This is to certify that the

dissertation entitled

A decomposition of smooth simply-connected h-cobordant four-manifolds

presented by

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has been accepted towards fulfillment
of the requirements for

Doctor of Philosophy degree in mathematics

[signature]
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Date Oct 1, 97

# A DECOMPOSITION OF SMOOTH SIMPLY-CONNECTED h-COBORDANT FOUR-MANIFOLDS

by

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1997

# ABSTRACT

## A DECOMPOSITION OF SMOOTH SIMPLY-CONNECTED h-COBORDANT FOUR-MANIFOLDS

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In [A] S. Akbulut found the simplest possible example of a non-product h-cobordism. It is relative to the boundary cobordism $A$ of two compact contractible 4-manifolds, which are diffeomorphic to each other (not relative to the boundary). Akbulut found his example as a subcobordism in a bigger non-product h-cobordism $W$ of two closed manifolds homotopy equivalent to $K3$ surface blown up once. Closure of the complement of $A$ in $W$ smoothly has a product structure.

In my dissertation I prove a theorem, which generalizes example of Akbulut. It states that given a (4+1)-dimensional smooth simply-connected h-cobordism, one can find a subcobordism between two compact, contractible, diffeomorphic (not relative to the boundary) manifolds, so that there is a product structure on the closure of the complement of this subcobordism. This, for example, implies that, given two homotopy equivalent, smooth, compact, simply-connected, not diffeomorphic manifolds, one can cut off a compact contractible submanifold from one of them and reglue it back via some diffeomorphism of the boundary to get another one. Another implication is that any four-dimensional homotopy sphere is a twisted double of some compact contractible manifold.

## ACKNOWLEDGMENTS

I would like to express my deep gratitude to my teacher Selman Akbulut who guided me in the wonderful world of 4-dimensional topology. During the last three years I have experienced his continuous guidance and encouragement. In the process of writing my dissertation I have received many helpful hints and ideas from Professor Akbulut. I'm grateful for his generosity in sharing these with me.

While working on my dissertation I had the opportunity to meet a number of mathematicians. Many interesting and stimulating conversations took place. I would like to thank Weimin Chen, Tedi Draghici and Gregory Mikhalkin for their interest and helpful suggestions.

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# Introduction

One of the main tools of classification up to diffeomorphism of higher–dimensional smooth manifolds is an $s$-cobordism theorem, which states that smooth $s$-cobordism $W$ between two smooth closed manifolds $M_1$ and $M_2$ of dimension grater then or equal to 5 is diffeomorphic to a product cobordism $M_1 \times I$ and hence manifolds $M_1$ and $M_2$ are diffeomorphic. If we restrict our attention to simply-connected manifolds then corresponding theorem is called $h$-cobordism theorem. It turns out that both $s$-cobordism and $h$-cobordism theorems fail for manifolds of dimension 4. This failure is one of the reasons of mysterious behavior of smooth 4-manifolds.

First examples of $s$-cobordant and, in fact, homeomorphic, but not diffeomorphic manifolds of dimension 4 were discovered by Cappel and Shaneson in [CS], see also [F] for the proof that exotic manifold found by Cappel and Shaneson is actually homeomorphic to $RP^4$; first simply-connected examples were found by S. Donaldson in [D]. Since then great many pairs of simply-connected $h$-cobordant but not diffeomorphic manifolds were found using invariants defined by S. Donaldson and Seiberg-Witten invariants.

Even before first counterexample to $h$-cobordism theorem was found, it was known that such counterexample would imply existence of exotic $R^4$, i.e. open smooth 4-manifold which is homeomorphic but not diffeomorphic to Euclidean space of dimension 4. In fact, it was shown that any simply-connected $h$-cobordism can be split into two sub-cobordisms between open manifolds, one has a product structure, and another is an $h$-cobordism between two fake $R^4$'s. R. Gompf and Ž. Bizaca build very simple (in terms of handlebody) $R^4$(see [BG]) starting with a non-trivial $h$-cobordism between two $K3$ surfaces.

