ON THE RADIATION OF THE BICONICAL ANTENNA

Thesis for the Degree of Ph. D.

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Joseph Alphonse Meier

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This is to certify that the

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Charles P. Wells
Major professor

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By

Joseph Alphonse Meier

ABSTRACT

Submitted to the School of Graduate Studies of Michigan
State University of Agriculture and Applied Science
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ABSTRACT

The Lebedev integral transform is applied to solve the mixed boundary value problem representing the radiation of a biconical antenna. The problem is formally solved by use of the conventional Wiener - Hopf technique, and the above transform. This method does not lead to an explicit solution of the problem but to an infinite system of linear equations for the representation of the unknown transform function.

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DEDICATION

To Anita

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INTRODUCTION

The symmetrical biconical antenna of small apex angle was devised by Schelkunoff as a model for the simple wire dipole antenna for which no exact theory has yet been provided. The biconical structure permits discussion of the solution of Maxwell's equations in spherical coordinates. Study of this problem showed that the portion of space between the two cones and bounded by their surfaces forms a transmission line, with principal and higher modes. Thus Schelkunoff reduces the radiation problem to what is essentially a circuit problem, viz. the terminated biconical transmission line "loaded by empty space" with a terminal impedance whose value depends on the apex angle of the cones, their slant height and the driving frequency (Schelkunoff-8).

Schelkunoff calculates approximate values for this impedance making use of the simplifications that arise in this problem when the apex angle is nearly zero. By a simple transmission line transformation back to the origin, i.e. common vertex of the cone, the input impedance at the point generator can then be found.

A problem of considerable mathematical interest - even if of less practical significance - is the one involving symmetric biconical antennas of arbitrary apex angles, i.e. of any value between zero and 180°. This is a mixed boundary value problem which can and has been attacked in spherical coordinates by the method of separation of variables using series expansions in terms of the appropriate sets of wave functions (Schelkunoff-7). Being a mixed boundary value problem one is led to an infinite linear system for the coefficients of the expansions rather than explicit values for them. The reason for this is lack of orthogonality over the matching surface between the open space and the transmission line space.

Recent advances in the theory of mixed boundary value problems indicate that such problems can be solved explicitly by use of integral transforms. We refer here to the Wiener - Hopf technique applied to dual integral equations applicable to two complementary parts of a boundary on which the boundary conditions are not the same. This method is surveyed critically by Karp (1).

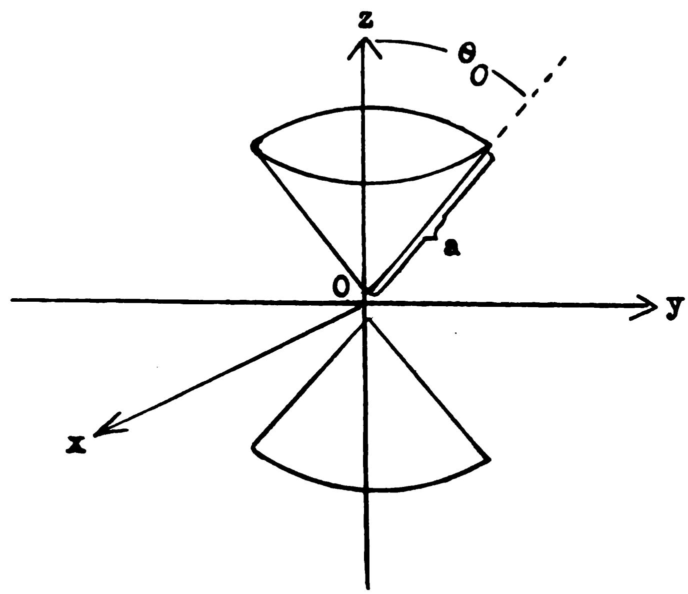
In the present problem we consider the antenna to be perfectly conducting finite conical sheets of equal apex angle and situated end to end on a common axis of symmetry. The boundary over which mixed conditions apply is the infinite double cone part of which is perfectly conducting the other part free space. In Schelkunoff's theory the conducting surfaces are capped by spherical surfaces. Here, in order to reduce the physical boundary to one coordinate surface we consider the uncapped structure.

The integral transform appropriate to the geometry of this structure is the Lebedev - Kontorovich transform (5). It has recently been applied to the problem of radiation from circular disks (Leitner, Wells - 3) which when formulated in spherical coordinates is also a mixed boundary value problem of the two part type. The present thesis is an extension of this work to the theory of biconical antennas, in the hope of obtaining simpler expressions for the input impedance of such antennas at the vertex.

II

STATEMENT OF THE PROBLEM

Consider a finite right biconical shell, c, shown in the figure below.



We assume that the generators of the cone make an angle $\theta = \theta_0 \quad (0 < \theta < \pi)$ with the z axis, and that the slant height of the cone is a .

Mathematically, we propose the following boundary value problem:

Find a function $H_\phi(r,\theta)$ satisfying the differential equation

$$(1) \qquad \nabla^2 H_\phi + \left(\frac{\omega^2}{c^2} - \frac{\csc^2\theta}{r^2}\right) H_\phi = 0$$

where ω is the frequency and c the speed of light, subject to the following conditions

(2) $\frac{\partial}{\partial \cos\theta}(\sin\theta\, H_\phi) = 0$, $r \leq a$, $\theta = \theta_0$

(3) H_ϕ is continuous across $\theta = \theta_0$, $r \geq a$

(4) $r\left(\frac{\partial H_\phi}{\partial r} + ikH_\phi\right)$ bounded as $r \rightarrow \infty$, $k^2 = \frac{\omega^2}{c^2}$,

(Sommerfeld radiation condition).

We seek a solution of the time independent Maxwell's equations (Stratton - 9)

$$\nabla \times \underline{E} + i\mu\omega \underline{H} = 0$$
$$\nabla \times \underline{H} - i\omega\varepsilon \underline{E} = 0$$

subject to the boundary conditions

(1) $\underline{E}_{\text{tangential}} = 0$, $\theta = \theta_0$

(2) Continuity of the field functions in free space,

(3) Sommerfeld radiation condition ;

μ and ε are inductive capacities, and ω the angular wave frequency.

It is well known that when symmetry with respect to ϕ is assumed, these equations separate into two independent sets, one containing E_r, E_θ, H_ϕ and the other E_ϕ, H_r, H_θ. We are interested only in the first set of equations, namely

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, H_\phi) = i\omega\varepsilon E_r$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ H_\phi) = -i\omega\varepsilon E_\theta\ ,$$

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(r\ E_\theta) - \frac{\partial E_r}{\partial \theta}\right\} = -i\omega\mu H_\phi\ .$$

It is from this set of equations that one can derive the equation

$$\frac{\partial^2 H_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial H_\phi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial H_\phi}{\partial \theta}\right) + \left(\frac{\omega^2}{c^2} - \frac{\csc^2\theta}{r^2}\right) H_\phi = 0$$

which is equation (1) cited above.

Physically, we have two conducting conical sheets fed by an alternating voltage of fixed frequency and amplitude between the two apices of the cone in the limit of zero gap. Such a method of excitation produces electrical currents on the conducting conical sheets which are purely radial. The electromagnetic field of the structure is transverse magnetic, i.e. the only component of magnetic field is H_ϕ where ϕ is the azimuthal variable about the axis of the antenna. For such an excitation it is also known, from physical considerations, that H_ϕ has azimuthal symmetry, and planar symmetry about the plane z = 0.

III

REPRESENTATION OF THE SOLUTION

Writing the differential equation in spherical coordinates we have

$$\frac{\partial^2 H_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial H_\phi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial H_\phi}{\partial\theta}\right) + \left(\frac{\omega^2}{c^2} - \frac{\csc^2\theta}{r^2}\right) H_\phi = 0.$$

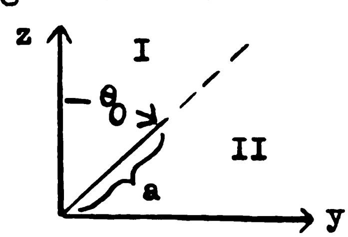
Application of the method of separation of variables yields

(5) $H_\phi =$

$$\sum_{n=0}^{\infty} (kr)^{-\frac{1}{2}} \left(a_n J_{n+\frac{1}{2}}(kr) + b_n Y_{n+\frac{1}{2}}(kr)\right)\left(c_n P_n^1(\cos\theta) + d_n P_n^1(-\cos\theta)\right)$$

where $J_{n+\frac{1}{2}}(kr)$, $Y_{n+\frac{1}{2}}(kr)$ and $P_n^1(\pm\cos\theta)$ are the Bessel, Neumann, and first order associated Lengendre functions, respectively. The coefficients a_n, b_n, c_n, and d_n are unknown constants whose values, once known, would yield a formal solution to the problem in terms of a series representation.

For physical reasons cited previously, the solution to the boundary value problem is known everywhere provided we obtain the solution to the problem in regions (I) and (II) indicated in the figure below.



Azimuthal symmetry and planar symmetry about the plane $z = 0$ provides the solution in the entire space.

We shall use a "function theoretic" method of solving the problem wherein we consider the functions in (5) as functions of their order rather than their argument. Through such an approach we will apply the Wiener-Hopf technique (1) which is essentially an application of the concepts of analytic continuation and Liouville's theorem.

From this point of view, the appropriate representation of the solution in the given regions is not a series representation, but an integral representation on the complex order plane. For the given set of boundary conditions the corresponding eigenfunctions do not form on orthogonal system. The usual methods for obtaining a formal solution in terms of a series representation can not be applied without infinite systems for the unknown quantities a_n to d_n in (5), whose coefficients are of a very complicated structure, as can be found in the work of Schelkunoff (7).

We consider the integral representations over a contour L, where L is a contour in a strip of finite width about the imaginary axis in the complex order plane, from $\sigma - i\,\infty$ to $\sigma + i\,\infty$, σ being a real number. The reason for such a contour becomes evident if one considers the Wiener-Hopf technique which is to be applied. This approach requires a study of the functions involved as functions of their order in overlapping half planes whose common region contains L.

