ABSTRACT

ON THE DIFFRACTION OF PLANE ELECTROMAGNETIC WAVES BY AN INFINITE SLIT

by Robert J. Spahn

The problem of the diffraction of plane polarized electromagnetic waves incident normally on an infinite slit of finite width is solved by the use of the Lebedev integral transform and the Wiener-Hopf technique. In particular, an expression for the ratio of the transmitted energy per unit area to the incident energy per unit area (transmission coefficient) is obtained for $a \ll \lambda$, where a is one-half of the slit width and λ is the wavelength.

ON THE DIFFRACTION OF PLANE ELECTROMAGNETIC WAVES BY AN INFINITE SLIT

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I. INTRODUCTION

In this thesis the exact solution of the problem of the diffraction of plane electromagnetic waves by an infinite slit of finite width in a perfectly conducting screen is discussed. The wave is normally incident and plane polarized with the electric vector parallel to the edge of the slit. The problem was first solved exactly using elliptic cylinder coordinates by Morse and Rubenstein[1] in 1938. The solution involves an infinite series of Mathieu functions. The analytical properties of these functions are even now insufficiently understood and in the exploitation of the solution, one is led almost exclusively to numerical work.

In our solution, we choose the circular cylindrical coordinate system to describe the electromagnetic field. In this coordinate system, the solution to the wave equation is an infinite series of Bessel functions of integer order multiplied by circular functions. We choose to represent the electromagnetic field by a contour integral in the complex order plane such that the infinite series of Bessel functions becomes the residue series of the contour integral.

In the circular cylindrical coordinate system, the boundary value problem is of the so-called two part variety and the boundary conditions lead to a dual set of homogeneous integral equations that we attempt to solve by the Wiener-Hopf technique. This technique leads to an infinite set of equations in an infinite number of unknowns which we solve by successive approximation. The solution thus obtained enables us to verify all of the terms except one and, in addition, to obtain a new term in the expression for the transmission coefficient. This quantity was first obtained by a different approximation method by Sommerfeld[2] (see, also, the review article by Bouwkamp[3]). It is defined as the ratio of the power transmitted per unit area to the power incident per unit area.

II. STATEMENT OF THE PROBLEM

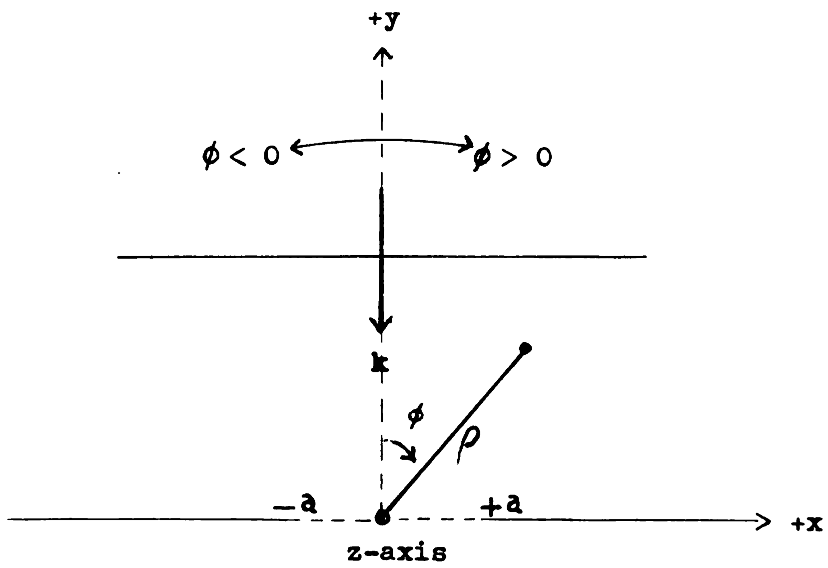

Figure 1

A plane electromagnetic wave, polarized with its electric vector parallel to the z-axis, is incident normally on an infinite slit of finite width, 2a, in a perfectly conducting screen. The slit lies in the xz plane with its length parallel to the z-axis; the screen extends to infinity both in the x and the z dimension (see Figure 1).

The time dependence is arbitrarily chosen as $e^{i\omega t}$, therefore, a plane wave of unit amplitude traveling in a given direction can then be written as $e^{-i(\mathbf{k}\cdot\mathbf{r})}$, where $\mathbf{k}$ is the

propagation vector.

The quantity E_z shall, in this thesis, be denoted by the symbol U. The total electric field U_a above the xz plane will be written as

$$U_a = U_0 + U_I \ ; \quad y \geqq 0 \tag{1}$$

where U_0 is the sum of an incident wave and a reflected wave as if there were no slit present in the perfectly conducting screen. The function U_0 is written as follows:

$$U_0 = e^{iky} - e^{-iky} \tag{2}$$

U_I is the perturbation in the total electric field caused by the presence of the slit.

Below the xz plane, the total electric field U_b is just that caused by the presence of the slit, viz.,

$$U_b = U_{II} \ ; \quad y \leqq 0 \tag{3}$$

Hereafter, U_I and U_{II} will be called the scattered fields above and below the slit, respectively.

A. Boundary Conditions

The functions U_I and U_{II} satisfy the scalar wave equation, viz.,

$$\nabla^2\psi + k^2\psi = 0 \ , \tag{4}$$

and are subject to the following conditions:

(a) $U_{I,II}(x,y,z) = U_{I,II}(-x,y,z)$; (U_I and U_{II} are symmetric about the yz plane).

(b) $U_I(x,y,z) = U_{II}(x,-y,z)$; (symmetry about the xz plane).

(c) $U_I = U_{II} = 0$, $\phi = \pi/2$, $\rho \geqq a$.

(d) $$\left.\frac{\partial U_{II}}{\partial \phi}\right|_{\phi \to \frac{\pi}{2}+} - \left.\frac{\partial U_I}{\partial \phi}\right|_{\phi \to \frac{\pi}{2}-} = -2ik\rho \;, \quad \rho \leqq a \;.$$

(e) $U_I = U_{II}$, $\rho \leqq a$, $\phi = \pi/2$.

(f) $$\lim_{\rho \to \infty} \left(\frac{\partial U_{I,II}}{\partial \rho} + ikU_{I,II} \right) \longrightarrow 0 \;;$$ (Sommerfeld's radiation condition).

(g) U_I, U_{II} and their first derivatives must be square integrable over all points (x,y), (Meixner's edge condition).

B. Discussion of the Boundary Conditions

Boundary conditions (a) and (b) are statements describing the symmetries of the wave field which are the result of the geometry of the diffracting obstacle.

Condition (c) says the surface currents in the conducting screen have such a direction as to cancel, exactly, the tangential component of the electric field incident on the screen.

Continuity of the magnetic field in the aperture is contained in condition (d). Also, continuity of the scattered field in the aperture is contained in (e).

Boundary condition (f) ensures that at great distances from the slit, the scattered field represents a divergent travelling wave.

The Meixner edge condition stated that only a finite amount of energy may be radiated by the singularity in the field at the edge of the screen (per unit length in the z direction).

C. Integral Representation of the Scattered Field

We would like to represent the solutions of the wave equation by integrals of the form

$$\int_L \mu \Lambda(\mu)\, H_\mu(k\rho)\, d\mu$$

where $H_\mu(k\rho)$ is a Hankel function and L is a contour in the complex order plane. However, in earlier work,[4] it was found convenient to discuss solutions of problems of this type for pure negative imaginary k, viz.,

$$k = -i\gamma\ , \quad \gamma > 0, \tag{5}$$

because this puts milder restrictions on the choice of a contour and allows the use of the Lebedev transform theorem. After obtaining the solution of the problem for pure negative imaginary k, we transform to real positive k. In the light of the above discussion, we represent the scattered fields above and below the slit as follows:

$$U_I = \int_{-i\infty}^{i\infty} \mu \Lambda(\mu)\, \cos\mu\phi\, K_\mu(\gamma\rho)\, d\mu\ ;$$

$$-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2} \tag{6}$$

$$U_{II} = \int_{-i\infty}^{i\infty} \mu \, \Lambda(\mu) \, \cos\{\mu(\pi - |\phi|)\} \, K_\mu(\gamma\rho) \, d\mu \, ;$$

$$\frac{\pi}{2} \leqq \phi \leqq \pi \, , \quad -\pi \leqq \phi \leqq -\frac{\pi}{2} \, . \tag{7}$$

Here, $K_\mu(\gamma\rho)$ is the McDonald function.

The difference in the forms of the angular functions in the representations for U_I and U_{II} is a consequence of the symmetry conditions (a) and (b).

