This is to certify that the

dissertation entitled

$C^{1,\alpha}$ Regularity of Interfaces for Solutions of the Degenerate Parabolic p-Laplacian Equation

presented by

Youngsang Ko

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Zhengfang Zhou
Major professor

Date 7/24/97

$C^{1,\alpha}$ Regularity of Interfaces for Solutions of the Degenerate Parabolic $p-$Laplacian Equation

By

Youngsang Ko

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1997

ABSTRACT

$C^{1,\alpha}$ Regularity of Interfaces for Solutions of the Degenerate Parabolic $p-$Laplacian Equation

By

Youngsang Ko

In this dissertation, we study interfaces of the parabolic p-Laplacian equation $u_t = \Delta_p u$ for $p > 2$. We establish their $C^{1,\alpha}$ regularity after a large time under some suitable conditions on the initial data. It is well known that the solution u and $\nabla u \in C^\alpha$ for all $t > 0, \quad x \in R^N$. Hence $\nabla u = 0$ on the interface, which unfortunately does not give information on the interface itself. The key idea is to study a new function $v = \frac{p-2}{p-1} u^{\frac{p-2}{p-1}}$ which has the same interface as u. The lower and upper bounds for v_t and $|\nabla v|$ can be estimated near the interface by constructing some lower and upper solutions. Consequently some standard theory on strongly parabolic equations can be used for v. Finally, the Hölder continuity of the normal direction of the interface can be obtained by using dilation arguments and some special monotonicity of v. In addition, some useful and interesting properties of v are also obtained along the way to prove the main theorem.

To my family, especially to my mother.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Professor Zhengfang Zhou, my dissertation advisor. All of my work has been done under his patient guidance and many valuable suggestions.

I would also like to express my appreciation to my committee members Professor D. R. Dunninger, Professor M. Frazier, Professor T. Parker and Professor C. Y. Wang for their time and valuable suggestions.

Finally I would like to thank my wife Jin-Sook Choi, daughter Sarah and son Sachan for sharing the difficulties while we were in East Lansing.

TABLE OF CONTENTS

CHAPTER 1

Introduction

Let $x = (x_1, \ldots, x_N)$ denote any point in R^N and let $\nabla u(x,t) = (u_{x_1}, \ldots, u_{x_N})$ be the usual gradient of u with respect to space. For notational simplicity we will use $\hat{\nabla} u(x,t)$ to denote $(u_{x_1}, \ldots, u_{x_N}, u_t)$. We consider the Cauchy problem for the degenerate parabolic p-Laplacian equation

$$u_t = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}), \quad \mathrm{x} \in \mathrm{R}^{\mathrm{N}}, \mathrm{t} > 0, \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x). \tag{1.2}$$

We will assume throughout this thesis that $p > 2$ and $u_0(x)$ is nonnegative, integrable, and of compact support. Equation (1.1) appears in a number of applications to describe the evolution of diffusion processes, in particular the flow of non-Newtonian fluids in a porous medium in which u stands for the density [20].

There are, in general, no classical solutions of (1.1) and (1.2) because of the degeneracy of the equation at points where $\nabla u = 0$. Nevertheless, it is well known [10] that the Cauchy problem has a unique weak solution $u \geq 0$ satisfying

(i) $u \in C([0,\infty), L^1(R^N)) \bigcap L^\infty(R^N \times [\tau,\infty))$ for any $\tau > 0$,

(ii) $u, \nabla u \in C^\alpha_{loc}$ for some $\alpha > 0$,

(iii) $u \in C^\infty(\Omega)$, where $\Omega = \{(x,t) \in R^N \times (0,\infty) | \nabla u \neq 0\}$.

Solutions of the Cauchy problem (1.1) and (1.2) have an important special property: finite propagation speed. If u_0 has compact support, so does $u(.,t)$ for each positive time t. Therefore there is an interface that separates the region where $u > 0$ from the region where $u = 0$. Let

$$\Omega = \{(x,t) \in R^N \times [0,\infty) \mid u(x,t) > 0\},$$

and $\Omega(\tau) = \Omega \bigcap \{t = \tau\} \subset R^N$. Then the interface is $\Gamma = \partial\Omega \bigcap \{t > 0\}$. The purpose of this thesis is to study the regularity of this interface Γ.

The regularity of Γ for this equation was studied by [9] and [20]. They proved that the interface is Lipschitz continuous for large times, and is globally Lipschitz if the initial data u_0 satisfies certain nondegeneracy conditions.

On the other hand, L.A. Caffarelli and N.J, Wolanski [6] have established the $C^{1,\alpha}$ regularity for the interface of the degenerate porous medium equation

$$\text{(1.3)} \qquad \begin{cases} u_t = \Delta(u^m), & m > 1, x \in R^N, t \in (0,\infty), \\ u(x,0) = u_0(x). \end{cases}$$

This equation is, at least formally, quite similar to (1.1). The idea of this thesis is to adapt the methods of L.A. Caffarelli and N.J, Wolanski to obtain $C^{1,\alpha}$ regularity

of the interface for the p-Laplacian equation (1.1).

Establishing $C^{1,\alpha}$ regularity is equivalent to showing that the normal direction $\nu(x,t)$ of the interface is a C^α function locally. The following important observation from [6] is a key.

Proposition 1.1 ([6]) Let v be a function in $B_2\times(-2,2)$ with a Lipschitz interface through $(0,0)$. If there exist positive constants J, S and a sequence of monotone cones $\hat{K}(\nu_k,\Theta_k)$ in $R^N\times R$ with axis in the direction of ν_k and aperture Θ_k such that $\Theta_k \geq (1-S)\Theta_{k-1}+S\frac{\pi}{2}$ and

$$D_\tau v(x,t)\geq 0 \quad \text{for } (x,t)\in B_{J^k}\times(-J^k,J^k), \tau\in\hat{K}(\nu_k,\Theta_k),$$

then the interface of v is $C^{1,\alpha}$ near $(0,0)$.

The idea of [6] is to start from a solution u of (1.3) and construct a new function v that has the same interface as u such that v satisfies the conditions of Proposition 1.1.

For the porous medium equation, the flux is given by $\nabla(u^m)$. Consequently the velocity of the particles is $\frac{m}{m-1}\nabla(u^{m-1})$. Naturally the velocity of the particles should tell us how the interface moves as time progresses. It is no surprise that a new function $v=\frac{m}{m-1}u^{m-1}$ was used in that case [6]. The interface for u is obviously the same as that for v. But v has better properties for the study of the regularity of the interface. For example, v is Lipschitz continuous while u is not. Furthermore, one can estimate the lower and upper bounds for $|\nabla v|$ and v_t near the interface by using a dilation argument: If $v(x,t)$ is a solution, so is $\frac{1}{h}v(hx,ht)$. Also after a suitable dilation, the equation for v becomes a nondegenerate equation near the boundary, so some estimates can be derived by the theory of nondegenerate parabolic equations. The most difficult and important step in [7] is obtaining that a positive lower bound on $|\nabla v|$ near the interface.

For the parabolic p-Laplacian, the regularity of its solutions is better than solutions of porous medium equation since u and ∇u are in C^α. One might expect that this alone might give better regularity of the interface for p-Laplacian. But a function $f \in C^\alpha$ with $\nabla f \in C^\alpha$ does not automatically have a $C^{1,\alpha}$ interface. For example, let g_1 and g_2 be two continuously differentiable functions on R^+ such that $g_1(t) > g_2(t)$ for all $t > 0$, $g_1'(t) > 1$ and $g_2'(t) < -1$ and $g_1'(t)$ is NOT Hölder continuous. Then

$$f(x,t) = \begin{cases} \min\{|x - g_1(t)|^{1+\alpha}, |x - g_2(t)|^{1+\alpha}\} & \text{if } g_2(t) < x < g_1(t), \\ 0 & \text{otherwise} \end{cases}$$

has an interface $x = g_1(t)$ that is C^1 but not $C^{1,\alpha}$.

It also should be pointed out that even though the solution u of Equation (1.3) is only C^α, its interface is $C^{1,\alpha}$ for suitable u_0. It is conjectured in [6] that the interface could even be smooth if $\Omega(0)$ is a smooth convex set and u_0 satisfies some extra technical conditions. One such example is given by Barenblatt solutions [10]. It was reported in the AMS Detroit meeting recently that C^∞ regularity of the interface for the porous medium equation has been established by Panagiota Daskalopoulous and Richard Hamilton.

For p-Laplacian, we will follow a similar strategy, taking v to be the function

$$v(x,t) = \frac{p-1}{p-2} u^{\frac{p-2}{p-1}}(x,t). \tag{1.4}$$

A straightforward computation shows that

$$v_t = \frac{p-2}{p-1} v \Delta_p v + |\nabla v|^p. \tag{1.5}$$

The motivation for this choice is that equation (1.5) has a nice dilation property and we can establish the lower and upper bounds for v_t and $|\nabla v|$ (this is done in chapter 2). The exponent $\frac{p-2}{p-1}$ comes from the fundamental solution(again, see chapter 2). This function v was also used in [9, 20].

Before we state our main result, let us make a definition to facilitate later statements.

Definition 1.1 *We say that a function $w : R^N \to R$ is in the class C^* if w satisfies the following conditions:*

(i) $D = \{x \mid w(x) > 0\}$ is a C^1 domain and $D \subset B_R$, where B_R denotes the ball of radius R centered at 0;

(ii) $w \in C^1(\overline{D})$;

(iii) There exist a strip $S \subset D$ along the boundary ∂D and positive constants k_1 and k_2 such that

$$k_1 \le |\nabla w| \le k_2 \quad \text{in } S;$$

(iv) There exists a constant $a > 0$ such that $w \ge a$ in $D \backslash S$;

(v) There exists a constant $k_0 > 0$ such that $(\partial_{ij}w) \geq -k_0 I$ in the sense of distributions.

The main result of this thesis is the following theorem.

Theorem 1.2 *Let $v_0 = \frac{p-1}{p-2}u_0^{\frac{p-2}{p-1}} \in C^*$, then there exists $T_0 \geq 0$ depending only on p, N and the constants in C^* with the following properties:*

(i) $u(x,t) > 0$ for all $(x,t) \in B_{1+R} \times [T_0, \infty)$; and

(ii) For any $T_0 < \underline{t} < t^0 < \bar{t}$, there is an $\alpha > 0$ depending only on $p, N, \underline{t}, \bar{t}$ and constants in C^* such that the interface near the point $(x^0, t^0) \in \Gamma$ is $C^{1,\alpha}$.

The thesis will be organized as follows. In Chapter 2, we will state and prove some preliminary results on the interface Γ by using a comparison principle. In particular, we will prove the Lipschitz continuity of Γ. In Chapter 3, we establish lower bounds for v_t and $|\nabla v|$ near the boundary by carefully studying the equation for v and by using the general theory on degenerate parabolic equations. Finally, Theorem 1.2 is proved in Chapter 4 by using the homogeneous structure of v and by constructing some lower and upper solutions.

CHAPTER 2

Preliminaries

In this chapter we state some preliminary results which will be useful later. Most of these were established in [9], [20]. Here we will summarize the relevant results and sketch some proofs; complete details can be found in [9, 20].

First, it is well known [20] that equation (1.1) has the following comparison principle.

Proposition 2.1 *Let u_1 and u_2 be two nonnegative solutions of (1.1) on S_T, where $S_T = R^N \times (0,T)$ with initial values $\varphi_1, \varphi_2 \in L^1(R^N)$ respectively. Then $\varphi_1 \leq \varphi_2$ on R^N implies $u_1 \leq u_2$ on S_T.*

PROOF. Let $\varphi_i^\delta = \varphi_i * \rho_\delta$, $i = 1,2$, where ρ_δ is a convolution kernel whose support is the ball of center 0 and radius δ, and let u_i^δ, $i = 1,2$ be the solutions of the following approximate problems of (1.1),(1.2)

$$\begin{cases} u_t = \operatorname{div}((|\nabla \mathrm{u}|^2 + \delta)^{\frac{p-2}{2}} \nabla \mathrm{u}), \\ u(x,0) = \varphi_i^\delta. \end{cases}$$

Then by the comparison principle for strongly parabolic equations, we have

$$u_1^\delta \le u_2^\delta \quad in \quad S_T. \tag{2.1}$$

By the uniqueness of solutions of (1.1),(1.2)

$$u_1 = \lim_{\delta\to 0} u_1^\delta, \quad u_2 = \lim_{\delta\to 0} u_2^\delta.$$

Hence Proposition 2.1 follows by letting $\delta \to 0$ in (2.1).

Remark **2.1** *The conclusion of Proposition 2.1 can be extended to a domain of the form* $\Omega \times (0,T)$ *with* $\partial\Omega$ *being smooth.*

There are two types of solutions which are frequently used for comparison functions for (1.1). One is a separable solution

$$u(x,t) = g(r)f(t) = (\frac{p-2}{p})^{\frac{p-1}{p-2}}[\frac{\lambda r^p}{T-t}]^{\frac{1}{p-2}} \tag{2.2}$$

for any $T > 0$, where $r = |x|$ is the usual norn in R^N, and

$$\lambda = \frac{1}{N(p-2)+p.} \tag{2.3}$$

The other is the Barenblatt solution

$$B_{k,\rho}(x,t,\bar{x},\bar{t}) = k\rho^N[S(t)]^{-\lambda N}[1-(\frac{|x-\bar{x}|}{S^\lambda(t)})^{p/(p-1)}]_+^{(p-1)/(p-2)}, \tag{2.4}$$

where λ is given in (2.3), and

$$S(t) = \frac{1}{\lambda}(\frac{p}{p-2})^{p-1}k^{p-2}\rho^{N(p-2)}(t-\bar{t}), \quad t \geq \bar{t}.$$

One can check easily that

$$B_{k,\rho}(x,\bar{t},\bar{x},\bar{t}) = M\delta(x-\bar{x})$$

where $\delta(x-\bar{x})$ as the Delta measure concentrated at $\bar{x}$ and

$$M = \int_{R^N} B_{k,\rho}(x,t,\bar{x},\bar{t})dx.$$

The next proposition gives the monotonicity of the support $\Omega(t)$.

Proposition 2.2 *The solution of (1.1) and (1.2) satisfies*

(2.5)
$$u_t \geq -\frac{u}{(p-2)t}$$

in the sense of distributions.

PROOF. Let $u_r(x,t) = ru(x, r^{p-2}t)$. Then u_r is a solution of (1.1) with initial value $u_r(x,0) = ru(x,0) = ru_0(x)$. Since $u_r(x,0) \leq u_0(x)$ for $r \in (0,1)$, by Proposition 2.1, we have $u_r(x,t) \leq u(x,t)$. It follows that

$$u(x,r^{p-2}t) - u(x,t) \leq (1-r)u(x,r^{p-2}t)$$

and therefore

$$\frac{u(x, r^{p-2}t) - u(x,t)}{(r^{p-2}-1)t} \geq \frac{r-1}{(1-r^{p-2})t} u(x, r^{p-2}t).$$

Letting $r \to 1^-$, we get

$$\lim_{s\to 0^-} \frac{u(x,t+s) - u(x,t)}{s} \geq \frac{-u(x,t)}{(p-2)t}.$$

Similarly, we can choose $r > 1$ and $r \to 1^+$ to obtain

$$\lim_{s\to 0+} \frac{u(x,t+s) - u(x,t)}{s} \geq \frac{-u(x,t)}{(p-2)t}$$

and the proposition is proved.

As a consequence of (2.5), the function $ut^{\frac{1}{(p-2)}}$ is non-decreasing for every $x \in R^N$. Hence $u(x,t^0) > 0$ and $t^1 > t^0$ imply $u(x,t^1) > 0$. Thus the support $\Omega(t)$ is monotonically increasing: $\Omega(t^0) \subset \Omega(t^1)$ whenever $t^1 > t^0$.

We also have the following monotonicity property for solutions of (1.1) and (1.2) for any fixed time t. The version for the porous medium equation was proved in [7].

Proposition 2.3 *Let R_0 be a positive number such that $\Omega(0) \subset B_{R_0}$. Then*

(1) For $x^1, x^2 \in R^N$ with $|x^1|, |x^2| > R_0$ and

$$\cos\langle x^2 - x^1, x^1\rangle \geq \frac{R_0}{|x^1|}, \tag{2.6}$$

$u(x^2,t) \leq u(x^1,t)$ for any positive t, where $\langle x^2 - x^1, x^1\rangle$ is the angle from the vector $x^2 - x^1$ to x^1;

(2) For every $r, t > 0$ we have

$$\inf_{B_r} u(x,t) \geq \sup_{\partial B_{r+2R_0}} u(x,t). \tag{2.7}$$

The proof of Proposition 2.3 requires the following Lemma.

Lemma 2.4 *Let $D = \text{supp} u_0(x)$ be compact and $D \subset \{x \in R^N | x_N > 0\}$. Then*

$$u(y, x_N, t) \geq u(y, -x_N, t) \quad for \quad all \quad y \in R^{N-1}, \quad x_N > 0, \quad t \in (0,T).$$

Proof. Set $v(x,t) = u(y, -x_N, t)$. Then $v(y,0,t) = u(y,0,t)$ for each $y \in R^{N-1}$ and $t \in (0,T)$, and

$$v(y, x_N, 0) = u(y, -x_N, 0) = 0 \leq u(y, x_N, 0)$$

for each $y \in R^{N-1}$ and $x_N > 0$. Hence by Remark 2.1, we obtain

$$u(y, -x_N, t) = v(y, x_N, t) \leq u(y, x_N, t) \quad in \quad R^{N-1} \times R^+ \times (0,T).$$

Proof of Proposition 2.3. Let H denote the hyperplane in R^N which bisects the line segment between x^1 and x^2 orthogonally. That is,

$$H = \{x \in R^N | (x, x^1 - x^2) = (\frac{1}{2}(x^1 + x^2), x^1 - x^2)\},$$

where $(\cdot,\cdot)$ denotes the usual scalar product in R^N. It is easy to verify that

$$\mathrm{dist}(H,0) = \frac{1}{2}\frac{|(x^1, x^1 - x^2) + (x^2, x^1 - x^2)|}{|x^2 - x^1|}.$$

Note that by (2.6),

$$(x^1, x^2 - x^1) = |x^1||x^2 - x^1|\cos\langle x^1, x^2 - x^1\rangle \geq R_0|x^2 - x^1|$$

and

$$(x^2, x^2 - x^1) \geq (x^1, x^2 - x^1) > R_0|x^2 - x^1|,$$

we have

$$\mathrm{dist}(H,0) > R_0.$$

Therefore x^1 and $\mathrm{supp} u_0(x)$ are in the same half-space with respect to H. Moreover, x^2 is the reflection of x^1 with respect to H. Thus by Lemma 2.4 and a suitable translation and a rotation,

$$u(x^1,t) \geq u(x^2,t).$$

For the second part, if $x^1 \in B_r$ and $x^2 \in \partial B_{r+2R_0}$, then by the same argument as above we have

$$\mathrm{dist}(H,0) \geq \frac{1}{2}\frac{(r+2R_0)^2 - r^2}{2(r+R_0)} \geq R_0.$$

Hence the same argument gives

$$u(x^1,t) \geq u(x^2,t).$$

Since x^1 and x^2 are arbitrary, (2.7) is established.

