and $b \neq 1$. Then n is the **logarithm of m to the base b**, written $n = \log_b m$, if and only if

$$b^n = m.$$



-UCSMP (1990b, p. 509)

The question they were considering was why the base of an logarithm had to be greater than zero -- in particular why the base could not be negative. One conjecture was that the output had to be positive, but that was shown to be problematic for inputs between zero and one, where the outputs were negative. They began to think about the consequences of allowing bases to be negative, using negative nine as a possible base. Because of the problem of taking "even" roots of negative numbers (which does not result in a real number), the ensuing conversation lead to a need for a definition of even and odd numbers.

M: Did everyone hear what S11 just said? He said that you can raise -9 to any power, you just can't take any even root of it?
S1: Can you repeat that?
M: You can raise -9 to any power, you just can't take any even root of it.
(Several students said they agreed.)
M: Are you agreeing only because S11 said it or do you agree?
(Many said S11, while others said they agreed.)
S3: I want to know if it is true, why it is true.
S8: I always get confused when you have decimals, whether they are even or odd.
M: Say a little bit more.
S8: Like 9.6, is that even or odd?
S2: One part is even and one part is odd.
S8: Would it be even because the decimal is even?
(Lots of students talking at once.)
M: Okay, hold on. Let's take a minute to think. This is a great question, and we need to get responses, but we can't talk over each other.
S3: If we could define even and odd.
S4: Can a fraction be even or odd?
S7: It takes two halves to make a whole, never mind.
M: How does this question relate to what we were talking about?
S7: If it is odd, you can't take the square root.
S8: If you have even roots, you can't find them.
S10: Is the definition of an even number, divisible by two?
S7: So you can divide 9 by two and get 4.5, so it is even?
M: We have to get this ironed out if we are going to know what an even root is.

S7: Do you know the answer?
M: I have *an* answer in mind, but I wouldn't consider it to be *the* answer.
This is what mathematicians have to do, make definitions and see what the implications are. S10 has given us a definition of even: An even number is a multiple of 2.
S10: That is what I said, but not what I want to say now. I think it has to be a whole number.
M: So, tell me if this is right. An even number is a whole number multiple of 2?
S10: You also have to come out with a whole number.
M: So give me your definition of an even number.
S10: An even number is a whole number, divisible by two that gives you a whole number. That takes care of negative
M: What assumption are you making when you say divisible by two? (He didn't write, "that gives you a whole number.")
S8: You have to write, "Gives you a whole number."
M: To me, divisible by two says that.
S7: An even number is a whole number when divided by 2 gives you a whole number. (Marv writes that on the board.)
S6: So decimals or fractions can't be even.
M: Is that an implication if we are going to accept this definition?
So, what does that say about S10's question about decimals?
S3: They can't be even.
S4: So if we come up with a definition for odd numbers, then we will know.
M: That's a good idea, let's get a definition for odd numbers.

One of the aspects of Marv's teaching that is most noticeably different is how he has dispersed the authority in his classroom among the textbook, himself and the students. This class did not use the textbook on a regular basis, but occasionally Marv handed books out, then asked the students to read certain sections or definitions and analyze the text by writing about it or just thinking about it. This was one of the occasions in which he asked them to think about this definition and be ready to talk about it. This transcript shows Marv as more of a participant in the discussion than as a focal point through which all students' comments went, as in the previous two vignettes. Here a discussion about a textbook definition of exponential functions led students to confront their understandings of even and odd numbers and eventually led to an inquiry into Peano's Axioms[1]. Marv doesn't feel compelled to control the conversation, and actually had not anticipated students' insistence on a definition for even and odd numbers nor their questioning the "bottom line" of definitions, which lead to the later discussion on axioms. By allowing the exponential functions to be the object of study, rather than the symbolic manipulation that would produce a predetermined answer, the students are confronted with big ideas in mathematics. Maybe even more significant, was Marv's response to a simple question that most mathematicians would call gibberish (S8: "I always get confused with decimals, whether they are even or odd") -- allowing it to become the subject of the study that gave rise to the students' confrontation with big ideas.

Method of Study and Overview

The study of Marv's teaching began with observations and conversations that were more for the purposes of studying work within the mathematics department at his school, in general, than studying his individual teaching. We had similar interests, as we both were thinking about how to teach high school mathematics to students not usually well connected to mathematics or how to make high school mathematics accessible to more students. Over a two year period, our conversations and my field observations, reflected dramatic changes in Marv's thinking and the enactment of instruction. During that time I had collected various sorts of data, including field notes of classroom observations, videos and interviews with Marv about his teaching. These experiences sparked my

[1] Peano's Axioms are one formal basis for constructing the number system.

interest in thinking about doing more in-depth study and writing about these changes or, more broadly, his professional life history.

I have been in contact with Marv over the past five years. This contact includes teaching parallel sections of introductory algebra down the hall from him in 1993-94, resulting in many informal conversations; providing documentation for professional development school (PDS) projects in his department during 1993-1996; conducting seven formal interviews interspersed throughout the five years; making more than fifty classroom observations, including several weeks of videotaping; and studying Marv's writing about his teaching (see Appendix B for a table containing dates and data details). In addition to the sources listed, I have had many informal conversations with Marv's colleagues, both at the school and at the university, who have served as secondary data sources. Throughout this time, I did not participate directly in the culture of Marv's classroom or have a direct influence on Marv or his teaching. Marv and I agree that my main influence was to serve as a sounding board of sorts as he worked at articulating his questions, findings, changes, frustrations and revelations. My questions could have pushed or challenged Marv, but that was not their intention nor did he received them in that way. Although we were in the search together, he was the one on the front line and we were both trying to make sense of his experiences and how he was thinking about them.

I have engaged in different levels of analysis of Marv's teaching, including writing various papers (Chazan, Ben-Haim, & Gormas, in press). I wrote papers for university classes about the algebra curriculum he was using, about the collaboration at his school, about the different kinds of understanding being value in his classroom, and about the changes in his practice. I then wrote a chronology of Marv's professional life history, following two very long interviews related to his school, and his learning and teaching experiences. My intention at this time was to write a teacher biography that would somehow identify and explain the changes in his practice, while also exploring the factors that led to them. As I studied the transcripts of his interviews and watched what was happening in his classes, it became clear that the transformations that took place in his thinking and instructional practice stood out as a culmination of his professional life. This professional transformation marked such a clear transition, it became important to me to try to come to a greater understanding of it. As I searched the literature, I found very few similar transformations in high school mathematics teachers, which also motivated me to dig deeper into this teacher's experience to discern why these changes seemed like a transformation, rather than a series of isolated changes. I eventually decided to treat the transformation that took place in this teacher's professional life as a case study and look more closely at this phenomenon.

As I worked at understanding Marv's professional transformation I was looking at the following three questions: 1) What were the factors that contributed to these changes? 2) What changed? and 3) What were the results of the changes? I looked for answers to these questions in what Marv said, what he wrote and what I was observing in his classroom. Since the study took place over such a long period of time, some of these answers were changing and developing continually. Then, as I searched for the evolution of his ideas, I could notice patterns and ask him questions that would identify his current thinking, while paying attention to how those ideas played out in the classroom. The changes that Marv experienced in his goals for his students' learning, his philosophy of mathematics and his changed view of the nature of teachings (discussed separately in Chapters 2-4) became evident to me as I was investigating what changed. As I became aware of how far-reaching the changes were and how they reinforced each other in ways that made it possible for his students to gain mathematical power, these changes stood out as the key to his professional transformation.

This study does not include a critique of the teacher's practice, since this was not a purpose of the study. I seriously considered this possibility during the analysis stage, but it did not seem appropriate given the dynamic nature of this teacher's thinking and practice. As I considered different aspects of his practice that might be critiqued, I noticed that they were sources of concern for him as well, and change seemed eminent. The changes that comprised his professional transformation and their result were much more pertinent to the intent of this study.

Part of my search to understand this teacher's experience was to write about my own related experiences. In an attempt to identify with his questions and recognize his students, it was helpful to do some autobiographical analysis. This identification and first-hand experience in the different roles in my life (mathematics teacher, teacher educator and parent) indirectly influenced the analysis by helping me recognize the difference between isolated changes and a professional transformation. Later, I decided not to include this writing in the dissertation since it did not contribute directly to this particular case study. It did invite me to think about changes in my thinking and practice that would have been important considerations if I were to experience a similar professional transformation as a high-school mathematics teacher.

As I studied the changes in Marv's teaching, it became important to investigate the validity or truth of his claims (Phillips, 1994; Donmoyer, 1990), by closely scrutinizing classroom observations and videos, and the consistency of his claims (Atkinson, 1997), by studying the transcripts of interviews and the various writing he has done over the past few years. Validity was also an issue that surfaced as I talked with others, including two professors that worked closely with this teacher, in an attempt to articulate my findings and obtain their readings of what was happening in his classroom and what he was saying. It was their influence that pushed me to go beyond the initial two years and investigate the tenacity of Marv's beginning attempts to teach differently than he was taught and ascertain if he had continued to change is similar or different ways over the next three years. This triangulation pushed my study beyond mere narrative, to a formal analysis of the narrative including a look at existing structures or frameworks and the social context of this teacher's professional life (Atkinson, 1997).

I began to identify patterns (Donmoyer, 1990) in the changes that revealed the fullness of what I came to call a professional transformation, as I studied the transcripts of the interviews, Marv's writings, my field notes and watched the videos of his teaching. In ways, this dissertation provides a vicarious experience, giving readers access to an experience they might not have otherwise had (Donmoyer), but at the same time refracting this experience through my, the researcher's, analytical eyes. By subdividing his professional transformation into the three main areas of a teacher's charges -- learners, subject matter and teaching (McDonald's [1992] "wild triangle" and Hawkins's [1974] "I, Thou, It"), I was able to separate the major transformations Marv experienced accordingly. I studied the transcripts of Marv's interviews for his perspective on the changes and the factors that led to the changes, comparing his analysis to what I was observing in his classes and what I noticed throughout this time.

Thinking about the changes according to the three aspects of learning, mathematics and teaching worked well for a deeper analysis of each of these areas, while at the same time posing problems because of the widespread overlap of these areas in a teacher's professional life. Thus the reader will notice that, for example, teaching comes up in the chapter on mathematics and mathematics is an important part of the chapter on learning mathematics. I decided that this overlap was both necessary and desirable since it is artificial to make this kind of separation within a teacher's professional life. Actually, it is the fact that the overlap came to represent the significance of Marv's changes that allows me to call it a transformation. Within each area, I looked for consistency between what he was saying and what I was noticing, while across areas the changes reinforced each other in ways that added up to a professional transformation, as discussed in Chapter 5. Marv is not the teacher he was five years ago.

As I related Marv's professional transformation to the field of mathematics education, the *NCTM Standards* (1989, 1991) stood out as very important documents to consider. One rarely reads mathematics education literature in which the *Standards* are not mentioned. The push in the *Standards* for teachers to provide mathematical power for all students drew my attention, as it had in 1989 when I was a high school mathematics teacher. Realizing that I was seeing Marv's students become mathematically empowered caused me to analyze the relationship between his professional transformation and this empowerment. This stood out to me as a very important result of such an encompassing and consistent transformation.

My intention in this work is to tell my story of Marv's story of his professional transformation as a mathematics teacher. As a storyteller and researcher, I am at times merely listening, hearing Marv, his students, and his colleagues talk about the teaching and learning that help define Marv's professional interactions in and out of the classroom, then translating and weaving his story of transformation into current discourses in mathematics education and the development of my own view of mathematics, learning and teaching. Throughout the story, I allow Marv the opportunity to speak from his own perspective, (Crapanzano, 1980), quoting him, giving him opportunity to read the transcript and continuing our dialogue throughout the writing and rewriting (Measer & Sikes, 1992). Yet, it is still my interpretation and analysis of Marv's transformation.

The changes that occurred in Marv's instructional practice and his thinking about learning, mathematics and teaching are so significant (so he feels, and I concur) that I assert he has undergone a professional transformation. This transformation is similar to the epistemological shifts that Nelson described above. The format I have chosen to tell this story is to begin by telling some biographical background in Chapter 1, to help the reader identify the context of Marv's professional life, including the opportunities his career have afforded him, some ordinary and some rather unique to his circumstances. Next, in Chapters 2-4, I detail the transformation, illustrated in a snapshot by the three vignettes in this chapter, by focusing on learning, mathematics and teaching, respectively, in each chapter. The transformation in Marv's view of and goals for his students' learning is discussed in Chapter 2, Marv's dramatically changing view of school mathematics and developments in his philosophy of mathematics provide the background for Chapter 3, and the transformation in Marv's perspective on the role of the teacher in a mathematics class is the subject of Chapter 4. Within each of these chapter I include ties to the *NCTM*'s definition of mathematical power, demonstrating the relationship between these dramatic changes and the development of mathematical power. The last chapter, Chapter 5, outlines the implications of such a transformation for the field of mathematics education from the perspectives of a mathematics teacher and a teacher educator.

Chapter One
Overview of Marv's Professional Life History

> **It (college math teaching) is a matter of presentation and evaluation. College was really ridiculous -- the instructional practices and everything. So in the dorms I ended up helping a lot of my friends with math and I found that I had some aptitude for that. (Marv, June 1996)**

Introduction

This chapter has been included in the dissertation to identify some of this teacher's background in learning and teaching, as well as the events, contacts and resources that may have contributed to the transformation that took place in his professional life. As an overview of Marv's professional life story, it will serve as a backdrop to help the reader organize and place incidents discussed in later chapters. A chronology of Marv's professional life appears in Appendix A.

My contacts and ability to learn about Marv's teaching and learning background has been enhanced by the fact that he attended and student-taught at the same high school where he now teaches, and earned his undergraduate degree and, just recently, his graduate degree at the university that has worked closely with this school over the years. Marv grew up in Hawkins and attended Hawkins High School, a midwestern suburban/rural high school located just outside the state capitol and about ten miles from the university. His thoughtfulness and tendency to question authorities are illustrated in a story he tells about a junior high science teacher. In Marv's view now, the teacher was actually teaching a misconception, while at the time he determined that the teacher's explanation just did not make sense.

> He hung two apples from a cord and he blew between them and it pulled them together. He said he just got the air molecules out of the way so they could pull together so the gravitational force could pull them together. Two things theoretically out in space should pull together. but he was saying it was just the air molecules that was keeping them, not the fact that he was creating rarefied air between them and the air pressure from outside. It wasn't that sophisticated. But I understood things well enough to sit there and think, "What's in the heck is this guy talking about?" It was like that all year. As I took a physical science minor in college and looked back on it, he created all of these misconceptions that when I was teaching physics at a high school level and I had to deal with them daily. I knew where they got them.(Spring 1996)

Yet, Marv's view of course work and school in high school were related to enjoying the recognition he received when he did well, knowing that he needed a scholarship to afford college and getting the work done so he could participate fully in sports. He played varsity football and wrestled with the varsity wrestling team all three years he was in high school. Marv said his parents were not directly involved in his schooling experiences, or other learning experiences, recalling that they were preoccupied a lot during his junior high and high school years in what he referred to as a long drawn out divorce. He said that he and his brother really did not need his parents to hound them about school -- they just did the work. If anything, related to school, he remembers them telling him to relax more. Marv was the first in his immediate family to attend college.

Influential High School Teacher

Marv's experiences in high school physics gave him a rich background in thinking about how the world works by formulating conceptual understandings using mathematics and scientific principles. Marv talked about his high school physics teacher, Mr. Ross, and his influence.

> He would throw us into a situation and say, 'All right, figure it out, make sense of it, using everything you have until now in your schooling to try to make sense of this.' Then later we would take a look at the formal stuff that all of those dead people figured out and wrote in textbooks. Learning did not feel like it did in other classes. It wasn't that hard mental crunch to memorize everything and spit it back. You had to think for yourself and make sense of it and write about it. (Marv, April 1996)

Marv was also influenced by his only high-school mathematics teacher, Mr. Stark, who later served as his supervising teacher during his student teaching.

> Mr. Stark had high expectations. He used the Socratic model, where he would ask us questions and direct them right to a student to make the flow of things go. He would lead us through a series of questions to the conclusion he wanted us to have. He helped us make connections within the mathematics we were doing. (Marv, April, 1996)

Marv decided that his high school mathematics instruction was phenomenal compared to the instruction he received in college as first an engineering major and later a mathematics major -- which he referred as nonexistent, mere presentation and evaluation. His high-school mathematics teacher was very interested in explaining and developing his understandings of the textbook content. According to Marv, his college instructors did not have a goal of helping anyone understand anything -- just rehash the textbook and send the students home to figure it out. Marv frequently found himself in the position of helping others figure out the mathematics, which he tributes back to his high school experience. According to Marv (April, 1996),

> I saw many people out there supposedly being teachers [at the university] that were not teaching anything. I had this experience of somebody [Mr. Stark] who really did help me understand things that were in a way much more so than the book had to offer. Once I had that comparison, by the end of my junior year [in college] I was thinking, 'I seem to have an aptitude for helping people with math and physical science stuff.' I saw someone who really made my life a lot easier. I was able to breeze through my college math classes because of what I had learned in high school. As I

started taking calculus, between the algebraic skill that I got from Mr. Stark and the conceptual ideas about graphs and what the areas under them meant and what the tangents to them meant, that I got from Mr. Ross, calculus and physics in college were a breeze.

Teacher Education

Marv later said that his teacher education program at the university planted "seeds of conflict" that would cause him to question the learning happening later in his classes. He had been in a program referred to as "Academic Learning" in the Department of Teacher Education. He remembered that there was an emphasis on helping students make connections with the subject matter. Marv has the following to say about his teacher education classes:

> Many of the instructors showed interest in disciplinary knowledge and how a teacher's expertise in that area really does affect the way he or she teaches and makes connections within the discipline. It was rather progressive at the time. It was not just teaching behavior models or things like that. It was really focusing on how knowledge is communicated to someone. This is rather strange that I am sitting here saying these things because at the time I thought it was boring and mundane. I think they really understood the fact you cannot teach someone what goes on in a classroom; they have to be in the classroom to understand that. What you can do is help someone start to form a perspective on teaching, structuring lessons and subject matter. ... We talked about creating lessons that made connections (I can't believe I am saying all of this as much as I bashed my education classes), but that is one of the things that first got me thinking differently about learning. Making these connections helped me make the transition from as a student, when I did school work to get it done to start thinking about really understanding it -- not just being able to do it, but have some understanding of it. (Marv, April 1996)

He dreaded the task of making concept maps, which the teacher candidates were required to construct, as they thought about the conceptual organization of ideas related to theirs subject matters. However, the concept maps forced him to think about the many connections within mathematics and reminded him of how his high school mathematics teacher had recognized connections and had worked at helping students formulate a better, more connected, understanding. Later conflicts arose as he became aware that his general mathematics students and introductory algebra were not making connections, and his connections did not seem to work for them.