The function $\Lambda(\mu)$ is the unknown function of our problem. A theorem, which will be stated in the next section, enables one to obtain necessary information concerning the poles and growth of $\Lambda(\mu)$.

D. Lebedev-Kontorovich Integral Transform Theorem

As was stated earlier, this powerful theorem is used to obtain necessary information about the unknown function $\Lambda(\mu)$ in terms of the boundary values. The transform theorem[5] will be stated without proof. It says the following: suppose we represent a function $g(\gamma\rho)$ by ("transform integral")

$$g(\gamma\rho) = \int_{-i\infty}^{i\infty} \mu \, \lambda(\mu) \, K_\mu(\gamma\rho) \, d\mu \, , \tag{8}$$

then ("inversion integral")

$$\frac{\lambda(\mu)}{\sin \pi\mu} = \frac{i}{\pi^2} \int_0^\infty g(\gamma\rho) \, K_\mu(\gamma\rho) \, \frac{d\rho}{\rho} \tag{9}$$

provided both integrals converge, $g(0) = 0$, and $\lambda(\mu)/\sin \pi\mu$

is an even function of μ and analytic in a strip of finite width containing the imaginary axis.

III. APPLICATION OF THE BOUNDARY CONDITIONS AND A DISCUSSION OF THE RESULTING INTEGRAL EQUATIONS

In this section, boundary conditions (c) and (d) will be imposed on (6) and (7). These will lead to a pair of integral equations that will contain the unknown function $\Lambda(\mu)$.

It will also be shown in this section that it is possible to discuss the overall properties of $\Lambda(\mu)$ on the complex μ-plane such as the location of its singularities and its growth as $|\mu| \longrightarrow \infty$. This is possible without actual knowledge of this function by discussions involving the boundary values of the scattered field on the screen and in the aperture.

Repeated use of Lebedev's transform theorem will be made in this discussion.

Application of boundary condition (c) to (6) and (7) yields a homogeneous integral equation, viz.,

$$\int_{-i\infty}^{i\infty} \mu\, \Lambda(\mu)\; K_\mu(\gamma\rho)\, \cos\frac{\mu\pi}{2}\, d\mu = 0\;;\; \rho \geqq a. \quad (10)$$

Application of boundary condition (d) to (6) and (7) yields an inhomogeneous integral equation, viz.,

$$\int_{-i\infty}^{i\infty} \mu^2 \Lambda(\mu) \sin \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu = -\gamma\rho ;$$

$$\rho \leqq a . \tag{11}$$

The two integral equations, (10) and (11), are fundamental in the solution of the problem and represent the known boundary values on the screen and in the aperture.

The complete representation of the boundary values is as follows:

$$\int_{-i\infty}^{i\infty} \mu \Lambda(\mu) \cos \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu =$$

$$= \begin{Bmatrix} U_{ap} & ; & \rho \leqq a \\ 0 & ; & \rho \geqq a \end{Bmatrix} \tag{12}$$

$$\int_{-i\infty}^{i\infty} \mu^2 \Lambda(\mu) \sin \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu =$$

$$= \begin{Bmatrix} -\gamma\rho & ; & \rho \leqq a \\ -\frac{\rho}{2} \left\{ \frac{\partial U_{II,I}}{\partial y} \right\} & ; & \rho \geqq a \end{Bmatrix} \tag{13}$$

where U_{ap} is the unknown value of the scattered field in the aperture, and

$$\left\{ \frac{\partial U_{II,I}}{\partial y} \right\} = \left.\frac{\partial U_{II}}{\partial y}\right|_{y\to 0^-} - \left.\frac{\partial U_I}{\partial y}\right|_{y\to 0^+}$$

is the discontinuity of the magnetic field across the xz plane (unknown for $\rho \geqq a$).

A. Properties of $\Lambda(\omega)$

It is now possible to study the properties of $\Lambda(\omega)$ without actually knowing $\Lambda(\omega)$. In order to perform this study, we apply the Lebedev theorem to equations (12) and (13) giving the unknown function $\Lambda(\omega)$ in terms of the boundary values.

By applying the Lebedev theorem to (13), we find, formally, that

$$\Lambda(\omega) = - \frac{2\gamma a \cos\frac{\mu\pi}{2}}{\pi^2\mu} \{K_\mu(x) \frac{ds_{0,\mu}(ix)}{dx} - s_{0,\mu}(ix) \frac{dK_\mu(x)}{dx} \}_{x=\gamma a} - \frac{i \cos\frac{\mu\pi}{2}}{\pi^2\mu} \int_a^\infty \{ \frac{\partial U_{II,I}}{\partial y} \} K_\mu(\gamma\rho)\, d\rho \,. \qquad (14)$$

Here, $K_\mu(x)$ is the McDonald function of complex order and $s_{0,\mu}(ix)$ is a Lommel function which arises in evaluating a factor of the inverse Lebedev transform integral for the function

$$\left\{ \begin{array}{ll} -\gamma\rho \;; & \rho \leqq a \\ 0 \;; & \rho \geqq a \end{array} \right\} ,$$

viz.,

$$\int_0^{\gamma a} K_\mu(x)\, dx \,.$$

It remains to be proven that the conditions of the Lebedev theorem are satisfied. Two of these, viz., $g(0) = 0$, and the convergence of the first transform integral obviously are satisfied by inspection of (13). The other conditions are that the inverse integral, viz., (14), converge uniformly in μ and that $\frac{\lambda(\mu)}{\sin \pi\mu} = \frac{\mu\Lambda(\mu)}{2\cos\frac{\mu\pi}{2}}$ is even and analytic in an infinite strip of finite width around the imaginary μ axis. That these are, indeed, satisfied will be shown later.

Now it can be shown that the function inside the curly brackets of (14) is an entire function of μ.

The integral in the second member on the right side of (14) is also an entire function of μ. First of all, $|K_\mu(\gamma\rho)| \sim$ $\sim e^{-\gamma\rho}\ \rho^{-1/2}$ as $\rho \longrightarrow \infty$ for all finite μ. Secondly, $\frac{\partial U_{II,I}}{\partial y}$ each satisfy Sommerfeld's radiation condition by virtue of boundary condition (f) and, therefore, also behave like $e^{-\gamma\rho}\rho^{-1/2}$ as $\rho \longrightarrow \infty$. Moreover, the integrand is continuous in ρ for all finite μ, except at $\rho = a$, the edge. However, $\partial U/\partial y$ which here denotes the magnetic vector behaves as $(\rho - a)^{-1/2}$ by virtue of the edge condition (g). Since $K_\mu(\gamma\rho)$ is continuous at $\rho = a$, the integral, although an improper integral, converges uniformly for all finite μ -- it is entire in μ.

Note that the proof in the last paragraph also verifies one of the two validity requirements (viz., convergence of the inversion integral, not shown before) in the use of Lebedev's theorem above. Thus, (14) gives us two properties of $\Lambda(\mu)$:

(i) odd function of μ

(ii) simple pole at $\mu = 0$.

We now consider (12). As it appears, the right side does not satisfy one of the conditions of the Lebedev theorem, viz., $g(0) = 0$, because the currents giving rise to the diffracted wave field must satisfy symmetry condition (a) and will interfere constructively along the z-axis, even in the aperture, i.e., $U_{ap}(0) \neq 0$.

The function $U_{ap}(\gamma\rho)$ possesses a Taylor expansion about the point $\rho = 0$ which converges for $0 \leqq \rho \leqq a$. This expansion is even in ρ because of boundary condition (a), i.e., the scattered field is symmetric with respect to the yz plane and in the aperture, x is ρ. Thus, we can write

$$U_{ap}(\gamma\rho) = U_{ap}(0) + b_2\rho^2 + b_4\rho^4 + \dots \quad ;$$
$$0 \leqq \rho \leqq a \tag{15}$$

where $U_{ap}(0)$ is the value of the scattered field at $\rho = 0$.