It follows immediately from (2.6) that $u(\cdot,t)$ is non-increasing along the ray $\{x = \lambda x^1, \lambda > 1\}$ if $|x^1| > R_0$, and the free boundary Γ can be represented by a spherical coordinates in the form

$$r = f(\theta, t) \quad for \quad \theta \in S^{N-1} \quad r > R_0.$$

It was also shown in [9] and [20] that the initial behavior of the interface is determined by the local properties of u_0 as follows.

Proposition 2.5 *Let*

(2.8) $$B(x) = \sup_{R>0}\left(R^{-N-\frac{p}{p-2}} \int_{B_R(x)} u_0(y)dy\right)$$

Then we have $u(x,t) > 0$ *for every* $t > 0$ *if and only if* $B(x) = \infty$. *Moreover, there exists a constant* $C = C(p,N) > 0$ *such that* $u(x,t) = 0$ *if*

$$0 < t < C[B(x)]^{2-p}.$$

PROOF. Suppose $B(x) = \infty$. From the Harnack principle (see Corollary 1 in [11]) we have that

$$R^{-N-\frac{p}{p-2}} \int_{B_R(x)} u_0(x)dx \le c\left(t^{-\frac{1}{p-2}} + t^{\frac{N}{p}} R^{-N-\frac{p}{p-2}} [u(x,t)]^{\frac{N(p-2)+p}{p}}\right).$$

Hence, if $u(x,t) = 0$ for some $t > 0$ then

$$B(x) \le ct^{-\frac{1}{p-2}};$$

this contradicts $B(x) = \infty$.

Now assume $B(x) < \infty$. From Theorem 1 in [12] we know that

$$\sup_{B_\rho} u(x,t) \le ct^{-\frac{N}{N(p-2)+p}} \rho^{\frac{p}{p-2}} [I_R(x)]^{\frac{p}{N(p-2)+p}}$$

for all $\rho > R > 0$ and $0 < t < T(R)$, where

$$I_R(x) = \sup_{\rho > R} \rho^{-N-\frac{p}{p-2}} \int_{B_\rho(x)} u_0(y)dy$$

and

$$T_R = c[B(x)]^{-p+2}.$$

Taking $R \to 0$ we are done.

In order to estimate the growth of the interface, we define the function

$$d(x) = \sup\{r > 0 | \int_{B_r(x)} u_0(y)dy = 0\}, \tag{2.9}$$

i.e., $d(x)$ measures the distance from x to the initial support of u. Since

$$B(x) \le |u_0|_{L^1(R^N)} (d(x))^{(-N+\frac{p}{p-2})},$$

Proposition 2.5 yields

Corollary 2.6 *Let $x \in R^N$ with $d(x) > 0$. Then*

$$u(x,t) = 0 \quad if \quad 0 \leq t \leq C_1(d(x))^{N(p-2)+p}|u_0|_{L^1}^{2-p},$$

where C_1 is a positive constant depending only on N and p.

Using Corollary 2.6, we see that if $\mathrm{supp} u_0 \subset B_{R_0}$, then for every $t > 0$, $\Omega(t)$ is contained in the ball $B_{R(t)}$ with

$$R(t) \leq C_2(|u_0|_{L^1}^{p-2}t)^{\frac{1}{N(p-2)+p}} + R_0, \tag{2.10}$$

which gives the upper bound of $\Omega(t)$. It turns out that this gives a very accurate information on the size of $\Omega(t)$. We set

$$R_M(t) = \sup\{|x| \mid x \in \Omega(t)\}, \quad R_m(t) = \inf\{|x| \mid x \in \partial\Omega(t)\}. \tag{2.11}$$

Proposition 2.7 *For the solution of (1.1)-(1.2), we have*

$$R_M(t) \leq R_m(t) + 2R_0, \tag{2.12}$$

$$R_M(t) \sim C(p,N,u_0)t^{\frac{1}{N(p-2)+p}}, \quad as \quad t \to \infty. \tag{2.13}$$

where $C(p,N,u_0)$ is a constant depending only on p, N and $|u_0|_{L^1}$.

Proof. Clearly the first inequality follows from part (2) of Proposition 2.3. Let

$$V(x,t,C) = t^{-\frac{1}{\mu}}C_0(p,N)\{C - (|x|t^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}$$

where $C > 0$ is a constant, $\mu = \frac{N(p-2)+p}{N}$,

$$C_0(p,N) = (\frac{p-2}{p})^{\frac{p-1}{p-2}}[N(p-2)+p]^{-\frac{1}{p-2}}.$$

Then $V(x,t,C)$ is a solution of (1.1) from (2.4). Without loss of generality, we may assume that $u_0(0) > 0$. Hence we can choose $t_0, C_1 > 0$ such that

$$t_0^{-\frac{1}{\mu}} C_0(p,N)\{C_1 - (|x| t_0^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}} \leq u_0(x).$$

It follows from Proposition 2.1 that

$$u(x,t) \geq (t+t_0)^{-\frac{1}{\mu}} C_0(p,N)\{C_1 - (|x|(t+t_0)^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}.$$

This implies the second part of Proposition 2.7.

Proposition 2.7 shows that if t is large, $\Omega(t)$ contains a ball of center 0 and radius of the order t^λ, $\lambda = \frac{1}{N(p-2)+p}$. Thus there exists a time $t(u_0) < \infty$ such that for $t > t(u_0)$ we have

$$\overline{B_{R_0}} \subset \Omega(t).$$

Let

$$T_0 = \inf\{\mathrm{t} > 0 \mid \overline{\mathrm{B}_{\mathrm{R}_0}} \subset \Omega(\mathrm{t})\}.$$

Then we have the following facts which were proven in [9] and [20].

Proposition 2.8 *The interface* Γ *can be represented as*

$$r = f(\theta, t) \quad for \quad t > T_0, \quad \theta \in S^{N-1},$$

where f is a Lipschitz continuous function of θ and t.

PROOF. First we can show that f is Lipschitz continuous with respect to θ. We know that there exists a large T_0 such that

$$B_{R_0}(0) \subset \Omega(t)$$

for all $t > T_0$. Then it follows from Proposition 2.3 that for every $(\overline{x}, t) \in \Gamma$, $|\overline{x}| > R > R_0$, $t > T_0$, we have $u(x,t) > 0$ for every x in a small cone K_ϵ of the form

$$K_\epsilon = \{x : |x - \overline{x}| < \epsilon \quad and \quad \cos\langle \mathrm{x} - \overline{\mathrm{x}}, \overline{\mathrm{x}} \rangle \leq -\frac{(1+\epsilon)\mathrm{R}}{|\overline{\mathrm{x}}|}\}$$

and $u(x,t) = 0$ in a cone K_ϵ' of the form

$$K_\epsilon' = \{x : |x - \overline{x}| < \epsilon \quad and \quad \cos\langle \mathrm{x} - \overline{\mathrm{x}}, \overline{\mathrm{x}} \rangle \geq \frac{\mathrm{R}}{|\overline{\mathrm{x}}|}\}.$$

Therefore, if $(x,t) \in \Gamma$ and $|x - \overline{x}| < \epsilon$, we have $x \in R^N - (K_\epsilon \cup K_\epsilon')$. This implies that f is Lipschitz continuous in θ.

To prove that $r = f(\theta, t)$ is Lipschitz continuous with respect to t, we need the following estimates which were proven in [9], [20].

Lemma 2.9 *There exists a constant $C(R_0, u_0, T_0) > 0$ such that*

$$|\nabla u(x,t)| \leq C(R_0, u_0, T_0) t^{-\frac{N+1}{N\mu}}$$

if $|x| \leq R_0, t \geq T_0$, where $\mu = \frac{N(p-2)+p}{N}$.

PROOF. We define a family of functions

$$u_k(x,t) = k^N u(kx, k^{N\mu}t) \quad k > 0.$$

Then u_k is a solution of (1.1) with initial value $u_k(x,0) = k^N u_0(kx)$. From the proof of Proposition 2.7, there exist constants $C_1, C_2, T_0 > 0$ such that

$$u(x,t) \le (t+T_0)^{-\frac{1}{\mu}} C_0(p,N)\{C_2 - (|x|(t+T_0)^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}$$

$$u(x,t) \ge (t+T_0)^{-\frac{1}{\mu}} C_0(p,N)\{C_1 - (|x|(t+T_0)^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}.$$

It follows that

$$u_k(x,t) \le k^N (k^{N\mu}t + T_0)^{-\frac{1}{\mu}} C_0(p,N)\{C_2 - (k|x|(k^{N\mu}t+T_0)^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}$$

$$u_k(x,t) \ge k^N (k^{N\mu}t + T_0)^{-\frac{1}{\mu}} C_0(p,N)\{C_1 - (k|x|(k^{N\mu}t+T_0)^{-\frac{1}{N\mu}})^{\frac{p}{p-1}}\}_+^{\frac{p-1}{p-2}}.$$

Thus there exist $t_2 > t_1 > 0, \alpha, \beta > 0$ independent of k such that if $t_1 \le t \le t_2, |x| \le R_0$, then

$$\alpha \le u_k \le \beta.$$

Hence by the interior estimates, we have

$$|\nabla u_k| \le C \quad if \quad t_1 \le t \le t_2, \quad |x| \le R_0$$

for some constant C independent of k. This means that

$$|\nabla u(x, k^{N\mu}t_1)| \leq Ck^{-(N+1)} \quad if \quad |x| \leq kR_0.$$

Putting $t = k^{N\mu}t_1$, we have

$$|\nabla u(x,t)| \leq C[\frac{t}{t_1}]^{-\frac{N+1}{N\mu}} \quad if \quad |x| \leq [\frac{t}{t_1}]^{\frac{1}{N\mu}} R_0.$$

This completes the proof.

By using the same family of solutions as in Lemma 2.9, we have the following estimate of sup$|\nabla$u$|$ in terms of u after a large time.

Corollary 2.10 *For every $\epsilon > 0$ there exists $T_1 = T_1(\epsilon, R_0, u_0)$ such that*

$$|\nabla u(x,t)| \leq \epsilon u(x,t)$$

for all $|x| \leq R_0$ and $t \geq T_1$.

PROOF. We consider again the family of solutions

$$u_k(x,t) = k^N u(kx, k^{N\mu}t)$$

as in Lemma 2.9. We know that u_k converge to a fundamental solution $\overline{u}$ uniformly with respect to $t \geq \tau$ for every τ(see Theorem 3 in [21]). Therefore there exists k_0 such that

$$c > 2\overline{u}(x,t) \geq u_k(x,t) \geq \frac{1}{2}\overline{u}(x,t) > \frac{1}{c} > 0$$

for $|x| \leq 2R_0$, $\frac{1}{2} \leq t \leq 2$, $k \geq k_0$, and for some c depending only on u_0, N and p. Since u_k are uniformly bounded in $R^N \times (\frac{1}{2}, 2)$, we obtain

$$|\nabla u_k| \leq c_1$$

for $t = 1, |x| \leq R_0$, and for some c_1, and this implies

$$|\nabla u(y, k^{N\mu}) \leq ck^{-(N+1)}$$

for all $|y| < R_0$. Note that $u(x, k^{N\mu}) \geq ck^{-N}$ for all $|x| \leq R_0$. Setting $t = k^{N\mu}$ we conclude that

$$|\nabla u(x,t)| \leq ct^{-\frac{1}{N\mu}} u(x,t)$$

if $|x| \leq R_0, t = k^{N\mu}$, and $k \geq k_0$. This completes the proof.

Next Lemma gives an estimate on u_t.

Lemma 2.11 *There exists a time $t_0 > 0$ such that $u_t \in L^\infty(R^N \times (\tau, \infty))$ for $\tau > t_0$ and on* $R^N \times (t_0, \infty)$

(2.14) $$2(t - t_0)u_t(x,t) + x \cdot \nabla u(x,t) - u(x,t) \leq 0.$$

PROOF. We consider the approximate problem of (1.1),(1.2)

(2.15) $$u_t = \operatorname{div}((|\nabla \mathrm{u}|^2 + \delta)^{\frac{p-2}{2}} \nabla \mathrm{u})$$

(2.16) $$u(x,0) = \varphi.$$

It is well known that (2.15),(2.16) has a solution u^δ with the property that for any $s > 0$, $u^\delta \in C^\infty(R^N \times (s,\infty)) \bigcap L^\infty(R^N \times (s,\infty))$ and

$$|u^\delta(x,t) - u(x,t)|_{C^1(K)} \to 0 \quad as \quad \delta \to 0,$$

for every compact subset K of S_T.

For every $\epsilon > 0$ we define a family of solutions of (2.15)

$$u^\delta_\epsilon(x,t) = \frac{1}{1+\epsilon} u^\delta((1+\epsilon)x, (1+\epsilon)^2 t + t_0),$$

where t_0 is a large time to be chosen below.

We want show first that for every $\epsilon > 0$ and $x \in R^N$

$$u^\delta_\epsilon(x,0) \le u^\delta_{\epsilon=0}(x,0) = u^\delta(x,t_0),$$

if t_0 is large enough. For this purpose we write the difference $u^\delta_\epsilon(x,0) - u^\delta(x,t_0)$ as

$$u^\delta_\epsilon(x,0) - u^\delta(x,t_0) = ((1+\epsilon)^{-1} - 1)u^\delta((1+\epsilon)x,t_0) + u^\delta((1+\epsilon)x,t_0),$$

$$\text{(2.17)} \qquad -u^\delta(x,t_0) = -\frac{\epsilon}{1+\epsilon} u^\delta((1+\epsilon)x,t_0) + u^\delta((1+\epsilon)x,t_0) - u^\delta(x,t_0).$$

Notice that Proposition 2.3 still holds for the solution u^δ. We have

$$u^\delta((1+\epsilon)x,t_0) \le u^\delta(x,t_0) \quad if \quad |x| > R_0.$$

Since $u^\delta \geq 0$, we conclude that

$$u_\epsilon^\delta(x,0) \leq u^\delta(x,t_0) \quad if \quad |x| \geq R_0.$$

In case $|x| < R_0$, by Lemma 2.9 for t_0 large enough we have

$$C > u(x,t_0) > \frac{1}{C}, \quad |\nabla u| \leq \frac{1}{2CR_0} \quad in \quad B_{R_0}(0).$$

It follows that there exists a constant C_1 such that

$$C_1 > u^\delta(x,t_0) > \frac{1}{C_1}, \quad |\nabla u^\delta| \leq \frac{1}{2C_1R_0} \quad in \quad B_{R_0}(0)$$

provided that δ is small enough. Therefore, since

$$|u^\delta((1+\epsilon)x,t_0) - u^\delta(x,t_0)| \leq \epsilon|x||\nabla u^\delta(\xi,t_0)| \leq \frac{\epsilon}{2C_1},$$

we obtain from (2.17)

$$u_\epsilon^\delta(x,0) \leq u_{\epsilon=0}^\delta(x,0).$$

Hence by the maximum principle, we conclude that

$$u_\epsilon^\delta(x,t) \leq u_{\epsilon=0}^\delta(x,t) \quad for \quad x \in R^N, \quad t > t_0.$$

Differentiation of $u_\epsilon^\delta(x,t)$ with respect to ϵ at $\epsilon = 0$ gives

$$u^\delta(x,t+t_0) - 2tu_t^\delta(x,t+t_0) - x \cdot \nabla u^\delta(x,t+t_0) \geq 0. \tag{2.18}$$

If we let $\delta \to 0$, then we obtain (2.14) for u. Combining (2.14) and Proposition 2.7, we obtain

$$u_t \in L^\infty(R^N \times (\tau, \infty)) \quad for \quad any \quad \tau > t_0.$$

Now we are ready to prove the remaining part of Proposition 2.8.