First Teaching Assignment

Marv seems to be a person who is willing to go against the odds, take a risk and continue to hope for the best. He writes about the influence of his grandfather (Autobiographical Sketch, Fall 1997), who taught him to work hard, but find ways to give to others while you are doing things that make you happy. Marv's first three years of teaching were in an urban school across the country, away from his friends and family, in an environment much different from his home town. His grandfather's words echoed in his ears and although it was not the easy route, he felt it was the right one. In California, Marv would have mountains to climb and bike, the ocean swells to conquer surfing and kayaking, and quiet ocean coves to snorkel -- all in the midst of mild and gorgeous weather patterns. In contrast, given Marv's midwestern suburban upbringing, teaching in a coastal urban school would present a new set of challenging, confusing and frightening circumstances and open Marv's eyes to a world about which he knew very little. It was an opportunity for Marv to give to others, following his grandfather's model and advice, while doing what made him happy. Following is Marv's description of his mathematics teaching experience there:

> They gave you a pamphlet that said what you taught on what days, including the problems you assign, the worksheets you give out and the tests you give. They were using a mastery learning cycle for testing -- you were to give three tests on everything. The program was called "The Achievement Goals Program." You would get a crate of material at the

> beginning of each quarter, including the warm-ups and everything you were to assign. There was a person that walked around with a cup of coffee, a 16-17 year veteran teacher, who regularly looked at your little chart to see where you were. This program was horrendous. You had no ability as a teacher to do anything very creative or conceptual because the administration mandated what you would do. They needed secretaries, not teachers. (Marv, June 1996)

In the experience that is described below, Marv is pushed into isolation within his profession. During Marv's first year of teaching, in California, he faced many challenges. He was assigned to the "brightest" students, mostly coming from privileged backgrounds in a magnet school for a physics class, to another group of students that were the least connected to school coming from the lowest-income neighborhood for a required general mathematics class and an honors physics class that had no prerequisites. Marv was thinking about one of these classes in relation to his former teacher education courses and wondering how he might make some connections to students' prior knowledge in introducing the topic. While still contemplating different approaches, and wanting to do it "right," he encountered another mathematics teacher in the copy room and asked her about approaches she had used in the past. Later that day the principal came by Marv's room to ask him if he was having trouble -- she had understood from a conversation with another teacher that Marv might not be able to handle the subject matter. Marv gave none of his colleagues the opportunity to question his subject matter knowledge the rest of the year. Teaching was turning into a lonely profession (Lortie, 1975).

Both summers Marv was in California, he attended NCTM Summer Mathematics Institutes that paved the way for him to continue to think about mathematics, making connections himself and considering how to teach in more interesting and creative ways.

Although Marv doesn't remember the specific mathematics he encountered during these workshops, he does recall them giving him a desire to know more mathematics. Marv mentioned that his clearest memory of the immediate difference the summer institutes made was that it gave them permission to talk about teaching mathematics differently. They didn't actually study the *NCTM Curriculum Standards* that had just been published, but the purpose of the institute was to expose the teachers to ideas that were consistent with these standards.

One of the challenges Marv faced during this time was related to his reputation for being able to control his classes: "The better I got at handling my classes, the more difficult students I was given." He remembers one student that had refused to leave the class when asked and when the teacher called the security guards into the classroom, "he took both of them out." To everyone's horror, instead of suspending or expelling him, they gave him a different schedule. He ended up in Marv's mathematics class and a violent situation ensued in Marv's class. Later, Marv talked with the young man and listened to his explanation. Marv's reflections included, "I could understand him and I knew it was time to leave. When I could understand that in the context of those kids' lives that he had to do that or his life would have been miserable or that kid and his buddies would have come at him again if he hadn't responded, then I knew, there was nothing I could do." His decision to return to his hometown and pursue a master's degree in mathematics at his university alma mater, was solidified when a part-time teaching position back at his old high school was offered to him.

Teaching at Hawkins High School

Hawkins High School had been participating as a Professional Development School (PDS), (Homes Group, 1990), entrenched in professional development work in conjunction with the university, for the previous three years. US News and World Report (1993) chose HHS as one of the top ten exemplary high schools in the nation the year before. One issue that had made the headlines in 1990 was the decision to have students come to school only 4 1/2 days per week, with the remaining half day devoted to teachers' professional development. Marv told me that for the first couple of years he taught at HHS he tended to feel out of place -- so many people thinking deeply about education and trying out neat ideas. He wondered if he fit in and if he would ever be able to participate in similar professional activities since he was "just teaching."

The HHS mathematics teachers were very excited about a textbook series (UCSMP, 1990a, 1990b) they were just beginning to use when Marv arrived on the scene. One of the things that pleased these teachers was the use of applications throughout the sections eliminating the question of, "When will I ever use this?" The fact that the mathematics had practical applications, highlighted in each section, gave the students assurance that the mathematics was worthwhile whether they, personally, would use it or not. During Marv's first year at HHS (1991-1992), he taught part-time, including introductory algebra and physics -- working with his old physics teacher in a parallel section. At the same time, he was a full-time graduate student in mathematics. The next year Marv became a full-time teacher at HHS, adding calculus to his schedule, and the graduate work took more of a back seat. During one semester that year, he shared teaching assignments with a first-year teacher, combining two small introductory algebra classes. One of the approaches they used in an attempt to impact their students' learning was to continually ask themselves, "What is it we want the students to know about this?" In the process, they began to take the first step in a journey toward confronting the big ideas, first in introductory algebra, and ultimately in mathematics as a whole.

Sharing Teaching Assignments

I developed a working relationship with Marv while documenting mathematics classes at HHS the third year he taught at HHS (1993-1994), his sixth year of teaching. During the professional development morning each week teachers at HHS normally spent one to two hours in department meetings, besides attending sessions that cut across subject matters. The mathematics teachers would divide the time between full department meetings and subject-matter small groups, representing different PDS projects and special interests in developing projects to supplement the textbook. I usually participated in and took notes for the department meeting and met with the algebra group.

The original plan for one of the PDS projects this year was to have one of the mathematics teachers, Sherry, who was teaching both mathematics and Spanish, team teach or share teaching assignments with Marv. Sherry had team taught with Del, the professor from the university, the previous two years, working on innovative curricular ideas and pedagogy for introductory algebra in an attempt to better understand the mathematical conceptions of students who had not experienced success in mathematics in the past. To carry out the team teaching plan with Marv, Sherry was to be released from one of her Spanish classes, but a release teacher could not be found. So, Sherry and Marv spent the first quarter trying to do similar things with separate classes. I released Marv from one of his algebra classes at the beginning of the second quarter, for the remainder of the school year, so he could share teaching assignments with Sherry. In this position, Marv was able to participate with Sherry in the development of curricula, discuss individual students' understandings and observe his students being taught. Since they were both assigned to the class, they shared interest in and responsibility for every aspect of the teaching and learning in the classroom.

This team-teaching experience marked an important transition in Marv's professional life. He wrote the following about it:

> Team Teaching has provided for me opportunities that have alleviated much of my frustration as a teacher. Traditional methods of professional development -- one day workshops, research articles, etc. -- were a large part of the frustrations I had felt as a teacher trying to improve my practice. For me, they helped foster a notion that to become an excellent teacher, I needed only see someone do it right and then copy their technique.... I know that for me and my colleagues at (Hawkins) High School teaching together on a daily basis (with the same courses and students) has helped us grow as professionals in the field of mathematics and teaching in innumerable ways. I now see good teachers as people who constantly reevaluate their understanding of the subject, constantly struggle with activities and lessons, and who seek their colleagues as resources. (Team Teaching Reflections, March 1996)

Marv felt that the collaboration with Sherry, involving listening to students and observing her teaching, gave him a new vision of the algebra curriculum, student learning, as well as a pedagogy that was student-centered. The experience was an "eye-opener" for Marv in many respects. First, Sherry did not use a textbook, nor did she have a planned curriculum, but instead she had some ideas about the direction of the curriculum and some worksheets that she and Del and developed over the past two years. She seemed poised to write new activities based what the students were saying and doing, and her curricular goals. The instruction seemed to center around these worksheets and the problems on them, often only one or two, that students discussed and presented their solutions to. It was a very different kind of pedagogy than Marv had every experienced. At times, Marv felt he was unsure of "where things were going" or if the students would end up knowing algebra or something else, but he was afforded the unique opportunity to observe students interact with the mathematics, observing in ways not possible when he was teaching on his own. He said it was like "watching your students get taught."

During the next two school years Marv was forced to articulate the mathematics, rationale and big ideas of a new curriculum that did not use a textbook or follow traditional formats in introductory algebra, while he shared teaching assignments with teachers not familiar with these ideas. It is ironic that these teachers were a former teacher and coach to Marv while he was in high school, but in this instance he was taking the lead, being the teacher more familiar with the new strategies. These team-teaching experiences gave him the opportunity to question the changes in the curriculum. He could also observe and reflect with colleagues regarding particular student's ideas and his mathematical and curricular goals.

Relationships within the Mathematics Education Community

The Mathematics Department at HHS has proven to be a productive venue for teachers interested in mathematics education. Teachers within the department have formed a community of mathematics educators intent on developing curricula, incorporating their students conceptions and questions, while challenging each other as they consider the goals and content, and their own questions. The atmosphere encourages risk-taking and asking hard questions, while teachers welcome input from the university and participate in mathematics-education seminars held there. Marv has become acclimated to this environment and even led the HHS mathematics teachers' presentation at the university, attended by graduate students and professors from the mathematics and teacher education Departments in the spring of 1977. During this presentation the teachers discussed their goals for mathematics learning at HHS and used a video from Marv's calculus class to illustrate the students' participation in their learning.

Marv has worked directly with professors within the Department of Teacher Education. He has felt like a "partner in inquiry" as they have investigated the teaching and learning high school mathematics. He talks about these relationships in terms of how and the professors have contributed to each other's thinking in important ways. He and his students' ideas have impacted the thinking of the university collaborators as they work with him enacting his changing vision of high school mathematics teaching. Marv reflects the many conversations with Del, the professor that introduced the innovative introductory algebra curricula ideas, how he connected Marv to a network of mathematics educators that had similar interests, and how he has challenged Marv's thinking about the nature of mathematics and its relationship to school mathematics.

Marv did some work with Michal Yerushalmy, a professor from Haifa University in Israel, who is also a developer of software and intensive technology for the mathematics curriculum, during the spring of 1996. Yerushalmy's work includes research into the use of technology for students to explore, make and test conjectures, while posing problems for students that require interpretations of what is displayed by the technology tool (Yerushalmy and Gilead, 1997). More than the technology or curriculum ideas, Marv says he was impacted by her emphasis on *using the students' work and conclusions* to design activities that might bring about deeper understandings, rather than designing a series of activities for students to step through. In the process, she has done extensive work in thinking about mathematics and big ideas students should encounter. One of the on-going conversations Marv has had with Yerushalmy was about the proof of the Chain-Rule recorded in textbooks for calculus students. Marv was struck that Yerushalmy, a renown mathematics educator, actually critiqued the textbook proof, (Marv says, "[In the past] I had criticized the proofs for their lack of coherence to students, but not for mathematical quality", 1998) and she shared his confusion for improving the mathematical quality of the proofs in the textbooks. Marv felt that his questions and ideas were met with understanding and thus began a long-term conversation around making the proof sensible to students, while also improving its mathematical quality. Yerushalmy has ended up playing an influential role in Marv's freedom and ultimate mathematical power, as he searched his own understandings and felt at liberty to critique the "canon."

Marv participated in a doctoral level class on the philosophy of mathematics during the fall of 1996. This class followed a summer in which Marv became engrossed in the idea that mathematics was actually a human construction rather than an infallible truth that resided in textbooks. The professor for this course, Bob, a tenured mathematics professor turned teacher educator, who was disillusioned by the facade that many mathematicians hide behind, introduced Marv to a philosophy of mathematics that contends that mathematics is socially constructed (Ernest, 1991). The following year (1997-98) Marv worked with this professor analyzing and creating calculus curriculum. Marv marvels at his openness and willingness to hear and value Marv's ideas and his students ideas, as they worked behind the scenes combining these ideas and their own knowledge of the subject matter to develop a reasonable story line for the course. According to Marv, this professor gave him "the confidence to question mathematics."

During the 1996-1997 school year, Marv began working with Ricardo Nemirovsky, a researcher with Technical Education Research Center (TERC). Nemirovsky (1994) has worked extensively to uncover students' perceptions of mathematical ideas. In his work Nemirovsky uses microcomputer-based lab tools that measure and model various relationships, particularly those among time, distance, velocity and acceleration, and transfer these relationships directly onto a graph. These tools present students with various phenomena that force them to analyze a variety of ways to think about these relationships. The resulting analysis uncovers many of the students conceptions of big idea in mathematics, while giving the students the opportunity to experiment with their ideas and conjectures and receive immediate feedback. Nemirovsky invited Marv to take a mechanical device that connected the movement of play cars with computer displays that graphed relationships, and use them in his classroom. According to Marv, "the software and hardware opened up so many possibilities that I began to think of ways of teaching and thinking about the content that could not come any other ways." Marv sees this technology opening up opportunities for the students to ask questions when they would not normally do so, (or they would just ask the teacher for the "right answer"). Marv contends that with the mechanical devices, the student can test their ideas, visualize them, determine what is right without the teacher or textbook deciding what is right and wrong, and then ask why and continue their investigation. The teacher can then help students with their exploration and participate with them rather than act only as an evaluator.

In 1996, Marv also met Jim Kaput, a professor at University of Massachusetts, Dartmouth. In January of 1998 Marv, along with other high school mathematics teachers, was invited to participate in a seminar related to SimCalc, a National Science Foundation (NSF) funded initiative to develop software to simulate mathematics of change. Kaput's (1995) goal of "democratizing access to big ideas" is consistent with the work being done at HHS, which began as a study to uncover the conceptions of students not well connected to algebra, using a curriculum based on the operations on functions as the objects of algebra (Bethell, Chazan, Hodges and Schnepp, 1995). Kaput's influence on Marv was of a more general nature, helping him realize that there are other people working on interesting ideas and finding it refreshing that some of those people who are doing important things in mathematics education, appreciate the work he is doing.

The Real Story

Although Marv has had unique opportunities within the mathematics-education community, experiences that many secondary mathematics teachers may not be afforded, the real story of his professional life history are the epistemological shifts plus the changes in his practice, evidenced by the three vignettes in the Introduction, that make up the professional transformation. As a case study of one teacher, this research does not reveal specific steps this teacher took, and certainly does not recommend steps all teachers should take to transform their practice. Nor does it prescribe specific changes that should occur. Instead, the real story of professional transformation, both to Marv and to myself, lies in the shifts that have occurred in Marv's thinking and perspective of 1) what it means to understand mathematics; 2) the development and content of mathematics (both in school and as a discipline); and 3) what it means to teach mathematics so that students are afforded the opportunity to become mathematically empowered, and how these changed perspectives impacted his practice. In the next three chapters, I will explicate these shifts by positioning Marv's experiences and his changed thinking within theoretical frameworks discussed in current mathematics-education literature, in attempt to identify the pertinent aspects of his professional transformation and the ways in which it led to mathematical power for his students.

Chapter Two

Shifts in Marv's Goals for Understanding Mathematics:

From Memorizing Rules and Algorithms to Connecting Big Ideas

> *I made mathematics appear to be a discipline full of imposed, mysterious and disjointed rules. (Marv, 1996)*

Two Types of Understanding

What does it mean to learn mathematics? A related but very different question is, What are some distinguishing characteristics of a person who excels at learning school mathematics? For many, learning mathematics is the equivalent of the ability to remember rules and procedures, then apply them correctly to produce right answers to textbook and test problems. Exploration, creativity, inspiration or intuition are not traditionally seen as attributes of mathematics learning in school. Logical reasoning in school mathematics **has traditionally** been seen as a skill related to being able to make proper and reasonable use of established definitions, theorems, proofs, algorithms and arguments as they appear in textbooks -- at the exclusion of developing one's own ideas. Again, the *NCTM Standards* (1989, 1991) advocate providing mathematical power for students. What does mathematical power look like? According to the *Standards* (1989), "This term [mathematical power] denotes an individual's ability to explore, conjecture and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems. ...[F]or each individual, mathematical power involves the development of personal self-confidence" (p. 5). How does this ability to explore, conjecture and reason logically relate to working out traditional textbook problems? The

NCTM's definition of mathematical power seems to be based on a very different kind of learning than has been traditionally attributed to school mathematics. One way to contrast the differences between traditional ways of learning mathematics versus those advocated by the NCTM's definition of mathematical power is to think about what is being understood in each case.

Richard Skemp (1978), a renowned mathematics-education researcher and author, discusses two different kinds of mathematical understandings: instrumental understanding and relational understanding. According to Skemp, instrumental understanding is "possessing and being able to use rules," while relational "understanding is knowing both what to do and why" (p. 9). Although an instrumental understanding is more likely to produce consistent quick right answers, it is less likely to be remembered or transferred to various applications. Obtaining a relational understanding is intrinsically motivating and builds confidence as the students develop a sense of schemas of mathematics that are both dynamic and organic in quality, "by which they seem to act as an agent of their own growth" (Skemp, 1978, p. 13). The connections necessary for building the schema are mentioned in the *NCTM Standards for Professional Teaching* (1991, p. 1) definition of mathematical power. The understandings that would allow one to make these connections have been referred to by others as a conceptual understanding rather than a procedural understanding (Davis, 1986; Hiebert & Lefevre, 1986). In the next section of this chapter, I will discuss Marv's transformation from valuing strictly an instrumental understanding of the mathematics for his students' learning, to recognizing, celebrating and setting goals for students' relational understanding of the mathematics.