In order to circumvent the difficulty appearing in (12), we subtract the value of the scattered field at the origin from both sides, viz.,

$$\int_{-i\infty}^{i\infty} \mu\Lambda(\mu) \cos\frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu - U_{ap}(0) =$$

$$= \begin{Bmatrix} U_{ap}(\gamma\rho) - U_{ap}(0) \;; & \rho \leqq a \\ -U_{ap}(0) \;; & \rho \geqq a \end{Bmatrix} \tag{16}$$

Now it can be shown that

$$1 = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} \cos\frac{\mu\pi}{2}\, K_\mu(\gamma\rho)\, d\mu .^{8} \tag{17}$$

Using (17), (16) can be written

$$\int_{-i\infty}^{i\infty} \mu\{\Lambda(\mu) - \frac{U_{ap}(0)}{\pi i \mu}\} \cos\frac{\mu\pi}{2}\, K_\mu(\gamma\rho)\, d\mu =$$

$$= V(\gamma\rho) \tag{18}$$

where

$$V(\gamma\rho) = \begin{Bmatrix} U_{ap}(\gamma\rho) - U_{ap}(0) \ ; & \rho \leqq a \\ -U_{ap}(0) \ ; & \rho \geqq a \end{Bmatrix} \tag{19}$$

$V(\gamma\rho)$ now has the proper behavior near $\rho = 0$ and one may use the Lebedev theorem to show that, formally,

$$\Lambda(\mu) = \frac{U_{ap}(0)}{\mu\pi i} + \frac{2i \sin\frac{\mu\pi}{2}}{\pi^2}\int_0^\infty V(\gamma\rho) K_\mu(\gamma\rho)\, \frac{d\rho}{\rho} \ . \tag{20}$$

The validity conditions of Lebedev's theorem again require verification in this application. Considering (18), it is obvious that the condition $g(o) = 0$ is satisfied and also that the transform integral, viz., (18), converges. We still have to prove that

$$\frac{\lambda(\mu)}{\sin\mu\pi} = \frac{\Lambda(\mu) - \frac{U_{ap}(0)}{\mu\pi i}}{2\sin\frac{\mu\pi}{2}} \tag{21}$$

is even and analytic in the strip and that the integral in (20) converges. This will be shown later.

The integrand in (20) is a continuous function of ρ in the entire infinite interval as follows from (19) and (15). As $\rho \longrightarrow \infty$, $V(\gamma\rho)$ remains finite and $|K_\mu(\gamma\rho)| \sim e^{-\gamma\rho}\rho^{-1/2}$ for all finite μ. However, near $\rho = 0$, $V(\gamma\rho) \sim \rho^2$ and $|K_\mu(\gamma\rho)| \sim \rho^{-|\mathrm{Re}\,\mu|}$, therefore, the integrand in (20) converges uniformly in μ only in the strip $-2 < \mathrm{Re}\,\mu < 2$.

At this point, notice that the result just obtained verifies the convergence of the inversion integral in the use of Lebedev's theorem on (18).

The function of μ represented by the integral in (20) can be continued outside this strip by a method whose steps we now outline: substitute the infinite series (15), and invert the order of integration and summation. This can be justified. Two types of integrals will result:

$$\int_0^a \rho^{2m+1} K_\mu(\gamma\rho)\, d\rho \quad ; \quad m = 0, 1, 2, \dots ,$$

and

$$\int_a^\infty \rho^{-1} K_\mu(\gamma\rho)\, d\rho \quad .$$

Each of these integrals lead to Wronskians of $K_\mu(\gamma\rho)$ and $s_{2m+1,\mu}(i\gamma\rho)$, $m = 0, 1, \dots$, and $s_{0,\mu}(i\gamma\rho)$, respectively. These expressions are analogous to the one in (14).

Now, these Lommel functions all possess simple poles at $\mu = \pm 2n$, $n = 1, 2, 3, \dots$.[7] The residues are unknown

because the coefficients in (15) are unknown, however, we do know where the singularities are located.

Now the factor $\sin\frac{\mu\pi}{2}$ of the integral in (20) has simple zeros at $\mu = \pm 2n$, $n = 1, 2, 3, \ldots$, therefore, the product is an entire function of μ. Thus, we again see from (20) that

(i) $\Lambda(\mu)$ has a simple pole at $\mu = 0$

(ii) $\Lambda(\mu)$ is an odd function of μ.

B. Growth of $\Lambda(\mu)$

In discussing the growth of $\Lambda(\mu)$, we consider (14) for:

(a) $|\mathrm{Re}\,\mu| \longrightarrow \infty$ and, (b) $|\mathrm{Im}\,\mu| \longrightarrow \infty$.

(a) $|\mathrm{Re}\,\mu| \longrightarrow \infty$

$$|\Lambda(\mu)| \leqq \left| \frac{2\gamma a \cos\frac{\mu\pi}{2}}{\pi^2\mu} \right| \left| K_\mu(\gamma a) \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} - \right.$$

$$\left. - s_{0,\mu}(i\gamma a) \frac{dK_\mu(\gamma a)}{d\gamma a} \right| +$$

$$+ \left| \frac{i \cos\frac{\mu\pi}{2}}{\pi^2\mu} \right| \left| \int_a^\infty \left\{ \frac{\partial U_{II,I}}{\partial y} \right\} K_\mu(\gamma\rho)\, d\rho \right| \leqq$$

$$\leqq A\left\{ \left| \frac{K_\mu(\gamma a)}{\mu} \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} \right| + \right.$$

$$\left. + \left| \frac{s_{0,\mu}(i\gamma a)}{\mu} \frac{dK_\mu(\gamma a)}{d\gamma a} \right| \right\} +$$

$$+ B \left|\frac{1}{\mu}\right| \left| \int_a^\infty K_\mu(\gamma\rho)\, d\rho \right| , \qquad (22)$$

where

$$A = \left| \frac{2\gamma a \cos\frac{\mu\pi}{2}}{\pi^2} \right| \tag{23}$$

and

$$B = \left| \frac{i \cos\frac{\mu\pi}{2} M}{\pi^2} \right| , \tag{24}$$

M being the maximum absolute value of $\left\{ \frac{\partial U_{II,I}}{\partial y} \right\}$.

Now when μ is large[7]

$$|s_{0,\mu}(i\gamma a)| \sim \left| \frac{(\gamma a)^{3/2}}{9/4 - \mu^2} \right| , \tag{25}$$

therefore,

$$\left| \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} \right| \sim \left| \frac{(\gamma a)^{1/2}}{9/4 - \mu^2} \right| . \tag{26}$$

Also,

$$\left| \frac{dK_\mu(\gamma a)}{d\gamma a} \right| \sim |\mu K_\mu(\gamma a)| . \tag{27}$$

By the same token,

$$\left| \int_a^\infty K_\mu(\gamma\rho)\, d\rho \right| \sim \left| \frac{K_\mu(\gamma a)}{\mu} \right| . \tag{28}$$

Combining these results, we can write

$$|\Lambda(\mu)| \leqq A\left\{ \left| \frac{K_\mu(\gamma a)}{\mu^3} \right| + \left| \frac{K_\mu(\gamma a)}{\mu^2} \right| \right\} + \\ + B \left| \frac{K_\mu(\gamma a)}{\mu^2} \right| . \tag{29}$$

Thus, as $|\mathrm{Re}\,\mu| \longrightarrow \infty$

$$|\Lambda(\mu)| \sim \left| \frac{K_\mu(\gamma a)}{\mu^2} \right| . \tag{30}$$

(b) $|\mathrm{Im}\mu| \longrightarrow \infty$

Again taking absolute values of both sides of (14), we write

$$|\Lambda(\mu)| \leqq \left| \frac{2\gamma a \cos\frac{\mu\pi}{2}}{\pi^2\mu} \right| \left| K_\mu(\gamma a) \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} - \right.$$

$$\left. - s_{0,\mu}(i\gamma a) \frac{dK_\mu(\gamma a)}{d\gamma a} \right| + \left| \frac{i \cos\frac{\mu\pi}{2}}{\pi^2\mu} \right| \cdot$$

$$\cdot \left| \int_a^\infty \left\{ \frac{\partial U_{II,I}}{\partial y} \right\} K_\mu(\gamma\rho)\, d\rho \right| \quad . \tag{31}$$

Now for large $|\mathcal{T}|$, it can be shown[9]

$$|K_\mu(\gamma a)| \sim \frac{e^{-(\pi/2)|\mathcal{T}|}}{|\mathcal{T}|^{1/2}} \tag{32}$$

where $\mathcal{T} = \mathrm{Im}\mu$. Also, it is easy to see that

$$\left|\cos\frac{\mu\pi}{2}\right| \sim e^{(\pi/2)|\mathcal{T}|} \quad . \tag{33}$$

Using (25) through (28), (32) and (33) in (31), we can write

$$|\Lambda(\mu)| \sim \frac{A'}{|\mathcal{T}|^{7/2}} + \frac{B'}{|\mathcal{T}|^{5/2}} + \frac{C'}{|\mathcal{T}|^{5/2}} \tag{34}$$

where A', B', and C' are constants (independent of $\mathcal{T}$).