In [A] S. Akbulut obtained an example of simplest possible non-trivial $h$-cobordism. It is a relative to the boundary cobordism between two compact contractible manifolds and it was found as a subcobordism in a bigger cobordism between two closed manifolds homotopy equivalent to $K3$ surface blown up once. In Akbulut's example that bigger cobordism splits into two parts: one is a product cobordism, another is a non-trivial cobordism relative to the boundary between two compact contractible 4-manifolds. Moreover, these two manifolds are diffeomorphic to each other, but they are not diffeomorphic relative to the boundary.

In my thesis I prove that any $h$-cobordism is a union of two compact sub-cobordisms, one of which has a product structure and another is an $h$-cobordism between two compact contractible manifolds. In addition, I show that this two manifolds are diffeomorphic. This result generalizes the example of Akbulut.

Similar result has been obtained by C.L. Curtis, M.H. Freedman, W.C. Hsiang and R. Stong in [CH0, CH1, CHFS]. Namely, they have shown that any two smooth simply-connected h-cobordant four-manifolds become diffeomorphic after removing compact contractible submanifold from each of them; moreover, remaining pieces are also simply-connected. It is not hard to see that their argument to show simply-connectedness can be merged with the proof below, thus proving that fundamental group of manifold $M$ (see main theorem below) is trivial.

# 1. Statement of the Main Theorem

Throughout these notes all maps and manifolds are smooth and immersions are in general position (or do their best if they have to obey some extra conditions). We also make the convention that if a star appears in place of a subindex, we consider a union of all objects in the family, where the index substituted by the star runs over its range. For example, $D_* \stackrel{\text{def}}{=} \bigcup_i D_i$.

Following is the statement of decomposition theorem we prove.

**Theorem.** *Let $U$ be a smooth, 5-dimensional, simply-connected h-cobordism with $\partial U = M_1 \sqcup (-M_2)$. Let $f : M_1 \to M_2$ be the homotopy equivalence induced by $U$.*

1. *There are decompositions*

$$M_1 = M \#_\Sigma W_1, \quad M_2 = M \#_\Sigma W_2$$

*such that $in_{2*}^{-1} \circ in_{1*} = f_* : H_2(M_1) \to H_2(M_2)$. Here $in_{2*}$, $in_{1*}$ are the maps induced in the second homology by embeddings of $M$ into $M_1$ and $M_2$ respectively, and $W_1$, $W_2$ are smooth, compact, contractible 4-manifolds, and $\Sigma = \partial W_1 = \partial W_2 = \partial M$.*

2. *These decompositions may be chosen so that $W_1$ is diffeomorphic to $W_2$.*

In fact, it can be seen from the proof that the whole cobordism can be decomposed into two subcobordisms, one is a product cobordism and one is diffeomorphic to $D^5$ (as a smooth manifold, without any additional structure).

We will also need the following

**Definition.** *We say that two collections $\{S_i\}_{i=1}^n$, $\{P_i\}_{i=1}^n$ of oriented 2-spheres immersed in oriented 4-manifold are algebraically dual if $\langle [S_i], [P_j] \rangle = \delta_{ij}$. Here, $[S]$ is a* *homology class of immersed sphere $S$ and $\langle \cdot\,, \cdot \rangle$ is the intersection form in the second homology of 4-manifold.*

*They are geometrically dual if they are algebraically dual and, moreover,* $\mathrm{card}(S_i \cap P_j) = \delta_{ij}$.

Note that is $\{S_i\}$ is a collection of immersed spheres in simply-connected manifold $X$, such that it has geometrically dual collection, then the complement of the first set of spheres $X \setminus S_*$ is also simply-connected.