The appropriate radial function for our solution, H_ϕ , is the Hankel function, $H_\mu^{(2)}(kr)$, where k is real. This is the function which satisfies the Sommerfeld condition cited above. However, in order that the integral representations of the solution do not diverge we must let $k = -i\gamma$, where γ is real. Such a substitution for k leads to the Macdonald function, $K_\mu(\gamma r)$. Since for $|\operatorname{Im}\mu| >> 1$, $H_\mu^{(2)}(kr) \sim e^{-\frac{\pi}{2}\operatorname{Im}\mu}$ and $K_\mu(\gamma r) \sim e^{-\frac{\pi}{2}|\operatorname{Im}\mu|}$, this change insures that our integrals converge along the contour L.

Such a substitution leads to a transition from a wave problem to one of "exponential - decay" with the same boundary conditions. The integral representations over the contour L would diverge if one returned from real γ to real k under the integral sign. However, these would converge if L were first deformed to surround semi-infinite portions of the real μ axis, and the consequent residue series leads to the correct results in the wave problem. This has recently been demonstrated in a paper by Oberhettinger (6), and verified in a paper by Leitner and Wells (3).

With these remarks we define our solutions in regions (I) and (II) as follows:

$$(6)\quad H_\phi(r,\theta) = \begin{cases} \dfrac{1}{\sqrt{r}}\displaystyle\int_L \mu\, A(\mu) K_\mu(\gamma r) P^1_{-\frac{1}{2}+\mu}(\cos\theta)\,d\mu, & 0 \le \theta < \theta_0, \\[2ex] \dfrac{1}{\sqrt{r}}\displaystyle\int_L \mu\, B(\mu) K_\mu(\gamma r) \left\{ \dfrac{P^1_{-\frac{1}{2}+\mu}(\cos\theta) + P^1_{-\frac{1}{2}+\mu}(-\cos\theta)}{\sin\frac{\pi}{2}(\mu - \frac{1}{2})} \right\} d\mu, & \theta_0 \le \theta \le \frac{\pi}{2}. \end{cases}$$

$A(\mu)$ and $B(\mu)$ are unknown functions to be determined for a formal solution of the problem.

The term $\sin \frac{\pi}{2}(\mu - \frac{1}{2})$ appearing in the second integral is introduced in order to explicitly indicate the proper eigenfunction corresponding to the principal mode $\mu = \frac{1}{2}$ of the biconical transmission line. In the limit $\mu = \frac{1}{2}$ the quantity in the bracket is indeterminate and has the value $\frac{1}{\sin\theta}$, which is the appropriate principal mode function.

Utilizing the relation (Magnus-4)

$$\frac{\partial}{\partial\cos\theta}\left\{\sin\theta\, P^{1}_{-\frac{1}{2}+\mu}(\pm\cos\theta)\right\} = \pm(\mu^2 - \tfrac{1}{4})P_{-\frac{1}{2}+\mu}(\pm\cos\theta)$$

one obtains from (6) the representation

$$(7)\quad \frac{\partial}{\partial\cos\theta}(\sin\theta\, H_{\phi}) =$$

$$\begin{cases} \dfrac{1}{\sqrt{r}}\displaystyle\int_L \mu\, A(\mu)(\mu^2 - \tfrac{1}{4})K_{\mu}(\gamma r)P_{-\frac{1}{2}+\mu}(\cos\theta)d\mu, & 0 \le \theta \le \theta_0 \\[2ex] \dfrac{1}{\sqrt{r}}\displaystyle\int_L \mu\, B(\mu)(\mu^2 - \tfrac{1}{4})K_{\mu}(\gamma r)\left\{\dfrac{P_{-\frac{1}{2}+\mu}(\cos\theta) - P_{-\frac{1}{2}+\mu}(-\cos\theta)}{\sin\frac{\pi}{2}(\mu - \frac{1}{2})}\right\}d\mu, & \theta_0 \le \theta \le \frac{\pi}{2}. \end{cases}$$

It follows from Maxwell's equations that

$$i\omega\varepsilon r E_r = -\frac{\partial}{\partial\cos\theta}(\sin\theta\, H_{\phi}).$$

Since the tangential component of E, the electric field, is to be continuous at the boundary between regions (I) and (II) cited previously we see that the jump in E_r, the tangential component of E, across $\theta = \theta_0$ for all r must be equal to zero. Computing the jump by use of (7) we have

$$\int_L \mu(\mu^2 - \tfrac{1}{4}) K_\mu(\gamma r) \left\{ \left[A(\mu) - \frac{B(\mu)}{\sin \frac{\pi}{2}(\mu - \frac{1}{2})} \right] P_{-\frac{1}{2}+\mu}(\cos\theta_0) + \right.$$

$$\left. + \frac{B(\mu)}{\sin \frac{\pi}{2}(\mu - \frac{1}{2})} P_{-\frac{1}{2}+\mu}(-\cos\theta_0) \right\} d\mu = 0$$

for $\theta = \theta_0$ and all r.

Since this is to be true for all r we must have the expression in brackets equal to zero and it follows that

$$(8) \qquad \frac{B(\mu)}{\sin \frac{\pi}{2}(\mu - \frac{1}{2})} = \frac{P_{-\frac{1}{2}+\mu}(\cos\theta_0)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} A(\mu).$$

Before enforcing condition (3) we rewrite (6) in the form

$$H_\phi(r,\theta) =$$

$$\begin{cases} -\dfrac{\sin\theta}{\sqrt{r}} \displaystyle\int_L \mu \, A(\mu) K_\mu(\gamma r) P'_{-\frac{1}{2}+\mu}(\cos\theta) d\mu \,, & 0 \le \theta \le \theta_0 \\[2em] -\dfrac{\sin\theta}{\sqrt{r}} \displaystyle\int_L \mu \frac{B(\mu)}{\sin \pi(\mu - \frac{1}{2})} K_\mu(\gamma r) \left\{ P'_{-\frac{1}{2}+\mu}(\cos\theta) - P'_{-\frac{1}{2}+\mu}(-\cos\theta) \right\} d\mu, & \theta_0 \le \theta \le \frac{\pi}{2} \end{cases}$$

where prime denotes derivative and where we have used the relations (Magnus-4)

$$P^1_{-\frac{1}{2}+\mu}(\pm \cos\theta) = \mp \sin\theta \, P'_{-\frac{1}{2}+\mu}(\pm \cos\theta) .$$

Now enforcing condition (3) which states that $H_\phi(r,\theta)$ must be continuous for $\theta = \theta_0$ and $r \geq a$ we have

$$\int_L \mu \, K_\mu(\gamma r) \left\{ \left[A(\mu) - \frac{B(\mu)}{\sin \frac{\pi}{2}(\mu - \frac{1}{2})}\right] P'_{-\frac{1}{2}+\mu}(\cos\theta_0) + \frac{B(\mu)}{\sin \frac{\pi}{2}(\mu - \frac{1}{2})} P'_{-\frac{1}{2}+\mu}(-\cos\theta_0) \right\} d\mu = 0$$

for $\theta = \theta_0$ and $r \geq a$.

With the aid of the relation between $A(\mu)$ and $B(\mu)$ given by (8) one may rewrite the integral expression just given in the form

$$\int_L \mu \, A(\mu) \, K_\mu(\gamma r) \, \frac{W[P_{-\frac{1}{2}+\mu}(\cos\theta_0), \, P_{-\frac{1}{2}+\mu}(-\cos\theta_0)]}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} \, d\mu = 0$$

where $W[P_{-\frac{1}{2}+\mu}(\cos\theta_0), \, P_{-\frac{1}{2}+\mu}(-\cos\theta_0)]$ is the Wronskian of the functions $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ and $P_{-\frac{1}{2}+\mu}(-\cos\theta_0)$, and has the value

$$-\frac{2}{\pi} \cdot \frac{1}{\sin^2\theta_0} \cos \pi\mu .$$

Hence, it follows that

$$(9) \quad \int_L \mu \frac{\cos \pi\mu \, A(\mu)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} K_\mu(\gamma r)d\mu = 0 \,, \qquad \theta = \theta_0; \; r \geq a \,.$$

Enforcing condition (2) which states that $\frac{\partial}{\partial \cos\theta}(\sin\theta \, H_\phi) = 0$

for $\theta = \theta_0$ and $r \leq a$

we have

$$(10) \quad \int_L \mu \, A(\mu)(\mu^2 - \tfrac{1}{4}) \, P_{-\frac{1}{2}+\mu}(\cos\theta_0) \, K_\mu(\gamma r)d\mu = 0 \,, \qquad \theta = \theta_0 \; r \leq a$$

The expressions (9) and (10) form a set of dual integral equations for the unknown function $A(\mu)$.

(1) Properties of $A(\mu)$

By use of certain general properties of the electromagnetic fields of radiating structures such as ours, it is now possible to discuss the behavior of $A(\mu)$ and certain related functions without explicitly knowing them. To do this one makes use of the Lebedev transform theorem. Such an analysis is necessary in order that the conventional Wiener-Hopf technique can be applied without too much difficulty in later paragraphs.

This theorem (5) we state as follows: Let $\Lambda(\mu)$ be analytic in a strip of finite width about the imaginary axis containing L and having decay at least as rapid as $I_\mu(\gamma r)$ in the distant parts of the strip. Then, provided both integrals converge,

$$\phi(\gamma r) = \int_L \mu \Lambda(\mu)\, I_\mu(\gamma r) d\mu$$

$$\Lambda(\mu) = \frac{1}{\pi i} \int_0^\infty \phi(\gamma r)\, K_\mu(\gamma r) \frac{dr}{r} .$$

Here $I_\mu(\gamma r)$ is the modified Bessel function whose growth for large $|\operatorname{Im}\mu|$ is $\dfrac{e^{+\frac{\pi}{2}|\operatorname{Im}\mu|}}{|\operatorname{Im}\mu|^{\frac{1}{2}}}$.