PROOF. We have already proved in that Γ can be represented as

$$r = f(\theta, t) \quad if \quad t > T_0,$$

and f is a Lipschitz continuous function of the variable θ. Thus it remains to prove that f is Lipschitz continuous in t uniformly in $\theta \in S^{N-1}$. Notice that by Proposition 2.3, $x \cdot \nabla u^\delta = r u_r^\delta < 0$ if $|x| > R_0$ and that $u_\delta \geq 0$. By (2.18), we have

$$\frac{d}{ds}\left(\frac{u^\delta(r,\theta,t)}{t - T_0}\right) \leq \frac{1}{2(t - T_0)}(r u_r^\delta + 2(t - T_0)u_t^\delta - u^\delta) \leq 0,$$

where $r = r_1 e^{s-t_1}$, $r_1 > R_0$, $t = T_0 + (t_1 - T_0)e^{s-t_1}$, $s \geq t_1 > T_0$ and u^δ is the solution of (2.15), (2.16). It follows that

$$\frac{u^\delta(r,\theta,t)}{t - T_0} \leq \frac{u^\delta(r_1,\theta,t_1)}{t_1 - T_0} \quad if \quad t > t_1$$

and

$$\frac{u(r,\theta,t)}{t - T_0} \leq \frac{u(r_1,\theta,t_1)}{t_1 - T_0} \quad if \quad t > t_1.$$

Hence if $u(r_1, \theta, t_1) = 0$, we get

$$f(\theta, t) \leq r_1 e^{s-t_1} = r_1 \frac{t - T_0}{t_1 - T_0} = f(\theta, t_1) \frac{t - T_0}{t_1 - T_0},$$

i.e.,

$$f(\theta, t) - f(\theta, t_1) \leq f(\theta, t_1) \frac{t - t_1}{t_1 - T} \quad if \quad t > t_1 > T > T_0.$$

Finally we define a family of solutions to (1.1) in $R^N \times (T_0, \infty)$:

$$u_\epsilon(x, t) = \frac{1}{(1+\epsilon)^{\frac{p-1}{p-2}}} u((1+\epsilon)x, (1+\epsilon)t + T)$$

for $\epsilon > 0$ and $T > T_0$. We want to show that for every $\epsilon \in (0, 1)$ and $x \in R^N$,

$$u_\epsilon(x, 0) \leq u(x, T).$$

To do this we write the difference $u_\epsilon(x, 0) - u(x, T)$ as

$$u_\epsilon(x, 0) - u(x, T) = \left[\frac{1}{(1+\epsilon)^{\frac{p-1}{p-2}}} - 1 \right] u((1+\epsilon)x, T) + u((1+\epsilon)x, T) - u(x, T).$$

If $|x| > R_0$, then from Proposition 2.3,

$$u((1+\epsilon)x, T) - u(x, T) \leq 0$$

and

$$u_\epsilon(x, 0) - u(x, T) \leq \left[\frac{1}{(1+\epsilon)^{\frac{p-1}{p-2}}} - 1 \right] u((1+\epsilon)x, T) \leq 0.$$

Now we consider the case $|x| \le R_0$. From Corollary 2.10, we have

$$u((1+\epsilon)x,T) - u(x,T) \le \epsilon|x||\nabla u(\xi x,T)| \le \epsilon R_0 \frac{1}{cR_0} u(x,T) \le \frac{\epsilon}{c} u(x,T)$$

with some constant c, where $\xi \in (1, 1+\epsilon)$. Hence we have

$$\frac{u_\epsilon(x,0) - u(x,T)}{\epsilon} \le 0$$

for all $\epsilon \in (0,1)$, and differentiating $u_\epsilon(x,t)$ with respect to ϵ we have

$$-\frac{p-1}{p-2} u(x,t+T) + x \cdot \nabla u(x,t+T) + t u_t(x,t+T) \le 0.$$

Replacing t by $t+T$ we obtain

$$(2.19) \qquad (t-T)u_t(x,t) \le \frac{p-1}{p-2} u(x,t) - x \cdot \nabla u(x,t).$$

Hence u_t is bounded. Now if h is a fixed positive constant such that $t - T \ge h$, (2.19) can be written in the form

$$\frac{d}{dt}[e^{-\frac{(p-1)t}{(p-2)h}} u(r_0 e^{\frac{t-t_1}{h}}, \theta, t)] \le 0$$

for $T + 1 \ge t > t_1 > T$ and θ fixed. Therefore, if $u(r_0, \theta, t_1) = 0$, then

$$u(r_0 e^{\frac{t-t_1}{h}}, \theta, t) = 0 \quad for \quad t > t_1.$$

This gives

$$f(\theta,t) \leq f(\theta,t_1)e^{\frac{t-t_1}{h}},$$

and we get for $T < t_1 < t \leq T+1$,

$$f(\theta,t) - f(\theta,t_1) \leq c_3 f(\theta,t_1)(t-t_1),$$

where $c_3 = c_3(h,T) = \frac{1}{h}e^{\frac{1}{h}(T-T_0+1)}$. This completes the proof of Proposition 2.8.

Next we state some results regarding the smoothness of $v = \frac{p-1}{p-2}u^{\frac{p-2}{p-1}}$. Note that v satisfies equation (1.5). It is shown that if v is a solution of (1.5) with an initial data $v_0 \in C^*$, then v_t and $|\nabla v|$ are bounded in $R^N \times (T,\infty)$ for some $T > 0$. Furthermore the following was established in [9], [20].

Proposition 2.12 *There exist constants $A, B > 0$ such that*

$$\frac{A-p}{p-1}v(x,t) + x \cdot \nabla v(x,t) + (At+B)v_t \geq 0 \tag{2.20}$$

in the sense of distributions.

PROOF. We consider a family of solutions

$$v^\epsilon(x,t) = \frac{(1+A\epsilon)^{\frac{1}{p-1}}}{(1+\epsilon)^{\frac{p}{p-1}}}v((1+\epsilon)x,(1+A\epsilon)t+B\epsilon). \tag{2.21}$$

The idea is to show that

$$v^\epsilon(x,t) \geq v(x,t)$$

for small ϵ, and then by differentiating v^ϵ with respect to ϵ we have

$$\frac{A-p}{p-1}v(x,t) + x \cdot \nabla v + (At+B)v_t \geq 0.$$

We approximate v_0 by

$$v_0^\delta = v_0 * \rho_\delta(x) + \delta^\alpha,$$

where $\rho_\delta(x)$ is a convolution kernel and α will be chosen later. Suppose that $v^\delta(x,t) = \frac{p-1}{p-2}(u^\delta(x,t))^{\frac{p-2}{p-1}}$ is the solution to

$$v_t^\delta = \frac{p-2}{p-1}v^\delta \mathrm{div}(|\nabla \mathrm{v}^\delta|^{\mathrm{p}-2}\nabla \mathrm{v}^\delta) + |\nabla \mathrm{v}^\delta|^{\mathrm{p}}$$

with $v^\delta(x,0) = v_0^\delta(x)$. We note that $v^\delta \geq \delta^\alpha > 0$ and $v^\delta \in C^\infty$. If there is no confusion, we omit δ in various expressions. If δ is sufficiently small, then

$$v(x,0) \geq \frac{a}{2} \quad in \quad \Omega_1 = \Omega \setminus S$$

and

$$|\nabla v(x,0)| \geq \frac{k_1}{4} \quad in \quad S.$$

In fact, this inequality is also true in a neighborhood $U_{c\delta}$ of $\partial\Omega$ of the form

$$U_{c\delta} = \{x \in R^N | \mathrm{dist}(\mathrm{x}, \partial\Omega) < \mathrm{c}\delta\}$$

for a constant $c \in (0, \frac{1}{2})$.

Now we consider several different regions.

(i) First we consider the region where $|\nabla v_0| > \frac{k_1}{4}$(in particular, $S \cup U_{c\delta}$). We have

$$\begin{aligned} I_\epsilon &\equiv \frac{1}{\epsilon}(v^\epsilon(x,0) - v(x,0)) \\ &\geq \frac{1}{\epsilon}\left[\left(1 + \frac{\epsilon}{2}\frac{A-p}{p-1}\right) v((1+\epsilon)x, B\epsilon) - v(x,0)\right] \end{aligned}$$

if ϵ is small. Hence from the mean value theorem,

$$I_\epsilon \geq \frac{1}{2}\frac{A-p}{p-1} v((1+\epsilon)x, B\epsilon) + Bv_t((1+\epsilon)x, \theta\epsilon) + x \cdot \nabla v(\xi, 0),$$

where $\theta \in (0, B)$ and ξ lies in the line segment from x to $(1+\epsilon)x$. If ϵ is small enough, then

$$v((1+\epsilon)x, B\epsilon) \cong v(x,0), \quad v_t((1+\epsilon)x, \theta\epsilon) \cong v_t(x,0), \quad \nabla v(\xi, 0) \cong \nabla v(x,0).$$

Therefore, there exists $c > 0$ depending only on N, p and v_0^δ such that

$$I_\epsilon \geq \frac{1}{2}\frac{A-p}{p-1} v(x,0) + Bv_t(x,0) + x \cdot \nabla v(x,0) - c\epsilon$$

for some c. Using the equation

$$v_t = \frac{p-2}{p-1} v \Delta_p v + |\nabla v|^p,$$

we get

$$I_\epsilon \geq \left(\frac{1}{2}\frac{A-p}{p-1} + \frac{p-2}{p-1}B\Delta_p v(x,0)\right) v(x,0) + B|\nabla v(x,0)|^p + x \cdot \nabla v(x,0) - c\epsilon$$

Since $|x| \leq R + \delta$ and $\Delta v_0 \geq -Nk_0$, we have

$$I_\epsilon \geq \left(\frac{1}{2}\frac{A-p}{p-1} - \frac{p-2}{p-1}BNk_0\right) v(x,0) + |\nabla v(x,0)| \left(B|\nabla v|^{p-1} - R - \delta\right) - c\epsilon.$$

If we choose A and B such that

$$B > \frac{4R}{k_1^{p-1}} \quad and \quad \frac{1}{2}\frac{A-p}{p-1} - \frac{p-2}{p-1}BNk_0 \geq 0,$$

then

$$I_\epsilon \geq \frac{k_1}{4}\left(\frac{Bk_1^{p-1}}{4} - R - \delta\right) - c\epsilon > 0$$

for small ϵ and δ.

(ii) Next, we consider the region $\Omega_1 = \Omega \setminus S$.

We only need to consider those points where

$$|\nabla v_0(x)| \leq \frac{k_1}{4}.$$

In this case we see that

$$I_\epsilon \geq \left(\frac{1}{2}\frac{A-p}{p-1} - \frac{p-2}{p-1}BNk_0\right)\frac{a}{2} - \frac{Rk_1}{4} - c\epsilon,$$

and if $\frac{A-p}{2(p-1)} - \frac{p-2}{p-1}BNk_0 \geq Rk_1/a$, then

$$I_\epsilon \geq 0$$

for small ϵ.

(iii) Next we consider the region $\Omega_3 = \{x \in R^N | \quad \text{dist}(\text{x}, \Omega) \geq \delta\}$.

In Ω_3, $v(x,0) = \delta^\alpha$. Since $v \geq \delta^\alpha$ from the maximum principle, we have

$$I_\epsilon \geq 0.$$

(iv) Finally, we consider the region $\Omega_4 = \{x \in R^N | c\delta \leq \text{dist}(\text{x}, \Omega) \leq \delta\}$ with $0 < c < 1$.

In this case we select a particular family of cutoff functions $\{\rho_\delta\}$ satisfying

$$\rho_\delta(x) = 0 \quad if \quad |x| \geq \delta,$$

$$\rho_\delta(x) = \rho_\delta(0) \quad if \quad |x| \leq \delta - \delta^{1+\gamma} \quad for \quad some \quad \gamma \in (0,1),$$

$$0 \leq \rho_\delta(x) \leq \rho_\delta(0), \quad \rho_\delta \in C^\infty.$$

Now suppose $\text{dist}(\text{x}, \Omega) \in (\delta - \delta^{1+\gamma}, \delta)$; then

$$\begin{aligned} |\nabla v_0^\delta(x)| &\leq \int_{B_\delta \cap \Omega} |\nabla v_0(y)| \rho_\delta(x-y) dy \\ &\leq ck_2 \int_{B_\delta \cap \Omega} \rho_\delta(x-y) dy. \end{aligned}$$

Now we observe that

$$\int_{B_\delta \cap \Omega} \rho_\delta(x-y)dy \le c\delta^{\gamma\frac{N+1}{2}}$$

and hence

$$|\nabla v_0^\delta(x)| \le ck_2\delta^{\gamma\frac{N+1}{2}}.$$

Thus

$$I_\epsilon \ge \left(\frac{1}{2}\frac{A-p}{p-1} - \frac{p-2}{p-1}BNk_0\right)\delta^\alpha - (R+2\delta)k_2\delta^{\gamma\frac{N+1}{2}} - c\epsilon,$$

In particular, if $0 < \frac{2\alpha}{N+1} < \gamma < 1$ and ϵ is small, then

$$I_\epsilon \ge 0.$$

Finally, we consider those points x such that

$$c\delta \le \mathrm{dist(x,\Omega)} \le \delta - \delta^{\gamma+1}.$$

Recall that

$$\nabla^2 v_0 \ge -k_0 I$$

in the sense of distributions. We know that

$$\begin{aligned} I_\epsilon \ \ge \ & \left[\frac{1}{2}\frac{A-p}{p-1} + \frac{p-2}{p-1}B|\nabla v|^{p-2}a_{ij}(\nabla v)v_{x_i x_j}\right]v_\delta \\ & + B|\nabla v(x,0)|^p - |x||\nabla v_\delta(x,0)| - c\epsilon, \end{aligned}$$

where

$$a_{ij}(\nabla v) = \delta_{ij} + (p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v(x,0)|^2}.$$

Since $v_0 \in W^{1,1}(S)$, we have that

$$\begin{aligned} |\nabla v(x,0)| &= \left|\int_{B_\delta\cap\Omega} \nabla v_0(y)\rho_\delta(x-y)dy\right| \\ &\geq k_1\int_{B_\delta\cap\Omega}\rho_\delta(x-y)dy \\ &\geq ck_1\delta^{\gamma\frac{N+1}{2}}. \end{aligned}$$

Also since $\nabla^2 v(x,0) \geq -k_0 I$, we have

$$\Delta v = \int_{B_\delta\cap\Omega}\Delta v_0(y)\rho_\delta(x-y)dy - \int_{\partial(B_\delta\cap\Omega)}(\nabla v_0\cdot\nu)\rho_\delta(x-y)d\sigma_y$$

and

$$\begin{aligned} |\Delta v| &\geq k_1\int_{B_\delta\cap\partial\Omega}\rho_\delta(x-y)d\sigma_y - Nk_0\int_{B_\delta\cap\Omega}\rho_\delta(x-y)dy \\ &\geq k_1\delta^{-1+\frac{\gamma(N-1)}{2}} - Nk_0\delta^{\frac{\gamma(N+1)}{2}}. \end{aligned}$$

Observe that at $x \in \partial\Omega$,

$$\nabla v_0 = c\nu.$$

Hence we obtain

$$(p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v|^2}\int_{B_\delta\cap\Omega} v_0(y)(\rho_\delta(x-y))_{x_ix_j}dy$$

$$
\begin{aligned}
&= (p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v|^2}\int_{B_\delta\cap\Omega}(v_0)_{x_ix_j}\rho_\delta(x-y)dy \\
&\quad -(p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v|^2}\int_{B_\delta\cap\partial\Omega}(v_0)_{x_i}\nu_j\rho_\delta(x-y)d\sigma_y \\
&\geq -ck_0\delta^{\gamma\frac{N+1}{2}}
\end{aligned}
$$

for some c. Therefore, combining all these together, we obtain

$$
\frac{p-2}{p-1}B|\nabla v|^{p-2}a_{ij}(\nabla v)v_{x_ix_j}v(x,0)-(R+2\delta)|\nabla v|
$$

$$
\begin{aligned}
\geq\ & \frac{p-2}{p-1}Bc\left[k_1\delta^{\gamma\frac{N+1}{2}}\right]^{p-2}\left(k_1\delta^{-1+\frac{\gamma(N-1)}{2}}-ck_0\delta^{\gamma\frac{N+1}{2}}\right)\delta^\alpha \\
& -(R+2\delta)\frac{k_1}{5}-c\epsilon.
\end{aligned}
$$

Hence for sufficiently small δ, we have

$$
I_\epsilon \geq Bck_1^{p-1}\delta^{-1+\alpha+\gamma(\frac{(N+1)(p-2)}{2}+\frac{N-1}{2})}+(R+2\delta)\frac{k_1}{5}-c\epsilon.
$$

Therefore if we choose $0<\alpha<1$ and γ so that

$$
\frac{2\alpha}{1+N}<\gamma<\frac{1-\alpha}{\frac{(N+1)p}{2}-\frac{N+3}{2}},
$$

then for sufficiently small ϵ compared to δ we get

$$
I_\epsilon \geq 0.
$$

This proposition immediately implies

Proposition 2.13 *Under the assumptions of Proposition 2.12, the function* $v(x(s), t(s))e^{\frac{A-p}{p-1}s}$ *is non-decreasing along the curves*

$$x(s) = x(0)e^s, \quad t(s) = \frac{1}{A}[(At(0) + B)e^{As} - B], \quad s \geq 0, \tag{2.22}$$

PROOF. Along these curves

$$x'(s) = x(s) \quad \textit{and} \quad t'(s) = At(s) + B.$$

Hence by (2.20) we have

$$\frac{d}{ds}\left[v(x(s), t(s))e^{\frac{A-p}{p-1}s}\right] = \left[\frac{A-p}{p-1}v + x \cdot \nabla v + (At + B)v_t\right] e^{\frac{A-p}{p-1}s} \geq 0.$$

Remark **2.2** *The proof of Proposition 2.12 shows that if we choose a particular origin* x^* *of coordinates such that* $D \subset B_R(x^*)$*, then the curves (2.22) can be written as*

$$x(s) = (x(0) - x^*)e^s + x^*, \quad t(s) = \frac{1}{A}[(At(0) + B)e^{As} - B].$$

We also have the following estimate on Γ from Proposition 2.13.