# 2. Proof of the First Part of the Theorem

We start with studying a handlebody decomposition of cobordism $U$. Observe that $U$ has a handlebody with no 1- and 4-handles. Let $N$ be the middle level of $U$ between 2- and 3-handles. Then we have two diffeomorphisms

$$\varphi_1 : M_1 \# S^2_{11} \times S^2_{12} \# \dots \# S^2_{n1} \times S^2_{n2} \to N$$
$$\varphi_2 : M_2 \# S^2_{11} \times S^2_{12} \# \dots \# S^2_{n1} \times S^2_{n2} \to N.$$

By enriching the handlebody of $U$ by 2-3 canceling pairs of handles and choosing $\varphi_1$ and $\varphi_2$, we can assume that $(\varphi_2^{-1} \circ \varphi_1)_*|_{H_2(M_1)} = f_*$ and $(\varphi_2^{-1} \circ \varphi_1)_*[S^2_{ij}] = [S^2_{ij}]$, $i = 1, \dots, n$, $j = 1, 2$. See [W1, W2]. Then the two embeddings $c_1 : \bigsqcup S^2_{i1} \vee S^2_{i2} \to N$, $c_2 : \bigsqcup S^2_{i1} \vee S^2_{i2} \to N$ of disjoint unions of wedges of 2-spheres, representing cores of products of spheres in decompositions $\varphi_1$ and $\varphi_2$, are homologous and hence homotopic, for $\pi_1(N) = \{1\}$. We have two algebraically dual collections of embedded 2-spheres $\{S_i = c_1(S^2_{i1}) \subset N\}_{i=1}^n$, $\{P_i = c_2(S^2_{i2}) \subset N\}_{i=1}^n$, and surgery along $\{P_i\}$ gives $M_1$; along $\{S_i\}$ gives $M_2$. Put $V_0 = \overline{\mathrm{Nd}_N(S_* \cup P_*)}$, the closed regular neighborhood of $S_* \cup P_*$ in $N$.

Here we give a sketch of the rest of the construction for the proof of the first part of the theorem and than work out the details.

1. Manifold $V_0$ has a free fundamental group and its group of second homology is generated by classes of spheres $S_i$ and $P_i$, $i = 1, \dots, n$. We enlarge $V_0$ to obtain the bigger manifold $V_1$ so that collections of spheres $\{S_i\}$ and $\{P_i\}$ have geometrically dual collections of immersed spheres inside of $V_1$ and the properties mentioned above are satisfied.

2. We add 1-handles and essential 2-handles to $V_1$ so that the fundamental group vanishes and no new two dimensional homology classes appear.

**3.** Surgery of the manifold obtained in step 2 along collections $\{S_i\}$ and $\{P_i\}$ gives two contractible submanifolds $W_1$ and $W_2$ of $M_1$ and $M_2$, respectively, and $\overline{M_1 \setminus W_1} \cong \overline{M_2 \setminus W_2}$.

The intersection points of embedded spheres $S_i$ and $P_j$ can be grouped in pairs of points of opposite sign with one extra point of positive sign when $i = j$, which we refer to as wedge points. Consider a disjoint collection of Whitney circles $l_1, l_2, \ldots, l_m$ in $S_* \cup P_*$, one for each pair of points considered above. Push these circles to the boundary of the neighborhood of $S_* \cup P_*$ in $N$ and call this new circles $l'_1, l'_2, \ldots, l'_n$ as in. Each of $l'_i$ is contractible in $N \setminus S_*$, as well as in $N \setminus P_*$, since the latter two are simply-connected manifolds. Thus, for each $l_i$ we can find two immersed disks $D_i$ and $E_i$ with $\partial D_i = \partial E_i = l_i$, coinciding along the collar of the boundary, and satisfying the following condition:

$$\overset{\circ}{D}_* \cap S_* = \emptyset, \quad \overset{\circ}{E}_* \cap P_* = \emptyset \tag{1}$$

were $\overset{\circ}{D}$ stands for interior of disc $D$.

Now we shall show that it is possible to find disks $D'_i$ and $E'_i$ so that they are homotopic rel(collar of the boundary) and the condition similar to (1) is satisfied.