With the aid of condition (2) and Maxwell's equations we may write

$$\left.\begin{array}{ll} r \leq a & 0 \\ r \geq a & -i\omega\varepsilon r\, E_r \end{array}\right\} =$$

$$= \frac{1}{\sqrt{r}} \int_L \mu\, A(\mu)(\mu^2 - \tfrac{1}{4})\, K_\mu(\gamma r)\, P_{-\frac{1}{2}+\mu}(\cos\theta_0) d\mu .$$

One may show that

---

(11) $$A(\mu)(\mu^2 - \tfrac{1}{4})\, P_{-\frac{1}{2}+\mu}(\cos\theta_0) =$$

$$= -\frac{\omega\varepsilon}{\pi} \int_a^\infty \sqrt{r}\, E_r(r,\theta_0)\, I_\mu(\gamma r) dr .$$

It is well known that $H_\phi(r,\theta)$ need not be continuous across the conducting conical sheets. If we denote by $[H_\phi(r,\theta_0)]$ the discontinuity in $H_\phi(r,\theta)$ when $\theta = \theta_0$ and $r \leq a$ we have

$$\left.\begin{array}{ll} r \leq a & [H_\phi(r,\theta_0)] \\ r \geq a & 0 \end{array}\right\} =$$

$$= \frac{2}{\pi\sqrt{r}\,\sin\theta_0}\int_L \mu \frac{\cos\pi\mu\, A(\mu)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} K_\mu(\gamma r)d\mu .$$

One may also show

$$(12)\quad \frac{A(\mu)\cos\pi\mu}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} = \frac{\sin\theta_0}{2i}\int_0^a \frac{[H_\phi(r,\phi_0)]}{\sqrt{r}} I_\mu(\gamma r)dr .$$

$E_r(r,\theta_0)$ is a continuous function of r for $a \leq r < \infty$ and has the behavior $\dfrac{1}{\sqrt{r-a}}$

as r approaches a, and $\dfrac{e^{-\gamma r}}{r^2}$ at infinity. $I_\mu(\gamma r)$ has a finite value at $r = a$ and behaves as $\dfrac{e^{\gamma r}}{\sqrt{r}}$ as r approaches infinity. Furthermore, $I_\mu(\gamma r)$ is an entire function of μ. With these facts and theorems concerning integral representations of entire functions it can be shown that the integral in the right member of (11) defines an entire function of μ.

Similarly, it is known that $[H_\phi(r,\theta_0)]$ behaves as the continuous function $\frac{1}{r}$ near $r = 0$ and is continuous elsewhere in the range of integration in (12). At the lower limit, zero, the integrand in the right member of (12) behaves essentially as $r^{\mu-3/2}$. Upon further consideration, it follows that the right member of (12) defines an analytic function of μ for $\mathrm{Re}\mu > \frac{1}{2}$.

Now by (11) $A(\mu)$ has, at most, simple poles at $\mu = \pm\frac{1}{2}$ and at the zeroes, for fixed θ_0, of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$. If $A(\mu)$ has no poles at any of these values then $\int_a^\infty \sqrt{r}\; E_r(r,\theta_0) I_\mu(\gamma r)dr$ must have zeroes there.

By (12) $A(\mu)$ at most, on the right half plane $\mathrm{Re}\mu > \frac{1}{2}$, has simple poles at $\mu = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$ and only there. If not, then $\int_0^a \frac{[H_\phi(r,\theta_0)]}{\sqrt{r}} I_\mu(\gamma r)dr$ has zeroes there.

Since $P_{-\frac{1}{2}+\mu}(\cos\theta_0) = 0$ for μ equal to certain irrational real numbers, $A(\mu)$ cannot have, on the right half plane $\mathrm{Re}\mu > \frac{1}{2}$, poles at the right hand zeroes of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$, since that would contradict (12). It follows that

$$\int_a^\infty \sqrt{r}\; E_r(r,\theta_0)\; I_\mu(\gamma r)dr = 0 \text{ at the zeroes of } P_{-\frac{1}{2}+\mu}(\cos\theta_0).$$

Since by (11) $A(\mu)$ does not have poles at $\mu = \frac{3}{2}, \frac{7}{2}, \frac{11}{2} \ldots$, it follows that

$\int_0^a \frac{[H_\phi(r,\theta_0)]}{\sqrt{r}} I_\mu(\gamma r)dr = 0$ at these values. In summary, we have shown that, at most, $A(\mu)$ may have poles at $\mu = \pm \frac{1}{2}$ and at the negative zeroes of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$, for fixed θ_0.

We will now show that with certain assumptions, integrals (9) and (10) can be closed on half planes, thus insuring a formal solution for this set of dual integral equations. Actually we must show that they can be closed on complementary half planes since this is essential to application of the Wiener-Hopf technique.

We make the assumption that as $|\mu| \rightarrow \infty$ in the right half plane $Re\mu > \frac{1}{2}$

$$\int_0^a \frac{[H_\phi(r,\theta_0)]}{\sqrt{r}} I_\mu(\gamma r)dr \approx \frac{\left(\frac{\gamma a}{2}\right)^\mu}{\Gamma(1+\mu)} \mu^{-\alpha}$$

where $Re\alpha > 0$.

This implies that in the distant portions of the strip we have the behavior $e^{+\frac{\pi}{2}|\tau|} |\tau|^{-\frac{1}{2} - Re\,\alpha}$

With this assumption $\frac{\mu \cos \pi\mu \, A(\mu)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} K_\mu(\gamma r)$

behaves as $\left(\frac{a}{r}\right)^\mu \mu^{-\alpha}$ when $|\mu| \rightarrow \infty$ and as $|\tau|^{-Re\,\alpha}$ on the strip. Hence, (9) converges and can be closed to the right when $r \geq a$.

In its present form (10) cannot be closed on either half plane and a decomposition

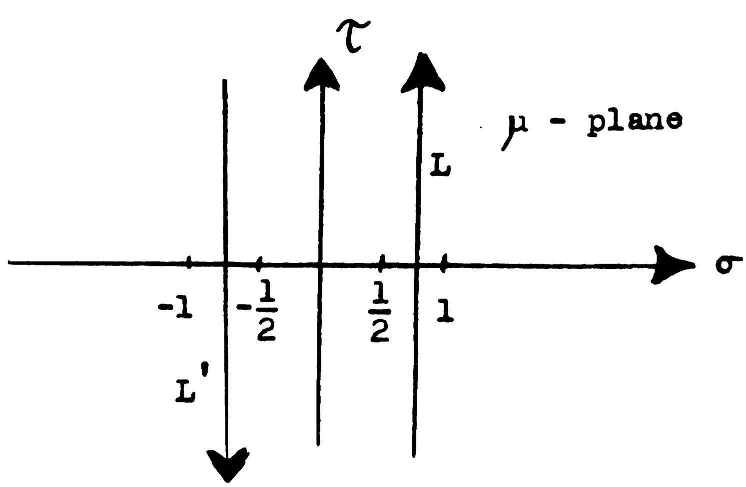
$$K_\mu(\gamma r) = \frac{\pi}{2} \frac{I_{-\mu}(\gamma r) - I_\mu(\gamma r)}{\sin \pi\mu}$$

which is the definition of $K_\mu(\gamma r)$ in terms of $I_\mu(\gamma r)$, is necessary. It follows that (10) can be written in the form

$$(13) \quad \frac{\pi}{2}\int_L \mu(\mu^2 - \tfrac{1}{4})\, A(\mu)\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)\, \frac{I_{-\mu}(\gamma r)}{\sin \pi\mu}\, d\mu$$

$$- \frac{\pi}{2}\int_L \mu(\mu^2 - \tfrac{1}{4})\, A(\mu)\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)\, \frac{I_\mu(\gamma r)}{\sin \pi\mu}\, d\mu = 0.$$

We mention at this time that these arguments, involved in closing (9) on the right half plane $\mathrm{Re}\,\mu > \frac{1}{2}$, restrict the contour L such that $\sigma > \frac{1}{2}$ but finite. In order to carry out the following arguments it is necessary to make another but final restriction that $\frac{1}{2} < \sigma < 1$.



The integrand of the second integral in (13) is free of poles in the strip $-1 < Re\mu < 1$. If we make the transformation $\mu = -\mu'$, and note that $(\mu^2 - \frac{1}{4})\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)$ is an even function of μ, we obtain

$$\frac{\pi}{2}\int_{L'} \mu(\mu^2 - \frac{1}{4})\, A(-\mu)\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)\, \frac{I_{-\mu}(\gamma r)}{\sin \pi\mu}\, d\mu$$

where L' is a contour in the complex order plane from $\sigma + i\infty$ to $\sigma - i\infty$ and $-1 < \sigma < -\frac{1}{2}$. (See figure above).

With the aid of (11) and the assumption that the integral involved behaves as $I_\mu(\gamma a)\, u^{-\beta}$, where $Re\beta > 0$, one can show that the integrand of the above integral approaches zero on distant parts of either end of the strip $-1 < Re\mu < 1$. It follows by reversing the sense of L', that the resulting contour may be deformed into L such that one may write (13) in the form

$$(14) \quad \int_L \mu(\mu^2 - \frac{1}{4})\, a(\mu)\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)\, I_{-\mu}(\gamma r)\, d\mu = 0 \qquad r \le a$$

where $a(\mu) = \frac{\pi}{2}\,\frac{A(\mu) - A(-\mu)}{\sin \pi\mu}$. By the use of (11) and the definition of $a(\mu)$ we have

$$(\mu^2 - \frac{1}{4})\, a(\mu)\, P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0) = \frac{\omega\epsilon}{\pi}\int_a^\infty \sqrt{r}\, E_r(r,\theta_0)\, K_\mu(\gamma r)\, dr\, .$$

$K_\mu(\gamma r)$ has a finite value at $r = a$ and behaves as $\dfrac{e^{-\gamma r}}{\sqrt{r}}$ at infinity. Furthermore $K_\mu(\gamma r)$ is an entire function of μ. Using these facts and those cited previously in connection with $E_r(r,\theta_0)$ one can show that the integral in the above expression converges and defines an entire function of μ. With regard to the value of this integral we assume

$$\int_a^\infty \sqrt{r}\; E_r(r,\theta_0)\; K_\mu(\gamma r)dr \approx K_\mu(\gamma a)\mu^{-\beta}$$

where $\mathrm{Re}\beta > 0$, which, in the distant portions of the strip becomes $\sqrt{\dfrac{2\pi}{|\tau|}}\; e^{-\frac{\pi}{2}|\tau|}|\tau|^{-\mathrm{Re}\beta}$.

With this assumption the integrand of (14) behaves as $\left(\frac{a}{r}\right)^\mu \mu^{-\beta}$ when $\mu \longrightarrow -\infty$ and as $|\tau|^{-\mathrm{Re}\beta}$ on the strip. Hence (14) can be closed to the left when $r \leq a$. Thus we have shown that this integral equation is formally satisfied.

