ABSTRACT

ON THE DIFFRACTION OF WAVES FROM A FINITE WEDGE

by Daniel Richard Killoran

The scattering of electromagnetic waves emitted by a line source and impinging on a wedge is examined by means of a combination of the Wiener-Hopf technique and a modification of the Lebedev-Kontorovitch integral transform. The wedge is considered infinite in the axial direction but finite in the plane perpendicular to the axis.

An infinite system of equations involving values of the unknown transform function and its derivative at special points is obtained, but the system is not solved. For the special case of a strip with the source at an infinite distance from the wedge, a simple assumption leads to agreement in the first order with the results of Sommerfeld, but produces disagreement in the second and subsequent orders.

For the symmetric finite wedge, the nature of the variation of the cross-section arising from a change in the wedge angle is determined qualitatively.

ON THE DIFFRACTION OF WAVES FROM A FINITE WEDGE

By

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I. Introduction

This thesis examines the problem of the scattering of electromagnetic waves emitted by a line source with the electric vector parallel to the axis of the source and impinging on a perfectly conducting strip or wedge of half width $\underline{b}$. The wedge angle $\underline{2\beta}$ and the location of the source with respect to the wedge are arbitrary, and the wedge is to be thought of as having infinite length in the direction parallel to the axis of the source.

The problem is treated by a combination of the Wiener-Hopf technique and a modification of the Lebedev integral transform [1]. An infinite system of equations involving the unknown transform function and its derivative is obtained, but the system is not solved. For the special case of a strip with the source at infinite distance from the wedge a simple assumption leads to agreement in the first order with results previously obtained [2], but produces disagreement in the second and subsequent orders.

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II. Statement of the Problem

Consider a wedge of infinitesimal thickness and of infinite extent along its axis (the z-direction). Let the length of one side of the wedge be b and the interior angle be 2β. Place a line source of waves (S) at the point (r_0, ϕ_0) with its axis parallel to that of the wedge.

Calling the z-component of the incident electric field U_i, then

$$U_i = i\pi\, H_0^1 \left\{ k\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)} \right\} \tag{1}$$

where k is the wave number and H_0^1 is the Hankel function of the first kind (see Figure 1). We will use the abbreviation

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)} \tag{2}$$

Then we have

$$U = U_s + i\pi\, H_0^1 \{ kR \} \tag{3}$$

We seek a solution (U) of the 2 dimensional wave equation in circular cylindrical coordinates

$$\text{div grad } U + k^2 U = 0 \tag{4}$$

valid everywhere and satisfying the boundary conditions:

(a) U, U_i and $U_s \rightarrow \frac{e^{ikr}}{\sqrt{r}}$ as $r \longrightarrow \infty$. (Sommerfeld radiation condition).

(b) $U = 0$, $r<b$, $\phi = \pm\beta$. Wave function is zero on the wedge.

(c) $\frac{\partial U_s}{\partial \phi}$ is continuous across $\phi = \pm\beta$ for $r>b$.

(d) U_s is continuous for all values of r.

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III. The Integral Representation

The representation chosen is the modified Lebedev transform:

$$F\{x\} = \int_L G\{y\} K_y\{x\} y dy \tag{5}$$

where $K_y\{x\}$ is the MacDonald function, or Hankel function of imaginary argument, defined by the relation

$$i \pi H^1_y\{ix\} = 2 e^{-i \pi \frac{y}{2}} K_y\{x\} \tag{6}$$

and the contour extends from $a - i \infty$ to $a + i \infty$.

This modified transform is chosen in preference to the more usual transform

$$F\{x\} = \int_L G\{y\} I_y\{x\} y dy$$

in order to simplify the representation of the source function. Moreover, if a residue series is extracted from this integral, each term will contain a MacDonald function which, upon returning to real wave number, will ensure that the solution satisfies boundary condition (a).

We must extablish the conditions under which equation (5) may be inverted. One form of Lebedev transform theorem [1] states that:

Theorem I. If

$$F\{x\} = \int_L G\{y\} I_y\{x\} y dy$$

then

$$i \pi G\{y\} = \int_0^\infty F\{x\} K_y\{x\} \frac{dx}{x}$$

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Provided that:

$G\{y\}$ -is analytic in a strip of finite width containing the imaginary axis.