Marv's Transformation to Relational Understanding Goals

When in high school, Marv was actually exposed to a teacher who valued a relational understanding. The subject was physics and the teacher, Mr. Ross. When Marv reflects back on that experience, he marvels at the fact that it didn't influence his teaching until much later. When Marv was a student teacher, he saw his high school physics teacher as an enigma. Early in his career, he did not know how Mr. Ross' unique way of teaching fit with the rest of what he was learning and experiencing. Only later did he realize it "was exactly the way he wanted to do things." Mr. Ross did not view his role as physics teachers as someone who should hand out knowledge to his students or even tell them what they needed to know. His practice was to set up situations for his students to explore and investigate, considering what they knew, using sources he made available, including himself. By being led by their questions and findings, and sharing them with each other, they were being given opportunities to learn physics. Marv has come to view Mr. Ross' model as an effective way to approach learning mathematics.

Marv's description of the mathematics teachers he had experienced who influenced the direction of his early teaching was much different.

> My previous experience had shown me teachers that could persuade children to engage in typical textbook curriculum by their wonderful personalities (my college methods instructor), their ability to instill fear (many of my former teachers), or out of respect for their long-standing reputation as an excellent teacher (my high school mathematics teacher). As a relative novice, I had no long-standing reputation, so I tried on the other two hats, but performed neither well. My attempts were embarrassing failures. I could not impose accountability standards that forced children into completing assignments; nor could I throw 'cool' problems and mathematical tricks at them to entice their curiosity. My disposition and

> approach to mathematics ran counter to both of these models. Unfortunately, I could not appreciate any other vision of what a successful teacher might look like in the classroom. It takes time to develop a significant appreciation of any teaching style or technique; sit-and-get workshops, snapshot in-services and professional teaching articles did not change my practice. They gave me ideas for a lesson or two only. The only time tested role models I had were my practice teaching mentor, university methods course instructor and teachers with whom I shared classrooms. (Marv, January 1996)

Marv's instruction and assessment before 1994 reflected a focus on helping his students develop an instrumental understanding. Referring to the first vignette in the introduction, Marv stressed rules, without giving students opportunities to know why the rules were sensible. This is typical of school mathematics, where students are asked to memorize the process for "carrying," "borrowing," "solving" linear and quadratic equations, "plugging" values into area and volume formulas, "writing" two column geometry proofs, etc., as spelled out in the textbook, without the supporting framework for analyzing or knowing why what they are doing works. Traditionally, a student like Marv, who is able to excel in an instrumental understanding of mathematics, is not only very successful in school mathematics, but encouraged to become a mathematics teacher.

What happens when students question the rules, or for whatever reason do not commit them blindly to memory and continue to wonder about the mathematics? Often they are told they are not ready for algebra and teachers continue to work on hammering the arithmetic rules into their minds. This was my experience when teaching at the high-school level. The test to decide if a student was ready for and able to learn algebra was an arithmetic test with a significant emphasis on operations of fractions. Students who did not pass the test were put into a pre-algebra course that stressed arithmetic operations, especially fractions. Students not well-connected to school mathematics were the population on whom Del and Sherry were focusing when developing the introductory algebra curriculum at HHS. Marv was frustrated by his work with these students and their seeming inability to develop an instrumental understanding of the mathematics.

As a student, Marv saw learning as equivalent to "doing school." His ability to "do school" successfully could explain his frustration and confusion over students that were not able to perform similarly. Consequently, education courses on learning theories seemed irrelevant to him. He reported that he had not even thought about or developed a conceptual understanding about what it meant to learn apart from doing school, making it difficult to consider different learning theories as an undergraduate in the teacher education program. During his first five years of teaching, his goal for students was to get them to "do school" successfully, which could be translated into developing an instrumental understanding of the mathematics. The success that really matters and that a classroom teacher can influence, in Marv's eyes at that time, was successful completion of assignments and doing well on tests and exams. Marv pointed this out to me when the topic of developing the students' self-esteem, a popular topic at HHS, would come up. Marv felt that his contribution to his students' self-esteem would be to help them become successful in his classroom.

This focus could explain some of Marv's inclination toward improving his mathematics instruction. He envisioned his job description in a more narrow sense than many teachers. He thought his relationship with students should deal exclusively with their interaction with the work of the class. This was his area of expertise, and to become a counselor or social worker to his students, he decided, would do them more of a disservice.

Focusing on their learning served Marv well as a motivation for improving his instructional practice. Essentially, it frustrated him that his students were not able to "do school" in the ways that he did. One of the major motivations to change was his frustrations with the students' performance or ability to apply and retain the material covered in class (i.e. developing an instrumental understanding). He talked about bringing up something covered a few weeks earlier, only to discover that there was nothing there -- students looked at him blankly. Marv's continued response to this frustration was to examine his teaching practice rather than blame the students. This tendency could well be related to his teacher education experience in placing responsibility for students' making connections on the teacher's planning and the experiences afforded the students, an emphasis Marv discussed.

Marv's claim that his teacher education course work did eventually impact his teaching is an interesting exception to research and the image of university education courses on teachers' learning to teach (Feiman-Nemser, 1983). Teachers are quick to report that only actual school experiences taught them anything about teaching. They learned to teach by teaching, and most maintain that university classes provided theory not connected to the real work of teaching. Marv might also have said something similar early in his career, because the ideas related to making connections and learning did not effect a change in his thinking at the time, but only later when he became cognizant of what kinds of learning his students were experiencing. His recognition that many students were not retaining an understanding that could be used later and applied in different settings was not something he easily dismissed as the students' problem. Recognizing the differences between what he had discussed in his teacher education course work and his experiences during his first few years of teaching, he says that "seeds of conflict" were sown during those courses. He recognized that if students were not making these connections, it had something to do with the way they were being taught. It is possible that the seeds planted during teacher education were able to grow because of Marv's experience sharing teaching assignments -- in a way modeling for Marv or providing him with a school or field experience that illustrated many of the ideas and methods Marv eventually embraced.

Marv's preoccupation with the students' instrumental understanding is demonstrated in the following clip from his classroom. The homework question was to solve a simple proportion problem in which 3/5 = x/100. My notes say,

> Amazingly enough, although they had gone through this process two days before, no one in the class knew how to solve it. So Marv told them to turn to page 242 in the book and read it. Several in the class expressed frustration over the reading. Marv addressed that problem and talked to them about getting more out of the reading -- asking questions, drawing diagrams, do some calculations. ...He then went on to show them how to solve proportions with unknowns. (Field Notes, December 1993)

Marv's main concern was that students didn't remember the process and that they couldn't follow the explanation and example in the textbook to reproduce the process. The textbook (UCSMP, 1990a, p. 242) introduces the "Means-Extremes Property" (see below).

Why does this property "work"? Why is it called the "Mean-Extremes Property"? These questions are not addressed in the text. The goal of this procedure appears to be an instrumental understanding, knowing both what to do and when. Even the term "Means-Extremes" was not discussed in the textbook, other than stating that "The numbers *b* and *c* are the **means** of the proportions" and "The numbers *a* and *d* are the **extremes** of the proportion" (p. 242). The fact that students could not remember material covered two weeks before and that many of them failed semester exams were sources of great frustration for Marv. According to Marv, part of the frustration could be traced back to his teacher education experience in which it was emphasized that students should be making connections -- within the mathematics and to their own understandings. This was not happening and it was not clear to Marv how to teach the material in the textbook and allow students to make connections.

Besides his own frustrations, another spark that seemed to ignite Marv's transformation in his goal for his students' learning was the engagement and connections he saw them making in the class he was sharing with Sherry. These were students who were in introductory algebra while in the tenth, eleventh and twelfth grade -- and for many this was not their first attempt to pass the class. Yet, he was hearing them and seeing them make vital mathematical connections and understandings that he wished more of his calculus students had made. One of those connections was the distinction between different types of rates of change.

Marv talked about feeling rather uncomfortable at first when Sherry would assign a single problem, spend a day or two on it (sometimes even longer) and not necessarily assign any homework, such as the one presented in the second vignette in the Introduction. In this example, the students worked on their ideas and the initial discussion during one class period. Although they had discussed less than half of the worksheet, many very interesting ideas had surfaced during the small group discussions and the whole class discussion that would give Marv some ideas about their conceptions of numbers, non-integers in particular. Prior to this, Marv thought that homework and doing a larger quantity of problems was a way of holding students responsible for their learning. Later he gave the following reflections:

> I started to see that these problems had so much to them that the kids could really dig into them. They could find alternative methods of solving them and thinking about them. If they had come out of the textbook, they could have flipped to the back and assumed there was only one correct answer. Even then if you talk about another way to approach it, they say it is confusing -- they just want you to show them one way. With these problems, a student could talk about it graphically, another one could be looking at the table and even their results could be different or it could be expressed differently. They could talk about what they were thinking and write about it. (Marv, Fall 1994)

Marv noticed that students at different levels of understanding could participate equally well and stay engaged. The goal was not to work toward a single right answer, but instead to give a forum for individual students or small groups of students to express their understandings to the class and participate in mathematical conversations. The goal or focus was a relational understanding of the mathematics -- knowing both what to do and why.

The conversations between the students, both in small groups and whole-class modes, took on a different dimension than Marv had experienced in his other algebra classes. Marv felt that the difference was related to the situations they faced. These "situations" represented the problems Sherry used to confront the students with -- questions or problems that might push their thinking related to the topics being discussed. The situations were either written by Sherry or ones that she and Del had used the two previous years while teaching the same course. As they analyzed these situations, the students had something to work on and figure out; they needed to bounce ideas off one another. Time spent presenting ideas did not have to be teacher centered. Students would go to the board to help the others understand how they were thinking about it.

Although the situations held the students' interest, Marv noticed that Sherry's focus was not the material but the students, as she listened to what they were saying and doing. They were both continually amazed by the sophisticated and thoughtful responses and mathematical ideas the students considered. Sherry recalled a time when Marv was talking with the class about the effect the rate of change had on the graph of a linear relationship. He held his pencil diagonally as a line and asked what would happen in the rate of change increased. The class indicated that he should tilt the pencil more. As he tilted it more and more, they indicated that the rate of change would be continually getting bigger, until he held it completely vertical. Someone said that would be impossible. He asked, "Why?" A student spoke up and suggested that the class think about the x-axis as "time" and y-axis as "distance." The student went on to say that a line that goes straight up and down would be like going somewhere in no time. Here was a student thinking about lines for what they are - - a *representation* for a *relationship* between an independent and dependent variable changing at a constant rate, developing a relational understanding of the mathematics rather than memorizing the rule the vertical lines have an undefined slope. This student noticed that a vertical line cannot represent a linear function and was able to articulate why using distance as it relates to time.

Marv contrasted the conversations happening in his other algebra class that was following the book. He said that comments from students in that class most often were, "How do you do this?," "I don't get this," or "What's the answer?" Other comments in class would be short single-word answers to his questions. Beyond that, the conversations were not related to mathematics. In his class with Sherry, the students would argue with each other or present alternative ideas. Marv was thrilled with the connections he saw students making -- for example, connections between tables of functions and the related algebraic expressions, which they called rules, that enabled them to make conjectures about the behavior of the graph. As they investigated the mathematics, they were using graphs, rules and tables as tools for a relational understanding, rather than merely as the goal of an instrumental understanding. One of the differences between these classes was the type of understanding being valued. In a class that values an instrumental understanding, knowing **only** the process and when to use it puts the students in the position of recipients of schemas already developed, left with a simple isolated schema that is merely a process or set of rules. In a class that values a relational understanding, students are building more complex and connected schemas through their present understandings, while being exposed to new experiences in the class.

Marv's changing view of the understanding he wanted for his students is illustrated in the second and third vignettes in the Introduction. In the second vignette, Marv allows the students the opportunity to use any method that makes sense to them to find the x-intercept. Here the students are invited to make connections to what they know, look for patterns and develop their own schemas that will help them find an answer. Marv never gives them a procedure or rule to follow. The students are thinking about the behavior of logarithmic functions, odd and even numbers and definitions in the third vignette. Again, Marv's goal is the development of their collective understandings around the graph and textbook definition of logarithmic functions, inviting them to build a sensible schema, not relying on himself or the textbook to be the sole authorities or disseminators of fixed procedures in the classroom. Consistent with Skemp's claim for relational understandings, the students in this class remember this conversation two years later, and refer to it in a calculus class, remembering even the details of the conversation.

This issue of authority in the classroom is discussed more in Chapter Four. Teaching with only the goal of an instrumental understanding is based on the authority of the textbook and the teacher, while the expectation of a relational understanding invites the students to assume some authority over their own understandings. This issue calls for a transformation of the teacher's role in the classroom, the topic of Chapter Four of this dissertation.

What follows is a students' explanation of a typical day in Marv's introductory algebra class near the end of the first year he taught without a textbook

> Well, usually we will do our work, then the next day we come and we will go over what we have done and we will, like, talk about it. He will ask us questions that we will need to answer and he will put things on the board

> and have us figure it out. He will never tell us the answer, and that gets frustrating. But we will say, "Tell us." And he'll say, "Figure it out." Then he will give us a new thing that is a little bit harder than what we had before to work on. He will go around the classroom and help us, then the next day we will go over it and we will move on to something a little bit harder. We usually work by ourselves, but talk a lot about it. We don't cheat which is really weird, because in math classes I have had before, people copy each other. We talk with our neighbors and stuff, helping each other out. Everybody participates. I guess it is because, like, Mr. Smith makes us participate, because he throws questions out at us all of the time. And it makes us think about it and everybody can give their ideas and, like, not really be afraid because like I said, we're all friends and nobody is afraid to say what they think. (Student #8, Student Interviews, 1995)

From this student's perspective, the students are very active in thinking, talking and participating in the work of the class. It is interesting that the student differentiates between copying each other's work and helping each other out. I overheard an interesting conversation between two students in Marv's class that supports this difference. One student questioned another student regarding a situation being presented on a worksheet. Although I did not hear how the question was phrased, the student being questioned said, "Yes, I answered that one. Tell me how *you* are thinking about it." The first student went on to talk about what she was thinking and what she found puzzling. They discussed different sides to the issue without once considering what might be the "right" answer or even the answer the teachers had in mind.

My notes reflect my amazement. My previous experience was similar to this student's: Students who "helped" other students told them what to do or how they did it, with little or no questioning or interaction. When the goal is an instrumental understanding, questions are about a process, either the most efficient process or the one being covered at that time, while in this case the goal was making sense of the issue, examining one's own ideas in relation to others' ideas, including tradition and convention, with the ultimate objective of developing a relational understanding.

Marv discusses the transformation in the similar terms. In reference to the first six years of his instructional practice, Marv (Synthesis Paper, unpublished) said,

> It involved lecturing about a concept that students had yet to think about, or demonstrating algorithms for which no need has been acknowledged by the intended learners. I made mathematics appear to be a discipline full of imposed, mysterious and disjointed rules.

An instrumental understanding, apart from a relational understanding, involves memorizing just such rules. Although Chapter Four will focus more directly on Marv's teaching, the learning goals affect the way one teaches. By "beginning in the learner's world," Marv sees his instruction as allowing the students' questions, interests and ideas to motivate an investigation of the material he designs to build the ideas of the course. When focusing on an instrumental understanding of the mathematics, investigations were not needed as much as practice. In the following quote Marv describes a different process when the goal is a relational understanding:

> With an inquiry based approach to new content, followed by a socially constructed analysis of the students' findings, students are drawn into a more interesting, connected study of mathematics that is far from superficial. Their study builds from ideas that they can identify and, the questions their mathematics seeks to answer are explicit -- because the students ask them. This process also adds credibility to a standard of mathematical rigor that is unappreciated by mathematics students whose teachers artificially impose this tradition or constantly provide the rigorous arguments themselves. As students work toward consensus, validating or rejecting results, thorough arguments become necessary and valued. In this process, generalizations are made, mathematical technique and structure are built, and established mathematics can be studied more critically than in a traditional setting. As a result, more students engage in a conscientious, coherent study of mathematics, which I believe enables them to continue to study the subject as well as apply it in other areas. (Marv, Synthesis Paper, unpublished)

In this process, the students' relational understandings of the content are the focus rather than specific content goals or benchmarks, as understood by the teacher or documented in the textbook or discipline.

Marv is now operating under that assumption that if mathematics is reasonable, students will eventually come to fairly similar conclusions or questions. This will either force them to push further or consult with what others have done with the same ideas. He is thinking about the difference between investigations initiated by the students and those orchestrated by the teacher -- seeing the need for both. Thus it is clear that Marv does not feel that a relational understanding should stand in opposition to an instrumental one, but instead they should complement one another, a view shared by Skemp (1978). Marv voices this perspective and concerns in the following quote.

> As students mathematize problem situations, they engage in the construction of mathematics. It may not always be the exact mathematics I know and intend, but I believe that the process itself is an effective vehicle for helping students better appreciate the mathematics developed by others. What I saw as problematic at one time -- students agreeing on and adapting alternative approaches, definitions, etc. -- actually provides a context for rigorous, mathematically sophisticated discussions and authentic reasoning from students as they study pre-existing mathematics. I see my students more willing to delve into, critically analyze and attempt to understand the workings of other mathematical systems after they have created their own approach or exhausted their own attempts at abstracting mathematics from related contexts. Managing this process is difficult. ...The dynamics of human interactions are complex as mathematical ideas move from one individual's suggestion, to face group criticism, and on to acceptance or rejection by the community at large. Relinquishing the traditional authority held by a teacher is not easy either. Finding a balance between giving input into student conversations and guarding against unintentional manipulation of the dialogue that might subvert the process is not easy either. (Marv, Fall 1996, "Social Constructivism Happens," TE 950)

Transforming one's teaching to value a relational understanding presupposes a view of mathematics that is also relational (Skemp, 1978). Relational mathematics involves developing a schema that is dynamic and connected rather than a series of skills that tend to be limited and isolated, as is the case in instrumental mathematics. Marv's transformations related to his views and understandings of mathematics will be discussed in Chapter Three of this dissertation.