To summarize, it has been shown that the dual integral equations are convergent and meaningful statements of the problem when they are written in the form

$$(15)\quad \int_L \mu \frac{\cos \pi\mu\; A(\mu)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} K_\mu(\gamma r)d\mu = 0 \qquad r \geq a$$

$$(16)\quad \int_L \mu(\mu^2 - \tfrac{1}{4})\; a(\mu)\; P_{-\frac{1}{2}+\mu}(\cos\theta_0)\; I_{-\mu}(\gamma r)d\mu = 0 \qquad r \leq a$$

where $a(\mu) = \dfrac{\pi}{2}\dfrac{A(\mu) - A(-\mu)}{\sin \pi\mu}$.

IV

SOLUTION OF THE PROBLEM

We are now in a position to attack the revised integral equations by function - theoretical techniques. In order to carry out this procedure it is more convenient to write them in the form

$$(15a)\quad \int_L D^+(\mu)\,\frac{\left(\frac{\gamma a}{2}\right)^{\mu}}{\Gamma(\mu)}\,K_{\mu}(\gamma r)\,d\mu = 0 \qquad r > a$$

$$(16a)\quad \int_L C^-(\mu)\,\frac{\Gamma(1-\mu)}{\left(\frac{\gamma a}{2}\right)^{-\mu}}\,I_{-\mu}(\gamma r)\,d\mu = 0 \qquad r \leq a$$

where

$$(17)\quad D^+(\mu) = \frac{\cos \pi\mu \quad A(\mu)}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)} \cdot \frac{\Gamma(1+\mu)}{\left(\frac{\gamma a}{2}\right)^{\mu}}$$

$$(18)\quad C^-(\mu) = u(\mu^2 - \tfrac{1}{4})\; a(\mu)\; P_{-\frac{1}{2}+\mu}(\cos\theta_0) \cdot \frac{\left(\frac{\gamma a}{2}\right)^{-\mu}}{(1-\mu)}.$$

$D^+(\mu)$ is a function analytic on the right half plane $R\ell\mu > \frac{1}{2}$ and of algebraic decay of the order $\mu^{-\alpha}$ in the strip, $\frac{1}{2} < R\ell\mu < 1$, and in all directions to its right. $C^-(\mu)$ is an entire function and of algebraic decay of the order $\mu^{-\beta}$ in the strip and in all directions to its left. The superscripts + and - are to denote "analytic" on a "plus" or "minus" half plane with the additional property of at least algebraic decay in those regions respectively.

With the aid of the definition of $a(\mu)$ we can now establish a relationship between $D^{+}(\mu)$ and $C^{-}(\mu)$ of the form

$$C^{-}(\mu) = (\mu^2 - \tfrac{1}{4})\, P_{-\frac{1}{2}+\mu}(\cos\theta_0)\,[P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)]\frac{1}{2\cos\pi\mu}\cdot$$

$$\left\{ D^{+}(\mu) - \left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)}\, D^{+}(-\mu) \right\}.$$

We now define $M_{-\frac{1}{2}+\mu}(\cos\theta_0) = P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)$

and utilize the relation $\dfrac{\pi}{\cos\pi z} = \Gamma(\tfrac{1}{2}+\mu)\Gamma(\tfrac{1}{2}-\mu)$

wherein we rewrite the previous expression in the form

$$(19)\quad C^{-}(\mu) = \frac{1}{2\pi}(\mu^2 - \tfrac{1}{4})\,\Gamma(\tfrac{1}{2}+\mu)\,\Gamma(\tfrac{1}{2}-\mu)\, P_{-\frac{1}{2}+\mu}(\cos\theta_0)\, M_{-\frac{1}{2}+\mu}(\cos\theta_0)\cdot$$

$$\left\{ D^{+}(\mu) - \left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)}\, D^{+}(-\mu) \right\}.$$

Our immediate purpose is to transform this equation into one whose left and right hand members, say, are minus and plus functions respectively. It is at this point where a modicum of investigation reveals that such an equation defines an analytic function in the entire plane and hence by Liouville's Theorem is identically zero. This will be illustrated precisely in later paragraphs. However, in order to carry out this procedure we need first to investigate some of the properties of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ and $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$.

(1) Factorization of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ - Asymptotic Behavior of the factors.

$P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ is known to possess a countable number of irrational simple zeroes which we shall denote by $\mu_{P,m}(\theta_0)$, $m = \cdots -2,-1,0,1,2,\cdots$ For fixed θ_0 it is an entire even function of μ and its zeroes are given asymptotically by the expression

$$(20)\qquad \mu_{P,m}(\theta_0) \sim \frac{\pi}{\theta_0}\left(m - \frac{1}{4}\right) + \frac{C(\theta_0)}{m}, \qquad 0 < \varepsilon \leq \theta_0 \leq \pi - \varepsilon .$$

It can be shown* that we may factor $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ as follows:

$$(21)\quad P_{-\frac{1}{2}+\mu}(\cos\theta_0) = $$

$$= P_{-\frac{1}{2}}(\cos\theta_0)\prod_{m=1}^{\infty}\left(1 + \frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{P,m}(\theta_0)}} .$$

$$\cdot\prod_{m=1}^{\infty}\left(1 - \frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{P,m}(\theta_0)}} .$$

Let us call the first infinite product appearing in (21)

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* See note 1

$k^+(\mu,\theta_0)$ and the second $k^-(\mu,\theta_0)$. In order to investigate $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ as a combination of "plus" and "minus" functions defined previously we will need to know the asymptotic behavior of both $k^+(\mu,\theta_0)$ and $k^-(\mu,\theta_0)$.

For the present we confine our attention to the growth of $k^+(\mu,\theta_0)$ as compared to the growth of the infinite product

$$L(\mu) = \prod_{m=1}^{\infty}\left(1 + \frac{\theta_0\mu}{\pi}\Big/\left(m - \frac{1}{4}\right)\right)e^{-\frac{\theta_0\mu}{\pi}\Big/\left(m-\frac{1}{4}\right)}.$$

We can express $L(\mu)$ in terms of gamma functions and obtain

$$(22)\quad L(u) = \frac{\Gamma(3/4)}{\Gamma\left(\frac{\theta_0\mu}{\pi} + \frac{3}{4}\right)} \cdot e^{\frac{\theta_0\mu}{\pi}\psi\left(\frac{3}{4}\right)}$$

To carry out this transformation we have made use of the expression (Magnus-4)

$$(23)\quad \frac{\Gamma(a+1)}{\Gamma(z+a+1)}\, e^{z\psi(a+1)} = \prod_{m=1}^{\infty}\left(1 + \frac{z}{m+a}\right)e^{-\frac{z}{m+a}}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$. To obtain (22) we take $z = \frac{\theta_0\mu}{\pi}$ and $a = -\frac{1}{4}$ in (23) and compare with the original form of $L(\mu)$.

Now form the ratio $R(\mu)$, of $k^+(\mu,\theta_0)$ and $L(\mu)$. We obtain

$$(24)\quad R(\mu) = \frac{k^+(\mu,\theta_0)}{L(\mu)} = \frac{\prod_{m=1}^{\infty}\left(1 + \frac{\mu}{\mu_{P,m}(\theta_0)}\right) e^{-\frac{\mu}{\mu_{P,m}(\theta_0)}}}{\prod_{m=1}^{\infty}\left(1 + \frac{\mu}{\frac{\pi}{\theta_0}(m - \frac{1}{4})}\right) e^{-\frac{\mu}{\frac{\pi}{\theta_0}(m - \frac{1}{4})}}} .$$

Taking the logarithm of both sides and adding corresponding terms we get

$$\log R(\mu) = \sum_{m=1}^{\infty} \log\left[\left(\frac{1 + \frac{\mu}{\mu_{P,m}(\theta_0)}}{1 + \frac{\mu}{\frac{\pi}{\theta_0}(m - \frac{1}{4})}}\right) e^{\mu\left(\frac{1}{\frac{\pi}{\theta_0}(m - \frac{1}{4})} - \frac{1}{\mu_{P,m}(\theta_0)}\right)}\right]$$

$$= \sum_{m=1}^{\infty} \log\left[\frac{1 + \frac{\mu}{\mu_{P,m}(\theta_0)}}{1 + \frac{\mu}{\frac{\pi}{\theta_0}\left(m - \frac{1}{4}\right)}}\right] + \left[\sum_{m=1}^{\infty}\left(\frac{\mu}{\frac{\pi}{\theta_0}\left(m - \frac{1}{4}\right)} - \frac{\mu}{\mu_{P,m}(\theta_0)}\right)\right]$$

since each series converges separately for all μ, as a consequence of the asymptotic property of $u_{P,m}(\theta_0)$ (See (20)).

It follows that

$$R(\mu) = \left\{\prod_{m=1}^{\infty}\left(\frac{1 + \frac{\mu}{\mu_{P,m}(\theta_0)}}{1 + \frac{\mu}{\frac{\pi}{\theta_0}(m - \frac{1}{4})}}\right)\right\}\left\{e^{\frac{\theta_0\mu}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{m - \frac{1}{4}} - \frac{1}{\frac{\theta_0}{\pi}\mu_{P,m}(\theta_0)}\right)}\right\} .$$

For large $|\mu|$, in the region $|\arg\mu| \geq \frac{\pi}{2}$, $R(\mu)$ becomes*

$$R(\mu) \sim \left[\prod_{m=1}^{\infty} \left\{ \frac{(m - \frac{1}{4})}{\frac{\theta_0}{\pi}\mu_{P,m}(\theta_0)} \right\} \right] e^{\left\{ \frac{\theta_0\mu}{\pi} \sum_{m=1}^{\infty} \left(\frac{1}{m - \frac{1}{4}} - \frac{1}{\frac{\theta_0}{\pi}\mu_{P,m}(\theta_0)} \right) \right\}}$$

or

$$R(\mu) \sim A(\theta_0) e^{\frac{\theta_0}{\pi} B(\theta_0)\mu}$$

where $A(\theta_0)$ denotes the infinite product and $B(\theta_0)$ the infinite sum.

From equations (22) and (24) we have then

$$(25) \quad k^{+}(\mu,\theta_0) \sim \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{\theta_0}{\pi}u + \frac{3}{4}\right)} A(\theta_0) e^{\frac{\mu\theta_0}{\pi}\left[B(\theta_0) + \psi\left(\frac{3}{4}\right)\right]}, \quad \mathrm{Re}\,\mu \geq 0 .$$

A similar treatment of $k^{-}(\mu,\theta_0)$ yields

$$(26) \quad k^{-}(\mu,\theta_0) \sim \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4} - \frac{\theta_0}{\pi}\mu\right)} \cdot A(\theta_0) e^{-\frac{\mu\theta_0}{\pi}\left[B(\theta_0) + \psi\left(\frac{3}{4}\right)\right]}, \mathrm{Re}\,\mu \leq 0 .$$

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*See note 2

(2) Factorization of $M_{-\frac{1}{2}+u}(\cos\theta_0)$ - Asymptotic Behavior of the Factors.