Corollary 2.14 *If* $r = f(\theta, t)$ *is the equation of the free boundary for* $t > T_0$*, then*

$$f(\theta, t) \geq f(\theta, T_0)\left(\frac{At + B}{AT_0 + B}\right)^{\frac{1}{A}}.$$

PROOF. The curves $(x(s), t(s))$ above can be written as

$$x(s) = x(0)e^s = x(0)\left(\frac{At+B}{AT_0+B}\right)^{\frac{1}{A}}.$$

Hence if $\theta = \frac{x(0)}{|x(0)|}$ and $r_0 = |x(0)|$, then $v(r_0, \theta, T_0) > 0$ implies

$$v\left(r_0\left(\frac{At+B}{AT_0+B}\right)^{\frac{1}{A}}, \theta, t\right) > 0.$$

Therefore we obtain

$$f(\theta, t) \geq r_0\left(\frac{At+B}{AT_0+B}\right)^{\frac{1}{A}}$$

for any r_0 such that $v(r_0, \theta, T_0) > 0$. Consequently

$$f(\theta, t) \geq f(\theta, T_0)\left(\frac{At+B}{AT_0+B}\right)^{\frac{1}{A}}.$$

Proposition 2.15 *Let $v_0 \in C^*$. Then for every point $(\overline{x}, \overline{t}) \in S_T$ there exist positive constants $A, B, C > 0$ depending only on $v_0, N, p, \overline{t}$ and $R_1 = \sup\{\mathrm{dist}(\overline{x}, y) | y \in D\}$ such that*

$$v(x,t) \geq v(\overline{x}, \overline{t})e^{-C(t-\overline{t})} \tag{2.23}$$

for every (x, t) satisfying $\overline{t} < t < \overline{t} + \epsilon$ for some $\epsilon = \epsilon(A, B)$ and

$$\frac{|x - \overline{x}|}{|t - \overline{t}|} \leq \frac{R_1}{A\overline{t} + B}.$$

PROOF. Let us take A, B and $C = \frac{A-p}{p-1}$ as in Proposition 2.12 with $R = 3R_1$. Let

$$U = \{(x,t) | \frac{|x-\overline{x}|}{t-\overline{t}} \leq \frac{R_1}{A\overline{t}+B}\}.$$

We first prove that for every $(x,t) \in U$, there exists a point $y \in R^N$ such that

$$x - y = (\overline{x} - y)\left(\frac{At+B}{A\overline{t}+B}\right)^{\frac{1}{A}} \tag{2.24}$$

and Ω is contained in the ball $B_R(y)$. Clearly, for every (x,t) there exists a point $y \in R^N$ such that (2.24) holds. We now prove that $D \subset B_R(y)$. By (2.24), if $(x,t) \in U$ then we have

$$\begin{aligned} |\overline{x} - y|\left(\frac{At+B}{A\overline{t}+B}\right)^{\frac{1}{A}} &\leq |x-\overline{x}| + |\overline{x}-y| \\ &\leq R_1\frac{t-\overline{t}}{A\overline{t}+B} + |\overline{x}-y|. \end{aligned}$$

Hence we have

$$|\overline{x} - y|\left[\left(\frac{At+B}{A\overline{t}+B}\right)^{\frac{1}{A}} - 1\right] \leq R_1\frac{t-\overline{t}}{A\overline{t}+B}.$$

Writing $\left(\frac{At+B}{A\overline{t}+B}\right)^{\frac{1}{A}}$ as a power of $t - \overline{t}$ and using the fact that ϵ is small, we get

$$\frac{1}{2}|\overline{x} - y|\frac{t-\overline{t}}{A\overline{t}+B} \leq R_1\frac{t-\overline{t}}{A\overline{t}+B}.$$

Thus $|\overline{x} - y| \leq 2R_1$, and this gives $|y - z| \leq 3R_1 = R$ for every $z \in D$. Notice that the curve (2.24) can be written in the form

$$x(s) = (\overline{x} - x^*)e^s + x^*, \quad t(s) = \frac{1}{A}[(A\overline{t} + B)e^{As} - B]$$

and it passes through $(\overline{x}, \overline{t})$. Thus (2.23) follows from Proposition 2.13.

Theorem 2.16 *Let $v_0 \in C^*$. Then the interface Γ of (1.1), (1.2) can be written as $t = S(x)$ for $x \in R^N \setminus \overline{D}$ and S is a Lipschitz continuous function.*

PROOF. We first show that Γ can be written as $t = S(x)$. If the assertion is not true, then there exists two points $(x^0, t^1), (x^0, t^2) \in \Gamma$ with $t^1 < t^2 < t^1 + \epsilon$. Then there exists a point $x^1 \in R^N$ such that $v(x^1, t^2) > 0$ and

$$(x^0, t^2) \in \{(x, t) | \frac{|x - x^1|}{t - t^1} \leq \frac{R_1}{At_1 + B}, \quad t^1 < t < t^1 + \epsilon\},$$

where $R_1 = \sup\{\mathrm{dist}(x^1, y) | y \in D\}$. Hence by Proposition 2.15, we have

$$v(x^0, t^2) \geq v(x^1, t^1)e^{-C(t^2 - t^1)},$$

which contradicts to the fact that $(x^0, t^2) \in \Gamma$. Thus Γ can be written as $t = S(x)$ for $x \in R^N \setminus \overline{D}$. We now prove that $t = S(x)$ is Lipschitz continuous.

Let $(x^0, t^0) \in \Gamma$. Then by Proposition 2.15, we have

$$v(x, t) = 0 \quad if \quad |x - x^0| \leq \frac{R_1}{At + B}|t - t^0|, \quad t^0 - \epsilon < t < t^0,$$

and

$$v(x,t) > 0 \quad if \quad |x - x^0| < \frac{R_1}{At^0 + B}|t - t^0|, \quad t^0 < t < t^0 + \epsilon,$$

Thus, $t = S(x)$ is a Lipschitz continuous function.

CHAPTER 3

Nondegeneracy of ∇v near the interface

This chapter will establish some nondegeneracy results by using the preliminaries in the previous section. First we have the following result which is similar to Lemma 3.3 of [7], and shows the non-degeneracy at the free boundary Γ.

Proposition 3.1 *Let* $v_0 \in C^*$*. Suppose for simplicity that* $\mathrm{supp} v_0 \supset B_1$*. Let* $(x^0, t^0) \in \Gamma$ *and* $|x^0| > 1$*. Then there exists a constant c depending on* $p, N, |x_0|, t^0$*, A and B of Proposition 2.12 such that*

$$\sup\{v(x,t) \mid |x - x^0| < h, \quad 0 \le t - t^0 \le t(x^0)h\} > ch \tag{3.1}$$

where $t(x^0) = \frac{\lambda}{2^p c^{p-1}} (\frac{p-1}{p})^{p-1} (1 - |x^0|^{-2})^{p/2}$.

PROOF. We will use the comparison principle to prove this proposition. Let $x^1 = (1+\frac{h}{2|x^0|})x^0$. Then for $x \in \Omega(t^0)$ we have

$$|x - x^1| \geq \frac{h}{2}(1 - |x^0|^{-2})^{1/2}. \tag{3.2}$$

In fact , if $|x| \leq 1$, then we obviously have

$$|x^1 - x| \geq |x_1| - |x| \geq \frac{h}{2} \geq \frac{h}{2}(1 - |x^0|^{-2})^{1/2}.$$

For $|x| > 1$, $x \in \Omega(t^0)$ implies that $v(x, t^0) > 0 = v(x^0, t^0)$. Hence by Proposition 2.3

$$\frac{1}{|x^0|} > \cos\langle \mathrm{x} - \mathrm{x}^0, \mathrm{x}^0\rangle, \mathrm{i.e.}, (\mathrm{x}^0, \mathrm{x} - \mathrm{x}^0) = |\mathrm{x}^0||\mathrm{x} - \mathrm{x}^0|\cos\langle \mathrm{x} - \mathrm{x}^0, \mathrm{x}^0\rangle \leq |\mathrm{x} - \mathrm{x}^0|.$$

Therefore

$$\begin{aligned}
|x - x^1|^2 &= |x - x^0 - \frac{h}{2}\frac{x^0}{|x^0|}|^2 \\
&= |x - x^0|^2 + \frac{h^2}{4} - h\frac{(x - x^0, x^0)}{|x^0|} \\
&\geq |x - x^0|^2 + \frac{h^2}{4} - h\frac{|x - x^0|}{|x^0|} \\
&= (|x - x_0| - \frac{h}{2|x^0|})^2 + \frac{h^2}{4}[1 - \frac{1}{|x^0|^2}] \\
&\geq \frac{h^2}{4}(1 - \frac{1}{|x^0|^2}).
\end{aligned}$$

which implies (3.2).

Now set

$$w(x,t) = \frac{p-1}{p}\left(\frac{\lambda|x-x^1|^p}{\frac{\lambda}{\alpha}-(t-t^0)}\right)^{\frac{1}{p-1}}$$

with

$$\alpha = \frac{2^p c^{p-1} h^{-1} (\frac{p}{p-1})^{p-1}}{(1-|x_0|^{-2})^{p/2}},$$

where c is a constant we will choose later. Then we know from (2.2) that w is a non-negative solution of

$$w_t = \frac{p-2}{p-1} w\Delta_p w + |\nabla w|^p, \quad 0 < t - t^0 < \frac{\lambda}{\alpha}.$$

We will use w as a comparison function. For $x \in \Omega(t^0)$,

$$\begin{aligned} w(x,t^0) &= \frac{p-1}{p}(\alpha|x-x^1|^p)^{\frac{1}{p-1}} \\ &\geq \frac{p-1}{p}\alpha^{\frac{1}{p-1}}(\frac{h}{2}(1-|x^0|^{-2})^{\frac{1}{2}})^{\frac{p}{p-1}} \\ &= \frac{p-1}{p}\{\frac{2^p c^{p-1}}{h}(\frac{p}{p-1})^{p-1}\}^{1/(p-1)} \\ &\quad \times \frac{h^{\frac{p}{p-1}}}{(1-|x^0|^{-2})^{\frac{p}{2(p-1)}}} \frac{1}{2^{\frac{p}{p-1}}}(1-|x^0|^{-2})^{\frac{p}{2(p-1)}} \\ &= ch. \end{aligned}$$

If $v(x,t^0) \leq ch$ for all $x \in \Omega(t^0) \bigcap B(x^0,h)$, we have $v(x,t^0) \leq w(x,t^0)$ for $x \in B(x^0,h)$.

When $|x-x^0| = h$,

$$|x-x^1| \geq |x-x^0| - |x^0-x^1| = \frac{h}{2} \tag{3.3}$$

and

$$
\begin{aligned}
w(x,t) &\geq \frac{p-1}{p}\left(\frac{\lambda(\frac{h}{2})^p}{\frac{\lambda}{\alpha}-(t-t^0)}\right)^{\frac{1}{p-1}} \\
&\geq \frac{p-1}{p}\left(\alpha(\frac{h}{2})^p\right)^{\frac{1}{p-1}} \\
&= \frac{p-1}{p}\left(\frac{2^p c^{p-1} h^{-1} p^{p-1}}{(p-1)^{p-1}(1-|x^0|^{-2})^{p/2}}(\frac{h}{p})\right)^{\frac{p}{p-1}} \geq ch.
\end{aligned}
$$

Hence if $v(x,t) \leq ch$ on $\{(x,t) \mid |x-x^0| = h, 0 < t-t^0 < \frac{\lambda}{\alpha}\} \cup \{(x,t^0) \mid |x-x^0| \leq h\}$, then we have $v(x,t) \leq w(x,t)$ in that set. Therefore the comparison principle concludes that

$$
(3.4) \qquad v(x,t) \leq w(x,t), \quad |x-x^0| \leq h, \quad 0 \leq t-t^0 < \frac{\lambda}{\alpha}.
$$

Since $w(x^1,t) = 0$ for $0 \leq t-t^0 < \frac{\lambda}{\alpha}$, we should have $v(x^1,t) = 0$ for $0 \leq t-t^0 < \frac{\lambda}{\alpha}$. Next, from Proposition 2.13 if $r = f(\theta,t)$ is the representation of Γ, then we have

$$
f(\theta_0,t) \geq f(\theta_0,t^0)\left(\frac{At+B}{At^0+B}\right)^{\frac{1}{A}} \quad (t > t^0).
$$

Let $t = t^0 + \frac{\lambda}{\alpha} = t^0 + \beta h$ with $\beta = \frac{\lambda}{\alpha h} = \frac{\lambda}{2^p c^{p-1}(\frac{p}{p-1})^{p-1}}(1-|x^0|^{-2})^{\frac{p}{2}}$

Then

$$
\frac{At+B}{At^0+B} = 1 + \frac{A}{At^0+B}(t-t^0) = 1 + \frac{A\beta}{At^0+B}h.
$$

Let $g(h) = 1 + \frac{A\beta}{At^0+B}h$ and $s(h) = (1+\frac{h}{2|x^0|})^A$. Then $g(0) = s(0) = 1$ and

$$
g'(h) = \frac{A\beta}{At^0+B}, \quad s'(h) = \frac{A}{2|x^0|}(1+\frac{h}{2|x^0|})^{A-1}
$$

Hence

$$\begin{aligned} g'(h) - s'(h) &= A\{\frac{\beta}{At^0 + B} - (1 + \frac{h}{2|x^0|})^{A-1}\frac{1}{2|x^0|}\} \\ &\geq A\{\frac{\beta}{At^0 + B} - [1 + \frac{2|x^0|}{2|x^0|}(2^{\frac{1}{A-1}} - 1)]^{A-1}\frac{1}{2|x^0|}\} \\ &= A\{\frac{\beta}{A|t^0| + B} - \frac{1}{|x^0|}\} \geq 0. \end{aligned}$$

if $h < 2|x^0|(2^{\frac{1}{A-1}} - 1)$ and $\beta \geq \frac{A|t^0|+B}{|x^0|}$. Therefore when $t = t^0 + \frac{\lambda}{\alpha}$,

$$f(\theta_0, t) > f(\theta_0, t^0)[g(h)]^{\frac{1}{A}} \geq f(\theta_0, t^0)[s(h)]^{\frac{1}{A}} = |x^0| + \frac{h}{2}. \tag{3.5}$$

Since $x^1 = (1 + \frac{h}{2|x^0|})x^0 = (|x^0| + \frac{h}{2})\theta_0, \quad \theta_0 = \frac{x^0}{|x^0|}$, we should have

$$v(x^1, t^0 + \frac{\lambda}{\alpha}) > 0, \tag{3.6}$$

which contradicts to our previous estimate (3.4). Therefore

$$\sup_{\substack{|x - x^0| \leq h \\ 0 \leq t - t^0 \leq t(x^0)h}} v(x, t) > ch$$

if $h < 2|x^0|(2^{\frac{1}{A-1}} - 1)$ and c is small enough.

Note that the only condition on c is that

$$\begin{aligned} \beta &= (\frac{p-1}{p})^{p-1}\frac{\lambda(1 - |x^0|^{-2})^{p/2}}{2^p}\frac{1}{c^{p-1}} \\ &> \frac{At_0 + B}{|x^0|} \end{aligned}$$

i.e.,

$$c < \frac{\lambda^{\frac{1}{p-1}}(1-|x^0|^{-2})^{\frac{p}{2(p-1)}}}{[(At^0+B)2^p|x^0|]^{\frac{1}{p-1}}}\frac{p-1}{p}$$

Therefore c can be chosen independent of h and (x,t) in regions of the form

$$|x| \geq R, \quad t \leq \bar{t}$$

and Proposition 3.1 will be true for $h \leq 2R(2^{\frac{1}{A-1}}-1)$.

Actually we have the following stronger version of Proposition 3.1.

Corollary 3.2 *Let $v_0 \in C^*$ and (x^1,t^1) be such that*

$$|x^1| \geq R > 1, \quad T_0 < \underline{t} \leq t^1 \leq \bar{t}.$$

If $f(\frac{x^1}{|x^1|},t^1) - |x^1| = h$ with $h \leq min\{1, 2R(2^{\frac{1}{A-1}}-1)\}$, then we have

$$v(x^1,t^1) \geq \bar{c}h, \tag{3.7}$$

where $\bar{c}$ depends only on $A, B, R, \bar{t}, \underline{t}, p$ and N.

PROOF. The idea first is to find a point (x^0,t^0) such that $v(x^0,t^0) = 0$ with $T_0 < t^0 < t^1$, such that $\frac{x^0}{|x^0|} = \frac{x_1}{|x_1|}$ and estimate the difference between $|x_0|$ and $|x^1|$. Then Proposition 3.1 concludes that $v(x^2,t^2) > \bar{c}h$ for some point in a small neighborhood of (x^0,t^0). Finally we will use Proposition 2.3 and Proposition 2.12 to see that $v(x^1,t^1) \geq \bar{c}v(x^2,t^2)$.

Let $\bar{\alpha} = \frac{1}{2}min(1, \underline{t} - T_0)$ and let c be the constant in Proposition 3.1 for the region $|x| \geq R$ and $t \leq \bar{t}$. Let us choose a constant k large enough so that

$$t^0 = t^1 - \lambda_1(1 - \frac{1}{R_M(\bar{t})^2})\frac{h}{k} \geq t^1 - \bar{\alpha}, \quad \lambda_1 = \frac{\lambda}{2^p c^{p-1}}(\frac{p-1}{p})^{p-1},$$

where $R_M(\bar{t})$ is given (2.11) and is estimated in Proposition 2.6 in terms of $\bar{t}$ and $\|u_0\|_{L^1}$. Note that k can be chosen independent of $h \leq 1$. Let now x^0 be such that $(x^0, t^0) \in \Gamma$ and $\frac{x^0}{|x^0|} = \frac{x_1}{|x_1|} = \theta_1$. That is, $|x_0| = f(\theta_1, t_0)$. Then we have $|x^0| - |x^1| \geq \frac{h}{k}$ if k is large enough independently of h. In fact, since $|x^1| = f(\theta_1, t^1) - h$, we have

$$|x^0| - |x^1| = f(\theta_1, t^0) - f(\theta_1, t^1) + h.$$

Note that f is non-decreasing in t from Proposition 2.13,we see that $|x^0| \leq 2|x^1|$ for $h \leq 1 \leq |x^1|$.

Recall Proposition 2.8 concludes that

$$f(\theta_1, t^1) - f(\theta_1, t^0) \leq \begin{cases} \frac{f(\theta_1, t^0)}{t^0 - T_0}(t^1 - t^0) & \text{if } t^0 > T_0 \\ c_3 f(\theta_1, t^0)(t^1 - t^0) & \text{if } T_0 + 1 \geq t^0 \geq \underline{\underline{t}} \end{cases}$$

where $\underline{\underline{t}} = \underline{t} - \frac{\underline{t} - T_0}{2}, c_3 = c_3(\underline{\underline{t}}, T_0)$.