The union of $D_i$ and $E_i$ with appropriate orientations gives us a class $[D_i \cup E_i]$ in the group of the second homology of $N$, which splits into the direct sum $H_2(N) = H_2(M_1) \oplus < [S_j], [P_j] >$. So, we can write $[D_i \cup E_i] = a + \sum \beta_k [S_k] + \sum \gamma_k [P_k]$, where $a \in H_2(M_1)$. Using the diffeomorphism $\varphi_1$ and the fact that $\pi_1(M_1) = \{1\}$ we can realize class $a$, considered as a class in $H_2(N)$, by an immersed sphere $A$ in $N$ disjoint from $S_*$. Classes $\sum \beta_k [S_k]$ and $\sum \gamma_k [P_k]$ can be realized by the immersed spheres $B$ and $C$ in $N$ disjoint from $S_*$ and $P_*$, respectively. Now taking the connected sums $D'_i = D_i \#(-A)\#(-B)$, $E'_i = E_i \#(-C)$ ('$-$' here denotes reversal of orientation) ambiently along carefully chosen paths, we obtain discs $D'_i$ and $E'_i$ satisfying property similar to (1) and homotopic rel(collar of the boundary). Here we again use $\pi_1(N) = \{1\}$.

Consider homotopy $F_t : \bigsqcup D_i^2 \to N$ rel(collar of the boundary), where $F_0(\bigsqcup D_i^2) = D'_*$ and $F_1(\bigsqcup D_i^2) = E'_*$. It can be also viewed as a homotopy of the union of the disks and spheres $S_*$ and $P_*$, where the spheres stay fixed during the homotopy.

**Lemma 1.** *Homotopy $F$ can be perturbed, fixing ends, $S_* \cup P_*$ and the collar of the boundary of disks in the disks, to homotopy $F'$ satisfying the following property: $F|_{[0,\frac{1}{2}]}$ can be decomposed in a sequence of simple homotopies each of which is either a cusp or finger move; analogously, $F|_{[\frac{1}{2},1]}$ can be decomposed in a sequence of inverse cusp moves and Whitney tricks.*

For definitions of cusp, finger move, inverse cusp move and Whitney trick see [FQ, K, GM].

Proof of Lemma 1 is given in the last section.

Assume, now, that $F$ has the property provided by Lemma 1 above. Consider $K_* = \bigcup_i K_i = F_{\frac{1}{2}}(\bigsqcup D_i^2)$. Each of $K_i$ may intersect both $S_*$ and $P_*$, but it is homotopic rel(collar of the boundary) to the disk with interior disjoint from $S_*$, via a homotopy decreasing number of intersection points. This means that the intersection points of $\overset{\circ}{K}_*$ and $S_*$ can be grouped in pairs so that for each pair there is an embedded Whitney disk $X_k$ with interior disjoint from the rest of the picture. The same argument shows that there are embedded disks $Y_k$ for intersection points of $\overset{\circ}{K}_*$ and $P_*$, see Figure 1.

Using embedded disks from collection $\{X_k\}$ we can push all disks $K_i$ off $S_*$, producing the new disks $K'_i$. Each $K'_i$ has an interior disjoint from $S_*$ and its boundary is a Whitney circle for a pair of points of opposite sign in $S_* \cap P_*$. Applying an immersed

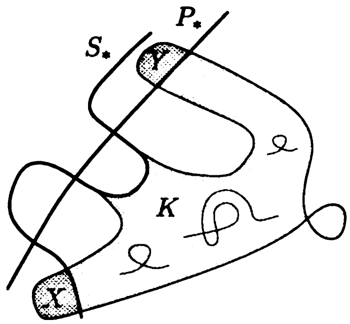


Figure 1: Disk $K$ may intersect both $S_*$ and $P_*$

Whitney trick to $P_*$ along $K'_*$ we eliminate all the intersections with $S_*$, except those required by algebraic conditions and produce the collection $\{S_i^\perp\}$ geometrically dual to $\{S_i\}$. Note that, since disks $K'_i$ are, in general, immersed and may have non-trivial relative normal bundle in $N$, we may have to introduce (self-)intersection points to spheres $\{S_i^\perp\}$ during this process. In the same way, using disks from collection $\{Y_i\}$ we can obtain the immersed Whitney disks $K''_i$ disjoint from $P_*$. They allow us to create a collection $\{P_i^\perp\}$ of immersed spheres dual to $\{P_i\}$. Note, that all described homotopies are supported in a regular neighborhood of $K_* \cup X_* \cup Y_*$.