Thus, as $|\mathcal{T}| \longrightarrow \infty$,

$$|\Lambda(\mu)| \sim |\mathcal{T}|^{-5/2} \tag{35}$$

Summarizing the properties of $\Lambda(\mu)$, we have

(i) odd function of μ

(ii) simple pole at $\mu = 0$ with residue $\dfrac{U_{ap}(0)}{\pi i}$

(iii) $|\mathrm{Re}\,\mu| \longrightarrow \infty$, $|\Lambda(\mu)| \sim \left| \frac{K_\mu(\gamma a)}{\mu^2} \right|$

(iv) $|\mathrm{Im}\,\mu| \longrightarrow \infty$, $|\Lambda(\mu)| \sim |\mathcal{T}|^{-5/2}$, $\mathcal{T} = \mathrm{Im}\,\mu$.

C. Validity of the Applications of Lebedev's Theorem

In section A, the proofs of the applicability of Lebedev's theorem to (12) and (13) were given except one, i.e., $\lambda(\mu)/\sin\mu\pi$ be even and analytic in the infinite strip of finite width containing the imaginary axis. This proof follows.

Referring to the theorem and to (13),

$$\frac{\lambda(\mu)}{\sin\mu\pi} = \frac{\mu\Lambda(\mu)}{2\cos\frac{\mu\pi}{2}} .$$

Two of the properties of $\Lambda(\mu)$ already determined were that $\Lambda(\mu)$ is odd and has a simple pole at $\mu = 0$. Therefore, $\mu\Lambda(\mu)$ is analytic at $\mu = 0$ and is even in μ. The circular function in the denominator $\cos\frac{\mu\pi}{2}$ is even in μ and has simple zeros at $\mu = \pm (2n + 1)$, $n = 0, 1, 2, \ldots$.

Since $|\Lambda(\mu)| \sim |\mathcal{T}|^{-5/2}$ as $|\mathcal{T}| \longrightarrow \infty$,

$$\left|\frac{\mu\Lambda(\mu)}{2\cos\frac{\mu\pi}{2}}\right| \sim \frac{e^{-(\pi/2)|\mathcal{T}|}}{|\mathcal{T}|^{3/2}}$$

as $|\mathcal{T}| \longrightarrow \infty$.

Thus, it has been determined that $\frac{\mu\Lambda(\mu)}{2\cos\frac{\mu\pi}{2}}$ is even and analytic in a strip $-1 < \mathrm{Re}\,\mu < 1$.

Referring now to (18),

$$\frac{\lambda(\mu)}{\sin\mu\pi} = \frac{\Lambda(\mu) - \frac{U_{ap}(0)}{\mu\pi i}}{2\sin\frac{\mu\pi}{2}} = \frac{\mu\Lambda(\mu) - \frac{U_{ap}(0)}{\pi i}}{2\mu\sin\frac{\mu\pi}{2}} .$$

By arguments, in terms of the properties of $\Lambda(\mu)$, analogous to those above, it is easy to see that the numerator and denominator are both even in μ , therefore, the ratio is even. Near $\mu = 0$,

$$\{\mu\Lambda(\mu) - \frac{U_{ap}(0)}{\pi i}\} \sim \mu,$$

also,

$$\sin\frac{\mu\pi}{2} \sim \frac{\mu\pi}{2} \quad ,$$

therefore, the ratio is analytic at $\mu = 0$.

The circular function $(\sin\frac{\mu\pi}{2})$ in the denominator ensures proper behavior for $|\Upsilon| \longrightarrow \infty$. Therefore, $\dfrac{\Lambda(\mu) - \frac{U_{ap}(0)}{\mu\pi i}}{2\sin\frac{\mu\pi}{2}}$

is even and analytic in an infinite strip $-2 < \mathrm{Re}\,\mu < 2$ containing the imaginary axis.

IV. SOLUTION OF THE PROBLEM

Before applying the Wiener-Hopf technique in the solution of this problem, it is necessary that we obtain two homogeneous integral equations over a contour L. The required properties of the integrands of these integral equations are:

(a) that each integrand be analytic on a half plane and that these two half planes be complementary with a common strip of overlap containing the contour L,

(b) that each integrand approach zero at least algebraically in all directions as the variable is allowed to go infinite in its respective half plane of analyticity.

A. The Dual Integral Equations and the Wiener-Hopf Technique

The two integral equations that we are concerned with are:

$$\int_{-i\infty}^{i\infty} \mu \Lambda(\mu) \cos \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu = 0 \;; \quad \rho \geqq a, \qquad (10)$$

and

$$\int_{-i\infty}^{i\infty} \mu \{\mu \sin \frac{\mu\pi}{2} \Lambda(\mu) - \sigma(\mu)\} K_\mu(\gamma\rho)\, d\mu = 0 \;;$$

$$\rho \leqq a \;, \qquad (36)$$

where $\sigma(\mu)$ is the Lebedev transform of the function

$$\begin{Bmatrix} -\gamma\rho \; ; & \rho \leqq a \\ 0 \; ; & \rho \geqq a \end{Bmatrix},$$

viz.,

$$\frac{\sigma(\mu)}{\sin\mu\pi} = -\frac{\gamma a}{\pi^2}\left\{K_\mu(x)\frac{ds_{0,\mu}(ix)}{dx} - \right.$$

$$\left. - s_{0,\mu}(ix)\frac{dK_\mu(x)}{dx}\right\}_{x=\gamma a} .$$

It can be shown by methods similar to those used in section III-B that $\frac{\sigma(\mu)}{\sin\mu\pi}$ is even in μ and analytic in a strip of width $-1 < \mathrm{Re}\,\mu < 1$. As $|\mathrm{Re}\,\mu| \longrightarrow \infty$, $\left|\frac{\sigma(\mu)}{\sin\mu\pi}\right| \sim \left|\frac{K_\mu(\gamma a)}{\mu}\right|$ and as $|\mathrm{Im}\,\mu| \longrightarrow \infty$, $\left|\frac{\sigma(\mu)}{\sin\mu\pi}\right| \sim e^{-(\pi/2)|\tau|}|\tau|^{-3/2}$ where $\tau = \mathrm{Im}\,\mu$.

Now the integrand in (36) is an entire function of μ. We would like to modify it so that an infinite semicircle may be added to the contour (here taken along the imaginary axis but it can be moved since the integrand is entire) without changing the value of the integral. To accomplish this, we substitute, for $K_\mu(\gamma\rho)$, the following identity, viz.,

$$K_\mu(\gamma\rho) = \frac{I_{-\mu}(\gamma\rho) - I_\mu(\gamma\rho)}{\frac{2}{\pi}\sin\mu\pi} . \tag{37}$$

With (37) substituted in (36), (36) becomes

$$\int_{-i\infty}^{i\infty}\frac{\mu}{\sin\mu\pi}\left\{\mu\sin\frac{\mu\pi}{2}\Lambda(\mu) - \sigma(\mu)\right\} I_\mu\, d\mu = 0 \; ;$$

$$\rho \leqq a . \tag{38}$$

One can see that the integrand in (38) is still entire. Its decay on a right half plane is such as to allow the addition of an infinite semicircle without changing the value of the integral. In showing this, one needs the following relation, viz.,

$$I_\mu(\gamma\rho) = \frac{(\gamma\rho)^\mu 2^{-\mu}}{\Gamma(1+\mu)}\{1 + O(\tfrac{1}{\mu})\} \tag{39}$$

as $\operatorname{Re}\mu \longrightarrow +\infty$. The remaining factor in the integrand has the following behavior, viz.,

$$\left|\frac{\mu}{\sin\mu\pi}\{\mu\sin\frac{\mu\pi}{2}\Lambda(\mu) - \sigma(\mu)\}\right| \sim$$

$$\sim \left|\frac{(\gamma a/2)^{-\mu}}{\Gamma(1-\mu)}\{1 + O(\tfrac{1}{\mu})\}\right|$$

as $\operatorname{Re}\mu \longrightarrow +\infty$. The product, therefore, behaves as

$\left|\frac{1}{\mu}(\frac{\rho}{a})^\mu\right|$ but since $\rho \leqq a$, this product goes to zero.

As $|\operatorname{Im}\mu| \longrightarrow \infty$,

$$I_\mu(\gamma\rho) \sim \frac{e^{(\pi/2)|\Upsilon|}}{\sqrt{2\pi|\Upsilon|}}$$

and

$$\frac{\mu}{\sin\mu\pi}\{\mu\sin\frac{\mu\pi}{2}\Lambda(\mu) - \sigma(\mu)\} \sim$$

$$\sim \frac{e^{-(\pi/2)|\Upsilon|}}{\sqrt{2\pi|\Upsilon|}}\sin(F(\Upsilon))$$

where $\Upsilon = \operatorname{Im}\mu$, and $F(\Upsilon)$ is a real function of Υ. It is easy to see that the product behaves as $\frac{\sin(F(\Upsilon))}{|\Upsilon|}$; this goes to zero for large $|\Upsilon|$.