$M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ has simple zeroes at the half odd integers $\mu = \pm(2m - \frac{3}{2})$, $m = 1,2,3,\cdots$ and, for fixed θ_0, at certain irrational values of μ which we shall denote by $\pm\mu_{M,m}(\theta_0)$, $m = 1,2,3,\cdots$. Furthermore, $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ is an entire function of μ. The irrational zeroes are given asymptotically by the expression

(27) $$\mu_{M,m}(\theta_0) = \frac{2m\pi}{\pi - 2\theta_0} + \frac{C'(\theta_0)}{m}.$$

Analogous to $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ just investigated, $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ has an infinite product representation of the form*

(28) $$M_{-\frac{1}{2}+\mu}(\cos\theta_0) = M_{-\frac{1}{2}}(\cos\theta_0)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{2m-\frac{3}{2}}\right)e^{-\frac{\mu}{2m-\frac{3}{2}}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{2m-\frac{3}{2}}\right)e^{\frac{\mu}{2m-\frac{3}{2}}}$$
$$\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{M,m}(\theta_0)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{M,m}(\theta_0)}}.$$

*See note 3

Let us denote the last two factors of (28) by $k_M^+(\mu,\theta_0)$ and $k_M^-(\mu,\theta_0)$ respectively. We now utilize (23) to rewrite the second and third factors in terms of gamma functions and it follows that we may write (28) in the form

(29) $M_{-\frac{1}{2}+\mu}(\cos\theta_0) =$

$$M_{-\frac{1}{2}}(\cos\theta_0)\frac{\Gamma^2(\frac{1}{4})}{\Gamma(\frac{\mu}{2}+\frac{1}{4})\,\Gamma(-\frac{\mu}{2}+\frac{1}{4})}e^{\frac{\mu}{2}\psi(\frac{1}{4})}e^{-\frac{\mu}{2}\psi(\frac{1}{4})}k_M^+(\mu,\theta_0)k_M^-(\mu,\theta_0)\,.$$

For future needs we now develop the asymptotic behavior of $k_M^+(\mu,\theta_0)$ and $k_M^-(\mu,\theta_0)$. We begin by considering the growth of $k_M^+(\mu,\theta_0)$ as compared to the growth of the infinite product

$$N(\mu) = \prod_{m=1}^{\infty}\left(1+\frac{\mu(\pi-2\theta_0)}{2m\pi}\right)e^{-\frac{\mu(\pi-2\theta_0)}{2m\pi}}\,.$$

A detailed discussion is not necessary in this case since the approach is exactly the same as that used to develop (25). We simply state the result which takes the form

$$(30)\quad k_M^+(\mu,\theta_0)\sim\frac{A'(\theta_0)}{\Gamma[\mu(\frac{1}{2}-\frac{\theta_0}{\pi})+1]}e^{\mu\left(\frac{1}{2}-\frac{\theta_0}{\pi}\right)\left\{B'(\theta_0)-C\right\}}\,.$$

A similar treatment of $k_M^-(\mu,\theta_0)$ yields

$$(31) \qquad k_M^-(\mu,\theta_0) \sim \frac{A'(\theta_0)}{\Gamma\left[1-\mu\left(\frac{1}{2}-\frac{\theta_0}{\pi}\right)\right]} e^{-\mu\left(\frac{1}{2}-\frac{\theta_0}{\pi}\right)\left\{B'(\theta_0)-C\right\}} .$$

C is the well known Euler constant and has the value .577215 ··· . $A'(\theta_0)$ is an infinite product given explicitly as

$$A'(\theta_0) = \prod_{m=1}^{\infty}\left\{\frac{2m\pi}{\mu_{M,m}(\theta_0)(\pi-2\theta_0)}\right\} .$$

$B'(\theta_0)$ is an infinite sum of the form.

$$B'(\theta_0) = \sum_{m=1}^{\infty}\left(\frac{1}{m} - \frac{2\pi}{\mu_{M,m}(\theta_0)(\pi-2\theta_0)}\right) .$$

Both expressions just mentioned are convergent and represent constants for a particular value of θ_0.

Using (21) and (28) we now rewrite (19) as

(32) $C^-(\mu) =$

$$\frac{1}{2\pi}(\mu^2 - \frac{1}{4})K^+(\mu,\theta_0)K^-(\mu,\theta_0) \cdot$$

$$\left\{D^+(\mu) - \left(\frac{\gamma a}{2}\right)^{-2\mu}\frac{\Gamma(1+\mu)}{\Gamma(1-\mu)}D^+(-\mu)\right\}$$

where

(33) $K^{+}(\mu,\theta_0) =$

$$\sqrt{P_{-\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\Gamma(\tfrac{1}{2}+\mu)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{2m-\frac{3}{2}}\right)e^{-\frac{\mu}{2m-\frac{3}{2}}}$$

$$\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{P,m}(\theta_0)}}\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{M,m}(\theta_0)}}e^{f(\mu)},$$

(34) $K^{-}(\mu,\theta_0) =$

$$\sqrt{P_{-\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\Gamma(\tfrac{1}{2}-\mu)\prod_{m=1}^{\infty}\left(1-\frac{\mu}{2m-\frac{3}{2}}\right)e^{\frac{\mu}{2m-\frac{3}{2}}}$$

$$\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{P,m}(\theta_0)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{M,m}(\theta_0)}}e^{-f(\mu)}$$

where $f(\mu)$ is so determined that the growths of $K^{+}(\mu,\theta_0)$ and $K^{-}(\mu,\theta_0)$ are algebraic in the proper regions.

To study the asymptotic behavior of $K^{+}(\mu,\theta_0)$ and $K^{-}(\mu,\theta_0)$ we refer to (25),(26),(29),(30),(31) and thus obtain

(35) $K^{+}(\mu,\theta_0) =$

$$\sqrt{-P_{\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}+\mu\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}\right)}\cdot$$

$$k^{+}(\mu,\theta_0)k_M^{+}(\mu,\theta_0)e^{\frac{\mu}{2}\psi\left(\frac{1}{4}\right)+f(\mu)},$$

(36) $K^{-}(\mu,\theta_0) =$

$$\sqrt{-P_{\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}-\mu\right)}{\Gamma\left(-\frac{\mu}{2}+\frac{1}{4}\right)}\cdot$$

$$k^{-}(\mu,\theta_0)k_M^{-}(\mu,\theta_0)e^{-\frac{\mu}{2}\psi\left(\frac{1}{4}\right)-f(\mu)}.$$

Now these factors are, by Stirling's formula*, asymptotically

(37) $K^{+}(\mu,\theta_0)\sim$

$$\sqrt{P_{-\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)A(\theta_0)A'(\theta_0)\mu^{-\frac{1}{2}},\quad \operatorname{Re}\mu \geq 0$$

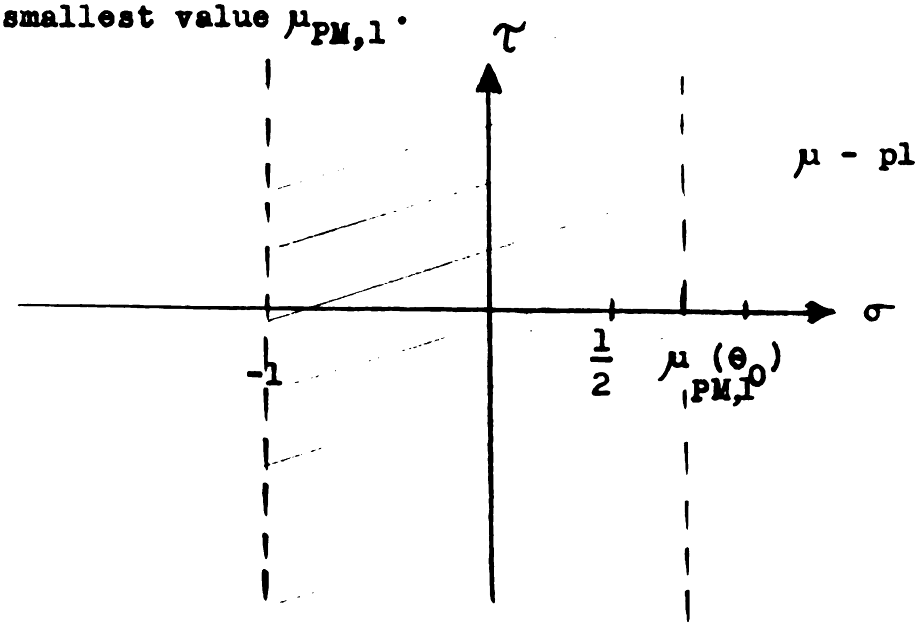
$K^{-}(\mu,\theta_0)\sim$

$$\sqrt{P_{-\frac{1}{2}}(\cos\theta_0)M_{-\frac{1}{2}}(\cos\theta_0)}\;\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)A(\theta_0)A'(\theta_0)(-\mu)^{-\frac{1}{2}},\quad \operatorname{Re}\mu \leq 0.$$

---

* $\Gamma(\mu)\sim\frac{\mu^{\mu-\frac{1}{2}}e^{-\mu}}{\sqrt{2\pi}}$ for $|\arg\mu| < \pi$

We now begin the task of transforming equation (32) into one whose left member consists of functions all of which are minus functions and whose right member consists entirely of plus functions. It is important that all these functions, as functions of their order, have a common strip of regularity. Reasons for this restriction were stated previously and will again be emphasized. As will be shown in the following pages this common region of regularity is the strip $-1 < Re\mu < \mu_{P1}(\theta_0)$ or $\mu_{M1}(\theta_0)$, taking the smallest of these two values. For future reference we shall call this smallest value $\mu_{PM,1}$.



We now write (32) in the form

$$(39)\ \frac{C^-(\mu)}{K^-(\mu,\theta_0)} = (\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)D^+(\mu) -$$

$$-(\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu}\frac{\Gamma(1+\mu)}{\Gamma(1-\mu)}D^+(-\mu).$$

(3) <u>Properties of functions in</u> (39).