(i) when $T_0 + 2\bar{\alpha} < t^1$, we have

$$\begin{aligned} |x^0| - |x^1| &= f(\theta_1, t^0) - f(\theta_1, t^1) + h \\ &\geq h - \frac{f(\theta_1, t^0)}{t^0 - T_0}(t^1 - t^0) \\ &\geq h - \frac{2|x^1|}{t^0 - T_0}(t^1 - t^0) \end{aligned}$$

$$
\begin{aligned}
&\geq h - \frac{2|x^1|}{t^1 - (T_0 + \overline{\alpha})}(t^1 - t^0) \\
&\geq h - \frac{2R(\overline{t})}{\overline{\alpha}}\lambda_1(1 - R(\overline{t})^{-2})\frac{h}{k} \\
&= h - \gamma\frac{h}{k} \geq \frac{h}{k}
\end{aligned}
$$

if $1 - \frac{\gamma}{k} \geq \frac{1}{k}$, where

$$
\gamma = \lambda_1 \frac{2R(\overline{t})}{\overline{\alpha}}(1 - R(\overline{t})^{-2}).
$$

(ii) when $t_1 \leq T_0 + 2\overline{\alpha} \leq T_0 + 1$, we have $t^0 \leq T_0 + 1$ and $t^0 \geq t^1 - \overline{\alpha} \geq \underline{t} - \frac{1}{2}(\underline{t} - T_0) = \underline{\underline{t}}$,

$$
\begin{aligned}
|x^0| - |x^1| &\geq h - [f(\theta_1, t^1) - f(\theta_1, t^0)] \\
&\geq h - c_3 f(\theta_1, t^0)(t^1 - t^0) \\
&\geq h - 2c_3|x^1|(t^1 - t^0) \\
&= h - 2c_3\lambda_1(1 - R(\overline{t})^{-2})\frac{h}{k}|x^1| \\
&\geq h - 2c_3\lambda_1 R(\overline{t})(1 - R(\overline{t})^{-2})\frac{h}{k} \\
&\geq h - \delta\frac{h}{k} \geq \frac{h}{k}
\end{aligned}
$$

if $1 - \frac{\delta}{k} \geq \frac{1}{k}$, where

$$
\delta = 2c_3\lambda_1 R(\overline{t})(1 - R(\overline{t})^{-2}).
$$

Both (i) and (ii) are satisfied for k large enough independent of h. As $|x^0| \geq |x^1| \geq R$ and $t^0 \leq t^1 \leq \overline{t}$, the constant c works for (x^0, t^0). Now we see that the ball of center x^0 and radius $\frac{h}{k}(1 - |x^1|^{-2})^{1/2}$ is contained in the cone with vertex

x^1 of directions in which v is decreasing. That is

$$(3.8) \qquad |x-x^0|^2 \le (\frac{h}{k})^2(1-|x^1|^{-2}) \quad implies \quad (x^1, x-x^1) \ge |x-x^1|.$$

In fact, if $(x^1, x-x^1) < |x-x^1|$, then

$$\begin{aligned} |x-x_0|^2 &= |x-x^1-\mu\frac{x^1}{|x^1|}|^2 \\ &> |x-x^1|^2+\mu^2-2\mu\frac{|x-x^1|}{|x^1|} \\ &= F(|x-x^1|) \end{aligned}$$

where $F(\lambda) = \lambda^2 - \frac{2\mu}{|x^1|}\lambda + \mu^2$ with $\mu = |x^0| - |x^1| \ge \frac{h}{k}$. Hence

$$|x-x^0|^2 \ge minF(\lambda) = F(\frac{\mu}{|x^1|}) = \mu^2(1-|x^1|^{-2}) \ge (\frac{h}{k})^2(1-|x^1|^{-2}).$$

Therefore (3.8) holds. Let now

$$\overline{h} = \frac{h}{k}(1-R^{-2})^{\frac{1}{2}} \le \frac{h}{k}(1-|x^1|^{-2})^{\frac{1}{2}} \le \frac{h}{k}$$

and let us apply proposition 3.1 in the cylinder

$$|x-x^0| \le \overline{h}, \quad 0 \le t-t^0 \le \lambda_1(1-|x^0|^{-2})^{\frac{p}{2}}\overline{h}.$$

Then we have a point (x^2, t^2) in that cylinder for which $v(x^2, t^2) > c\overline{h}$. Since

$$t^2 - t^0 \le \lambda_1(1-|x^0|^{-2})^{\frac{p}{2}}\overline{h} \le \lambda_1(1-R_M(\overline{t})^{-2})\frac{h}{k} = t^1 - t^0,$$

and $|x^0| = f(\theta_1, t^0) \le f(\theta_1, \bar{t}) \le R_M(\bar{t})$, we have $t^2 \le t^1$, and $|x^2| \ge |x^0| - |x^2 - x^0| \ge |x^0| - \bar{h} \ge |x^1| > R > 1$.

Recall that Proposition 2.3 implies that

$$x \cdot \nabla v = r \frac{\partial v(x^2, t)}{\partial r} \le 0.$$

Hence Proposition 2.12 concludes that $\frac{\partial}{\partial t}(v(x^2, t)e^{\frac{A-p}{p-1}t}) \ge 0$. Hence

$$\begin{aligned} v(x^2, t^1) &\ge v(x^2, t^2)e^{-\frac{A-p}{p-1}(t^1 - t^2)} \\ &\ge v(x^2, t^2)e^{\frac{A-p}{p-1}(\bar{t} - \underline{t})} \ge \bar{c}h. \end{aligned}$$

Since x^2 belongs to the cone with vertex x^1 of directions where v is decreasing, we have

$$v(x^1, t^1) \ge v(x^2, t^1) \ge \bar{c}h,$$

where $\bar{c}$ depends only on $A, p, N, R, \bar{t}$ and $\underline{t}$.

The Hölder continuity of gradient of solutions of (1.1) and the following general class of quasilinear equations

$$u_t - \mathrm{div}\mathbf{A}(\mathrm{x}, \mathrm{t}, \nabla \mathrm{u}) = \mathbf{B}(\mathrm{x}, \mathrm{t}, \mathrm{u}, \nabla \mathrm{u}) \quad \text{in} \quad \Omega_{\mathrm{T}}, \tag{3.9}$$

was discussed in [8],[10] and [24], where the functions

$$\mathbf{A} : \Omega_T \times R^N \to R,$$

$$\mathbf{B} : \Omega_T \times R \times R^N \to R,$$

satisfy the structure conditions:

(S_1) $C_0|Du|^p - \varphi_0 \le \mathbf{A} \cdot Du \le C_1|Du|^p + \varphi_1,$

(S_2) $\frac{\partial \mathbf{A}}{\partial u_{x_k}} u_{x_k x_j} \cdot Du_{x_j} \ge C_0 |Du|^{p-2} \sum_{j=1}^N |Du_{x_j}|^2 - \varphi_0,$

(S_3) $\sum_{k=1}^N |\frac{\partial \mathbf{A}}{\partial u_{x_k}}| \le C_1 |Du|^{p-2} + \varphi_1,$

(S_4) $\sum_{k=1}^N |\frac{\partial \mathbf{A}}{\partial x_k}| \le C_1 |Du|^{p-1} + \varphi_2,$

(S_5) $|\mathbf{B}| \le C_1|Du|^{p-1} + \varphi_2$, where $C_i, i = 0, 1$ are given positive constants and $\varphi_i, i = 0, 1, 2$ are given non-negative functions satisfying

(S_6) $\varphi_0 + \varphi_1^{\frac{p}{p-1}} + \varphi_2^2 \in L^q_{loc}(\Omega_T), \quad q > \frac{N+2}{2}.$

It was shown that for any compact subset K of Ω_T, the Hölder coefficient and exponent of ∇u depend only on K, p and N. In particular from [10] we have

Proposition 3.3 *(The degenerate case $p > 2$). Let u be a weak solution in Ω_T of (3.9). Then there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of N and p such that, for every compact subset K of Ω_T,*

$$|\nabla u(x^0, t^0) - \nabla u(x^1, t^1)| \le \gamma L \left(\frac{|x^0 - x^1| + max\{1, L^{\frac{p-2}{2}}\}\sqrt{|t^0 - t^1|}}{dist(K; \Gamma)} \right)^\alpha,$$

for every pair of points $(x^i, t^i) \in K, i = 0, 1,$ where $L = \sup_{\Omega_T} |\nabla u|$.

But it can be seen from the proof that L can be replaced by $\sup_K |\delta u|$.

Now consider the following equation which by a straightforward computation is equivalent to (1.4)

$$v_t = \mathrm{div}\left(\frac{p-2}{p-1} v|\nabla v|^{p-2}\nabla v\right) + \frac{1}{p-1}|\nabla v|^p \tag{3.10}$$

Let

$$\mathbf{A}(x,t,\nabla v) = \frac{p-2}{p-1} v|\nabla v|^{p-2}\nabla v, \quad \mathbf{B}(x,t,v,\nabla v) = \frac{1}{p-1}|\nabla v|^p.$$

Then since $|\nabla v| \leq L < \infty$, $\mathbf{A}, \mathbf{B}$ clearly satisfy the structure conditions and ∇v is Hölder continuous in a compact subset K of Ω, where $0 < c_1 \leq v \leq c_2 < \infty$ for some constant c_1, c_2.

Theorem 3.4 *Let $v_0 \in C^*$. Then there exists a neighborhood of Γ in each strip of the form $T_0 < \underline{t} \leq t \leq \bar{t}$ and a positive constant β depending on the strip and on the initial value v_0 such that*

$$|\nabla v(x,t)|, v_t \geq \beta$$

if $v(x,t) > 0$ and (x,t) belongs to that neighborhood.

PROOF. Let $(x^1, t^1) \in \Gamma$ and let $W = B(x^1, 6h) \times (t^1 - 6h, t^1 + 6h)$ for some small h be a neighborhood of (x^1, t^1) such that there exists a cone $\underline{C}$ of directions in which v is increasing for each point in W (from 3.8). We may assume that the axis of the cone is the x_N-axis. Then there is a function $f(\overline{x}, t)$ such that $\Gamma \cap W$ can be represented by

$$x_N = f(\overline{x}, t)$$

where $\overline{x} = (x_1, \cdots, x_{N-1})$ with f Lipschitz. Let $(x^0, t^0) \in W$ be such that $x_N^0 - f(\overline{x}^0, t^0) = h$. Since the cone $\underline{C}$ can be taken to contain the radial direction of each point in W, $v(x^0, t^0) \geq \bar{c}h$ by Corollary 3.2. Note that $|\nabla v| \leq L < \infty$ in $R^N \times (T, \infty)$ for some $T > 0$ see([13]).

$v_{x_N}(\overline{x^0}, f(\overline{x^0}, t^0) + y, t^0) \geq \frac{\bar{c}}{2}$ for some $y \in (\frac{\bar{c}}{2L}h, h)$.

In fact, suppose on the contrary that $v_{x_N}(\overline{x^0}, f(\overline{x^0}, t^o) + y, t^0) < \frac{\bar{c}}{2}$ for every $y \in (\frac{\bar{c}}{2L}h, h)$. Let

$$v_h(x,t) = \frac{1}{h}v(x^0 + hx, t^0 + ht) = \frac{1}{h}v(\overline{x^0} + h\overline{x}, x_N^0 + hx_N, t^0 + ht).$$

Then $v_h(0,0) = \frac{1}{h}v(x^0, t^0)$, $v_h(0,-1,0) = \frac{1}{h}v(\overline{x^0}, f(\overline{x^0}, t^0), t^0) = 0$, $v_h(0, -1 + \frac{y}{h}, 0) = \frac{1}{h}v(\overline{x^0}, y + f(\overline{x^0}, t^0), t^0)$. Note $\frac{\partial v_h}{\partial x_N}(x,t) = v_{x_N}(x^0 + hx, t^0 + ht)$. Then, $\frac{\partial v_h}{\partial x_N}(0, \eta, 0) < \frac{\bar{c}}{2}$, for all $\eta \in (-1 + \frac{\bar{c}}{2L}, 0)$ by assumption. Therefore

$$\begin{aligned}
\bar{c} &\leq v_h(0,0,0) - v_h(0,-1,0) \\
&= \int_{-1}^{0} \frac{\partial v_h}{\partial x_N}(0,\eta,0)d\eta \\
&= \int_{-1}^{-1+\frac{\bar{c}}{2L}} \frac{\partial v_h}{\partial x_N}(0,\eta,0)d\eta + \int_{-1+\frac{\bar{c}}{2L}}^{0} \frac{\partial v_h}{\partial x_N}(0,\eta,0)d\eta \\
&\leq L \cdot \frac{\bar{c}}{2L} + \frac{\bar{c}}{2}(1 - \frac{\bar{c}}{2L}) \\
&= \frac{\bar{c}}{2} + \frac{\bar{c}}{2} - \frac{\bar{c}^2}{4L} < \bar{c}.
\end{aligned}$$

Hence there exists $\eta \in (-1 + \frac{\bar{c}}{2L}, 0)$ such that $\frac{\partial v_h}{\partial x_N} \geq \frac{\bar{c}}{2}$, i.e., there exists $y \in (\frac{\bar{c}h}{2L}, h)$ such that $v_{x_N}(\overline{x^0}, f(\overline{x^0}, t^0) + y, t^0) \geq \frac{\bar{c}}{2}$. Let $(\overline{x^0}, f(\overline{x^0}, t^0) + y, t^0) = (z^0, t^0)$. Then

$$\frac{\bar{c}h}{2L} \leq z_N^0 - f(\overline{x^0}, t^0) = y.$$

Now choose $M > 0$ large enough so that $\gamma(\frac{1}{M})^\alpha \leq \frac{1}{2}$ where γ and α are constants in Proposition 3.3. Let $K_1 = \{(x,t) : |x - z^0| \leq \lambda h, |t - t^0| \leq sh\}$ where λ and s are small enough so that $K_1 \subset \Omega_T$ and $dist(K_1; \Gamma) \geq (2\lambda h + max(L^{\frac{p-2}{2}}, 1)\sqrt{sh}))M$.

Let $\max_{K_1}|\nabla v| = L_1$ and let $|\nabla v(x^1,t^1)| = L_1$, where $(x^1,t^1) \in K_1$. Then $L_1 \geq \frac{\bar{c}}{2}$ and

$$||\nabla v(\overline{x^0}, z_N^0 + \lambda h, t^0)| - |\nabla v(\overline{x^1}, x_N^0, t^1)|| \leq \gamma L_1 (\frac{2\lambda h + \max(1, L_1^{\frac{p-2}{2}})\sqrt{sh}}{dist(K_1;\Gamma)})^\alpha$$

Therefore $|\nabla v(\overline{x^0}, z_N^0 + \lambda h, t^0)| \geq \frac{1}{2}L_1 \geq \frac{1}{2^2}\bar{c}$, and $z_N^0 + \lambda h - f(\overline{x^0}, t^0) = y + \lambda h$.

Next, let $(z^1, t^0) = (\overline{x^0}, z_N^0 + \lambda h, t^0)$ and $K_2 = \{(x,t) : |x - z^1| \leq \lambda h, \quad |t - t^0| \leq sh\}$. Then $K_1 \cap K_2 \neq \phi$, $\mathrm{meas K_1} = \mathrm{meas K_2}$ and $\mathrm{dist(K_2;\Gamma)} \geq \mathrm{dist(K_1;\Gamma)}$. Let $\max_{K_2}|\nabla v| = L_2$. Then $L_2 \geq \frac{\bar{c}}{2^2}$.Let $|\nabla v(x^2,t^2)| = L_2$ for some $(x^2,t^2) \in K_2$. Then again by Proposition 3.3 we have

$$|\nabla v(\overline{x^0}, z_N^0 + 2\lambda h, t^0)| \geq \frac{L_2}{2} \geq \frac{\bar{c}}{2^3}.$$

And $z_N^0 + 2\lambda h - f(\overline{x^0}, t^0) = y + 2\lambda h$. Hence inductively, for each positive integer n we can define a sequence of compact sets K_n with $\mathrm{dist(K_n;\Gamma)} \geq (2\lambda \mathrm{h} + \max(\mathrm{L}^{\frac{p-2}{2}}, 1)\sqrt{\mathrm{sh}}))\mathrm{M}$ and a point $(z^n, t^0) = (\overline{x^0}, y + n\lambda h + f(\overline{x^0}, t^0), t^0)$ which belongs to that set.

Now let n_0 be the largest integer such that $n_0\lambda \leq 1 - \frac{\bar{c}}{2L}$. And let K_{n_0} and $(z^{n_0-1}, t^0) \in K_{n_0}$ be the corresponding compact set and a point respectively. Let $L_{n_0} = \max_{K_{n_0}}|\nabla v|$. Then $L_{n_0} \geq \frac{\bar{c}}{2^{n_0+1}}$. And let (x^{n_0}, t^{n_0}) be a point in K_{n_0} such that $|\nabla v(x^{n_0}, t^{n_0})| = L_{n_0}$. Then $|\nabla v(x^0, t^0)| \geq \frac{\bar{c}}{2^{n_0+1}} = \beta_1$. And this inequality is valid for every (x^0, t^0) in a neighborhood of (x^1, t^1) contained in W.

Thus it remains to show that v_t is bounded away from zero. By Proposition 2.12 we have

$$\frac{A-p}{p-1}v(x,t) + x \cdot \nabla v(x,t) + (At+B)v_t \geq 0$$

for some constants A and B. Also by (2.6) and (3.8) we have

$$D_{-r}v = |\nabla v| cos\langle -r, \nabla v\rangle \geq \frac{\beta_1}{R+1}\sqrt{(R+1)^2 - R^2} = c > 0.$$

Therefore

$$\begin{aligned} v_t &\geq \frac{-rv_r - \frac{A-p}{p-1}v}{At+B} \\ &\geq \frac{Rc - \frac{A-p}{p-1}C}{At+B}, \end{aligned}$$

where C is a bound of v in W. Then since for $(x,t) \in W = B(x^1, 6h) \times (t^1 - 6h, t^1 + 6h)$, $v(x,t) = v(x,t) - v(\overline{x}, f(\overline{x},t), t) \leq |\nabla v| 6h \leq 6Lh$. Hence $C \leq 6Lh$. Therefore, $v_t \geq \beta_2 > 0$ in W if h is small enough. Hence $|\nabla v|, v_t \geq \beta = min(\beta_1, \beta_2)$ in W.