Now that (38) defines a right half plane, we must be able to define (10) properly on a left half plane. In the present form of the integrand, this cannot be done. It is necessary to make the following "split":

$$\mu \Lambda(\mu) \cos\frac{\mu\pi}{2} = \theta(\mu) + \theta(-\mu) \tag{40}$$

where $\theta(\mu)$, obtained from (20), is the following expression:

$$\begin{aligned}\theta(\mu) = \frac{i\mu}{2\pi}\int_0^a \{U_{ap}(\gamma\rho) - U_{ap}(0)\}\, I_{-\mu}(\gamma\rho)\,\frac{d\rho}{\rho} + \\ + \frac{iU_{ap}(0)\mu}{\pi}\left\{\frac{\gamma a}{2}\right\}\Big\{I_{\mu}(\gamma a)\ \frac{ds_{-1,\mu}(-i\gamma a)}{d\gamma a} - \\ - s_{-1,\mu}(-i\gamma a)\ \frac{dI_{-\mu}(\gamma a)}{d\gamma a}\Big\}\ .\end{aligned} \tag{41}$$

Analysis of (41) by methods similar to those used previously shows that $\theta(\mu)$ is analytic to the left of the line $Re\mu = +2$, and that for large μ, it behaves as $|I_{-\mu}(\gamma a)|$. Utilizing these properties of $\theta(\mu)$, (10) can be modified and written as follows:

$$\int_{-i\infty}^{i\infty} \theta(\mu)\, K_{\mu}(\gamma\rho)\, d\mu = 0\ ;\quad \rho \geqq a\ . \tag{42}$$

Now as $Re\mu \longrightarrow -\infty$, $\theta(\mu) \sim \dfrac{(\gamma a/2)^{-\mu}}{\Gamma(1-\mu)}$ and $K_{\mu}(\gamma\rho) \sim$

$\dfrac{(\gamma\rho/2)^{\mu}}{\Gamma(1+\mu)}$, therefore, the product behaves as $\left|\frac{1}{\mu}\,(\rho/a)^{\mu}\right|$.

Since $Re\mu < 0$, this product goes to zero for $\rho \geqq a$ as $Re\mu \longrightarrow -\infty$.

As $|\mathrm{Im}\,\mu| = |\tau| \longrightarrow \infty$, $\Theta(\mu) \sim \dfrac{e^{+(\pi/2)|\tau|}}{|\tau|^{1/2}}$ and $K_\mu(\gamma\rho) \sim$

$\sim \dfrac{e^{-(\pi/2)|\tau|}}{|\tau|^{1/2}} \sin(F(\tau))$, therefore, the product again goes to zero. Thus, we may add an infinite semicircle to the left of the contour and, since the integrand is analytic to the left of $\mathrm{Re}\,\mu = +2$, the value of the integral is not changed.

Summarizing, we now have a dual set of integral equations satisfying the properties stated at the beginning of this chapter and, also, a so-called functional relation connecting the two integrands. They are:

$$\int_{-i\infty}^{i\infty} \frac{\mu}{\sin\mu\pi} \{\mu \sin\frac{\mu\pi}{2}\, \Lambda(\mu) - \sigma(\mu)\} I_\mu(\gamma\rho)\, d\mu = 0 \;;$$
$$\rho \leqq a \;, \tag{38}$$

$$\int_{-i\infty}^{i\infty} \Theta(\mu)\, K_\mu(\gamma\rho)\, d\mu = 0 \;; \quad \rho \geqq a, \tag{42}$$

and

$$\mu \Lambda(\mu) \cos\frac{\mu\pi}{2} = \Theta(\mu) + \Theta(-\mu) \;. \tag{40}$$

We now apply the Wiener-Hopf technique beginning with (38) by writing the integrand in terms of a "plus function," i.e., a function that is analytic on a right half plane with algebraic decay in all directions on that half plane, viz.,

$$\frac{(\gamma a/2)^\mu}{\mu \Gamma(\mu)} \{ \frac{\mu \sin\frac{\mu\pi}{2}\, \Lambda(\mu) - \sigma(\mu)}{\sin\mu\pi} \} = g^+(\mu) \;; \tag{43}$$

$g^+(\mu)$ has a decay built into it which is at least of order

$\frac{1}{\mu^2}$. Using (40), (43) can be written

$$g^+(\mu) = \frac{(\gamma a/2)^{\mu}}{\mu\Gamma(\mu)}\left\{\frac{\tan\frac{\mu\pi}{2}\,(\Theta(\mu) + \Theta(-\mu)) - \sigma(\mu)}{\sin\mu\pi}\right\} \quad . \tag{44}$$

We now define $h^-(\mu)$ by

$$\Theta(\mu) = \frac{(\gamma a/2)^{-\mu}\mu\, h^-(\mu)}{\Gamma(1-\mu)} \tag{45}$$

or

$$h^-(\mu) = \frac{(\gamma a/2)^{\mu}\,\Gamma(1-\mu)\Theta(\mu)}{\mu} \quad . \tag{46}$$

One can easily see that $h^-(\mu)$ is analytic to the left of the imaginary axis with algebraic decay, e.g., a "minus function." Also, $h^-(-\mu)$ (a corresponding "plus function") would be analytic to the right of the imaginary axis and its value would be

$$h^-(-\mu) = \frac{(\gamma a/2)^{-\mu}\Gamma(1+\mu)\Theta(-\mu)}{(-\mu)} \quad . \tag{47}$$

When (45) and (47) are inserted in (44), we get

$$g^+(\mu) = \left\{\frac{\tan\frac{\mu\pi}{2}}{\pi}\right\} h^-(\mu) - \frac{(\gamma a/2)^{2\mu}\, h^-(-\mu)}{2\Gamma(\mu)\Gamma(1+\mu)\cos^2\frac{\mu\pi}{2}} -$$

$$- \frac{(\gamma a/2)^{\mu}\sigma(\mu)}{\Gamma(\mu)\sin\mu\pi} \quad . \tag{48}$$

One more step is to write $\sigma(\mu)$ in terms of a "plus function" and a "minus function." We begin by writing

$$\sigma(\mu) = Q_1(\mu) + Q_2(\mu) \tag{49}$$

where

$$Q_1(\mu) = -\frac{\gamma a}{2\pi}\{I_{-\mu}(\gamma a)\ \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} -$$

$$- s_{0,\mu}(i\gamma a)\ \frac{dI_{-\mu}(\gamma a)}{d\gamma a}\} \tag{50}$$

and

$$Q_2(\mu) = \frac{\gamma a}{2\pi}\{I_{\mu}(\gamma a)\ \frac{ds_{0,\mu}(i\gamma a)}{d\gamma a} -$$

$$- s_{0,\mu}(i\gamma a)\ \frac{dI_{\mu}(\gamma a)}{d\gamma a}\}\ . \tag{51}$$

Now, it can be shown that Q_1 and Q_2 are entire functions and that $Q_2(-\mu) = -Q_1(\mu)$. We introduce "plus" and "minus" functions into $\sigma(\mu)$ by writing

$$Q_1(\mu) = \frac{(\gamma a/2)^{-\mu}}{\Gamma(1-\mu)}\, q_1^-(\mu) \tag{52}$$

and

$$Q_2(\mu) = \frac{(\gamma a/2)^{\mu}}{\Gamma(1+\mu)}\, q_2^+(\mu)\ . \tag{53}$$

Notice also that $q_2^+(-\mu) = -q_1^-(\mu)$. Now by inserting (49), (52), and (53) into (48), there results

$$g^+(\mu) = \underbrace{\{\frac{\tan\frac{\mu\pi}{2}}{\pi}\}h^-(\mu)}_{\text{i}} - \underbrace{\frac{(\gamma a/2)^{2\mu}h^-(-\mu)}{2\Gamma(\mu)\Gamma(1+\mu)\cos^2\frac{\mu\pi}{2}}}_{\text{ii}} - \underbrace{\qquad}_{\text{iii}}$$

$$- \underbrace{\frac{q_1^-(\mu)}{\mu\pi}}_{\text{iv}} - \underbrace{\frac{(\gamma a/2)^{2\mu}\, q_2^+(\mu)}{\Gamma^2(\mu+1)\sin\mu\pi}}_{\text{v}}\ . \tag{54}$$

Considering (54), we would now like to collect plus terms on one side and minus terms on the other. Before doing this, we must discuss where the poles lie in each of the terms (i), (ii), (iii), (iv) and (v) in (54):

(i) is an entire function of μ having algebraic decay in all directions,

(ii) -- remove its simple poles at $\mu = -1, -3, \ldots, -(2n+1)$, $n = 0, 1, 2, \ldots$ by subtracting an infinite partial fraction series; then this function minus this partial fraction series will be a minus function to the left of $\mathrm{Re}\,\mu = +1$,

(iii) -- remove simple and double poles at $\mu = (2n+1)$, $n = 0, 1, 2, \ldots$ by infinite partial fraction series, then this function minus its partial fraction series will be a plus function to the right of $\mathrm{Re}\,\mu = -1$,

(iv) is a minus function to the left of $\mathrm{Re}\,\mu = 0$,

(v) -- remove simple poles at $\mu = 0, 1, 2, \ldots$, then this function minus its partial fraction series will be an entire function of μ.