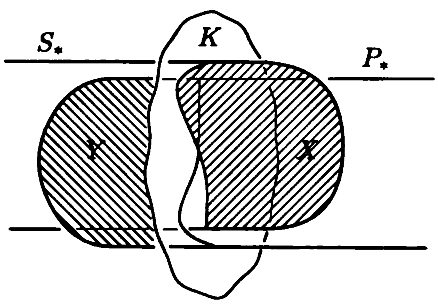


Figure 2: Boundaries of discs $X_i$ and $Y_j$ are not necessarily disjoint

Thus, one can see that the manifold $V_1 = V_0 \cup \overline{\mathrm{Nd}_N(K_* \cup X_* \cup Y_*)}$ has the following properties:

1. $H_2(V_1)$ is generated by homology classes of embedded spheres $S_i$, $P_i$.

2. The collection of spheres $\{S_i\}$ has the geometrically dual collection $\{S_i^\perp\}$ of immersed spheres in $V_1$. Similarly, collection of spheres $\{P_i\}$ has geometrically dual collection $\{P_i^\perp\}$.

3. $\pi_1(V_1)$ is a free group.

Consider a handlebody of $N$ starting from $V_1$. Put $V_2 = V_1 \cup$(union of 1-handles). Manifold $V_2$ still satisfies properties similar to (1), (2), (3) for $V_1$.

Let $g_1, \dots, g_l$ be free generators of $\pi_1(V_2)$. If we fix paths from a basepoint to attaching circles of 2-handles in the handlebody of $N$, then these circles represent elements of $\pi_1(\partial V_2)$, say $h_1, \dots, h_L$. Since $V_2 \cup$ (all 2-handles) is a simply connected manifold, each $g_i$ has a lift $\tilde{g}_i$ in $\pi_1(\partial V_2)$, such that it belongs to the normal subgroup of $\pi_1(\partial V_2)$ generated by $h_1, \dots, h_L$. In other words $\tilde{g}_i = (\alpha_1 h_{i_1}^{\pm 1} \alpha_1^{-1})(\alpha_2 h_{i_2}^{\pm 1} \alpha_2^{-1}) \dots (\alpha_k h_{i_k}^{\pm 1} \alpha_k^{-1})$. Duplicating 2-handles, equipping them with appropriate orientations and choosing paths joining attaching circles of 2-handles to the basepoint we may write $\tilde{g}_i = h'_1 h'_2 \dots h'_k$, where $h'_1, h'_2, \dots, h'_k$ are elements of $\pi_1(\partial V_2)$ obtained from attaching spheres of new handles with new paths to the basepoint. Now, handles in the decomposition of $\tilde{g}_i$ are all distinct and we can slide the first handle over all others along paths joining feet of the handles to the basepoint. The attaching circle of the resulting handle, call it $G_i$, is freely homotopic to $g_i$ in $V_2$. Adding such $G_i$'s to $V_2$ for each generator of $\pi_1(V_2)$ we obtain the simply-connected manifold $V_3$. Since $g_1, \dots, g_l$ are free generators of the fundamental group, we do not create any additional two-dimensional homology classes.

Surgery of $V_3$ along collections of embedded spheres $\{S_i\}$ and $\{P_i\}$ gives two contractible sub-manifolds $W_1$ and $W_2$ of $M_1$ and $M_2$, respectively. Put $M \stackrel{\text{def}}{=} \overline{N \setminus V_3} \cong$

$\overline{M_1 \setminus W_1} \cong \overline{M_2 \setminus W_2}$. So we have the decompositions:

$$M_1 = M \#_\Sigma W_1, \quad M_2 = M \#_\Sigma W_2$$

The property of induced maps in the homology stated in Theorem is obvious. The first part of Theorem is proved.

# 3. Proof of the Second Part of the Theorem

Proof of the second part is based on the following Lemma 2.