-decays at least as fast as $|y|^{-\frac{1}{2}} \exp\{-|y| \frac{\pi}{2}\}$ as $y \rightarrow s \pm i\infty$ where s lies in the strip.

$F\{x\}$ -goes to zero like some positive power of $\underline{x}$ as $x \rightarrow 0$.

-is bounded by $|x|\ e^x$ as $x \rightarrow \infty$.

L -is some contour from $a - i\infty$ to $a + i\infty$ lying entirely within the strip.

It can be seen that $G\{y\}$ is even. Then if we break the first integral up into an integral along the top half of the imaginary axis plus an integral along the bottom half, change y to minus y in the second integral, and recombine, we get

$$F\{x\} = \int_0^{i\infty} G\{y\} \left(I_{-y}\{x\} - I_y\{x\}\right) y\ dy$$

Using the definition of the MacDonald function

$$K_y\{x\} = \frac{\pi}{2} \frac{I_{-y}\{x\} - I_y\{x\}}{\sin y\pi} \tag{7}$$

and the fact that the resulting integrand is even, we can again extend the integration over the original contour. Then

$$F\{x\} = \frac{-1}{\pi} \int_L G\{y\} K_y\{x\} \sin y\pi\ \ y\ dy$$

We may now redefine $G\{y\}$ so that

$$G\{y\} \text{ becomes } -\pi \frac{G\{y\}}{\sin y\pi}$$

Using this new definition of $G\{y\}$, we have:

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Theorem II.

If $$F\{x\} = \int_L G\{y\}\ K_y\{x\}\ y\ dy \tag{8}$$

then $$\frac{\pi^2}{i}\ \frac{G\{y\}}{\sin\ y\pi} = \int_0^\infty F\{x\}\ K_y\{x\}\ \frac{dx}{x} \tag{9}$$

Provided that:

$G\{y\}$	-is analytic in a strip of finite width containing the imaginary axis.
	-grows no faster than $\lvert y\rvert^{-\frac{1}{2}} \exp\left\{ \lvert y\rvert \frac{\pi}{2} \right\}$ as $y \rightarrow a \pm i\infty$ where a lies in the strip.
$F\{x\}$	-goes to zero like some positive power of x as $x \rightarrow 0$.
	-is bounded by $\lvert x\rvert\ e^x$ as $x \rightarrow \infty$.
L	-is some contour from $a - i\infty$ to $a + i\infty$ within the strip.

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IV. The Representation of the Scattered Field

In order to use the Lebedev transform effectively, it is desirable to consider the wave number to be imaginary, a method due to Oberhettinger [3]. Let

$$k = ig \tag{10}$$

where g is to be real and positive, so that the functions U, U_i, and U_s decay exponentially rather than algebraically at infinity.

It will be necessary to use different representations in the three regions of the (r,ϕ)-plane (see Figure 1) in order to ensure continuity. Let

$$U_s = \int_L (F_1\{\mu\} \cos\mu\phi + F_2\{\mu\} \sin\mu\phi)\, K_\mu\{gr\}\, \mu d\mu \tag{11}$$

in Region I

$$U_s = \int_L (M_1\{\mu\} \cos\mu\{\pi-\phi\} + M_2 \sin\mu\{\pi-\phi\})K_\mu\{gr\}\, \mu d\mu \tag{12}$$

in Region II

$$U_s = \int_L (M_1\{\mu\} \cos\mu\{\pi+\phi\} - M_2\{\mu\} \sin\mu\{\pi+\phi\})K_\mu\{gr\}\, \mu d\mu \tag{13}$$

in Region III

We note that we already have continuity for $\phi \rightarrow \pm\pi$.