Relational Understanding and Mathematical Power

Memorizing procedures or algorithms implies that the subject matter is static -- given the right steps and skills, problems are solved and right answers are found. Often these steps involve mathematical "tricks" that must be memorized or algorithms invented by someone else that are guaranteed **always to** "work." When this is the major focus, it often results in a "learning situation that leads to the quickest most comfortable route to mastered habit and attitude, using prescribed and applied knowledge (Schwab, 1978, p. 173)." When this represents the learning goals of the teacher, this habit and attitude inhibits the application of ideas beyond the immediate situation, while stifling sense-making and creativity that allows one to engage in reasoned logic or gain mathematical power.

Often, textbooks and curriculum packages are designed with this type of learning in mind. Students are led, either by examples shown by the teacher or those in the textbook, to find answers, then they are given problems to which they must find the answers using similar strategies. The teacher's role is to understand the examples and help the students solve the problems. In most cases, teachers do not need to understand the mathematics beyond what is presented in the material, thus they are basing their instruction on an instrumental understanding of the material. The expectation is that the students would become programmed and molded to produce someone else's answers using someone else's ideas to answer someone else's questions.

"Perhaps the most important single reason why students give illogical answers to problems with irrelevant questions or irrelevant data is that those students believe that mathematics doesn't make sense" (Frankenstein & Powell, 1994). Students would come to believe that mathematics does not make sense if they are never expected to use their sense-making abilities (or ignore them) to think about or engage in the problems. This is the type of knowledge, an instrumental understanding, that was valued in Marv's school mathematics classes and university mathematics experiences and that he valued in his beginning years of teaching.

Relational (Skemp, 1978) or conceptual understanding (Hiebert & Lefevre, 1986) is based on the idea that students need to know both the how and why behind mathematical concepts and procedures. Here students build their understandings by making connections to their current understandings and related topics and building a schema that is transferable to other situations (Skemp, 1978, p. 13). There is not a predetermined procedure to follow; instead, given a situation students begin with their ideas, and then are challenged by other's ideas, convention, tradition and their continued thinking. Marv observed this kind of understanding being sought and happening in Sherry's class. More emphasis is placed on the process of developing mathematical arguments and articulating mathematical ideas when one is in search of a relational or conceptual understanding. Marv came to value relational or conceptual understanding as he transformed the focus of his instruction to the students and their ideas, with curriculum goals providing the background and guiding his decisions in the use of student ideas and concerns and in the design of mathematically rich situations.

One of the results of Marv's transformed learning goals for his students was an atmosphere that encourages and provides for students to gain mathematical power. Students are not likely to gain mathematical power, according to the definition in the *NCTM Curriculum and Evaluation Standards*, when the goal of their learning is an instrumental understanding. Apart from a relational understanding, understanding mathematics is memorizing a complex network of rules, many of which have been developed merely for convention, creating an internal consistency within mathematical operations (e.g., a negative times a negative is positive; $a^0 = 1$). Relational understanding, born out of the students' opportunity to explore, question and develop a schema of the mathematics, in community with other students and in light of traditional mathematics and the historical development of mathematics, and includes an instrumental understanding, is much more likely to contribute to the NCTM's definition of mathematical power.

Chapter Three

Shifts in Marv's Philosophy of Mathematics: From Infallible Truth to Human Construction

As I struggled with these classroom experiences, continued to collaborate with my peers and read educational literature, I came to the conclusion that it was my mathematical philosophy that needed questioning, not my intended approach to teaching and learning. (Marv, December 1996)

Mathematics can be understood to be an existing body of knowledge, on which mathematicians continue to build and it can be thought of as the process of developing and analyzing these and other quantitative and spatial relationships. School mathematics (the content that students encounter in grades K-16) has traditionally been presented as an existing body of knowledge. It is comprised of isolated skills and algorithms that students must memorize and become proficient at, while applying them in specific situations, developing an instrumental understanding. The relational understanding of mathematics that Marv began to value is dependent on a view of mathematics that is relational.[2] It might be helpful in this section to draw a distinction between school mathematics and mathematics as the classical discipline and existing body of knowledge. As a student and in his beginning years of teaching Marv saw these two as one and the same, school mathematics merely providing the skills that could prepare one for the study of mathematics as a discipline. At a certain point, Marv recognized that school mathematics

[2] Skemp (1978) introduces the idea of relational mathematics and says that "learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point (p. 14)."

was more flexible, ideas could be investigated in a variety of ways, but the goal still needed to be an established set of rules and skills that make up the discipline of mathematics. As Marv's philosophy of mathematics changed, viewing it as a human construction rather than an infallible truth that was discovered, he adjusted his enactment of school mathematics accordingly. He came to realize that just as mathematics was socially constructed over time, his students could construct their understandings of the content over time, and that because it was reasonable, his students would come to many of the same conclusions. And, just as mathematicians construct mathematics by pursuing their questions, students can develop a more connected understanding of the content as they pursue their questions. Thus, Marv's transformed philosophy of mathematics directly affected the content to be considered and mathematical activity that went on in his classroom.

Framework for Teacher Shifts in their Conception of School Mathematics

As teachers are confronted with reform ideas of teaching mathematics differently than they were taught, emphasizing a different kind of understanding, they must rethink their own understanding of the mathematics and the content of school mathematics. A teacher's philosophy of mathematics affects the activity of a mathematics classroom and ultimately the opportunity the students have to gain mathematical power. "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, quoted in Ernest, 1991, p. xiii). Deborah Schifter (1995), who currently works at the Center for the Development of Teaching at Education Development Corporation, where she directs an National Science Foundation funded project entitled "Teaching to the Big Ideas," has set up a framework for interpreting teachers' changes in the conception of school mathematics as they move through reforming their practice. In some respects, it represents a move from instrumental mathematics to relational mathematics. A summary of each stage is as follows:

> First Stage: School mathematics is facts and procedures; discrete and ungrounded; learned mechanically and mechanically applied. Truth is simply incarnate in textbooks and instructor
>
> Second Stage: School mathematics includes students finding patterns, solving problems and making conjectures
>
> Third Stage: Students follow up their ideas by investigating the reasonability, generalizability, and validity.
>
> Fourth Stage: Teaching is organized so students confront the "big ideas" of the mathematics curriculum -- the organizing principles of mathematics.

It is clear from my data that Marv went through similar stages in his transformation, perhaps with the addition of perhaps a fifth stage during which Marv gives himself and his students permission to critique existing bodies of mathematics, definitions and school mathematics curricula. In describing Marv's changing views of school mathematics I will reference Schifter's stages.

While changing his view of school mathematics, Marv faced a tension in his thinking and enacting his practice between staying true to the discipline of mathematics, and at the same time, valuing and pursuing his students' ideas and questions (Dewey, 1902; Elbow, 1986). This was partially a question of coverage (preparing students for standardized tests and future course work), but also it was a personal issue of integrity for Marv. It was important to him that the students not be compromised in their mathematics knowledge. This tension was not resolved for Marv until he changed his philosophy of mathematics. He changed from an absolutist philosophy of mathematics to what Paul Ernest (1991), a renowned mathematics education theorist and author, coins a social constructivism philosophy of mathematics. According to Ernest, in the absolutist philosophy, mathematical knowledge "consists of certain and unchangeable truths" (p. 7), while the social constructivism philosophy "draws on conventionalism, in accepting that human language, rules and agreement play a role in establishing and justifying the truths of mathematics. ... mathematics knowledge and concepts develop and change" (p. 42). This shift in Marv's thinking seemed important in order for him to enact changes in school mathematics content and activities consistently and in good conscience.

Marv's Transformation in His Conception of Mathematics

As illustrated by the first vignette in the Introduction, for the first five years of Marv's teaching career, he followed closely the order and content of the textbook. He later claimed that as he went through school, college and the beginning years of his career, he was convinced that the mathematics ***was*** what was in the textbook.

> I remember having the anxiety related to people being out there still creating mathematics, so the body is going to just keep getting bigger and bigger. So mathematics was just the stuff that was stored away in textbooks. That is the stuff you had to learn if you were going to learn math. It was completely a formalist, absolutist kind of view that said that the stuff in the textbooks weren't chosen conventions and techniques and things, but that is the way the world is and that is the way the world works. That was my conception coming out of college and for quite a while teaching. (February 1998)

This closely parallels Schifter's first stage. Marv's vision of an effective mathematics teacher was one who developed and executed clear explanations of the content in textbooks. Marv's teacher education courses and his high school mathematics teacher left him with images of students making connections, but he saw this being a result of the teacher's clear explanations of procedures (essentially the teacher's connections), not something initiated by or directly related to students' ideas.

Marv's first experience that would challenge his view of mathematics was the NCTM Summer Institutes in California. He later reported that the major impact these institutes had on his thinking was related to recognizing the patterns in mathematics, giving him a desire to know more mathematics. He envisioned graduate mathematics classes giving him the opportunity to pursue his questions and make deeper connections himself. It is possible that these NCTM institutes caused him to consider both the second and third stages of Schifter's framework, recognizing that if he was going to have his students investigate patterns and their ideas, he must have that experience himself first. Later, Marv was greatly disappointed by his experiences in graduate school mathematics, where once again his questioned were not considered, it was more presentation, practice and evaluation of an existing body of knowledge, according to him. He was not given the opportunity to become mathematically empowered, to "develop personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions" (see definition at the beginning of the Introduction) in this setting.

Marv began to seriously consider the content of school mathematics, in particular, introductory algebra while sharing teaching assignments with Sherry, in the midst of being introduced to new curriculum ideas for introductory algebra. It might be helpful to consider again the two vignettes appearing in the Introduction, only this time focusing on the mathematics. In the first scene from 1993, Marv is having the students find the value of an unknown quantity in the linear equation, $1/8 = 1/50\ y$. The students have been presented with a set procedure to find unknown quantities, often referred to as "solving linear equations." Here Marv's conception of school mathematics is clearly in line with Schifter's first stage where the mathematics is "facts and procedures ... learned mechanically and mechanically applied." In traditional introductory algebra textbooks, the first half of the textbook is devoted to working with unknown quantities in linear situations, while in the second half of the text, the unknown becomes a variable in a function. The first half is, in many ways, a continued review of arithmetic operations with an emphasis on developing symbolic manipulation skills, presumably to go from the specific to the more general. However, when "x" or "y", in this case, is an unknown quantity it is actually a very specific question, much like arithmetic problems in earlier grades.

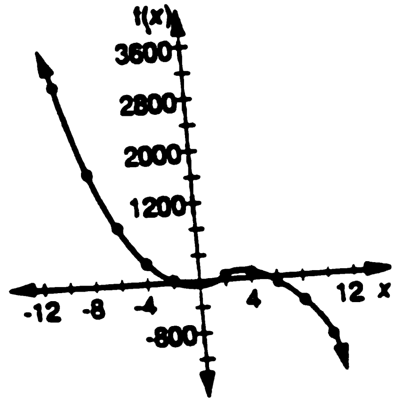
The approach to introductory algebra being developed at HHS (Chazan, 1993; 1996), is different in fundamental ways. In this approach, the unknown quantity "x" is a treated primarily as an independent variable (rather than a specific unknown quantity), whose changing value affects the dependent variable, "y," or the output. The objective is to be able to identify attributes about the relationships between quantities that are changing. The resulting functions become the objects of study and can be studied for patterns, tendencies and classified accordingly using their rule, table and graph as tools in the investigations. Thus in the second vignette, the students are working with the situation, "Take the input, multiply by -3 and then add 2," with the majority of the discussion centering around the question of the inputs that will make the output equal to zero. This could have been written similar to above, $-3x + 2 = 0$, with the directions being to solve for x. However, in this activity the goal is much different than finding this unknown value. The students end up studying this relationship and paying attention to what happens to the outputs as the inputs vary, while noticing patterns and general tendencies that might transfer to other functions, by using tables and graphs, as well as the rule. In this vignette, the students are using these tools to study the relationships, while exploring the mathematics and utilizing logical reasoning -- actually becoming mathematically empowered. The mathematics in the second vignette is closer to the mathematics found in Schifter's second or third stage, where some students are finding patterns, solving problems and making conjectures, while some are investigating the validity and reasonability of their ideas. It is also designed so students will confront a big idea in mathematics -- reasoning by continuity, consistent with Schifter's fourth stage.

The design of the algebra curriculum at HHS, which does not reside in a textbook or curriculum package, is related to Schifter's fourth stage -- considering functions and related relationships as a big idea in mathematics. The teachers at HHS are building a data base of problem situations and activities they have used in their quest to help students confront the big ideas of algebra. The third vignette in the Introduction demonstrates Marv's openness to Shifter's fourth stage, where he allows the students to inquire, conjecture and investigate first, the definition of even numbers and eventually "definitions" -- an organizing principle of mathematics.

Rather than talking about right and wrong answers, the norm in Sherry's and Marv's class was to notice different answers and discuss how and why they were different. This approach is consistent with Schifter's third stage in which students are investigating the validity of their ideas. For example in a class I observed, the students were discussing the classification of functions. Each student had been given a different rules for various functions, including linear, quadratic, exponential, cubic and quartic functions, and asked to make a table and graph it. The students then presented their graphs and tables to the class mentioning things they noticed, including the shape of the graph, whether it was increasing or decreasing and anything about the table they found interesting. Later the students, in groups of three or four, classified the graphs in groups that made sense to them. The final step was to agree as a class how to classify the functions. This provided the opportunity for students to argue about what they noticed and attempt to convince others. Throughout the work on classifying functions, there was no emphasis on right or wrong, yet the students were thinking about ideas that were more sensible or reasonable. One student noticed that the graphs were not always accurate since they were sketched by hand. She was able to convince the class, by looking at the table, that a cubic graph, see the graph and table below, was both increasing and decreasing, although the graph that had been sketched looked like it was only increasing with the middle part having no changes. Some students were convinced, others questioned the table. They continued to investigate.

$f(x) = -2x^3 + 11x^2 - x - 8$

X	Y
-10	3102
-8	1728
-6	826
-4	300
-2	54
0	-8
2	18
4	36
6	-50
8	-336
10	-918



(UCSMP, 1990b, p. 636, 906)

When many of the students changed their minds, they received affirmation from Sherry -- emphasizing her view that changing one's mind is evidence of learning.

Marv's reflections show that he had wrestled with this idea of discussing incorrect answers in the past.

> In math conferences and education courses you always hear about how fruitful it can be to discuss incorrect answers or incorrect processes. Yet, when you have something like $2x + 5 = 9$, a wrong value for x does not give you a whole lot to discuss. Students are not that willing to talk about the fact that they made a mistake somewhere. Yet they are very interested in trying to figure out why the function $2x + 5$ yields different outputs than $2(x+5)$ and start looking for reasons for the differences. (Spring 1995)

When the goal of mathematics is finding the correct answer and attempts are either right or wrong, as it is in Schifter's first stage, students become preoccupied with strategies that will prevent them from failing (Holt, 1982). These strategies prohibit them from thinking about the mathematics and what makes sense.

Marv talks about a specific series of events that spurred a transition in his own thinking related to school mathematics curricula. Sherry was out for a couple of weeks, for a medical situation, and Marv was left on his own with the algebra class and no textbook. He had been doing some planning with Sherry so he had ideas about how she and Del made curricular decisions. He knew he needed to consider old problems, think about the goals of the course, and then write problems or situations that made sense to him -- considering where the students were and the direction in which they needed to go. This represents a major shift in Marv's thinking related to school mathematics content and related activities for students. Based on the format of the class, Marv felt free to create curriculum based on his understanding of the concepts being studied and his students' sense making. As he tells it, a story of liberation emerged.

> It was amazing. I was actually in control of what was going on, rather than having this textbook guideline that I was chasing behind trying to sort through it all. I could think about what I wanted the kids to be able to do or get out of a problem, then design a problem accordingly. (Spring 1995)

Here Marv's thinking is beginning to resemble Schifter's fourth stage in which the teacher is considering how to organize the teaching so that students will be confronted with the mathematics he sees as important. Now when Marv thinks back about the comfort he and his students found in having a textbook the year before, he equates it to an abused spouse becoming used to the abuse. He said, "Kids hate textbooks, yet they will cry for them if you take them away from them." According to Marv,

> The thinking is taken away from the kid. The authors can say what they want, but as soon as you lay out the mathematics and say, 'Here, practice it,' or ask a question about how their model fits or how they set it up, ***they*** are doing the thinking. The questions that are about thinking are always based on your accepting the theorems and definitions in that section, with the only extension of the author's ideas being an application. It's nothing

> like, 'Here is a situation, write about the mathematics here and the patterns you notice.' (Marv, June 1996)

In addition to taking thinking away from the kids, at least one study has shown that pre-packaged curriculum programs can also take thinking away from the teacher. The project designers of IMPACT, a project designed to influence and support teacher change in teaching mathematics (Campbell & White, 1997) happened onto a very interesting phenomenon that addresses this subject. The project began with two sessions of a summer institute to help teachers take a deeper look at teaching mathematics for understanding, as opposed to helping students develop mathematical skills. When the first session began, the project designers did not have any "ready-to-use" materials. They focused on concepts and then helped teachers find or design material that would help students make sense of the concepts. The teachers were involved in curriculum design. For the second session of the same content, the new group of teachers were presented with "ready-to-use" materials. In the authors' words, "This was a mistake." They discovered that the teachers did not engage in the mathematical concepts, were not able to recognize the depth of the mathematical ideas and did not understand the mathematics when they weren't involved in the design of the materials. This consideration of the big ideas of related concepts is, again, Stage Four of Schifter's framework.

The freedom that Marv experienced in designing curriculum alleviated many of the frustrations he had experienced in trying to help students think beyond finding a single correct answer or memorize a procedure that did not necessarily make sense to them. As the designer, he could present the situation in ways that would encourage different responses, while inviting them to use familiar mathematical tools and representations to investigate and push their thinking. In the process of designing the situations, Marv needed to think through the mathematics himself, investigate the connections and his understandings.