**Lemma 2.** *If $W_1$, $W_2$ are homotopy balls built in the proof of the first part of Theorem and $S^4$ is a 4-dimensional sphere with standard smooth structure, then*

$$W_1 \#_\Sigma W_1 \cong S^4, \qquad W_1 \#_\Sigma W_2 \cong S^4.$$

Denote the boundary connected sum by '♮'. Then we can write

$$M_1 = M\#_\Sigma W_1 \cong (M\#_\Sigma W_1)\#(W_1\#_\Sigma W_2) \cong (M\natural W_1)\#_{\Sigma\#\Sigma}(W_1\natural W_2)$$

and

$$M_2 = M\#_\Sigma W_2 \cong (M\#_\Sigma W_2)\#(W_1\#_\Sigma W_1) \cong (M\natural W_1)\#_{\Sigma\#\Sigma}(W_2\natural W_1).$$

Decompositions above are shown on Figure 3.

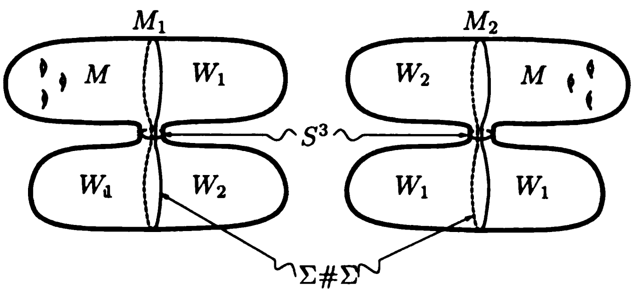


Figure 3: Making contractible parts in decompositions diffeomorphic

In order to proof Lemma 2 we will use the art of Kirby calculus.

First, we build a handlebody of $V_3$.

Consider $V_0'$, a closed neighborhood of $S_* \cup P_* \cup$ (arcs joining wedge points in $S_i \cap P_i$ and $S_{i+1} \cap P_{i+1}$). If there were no intersections between $S_*$ and $P_*$, except a wedge point for each pair $S_i$, $P_i$, then the handlebody would look like as shown on Figure 4.

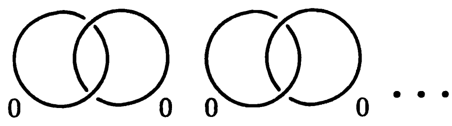


Figure 4: Handlebody of the neighborhood of wedges of spheres

Introducing a pair of intersection points of opposite signs corresponds to the move in Kirby calculus shown in Figure 5.

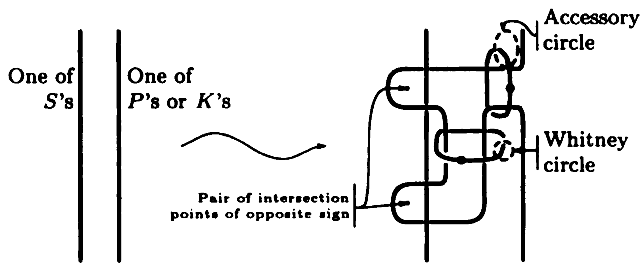


Figure 5: Finger move

Here, we introduce two 1-handles, one corresponding to a Whitney circle, another to an accessory circle of a newly introduced pair of intersections. The handlebody of $V_0'$ is obtained by applying several moves shown in Figure 5 to the picture in Figure 4. Again, if disks $K_i$ were embedded and disjoint from $S_* \cup P_*$, then to obtain the handlebody of $V_0 \cup \overline{\mathrm{Nd}(K_*)}$ one has to attach 2-handles to Whitney circles for each pair of intersection points of $S_*$ and $P_*$. Then we have to introduce intersections between $K_*$ and $S_* \cup P_*$ and self-intersections of $K_*$. Corresponding moves are shown on Figures 5 and 6.

Introducing a self-intersection corresponds to adding one 1-handle to the picture, as shown on Figure 6. Addition of disks from collections $\{X_i\}$, $\{Y_i\}$ corresponds

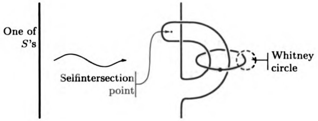


Figure 6: Cusp move

to attaching 2-handles to the circles linked once to the 1-handles corresponding to the Whitney circles, and unlinked from other 1-handles. They may link each other according to the fact that boundaries of $X$'s and $Y$'s are not necessarily disjoint. This phenomenon is illustrated on Figure 2. Then, as in previous steps, add 1-handles to the picture reflecting intersections of $X_*$ and $Y_*$, as in Figure 7.