Now we require continuity for $\phi \longrightarrow \pm\beta$. The integrands of the expressions on either side of the wedge must be identical, and the resulting set of equations gives

$$M_1 \cos\mu\{\pi-\beta\} = F_1 \cos\mu\beta \tag{14}$$

$$M_2 \sin\mu\{\pi-\beta\} = F_2 \sin\mu\beta \tag{15}$$

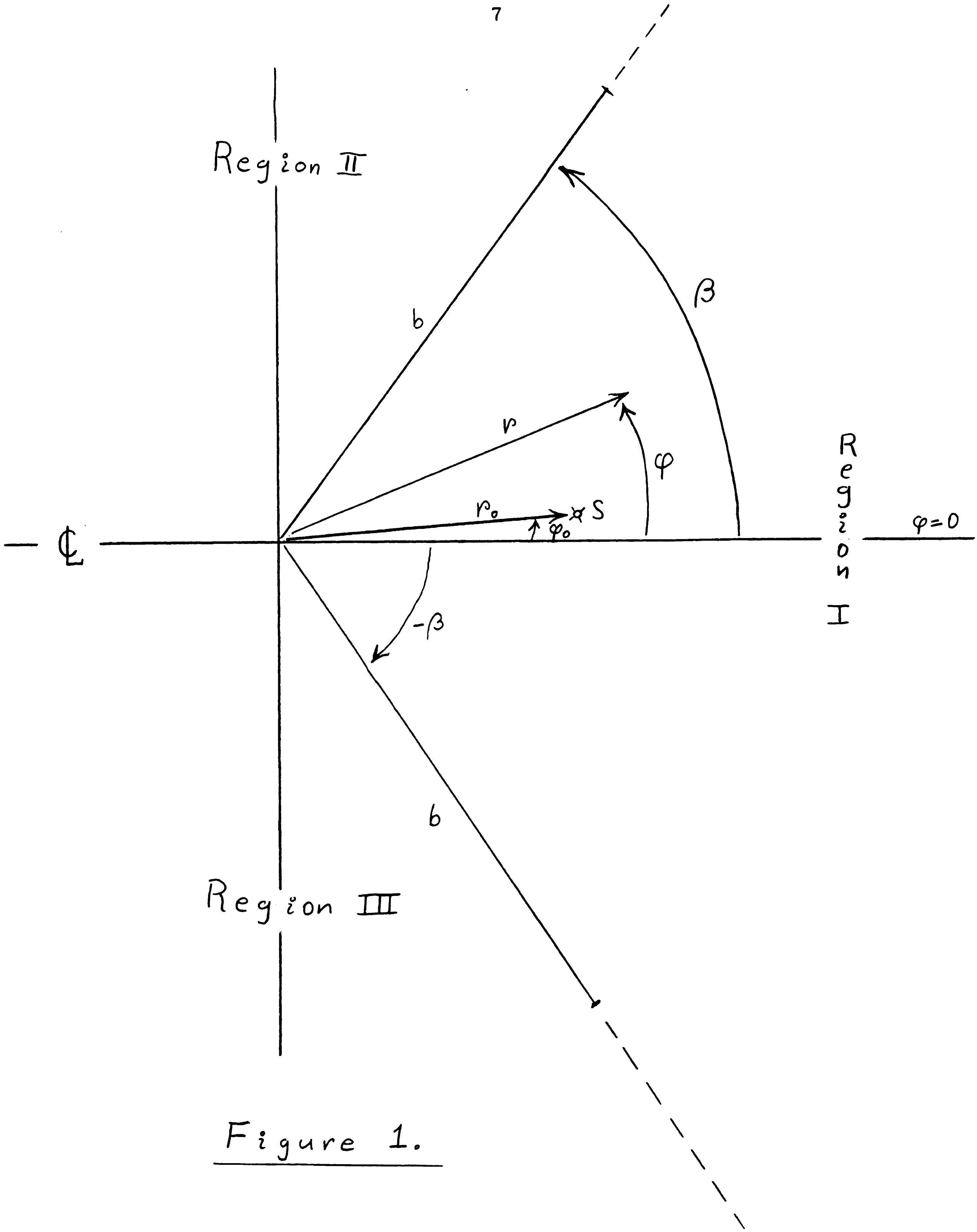

Figure 1.