Marv had moved into Schifter's fourth stage as he began to reconsider his own ideas about the curriculum of introductory algebra and how to design appropriate situations for the students. He actually changed his mind about the essence of the mathematics in school mathematics as evidenced by the following quote:

> The course called algebra to me was teaching a bunch of techniques for solving equations and simplifying equations. ... an assembly line of stuff that kids had to become proficient at before they could move through the curriculum. Now I see these techniques as tools in a process of studying different kinds of patterns and relationships. It's the patterns and relationships that I see are the fundamental ideas right now. I love the symbolic theoretical mathematics that comes out of stuff. I think they are very cool and elegant. I think they make a great model and structure. But, that is what it is now. Before I thought it was infallible -- this is the way it is. With that kind of attitude I can see why people get mystical about mathematics. The Pythagoreans worshipped it. The fundamental laws, for example, circumference equals pi times diameter, are a lot more interesting when you think about the distance around a circle is always in a constant ratio to the distance across the circle. I'd rather a kid know that and if they know that they are going to experience pi, even though they might not know $c=\pi d$, or c equals two times that little Greek thing times r. ...Mathematics is just a model and a structure with which you should be free to play. (Marv, Spring 1995)

Here Marv is beginning to consider mathematical concepts that cut across different mathematics courses and the ideas that form the connections. He is questioning his past practice of marching through a textbook, section by section, implying that this approach will leave students without connections, while providing only isolated skills put in short term memory. Marv feels that one of the most important lessons he has learned over the past few years has been "if the teacher does not have an understanding of the way their content fits together -- the content is useless to students" (February, 1998).

Yet, he still wrestled with ideas related to how students would discover or uncover what is found in textbooks. Moving away from the idea that the mathematics was in the textbooks was a very hard transition for Marv. One of the major dilemmas for Marv was helping students make the connections he expected them to make. He knew now how to get students thinking about the mathematics and noticing interesting phenomena, but he felt that he often had to force his ideas and direction on them in order for them to cover the material or reach the conclusions that would prepare them for the next step. This question became more intense for Marv as he took on the role of lead teacher in the fall of 1994, during which he was expected to model teaching the emerging introductory algebra curriculum, examining linear functions while treating x primarily as a variable, without a textbook.

It was the year after Marv shared teaching assignments with Sherry, and he was assigned to teach introductory algebra with a teacher I will call Sherman, a veteran teacher at HHS. Sherman had taught mechanical drawing many years at HHS and had even been Marv's teacher for a shop class when Marv was a freshman in high school. Throughout his years at HHS, Sherman had also taught one or two mathematics classes -- introductory algebra or general mathematics each year. He was intent on pushing his students to be responsible and had developed an accountability system in which he had established weekly contact with many of the parents of students in his algebra classes. Sherman saw a connection to success in mathematics, and ultimately success in life, and a high work ethic.

Del had suggested that Sherman share his work with parents with Marv, while Marv share the new curriculum ideas with him. Meanwhile, Sherry was sharing teaching assignments in both introductory and advanced algebra with Matte, the department chair at HHS. I was involved in documenting these team teaching situations for PDS work, a study of the benefits or tensions in team teaching situations (Chazan, Ben-Haim & Gormas, 1998).

Many of the conversations Marv and Sherman shared for the first half of the year were centered around content goals and coverage. They were both concerned that students would leave the class not prepared for traditional geometry or advanced algebra -- as these subjects were presented traditionally, from the canon of the textbooks. Sherman was feeling disoriented because he could not go to a book or a file of materials that would outline the content of the course. Consequently, he began to gather the material for each unit of study into a notebook, organizing the activities in the order they were presented to the students and even working ahead. lining up activities from the previous year.

Later in the year, Marv tells about his discomfort with the direction of the course, as he began realize that he had fallen into a pattern of trying to force students into ***his*** connections and rich understandings of the material, so they would be prepared for the next activity. "I would have a vision of where I wanted things to go and I would hold to it steadfastly for a while, not realizing how stifling that is to students." Marv's recognition that he was pushing the students through the curriculum, compiled from the previous year, came first by looking at his students' faces, seeing blank looks, and then listening to what they were saying.

> When I was doing that, I could see it in the faces of the kids that it was not natural. I was being hypocritical about what I was saying. I don't know if they were thinking that deeply, but they would become frustrated and start

> using the same vocabulary that they did when I was using the book, like, "I don't get it -- period."

The revelation of what was happening came one day when Marv was handing out an activity to the students. He glanced down while one of the students read the activity out loud and realized, to his embarrassment, that the names of the students used in the examples were from the algebra class the previous year, and he realized that the activity was based on the students' ideas from the that class and not the current class. This was a subtle, but clear, message to Marv that he was not using his students' ideas or insight, but was instead expecting them to chase after his ideas and come to his understandings, or to be at the exact same place as the class the previous year.

Even though Marv and Sherman reached the place in their conversations that they discussed what the students were doing and saying rather than the curriculum content (Marv, Spring 1995), the dilemma persisted throughout the year for Marv. How do you help students discover what they need to know for school mathematics -- essentially textbook content? He found himself allowing the students time and space to discuss the mathematics coming out of situations, and he knew learning was happening. But, ultimately, if they didn't "discover" what he wanted them to get out of it, he would tell them and have them practice it. The whole idea of students making discoveries bothered him. He said, "I don't think kids are going to sit there and have their own little epiphanies about mathematics - like 'I don't want to keep writing 4 x 3 x 2 x 1 so I'll put an exclamation point at the end.' I think some people have some vision that it (exploration or discovery) is like that." At the same time, he wasn't sure how to get kids to "discover" what the curriculum was expecting. Eventually, Marv decided that this was a futile and unnecessary expectation.

By the end of the 1995-1996 school year, Marv, at his own admission had become fairly good at designing interesting situations that would engage students in pertinent mathematics, in-depth enough to result in lively class discussions. This was an indication that Marv's conception of school mathematics had moved along in many respects to Shifter's third stage, and often the fourth stage, in which students were finding patterns and making conjectures and then debating their ideas using reasoning skills. These class discussions, at every level from introductory algebra through calculus, would involve the students in varying aspects and perspectives of the mathematics that would be rich in content, but often left Marv wondering what to do with the students' ideas. What good were the conversations if when all was said and done, the students needed to be herded into the traditional structure? Marv's reflections about this dilemma are summed up in the following quote:

> I knew that in order for them to have any hope of acquiring anything useful or carrying anything with them, they had to spend time making sense of the stuff before I showed them anything or taught them anything directly. But I was still kind of unsure about the conversations. I wasn't really sure what curriculum was all about. I thought it was about writing textbooks or deciding just what the kids were doing, then the teaching was something else. Looking back now, there was some conflict between what figuring out I was suppose to be doing with the kids in the classroom and the content of what I was suppose to be doing - pedagogy versus curriculum. I don't think I would make that distinction any more. (February 1998)

Del, the university professor working with HHS math teachers, responded to Marv's dilemma by giving him some books to read, one of which was Proofs and

Refutations: The Logic of Mathematical Discovery, by Mire Lakatos (1976). That summer, while hiking in Ireland with his wife, Marv read the book and found it amazing.

> When I read Proofs and Refutations in Ireland that one summer, I was walking around I in a daze, thinking, 'Wait a minute this is all just a house of cards. People just put stuff together and you can change that.' It is not just changing how you make sense of it or change how you prove this or prove that. There is not just one proof for anything -- which is where I was at before that. You can construct something completely different , depending on your definitions and axioms. It is completely absurd for mathematicians to say that there is an absolute foundation that they go back to or build off from. (February, 1998)

In this famous work, Lakatos writes a fictional dialogue between a teacher and students around the Ellen's polyhedral formula; V - E + F = 2. In the dialogue, the teacher presents a "thought-experiment" which he poses could serve as a proof of the formula. The students caste doubts and find exceptions to the teacher's proof, while also developing cases for or against each other's ideas. In Lakatos' introduction to the story, he points out the pattern of proof, then refutation of that proof, resulting in a "decomposition of the original conjecture into sub-conjecture," which represents a more accurate view of mathematical proof, according to Lakatos, than a deductive guarantee of certain truth (Lakatos, p. 13). The heart of the story is tied to the footnotes in which Lakatos outlines the historical significance and the historical mathematicians represented by the arguments in the discussion.

Lakatos' purpose was to discuss the philosophy of mathematics or the *methodology of mathematics* (p.3). Yet, he does this using people rather than symbols or formal proofs. He says that the core of this case-study is to "challenge mathematical formalism" (p. 5). One quickly recognizes the fallibility of mathematics inherent in its development. Definitions and axioms were adjusted to purposely restrict the situation and leave out things so mathematicians could agree on a coherent structure.

Even after realizing the importance of allowing students the opportunity to make sense of concepts, Marv had still felt uncomfortable when students veered too far away from established mathematics. Marv describes the change in his thinking as follows:

> After "Proofs and Refutations" and some other stuff, I kind of faded away from the learning thing and starting thinking about if I am having kids discuss then it is perfectly okay and perfectly reasonable and perfectly legal for them to come up with their own definitions, conceptions and structures of axioms and everything else. (February 1998)

The "learning thing" that Marv is referring to above was his tendency to hold onto specific learning goals for his students. The third vignette in the Introduction of this dissertation is a wonderful example of students coming up with their own definition, as they wrestled with the definition of an even number. During this scenario, Marv is allowing is students the opportunity to develop mathematical power and exhibit to him the value of organizing his teaching in ways that students are allowed to question and confront what Shifter refers to as "the organizing principles of mathematics" in her Fourth Stage. The students' quest for understanding and insight remained a major factor in Marv's professional transformation. It is interesting that this teaching event happened prior to Marv's reading Lakatos. Marv claims that Lakatos would not have made sense, in the context of the classroom, if he had not experienced his students having very similar arguments and developing proofs and refutations, or agreeing on definitions.

An interesting confirmation that his view of mathematics had changed from something 'out there to be discovered' was solidified in the Fall of 1996 when an intern was observing Marv's calculus class.

> When I really appreciated that was last year when an intern was sitting in on a calculus class and he said, "Oh my goodness, they almost discovered the derivative." No, they were doing something that is perfectly logical. They were looking at rates of change over smaller and smaller intervals and they realized it would be a pain to keep having to do that at every single point, it would be a whole lot easier if we could generalize this in some way so that we could get a function that would do it. That is not discovering the derivative. The 'derivative' isn't matter or some kind of energy. I don't think there is anything like that. It's not something hanging out there in space that whoever runs the universe, God or whatever, has as part of his operating principles. It is a human construction. Once he said that, that was after I read "Proofs and Refutations," I started looking at a lot of other things. The definition of continuity -- you are not saying that there is a special class of functions out there that we can do all of this wonderful mathematics on. It is 'we want to get rid of all of this stuff that is hard and confusing and causes us problems, so that we can do mathematics on it'. That is what the definition of continuity if all about -- to kick all of the other stuff out of your conversation. That is a really different perspective on what that is. (February 1998)

This major transition in Marv's thinking provided insight into the question that had bothered him for several years, "How do you get kids to discover what is in the textbooks?" After having spent the summer considering that what was in the textbooks was just a particular view of a piece of humanly constructed mathematics, Marv was thinking more about the concepts and why people had agreed to construct it in certain ways. Realizing that the process of finding a derivative is not a truth to be discovered, but instead is one way of talking about the phenomena of changing rates, represented a new perspective on mathematics. This experience helped Marv realize what he came to see as the similarities between school mathematics and school mathematics activity and the discipline of mathematics and its development.

During the fall of 1996 Marv took a doctorate level teacher education course on the philosophy and epistemology of mathematics at the university This course served to push Marv's developing philosophy of mathematics and challenged him to transfer his ideas of mathematics being fallible into his classroom and affected both his curricula and pedagogy. By the end of the course, Marv had developed some new theories about his students' mathematical discussions and their construction of mathematical structures. Although still concerned about managing the process, Marv was beginning to value the students' alternatives structures and how that process will open up traditional structures to the student's world of understanding.

> As students mathematize problem situations, they engage in the construction of mathematics. It may not always be the exact mathematics I know and intend, but I believe that the process itself is an effective vehicle for helping students better appreciate the mathematics developed by others. What I saw as problematic at one time -- students agreeing on and adapting alternative approaches, definitions, etc. -- actually provides a context for rigorous, mathematically sophisticated discussions and authentic reasoning from students as they study pre-existing mathematics. I see my students more willing to delve into, critically analyze and attempt to understand the workings of other mathematical systems after they have created their own approach or exhausted their own attempts at abstracting mathematics from related contexts. Managing this process is difficult. ... The dynamics of human interactions are complex as mathematical ideas move from one individual's suggestion, to face group criticism, and on to acceptance or rejection by the community at large. Relinquishing the traditional authority held by a teacher is not easy either. Finding a balance between giving input into student conversations and guarding against unintentional manipulation of the dialogue that might subvert the process is not easy either. (Marv, Fall 1996, "Social Constructivism Happens," TE 950)

Marv became intrigued by the idea that mathematics is socially constructed (Ernest, 1991) and began to imagine how that might affect his work with students and school mathematics. In his own words, "I read Ernest and social constructivist stuff and got a better handle on it being perfectly okay if the kids are talking about some stuff and my role was to help guide them to come to some kind of a consensus and conclusions and build their own mathematical structure - construct their own mathematics." Marv is now quick to say that mathematics is a human construction, which gives him and his students permission to take part in the process, while taking into consideration established mathematics. According to Ernest, "The social constructivist view is that the objects of mathematics are social constructs or cultural artifacts" (p. 57). Ernest confronts the notion that mathematics is merely scientific proven objective knowledge and explains his position regarding the social construction of mathematics in the following passage:

> The social constructivist view is that objective knowledge of mathematics is social, and is not contained in texts or other recorded materials, nor in some ideal realm. Objective knowledge of mathematics resides in the shared rules, conventions, understandings, and meanings of the individual members of society, and in their interactions (and consequently, their social institutions). Thus objective knowledge of mathematics is continually recreated and renewed by the growth of subjective knowledge of mathematics, in the minds of countless individuals. (p. 82)

Realizing that his students did not have to reproduce exactly what was in the textbooks was emancipator for Marv. Marv tells of an interesting example of his decision to not push them to textbook conclusions, but depend first of all on his own growing understanding and experience of the content and curriculum, then value students' ideas that differ from convention. Marv had decided to postpone study of the Product Rule in his calculus class until late in the year, deciding to focus first on the Chain Rule in an attempt to let this concept "sit in their minds for a period of time." While working with Michal Yerushalmy, he had been considering the ramifications of allowing the students to gain a more grounded understanding of the Chain Rule. He consciously held off introducing the Product Rule until it could be connected to 'Integration by Parts.' Later, when he could hold off no longer, he was introducing the traditional proof of the Product Rule, which included some "magical steps" (steps that are often referred to as "tricks" that someone sometime put together to make the proof work). One of his students then spoke up, noticing that by taking the natural log and using the Chain Rule, the Product Rule made perfect sense. Here was a proof that was reasonable to the entire class, with no magical steps. This experience confirmed Marv's belief that "students learn more and analyze other people's mathematics much more thoroughly if they are first given a chance to tackle situations with mathematics they already know or create new mathematics" (Feb. 1998).

Marv does feel a responsibility to be constantly aware of the differences between his students' constructions and traditional constructions and conventions. Then only if the differences seemed significant does he expose the students to historical and traditional thought on the subject, avoiding setting up tradition as the ultimate authority. Marv summed up his emerging view of students' mathematical constructions as follows: "I want kids to spend time forming conceptions about stuff before they try to tackle any kind of traditional stuff. A lot of times we don't have to do that (tackle the traditional stuff) because whatever they come up with is close enough. I use to think you had to mold it into that, I don't think you have to do that anymore."

In order for Marv to move from Schifter's stage three to stage four in his conceptions of school mathematics, he needed to consider his philosophy of mathematics and what counts as mathematical knowledge. He began to separate the existing body of knowledge and the processes that were included in its development, realizing that rather than an infallible truth it was a human construction. This step gave him permission to step outside of tradition and study the big ideas or organizing principles of mathematics without being totally bound by existing textbooks. He could then take his students' ideas more seriously, as he developed his understandings of the content around these principles. In the following quote, quoted earlier when considering Marv's goal for students' understanding, Marv describes his understanding of the social construction of mathematics in his classroom:

> With an inquiry based approach to new content, followed by a socially constructed analysis of the students' findings, students are drawn into a more interesting, connected study of mathematics that is far from superficial. Their study builds from ideas that they can identify and, the questions their mathematics seeks to answer are explicit -- because the students ask them. This process also adds credibility to a standard of mathematical rigor that is unappreciated by mathematics students whose teachers artificially impose this tradition or constantly provide the rigorous arguments themselves. As students work toward consensus, validating or rejecting results, thorough arguments become necessary and valued. In this process, generalizations are made, mathematical technique and structure are built, and established mathematics can be studied more critically than in a traditional setting. As a result, more students engage in a conscientious, coherent study of mathematics, which I believe enables them to continue to study the subject as well as apply it in other areas. (Marv's writing, 1997)

In this process, the students' understandings of the content are the focus rather than specific content goals or benchmarks, as understood by the teacher or documented in the textbook or discipline. An example of this process for Marv, described below, was his recent development of curriculum for his calculus class.

Over the summer of 1997 Marv spent significant time thinking about the big ideas of calculus and how they were connected. Marv has determined that his role as creator of curriculum is more of "getting the ball rolling" than setting ***specific*** goals that his students must reach. The content then becomes the investigation that ensues, initiated by the problem being studied. This would compare to an assertion made by Ernest (1991), that while problem solving is a convergent activity, mathematical investigations are divergent (p. 285) . Ernest goes on to propose that the problem and the investigation make up the content of school mathematics.

In consultation with a mathematics professor, Bill Rosenthal, who published a textbook on calculus (Rosenthal, 1989), Marv decided to have his students encounter rates and accumulated quantities in a context before doing any of the formal established algorithms associated with calculus. In the design of the first few activities of calculus the next year, Marv was influenced by a question that Ricardo Nemirovsky asked him related to the mechanical device mentioned in the Introduction. in which two cars are controlled mechanically while their distances, rates, etc. are recorded electronically, and displayed graphically. Nemirovsky gave one car a variable velocity and asked, "How fast would the other have to travel in order to reach the same place in the same time, if it moved at a constant pace?" The implications of such a task are very interesting and for Marv, "the thinking it initiated can be credited for a good deal of my new personal understanding of calculus." He began the year in calculus by riding a stationary bike, giving the students his pedaling RPM every 15 seconds and then asking them to figure out how many times his legs rotated.