To obtain the handlebody of $V_2$ we have to attach several 1-handles. The link on Figure 7 shows all possible phenomena which can occur. Thus, the handlebody of $V_2$ has 1-handles of six types coming from

1. Whitney circles of intersections of $S_*$ and $P_*$;

2. accessory circles of intersections of $S_*$ and $P_*$;

3. Whitney circles of intersections of $K_*$ and $S_* \cup P_*$;

4. accessory circles of intersections of $K_*$ and $S_* \cup P_*$;

5. Whitney and accessory circles of self-intersections of $K_*$;

6. Extra 1-handles.

For each 1-handle of type 1 and 3 there is a 2-handle attached to the circle linked to this 1-handle geometrically once and algebraically zero times to other 1-handles.

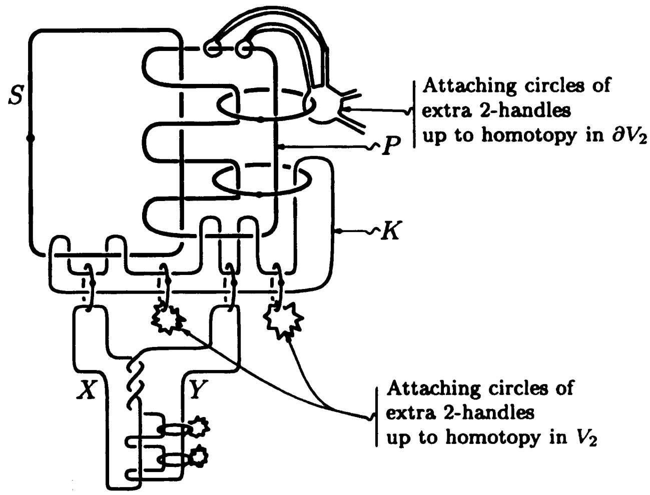


Figure 7: Handlebody of $W_1$

The attaching circles $h_1, \ldots, h_l$ of other 2-handles are homotopic to free generators of $\pi_1(V_2)$. As generators we may choose the cores $g_1, \ldots, g_l$ of 1- handles of type 2, 4, 5 and 6.

Consider the homotopy $F : S^1 \times I \to V_2$, $F(\cdot, 0) = g_i$, $F(\cdot, 1) = h_i$. We can make it disjoint from $S_* \cup K_* \cup X_* \cup Y_*$. First, intersections of the image of the homotopy with $X_*$ and $Y_*$ can be turned into intersections with $K_*$ by pushing them toward the boundary of $X_* \cup Y_*$. Then, in the same way, we can avoid intersections with $K_*$ by cost of new intersections with $P_*$. Finally, intersections with $S_*$ can be removed using the geometrically dual collection of immersed spheres $\{S_i^\perp\}$. Call this new homotopy $F'$ and let $\{x_1, x_2, \ldots, x_k\} = F'^{-1}(P_*)$. If we remove a small neighborhood of a union of disjoint arcs joining each $x_i$ to the point on $S^1 \times \{0\}$ from $S^1 \times I$, then restriction of $F'$ on this set is a homotopy of $h_i$ to the curve ${g'}_i$ which is a band connected sum of $g_i$ and meridians of $P_*$.

This homotopy is disjoint from $S_* \cup P_* \cup K_* \cup X_* \cup Y_*$ and can be pushed to the boundary of $V_2$.

We obtain the handlebody of $W_1$ by attaching 2-handles to $h_1, \dots, h_l$ and performing surgery along $S_*$, which corresponds to putting dots on the circles representing $S_*$ in Kirby calculus.