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V. The Representation of the Source

According to equation (10) given by Oberhettinger [4]

$$K_o\{gR\} = \frac{2}{i\pi} \int_o^{i\infty} K_\mu\{gr\} \; K_\mu\{gr_o\} \cos \mu\{\pi - |\phi - \phi_o|\} \, d\mu \quad (16)$$

The integrand of the above is even in μ, so the integration can be extended over the entire imaginary axis.

$$K_o\{gR\} = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} K_\mu\{gr\} \; K_\mu\{gr_o\} \cos \mu\{\pi - |\phi - \phi_o|\} \, d\mu \quad (17)$$

Since the integrand is entire, we may deform this contour to coincide with L, the contour used in the representation of U_s. Then, using equation (6), we have

$$i\pi H_o^1\{igR\} = \frac{2}{i\pi} \int_L K_\mu\{gr\} \; K_\mu\{gr_o\} \cos \mu\{\pi - |\phi - \phi_o|\} \, d\mu \quad (18)$$

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VI. The Second Boundary Condition and the First Integral Equation

For the total wave function U we now have

$$U = \int_L \left(F_1 \cos\mu\phi + F_2 \sin\mu\phi + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\{\pi - |\phi-\phi_o|\}\right) \cdot K_\mu\{gr\}\ \mu d\mu \tag{19}$$

We will assume for convenience that $\beta > |\phi_o|$ (see Figure 1). Then, by the second boundary condition we require that $U = 0$ for $\phi = \beta$.

$$0 = \int_L (F_1 \cos\mu\beta + F_2 \sin\mu\beta + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\{\pi-\beta+\phi_o\}) K_\mu\{gr\}\ \mu d\mu \tag{20}$$

and also at $\phi = -\beta$.

$$0 = \int_L \left(F_1 \cos\mu\beta - F_2 \sin\mu\beta\ \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\{\pi-\beta-\phi_o\}\right) K_\mu\{gr\}\ \mu d\mu \tag{21}$$

Adding the above equations, and reducing the sum of cosines, we have

$$0 = \int_L (F_1 \cos\mu\beta + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\phi_o \cos\mu\{\pi-\beta\}) K_\mu\{gr\} \mu d\mu \qquad \underline{r<b} \tag{22}$$

Subtracting the equations,

$$0 = \int_L (F_2 \sin\mu\beta - \frac{2}{i\pi\mu} K_\mu\{gr_o\} \sin\mu\phi_o \sin\mu\{\pi-\beta\}) K_\mu\{gr\} \mu d\mu \qquad \underline{r<b} \tag{23}$$

For brevity, the following abbreviations will be used

$$J_1 = F_1 \cos\mu\beta + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\phi_o \cos\mu\{\pi - \beta\} \tag{24}$$

$$J_2 = F_2 \sin\mu\beta - \frac{2}{i\pi\mu} K_\mu\{gr_o\} \sin\mu\phi_o \sin\mu\{\pi-\beta\} \tag{25}$$

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Then we have the first integral equations for $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively.

$$0 = \int_L J_1\{\mu\} \; K_\mu\{gr\} \; \mu d\mu \qquad \underline{r<b} \tag{26}$$

$$0 = \int_L J_2\{\mu\} \; K_\mu\{gr\} \; \mu d\mu \qquad \underline{r<b} \tag{27}$$

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VII. The Third Boundary Condition and the Second Integral Equation

Since the wedge does not extend beyond r = b, the ϕ -derivative of the scattered wave must be continuous for $\phi = \pm\ \beta$, r>b. Differentiating equations (11), (12), and (13) under the integral sign, and letting $\phi \rightarrow \pm\ \beta$

$$\frac{\partial U_s}{\partial \phi}\bigg|_{\pm\beta} = \int_L (-F_1 \sin \pm\mu\beta + F_2 \cos \pm\mu\beta)\ K_\mu\{gr\}\ \mu^2 d\mu \tag{28}$$

in Region I

$$\frac{\partial U_s}{\partial \phi}\bigg|_{+\beta} = \int_L (M_1 \sin\mu\{\pi - \beta\} - M_2 \cos\mu\{\pi - \beta\})K_\mu\{gr\}\ \mu^2 d\mu \tag{29}$$

in Region II

$$\frac{\partial U_s}{\partial \phi}\bigg|_{-\beta} = \int_L (-M_1 \sin\mu\{\pi - \beta\} - M_2 \cos\mu\{\pi - \beta\})\ K_\mu\{gr\}\ \mu^2 d\mu \tag{30}$$

in Region III

Requiring continuity for $\phi = \beta$.