(a) The function $\dfrac{C^-(\mu)}{K^-(\mu,\theta_0)}$.

Recall that according to our assumptions $C^-(\mu)$ is an entire function with algebraic decay of the order $\mu^{-\beta}$, $Re\beta > 0$, in the left half plane. $\dfrac{C^-(\mu)}{K^-(\mu)}$ has poles, see (34), at $\mu_{P,m}(\theta_0)$ and $\mu_{M,m}(\theta_0)$, m = 1,2,3, $\cdots$. $\dfrac{C^-(\mu)}{K^-(\mu)}$ is a function analytic on the left half μ plane $Re\mu < \mu_{PM,1}(\theta_0)$ and of algebraic decay of the order $u^{\frac{1}{2}-\beta}$ in the strip, $-1 < Re\mu < \mu_{PM,1}(\theta_0)$, and in all directions to its left. It is clear that, provided $Re\beta > \frac{1}{2}$, $\dfrac{C^-(\mu)}{K^-(\mu)}$ is a minus function.

b) <u>The function $(\mu - \frac{1}{2})(\mu + \frac{1}{2})K^+(\mu)D^+(\mu)$</u>

$D^+(\mu)$ has simple poles at $\mu = \pm\frac{1}{2}$, $-n$, $-\mu_{P,n}(\theta_0)$, $-\mu_{M,n}(\theta_0)$, n = 1,2,3, $\cdots$, $K^+(\mu,\theta_0)$ has simple zeroes at $\mu = -\mu_{P,n}(\theta_0)$, $-\mu_{M,n}(\theta_0)$, n = 1,2,3, $\cdots$ and simple poles at $\mu = -(2n - \frac{1}{2})$, n = 1,2,3, $\cdots$. It follows that $(\mu - \frac{1}{2})(\mu + \frac{1}{2})K^+(\mu)D^+(\mu)$ has simple poles at $\mu = -n, -(2n - \frac{1}{2})$, n = 1,2,3, $\cdots$ and thus is a function analytic on the right half plane $Re\mu > -1$. Furthermore, it is of algebraic decay of the order $\mu^{3/2-\alpha}$ in the strip, $-1 < \mathrm{Re}\,\mu < \mu_{PM,1}(\theta_0)$, and in all directions to its right provided $R\ell\,\alpha > \frac{3}{2}$.

(c) The function $(\mu + \frac{1}{2})(\mu - \frac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} D^+(-\mu)$.

Using information already stated in part (b) and properties of the gamma function we see that

$(\mu + \frac{1}{2})(\mu - \frac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2u} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} D^+(-\mu)$ has simple

poles at $\mu = \mu_{M,n}(\theta_0), \mu_{P,n}(\theta_0), -n, -(2n - \frac{1}{2})$, $n = 1,2,3, \cdots$.

It behaves as $|\tau|^{2\sigma + \frac{3}{2} - \alpha}$ in the strip, where σ is equal to to the real part of μ.

If we let $\mu = \rho e^{i\theta}$ and apply Stirling's formula we find

$(\mu + \frac{1}{2})(\mu - \frac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} D^+(-\mu)$ behaves

asymptotically as, $\rho^{2\rho\cos\theta}\, e^{-\rho(2\theta - \pi)\sin\theta - 2\rho\cos\theta}$.
Hence, it decays in the left half plane and would be a minus function if it did not have simple poles at $\mu = -n, -(2n - \frac{1}{2})$ $n = 1,2,3, \cdots$, We have found no way in which this mixed term can be avoided such that separation into half planes of analyticity is possible. However, this mixed function can be split into two parts by adding and subtracting Mittag-Leffler series which have the same residues, of the form

$$(40)\quad \sum_{n=1}^{\infty} \frac{R_n}{\mu + n} \qquad , \qquad \sum_{n=1}^{\infty} \frac{\tau_n}{\mu + 2n - \frac{1}{2}}$$

where R_n and τ_n are residues of

$$(\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^{+}(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1 + \mu)}{\Gamma(1 - \mu)} D^{+}(-\mu)$$

at the simple poles $\mu = -n$ and $\mu = -(2n - \frac{1}{2})$, $n = 1,2,3,\cdots$, respectively. These residues are

$$R_n = \frac{-\pi(n^2 - \frac{1}{4})g(-n)\left(\frac{\gamma a}{2}\right)^{2n}}{\Gamma(n)\Gamma(\frac{1}{2} + n)n!} D(n)$$

(41)

$$\tau_n = \frac{2n\ \pi\ g(-2n + \frac{1}{2})\left(\frac{\gamma a}{2}\right)^{4n-1}}{(2n - 2)!(2n - \frac{1}{2})\Gamma^2(2n - \frac{1}{2})} D(2n - \tfrac{1}{2})$$

where $$D(\mu) = \frac{\Gamma(1 + \mu)}{\left(\frac{\gamma a}{2}\right)^{\mu}} \frac{\cos \pi\mu\ A(\mu)}{M_{-\frac{1}{2}+\mu}(\cos\theta_0)}$$

and $$g(\mu) = \prod_{m=1}^{\infty}\left(1 + \frac{\mu}{2m - \frac{3}{2}}\right) e^{-\frac{\mu}{2m - \frac{3}{2}}} .$$

$$\prod_{m=1}^{\infty}\left(1 + \frac{\mu}{\mu_{P,m}(\theta_0)}\right) e^{-\frac{\mu}{\mu_{P,m}(\theta_0)}} \prod_{m=1}^{\infty}\left(1 + \frac{\mu}{\mu_{M,m}(\theta_0)}\right) e^{-\frac{\mu}{\mu_{M,m}(\theta_0)}} .$$

Using (41) and our assumptions about $D(\mu)$ it can easily be shown that both series in (40) are uniformly convergent in μ. If we consider the functions representing each series in (40) we find that these functions are free of poles on the right half plane $Re\mu > -1$. The behavior of these functions or their series representations as $|\mu| \to \infty$ is a difficult question. Their nature strongly suggests that they approach zero as $|\mu| \to \infty$ and we will assume this fact and consider both series in (40) as plus functions.

The difference between

$$(\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} D^+(-\mu)$$ and the two

Mittag-Leffler series is pole free on the left half plane, $Re\mu < \mu_{PM,10}(\theta_0)$, and because of previous considerations is in fact a minus function. Hence it may be transposed to the left hand side of (39). We have

$$(42)\quad \frac{C^-(\mu)}{K^-(\mu,\theta_0)} + (\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)\left(\frac{\gamma a}{2}\right)^{-2\mu} \frac{\Gamma(1+\mu)}{\Gamma(1-\mu)} D^+(-\mu) +$$

$$+\sum_{n=1}^{\infty} \frac{R_n}{\mu + n} + \sum_{n=1}^{\infty} \frac{\tau_n}{\mu + 2n - \frac{1}{2}} =$$

$$= (\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)D^+(\mu) + \sum_{n=1}^{\infty} \frac{R_n}{\mu + n} + \sum_{n=1}^{\infty} \frac{\tau_n}{\mu + 2n - \frac{1}{2}} .$$

The separation of functions is now complete. The resulting left hand side is a minus function and the resulting right hand side a plus function. The equality states that the two sides are analytic continuations of the same function defined in the common strip, $-1 < \mathrm{Re}\,\mu < \mu_{PM,1}(\theta_0)$. Since these continuations decay as $\mu \to \infty$ in the respective half planes, the function so defined has by Liouville's Theorem the value zero.

Hence we have

$$(43)\quad (\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})K^+(\mu,\theta_0)D^+(\mu) + \sum_{n=1}^{\infty} \frac{R_n}{\mu + n} + \sum_{n=1}^{\infty} \frac{\tau_n}{\mu + 2n - \frac{1}{2}} = 0$$

where $D^+(\mu) = \dfrac{\Gamma(1+\mu)}{\left(\frac{\gamma a}{2}\right)^{\mu}} \; \dfrac{\cos\pi\mu\, A(\mu)}{M_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)}$, and

$A(\mu)$ is the unknown function to be determined.

When μ is set equal to $1, \frac{3}{2}, 2, 3, \frac{7}{2}, \ldots$, it is clear that there arises an infinite system of equations. Hence, (43) does not allow explicit evaluation of the unknown function $A(\mu)$ but only at the values $\mu = 1, \frac{3}{2}, 2, 3, \frac{7}{2}, 4, 5, \frac{11}{2}, \ldots$ which are themselves the roots of the infinite system mentioned above. This set of linear equations may possess a simple explicit inversion, but the present author has not studied this question in detail.

V.

INPUT IMPEDANCE AT THE ORIGIN OF THE BICONICAL ANTENNA

The input impedance, $Z(\theta_0, ka)$, at the origin of the biconical antenna is defined as follows:

$$(44) \qquad Z(\theta_0, ka) = \lim_{r \to 0} \frac{V(r;\theta_0, ka)}{I(r;\theta_0, ka)}$$

where

$$(45) \qquad V(r;\theta_0, ka) = 2\int_{\theta_0}^{\frac{\pi}{2}} E_\theta r\, d\theta = \frac{2i}{\omega\varepsilon}\int_{\theta_0}^{\frac{\pi}{2}} \frac{\partial}{\partial r}(rH_\phi)\, d\theta$$

and

$$(46) \qquad I(r;\theta_0, ka) = 2\pi r \sin\theta_0 \, H_\phi(r, \theta_0).$$

The relation needed to derive the value of the voltage and current at the origin of the antenna is given by

$$(47) \qquad \sqrt{r}\, H_\phi(r, \theta;\theta_0, ka) =$$

$$= \left\{ \begin{array}{l} -\dfrac{\pi}{4 \sin\theta \log \tan\frac{\theta_0}{2}} \left\{ I_{\frac{1}{2}}(\gamma r) \operatorname{Res} A(-\tfrac{1}{2}) + I_{-\frac{1}{2}}(\gamma r) \operatorname{Res} A(\tfrac{1}{2}) \right\} \\ -\displaystyle\int_L \mu\, I_\mu(\gamma r) a(\mu) P_{-\frac{1}{2}+\mu}(\cos\theta_0) \frac{\left\{ P^1_{-\frac{1}{2}+\mu}(\cos\theta) + P^1_{-\frac{1}{2}+\mu}(-\cos\theta) \right\}}{P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0)}\, d\mu . \end{array} \right.$$

The above expression is derived from

$$(47\text{-a})\quad H_\phi = \frac{1}{\sqrt{r}}\int_L K_\mu(\gamma r)\mu A(\mu)\, P_{-\frac{1}{2}+\mu}(\cos\theta_0)\frac{\left\{P^1_{-\frac{1}{2}+\mu}(\cos\theta)+P^1_{-\frac{1}{2}+\mu}(-\cos\theta)\right\}}{P_{-\frac{1}{2}+\mu}(\cos\theta_0)-P_{-\frac{1}{2}+\mu}(-\cos\theta_0)}\,d\mu,$$

$$\theta_0 \leq \theta \leq \frac{\pi}{2},$$

which is obtained from (6) using the representation in the region $\theta_0 \leq \theta \leq \frac{\pi}{2}$ and the relation (8). In order that (47-a) be a convergent and meaningful expression in the right half μ-plane requires a transition from $K_\mu(\gamma r)$ to $I_\mu(\gamma r)$. As in previous arguments this synthesis leads to a kernel involving $a(\mu)$ instead of $A(\mu)$. However, in our present case the transformation of contour leads to residues due to simple poles at $\mu = \pm\frac{1}{2}$. The two residue terms signify the outgoing principal biconical wave and its reflection from the terminus of the antenna.