Then by Theorem 3.4 and [13], we have

Corollary 3.5 *Let u is a solution of (1.1), (1.2) in $B_6 \times (-6,6)$. Let $(0,0) \in \Gamma$. Then for $v = \frac{p-1}{p-2}u^{\frac{p-2}{p-1}}$ the following statements hold:*

(i) v is Lipschitz continuous. That is, there exists $L > 0$ such that $|\nabla v|, v_t \leq L$.

(ii) $D_{\overline{\nu}}v, D_t v \geq \beta > 0$ if $v > 0$, where $\overline{\nu}$ is a unit direction in R^N.

(iii) $\Delta_p v \geq -C_1$ in $B_6 \times (-6,6)$.

PROOF. Let $(x^1, t^1) \in \Gamma$ with $|x^1| \geq R+1, T_0 < \underline{t} \leq t^1 \leq \bar{t} < \infty$. Let $6h_0 = min(t^1 - \underline{t}, \bar{t} - t^1)$. Then there exist constants $h \leq h_0, \beta, L > 0$ depending only on h_0 and a direction $\bar{\nu}$ in R^N such that for $v_h(x,t) = \frac{1}{h} v(x^1 + hx, t^1 + ht)$,

$$D_{\bar{\nu}} v_h(x,t), \frac{\partial}{\partial t} v_h \geq \beta, \quad |\nabla v_h|, \frac{\partial}{\partial t} v_h \leq L$$

in $B_6 \times (-6, 6)$ by Theorem 3.4 and [13]. Also it was shown that there is a constant C_1 depending only on T_0 such that

$$\Delta_p v \geq -C_1 \quad if \quad t \geq T_0.$$

In particular,

$$\Delta_p v_h(x,t) = h \Delta_p v(x^1 + hx, t^1 + ht) \geq -C_1 h \geq -C_1$$

in $B_6 \times (-6, 6)(h \leq 1)$.

Next, we notice that equation (1.4) can be written as the following equation:

$$\begin{aligned} v_t &= \frac{p-2}{p-1} v(|\nabla v|^{p-2} \Delta v + \nabla(|\nabla v|^{p-2} \nabla v) + |\nabla v|^p \\ &= \frac{p-2}{p-1} v |\nabla v|^{p-2} (\delta_{ij} + (p-2) \frac{v_{x_i} v_{x_j}}{|\nabla v|^2}) v_{x_i x_j} + |\nabla v|^p. \end{aligned}$$

Hence if v is a solution of (1.4) then v is also a solution of

$$w_t = \frac{p-2}{p-1} v |\nabla v|^{p-2} (\delta_{ij} + (p-2) \frac{v_{x_i} v_{x_j}}{|\nabla v|^2}) w_{x_i x_j} + |\nabla v|^2. \tag{3.11}$$

Therefore in any compact subdomain Ω_1 of Ω where v and $|\nabla v|$ are bounded above and bounded away from zero (since the coefficients and free terms are bounded and continuous) (3.11) becomes a uniformly parabolic equation and

$$\|D_{ij}v\|_{L^\infty(\Omega_1)} < \infty$$

if v is a solution of (1.4) in Ω_1 by Theorem 12.2 on page 224 of [23].

CHAPTER 4

Proof of Main Theorem

To prove our main theorem we need the following sequence of propositions.

Proposition 4.1 *Let v be as in Corollary 3.5 and $\nu = \frac{1}{\sqrt{2}}\overline{\nu} + \frac{1}{\sqrt{2}}e_{N+1}$. Then there exists $\Theta_0 > 0$ such that $D_\tau v \geq 0$ in $B_6 \times (-6,6)$ for any τ in $\hat{K}(\nu, \Theta_0)$, where*

$$\hat{K}(\nu, \Theta_0) = \{\tau \mid \langle \tau, \nu \rangle \leq \Theta_0\}$$

PROOF. Since by Corollary 3.5,

$$D_\nu v = |\hat{\nabla} v| \cos\langle \nu, \hat{\nabla} \mathrm{v} \rangle \geq \beta$$

for some $\beta > 0$ and $|\hat{\nabla} v| \leq \sqrt{2}L$, we have

$$\cos\langle \nu, \hat{\nabla} \mathrm{v} \rangle \geq \frac{\beta}{\sqrt{2}\mathrm{L}} = \beta_1 > 0.$$

Let $\Theta_0 = \arcsin\beta_1$. If $\langle \nu, \tau \rangle \le \Theta_0$ then $\langle \tau, \hat{\nabla} v \rangle \le \frac{\pi}{2}$. Therefore

$$D_\tau v(x,t) \ge 0 \quad (x,t) \in B_6 \times (-6,6), \quad \tau \in \hat{K}(\nu, \Theta_0).$$

Proposition 4.2 *Let v be as before and τ be such that $D_\tau v(x,t) \ge 0$ for $(x,t) \in B_4 \times (-4,4)$. Then*

$$v((x,t) + \gamma\tau) \ge (1+\delta) v(x,t)$$

for $(x,t) \in B_2 \times (-\frac{4(L+1)}{\beta p}\epsilon, \frac{4(L+1)}{\beta p}\epsilon) \bigcap (v = \epsilon)$, where ϵ, $\gamma > 0$ are sufficiently small and $\delta = C\gamma\epsilon^{-1} cos\langle \tau, \hat{\nabla} v(\overline{\nu}, -\frac{4(L+1)}{\beta p}\epsilon)\rangle$.

PROOF. It suffices to show that there exists $C > 0$ such that

$$D_\tau v \ge C\cos\langle \tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle$$

in $B_2 \times (-2r, 2r) \bigcap (v = \epsilon)$ with $r = \frac{4(L+1)}{\beta p}\epsilon$. Now consider a non-negative function $g = D_\tau v$ in $B_4 \times (-4,4)$. Then g is a solution of the equation

$$\begin{aligned} g_t \;=\; & \frac{p-2}{p-1} v |\nabla v|^{p-2} [\delta_{ij} + (p-2)\frac{v_{x_i} v_{x_j}}{|\nabla v|^2}] g_{x_i x_j} \\ & + [\frac{(p-2)^2}{p-1} v |\nabla v|^{p-4} \Delta v + \frac{(p-2)(p-4)}{p-1} v |\nabla v|^{p-6} (\nabla^2 v \nabla v) \cdot \nabla v] \nabla v \cdot \nabla g \\ & + p |\nabla v|^{p-2} \nabla v \cdot \nabla g + \frac{2(p-2)}{p-1} |\nabla v|^{p-4} (\nabla^2 v \nabla v) \cdot \nabla g + \frac{p-2}{p-1} (\Delta_p v) g. \end{aligned}$$

This is a uniformly parabolic equation with bounded coefficients in any bounded region where v and $|\nabla v|$ is bounded away from zero. We choose a region in such a way that it contains the set $(v = \epsilon)$ and use Harnack's inequality there [23].

Consider a new system of coordinates in R^N such that $e_N = \overline{\nu}$. Since $D_{x_N} \geq \beta > 0$, there exists a Lipschitz continuous representation of Γ,

$$x_N = f(\overline{x}, t), \qquad \overline{x} = (x_1, \cdots, x_{N-1}). \tag{4.1}$$

We use c to denote a Lipschitz constant for f. Let

$$\Omega = \begin{cases} \frac{\epsilon}{2L} \leq x_N - f(\overline{x}, 0) \\ |x| \leq 4 \\ |t| \leq \frac{8(L+1)}{\beta p}\epsilon \end{cases}$$

and

$$\Omega_1 = \begin{cases} \frac{\epsilon}{L} \leq x_N - f(\overline{x}, 0) \\ |x| \leq 2 \\ |t| \leq \frac{4(L+1)}{\beta p}\epsilon \end{cases}$$

We see that $B_2 \times (-\frac{4(L+1)}{\beta p}\epsilon, \frac{4(L+1)}{\beta p}\epsilon) \bigcap (v \geq \epsilon)$ is contained in Ω_1. In fact, for $|t| \leq \frac{4(L+1)}{\beta p}\epsilon$, $|x| \leq 2$ and $x_N - f(\overline{x}, t) < \frac{\epsilon}{L}$ we have, by the mean value theorem,

$$\begin{aligned} v(\overline{x}, x_N, t) &= v(\overline{x}, x_N, t) - v(\overline{x}, f(\overline{x}, t), t) \\ &= v_{x_N}(\overline{x}, \psi, t)(x_N - f(\overline{x}, t)) \leq L \cdot \frac{\epsilon}{L} = \epsilon \end{aligned}$$

To apply Harnack's inequality, we need a lower and an upper bound for v in Ω. For $(x, t) \in \Omega$,

$$v(x, t) = v(\overline{x}, x_N, t) - v(\overline{x}, f(\overline{x}, t), t)$$

$$\begin{aligned} &= v_{x_N}(\overline{x}, \psi, t)(x_N - f(\overline{x}, t)) \\ &\geq \beta \cdot \frac{\epsilon}{L} \end{aligned}$$

and

$$v(x,t) = v(x,t) - v(x,0) + v(x,0) - v(0,0) \leq L|t| + L|x| \leq L(4 + \frac{8(L+1)}{\beta^p}\epsilon).$$

Now applying Harnack's inequality to the function $g = D_\tau v$ in Ω yields, noting that $(e_N, -\frac{4(L+1)}{\beta^p}\epsilon) \in \Omega_1$ when ϵ is small,

$$D_\tau v(x,t) \geq \Lambda D_\tau v(e_N, -\frac{4(L+1)}{\beta^p}\epsilon).$$

Therefore

$$\begin{aligned} D_\tau v(x,t) &\geq \Lambda D_\tau v(e_N, -\frac{4(L+1)}{\beta^p}\epsilon) = \Lambda\langle\tau, \hat{\nabla} v((e_N, -\frac{4(L+1)}{\beta^p}\epsilon) \\ &= \Lambda|\hat{\nabla} v((e_N, -\frac{4(L+1)}{\beta^p}\epsilon)|cos\langle\tau, \hat{\nabla} v((e_N, -\frac{4(L+1)}{\beta^p}\epsilon)\rangle \\ &\geq \Lambda\beta cos\langle\tau, \hat{\nabla} v((e_N, -\frac{4(L+1)}{\beta^p}\epsilon)\rangle. \end{aligned}$$

To show that $v((x,t) + \gamma\tau) \geq (1+\delta)v(x,t)$ for $v(x,t) < \epsilon$, we need a technical lemma.

Lemma 4.3 *Let v be a solution of (1.5) satisfying Corollary 3.5 in $B_5 \times (-5, 5)$. Then there exists $C_2 > 0$ such that*

$$vD_{ij}v \geq -C_2, \quad i,j = 1, \cdots, N,$$

in $B_2 \times (-2r, 2r)$ if $r < min(c, 1)$, c being the Lipschitz constant of f in (4.1).

PROOF. Let f be the representation of Γ in (4.1). And let $(x^0, t^0) \in B_2 \times (-2r, 2r) \bigcap (v > 0)$ and let

$$\begin{aligned} h &= x_N^0 - f(\overline{x^0}, t^0) \\ &= x_N^0 - f(\overline{x}^0, t^0) + f(\overline{x^0}, 0) - f(\overline{x^0}, 0) - f(0, 0). \end{aligned}$$

Then $h \leq 2(1 + c + cr) \leq (1 + c)(1 + r)$. Let us consider the region

$$R_h = \begin{cases} |(x_N - f(\overline{x}, t^0)) - (x_N^0 - f(\overline{x^0}, t^0))| & \leq \frac{h}{2(1+c)^2}, \\ |\overline{x} - \overline{x^0}| \ \leq \frac{h}{2(1+c)^2}, \\ |t - t^0| \ \leq \frac{h}{4(1+c)^2}. \end{cases}$$

It is straightforward to check that R_h is contained in $B_5 \times (-5, 5)$. By the change of the variables

$$x = \frac{y - x^0}{h}, \quad t = \frac{s - t^0}{h}, \quad (y, s) \in R_h,$$

the region R_h is transformed into

$$\Omega = \begin{cases} |(x_N - g(\overline{x})| \ \leq \frac{1}{2}, \\ |\overline{x}| \ \leq \frac{1}{2}, \\ |t| \ \leq \frac{1}{4c}, \end{cases}$$

where $g(\overline{x}) = \frac{1}{h}\{f(\overline{x^0} + h\overline{x}, t^0) - f(\overline{x^0}, t^0)\}$. Let

$$v_h(x,t) = \frac{1}{h}v(x^0 + hx, t^0 + ht), \quad (x,t) \in \Omega.$$

Then $v_h(x,t) \geq \frac{1}{4}\beta$ in Ω. In fact, let $(x,t) \in \Omega$ and $(y,s) \in R_h$ be such that

$$y = x^0 + hx, \qquad s = t^0 + ht.$$

Then

$$\begin{aligned} y_N - f(\overline{y}, s) &= y_N - f(\overline{y}, t^0) + f(\overline{y}, t^0) - f(\overline{t}, s) \\ &\geq x_N^0 - f(\overline{x^0}, t^0) - \frac{1}{2}h + f(\overline{y}, t^0) - f(\overline{y}, s) \\ &\geq \frac{1}{2}h - \frac{1}{4}h \\ &= \frac{1}{4}h. \end{aligned}$$

Thus $v(y,s) \geq \frac{1}{4}\beta h$ and consequently

$$v_h(x,t) \geq \frac{1}{4}\beta \quad in \quad \Omega.$$

Since v_h is also a solution of a uniformly parabolic equation

$$\begin{aligned} v_t &= \frac{p-2}{p-1} v \mathrm{div}(|\nabla \mathrm{v}|^{\mathrm{p}-2}\nabla \mathrm{v}) + |\nabla \mathrm{v}|^{\mathrm{p}} \\ &= \frac{p-2}{p-1} v|\nabla v|^{p-2}\{\delta_{ij} + (p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v|^2}v_{x_i x_j}\} + |\nabla v|^p. \end{aligned}$$

Since $v|\nabla v|^{p-2}\{\delta_{ij} + (p-2)\frac{v_{x_i}v_{x_j}}{|\nabla v|^2} v_{x_i x_j}\}$, $|\nabla v|^p \in C^{\alpha,\frac{\alpha}{2}}(R)$ (see [23]),

$$\| D_{ij}v_h \|_{L^\infty(R)} \leq C$$

for some constant C depending on β, L. Hence

$$|D_{ij}v(x^0,t^0)| \leq \frac{C}{h}.$$

Since $v(x^0,t^0) \leq Lh$ we have

$$|v(x^0,t^0)D_{ij}v(x^0,t^0)| \leq CL$$

for every $(x^0,t^0) \in B_2 \times (-2r,2r) \bigcap (v>0)$, which implies that

$$vD_{ij}v \geq -CL = C_2$$

in the sense of distributions in $B_2 \times (-2r,2r)$.

Lemma 4.4 *Let v be a solution of (1.5) in $B_4 \times (-4,4)$. Let k be a C^1 function defined in B_2 such that $k \equiv 0$ in B_1; $|\nabla k| \leq 2\epsilon, -2\epsilon I \leq \nabla^2 f \leq 2\epsilon I$ (I : unit $N \times N$ matrix), $k \geq 0$ and $k \equiv \epsilon$ if $|x| = 2$. Let $\delta < 1$ and $y(x,t) = v(x,t) + \delta[v(x,t) + \frac{\beta^p}{2(L+1)}(t+\alpha) - k(x)]_+$ if $|x| \leq 2, t \in (-\frac{8(L+1)}{\beta^p}\epsilon, -\alpha)$. Then y is a subsolution of (1.4) in $B_2 \times (-\frac{8(L+1)}{\beta^p}\epsilon, -\alpha) \bigcap (v \leq \epsilon)$ when ϵ is small enough.*

PROOF. Let $g(s) = \delta s_+$. Then $g(s) \leq s_+ g'(s)$ for $s \leq \epsilon$, $g'(s) \leq \delta < 1$ and $g''(s) \geq 0$ in the sense of distributions. Set

$$Ly = \frac{p-2}{p-1} y \Delta_p y + |\nabla y|^p - y_t.$$

We want to show that $Ly \geq 0$. Note that

$$\begin{aligned}
y_t &= (1+g')v_t + \frac{\beta^p}{2(L+1)} g' \\
y_{x_i} &= (1+g')v_{x_i} - g' f_{x_i} \\
\nabla y &= (1+g')\nabla v - g' \nabla f, \\
|\nabla y|^2 &= (1+g')^2 |\nabla v|^2 - 2g'(1+g')\nabla v \cdot \nabla f + (g')^2 |\nabla f|^2, \\
y_{x_i x_j} &= g'' v_{x_i} v_{x_j} - g'' v_{x_i} f_{x_j} + (1+g') v_{x_i x_j} + g'' f_{x_i} f_{x_j} - g'' v_{x_j} f_{x_i} - g' f_{x_i x_j} \\
\Delta y &= (1+g')\Delta v - g' \Delta f + g'' |\nabla (f-v)|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
|\nabla y|^2 \Delta y &= \{(1+g')^2 |\nabla v|^2 - 2g'(1+g')\nabla v \cdot \nabla f + (g')^2 |\nabla f|^2\} \\
&\quad \times \{(1+g')\Delta v - g' \Delta f + g''(|\nabla (v-f)|^2\} \\
&= (1+g')^3 |\nabla v|^2 \Delta v + g''(1+g')^2 |\nabla v|^2 |\nabla (v-f)|^2 \\
&\quad + (1+g')\{-2g'(1+g')\nabla v \cdot \nabla f + (g')^2 |\nabla f|^2\} \Delta v + g' G.
\end{aligned}$$

where $G = 0(\epsilon)$. And

$$\begin{aligned}
y_{x_i} y_{x_j} y_{x_i x_j} &= \{(1+g')v_{x_i} - g' f_{x_i}\}\{(1+g')v_{x_j} - g' f_{x_j}\} \\
&\quad \times \{g''(v_{x_i} v_{x_j} - v_{x_i} f_{x_j} - v_{x_j} f_{x_i} + f_{x_i} f_{x_j}) + (1+g') v_{x_i x_j} - g' f_{x_i x_j}\} \\
&= \{(1+g')^2 v_{x_i} v_{x_j} - g'(1+g')(v_{x_i} f_{x_j} + v_{x_j} f_{x_i}) + (g')^2 f_{x_i} f_{x_j}\}
\end{aligned}$$

$$
\begin{aligned}
&\times\{g''(v_{x_i}v_{x_j} - v_{x_i}f_{x_j} - v_{x_j}f_{x_i} + f_{x_i}f_{x_j}) + (1+g')v_{x_ix_j} + g'G\} \\
\geq\ & (1+g')^3 v_{x_i}v_{x_j}v_{x_ix_j} + g''(1+g')^2(\beta^4 - \epsilon M) \\
&-g'(1+g')v_{x_ix_j}\{(1+g')(v_{x_i}f_{x_j} + v_{x_j}f_{x_i}) - g'f_{x_i}f_{x_j}\} + g'G \\
\geq\ & (1+g')^3 v_{x_i}v_{x_j}v_{x_ix_j} - g'(1+g')v_{x_ix_j}\{(1+g')(v_{x_i}f_{x_j} + v_{x_j}f_{x_i}) \\
&+g'f_{x_i}f_{x_j}\} + g'G
\end{aligned}
$$

if ϵ is small enough , where $M = M(N, L)$ is a finite constant. Therefore

$$
\begin{aligned}
\Delta_p y &= |\nabla y|^{p-4}\{|\nabla y|^2 \Delta y + (p-2) y_{x_i} y_{x_j} y_{x_ix_j}\} \\
&\geq |\nabla y|^{p-4}(1+g')^3\{|\nabla v|^2 \Delta v + (p-2) v_{x_i}v_{x_j}v_{x_ix_j}\} \\
&\quad -g'(1+g')|\nabla y|^{p-4}\{[2(1+g')\nabla v\cdot\nabla f - g'|\nabla f|^2]\Delta v \\
&\quad +(p-2)[(1+g')(v_{x_i}f_{x_j} + v_{x_j}f_{x_i}) - g'f_{x_i}f_{x_j}]v_{x_i}v_{x_j}v_{x_ix_j}\} \\
&\quad +g'G.
\end{aligned}
$$