$$0 = \int_L (-F_2 \sin\mu\beta + F_2 \cos\mu\beta - M_1 \sin\mu\{\pi-\beta\} + M_2 \cos\mu\{\pi-\beta\}) \cdot K_\mu\{gr\}\ \mu^2 d\mu \tag{31}$$

Requiring continuity for $\phi = -\beta$.

$$0 = \int_L (F_1 \sin\mu\beta + F_2 \cos\mu\beta + M_1 \sin\mu\{\pi - \beta\} + M_2 \cos\mu\{\pi - \beta\}) \cdot K_\mu\{gr\}\ \mu^2 d\mu \tag{32}$$

Subtracting (31) from (32)

$$0 = \int_L F_1\{\mu\} \frac{\sin\mu\pi}{\cos\mu\{\pi-\beta\}} K_\mu\{gr\}\ \mu^2\ d\mu \qquad \underline{r>b} \tag{33}$$

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Adding (31) and (32)

$$0 = \int_L F_2\{\mu\} \frac{\sin\mu\pi}{\sin\mu\{\pi-\beta\}} K_\mu\{gr\} \mu^2 \, d\mu \qquad \underline{r>b} \qquad (34)$$

which will be referred to as the second integral equations for F_1 and F_2 respectively.

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VIII. The First Integral for $F_1\{\mu\}$ and the Function $C\{\mu\}$

We must now determine whether equation (26) can be closed on either half-plane, and whether its integrand has any singularities. Letting $\phi = \beta$ in equation (19)

$$U\{\beta\} = \int_L (F_1 \cos\mu\beta + F_2 \sin\mu\beta + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\{\pi - \beta + \phi_o\}) \cdot K_\mu\{gr\} \mu d\mu \tag{35}$$

Letting $\phi = -\beta$ in equation (19)

$$U\{-\beta\} = \int_L (F_1 \cos\mu\beta - F_2 \sin\mu\beta + \frac{2}{i\pi\mu} K_\mu\{gr_o\} \cos\mu\{\pi - \beta - \phi_o\}) \cdot K_\mu\{gr\} \mu d\mu \tag{36}$$

Adding and subtracting, cf equation (24), (25)

$$\frac{U\{\beta\} + U\{-\beta\}}{2} = \int_L J_1\{\mu\} \quad K_\mu\{gr\} \mu d\mu, \tag{37}$$

$$\frac{U\{\beta\} - U\{-\beta\}}{2} = \int_L J_2\{\mu\} \quad K_\mu\{gr\} \mu d\mu \tag{38}$$

Applying the inverse transform

$$\frac{\pi^2}{i \sin\mu\pi} \quad J_1\{\mu\} = \int_b^\infty \frac{U\{\beta\} + U\{-\beta\}}{2} K_\mu\{gr\} \frac{dr}{r} \tag{39}$$

and

$$\frac{\pi^2}{i \sin\mu\pi} \quad J_2\{\mu\} = \int_b^\infty \frac{U\{\beta\} - U\{-\beta\}}{2} \quad K_\mu\{gr\} \quad \frac{dr}{r} \tag{40}$$

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Since the integrals on the right converge absolutely, and uniformly in μ on the entire finite μ plane, the right hand side of equations (39) and (40) must be entire in μ. Therefore, a fortiori, J_1 and J_2 must be entire functions of μ. The validity of this proof depends upon the applicability of the inverse transform. It can be seen, moreover, that the only condition of Theorem II not obviously satisfied is the restriction on the growth of $J\{\mu\}$. It can be shown, by a method presently to be exhibited, that the integrals on the right hand side of (39) and (40) have the asymptotic form (const.)$\cdot$ $K_\mu\{gb\}/\mu$ as $|\mu| \rightarrow \infty$. It then becomes apparent that

$$J_1\{\mu\} \approx J_2\{\mu\} \approx |\mu|^{-\frac{1}{2}} \exp\left\{|\mu| \frac{\pi}{2}\right\} \tag{41}$$

$$\text{as } |I_m\mu| \rightarrow \infty$$

Examination of equations (24) and (25) then shows that

$F_2\{\mu\} \sin\mu\beta$ is entire

and (42)

$F_1\{\mu\} \cos\mu\beta$ has but one singularity; at zero, possessing residue

$$F_1\{\mu\} \rightarrow -\frac{2}{i\pi\mu} K_o\{gr_o\} \qquad \text{as } \mu \rightarrow 0 . \tag{43}$$

However, since J_1 and J_2 do not decay at infinity on either side, the integrals (26) and (27) cannot be closed. We must modify the function in the integrand somehow to provide suitable decay. We will try to split the MacDonald function.