We summarize all the information about the handlebody of $W_2$ in Table 1. For 2-handles we use same notation as for the corresponding object in construction of $W_1$ above.

| | 1-handles | 2-handles |
|---|---|---|
| 1. | Surgery of $S_*$. | $P_*$; attaching circles go geometrically once through corresponding 1-handles $S_*$. |
| 2. | Whitney circles of intersections of $S_*$ and $P_*$. | $K_*$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table. |
| 3. | Whitney circles of intersections of $K_*$ and $S_* \cup P_*$. | $X_*$, $Y_*$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table. |
| 4. | Accessory circles of all intersections, Whitney circles of (self-)intersections of $K_*$, Whitney circles of intersections of $X_*$ and $Y_*$, extra 1-handles. | Extra 2-handles $H_*$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table and homotopic to band connected sum of 1-handles with meridians of $P_*$. |

Table 1: Handles of $W_1$

Taking a double of $W_1$ corresponds to attaching zero-framed 2-handles to the meridians of existent 2-handles, and as many 3-handles as there are 1-handles in the handlebody of $W_1$.

Recall that attaching circles $h_i$ of 2-handles $H_i$ are homotopic to $g_i'$. Sliding $H_*$ over dual handles $H_*^*$ we can obtain handles $H_*'$ attached to $g_*'$. Now $g'_1, \dots, g'_l$ may be linked to the meridians of handles $P_*$, $K_*$, $X_*$ and $Y_*$, but situation can be improved by sliding handles dual to $P_*$, $K_*$, $X_*$ and $Y_*$ over handles dual to $H'_*$. Further we may slide $H'_*$ over handles dual to $P_*$ to obtain handles attached to meridians of 1-handles from the fourth row of the table, so they can be canceled. The framings of $H'_*$ result in a twist of attaching circles of remaining 2-handles.

We undo this twist using dual handles.

With the next step we unlink handles $X_*$ and $Y_*$ from each other using dual 2-handles and cancel them with 1-handles of type 3. Now $K_i$ are attached to meridians of 1-handles of type 1 and also can be canceled. After canceling $P_*$ with 1-handles from surgery of $S_*$ we end up with unknotted, unlinked, zero framed handles and the same number of 3-handles, which is clearly $S^4$.

We need only small changes in our construction to prove the second diffeomorphism in Lemma 2. The handlebody of $W_2$ differs from $W_1$ by putting dots on the attaching circles of $P_*$ rather than $S_*$. Thus, the handlebody of $W_1 \#_\Sigma W_2$ is obtained from $W_1$ by attaching 2-handles to meridians of 1-handles, coming from surgered $S_i$'s, and to meridians of all 2-handles, except $P_i$'s. Using the same trick we can make the homotopy of $h_*$ to $g_*$ disjoint from $P_* \cup K_* \cup X_* \cup Y_*$, rather than $S_* \cup K_* \cup X_* \cup Y_*$. Applying the same procedure to $H_*$ gives us 2-handles $G''_*$ attached to the band connected sum of meridians of 1-handles corresponding to generators of $\pi_1(V_2)$ and meridians of $S_*$. But now 1-handles corresponding $S_*$ have dual 2-handles, so sliding $H_*$ over them gives handles dual to 1-handles generating $\pi_1(V_2)$. So we may apply the same procedure to simplify the handlebody and end up with $S^4$. This finishes the proof of Lemma 2 and the second part of Theorem.

# 4. Proof of Lemma 1

It is a simple consequence of singularity theory that a homotopy of a surface in 4-manifold can be decomposed (after small perturbation) in a sequence of finger moves, cusps, Whitney tricks and inverse cusp moves. In our case we have homotopy of 2-complex consisting of the union of surfaces with double points and collection of discs attached to this union along some embedded loops. Thus, one has to consider, in addition, a finger move along a path in one of the disks joining (self-)intersection of the disks with a point on its boundary. The inverse of this move is a Whitney trick with a Whitney circle intersecting the boundary of the disk. We have to show that it is possible to reorder these simple homotopies so that those increasing number of intersections come first. This is obvious after the following consideration: cusp birth happens in a small neighborhood of the point in the surface and we can assume that part of the surface in this neighborhood stays fixed during the part of homotopy preceding this cusp birth, so we can push this cusp up to the beginning of the homotopy. A finger move can be localized in the neighborhood of an embedded arc joining two points in the surface, thus we may apply the same argument and this finishes the proof of Lemma 1.

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