Now

$$
|\nabla y| = |(1+g')\nabla v - g'\nabla f| \geq (1+g')|\nabla v| - g'|\nabla f| = |\nabla v| + g'(|\nabla v| - |\nabla f|).
$$

Hence $|\nabla y| \geq |\nabla v|$ if ϵ is small enough and also we have

$$
\begin{aligned}
|\nabla y| &\leq (1+g')|\nabla v| + g'|\nabla f| \leq (1+g')|\nabla v| + 2g'\epsilon \\
&\leq (1+g')|\nabla v| + 2\epsilon.
\end{aligned}
$$

Then by Lemma 4.3 and $g \le \delta v^+$, we have

$$\begin{aligned} Ly &= \frac{p-2}{p-1}(v+g)\Delta_p y + |\nabla y|^p - (1+g')v_t - \frac{\beta^p}{2(L+1)}g' + g'G \\ &\ge \frac{p-2}{p-1}\frac{|\nabla y|^{p-4}}{|\nabla v|^{p-4}}(1+g')^3 v\Delta_p v - (1+g')v_t + |\nabla y|^p + \frac{p-2}{p-1}g\Delta_p y \\ &\quad -\frac{\beta^p}{2(L+1)}g' + g'G. \end{aligned}$$

Now if $v - f + \frac{\beta^p}{2(L+1)}(t+\alpha) < \epsilon$, we have $g \le g'\epsilon$ and

$$\begin{aligned} g\Delta_p y &\ge g(1+g')^3[\frac{|\nabla y}{|\nabla v|}]^{p-4}\Delta_p v + g'G \\ &\ge -C_1 g(1+g')^3[\frac{|\nabla y}{|\nabla v|}]^{p-4} + g'G \\ &\ge g'G. \end{aligned}$$

Hence

$$\begin{aligned} Ly &\ge \frac{|\nabla y|^{p-4}}{|\nabla v|^{p-4}}(1+g')^3(\frac{p-2}{p-1}v\Delta_p v - v_t) + |\nabla y|^p \\ &\quad +\{\frac{|\nabla y|^{p-4}}{|\nabla v|^{p-4}}(1+g')^2 - 1\}(1+g')v_t - \frac{\beta^p}{2(L+1)}g' + g'G. \end{aligned}$$

Now

$$\begin{aligned} |\nabla y|^p &= |\nabla y|^{p-4}|\nabla y|^4 = |\nabla y|^{p-4}\{(1+g')^4|\nabla v|^4\} + g'G \\ &= |\nabla y|^{p-4}\{g'(1+g')^3|\nabla v|^4 + (1+g')^3|\nabla v|^4\} + g'G \\ &= \frac{|\nabla y|^{p-4}}{|\nabla v|^{p-4}}(1+g')^3|\nabla v|^p + g'(1+g')^3|\nabla y|^{p-4}|\nabla v|^4 + g'G. \end{aligned}$$

Hence

$$\begin{aligned}
Ly &\geq \{\frac{|\nabla y|}{|\nabla v|}\}^{p-4}(1+g')^3(\frac{p-2}{p-1}v\Delta_p v + |\nabla v|^p - v_t) \\
&+\{(\frac{|\nabla y|}{|\nabla v|})^{p-4}(1+g')^2 - 1\}(1+g')v_t + g'(1+g')^3|\nabla y|^{p-4}|\nabla v|^4 \\
&-\frac{\beta^p}{2(L+1)}g' + g'G \\
&= \{(\frac{|\nabla y|}{|\nabla v|})^{p-4}(1+g')^2 - 1\}(1+g')v_t + g'(1+g')^3|\nabla y|^{p-4}|\nabla v|^4 \\
&-\frac{\beta^p}{2(L+1)}g' + g'G.
\end{aligned}$$

Now let us consider the following two cases:

(i) $p \geq 4$. Since $\frac{|\nabla y|}{|\nabla v|} \geq 1, v_t \geq 0$,

$$Ly \geq g'\{(1+g')^3\beta^p - \frac{\beta^p}{2(L+1)}g' + G\} \geq 0$$

if ϵ is small enough.

(ii) $2 < p < 4$. Since

$$1 \leq \frac{|\nabla y|}{|\nabla v|} \leq \frac{(1+g')|\nabla v| + 2g'\epsilon}{|\nabla v|} \quad \textit{and} \quad 0 < 4-p < 2,$$

we have

$$(\frac{|\nabla y|}{|\nabla v|})^{p-4} \geq \{\frac{|\nabla v|}{(1+g')|\nabla v| + 2g'\epsilon}\}^2.$$

Hence

$$Ly \geq \{(\frac{|\nabla y|}{|\nabla v|})^{p-4}(1+g')^2 - 1\}(1+g')v_t + g'(1+g')^3|\nabla y|^{p-4}|\nabla v|^4$$

$$
\begin{aligned}
& -\frac{\beta^p}{2(L+1)}g' + g'G \\
\geq\ & \{\frac{((1+g')|\nabla v|)^2}{((1+g')|\nabla v| + 2g'\epsilon)^2} - 1\}(1+g')v_t + g'(1+g')^3|\nabla y|^{p-4}|\nabla v|^4 \\
& -\frac{\beta^p}{2(L+1)}g' + g'G \\
=\ & \frac{2\{(1+g')|\nabla v| + g'\epsilon\}}{(1+g')|\nabla v| + 2g'\epsilon}\frac{-2g'\epsilon}{(1+g')|\nabla v| + 2g'\epsilon}(1+g')v_t \\
& + g'\{(1+g')^3|\nabla y|^{p-4}|\nabla v|^4 - \frac{\beta^p}{2(L+1)}g' + G\} \\
\geq\ & g'\{(1+g')^3|\nabla v|^p\frac{1}{[(1+g')|\nabla v| + 2g'\epsilon]^2} - \frac{\beta^p}{2(L+1)} + G\} \\
\geq\ & g'\{(1+g')^3\frac{\beta^p}{[(1+g')L + 2g'\epsilon]^2} - \frac{\beta^p}{2(L+1)} + G\} \geq 0
\end{aligned}
$$

if ϵ is small enough. Hence in the set where $v + \frac{\beta^p}{2(L+1)}(t+\alpha) - k$ is less than ϵ, $Ly \geq 0$ if ϵ is small enough. Since in the set $B_2 \times (-\frac{8(L+1)}{\beta^p}\epsilon, -\alpha) \bigcap (v < \epsilon), v + \frac{\beta^p}{2(L+1)}(t+\alpha) - k < \epsilon$, and we obviously have the relation $Ly \geq 0$ is valid there.

Proposition 4.5 *Let v and w be Lipschitz continuous solutions of*

$$v_t = \frac{p-2}{p-1}v\Delta_p v + |\nabla v|^p \quad in \quad B_4 \times (-4,4).$$

with v satisfying Corollary 3.5 in $B_4 \times (-4,4)$. Suppose that $w \geq v, 0 < \delta < 1$ and $w \geq (1+\delta)v$ in $B_2 \times (-\frac{8(L+1)}{\beta^p}\epsilon, \frac{8(L+1)}{\beta^p}\epsilon) \bigcap (v = \epsilon)$. Then if ϵ is small enough,

$$w \geq (1+\delta)v \quad B_1 \times (-\frac{4(L+1)}{\beta^p}\epsilon, \frac{4(L+1)}{\beta^p}\epsilon) \bigcap (v \leq \epsilon).$$

PROOF. Let $y = v + \delta(v + \frac{\beta^p}{2(L+1)}(t+\alpha) - k)_+$ with k as in Lemma 4.4 and $\alpha \in (-\frac{4(L+1)}{\beta^p}\epsilon, \frac{4(L+1)}{\beta^p}\epsilon)$. Then since for $t = -\frac{8(L+1)}{\beta^p}\epsilon, v(x,t) + \frac{\beta^p}{2(L+1)}(t+\alpha) - k \leq 0$

in B_2, by the assumption we have $w \geq y$ on the parabolic boundary of the set $B_2 \times (-\frac{8(L+1)}{\beta^p}\epsilon, -\alpha) \bigcap (v \leq \epsilon)$,

$$w \geq y \quad in \quad B_2 \times (-\frac{8(L+1)}{\beta^p}\epsilon, -\alpha) \bigcap (v \leq \epsilon).$$

Since $k \equiv 0$ in B_1,

$$w(x, -\alpha) \geq (1+\delta)v(x, -\alpha) \quad in \quad B_1 \bigcap (v \leq \epsilon).$$

Since $\alpha \in (-\frac{4(L+1)}{\beta^p}\epsilon, \frac{4(L+1)}{\beta^p}\epsilon)$ is arbitrary,

$$w \geq (1+\delta)v \quad in \quad B_1 \times (-\frac{4(L+1)}{\beta^p}\epsilon, \frac{4(L+1)}{\beta^p}\epsilon) \bigcap (v \leq \epsilon).$$

Proposition 4.6 *Let v and w be as in Proposition 4.5 and $v \leq \epsilon \quad in \quad B_{2r} \times (-2r, 2r)$ with $0 < r < min\{\frac{1}{2}, \frac{2(L+1)}{\beta^p}\epsilon\}$. Then, for any vector $\overline{\eta} \in R^N$,*

$$w(x,t) \geq (1+\delta)v(x + (t+2r)\phi(x)\overline{\eta}, t) \quad in \quad B_{2r} \times (-2r, 2r)$$

if ϕ is a smooth function such that $\mathrm{supp}\phi \subset \mathrm{B}_{2r}; \phi, |\nabla\phi|, |\nabla^2\phi| \leq \mu\delta; \phi \geq 0$ with μ small enough.

PROOF. Let $y(x,t) = (1+\delta)v(x+(t+2r)\phi(x)\overline{\eta}, t)$. Then by Proposition 4.5, $w \geq y$ on the parabolic boundary of $B_{2r} \times (-2r, 2r)$. (We recall that $B_{2r} \times (-2r, 2r) \subset B_1 \times ((-\frac{4(L+1)}{\beta^p}\epsilon, \frac{4(L+1)}{\beta^p}\epsilon) \bigcap (v \leq \epsilon))$. We only need to prove that y is a subsolution of (1.4) in $B_{2r} \times (-2r, 2r)$ if μ is small enough. This proves our proposition. Let

$e_N = \overline{\eta}$. Then

$$\begin{aligned}
y_t &= (1+\delta)\{v_t + v_{x_N}\phi\}, \\
y_{xi} &= (1+\delta)(v_{x_i} + v_{x_N}(t+2r)\phi_{x_i}) \\
\nabla y &= (1+\delta)\{\nabla v + v_{x_N}(t+2r)\nabla\phi\} \\
\Delta y &= (1+\delta)\{\Delta v + 2(t+2r)\nabla v_{x_N}\cdot\nabla\phi \\
&\quad + v_{x_N x_N}(t+2r)^2|\nabla\phi|^2 + v_{x_N}(t+2r)\Delta\phi\} \\
|\nabla y|^2 &= (1+\delta)^2\{|\nabla v|^2 + v_{x_N}^2(t+2r)^2|\nabla\phi|^2 + 2(t+2r)v_{x_N}\nabla v\cdot\nabla\phi\} \\
y_{x_i x_j} &= (1+\delta)\{v_{x_i x_j} + v_{x_i x_N}(t+2r)\phi_{x_j} + v_{x_N}(t+2r)\phi_{x_i x_j} \\
&\quad + v_{x_N x_j}(t+2r)\phi_{x_j} + (t+2r)^2\phi_{x_i}\phi_{x_j}v_{x_N x_N}\}
\end{aligned}$$

$$\begin{aligned}
|\nabla y|^2\Delta y &= (1+\delta)^3\{|\nabla v|^2 + v_{x_N}^2(t+2r)^2|\nabla\phi|^2 + 2(t+2r)v_{x_N}\nabla v\cdot\nabla\phi\} \\
&\quad \times\{\Delta v + (t+2r)[2\nabla v_{x_N}\cdot\nabla\phi + v_{x_N x_N}(t+2r)|\nabla\phi|^2 + v_{x_N}\Delta\phi]\} \\
&= (1+\delta)^3|\nabla v|^2\Delta v + \delta G \\
&\quad + (1+\delta)^3(t+2r)\{v_{x_N}\Delta v[(t+2r)v_{x_N}|\nabla\phi|^2 + 2\nabla v\cdot\nabla\phi] + (t+2r) \\
&\quad \times v_{x_N x_N}|\nabla\phi|^2[|\nabla v|^2 + v_{x_N}^2(t+2r)^2|\nabla\phi|^2 + 2(t+2r)v_{x_N}\nabla v\cdot\nabla\phi]\}
\end{aligned}$$

where $G = 0(\mu)$. And similarly

$$\begin{aligned}
y_{x_i}y_{x_j}y_{x_i x_j} &= (1+\delta)^3(v_{x_i} + v_{x_N}(t+2r)\phi_{x_i})\cdot(v_{x_j} + v_{x_N}(t+2r)\phi_{x_j}) \\
&\quad \cdot[v_{x_i x_j} + (t+2r)v_{x_i x_N}\phi_{x_j} + (t+2r)v_{x_N x_j}\phi_{x_i}] \\
&\quad + (1+\delta)^3\{(t+2r)^2\phi_{x_i}\phi_{x_j}v_{x_N x_N} + (t+2r)v_{x_N}\phi_{x_i x_j}\}
\end{aligned}$$

$$
\begin{aligned}
&= (1+\delta)^3 v_{x_i} v_{x_j} v_{x_i x_j} + \delta G \\
&\quad + (1+\delta)^3 v_{x_i x_j}(t+2r)[v_{x_i} v_{x_N} \phi_{x_j} + v_{x_j} v_{x_N} \phi_{x_i} + v_{x_N}^2 \phi_{x_i} \phi_{x_j} \phi_{x_i x_j}]\} \\
&\quad + (1+\delta)^3 (t+2r)(v_{x_i} + v_{x_N}(t+2r)\phi_{x_i}) \cdot (v_{x_j} + v_{x_N}(t+2r)\phi_{x_j}) \\
&\quad \times [v_{x_i x_N} \phi_{x_j} + v_{x_j x_N} \phi_{x_i} + (t+2r)\phi_{x_i} \phi_{x_j} v_{x_N x_N}].
\end{aligned}
$$

Now

$$
\begin{aligned}
|\nabla v| &\le |\nabla v| - L(t+2r)\mu\delta + \delta)[|\nabla v| - L(t+2r)\mu\delta] \\
&= (1+\delta)[|\nabla v| - L(t+2r)\mu\delta] \\
&\le |\nabla y| \\
&\le (1+\delta)|\nabla v|[1 + (t+2r)\mu\delta].
\end{aligned}
$$

if μ is small enough. Hence by Lemma 4.3

$$
\begin{aligned}
y\Delta_p y &= |\nabla y|^{p-4}\{y|\nabla y|^2 \Delta y + y(p-2) y_{x_i} y_{x_j} y_{x_i x_j}\} \\
&\ge |\nabla y|^{p-4}(1+\delta)^4 v\{|\nabla v|^2 \Delta v + (p-2)(\nabla^2 v \nabla v)\nabla v\} + \delta G.
\end{aligned}
$$

But, $|\nabla y|^p = |\nabla y|^{p-4}|\nabla y|^4 = |\nabla y|^{p-4}(1+\delta)^4 |\nabla v|^4 + \delta G$. Therefore

$$
\begin{aligned}
Ly &= \frac{p-2}{p-1} y\Delta_p y + |\nabla y|^p - y_t \\
&= \frac{p-2}{p-1}(1+\delta)^4 v|\nabla y|^{p-4}\{|\nabla v|^2 \Delta v + (p-2)(\nabla^2 v \nabla v)\nabla v\} \\
&+ \ (1+\delta)^4 |\nabla y|^{p-4}|\nabla v|^4 - (1+\delta) v_t + \delta G \\
&= \frac{p-2}{p-1}(1+\delta)^4 v|\nabla y|^{p-4}|\nabla v|^{4-p}|\nabla v|^{p-4}\{|\nabla v|^2 \Delta v + (p-2)(\nabla^2 v \nabla v)\nabla v\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (1+\delta)^4|\nabla y|^{p-4}|\nabla v|^{4-p}|\nabla v|^p - (1+\delta)v_t + \delta G \\
&= \quad (1+\delta)^4[\frac{|\nabla y|}{|\nabla v|}]^{p-4}(\frac{p-2}{p-1}v\Delta_p v + |\nabla v|^p - v_t) \\
&+ \quad (1+\delta)\{(1+\delta)^3[\frac{|\nabla y|}{|\nabla v|}]^{p-4} - 1\}v_t + \delta G \\
&= \quad \{(1+\delta)^3[\frac{|\nabla y|}{|\nabla v|}]^{p-4} + \delta(1+\delta)^3[\frac{|\nabla y|}{|\nabla v|}]^{p-4} - (1+\delta)\}v_t + \delta G \\
&\geq \quad \delta(1+\delta)^3[\frac{|\nabla y|}{|\nabla v|}]^{p-4}\beta + (1+\delta)[(1+\delta)^2\frac{|\nabla y|}{|\nabla v|}]^{p-4} - 1]v_t + \delta G
\end{aligned}
$$

Hence for $p \geq 4$,

$$
[\frac{|\nabla y|}{|\nabla v|}]^{p-4}(1+\delta)^2 \geq (1+\delta)^2 \geq 1.
$$

For $2 < p < 4$, since

$$
(1+\delta)^2[\frac{|\nabla y|}{|\nabla v|}]^{p-4} \geq (1+\delta)^2[\frac{|\nabla v|}{|\nabla y|}]^2 \geq \frac{1}{[1+(t+2r)\mu\delta]^2},
$$

$$
(1+\delta)^2[\frac{|\nabla y|}{|\nabla v|}]^{p-4} - 1 \geq \frac{1}{[1+(t+2r)\mu\delta]^2} - 1 = \frac{2+(t+2r)\mu\delta}{1+(t+2r)\mu\delta}\frac{-(t+2r)\mu\delta}{1+(t+2r)\mu\delta} = \delta G.
$$

Hence

$$
Ly \geq \delta\{(1+\delta)^3[\frac{|\nabla y|}{|\nabla v|}]^{p-4}\beta + G\} \geq 0
$$

if μ is small enough.