Using the definition of the MacDonald function (7), equation (26) becomes

$$0 = \int_L J_1\{\mu\}\frac{\pi}{2}\frac{I_{-\mu}\{gr\}}{\sin\mu\pi}\mu d\mu - \int_L J_1\{\mu\}\frac{\pi}{2}\frac{I_\mu\{gr\}}{\sin\mu\pi}\mu d\mu \tag{44}$$

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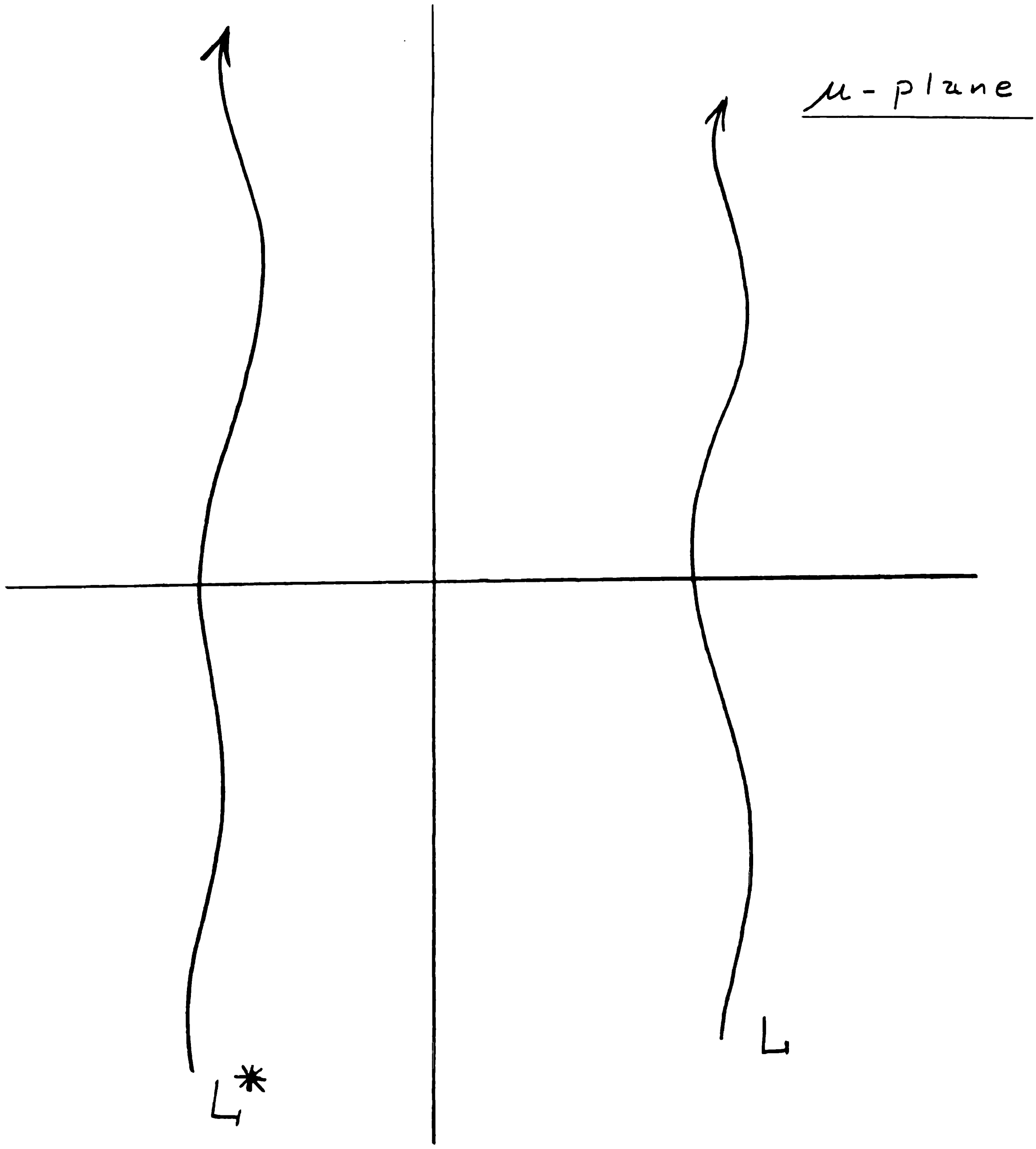

Figure 2. Contours in the μ-plane.