When each term in (54) is modified according to the preceeding prescription, "plus functions" are collected on one side of the equation and "minus functions" on the other. The properties of the "minus" side of the resulting equation are

(1) analytic to the left of the line $\mathrm{Re}\,\mu = 0$,

(2) algebraic decay in all directions in that half plane.

The properties of the "plus" function side are

(1) analytic to the right of the line $\operatorname{Re}\mu = -1$,

(2) algebraic decay in all directions in that half plane.

Thus, the "plus" side and the "minus" side have a common strip of overlap, each side is analytic in its respective half plane and has algebraic decay in that half plane, therefore, by Liouville's theorem, each side is equal to a constant, zero.

Letting $\mu = -2m - 1$, $m = 0, 1, 2, \ldots$, the minus side of the above discussed equation (dropping the superscript) can be written

$$- \frac{2}{\pi^2} h'(-2m - 1) + \frac{1}{\pi^2} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{h(-2n - 1)}{n - m} -$$

$$- \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{(\gamma a/2)^{4n+2} h(-2n - 1)}{(2n)!(2n + 1)!(m + n + 1)^2} +$$

$$+ \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{(\gamma a/2)^{4n+2} h(-2n - 1)}{(2n)!(2n + 1)!(m + n + 1)} \{\log(\frac{\beta a}{2}) -$$

$$- \psi(2n + 1) - \frac{1}{4n + 2}\} - \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{(\gamma a/2)^{4n+2} h'(-2n-1)}{(2n)!(2n+1)!(m+n+1)} =$$

$$= \frac{q(2m + 1)}{\pi(2m + 1)} - \sum_{n=0}^{\infty} \frac{(-1)^n q(n) (\gamma a/2)^{2n}}{\pi (n!)^2 (2m + n + 1)} . \qquad (55)$$

As can be seen by inspection, this infinite system involves the unknown $h(-2n - 1)$ and its derivative. It will be seen in the next section that the scattered fields can be written in terms of this unknown.

B. The Scattered Field Expressed as a Series Expansion

Referring to (7), the scattered field, U_{II}, for $y \leqq 0$ (below the slit) is

$$U_{II} = \int_{-i\infty}^{i\infty} \mu\Lambda(\mu) \cos\{\mu(\pi - |\phi|)\} K_\mu(\gamma\rho) \, d\mu \quad . \tag{7}$$

We wish to express this integral representation as a residue series. To accomplish this, it is necessary to use (40), viz.,

$$\mu\Lambda(\mu) \cos\frac{\mu\pi}{2} = \Theta(\mu) + \Theta(-\mu) \quad . \tag{40}$$

Inserting (40) in (7) yields

$$U_{II} = 2\int_{-i\infty}^{i\infty} \frac{\Theta(\mu) \cos\{\mu(\pi - |\phi|)\} K_\mu(\gamma\rho) \, d\mu}{\cos\frac{\pi\mu}{2}} \quad . \tag{56}$$

Since $\Theta(\mu)$ is analytic to the left of $\mathrm{Re}\,\mu = +2$ and behaves as $|I_{-\mu}(\gamma a)|$, an infinite semicircle can only be added on a left half plane.

The poles of the integrand are the simple poles of $\cos\frac{\mu\pi}{2}$ which occur at $\mu = \pm(2p + 1)$, $p = 0, 1, 2, \ldots$. Therefore, applying the Cauchy Residue Theorem yields

$$U_{II} = -8i \sum_{n=0}^{\infty} \Theta(-2n - 1) \, K_{2n+1}(\gamma\rho)(-1)^n \cdot$$

$$\cdot \cos\{(2n + 1)|\phi|\} \quad . \tag{57}$$

Now

$$\theta(-2n - 1) = -\frac{(ika/2)^{2n+1} h(-2n - 1)}{\Gamma(2n + 1)} \tag{58}$$

and

$$K(ik\rho) = -\frac{i\pi}{2} e^{-(i\pi/2)(2n+1)} H^{(2)}_{2n+1}(k\rho) \quad . \tag{59}$$

Substituting (58) and (59) into (57), U_{II} becomes

$$U_{II} = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n (ka/2)^{2n+1} h(-2n - 1)}{\Gamma(2n + 1)} \cdot$$

$$\cdot \quad \cos\{(2n + 1)|\phi|\} H^{(2)}_{2n+1}(k\rho) \quad . \tag{60}$$

Note that the transformation back to real positive k has been made via $\gamma = ik$.

C. The Transmission Coefficient

The transmission coefficient, $\mathcal{T}$, is defined as the ratio of the power transmitted by the slit per unit length along z to the incident power on the slit per unit length along z. In terms of the complex Poynting vector, it can be shown that this quantity may be written

$$\mathcal{T} = -\frac{1}{ak} \int_{\pi/2}^{\pi} \mathrm{Re}(iU_{II} \; \frac{\partial U_{II}}{\partial \rho})\rho \, d\phi \quad . \tag{61}$$

When equation (60) is substituted into (61) and the integral is evaluated for large ρ, the following expression is obtained:

$$\mathcal{T} = 2\pi^2 ka\{|h(-1)|^2 + \frac{k^4a^4}{64} |h(-3)|^2 + \ldots\} \quad . \tag{62}$$

Reference is now made to equation (46), viz.,

$$h(\mu,\varepsilon) = \frac{\varepsilon^{\mu}\Gamma(1-\mu)\Theta(\mu,\varepsilon)}{\mu} \tag{46}$$

where

$$\varepsilon = \frac{\gamma a}{2}\,, \quad k = -i\gamma \quad .$$

Now it can be shown that equation (41) can be expressed as a double series, viz.,

$$\Theta(\mu,\varepsilon) = \frac{i\mu}{2\pi}\sum_{j=0}^{\infty} b_{2j}\varepsilon^{2j-\mu}\sum_{p=0}^{\infty}\frac{\varepsilon^{2p}}{(-\mu+2p+2j)p!\Gamma(p+1-\mu)} \tag{63}$$

where the b_{2j} are the same as in (15).

Combining (46) and (63), we arrive at

$$h(\mu,\varepsilon) = \frac{i}{2\pi}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\sum_{p=0}^{\infty}\frac{\varepsilon^{2p+1}}{(2p-\mu+2j)p!(1-\mu)_p} \tag{64}$$

where

$$d_{2j}(\varepsilon) = b_{2j}\varepsilon^{2j-1}$$

and

$$(\alpha)_p = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+p-1)\ ;$$

$$\alpha_0 = 1, \quad p = 0, 1, \ldots .$$

Setting $\mu = -2n - 1$ in (44) yields

$$h(-2n-1,\varepsilon) = \frac{i}{2\pi}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\cdot$$

$$\cdot\sum_{p=0}^{\infty}\frac{\varepsilon^{2p+1}}{(2p+2j+2n+1)p!(2n+2)_p}\ . \tag{65}$$