By substitution of (47) into (46) and taking the limit as r approaches zero one obtains

$$(48)\quad \lim_{r\to 0} I(r;\theta_0,ka) = -\frac{\pi^2}{2}\,\frac{\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\log\tan\frac{\theta_0}{2}}\,\mathrm{Res}\,A(\tfrac{1}{2}).$$

The same substitution in (45), assuming that the process of integration and taking the limit can be interchanged, leads to

$$(49)\quad \lim_{r \to 0} V(r;\theta_0,ka) = \frac{2i}{\omega\varepsilon}\,\frac{\pi}{4}\,\frac{\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\,\text{Res } A(-\tfrac{1}{2}).$$

Using these results (44) takes the form

$$(50)\quad Z(\theta_0,ka) = \frac{\gamma i}{\pi\omega\varepsilon}\,\log\cot\frac{\theta_0}{2}\,\frac{\text{Res } A(-\frac{1}{2})}{\text{Res } A(\frac{1}{2})}.$$

One can write the system (43) in the more convenient form

$$(51)\quad \delta_\mu - \frac{1}{\theta_\mu}\sum_{n=1}^{\infty}\frac{\delta_n}{\mu + n} + \frac{1}{\theta_\mu}\sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{\mu + 2n - \frac{1}{2}} = 0,$$

$$\mu = 1, \tfrac{3}{2}, 2, 3, \tfrac{7}{2}, \ldots$$

where

$$(52)\quad \theta_\mu = \frac{\mu}{\pi}\,\frac{\Gamma^2(\mu + \frac{1}{2})\,\Gamma^2(\mu)}{\left(\frac{\gamma a}{2}\right)^{2\mu}}\,\frac{g(\mu)}{g(-\mu)}$$

and

$$(53)\quad \delta_\mu = \frac{\pi(\mu + \frac{1}{2})g(-\mu)\left(\frac{\gamma a}{2}\right)^{2\mu} D^+(\mu)}{\Gamma(\mu - \frac{1}{2})\,\mu\,\Gamma^2(\mu)}.$$

Recalling an earlier relation between $A(\mu)$ and $D^+(\mu)$ given by

$$A(\mu) = \frac{(\frac{\gamma a}{2})^\mu}{\Gamma(1 + \mu)}\,\frac{M^1_{-\frac{1}{2}+u}(\cos\theta_0)}{\cos\pi\mu}\,D^+(\mu)$$

and using (53) we have

$$(54)\quad A(\mu) = \left(\frac{ika}{2}\right)^{\mu} \frac{M_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)\left\{\sum_{n=1}^{\infty}\frac{\delta_n}{\mu+n} - \sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{\mu+2n-\frac{1}{2}}\right\}}{\mu(\mu^2-\frac{1}{4})\,\Gamma(\mu)\,\Gamma(\mu+\frac{1}{2})\,g(\mu)\cos\pi\mu} .$$

Using (54) we can now write an explicit relation giving the quotient of residues in (50). We observe that the simple pole of $A(\mu)$ at $\mu = \frac{1}{2}$ is due to the factor $(\mu - \frac{1}{2})$, and the simple pole at $\mu = -\frac{1}{2}$ due to a zero of $g(\mu)$ at $\mu = -\frac{1}{2}$. Computing residues at these values we have

$$(55)\quad \frac{\operatorname{Res} A(-\frac{1}{2})}{\operatorname{Res} A(\frac{1}{2})} = \frac{i}{(ka)}\, g(\tfrac{1}{2}) \operatorname{Res}\left\{\frac{1}{g(-\frac{1}{2})}\right\} \frac{\sum_{n=1}^{\infty}\frac{\delta_n}{n-\frac{1}{2}} - \sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{2n-1}}{\sum_{n=1}^{\infty}\frac{\delta_n}{n+\frac{1}{2}} - \sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{2n}} .$$

In view of this relation, and the fact that

$\frac{1}{\omega\varepsilon a} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{1}{ka}$ (50) becomes

$$(56)\quad Z = \frac{-\pi i}{ka}\sqrt{\frac{u}{\varepsilon}}\, g(\tfrac{1}{2}) \operatorname{Res}\left\{\frac{1}{g(-\frac{1}{2})}\right\} \frac{\sum_{n=1}^{\infty}\frac{\delta_n}{n-\frac{1}{2}} - \sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{2n-1}}{\sum_{n=1}^{\infty}\frac{\delta_n}{n+\frac{1}{2}} - \sum_{n=1}^{\infty}\frac{\delta_{2n-\frac{1}{2}}}{2n}} .$$

This is a general formula for the impedance at the origin of the biconical antenna.

VI.

DISCUSSION OF THE INFINITE SYSTEM

On cursory inspection of the system given by (51) one finds that the convergence in the matrix of coefficients is strong in columns and slow in rows. To see that this is true computations were performed of the θ_μ for certain values of the parameters γa and θ_0. This computing project requires the calculation of certain infinite products involved in the definition of $g(\mu)$, see eq. 41. These products contain the zeros of $P_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)$ and $M_{-\frac{1}{2}+\mu}^{1}(\cos\theta_0)$. A table of these zeroes for the angles $\theta_0 = 30°$ and $60°$ is given in note 4 of the appendix.

Asymptotically θ_μ takes the form

$$(57) \qquad \theta_\mu \sim \mu^{2\mu} .$$

With this behavior one hopes that (51) can be concluded after a few terms and that the infinite system can be approximated by a finite one. Actual computations show that quite a number of terms are needed. The function θ_μ does not grow fast enough for initial values of the set $1, \frac{3}{2}, 2, 3, \frac{7}{2} \ldots$ For example, computations for $\theta_0 = 60°$ show that

$$\theta_{\frac{19}{2}} \approx \frac{.023976}{(\gamma a)^{19}}$$

Thus numerical exploitation of the system (51) for arbitrary values of θ_0 without automatic computing equipment is a very tedious project in itself.

In the belief that for the limiting case of small θ_0 the computations simplify, we chose for numerical approximation

$$\theta_0 = 1 \text{ degree} , \qquad \gamma a = \frac{1}{10}$$

i.e. a thin and short antenna. In this case the θ_μ increase monotonically and rapidly as μ increases. The system (51) was cut off, successively, at 2, 3, 4, and 5 square. The results are given in the following table.

	2-square	3-square	4-square	5-square
$\delta_{\frac{3}{2}} / \delta_1$	$-(2)10^{-4}$ $.185850\ i$	$-(1.61369)10^{-4}$ $.188016\ i$	$-(5.59)10^{-4}$ $.192504\ i$	$-(1.025)10^{-4}$ $.193572\ i$
$\delta_{\frac{3}{2}} / \delta_1$	24.2119 $.297361\ i$	$(4.5702)10^{-4}$ $(1.4)10^{-5}\ i$	$-(2.16)10^{-4}$ $(1)10^{-6}\ i$	$(.4)10^{-3}$ $(1)10^{-6}\ i$
$\delta_{\frac{7}{2}} / \delta_1$		-27.2378 $-.338419\ i$	$(.25)10^{-4}$ $(1)10^{-6}\ i$	$(.152615)10^{-3}$ $(1)10^{-6}\ i$
$\delta_{\frac{5}{2}} / \delta_1$			36.3174 $.462011\ i$	$-(.32799)10^{-4}$ $(3)10^{-8}\ i$
$\delta_{\frac{11}{2}} / \delta_1$				-39.3437 $-.503287\ i$

(In this case δ_2, δ_4, i.e. all δ's of even subscripts were negligible).

As more and more unknowns of the system are included, the first unknowns (in this case $\delta_{3/2}/\delta_1$ is the only significant one) seem to converge to some value; presumably the root of the infinite system. The root for the last unknown of each system is in complete contradiction with the root for the corresponding unknown in the systems of higher order of approximation. An explanation of this is not available at the present time.

On the basis of these results the value for the input impedance of the antenna with $\gamma a = \frac{1}{10}$, $\theta_0 = 1^0$ was found to be

$$Z = 219 - 2311\, i \text{ ohms} .$$

Clearly much more numerical work must be done to establish the properties of this infinite system.

We also attempted to extract approximate solutions of this system for the limiting case of small or large γa, by use of ascending or descending power series expansions. In all these attempts the coefficients of these series are themselves roots of infinite systems and not of finite systems. This is contrary to situations in many other problems from the field of mixed boundary value problems (Leitner and Wells -3).

Thus we conclude that at present a complete check on the assumptions made previously in connection with the behavior of certain integrals cannot be made. A great amount of further study of the infinite system is required before they can be established beyond all doubt.

APPENDIX

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APPENDIX

Note I: An Infinite Product Expansion of the Legendre Functions of the First Kind.

An entire function $f(z)$ is said to be of finite order if there is a positive number A such that as

$$|z| = r \longrightarrow \infty$$

$$f(z) = O(\rho^{r^A}).$$

The lower bound ρ of numbers A for which this is true is called the order of the function. Thus, if $f(z)$ is of order ρ

$$f(z) = O(e^{r^{\rho+\varepsilon}})$$

for every positive value of ε, but not for every negative value (Titchmarsh - 10).