In the first integral we change the variable to $-\mu$ and reverse the direction of integration.

$$0 = \int_{L^*} J_1\{-\mu\}\frac{\pi}{2} \frac{I_\mu\{gr\}}{\sin\mu\pi}\,\mu d\mu - \int_L J_1\{\mu\}\frac{\pi}{2} \frac{I_\mu\{gr\}}{\sin\mu\pi}\,\mu d\mu \quad (45)$$

Now J_1 has no singularities; the pole at zero caused by the sine in the denominator is removed by the μ in the numerator, and all the other poles caused by the sine are removed by the zeroes of J_1, so we may deform the contour to coincide with the original contour L and combine the integrals.

$$0 = \int_L \frac{J_1\{-\mu\} - J_1\{\mu\}}{\sin\mu\pi} \frac{\pi}{2} I_\mu\{gr\}\,\mu d\mu \quad (46)$$

Now define

$$C\{\mu\} \equiv \frac{\pi}{2} \frac{J_1\{-\mu\} - J_1\{\mu\}}{\sin\mu\pi} \quad (47)$$

So we have

$$0 = \int_L C\{\mu\}\, I_\mu\{gr\}\,\mu d\mu \qquad \underline{r<b} \quad (48)$$

Similarly

$$0 = \int_L D\{\mu\} I_\mu\{gr\}\,\mu d\mu \qquad \underline{r<b} \quad (49)$$

where

$$D\{\mu\} \equiv \frac{\pi}{2} \frac{J_2\{-\mu\} - J_2\{\mu\}}{\sin\mu\pi} \quad (50)$$

In order to determine whether we may close integrals (48) and (49) we must examine the asymptotic behaviour of the functions $C\{\mu\}$ and $D\{\mu\}$ for large values of μ. Since the treatment is substantially the same for both functions, it will be done for $C\{\mu\}$ only. It follows from equation (40) that

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$$C\{\mu\} = \frac{1}{2\pi i} \int_b^{\infty} (U\{-\beta\} + U\{\beta\}) \quad K_\mu\{gr\} \frac{dr}{r} \tag{51}$$

Since both U and $K_\mu\{gr\}$ decay exponentially as $r \rightarrow \infty$, the integral converges at the upper limit. At the lower limit the U's may possess singularities, but by Meixner's [5] edge condition, the singularity can be no worse than

$$U\{\pm\beta\} \rightarrow (r-b)^{-\frac{1}{2}} \tag{52}$$

which is integrable. Moreover, $K_\mu\{gr\}$ is an entire function of μ, therefore the integral is an entire function of μ. Then $C\{\mu\}$ is entire.

A. Treatment of the local field

For future reference, and in particular to show that $U\{\beta\}$ can be expanded as a power series in r, we must examine the expected behaviour of the solution at metallic boundaries for very small distances from the surface. In particular, we are interested in the solution near a corner.

For the case we are considering (U = 0 on the boundary) U corresponds to the axial (z) component of the electricfield vector.

Then let us assume that

$$U \rightarrow \rho^{\delta} \quad \text{as} \quad \rho \rightarrow 0 \tag{53}$$

where ρ is the distance from the edge or corner. Then the requirement that the volume integral of the energy density be finite over any volume, however small, and that it must approach zero as the volume goes to zero even at an edge gives us the conditions.

$$\begin{aligned} &\int E_z^2 \rho \, d\rho \approx \int U^2 \rho \, d\rho \approx \int \rho^{2\delta+1} d\rho \approx \rho^{2\delta+2} \\ &\int H_\phi^2 \rho \, d\rho \approx \int \left(\frac{\partial U}{\partial \rho}\right)^2 \rho \, d\rho \approx \int \rho^{2(\delta-1)+1} d\rho \approx \rho^{2(\delta-1)+2} \\ &\int H_\rho^2 \rho \, d\rho \approx \int \left(\frac{1}{\rho} \frac{\partial U}{\partial \phi}\right)^2 \rho \, d\rho \approx \rho^{2\delta-2+2} \end{aligned} \tag{54}$$

all of which must go to zero with ρ. Therefore $\delta > 0$.