Taking the derivative of (64) and letting $\mu = -2m - 1$ gives

$$\left.\frac{dh}{d\mu}\right|_{\mu=-2m-1} = \frac{i}{2\pi}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\sum_{p=0}^{\infty} \frac{\varepsilon^{2p+1}}{p!(2p+2m+2j+1)(2m+2)_p}\cdot$$

$$\cdot\left\{\frac{1}{2m+2j+2p+1}+\frac{1}{2m+2}+\right.$$

$$\left.+\frac{1}{2m+3}+\cdots+\frac{1}{2m+1+p}\right\} . \qquad (66)$$

When (65) and (66) are substituted into (55) and the right side of (55) is expanded in terms of powers of ε, there results the following expression:

$$-\frac{i}{\pi^3}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\sum_{p=0}^{\infty}\frac{\varepsilon^{2p+1}}{p!(2p+2j+2m+1)(2m+2)_p}\cdot$$

$$\cdot\left\{\frac{1}{2p+2j+2m+1}+\frac{1}{2m+2}+\frac{1}{2m+3}+\cdots+\right.$$

$$\left.+\frac{1}{2m+1+p}\right\} +$$

$$+\frac{i}{2\pi^3}\sum_{\substack{n=0\\ n\neq m}}^{\infty}\frac{1}{n-m}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\sum_{p=0}^{\infty}\frac{\varepsilon^{2p+1}}{(2p+2n+1+2j)p!(2n+2)_p} -$$

$$-\frac{i}{4\pi^3}\sum_{n=0}^{\infty}\frac{\varepsilon^{4n+2}}{(2n)!(2n+1)!(m+n+1)^2}\sum_{j=0}^{\infty} d_{2j}(\varepsilon)\cdot$$

$$\cdot\sum_{p=0}^{\infty}\frac{\varepsilon^{2p+1}}{(2p+2n+1+2j)p!(2n+2)_p} +$$

$$+\frac{i}{\pi^3}\sum_{n=0}^{\infty}\frac{\varepsilon^{4n+2}}{(2n)!(2n+1)!}\,\frac{1}{(m+n+1)}\ \{\log\varepsilon -$$

$$- \psi(2n + 1) - \frac{1}{4n + 2}\} \sum_{j=0}^{\infty} d_{2j}(\varepsilon) \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{\varepsilon^{2p+1}}{(2p + 2n + 1 + 2j)p!(2n + 2)_p} -$$

$$- \frac{i}{2\pi^3} \sum_{n=0}^{\infty} \frac{\varepsilon^{4n+2}}{(2n)!(2n + 1)!} \frac{1}{(m + n + 1)} \sum_{j=0}^{\infty} d_{2j}(\varepsilon) \cdot$$

$$\cdot \sum_{p=0}^{\infty} \frac{\varepsilon^{2p+1}}{p!(2p + 2n + 2j + 1)(2n + 2)_p} \cdot$$

$$\cdot \{\frac{1}{2p + 2n + 2j + 1} + \frac{1}{2n + 2} + \cdots + \frac{1}{2n + 1 + p}\} =$$

$$= \frac{i\varepsilon}{(2m + 1)\pi^2} \{(1 - \delta_{m,0}) \frac{m}{2} \frac{1}{m(m + 1)} - 1\} -$$

$$- \frac{i\varepsilon^3}{2(2m + 1)\pi^2} \{- \frac{2m - 1}{3(2m + 2)} - \frac{1}{2}(1 - \delta_{m,0}) \cdot$$

$$\cdot \{\frac{1}{m(m + 1)} - \frac{(1 - \delta_{m,1})(2m - 2)}{(m + 2)(m + 1)m(m - 1)}\} + \ldots\} . \quad (67)$$

We now seek a solution of (67) for the $d_{2j}(\varepsilon)$ valid at and in the neighborhood of $\varepsilon = 0$ by successive approximation. Comparison of the magnitude of terms in (67) leads us to conclude that $d_{2j}(\varepsilon)$ must have the following form:

$$d_{2j}(\varepsilon) = d_{2j,0,0} + \varepsilon^2\{(\log \varepsilon - \psi(1) - \tfrac{1}{2})d_{2j,1,0} +$$

$$+ d_{2j,1,1}\} + \varepsilon^4\{(\log \varepsilon - \psi(1) - \tfrac{1}{2})^2 d_{2j,2,0} +$$

$$+ (\log \varepsilon - \psi(1) - \tfrac{1}{2})d_{2j,2,1} + d_{2j,2,2}\} +$$

$$+ \varepsilon^6\{ \ldots \} + \ldots \quad . \tag{68}$$

Before proceeding further in the solution of (67), let us utilize (68) to obtain a more appropriate expression for the transmission coefficient. First, the transition back to real k is made via $\gamma = ik$ and since $\varepsilon = \gamma a/2$, this is equivalent to replacing ε by $ika/2$. With this modification, one can write (64), using (68), as follows:

$$h(\mu,ka) = -\frac{1}{2\pi}\left\{\frac{ka}{2}\sum_{j=0}^{\infty}\frac{d_{2j,0,0}}{2j-\mu} - \frac{k^3a^3}{8}\cdot\right.$$

$$\cdot\left\{\sum_{j=0}^{\infty}\frac{d_{2j,0,0}}{(2j+2-\mu)(1-\mu)} +\right.$$

$$+ \left(\log\frac{\beta ka}{2} - \frac{1}{2} + \frac{i\pi}{2}\right)\sum_{j=0}^{\infty}\frac{d_{2j,1,0}}{2j-\mu} +$$

$$+ \left.\sum_{j=0}^{\infty}\frac{d_{2j,1,1}}{2j-\mu}\right\} + \frac{k^5a^5}{32}\left\{\left(\sum_{j=0}^{\infty}\frac{d_{2j,0,0}}{2!(2j+4-\mu)}\cdot\right.\right.$$

$$\cdot\left.\frac{1}{(1-\mu)(2-\mu)}\right) + \left(\log\frac{\beta ka}{2} - \frac{1}{2} + \frac{i\pi}{2}\right)\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{d_{2j,1,0}}{(2j+2-\mu)(1-\mu)} +$$

$$+\sum_{j=0}^{\infty}\frac{d_{2j,1,1}}{(2-\mu+2j)(1-\mu)} +$$

$$+ \left(\log \frac{\beta ka}{2} - \frac{1}{2} + \frac{i\pi}{2}\right)^2 \sum_{j=0}^{\infty} \frac{d_{2j,2,0}}{2j - \mu} +$$

$$+ \left(\log \frac{\beta ka}{2} - \frac{1}{2} + \frac{i\pi}{2}\right) \sum_{j=0}^{\infty} \frac{d_{2j,2,1}}{2j - \mu} +$$

$$+ \sum_{j=0}^{\infty} \frac{d_{2j,2,2}}{2j - \mu} \} + O(k^7a^7) \qquad (69)$$

where $\psi(1) = -\log\beta$.

When (69) is squared and μ takes on the values -1, -3, -5, ..., the transmission coefficient can be written as follows:

$$\mathcal{T} = \frac{k^3a^3}{8}\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right)^2 - \frac{k^5a^5}{16}\{(\log \frac{\beta ka}{2} - \frac{1}{2}) \cdot$$

$$\cdot\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j+1}\right) + \frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+3}\right) \cdot$$

$$\cdot\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right) + \left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,1}}{2j+1}\right)\} +$$

$$+ \frac{k^7a^7}{64}\{(\log \frac{\beta ka}{2} - \frac{1}{2})^2 \{\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right) \cdot$$

$$\cdot\left(\sum_{j=0}^{\infty} \frac{d_{2j,2,0}}{2j+1}\right) + \frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j+1}\right)^2\} +$$

$$+ (\log \frac{\beta ka}{2} - \frac{1}{2})\{\frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j+3}\right) +$$

$$+ \frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j + 3}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j + 1}\right) +$$

$$+ \left(\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j + 1}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,2,1}}{2j + 1}\right) +$$

$$+ \left(\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j + 1}\right)\left(\sum_{j=0}^{\infty} \frac{d_{2j,1,1}}{2j + 1}\right)\} + \dots \} +$$

$$+ O(k^9 a^9) \, . \tag{70}$$

In order to obtain expressions allowing solutions for the unknown d's, we substitute (68) into (67) and set coefficients of like orders of magnitude in ε equal to each other. Only the systems of equations giving solutions for the $d_{2j,0,0}$, $d_{2j,1,0}$, $d_{2j,1,1}$, $d_{2j,2,0}$, and $d_{2j,2,1}$ will be listed:

$$\sum_{j=0}^{\infty} d_{2j,0,0}\left\{ \frac{1}{(2j + 2m + 1)^2} - \frac{1}{2}\sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{1}{(n-m)(2j+2n+1)}\right\} =$$

$$= - \frac{\pi}{(2m + 1)} \left\{(1 - \delta_{m,0}) \frac{m}{2m(m + 1)} - 1\right\} - \frac{\pi}{2} \delta_{m,0} \tag{71}$$

$$\sum_{j=0}^{\infty} d_{2j,1,0}\left\{ \frac{1}{(2j + 2m + 1)^2} - \frac{1}{2}\sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{1}{(n-m)(2n+2j+1)}\right\} =$$

$$= \frac{1}{m + 1} \sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{(2j + 1)} \tag{72}$$

$$\sum_{j=0}^{\infty} d_{2j,1,1}\{ \frac{1}{(2j + 2m + 1)^2} - \frac{1}{2}\sum_{\substack{n=0\\n\neq m}}^{\infty} \frac{1}{(n-m)(2n+2j+1)}\} =$$

$$= \sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{(2j + 2m + 3)}\{\frac{1}{4j + 2} (2\log 2 - 2 - \frac{2}{3} - \cdots$$

$$- \frac{2}{2j + 1}) - \frac{1}{4m + 4} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m})\} -$$

$$- \frac{\pi}{8(m + 1)^2} - \frac{1}{2(m + 1)} \sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{(2j + 1)^2} - \frac{\pi}{8} \delta_{m,0} +$$

$$+ \frac{\pi}{24} \delta_{m,1} + \frac{\pi}{2(2m + 1)}\{- \frac{(2m - 1)}{3(2m + 2)} -$$

$$- \frac{(1 - \delta_{m,0})}{2} (\frac{1}{m(m + 1)} - (1 - \delta_{m,1}) \frac{(2m - 2)}{(m+2)(m+1)m(m-1)})\}$$

$$(73)$$

$$\sum_{j=0}^{\infty} d_{2j,2,0}\{ \frac{1}{(2j + 2m + 1)^2} - \frac{1}{2}\sum_{\substack{n=0\\n\neq m}}^{\infty} \frac{1}{(n-m)(2n+2j+1)}\} =$$

$$= \frac{1}{m + 1} \sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j + 1} \qquad (74)$$