If $f(\mu)$ is an even function of μ and if the order of $f(\mu)$ is such that $0 \leq \rho \leq 1$ then the Hadamard Factorization theorem permits the writing of $f(\mu)$ as an infinite product in the form

$$\text{(a)}\quad f(\mu) = f(0)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m}\right)\exp\left(-\frac{\mu}{\mu_m}\right)\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_m}\right)\exp\left(\frac{\mu}{\mu_m}\right)$$

where it is assumed that $f(0) \neq 0$. The μ_m are of course the roots of $f(\mu)$.

Now we know (Magnus and Oberhettinger - 4) for $\left|\mu - \frac{1}{2}\right| >>> 1$ and $\left|\arg \mu - \frac{1}{2}\right| < \pi$ that

(b) $$P_{-\frac{1}{2}+\mu}(\cos\theta_0) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)} \frac{\cos[\mu\theta_0 - \frac{1}{4}]}{\sqrt{2\sin\theta_0}} \left[1 + 0(\frac{1}{\mu})\right].$$

It can easily be shown that the order ρ of $\cos[\mu\theta_0 - \frac{\pi}{4}]$ is 1. By use of Stirling's approximation formula it can be shown that in the region $\arg\left|\mu - \frac{1}{2}\right| < \pi$ the quotient of gamma functions in (b) behaves asymptotically like $\mu^{-1/2}$ and hence does not affect the order of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$. Thus we can say in the region $\left|\mu - \frac{1}{2}\right| \ggg 1$ and $\left|\arg\mu - \frac{1}{2}\right| < \pi$, $P_{-\frac{1}{2}+\mu}(\cos\theta_0) = 0(e^{|\mu|})$. Since $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ is even in μ it is of order 1 and we may apply equation (a) to arrive at

$$P_{-\frac{1}{2}+\mu}(\cos\theta_0) =$$

$$= P_{-\frac{1}{2}}(\cos\theta_0)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{P,m}(\theta_0)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_{P,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{P,m}(\theta_0)}}$$

which was to be shown. It might be added that the above expression is a meaningful one in the sense that all infinite products involved converge.

Note 2: The Asymptotic Behavior of a Certain Infinite Product.

We wish to show that

$$R(\mu) = \prod_{m=1}^{\infty}\left(\frac{1 + \mu/\mu_{P,m}(\theta_0)}{1 + \mu/ \frac{\pi}{\theta_0}(m - \frac{1}{4})}\right)$$

for $\left|\arg \mu\right| \leq \frac{\pi}{2}$ approaches the constant $A(\theta_0)$, as $|\mu| \rightarrow \infty$, where

$$A(\theta_0) = \prod_{m=1}^{\infty}\left\{\frac{(m - \frac{1}{4})}{\frac{\theta_0}{\pi}\mu_{P,m}(\theta_0)}\right\}.$$

We make the substitution

$$\zeta = \frac{1}{\mu}$$

and examine the behavior of $R(\frac{1}{\zeta})$ in the neighborhood of $\zeta = 0$ for $\left|\arg \zeta\right| \leq \frac{\pi}{2}$.

$R\left(\frac{1}{\zeta}\right)$ can be written as

$$\text{(a)}\quad R(\tfrac{1}{\zeta}) = \prod_{m=1}^{\infty}\left\{\frac{\frac{\pi}{\theta_0}(m - \frac{1}{4})}{\mu_{P,m}(\theta_0)}\right\}\left\{\frac{\zeta\mu_{P,m}(\theta_0) + 1}{\zeta(m - \frac{1}{4})\frac{\pi}{\theta_0} + 1}\right\}.$$

As stated previously, asymptotically (Karp - 2)

$$\text{(b)}\quad \mu_{P,m}(\theta_0) \sim \frac{\pi}{\theta_0}(m - \frac{1}{4}) + \frac{C(\theta_0)}{m}, \qquad 0 < \theta_0 < \frac{\pi}{2}.$$

From this fact, it follows that the infinite product

$$\prod_{m=1}^{\infty}\frac{\frac{\pi}{\theta_0}(m - \frac{1}{4})}{\mu_{P,m}(\theta_0)} \equiv A(\theta_0)$$

converges.

If we call the infinite product of the second factors in (a) $N(\zeta)$, that is

$$N(\zeta) = \prod_{m=1}^{\infty}\left\{\frac{\zeta^{\mu_{P,m}(\theta_0)} + 1}{\zeta^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} + 1}\right\},$$

it can be shown that $N(\zeta)$ converges uniformly in all ζ in the region $\left|\arg \zeta\right| \leq \frac{\pi}{2}$. For, $N(\zeta)$ can be written as

$$\text{(c)} \quad N(\zeta) = \prod_{m=1}^{\infty}\left\{1 + \frac{\zeta^{\left[-\frac{\pi}{\theta_0}(m - \frac{1}{4}) + \mu_{P,m}(\theta_0)\right]}}{\zeta^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} + 1}\right\}.$$

If we define $f_m(\zeta)$ as

$$f_m(\zeta) = - \frac{\zeta^{\left[\frac{\pi}{\theta_0}(m - \frac{1}{4}) - \mu_{P,m}(\theta_0)\right]}}{\zeta^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} + 1}$$

then $N(\zeta)$ converges uniformly in the region $\left|\arg \zeta\right| \leq \frac{\pi}{2}$ if and only if the series

$$\text{(d)} \qquad \sum_{m=1}^{\infty}\left|f_m(\zeta)\right|$$

converges uniformly in this region.

By use of (b), we have for large m

$$\left|f_m(\zeta)\right| \leq \frac{\left|\zeta\right| \cdot \frac{\left|C(\theta_0)\right|}{m}}{\left|\zeta^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} + 1\right|}.$$

However,

$$\left|\zeta^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} + 1\right| \geq \left|\zeta\right|^{(m - \frac{1}{4})\frac{\pi}{\theta_0}} \quad \text{for } \left|\arg \zeta\right| \leq \frac{\pi}{2}.$$

It follows,

$$\left| f_m(\zeta) \right| \leq \frac{\theta_0 \, C(\theta_0)}{\pi \, m \, (m - \frac{1}{4})} \text{ .}$$

We have shown that the series (d) converges uniformly for all ζ , $\left| \arg \zeta \right| \leq \frac{\pi}{2}$, and hence so does the right hand side of (c).

Since $R(\frac{1}{\zeta}) = A(\theta_0) \, N(\zeta)$

we have that

$$\lim_{|\zeta| \to 0} R(\frac{1}{\zeta}) = A(\theta_0) \text{ ,} \qquad \left| \arg \mu \right| \leq \frac{\pi}{2} \text{ ,}$$

as was to be shown. The last statement follows from the fact that $N(\zeta)$ is continuous for all ζ , $\left| \arg \zeta \right| \leq \frac{\pi}{2}$.

Note 3: An Infinite Product Expansion of $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$.

Refering to the discussion in Note I, we need only show that the order ρ of $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ is such that $0 \leq \rho \leq 1$.

Recall that

$$M_{-\frac{1}{2}+\mu}(\cos\theta_0) = P_{-\frac{1}{2}+\mu}(\cos\theta_0) - P_{-\frac{1}{2}+\mu}(-\cos\theta_0).$$

We have (Note 1 - (b)) that for $\left|\mu - \frac{1}{2}\right| >>> 1$ and $\left|\arg \mu - \frac{1}{2}\right| < \pi$

$$P_{-\frac{1}{2}+\mu}(\cos\theta_0) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{\cos[\mu\theta_0 - \frac{\pi}{4}]}{\sqrt{2 \sin\theta_0}} [1 + O(\tfrac{1}{\mu})].$$

It follows that

$$P_{-\frac{1}{2}+\mu}(-\cos\theta_0) = P_{-\frac{1}{2}+\mu}[\cos(\pi - \theta_0]=$$

$$\frac{2}{\sqrt{\pi}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{\cos[\mu(\pi - \theta_0)- \frac{\pi}{4}]}{\sqrt{2 \sin\theta_0}} [1 + O(\tfrac{1}{\mu})].$$

Hence,

$$M_{-\frac{1}{2}+\mu}(\cos\theta_0) =$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)\sqrt{\sin\theta_0}} \cos \tfrac{1}{2}(\mu\pi - \tfrac{\pi}{2}) \cos \tfrac{\mu}{2}(2\theta_0 - \pi)[1 + O(\tfrac{1}{\mu})].$$

It can easily be shown that the order of both $\cos[\mu\theta_0 - \frac{\pi}{4}]$ and $\cos[\mu(\pi - \theta_0) - \frac{\pi}{4}]$ is 1. Recalling the behavior of the gamma functions from Note 1 we have in the region $\left|\mu - \frac{1}{2}\right| \ggg 1$ and $\left|\arg \mu - \frac{1}{2}\right| < \pi$

$$M_{-\frac{1}{2}+\mu}(\cos\theta_0) = O(e^{|\mu|}).$$

Since $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ is even in μ it is of order 1 and we may write

$$M_{-\frac{1}{2}+\mu}(\cos\theta_0) = M_{-\frac{1}{2}}(\cos\theta_0)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{2m-\frac{3}{2}}\right)e^{-\frac{\mu}{2m-\frac{3}{2}}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{2m-\frac{3}{2}}\right)e^{\frac{\mu}{2m-\frac{3}{2}}}\cdot$$

$$\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{-\frac{\mu}{\mu_{M,m}(\theta_0)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_{M,m}(\theta_0)}\right)e^{\frac{\mu}{\mu_{M,m}(\theta_0)}}.$$

This is a meaningful expression since all the infinite products involved converge.

Note 4: Zeros of $P_{-\frac{1}{2}+\mu}(\cos\theta_0)$ and $M_{-\frac{1}{2}+\mu}(\cos\theta_0)$ for $\theta_0 = 30°, 60°$.

$\theta_0 = 30°$

m	$\mu_{P,m}$	$\mu_{M,m}$
1	4.583687609	2.9160411
2	10.538550385	5.9631520
3	16.5249	8.975471
4	22.5183	11.981981
5	28.5145	
6	34.5119	
7	45.51020	
8	46.50888	
9	52.50786	
10	58.50706	

$\theta_0 = 60°$

1	2.27728827	5.9773804
2	5.2627794	11.988560671
3	8.25825872	17.992359
4	11.25608	23.994251
5	14.25481	
6	17.25398	
7	20.25339	
8	23.25295	
9	26.25262	
10	29.25235	

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