Corollary 4.7 *Let v and w be as in Proposition 4.6. Then the distance between their free boundaries is at least $\frac{\beta r\theta}{C_0} > 0$ in $B_r \times (-r, r)$ where θ is such that $\phi \geq \theta$ in B_r and C_0 is a bound of $|\hat{\nabla} w|$. Moreover, the distance between their level surface is at least $\frac{\beta r\theta}{C_0}$ and θ can be estimated from below by $\frac{3}{8}r^2\mu\delta$ with μ as in Proposition 4.6.*

PROOF. Suppose $v(x,t) = \lambda$; then

$$\begin{aligned} w(x,t) &> v(x+t+2r)\phi(x)\overline{\nu},t) \geq v(x,t) + \beta(t+2r)\phi(x) \\ &\geq \lambda + \beta r\theta \quad if \quad t \geq -r, x \in B_r. \end{aligned}$$

Suppose $dist((x,t);(w=\lambda)) \leq \frac{\beta r\theta}{C_0}$; then $w(x,t) \leq \lambda + \beta r\theta$, which implies that

$$dist((v=\lambda);(w=\lambda) \geq \frac{\beta r\theta}{C_0}.$$

Since we can assume $\phi \geq \frac{3}{8}r^2\mu\delta$ in B_r, the second assertion follows.

Now we are ready to prove our main proposition.

Proposition 4.8 *Let v be a solution of (1.5) satisfying Corollary 3.5 in $B_5 \times (-5,5)$. Then there exist $J > 0, S > 0$ depending on the constants in Corollary 3.5 and a monotone family of cones $\hat{K}(\nu_k, \Theta_k)$ such that $\Theta_k \geq (1-S)\Theta_{k-1} + S\frac{1}{2}\pi$ and for any $\tau \in \hat{K}(\nu_k, \Theta_k), (x,t) \in B_{J^k} \times (-J^k, J^k)$,*

$$D_\tau v(x,t) \geq 0.$$

PROOF. Let $w(x,t) = v((x,t)+\gamma\tau)$ with $\tau \in \hat{K}(\nu, \Theta_0)$,where $\hat{K}(\nu, \Theta_0)$ is the cone defined in Proposition 4.1. Then v and w satisfy the hypothesis of Proposition 4.5 with

$$\delta = C\gamma\epsilon^{-1}\mathrm{cos}\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r}),$$

with ϵ and r as determined by Proposition 4.6 and Lemma 4.3. Therefore, by Corollary 4.7 for any $\lambda > 0$,

$$\mathrm{dist}((\mathrm{v(x,t)} + \gamma\tau) = \lambda); (\mathrm{v(x,t)} = \lambda)) \geq \frac{\beta}{\mathrm{L}}\mathrm{r}\theta$$

in $B_r \times (-r, r)$. We know that $\theta \geq \frac{3}{8}r^2\mu\delta$, where μ is the constant in Proposition 4.6. Thus

$$\mathrm{dist}((\mathrm{v(x,t)} + \gamma\tau) = \lambda); (\mathrm{v(x,t)} = \lambda)) \geq \overline{\mathrm{C}}\gamma\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle,$$

where $\overline{C}$ depends only on β, L and C_1. So,

$$\begin{aligned} D_\tau v(x,t) &= \lim_{\gamma\to 0}\frac{v((x,t)+\gamma\tau) - v(x,t)}{\gamma} \\ &\geq \beta\overline{C}\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle \end{aligned}$$

in $B_r \times (-r, r)\bigcap(v > 0)$. Let $H = \{(x,t)|\langle(x,t), \hat{\nabla}v(\overline{\nu}, -2r)\rangle \geq 0\}$. Then $\partial H = \{(x,t)|\langle(x,t), \hat{\nabla}v(\overline{\nu}, -2r)\rangle = 0\}$. So $\mathrm{dist}(\tau, \partial\mathrm{H}) = \cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle$, and

$$D_\tau v(x,t) \geq C\mathrm{dist}(\tau, \partial\mathrm{H}),$$

where C is a constant depends only on β, L and C_1.

Let $\rho(\tau) = \tilde{C}|\tau|\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle$. If $q \in B(\tau, \rho(\tau))$, the ball with center τ and radius $\rho(\tau)$, then $D_q v(x,t) \geq 0$ for any $(x,t) \in B_r \times (-r, r)$. In fact let $\tilde{C}$ be such that

$$\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, \mathrm{t})\rangle \geq \tilde{\mathrm{C}}\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle$$

in $B_r \times (-r,r)$ $(\tilde{C} = C/L)$. Then if $q \in B(\tau, \rho(\tau))$ then $\sin\langle\tau, \mathrm{q}\rangle \leq \frac{\rho(\tau)}{|\tau|} \leq$ $\tilde{\mathrm{C}}\cos\langle\tau, \hat{\nabla}\mathrm{v}(\overline{\nu}, -2\mathrm{r})\rangle$. Hence $\sin\langle\tau, \mathrm{q}\rangle \leq \cos\langle\tau, \hat{\nabla}\mathrm{v}(\mathrm{x}, \mathrm{t})\rangle$. Hence

$$D_q v(x,t) = |\hat{\nabla} v| \cos\langle \mathrm{q}, \hat{\nabla}\mathrm{v}(\mathrm{x},\mathrm{t})\rangle \geq |\hat{\nabla}\mathrm{v}| \sin\langle\tau, \mathrm{q}\rangle \geq 0.$$

So let $S_{\tilde{C}} = \bigcup_{\tau \in K(\nu,\Theta_0)} B(\tau, \rho(\tau))$; then

$$q \in S_{\tilde{C}} \Rightarrow D_q v(x,t) \geq 0 \quad in \quad B_r \times (-r,r).$$

Then by Lemma 16 of [5],$S_{\tilde{C}}$ contains an intermediate cone $\hat{K}(\nu_1, \Theta_1)$. That is there exist $\nu_1 \in R^{N+1}, \Theta_1 \geq \Theta_0 + S(\frac{\pi}{2} - \Theta_0)$ such that

$$\hat{K}(\nu, \Theta_0) \subset \hat{K}(\nu_1, \Theta_1) \subset S_{\tilde{C}}.$$

where S depends only on $\tilde{C}$ and Θ_0 and we can choose it in such a way that it will be the same as we obtain by replacing Θ_0 by any $\Theta \geq \Theta_0$ in the definition of $S_{\tilde{C}}$.

Let $v_{r/5}(x,t) = \frac{5}{r} v(\frac{1}{5}rx, \frac{1}{5}rt)$ for $(x,t) \in B_5 \times (-5,5)$; Then $v = v_{\frac{r}{5}}$ satisfies equation (1.4) and the conditions of Corollary 3.5 with the same constants β, L and C_1. Also, for any $\tau \in \hat{K}(\nu_1, \Theta_1)$,

$$D_\tau v_{\frac{r}{5}}(x,t) \geq 0 \quad in \quad B_5 \times (-5,5).$$

So we can repeat the argument above and deduce that, in $B_r \times (-r,r)$,

$$D_q v_{\frac{r}{5}}(x,t) \geq 0$$

for any $q \in \hat{K}(\nu_2, \Theta_2) \supset \hat{K}(\nu_1, \Theta_1)$ with $\Theta_2 \geq \Theta_1 + S(\frac{\pi}{2} - \Theta_1)$; That is $D_\tau v(x,t) \geq 0$ in $B_{(\frac{r}{5})^2} \times (-(\frac{r}{5})^2, (\frac{r}{5})^2)$ for any $\tau \in \hat{K}(\nu_2, \Theta_2)$.

Rescaling and repeating the argument above we end up with a monotone family of cones $K(\nu_k, \Theta_k)$ such that $\Theta_k \geq \Theta_{k-1} + S(\frac{\pi}{2} - \Theta_{k-1})$ and $D_\tau v(x,t) \geq 0$ in $B_{(\frac{r}{5})^k} \times (-(\frac{r}{5})^k, (\frac{r}{5})^k)$ for any $\tau \in K(\nu_k, \Theta_k)$.

Proof of Theorem 1.2. Let (x^0, t^0) be the point as in Theorem 1.2. Then by dilation we may assume $(x^0, t^0) \in \Gamma \bigcap B_1 \times (-1, 1)$. Let

$$w(x,t) = v_h(x^0 + x, t^0 + t).$$

Then $w(0,0) = v_h(x^1, t^1)$ and w satisfies the conditions of Corollary 3.5 in $B_5 \times (-5, 5)$ and by Proposition 4.8, there exists a monotone family of cones $\hat{K}(\nu_k, \Theta_k)$ such that $D_\tau \bar{v} \geq 0$ in $B_{J^k} \times (-J^k, J^k)$ if $\tau \in \hat{K}(\nu_k, \Theta_k)$ with $\Theta_k \geq \Theta_{k-1} + S(\frac{1}{2}\pi - \Theta_{k-1})$. J and $S > 0$ depend only on constants in Corollary 3.5.

Since $\Theta_k \to \frac{1}{2}\pi$ as $k \to \infty$, the free boundary of w is differentiable at $(0,0)$ i.e., Γ is differentiable at (x^0, t^0). In fact,

$$\begin{aligned}
\Theta_k &\geq \frac{S}{2}\pi + (1-S)\Theta_{k-1} \geq \frac{S}{2}\pi + (1-S)(\frac{S}{2}\pi + (1-S)\Theta_{k-1}) \\
&= \frac{S}{2}\pi + \frac{S(1-S)}{2}\pi + (1-S)^2\Theta_{k-2} \geq \cdots \\
&\geq \frac{S}{2}\pi(1 + (1-S) + \cdots + (1-S)^{k-1}) + (1-S)^k\Theta_0 \\
&= \frac{\pi}{2}S\frac{1-(1-S)^k}{1-(1-S)} + (1-S)^k\Theta_0 \\
&= \frac{\pi}{2}(1-(1-S)^k) + (1-S)^k\Theta_0 \\
&= \frac{\pi}{2} - (1-S)^k(\frac{\pi}{2} - \Theta_0)
\end{aligned}$$

Also,

$$\begin{aligned} |\nu_{k+1} - \nu_k|^2 &= 2 - 2(\nu_{k+1}, \nu_k) = 2(1 - \cos\theta) \\ &\leq 2\theta^2 \leq 2(\frac{\pi}{2} - \Theta_k)^2 \\ &\leq 2(\frac{\pi}{2} - \Theta_0)^2(1 - S)^{2k}. \end{aligned}$$

Let us write $\nu(x^0, t^0) = \lim_{k\to\infty} \nu_k$. Then we have

$$|\nu(x^0, t^0) - \nu_k| \leq C\lambda^k \quad \textit{with} \quad 0 < \lambda < 1. \tag{4.2}$$

Assume now $(x, t) \in \Gamma$ and suppose

$$|x - x^0| < (\frac{r}{5})^k, \quad |t - t^0| < (\frac{r}{5})^k.$$

Then $D_\tau w(y, s) \geq 0$ if $w > 0$ for any $\tau \in \hat{K}(\nu_k, \Theta_k)$ and (y, s) close to (x, t). Thus $\langle \hat{\nabla} v(y, s), \nu_k \rangle \leq \frac{1}{2}\pi - \Theta_k$. Since

$$\frac{\hat{\nabla} v(y, s)}{|\hat{\nabla} v(y, s)|} \to \nu(x, t) \quad as \quad (y, s) \to (x, t),$$

we deduce that

$$\langle \nu(x, t), \nu_k \rangle \leq \frac{1}{2}\pi - \Theta_k \leq (1 - S)^k(\frac{1}{2}\pi - \Theta_0).$$

Consequently $|\nu(x,t) - \nu_k| \leq C\lambda^k$ with the same constants C and λ as in (4.2). Therefore

$$|\nu(x,t) - \nu(x^1,t^1)| \leq 2C\lambda^k.$$

This implies that there exists $K > 0$ and $0 < \alpha < 1$ depending only on β, L and C_1 such that

$$|\nu(x,t) - \nu(x^0,t^0)| \leq K(|x - x^1| + |t - t^1|)^\alpha$$

for any $(x^0,t^0) \in \Gamma \bigcap B_1 \times (-1,1)$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] N. D. Alikakos and L. C. Evans, Continuity of the gradient of the solutions of certain degenerate parabolic equations, *J. Math. Pures et Appl.* **62(9)** (1983), 253-268.

[2] D. G. Aronson, *Nonlinear Diffusion Problems, Free Boundary Problems: Theory and Applications*, vol. I, Pitman, 1983.

[3] D. G. Aronson and J. L. Vazquez, Eventual C^∞-regularity and concavity for flows in one-dimensional porous media, *Arch. Rational Mech. Anal.* **99(4)** (1987), 329-348.

[4] L. A. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, *Ind. Univ. Math. J.* **29** (1980), 361-381.

[5] L. A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries 1. Lipschitz free boundaries are $C^{1,\alpha}$, *Rev. Mat. Iber.* **3** (1987) 139-162.

[6] L. A. Caffarelli and N. J. Wolanski, $C^{1,\alpha}$ regularity of the free boundary for the N-dimensional porous medium equation, *Comm. Pure Appl. Math.* **43** (1990), 885-902.

[7] L. A. Caffarelli, J. L. Vázquez and N.I. Wolanski, Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation,*Ind. Univ. Math. J.* **36** (1987), 373-401.

[8] Y. Chen, Hölder continuity of the gradient of the solutions of certain degenerate parabolic equations, *Chin. Ann. of Math.* **8B** (1987), 343-356.

[9] H. J. Choe and J. Kim, Regularity for the interfaces of evolutionary p-Laplacian functions, *Siam J. Math. Anal.* **26 (4)** (1995), 791-819

[10] E. DiBenedetto, *Degenerate Parabolic Equations*, Spring-Verlag, 1993.

[11] E. DiBenedetto, Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations, *Arch. Rational Mech. Anal.* **100** (1988), 129-147

[12] E. DiBenedetto and M. A. Herrero, On the Cauchy problem and initial traces for a degenerate parabolic equation, *Trans. Amer. Math. Soc.* **314 (1)** (1989), 187-224

[13] J. R. Esteban and J. Vazquez, R*é*gularit*é* des solutions positives de l'*é*quation parabolique p-laplacienne, *C.R. Acad. Sci. Paris Ser. I Math.* **310** (1990), 105-110.

[14] L. C. Evans, A new proof of local $C^{1,\alpha}$ regularity for the solutions of certain degenerate elliptic P. D. E., *J. Diff. Eqs.* **45** (1982), 356-373.

[15] K. Höllig and H. O. Kreiss, C^{∞}-Regularity for the porous medium equation, *Math. Z.* **192** (1986) 217-224.

[16] B. F. Knerr, The porous medium equation in one dimension, *Trans. Amer. Math. Soc,* **234** (1977), 381-415

[17] H. Yuan, Hölder continuity of interfaces for the porous medium equation with absorption, *Comm. Partial Differential Equations* **18(5,6)** (1993), 965-976

[18] H. Yuan, Regularity of the free boundary for general filtration equations, *Northeastern Math. J.* **9(2)** (1993), 197-210.

[19] H. Yuan, Existence and nonexistence of interfaces of weak solutions for non-linear degenerate parabolic systems, *J. Partial Diff. Eqs.* **9** (1996), 177-185.

[20] J. Zhao and H. Yuan, Lipschitz continuity of solutions and interfaces of the evolution p-Laplacian equation, *Northeastern Math. J.* **8(1)** (1992), 21-37.

[21] J. Zhao, The asymptotic behavior of solutions of a quasilinear degenerate parabolic equation, *J. Differential Equations* **103** (1993), 33-52.

[22] J. Zhao, Lipschitz continuity of the free boundary of some nonlinear degenerate parabolic equations, *Nonlinear Anal.* **28(6)** (1997), 1047-1062.

[23] O. A. Ladyzhenskaya, N.A. Solonnikov and N.N. Uraltzeva, Linear and quasilinear equations of parabolic type, *Trans. Math. Monographs,* **23**, Amer. Math. Soc., Providence, R. I., 1968.

[24] M. Wiegner, On C^α - regularity of the gradient of solutions of degenerate parabolic systems, *Ann. Mat. Pura Appl.* **4 (145)** (1986), 385-405.