Later, Marv was excited to share about a group that found a method for taking rate information and getting an accumulation. Marv related that the group felt really comfortable and brought up arguments along the line, "you need to average the beginning and the end rate, and multiply by the interval and then add those all together." Marv said he was tempted to push that into trapezoids, but instead marveled at their insight and at how quickly they applied and made sense of the method they developed.

Philosophy of Mathematics and Mathematical Power

Marv's transformed view of the nature of mathematics and school mathematics puts him in the position to help his students develop mathematical power, as he plans his teaching around confronting students with the organizing principles of mathematics. As in the NCTM's definition of mathematical power, noted on the first page of this document, his calculus students in the above example were being given the opportunity to "explore, conjecture, and reason logically; to solve non-routine problems; to communicate about and through mathematics; and to connect ideas within mathematics and other intellectual activity." Rather than rote memorization of symbolic manipulations, these students are being asked to think about the content and put together reasonable conjectures. Then they must work through their ideas, using reasoning skills to address the non-routine situations Marv presents to them. This is possible ***because*** Marv has changed his philosophy of mathematics, recognizing it as a human construction, rather than a compilation of proven indisputable facts and procedures.

Armed with a new perception of the nature of mathematics, Marv has become more mathematically empowered himself. He now feels free to follow his ideas and pursue his questions, building his understanding or doing mathematics. He had a renewed desire to know mathematics and understand the connections and big ideas. Marv became more flexible in his teaching as he created and enacted curriculum and as he turned his attention to students' ideas and used students ideas; and the activity in the classroom changed from individual seat work to dynamic conversations. In the following paragraphs, I will expand these ideas that connect Marv's transformed view of mathematics and mathematical power, to his ability to provide opportunities for his students to become mathematically empowered.

"The issue, then, is not, What is the best way to teach? but, What is mathematics really all about?...Controversies about...teaching cannot be resolved without confronting problems about the nature of mathematics" (Hersh, 1979, p. 34). As Marv adjusted his philosophy of mathematics, recognizing it as a human construction, he also changed his philosophy of mathematics education, deciding that mathematics is socially constructed, both historically and in his classroom. These changes challenged him to work on his own sense making, a necessity for teachers according to Skemp (1978, p. 13), "So nothing else but relational understanding can ever be adequate for a teacher." Marv has a changed view of himself as a learner. He now sees learning as a disposition, a curiosity of sorts that puts you in position to make sense of the world, rather simply a matter of searching for one correct answer. He says he is more apt to think about connections and he is not content just being able to do things, because he wants to know why they work or if there is a more appropriate representation. Marv continues to show these signs of mathematical empowerment. According to Marv (1997, unpublished Synthesis Paper), "The teacher must know mathematics very well -- content, history, various branches, applications and so on." Marv believes that mathematical integrity must be maintained and could be sacrificed if mathematics teachers were to allow their own understandings to become static rather than continually growing.

Marv's students show many signs of being mathematically empowered. Marv noticed that when students keep their ideas for as long as possible, meaning that their ideas are seen as conceptions rather than misconceptions, they are in the best position to appreciate established mathematics. Their empowerment seems to be a direct outcome of Marv's freedom to think about and investigate the mathematics himself before designing curricula, and allowing the story to unfold for his students, accordingly. This freedom came about as a result of his challenging and changing his philosophy of mathematics, coming to see it as a social construct and then engaging in the construction of his ideas with colleagues. This idea that teachers must be involved in learning if they are to pass this onto their students is supported by Cochran-Smith (1991), in the following quote:

> One key assumption that underlies our process of alternative image building is that, for schools to be positive learning environments for students, they must be positive learning environments for teachers as well. Teachers who are not free to construct their own activities, and assess their own competence will be unable to create those possibilities for students. Teachers who do not have self-esteem and a sense that they can control their own destinies will find it difficult to foster those beliefs in others (p. 223).

Marv felt much more control of his students' opportunities to learn and much more positive about his own involvement in that learning process when he began to design activities for the students based on his perceptions of the students' connections and needs with respect to his developing curricular goals.

Moving away from school mathematics content being isolated skills (symbolic manipulation) to ideas described by patterns and relationships, with the symbols merely providing structure, marked a major shift in Marv's thinking. According to Polya (1950), "Mathematics has two faces. Presented in a finished form, mathematics appears as a purely demonstrative science, but mathematics in the making is a sort of experimental science. ... A mathematical theorem must be guessed before it is proved (p. 512)." On the other hand, "The school curriculum treats mathematics as a collection of discrete bits of procedural knowledge" (Ball, 1991). These differing views of the epistemology of mathematics or teacher's conceptions of mathematics is a powerful influence that informs a teacher's practice (Thompson, 1992). According to Hersh (1979), "Mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas which may be represented or suggested by physical objects" (p. 22). In Marv's case, a critical piece of his transformation was related to his thinking about the nature of school mathematics, allowing him the flexibility to create learning situations that often resulted in mathematically empowering his students.

The mathematical activity in Marv's classes has changed in many ways. One of the attributes of mathematical power is the ability "to communicate about and through mathematics." Marv came to accept the view that mathematics is socially constructed, a human construct, which has caused Marv to realize that his students' mathematical constructions are a valid and an imperative part of their learning mathematics. Given this freedom, as Marv says, "they proceed 'naturally' to do mathematics." According to Marv, this happens in his classes during the conversations. This is evidenced by the conversation in the third vignette in the Introduction. As the students work toward a consensus of meaning, they find it necessary to define even numbers. Then the conversation produces a forum for what Marv calls "epistemological shifts and balances" amongst the students and himself. This manifestation of mathematical power is possible because of Marv's transformed philosophy of mathematics in which he now views mathematically as fallible and socially constructed.

As Marv's focus moved from the textbook contents to the big ideas or organizing principles of mathematics he became much more attentive to student ideas. Marv's emerging view of mathematics as fallible and as a human construction has given him the permission he needs to investigate mathematics historically based on his own understandings, analyze textbooks critically, and design curricula for his students that are consistent with his own understandings. At the same time, Marv is working toward addressing questions his students are posing and inviting them to pose new questions. When a teacher like Marv becomes more directly involved in the teaching and learning process, such as designing the activities based on students' questions and sense making, as well as engaging students in their own relational learning, it represents a move to "abandon the ideal goal of deposit-making and replace it with the posing of the problems of men in their relations with the world" (Freire, 1970, p. 66). Marv's analogy of feeling like he was hitting his students with a stick when making them do textbook assignments gives life to the following statement about a violent situation:

> Any situation in which some men prevent others from engaging in the process of inquiry is one of violence. The means used are not important; to alienate men from their own decision-making is to change them into objects. (Freire, 1970, p. 73)

When teachers feels like they are chasing after a textbook that contains the canon of mathematics, followed by little evidence of learning, they ends up victimizing students. The experience of being out of control, as a pawn, under the control of faceless textbook writers that know neither of the teacher nor of the students, yet who were making all of his curricular decisions, is a picture of oppression. The liberation and mathematical power that Marv experienced as he began to think deeply about the mathematics and about the nature of mathematics was mirrored in his students' deep thinking and interest in pursuing mathematics questions, signs of developing mathematical power.

Chapter Four

Shifts in Marv's View of the Nature of Teaching Mathematics: From Listening for Right Answers to Hermeneutic Listening

> *Listening is a part of what I do and I did not think that it needed to be, other than listening to see if they are getting it or not. I have to listen more critically now to know what they are saying, to know about what they are becoming confused or what might be an interesting thing to pick up and run with. (Marv, June 1996)*

What Does it Mean to Teach Mathematics?

Another area in which Marv has gone through a major professional transformation is his vision of what it means to teach mathematics. Paul Ernest (1989) says that a "model or view of the nature of mathematics teaching [is one of the] key belief components of the mathematics teacher" (p. 250).[3] Early in his career, Marv's view of teaching was explaining procedures in a thorough and understandable manner. The procedure or idea being explained was his sense of it or his interpretation of the textbook's presentation. After spending time listening to the students while sharing teaching assignments with Sherry, Marv began to see the importance of finding out what the students understand and how they are making sense of the material as an important aspect of teaching. In Marv's words,

> I used to think that I was up here (indicates a height above his head) and the kids were down here (lowers his hand to lower than his waist). My job was to bring them up to where I was. Now I do not see it that way at all. I think I need to locate them, then come along side of them and travel with them to a place of greater understanding. I often end up taking a route I hadn't previously considered, stopping off at different points and reaching a variety of destinations. (Marv, Fall, 1996)

[3] The other two Ernest mentions are the teacher's "view or conception of the nature of mathematics" and the teacher's " model or view of the process of learning mathematics" (p. 250).

The nature of teaching has changed for Marv. It seems that the difference pointed out in his quote above is an important distinction if students are to be given the opportunity to gain mathematical power. If the teacher's job is to "bring students up" to his or her place of superior understandings, what is the place of exploration, conjecturing or logical reasoning, as spelled out in the NCTM's definition of mathematical power? If instead the teacher needs to "locate students," "come alongside students" and "travel with them," even taking routes or reaching places not previously considered, teaching is a much different undertaking. There is a much greater emphasis in this picture of teaching on what the students are saying, thinking and concluding as an invaluable and critical piece of teaching. Whereas, in Marv's previous understanding of what it meant to teach, the emphasis was to merely evaluate the students' responses, according to what the he, as their teacher, had explained and their ability to get a single correct answer.

Eventually, Marv saw teaching in terms of interacting with students in ways that help students appreciate established mathematics in the process of socially constructing mathematics the class will use and upon which they will build. As stated earlier, Marv marvels at what he calls the "epistemological checks and balances that go on in the conversations" in his classes, as the students build on, argue against and investigate each other's ideas. One way to think about the transformation in Marv's view of teaching is to analyze the changes in his enactment of the role of teacher in the classroom, while also noting what he says about his vision of teaching. One of the most dynamic ways in which Marv has demonstrated this transformation in his thinking about the nature of teaching has been the changing role that listening has played in his classroom. As he envisioned his role differently, it became imperative that he hear and understand his students' sense making. In the next section I will analyze the transformation that took place in Marv's enactment of his role as mathematics teacher in terms of the changes in his listening patterns in his classroom.

The Role of Listening in Teaching

In a recent work, Brent Davis (1997), an assistant professor of curriculum studies at the University of British Columbia, uses listening as a lens to view mathematics teaching. Davis divides the ways teachers listen into the following three categories: 1) Evaluative listening; 2) Interpretive listening; and 3) Hermeneutic listening. Evaluative listening is listening for the purpose of judging the rightness of an answer or listening for a certain response. Interpretive listening happens when a teacher asks questions to gain information about the sense students are making. The students' responses, right or wrong, are then used by the teacher to help students come to the same understanding as the teacher. In the third category, hermeneutic listening, the teacher becomes a participant, interacting with the students in building an understanding. Authority is dispersed amongst the class while the teacher and students jointly explore mathematical issues.

Marv now sees his practice as dependent on all three kinds of listening, whereas in the past he had only recognized evaluative listening as an important part of his job. At times he is merely trying to assess or evaluate how his students are understanding certain concepts; at other times he is listening closely to what they say and interacting in ways that might affect their understandings; and at still other times Marv is joining the students in their quest for understanding, allowing their ideas to develop, learning with and from them.

In Marv's description above, he indicates the changes in his perception of what it means to teach mathematics by thinking about the teacher's role relative to his or her students. His comment illustrate the role of listening in his changed view of teaching -- now he feels he needs to "locate students" (evaluative listening), "travel with them" (interpretive listening) and "take a route he hadn't previously considered" (hermeneutic listening). Several months earlier, he made the following comments about his job as teacher:

> I see my job as something completely different now. Going in, I thought it was telling kids and doing it in an understanding or clear or creative or fun way -- that was what my job was to be. My expectation of what I would consider a good classroom is completely different. I now view myself as someone who has to set up situations for kids to grow with and pull stuff out of, with an overall backdrop of topics and concepts the mathematics department has decided that each class would discuss. I see my role as finding ways to put kids in situations where they are going to be able to really talk about the material and learn. (Marv, June 1996)

A current goal of Marv's instruction is to create situations, as discussed in the last chapter, that help students confront the big ideas of the course, while at the same time enabling the students to "really talk about the material and learn." In order to create these situations, Marv must hear and understand the students' conversations. Marv's transformation in his vision of teaching mathematics changed from strictly an evaluative listener to include interpretive and hermeneutic listening. When his focus was on explaining the procedures, he was only listening to hear if the students' answers were right or wrong. Listening to interpret the students' understandings is similar to when Marv said he could get the students to talk, but didn't know what to do with what they were saying. As Marv began to hear and interact with his students and work with his students in the process of socially constructing mathematics, his view of teaching was transformed, as is evidenced by the hermeneutic listening going on in his room (see the third vignette in the Introduction).

Sometime into the second semester of sharing a teaching assignment with Sherry, Marv would repeatedly have an exciting story to tell about what was happening with students in their class. He related these stories to me when I came to teach the algebra class down the hall. He found the thinking the students were doing amazing -- tenth and eleventh graders, tracked into an introductory algebra course -- making connections to calculus concepts! On one occasion, they were discussing different possibilities for the rate of change of polynomials of higher degree, eventually agreeing that this concept only made sense at specific points, essentially referring to an instantaneous rate of change or the first derivative in calculus. Being in the position of sharing a teaching assignment, and with Sherry leading at this point, Marv was able to observe and listen to the students' ideas.

It is interesting to note that Marv's excitement and interest in changing his approach to teaching introductory algebra was partly motivated by hearing students' sense making. One aspect of Marv's role that he had not seriously considered in the first few years of his practice was listening.

> Listening is a part of what I do and I did not think that it needed to be, other than listening to see if they are getting it or not. I have to listen more critically now to know what they are saying, to know about what they are becoming confused or what might be an interesting thing to pick up and run with. (Marv, June 1996)

These comments by Marv show his transition in his conception of the role of the teacher from strictly evaluative listening, as defined by Davis (1997), to hermeneutic listening in which teachers become participants in the investigation, with that participation informing their instructional decisions. Hermeneutic listening has become a critical aspect of Marv's teaching role. He must hear his students' ideas, connections and current understandings in order to develop strategies to give the students appropriate mathematical experiences to build their conceptions of the mathematics. John Dewey writes in similar ways about the place of experiences in teaching.

> [W]hat concerns him [sic] as a teacher is the ways in which that subject may become a part of experience; what there is in the child's present that is usable with reference to it; how such elements are to be used; how his own knowledge of the subject matter may assist in interpreting the child's needs and doings and determine the medium in which the child should be placed in order that his growth may be properly directed. (1902, p. 201)

Teachers could come to know this "medium" only by actively listening to the child while the child is involved in an interactive environment around situations and problems that can be discussed and analyzed. Mathematically "rich" problems by themselves, apart from hermeneutic listening, will not put the teacher in a position to help the child grow mathematically.

Marv tells a couple of stories that illustrate his impression of the negative impact teaching mathematics by telling or informing, with only evaluative listening, can have on students. Del, the professor who taught at HHS, told one story that fascinated Marv and which he repeats. He recalls that Del was teaching the same group of students for algebra and Bible at a Jewish school. What was amazing was that when the students were in Bible class they were much more willing to debate and discuss issues, while in algebra class, the book was the complete authority. In reaction to this story, Marv said, "I thought that was so interesting that math teachers are so successful at beating kids into submission that there is this higher power -- higher than everything else and it is this answer in the back of the mathematics textbook."

In the following tale, Marv gives an interesting analogy to teaching mathematics by telling:

> It's kind of like when we were in Greektown going to the Detroit Institute of Art with my wife's family. My father-in-law said, "Okay, follow me." He jumped in the car and off he went. He knew how to get there; he knew the short cuts, how to drive in Detroit traffic so you do not get stopped at lights. I could see him doing it and I was following, but I did not understand what or why he was doing it. I think that is what goes on in the classroom sometimes. How many times have you sat in a lecture hall and someone is just blabbing away? They are experts in the field and they know about what they are talking. You hear their words, but you cannot see their images. (June 1996)

Marv's desire to allow students the authority to build their own images is evidenced by the following exchange in which his listening could be interpreted as hermeneutic listening. The students were working on a situation in which grandparents set up a savings account for their grandchild with $1000 and added $50 for each birthday.

> S_1: Mine was different, I had (0, 1050) because they got $50 the first year.
> S_2: Zero is when they started with $1000 and one is their first birthday when they got $50.
> M: Do you agree? Did you want to change your mind?
> S_1: No, I still think I am right.
> M: Can you convince him?
> S_1: No, I can't say it but I know what I am thinking.
> M: Okay, keep thinking about it and try to think of a way to verbalize it. We will come back to it.
> (Video, Marv's Algebra 1, May 1995)

It seems that Marv's intent here to allow this student to think for himself. Marv respects the student's idea although he is unable to verbalize his argument. I have noticed while observing Marv's classes that students and their ideas are not discounted because they are not what he was thinking or what he has seen in the past. Instead, he seems to believe that the students are operating with reason and that since mathematics is assumed to be reasonable, their ideas will be reconciled with the discipline without their being forced to memorize someone else's answers. This vignette illustrates Marv's willingness to share the authority in the classroom as he engages in hermeneutic listening, expecting to learn himself as he listens to the students' reasoning.

It is difficult, if not impossible, to clearly differentiate between curriculum and instruction, as they are so very dependent on each other. How a teacher teaches is very dependent on the content and what counts as knowledge. Marv's method of instruction before 1994 was to clearly explain the processes, concepts and steps needed to reproduce what he had said or what was in the textbook, with the only listening being evaluative. As Marv began to investigate and develop algebra and later calculus curricula, he also began to develop his instruction in ways that put the students' conceptions in the center ring, forcing him to engage in interpretive listening and then hermeneutic listening. The textbooks, mathematics traditions and conventions, and his own understandings became sideshows, important factors to be considered and compared with his students' understandings, but these were not under the main spot light during classroom instruction. Instead, his students' ideas and understandings have taken center stage.