Variation of $\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j+1}$ as a Function

of the Order of the Approximation

4x4

$d_{0,0,0} = 2.0187102$

$d_{2,0,0} = -1.0672165$

$d_{4,0,0} = 0.20245745$

$d_{6,0,0} = -0.73563349$

$$\sum_{j=0}^{3} \frac{d_{2j,0,0}}{2j+1} = 1.5983724$$

6x6

$d_{0,0,0} = 2.0077678$

$d_{2,0,0} = -1.0385090$

$d_{4,0,0} = 0.35200261$

$d_{6,0,0} = -2.7603063$

$d_{8,0,0} = 4.4203099$

$d_{10,0,0} = -2.7091091$

$$\sum_{j=0}^{5} \frac{d_{2j,0,0}}{2j+1} = 1.5825320$$

8x8

$d_{0,0,0} = 2.0054084$

$d_{2,0,0} = -0.89439463$

$d_{4,0,0} = -2.3352320$

$d_{6,0,0} = 14.149479$

$d_{8,0,0} = -43.945776$

$d_{10,0,0} = 66.266934$

$d_{12,0,0} = -47.718345$

$d_{14,0,0} = 12.717386$

$$\sum_{j=0}^{7} \frac{d_{2j,0,0}}{2j+1} = 1.580171$$

10x10

$$d_{0,0,0} = 2.0029674$$
$$d_{2,0,0} = -1.1958266$$
$$d_{4,0,0} = 3.8505777$$
$$d_{6,0,0} = -28.299519$$
$$d_{8,0,0} = 81.291771$$
$$d_{10,0,0} = -87.507227$$
$$d_{12,0,0} = -38.883055$$
$$d_{14,0,0} = 163.80438$$
$$d_{16,0,0} = -124.17881$$
$$d_{18,0,0} = 29.265336$$

$$\sum_{j=0}^{9} \frac{d_{2j,0,0}}{2j + 1} = 1.57383$$

12x12

$$d_{0,0,0} = 2.0026802$$
$$d_{2,0,0} = -1.0015692$$
$$d_{4,0,0} = -0.30395843$$
$$d_{6,0,0} = 1.1841691$$
$$d_{8,0,0} = -7.7028185$$
$$d_{10,0,0} = 18.972307$$
$$d_{12,0,0} = -24.769373$$
$$d_{14,0,0} = 25.040875$$
$$d_{16,0,0} = -26.083804$$
$$d_{18,0,0} = 7.5634341$$
$$d_{20,0,0} = 17.476718$$
$$d_{22,0,0} = -12.216622$$

$$\sum_{j=0}^{11} \frac{d_{2j,0,0}}{2j + 1} = 1.574942$$

14x14		16x16	
$d_{0,0,0}$	$= 2.0025757$	$d_{0,0,0}$	$= 2.0023806$
$d_{2,0,0}$	$= -0.98783842$	$d_{2,0,0}$	$= -0.97961739$
$d_{4,0,0}$	$= -0.53010458$	$d_{4,0,0}$	$= -0.65507263$
$d_{6,0,0}$	$= 2.1633397$	$d_{6,0,0}$	$= 2.5709285$
$d_{8,0,0}$	$= -7.5475131$	$d_{8,0,0}$	$= -6.9332251$
$d_{10,0,0}$	$= 9.1255387$	$d_{10,0,0}$	$= 5.6550704$
$d_{12,0,0}$	$= -2.0882360$	$d_{12,0,0}$	$= -2.6968822$
$d_{14,0,0}$	$= 6.0644652$	$d_{14,0,0}$	$= 15.213107$
$d_{16,0,0}$	$= -16.436335$	$d_{16,0,0}$	$= -13.678365$
$d_{18,0,0}$	$= -12.438990$	$d_{18,0,0}$	$= -24.266156$
$d_{20,0,0}$	$= 43.091401$	$d_{20,0,0}$	$= 36.726481$
$d_{22,0,0}$	$= -20.321572$	$d_{22,0,0}$	$= -31.944528$
$d_{24,0,0}$	$= -3.8012758$	$d_{24,0,0}$	$= 31.938553$
$d_{26,0,0}$	$= 1.8655841$	$d_{26,0,0}$	$= 18.111970$
		$d_{28,0,0}$	$= -52.795791$
		$d_{30,0,0}$	$= 21.888138$

14x14:

$$\sum_{j=0}^{13} \frac{d_{2j,0,0}}{2j+1} = 1.574912$$

16x16:

$$\sum_{j=0}^{15} \frac{d_{2j,0,0}}{2j+1} = 1.574668$$

$$\sum_{j=0}^{\infty} d_{2j,2,1} \left\{ \frac{1}{(2j + 2m + 1)^2} - \frac{1}{2} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{1}{(n-m)(2j+2n+1)} \right\} =$$

$$= - \frac{1}{4(m + 1)^2} \sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j + 1} + \frac{1}{m + 1} \sum_{j=0}^{\infty} \frac{d_{2j,1,1}}{2j + 1} -$$

$$- \frac{1}{2(m + 1)} \sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{(2j + 1)^2} \quad . \tag{75}$$

In all of the above expressions, $m = 0, +1, +2, +3, \ldots$.

It is to be pointed out that the solutions for $d_{2j,0,0}$, $d_{2j,1,0}$ and $d_{2j,2,0}$ are identical.

One also notices that the matrix of the coefficients of all the d's is the same for each expression, i.e., (71) through (75).

The method used to solve (71) and (73) was that of successive approximations. The order of the matrix and its corresponding solution is listed in Table I.

From this table, one notices that the $\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j + 1}$ appears to approach $\pi/2$. This value is necessary in order to have agreement with Sommerfeld's result[2]. (See also Bouwkamp[3]).

Granting that $\sum_{j=0}^{\infty} \frac{d_{2j,0,0}}{2j + 1}$ approaches $\pi/2$, then also do

$$\sum_{j=0}^{\infty} \frac{d_{2j,1,0}}{2j+1} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{d_{2j,2,0}}{2j+1} \quad \text{approach} \quad \pi/2.$$

In obtaining a solution to (73), it was necessary to use the numbers $d_{0,0,0}$, $d_{2,0,0}$, $d_{4,0,0}$, ... But as can be seen from Table I, the only numbers that were determined were $d_{0,0,0}$ and $d_{2,0,0}$ perhaps to the tenth place. Thus it was not possible to obtain even an approximate solution to the first coefficient $d_{0,1,1}$. However, using the results that have been obtained above, one can express the transmission coefficient, as far as we have gone in the expansion, as

$$\mathcal{T} = \frac{\pi^2 k^3 a^3}{32} - \frac{\pi^2 k^5 a^5}{64}\left(\log\frac{\beta ka}{2} - \frac{1}{2}\right) + $$
$$+ \frac{3\pi^2 k^7 a^7}{512}\left(\log\frac{\beta ka}{2} - \frac{1}{2}\right)^2 + \text{unknown terms} .$$

V. DISCUSSION AND CONCLUSIONS

The Lebedev transform, when applied to the problem of the slit, yields two homogeneous integral equations. These two equations are solved by the Wiener-Hopf technique resulting in a double infinity of linear algebraic equations in terms of the unknowns $h(-2n - 1)$ and $h'(-2n - 1)$, $n = 0, 1, 2, \ldots$. The method of successive approximations was applied to these equations.

An expression was obtained for the transmission coefficient involving the unknowns $h(-2n - 1)$ and $h'(-2n - 1)$. It was possible to obtain the first few terms in this expression. There was exact agreement between our result and the known result[3] except for a constant which we have not been able to ascertain because of the slow convergence of the series involved. However, a new term has been obtained.

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