In a video of Marv's calculus class, the students are considering a question that ends up leading them to ideas related to the Fundamental Theory of Calculus. Marv is seated at his desk in the corner interjecting his questions and insight, while the students are engaged in debates, making presentations and pursuing each other's ideas, building a mathematically sound case for their ideas. Traditionally, this theorem is presented, explained, examples are demonstrated by the teacher and students practice, applying the resulting algorithm to more examples. The risk is kept at a minimum, especially at this stage, when all but a small handful are happy to accept "truth" without question. As Marv said in relation to his first teaching assignment -- a trained secretary could perform the job.

When a teacher controls the curriculum and not the students, a higher level of intellectual engagement is required. The teacher must continually dig deeper into her or his own understandings, the discipline and the students' conceptions -- looking for connections and building more complex and involved structures. This process involves hermeneutic listening, listening that is searching for meaning with the students. Teaching that moves in this direction trusts that students and mathematics are both reasonable and will interact accordingly. Teaching that trusts students and pursues an intellectually sound basis for the content is motivated by love for all students, believing that they deserve an education, an opportunity to build their mathematical understandings and choices in society according to their desires and interests. It is inconceivable that Marv would say that he did not listen before his transformation. But he listened for what he wanted to hear: first the correct answer (evaluative listening), and then later for what the students might say that could help him move the class along to his goals for the content (interpretive listening). Marv is very explicit in the following quote that he has "transformed his role to a listener." Here Marv is discussing his use of hermeneutic listening as the students' ideas change the direction of the course and impact his thinking and planning.

> To equip students with a mathematical knowledge base that can give them cultural capital in our technological society, I must see to it that student ideas move each class closer and loser to an understanding of the world of the practicing mathematicians and people who require experience with formal mathematics. These considerations have transformed my role to a listener. As students deliberate and work toward resolution of problems, I must continually assess their understanding, note questions or difficulties, and use my observations artfully to develop subsequent lessons that will keep the class' progress centered on curricular goals. These must include traditional mathematical structures so that students are not disadvantaged when they move to settings that expect conventional structure. (Marv, Synthesis Paper, 1998)

Marv is discussing a balance between allowing the class to be completely student-centered and remaining true to the discipline. He feels that the key for him, in his role as teacher, is to listen in a participatory manner, so he can truly give the students the opportunities they need to both build their understandings and to understand the existing structures in the field.

> "He never answers our questions."
> "He never tells us anything. He just wants us to figure it out."
> "Instead of answering our questions, he just asks us more questions."
> "His questions make us think." (Student Interviews, 1995)

These comments have become normal occurrences in Marv's math classes. It is unsettling for some students when a mathematics teacher is resolute in not explaining procedures for getting right answers. Eventually, Marv's students realize that this technique is not a fluke; he really is not going to answer their questions and tell them exactly what to do to get the answers he expects. He never even tells them exactly what he expects, and he seems satisfied with whatever they decide to try or investigate. He does find some things they do more interesting than others and he does share insight from classical mathematics, but he does not present himself as the person with all the right answers or the sole authority. Harking back again to his teaching before 1994, this was quite a change for Marv, as is illustrated with the vignette in the Introduction from the fall of 1993. At that time he was intent on finding ways to make his explanations and understandings clear to his students so that they could produce similar results. For the first five years of his career, Marv was only an evaluative listener.

Listening is also a critical part of Marv's role as it relates to assessment. The mode of assessment that Marv likes to talk about rests on his listening carefully to how the students understand the situations and related mathematics. He recognizes assessment for the purposes of evaluation or assigning a grade as part of his role, but still struggles with creating authentic assessment tasks that would indicate what the students have learned and can do. He thinks about assessment daily rather than only on review or test days as he did in the past. He says, "The assessment I put my faith in is what kids tell me when we talk." Again, hermeneutic listening is the key for Marv. Following are some of the questions Marv frequently asks his students:

> "What do you think you might want to do?"
> "How did you or are you thinking about that?"
> "Why?"
> "Why does that make sense to you?"
> "Why do you think that is true?" (Field Notes, 1995-96)

These questions are asked in a personal sense. Each of the "you's" is addressed to a specific person, as Marv listens hermeneutically and works with the students to understand their thinking. Marv's questions invite students to become involved in the thinking and developing their mathematical conceptions by making conjectures and developing reasonable arguments. Students' responses give him vital information about how they are making sense of the ideas. This is teaching as listening, listening as teaching. It has resulted in students becoming increasingly mathematically empowered, both as the year progresses and as they advance in mathematics classes. Marv is amazed at how confident and thoughtful his calculus classes are compared to a few years earlier. He also marvels at how his calculus students recall conversations from his mathematics classes two years previously.

Teaching through Hermeneutic Listening and Mathematical Power

John Holt (1982) says, "I doubt very much if it is possible to ***teach*** anyone to understand ***anything***, that is to say, to see how various parts of it relate to all other parts, to have a model of the structure in one's mind. We can give other people names and lists, but we cannot give them our mental structures; they must build their own" (p.145). Can we give others our cognitive maps? In the book Making Connections: Teaching and the Human Brain (Caine & Caine, 1991) (a book that Marv claims has greatly influenced his thinking), the authors claim that cognitive maps are always very complex and personal, being dependent on one's past and present experiences. So, although we can give people opportunities for their own experiences, we cannot give them our experiences or the understandings we have developed. Teachers cannot give mathematical power to their students. However, by participating with students in conversations that include hermeneutic listening, students are afforded opportunities to gain mathematical power. Marv has learned the value of including students in his instruction by listening carefully to their sense making and adjusting his instruction accordingly. Giving students the authority to build their own mathematical schema, while using the teacher's expertise to create situations that will confront them with the big ideas is made possible by hermeneutic listening.

The changing view of the role of a teacher that seems to pervade in Marv's story moves in the direction of an intellectual/moral obligation rather than a clerical/procedural enactment of a job (Hansen, 1995). Like Giroux's (1997) cry for educators to move away from the language of critique related to social injustices and redefine their practices in a language of transformation and hope, this story illustrates a move toward seeing the need for school mathematics reform discourse to be clothed in transformation and hope. Marv's story is a beautiful and explicit example of what happens when a teacher shifts control, from trying to control student behavior to controlling curriculum and treats his or her students as intelligent and capable human beings. As the teacher has more control of the curriculum, they are in a better position to both hear the students and incorporate the students' ideas and ultimately influence the students.

One very interesting result is that students are given opportunities to gain mathematical power and confidence as they pursue their own questions and those of their classmates and teacher. In Marv's case, these investigations were directly linked to the way he now he listens to his students. He no longer feels obligated to lead them or facilitate their conversations to reach specific goals; instead, he participates in ways that turn his classroom into an intellectual forum in which he is a listener and a learner, while creating the situations that will develop a story line of the subject and helping students gain a relational understanding of mathematics.

Marv's vision of what it means to teach students mathematics has shifted from his performance in explaining the procedures to designing educative experiences, by being a student of the subject matter and an aggressive listener to his students. Although listening is one of the things he ***does*** differently, it is an indication of the epistemological shift that now says that teaching must be directly related to learning, not merely "presentation and evaluation" as Marv described his experience in his college mathematics courses. In a mathematics classroom that values giving students opportunities to gain mathematical power, the students are engaged in "exploring, making conjectures and participating in logical reasoning," yet they will not create something out of nothing (Dewey, 1902). So, the responsibility of the teacher, according to Marv, is to use their ideas and his expertise to create mathematics, while analyzing, critiquing and coming to better understand classical mathematics. Listening closely to students ideas is an important role of the teacher in this scheme.

Chapter Five
Implications for Developing Mathematical Power through a Professional Transformation

> ***Mathematical power*** *includes the ability to explore, conjecture, and reason logically; to solve non-routine problems; to communicate about and through mathematics; and to connect ideas within mathematics and other intellectual activity.* ***Mathematical power*** *also involves the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power.* (National Council of Teachers of Mathematics (NCTM), 1991, p. 1)

Summary of Results

The shifts in Marv's view of learning, mathematics and his role as a teacher, taken together with the changes he has made in his practice (as evidenced by the three vignettes in the Introduction), describe Marv's professional transformation. This sort of professional transformation is very rare and it cannot be reduced to a concrete set of actions or changes to practice. Instead, the fundamental components of Marv's transformation were shifts in his beliefs and the way he thought about his work as a teacher, together with his changed practice. These shifts put Marv in a position to provide opportunities for his students to gain mathematical power.

In particular, holding a goal for student learning that includes developing a relational understanding of the mathematics sets an agenda that fosters the ability to connect ideas within mathematics, a component of mathematical power. Exploring and making conjectures around mathematical ideas are skills that reflect mathematical power. They make sense only in the context of a perspective that views mathematics as a human construction rather than absolute truth. The *Standards* emphasizes communication as an important part of mathematical power, which implies a kind of interaction made possible by participating in hermeneutic listening rather than seeing mathematics teaching as a mere demonstration of the "right" way or the most "efficient" way. Thus the key components in Marv's professional transformation are central in developing mathematical power.

Fostering the abilities spelled out in the above definition of mathematical power (exploration, making conjectures, logical reasoning, mathematical communication and connections) in a group of high school mathematics students was made possible in Marv's classes as a result of his professional transformation. As many mathematics teachers have discovered, allowing students the freedom to explore mathematics does not mean they will happen onto conventions and ideas as they appear in textbooks. The intention and goal for having students explore mathematics needs to be consistent with the teacher's goal for the students' learning and the teacher's philosophy of mathematics. In Marv's case, the fact that his calculus students were allowed to explore their ideas related to accumulated rates was provided because of Marv's belief that students need to build a relational understanding of the mathematics, together with his belief that their understanding does not have to look exactly like what is recorded in textbooks. Prior to Marv's transformed goal for his students' learning and view of mathematics, exploration did not have an apparent purpose and was frustrating to students and the teacher, as Marv could only allow them to go so far and then he needed to tell the students what he wanted them to know.

As students are exploring ideas related to problem situations or following up on problems they have posed, they are positioned to make conjectures. An example of a conjecture from one of Marv's students (as it appears in the third vignette), is the student who suggested that "You can raise -9 to any power, you just can't take any even root of it." This conjecture led to a discussion during which students were communicating their mathematical understandings, using logical reasoning. This was possible for several reasons. First, Marv was having the students analyze and critique a textbook definition rather than just accepting it at face value. Second, he conceived his role as being responsible for listening carefully to their ideas and thinking through the mathematics with them, while addressing their ideas and questions. And third, his goal for their learning was to confront them with organizing principles of mathematics, (functions as objects of studies, domain, as a attribute of a function, which led to an investigation of the students' number sense) so they could develop their own relational understanding of the mathematics. When the learning goal is strictly an instrumental understanding, the teacher would accept the textbook definition at face value, have the students memorize it and encourage them to look at the graph to identify the domain represented. This combination and coordination of Marv's transformed beliefs and the resulting instructional enactment created a forum for students to become mathematically empowered.

Earlier I recounted the time when one of Marv's calculus students, using his understanding of mathematics, suggested a proof for the Product Rule for Differentiation that was both original and reasonable. This student's inventiveness and confidence (aspects of developing mathematical power) in proposing a proof that was different than the textbook proof, was an illustration of the mathematical power being realized in Marv's classes. Again, this was possible because Marv had come to realize that the proof was dependent on logical reasoning and was a human construction, rather than seeing the textbook proof representing indisputable truth. He followed the student's suggestion and incorporated his idea because he had become a hermeneutic listener.

Marv's students often spend several days investigating one problem or one mathematical question. They have learned that thinking and examining their ideas is much more desirable and valuable than quick instant answers. As his students construct mathematics, they exhibit perseverance in their search for understanding, a characteristic of mathematical power. These exhibitions of mathematical power seem to be directly related to Marv's professional transformation. As he continues to analyze and develop curricula for his classes based on his changed view of the nature of school mathematics, he positions himself and his students to be confronted with the story line he has developed based on classical mathematics. At the same time, Marv is allowing the class the opportunity to construct their own relational understanding of the mathematics, as he enacts his transformed role by engaging in hermeneutic listening.

As I searched to understand Marv's professional transformation, the encompassing nature of the changes in his thinking and the ways these dramatic changes affected his instruction played important roles. He was considering the big picture, perspectives of his profession, including his students and their understandings, the discipline and nature of school mathematics and how one teaches consistently with one's philosophy of mathematics. He was not influenced by a program of reform or a specific result, but instead by his own changing vision and a desire to stay true to what he believed. The relationships between the changes spelled out in the middle three chapters are so intertwined, it was challenging to tease them apart and analyze each separately. Yet, they represent very different aspects of his instructional practice. I would maintain that it was their relatedness or the consistency represented in his transformed goals for students' learning, his philosophy of mathematics and his view of teaching and the related changes in his practice that added up to a professional transformation. The changes actually reinforced each other. Isolated changes or changes that are inconsistent with one's practice would not qualify as a transformation in the same way. For example, one sees many instances in which teachers incorporate reform ideas in their classrooms (e.g. group work or allowing the students to talk about their interests or use of innovative curricula), but because the learning goal is still an instrumental understanding or students are presented with a predetermined process to reach the answer in the back of the book, mathematical power is not being fostered.

Marv seemed to gain confidence in his own relational understandings of the mathematics, and became more willing to explore and search for a connected understanding of mathematics -- all of which are signs of his increased mathematical empowerment as he transformed his philosophy of mathematics and goal of learning mathematics. After experiencing the freedom of pursuing his questions and critiquing conventional mathematics, Marv was anxious to give his students similar experiences.

Marv's desire for this students to have a rich connected understanding of the material challenged his philosophy of mathematics and his understanding of the nature of school mathematics. His feeling that the students should own their understandings and not merely replicate what he thought or what was written in the text pushed him to consider his role as their teacher and again his philosophy of mathematics. If mathematics was what was in the textbook, Marv was confused about what to do with students' various understandings and ideas. By challenging his beliefs and practices in all three areas, and doing it in a heart-felt way that impacted the enactment of his instruction and his basic belief system about the subject matter and profession, Marv was in a position to experience a professional transformation and position his students to gain mathematical power.

What does this imply for "non-Marv's", professional development or teacher education, as we try to support teachers and teacher candidates in ways that they might experience a similar transformation and subsequently teach differently than they were taught. Fundamentally, it seems that ***all*** of the following three questions are very important for teachers to consider, either explicitly or implicitly, realizing that their beliefs in each area will impact the other areas.

1) What is the goal for student learning?

2) What is mathematics?

3) What does it mean to teach mathematics?

One intention of this kind of reflection and investigation would be to move teachers and prospective teachers out of the realm of influence that motivates them to teach like they were taught or maybe more accurately, how they perceive they were taught. By articulating their beliefs and understandings they will come face to face with inconsistencies, both between their answers to the different questions and between their beliefs and practice. Throughout the rest of this chapter, I will consider different contexts, activities and circumstances in which practicing teachers and teacher candidates might be, either directly or indirectly, confronted with these questions. I will consider ways in which teachers might be supported in considering their beliefs and the related shifts in their thinking necessary to experience a professional transformation and position themselves to provide more students with more opportunities to gain mathematical power.

School/University Collaboration

The teacher's learning community, a key factor in Marv's professional transformation, presents us with a complex and important set of circumstances for the professional life of a teacher. Teachers typically spend many hours a day interacting with groups of students, with little time structured around professional collaboration. The half-day built into the weekly schedule at HHS provided a window of opportunity for professional development work, including confronting big ideas in mathematics, curricula development, and discussing students sense making with colleagues. In Marv's case, the community expanded beyond the walls of the school and included the university and even international mathematics education scholars. The resources provided by this greater community impacted his practice and ensuing professional transformation. Encouraging teacher candidates and practicing teachers to pursue relationships with university teacher educators and education scholars is one way to work at removing these walls and bringing the two worlds together.

The collaborative culture within HHS has been developed, at least in part, through the alliance formed with the university in relation to its standing as a PDS (Professional Development School). As a well-functioning PDS, the intentions of this design have been carried out at HHS. These intentions include "a school for the development of novice professionals, for continuing development of experienced professionals and for the research and development of the teaching profession" (Holmes, 1990, p.1). These ongoing relationships carried out as a partnership, rather than top-down directives or projects, have served both communities in pursuing serious inquiry related to policy, subject matters, teaching and learning. Marv feels that he is a "partner in inquiry with the university", and as a teacher, he represents a perspective that university researchers need. At the same time, he feels that his interactions with Del, who brought the curriculum ideas to HHS and has stayed in communication with Marv, and the network that has followed, has afforded him unique opportunities to learn how others are thinking about mathematics education. More than anything else, these contacts gave Marv permission and confidence to pursue his questions and ideas -- the opportunity to gain mathematical power.

One possible lesson learned from Marv's experience is that researchers and people in academia need to take teachers seriously. Is it possible that many teachers have frustrations and questions similar to Marv's, but no one is listening to them or they do not feel they have the right or opportunity to question their practice? It seems important to find out the questions that plague practicing teachers and give them opportunities to pursue them, to dig deeply and confront their beliefs in collaboration with other educators with similar concerns. Hargreaves and Dawe (1990) came to a similar conclusion in their study, implicating that strategies for professional development should "shift away from university or college-based courses targeted at the individual teacher and intended to raise his or her level of intellectual awareness and ability to reflect, to more school-centered forms of professional development which recognize, bring together, and build upon the skills, experience, and insights that teachers already have" (p. 229). Marv was the active agent in the professional transformation he experienced, not the curriculum, technology, outside resources or opportunities, etc.. Although they all made important contributions to the changes, the transformation did not happen in isolation in any one of those areas.

Sharing Teaching Assignments with a Teacher Asking Similar Questions

Having the same teachers teach the same students at the same time offered many advantages. Marv summarizes his reaction to the experience as follows:

> Teaching with someone gave me a common experience in mathematics so that we had a common language to develop our ideas. We could talk about what the students had said and the ideas that had surfaced and because we were both there we didn't have to fill in every detail or try to picture the context. Otherwise, you are hearing about someone else's experience and in your mind placing it in the context of your own classroom and practice. Team teaching pushes you to think about your teaching and why you are doing things. It gives you time to sit back and watch your students being taught, while you listen and think. You have opportunities for someone else to tell you what they saw and were thinking while you were teaching. The greatest professional benefit that I see is that in an attempt to understand a different way of teaching or the benefits of a different curriculum, you have the opportunity to see what goes on in the classroom. Looking at test scores or professional education articles gives you rather limited information. It is the change in attitude that the kids have about mathematics, what they are doing in the classroom, and how they understand the material. ... I was talking with one of our special ed. teachers who was at an elementary school and we were talking about teaching mathematics differently. She said she talked about it and talked about it with teachers, but did not get it until she team taught. That is the same conclusion we have come to. (Marv, spring 1995)

Sharing teaching assignments has encouraged collaborative in-depth study of the mathematics within the department at HHS. According to Marv, "I think that one of the best things about team teaching is that you're getting together with other math teachers and just being able to talk about math and talk about new ideas (Fall 94) . " This collegiality has made curricula creation and development a reality at HHS, as the teachers have worked together to think about the big ideas they want their students to encounter at each level and then share activities or situations they have developed to provide opportunities for that to happen. Teacher learning and professional collaborative relationships are tied together in much of the literature on collegiality (Cohn & Kottkamp, 1993; Engestrom, 1994; Rosenholtz, 1989; Little, 1990, 1993; Hargreaves & Dawe, 1990; Griffin, 1991; Richert, 1991).

Teacher Candidates' School Experiences

Marv does not hesitate to talk about the important transition in his thinking and instruction that took place while sharing teaching assignments with a teacher who was modeling a type of instruction Marv found compelling, while developing curricula based on a philosophy of mathematics that allowed for ambiguity and diversity. While in this situation, Marv was able to develop a vocabulary and a sense of teaching mathematics differently than he had been taught or than he had seen. The value of this experience for Marv implies that sharing classrooms with teachers modeling the kinds of teaching educators are endorsing could prove invaluable to teacher candidates' development

What message are we sending to teacher education students when we talk to them about teaching for a relational understanding and then expose them to teaching practices that focus only on an instrumental understanding? Is it possible for these teacher candidates to transfer what they are learning to an unknown or imagined setting and make sense of it? What picture comes to their minds when we talk about mathematics students engaging in conversations around the mathematics concepts if, as in most cases, it is a phenomena they have never experienced? Even when we give them similar experiences in a university classroom, they assume that holding conversations about mathematics is a reasonable activity for a group of mathematics majors, but not so for a group of freshmen studying introductory algebra. To most teacher candidates this is a completely foreign idea and will become a perceived reality only if they experience it firsthand. If we place teacher candidates with practicing teachers who are considering these questions, we can expect that they also might reconsider their beliefs. Or, in some cases, this collaboration can work the other way around, where interns asking these questions can impact practicing teachers.

A colleague and I were able to study the effect of contrasting field experiences recently when a secondary mathematics teacher candidate divided his field time during his senior year methods class between two very different settings (Gormas & Rosenthal, 1997). The candidate said that the conversations and experiences we had in the method's class about teaching mathematics "did not make sense" until he experienced it in the second placement, during which he listened to students share their ideas (making conjectures), disagree with each other (take part in mathematical reasoning while communicating about and through mathematics) and reach a consensus (problem solve using logic) about mathematical concepts. What really amazed the education student was that these high school students were 10th-12th graders in an introductory algebra course, most qualifying for "special education" case loads, necessitating a special education teacher's presence during the instruction.

Creating Curricula Considering a Philosophical and Historical Perspective

How does a mathematics teacher's philosophy of mathematics affect his or her view of curricula? If, as Marv and I both had thought, mathematics is restricted to the information, skills and processes presented in the textbook, there is not much work or thinking to do apart from describing the contents of the textbook. On the other hand, if mathematics includes the development, connections and ideas related to the various concepts, as Marv came to perceive mathematics, it is a dynamic field that includes the cultural, historical and on-going construction of these concepts.

Should teachers serve merely as disseminators of prepared curricula? If teachers are to play a more active role in the classroom, facilitating discourse and on-going assessment, should they be engaged in creating curricula for their students? This approach assumes that teachers have been confronted with the big ideas of mathematics and have engaged in the development of their own relational understandings. Marv's professional transformation included serious study of the mathematics content, as he considered the philosophy and history of mathematics and the big ideas within the subjects he was responsible for teaching. What resources do teachers need to partake in this type of activity? What are the implications of this for non-Marv's?

As teacher educators recognize the importance of helping teacher candidates move into Schifter's fourth stage of changing their conception of school mathematics, they might consider organizing ***their*** teaching so that the teacher candidates encounter the big ideas of mathematics. If we challenge teachers and teacher candidates to consider story lines of algebra, geometry and calculus, they will be in a position to consider the relationships, connections and build schema of the mathematics intended for students to learn. This kind of thinking becomes foundational if we ask teachers to create curricula that will help their students understand relational mathematics. Marv's experience in struggling with handling students' ideas in light of the intended curriculum suggests that teacher candidates could benefit greatly from investigating their philosophy of mathematics as well as the historical context of the mathematics. As long as people consider mathematics strictly as an infallible truth existing in a canon somewhere, rather than a human construction developed over time, memorizing and mimicking makes more sense than constructing an understanding, that is relational, as well as instrumental. Paying attention to the historical development of mathematics could help teacher candidates realize the flexibility, diversity and multiple approaches to mathematics that are viable and reasonable.

An important piece of Marv's professional transformation that is interesting to consider is the historical and cultural development of mathematics. Marv's reading of Lakatos (1976) introduced him to a philosophy of mathematics that views the construction of the discipline of mathematics as a human development, and therefore fallible. While investigating the history of mathematics, Marv began to recognize the many approaches to mathematics that are viable and put him in a position to both appreciate the conventions and traditions, as well as the freedom to critique them. This brand of mathematical power was then passed on to his students as he gave them similar opportunities based on their questions and discoveries. Considering the historical development of subject matter could be an important element in confronting the three big questions mentioned at the beginning of this chapter.

Developing a Relational Understanding of the Mathematics

The mathematics teachers at HHS have worked hard at developing their relational understandings of the mathematics they teach. The mathematical depth of their conversations seems related to their perceived responsibility for creating and developing mathematics curricula for their students (Hargreaves, 1992). As they consider the story line for each subject matter, they confront the big ideas themselves and then work on creating situations to help their students confront these big ideas (Schifter, 1995). An intern I worked with recently said, "My CT (collaborating teacher) ***never*** talks about the mathematics, only schedules and management issues. Even in department meetings, the discussions are centered around purchasing materials and attending conferences." This seems reasonable if the mathematics instruction is restricted to presentation and evaluation of the information presented in the textbook.

How can we help pre-service and in-service teachers build their own relational understandings? The idea that mathematics is more than and different than what appears in textbooks causes concern for mathematics teacher educators. According to Ball (1991), learner-focused mathematics teaching requires a broad knowledge base in mathematics to recognize and capitalize on opportunities that arise or apply mathematical ideas and procedures. Most mathematics teachers see mathematics only in the realm of school mathematics -- arithmetic, algebra and geometry textbooks (Thompson, 1992). A high school mathematics teacher recently said to me, "Mathematicians and textbook writers see high school teachers as too dumb and incapable of thinking deeply about the mathematics, so they encourage us to demonstrate watered-down, simplistic skills that will somehow help high school students succeed at higher level mathematics without understanding any of it." It isn't clear what has caused this situation, but it is clear that mathematics teachers, for the most part, seem comfortable in this position, while at the same time frustrated when students progress is slowed down or halted by their lack of understanding. One inference from Marv's early view of algebra being a series of skills that students need to get proficient at before moving on, is that teachers tend to see the mathematics only in the context of textbook examples and skills. Marv's experience implies that getting teachers involved in creating curriculum may push their thinking beyond amassing skills, to a relational understanding of the mathematics. Designing curricula, that takes into considerations big ideas and organizing principles of mathematics, seems like an important foundation for teacher candidates.

Focus on Student Learning

One way of thinking about this issue is in terms of authority. How should authority be dispersed in a classroom between tradition and convention (as represented in the textbook), the teacher and the students? The question of authority or criticism of the idea that the teacher is the sole holder of knowledge (Freire, 1970) in the classroom is at the heart of many subject matter reform issues and professional development efforts. Part of Marv's transformation in his practice has been to alter his teaching strategies from telling as the expert, to listening to students' thinking and ideas -- putting a high premium on their knowledge and sense making abilities. According to Marv's calculus students' comments on an end-of-the-year evaluation (Spring 1995), students find learning mathematics much more satisfying and enjoyable and feel prepared to continue in higher level mathematics after this type of mathematics experience. In their evaluations, they discuss the difference between trying to figure out the textbook and teacher's meanings to discussing their understandings, sharing the authority, and find the latter much more helpful in making sense of the concepts. The classical view of mathematics teaching, clearly explaining procedures and facts from mathematics textbooks, will not empower students mathematically if mathematical power is developed through conjectures, exploration, analysis and proof (NCTM, 1991). These processes invite teachers and students to actively participate in the discipline as well as study what others have developed and the existing body of mathematics knowledge.

One type of communication available to teachers and teacher candidates in all school settings is listening to students, as a type of teaching method, rather than focusing on their own performance or even the collaborating teacher's performance. As the teacher candidates hear the students' sense making, they are enacting on-going assessment and having the opportunity to observe that the learners are capable of reasonable, logical, mathematical thinking. This practice of listening might begin with more evaluative listening during the teacher candidate's time of mere observation and can be expanded to include hermeneutic listening that is more conducive to learning as the teacher candidate takes on more teaching responsibilities. Marv's transformation from strictly evaluative listening, to including interpretive and hermeneutic listening implies that teachers need to develop in this area.

Teacher educators have the unique job of addressing practical and theoretical aspects of the profession simultaneously (Labaree, 1996). Teacher education, as it is now enacted, even in the best programs, places an inordinate amount of focus and energy on teacher performance. Final evaluations for teacher candidates are based on performance -- their ability to lead and manage a class in "active" learning experiences over a period of time. Perhaps part of the implication of Marv's story is imagining a change in focus in teacher education programs toward discovering what it means to learn something by studying learners in context and making connections to their conceptions and the discipline by listening. It is significant that Marv found himself taking on the disposition of an on-going learner to become a better teacher rather finding better ways to perform. Dewey (1904) states the necessity of this type of disposition if a teacher is to expect intellectual engagement from his or her students.

> Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect the mental integrity and force of children (p. 329).

Marv discovered that it takes time to develop a Deweyan sense of 'adequate and genuine intellectual activity' on the job. Even as a veteran teacher, he was well into the second semester of a class in which student learning was the focus, before he was actually hearing the students and feeling comfortable with his own understanding of the curriculum.

Transformation Takes Time

I find it very encouraging and compelling that seeds planted during Marv's teacher education program were able to take root and contribute to his professional transformation. What are the delayed effects of teacher education courses? Teacher candidates desire answers similar to what they have gained in the rest of their education -- concrete procedures that will produce specific results. Marv argues that this is perfectly reasonable, based both on their previous schooling and their experiences related to learning. Changing one's perceptions constitutes challenging assumptions and offering alternative experiences. In Marv's case, when the idea was considered over time, as a teacher, it became something with which to reckon. This brings attention to the importance of in-service teacher education continuing the work of helping teachers make epistemological shifts (Nelson, 1995), building on their experiences in the classroom, while challenging their assumptions related to teaching and learning. These experiences give the teacher candidates opportunities to "change their beliefs about the nature of knowledge and learning, deepen their knowledge of mathematics, and reinvent their classroom practice from within the new conceptual framework" (Nelson, p.2).

People who enter the teaching profession do so after having gone through over a thousand hours of an "apprenticeship of observation" (Lortie, 1975). Their decision to teach is based on their school experience and, in making that decision, they are imagining teaching as they were taught. In the case of mathematics teaching, there is an enduring tradition of following a textbook section by section. Instruction has also followed a cycle which involves demonstrating how to do complete problems in the next section, assigning homework problems from the textbook and answering homework question. *The NCTM Curriculum, and Teaching Standards* (1989, 1991) were written to change this pattern and reform school mathematics, in an attempt to empower all students mathematically, (see definition in the Introduction). This case study has addressed ***how a teacher changed,*** or transformed his practice, including his basic beliefs about it, in ways that empowered him to first think about and then teach mathematics differently than his own school experiences. The changes that resulted in a transformation happened over a period of time in the midst of relevant experiences.

Final Thoughts

Marv's story is a story of transformation -- a kind of transformation that has been rare, especially in mathematics and especially at the high school level (Cuban, 1993). Teachers have a view of teaching, learning and mathematics that has been handed down to them by the society in which we live. Nearly everyone has a deep loyalty related to how school should be conducted and what it means to teach -- both in and out of school, which pushes one toward traditional ways of teaching (Cohen, 1988). Society is also interested in controlled results, which can only come from controlled practices. Sharing the authority with students, framing problems and being forced to provide intellectual rationales rather than being a voice for the text makes reform teaching a high risk and taxing adventure (Cohen). Yet, if we continue the same patterns without somehow confronting teachers with their beliefs and providing opportunities for transformations, we are perpetuating a problem that affects the depth to which the mathematics students have an opportunity to learn and the number of students who participate in mathematics instruction. The majority of our teachers and students will remain mathematically disempowered. The truth is "as a society, we seem to be educating only a tiny fraction of the population to be mathematically literate" (Ball, 1997). By recognizing that the structure and format of traditional mathematics instruction should not serve to sort students, this story of Marv's professional transformation is a testimony -- "bearing witness to motivate listeners to participate in the struggle against injustice" (Giroux, 1997). Here the injustice is against the many students who do not have the opportunity to become mathematically empowered.

This case of professional transformation leading to mathematical power was not merely a series of changed teaching techniques, although many changes occurred in Marv's practice, but was instead a transformed view of the profession as a whole, including the important aspects of his role within it. Marv transformed his basic belief system, related to the teaching and learning of mathematics. Through a variety of particular opportunities and circumstances that led to this professional transformation, Marv gained greater mathematical power, and was then positioned, as a teacher, to give his students opportunities to gain mathematical power.

So, is a transformation like this necessary for teachers to teach for mathematical power? That is a good question to consider, but not within the scope of this work, since it is a single case study. Yet, there is much to be learned from this transformation and the resulting mathematical power. First and foremost is that a teacher can and did experience a professional transformation of this nature. Second, the goal of each student having the opportunity to develop mathematical power, which doesn't seem feasible under a traditional instructional paradigm or techniques, became a reality as a result of this teacher's professional transformation. Third, the changes that have gone on in this teacher and his practice are heart-felt, not experimental or outwardly imposed, and therefore, there does not seem to be much chance he will revert back to the teacher he was, which says something about the long term affect of this kind of a transformation.

APPENDICES

Appendix A
Marv's Professional Chronology

Spring of 1984 - Graduated from HHS

Fall of 1984 - Entered university - School of Engineering

1987-88 Accepted to university's College of Education - Academic Learning Program; Student taught at HHS

Summer of 1988 Graduated from university - B.S. in mathematics, minor in physical science; Secondary Teaching Certificate

1988-89 1st year teaching - California: two physics, an extremely remedial 7th grade arithmetic and 8th grade general life science

1989-90 2nd year teaching - California: physics and math

1990-91 3rd year teaching - California: all math

1991-92 4th year teaching - part time at HHS: algebra one and physics and full time mathematics graduate student at university

1992-93 5th year teaching - HHS: physics, calculus and algebra one, first semester released to tutor algebra in after school program, second semester team taught algebra with first year teacher, Katie

1993-94 6th year teaching - HHS: algebra 1's, statistics, calculus; released from an algebra one class to team teach with Sherry
-The year I met Marv

1994-95 7th year teaching - HHS: team teaching algebra one with Sherman (former junior high school shop teacher), algebra two, calculus

1995-96 8th year teaching - HHS: team teaching algebra one with Brad (former high school coach), algebra two, calculus; university intern in algebra one class; took Master's level mathematics education course through university's Mathematics Department, visited Boston to learn about algebra software

1996-97 9th year teaching - HHS: algebra one, algebra two and calculus; university senior 4 hours/week, took mathematics education doctoral level course through university's College of Education (Mathematical Ways of Knowing with Mr. Bill Rosenthal); started work with mechanical devices in project with TERC, Ricardo Nemirovsky

1997-98 10th year teaching - HHS: algebra two, calculus, released one hour to work on TERC project; two university seniors; taking two master's level courses through university's College of Education

Appendix B
Data

Over the past four years I have gathered much information in an attempt to make sense of Marv's story. The main source for this information has been Marv himself, including interviews, his own writings, field notes and videos of his classes (see the table that follows). I have also interviewed his students, collected student work and their written evaluations of his classes, talked with his colleagues and collected writings from perspective teachers observing and participating in his classes.

	1993-1994	1994-1995	1995-1996	1996-1997	1997-1998
Field Notes (Each date represents one class period)	9/8; 9/13 (1st period); 9/13 (6th period); 9/17; 9/23; 9/28; 10/12; 11/3; 11/17; 11/18; 11/19; 12/2; 3/18 (2nd period); 3/14 - 3/18 (6th period); 3/21	9/28; 11/16; 11/18; 11/30; 12/21; 1/25; 3/6; 3/29	9/18; 9/29; 10/4; 10/5; 11/1; 11/16; 11/17; 11/30; 1/17; 2/20; 2/26; 2/27; 3/18; 3/27; 4/9; 4/17; 5/6; 5/7; 5/31		
Student Work		Calculus class evaluations	Alg 2 Tests		
Videos	3/14-3/18 (6th period)		5/20 - 5/24 (First and third class periods)	5/12 - 5/16 (Fourth period)	
Interviews	5/6	10/10; 4/6	4/10; 6/15		2/27;
Marv's Writings			Team Teaching (Fall)	Social Constructivism (Dec.) Philosophy of Teaching (Spring)	Synthesis Paper

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