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In other words, for this (Transverse Electric) case, the field U may not possess singularities at a corner, even if it is an edge.

B. Asymptotic behaviour of $C\{\mu\}$

Since U is non-singular at r = b, it may be expanded in a series of ascending powers of (r-b). Expanding each power binomially, and collecting like powers of r, we obtain a series expansion of $U \pm \beta$ in powers or r. This series will have an infinite radius of convergence, except for the special case $\phi_o = \pm \beta$, which we ignore. Then we can write

$$U\{-\beta\} + U\{+\beta\} = \sum_{n=0}^{\infty} a_n (gr)^{n+\alpha} \tag{55}$$

where α is some positive real number.
Then

$$C\{\mu\} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n \int_{gb}^{\infty} (gr)^{n+\alpha-1} K_\mu\{gr\} \, d(gr) \tag{56}$$

Since K_μ decays exponentially at infinity, the integrals all converge absolutely and the inversion of the order of integration and summation is justified.

Moreover

$$\int x^n H^1_\mu\{x\} \, dx = (n+\mu-1)\, x \, H^1_\mu\{x\} S_{n-1,\mu-1}\{x\} - x \, H^1_{\mu-1}\{x\} S_{n,\mu}\{x\} \tag{57}$$

where $S_{n,\mu}\{x\}$ is Lommel's function [6]. Let iz = x. Then

$$H^1_\mu\{iz\} = \frac{2}{i\pi} \exp\left\{-i \frac{\pi}{2} \mu\right\} K_\mu\{z\} \tag{58}$$

$$\int \exp\left\{i \frac{\pi}{2} n\right\} z^n K_\mu\{z\} \, dz = (n+\mu-1)\, z \, K_\mu\{z\} S_{n-1,\mu-1}\{iz\} - z \exp\left\{i \frac{\pi}{2}\right\} K_{\mu-1}\{z\} S_{n,\mu}\{iz\} \tag{59}$$

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Using the relations between $K_\mu\{z\}$ and $K_{\mu-1}\{z\}$ and $\partial K/\partial z$ and the similar relation between S and its derivative, we have

$$\int z^n K_\mu\{z\}\,dz = e^{-i\frac{\pi}{2}n}\left(\frac{z}{i}\right)\left\{K_\mu\{z\}\frac{S_{n,\mu}\{iz\}}{z} - S_{n,\mu}\{iz\}\frac{K_\mu\{z\}}{z}\right\} \quad (60)$$

But, according to Erdelyi, et al. [6]

$$S_{n,\mu}\{iz\} = s_{n,\mu}\{iz\} + A\{n,\mu\}\,J_\mu\{iz\} + B\{n,\mu\}\,J_{-\mu}\{iz\} \quad (61)$$

But when this expression is substituted into equation (59), the terms involving the Bessel functions become Wronskians, which are proportional to $1/z$. Then the factor z makes these terms constants, and they will cancel out when we evaluate the integral at the limits. Therefore

$$\int_{gb}^{\infty} z^n K_\mu\{z\}\,dz = \exp\left\{-i\frac{\pi}{2}n\right\}\left(\frac{z}{i}\right)\left[K_\mu\frac{\partial s_{n,\mu}}{\partial z} - s_{n,\mu}\frac{\partial K_\mu}{\partial x}\right|_{gb}^{\infty} \quad (62)$$

But at the upper limit, $K_\mu\{z\}$ decays exponentially, while

$$s_{n,\mu}\{z\} \approx z^{n-1}\left(1 + 0\left\{\frac{1}{z}\right\}\right) \quad (63)$$

which is merely algebraic, so the expression in the brackets goes to zero at the upper limit. At the lower limit

$$\int_{gb}^{\infty} z^n K_\mu\{z\}\,dz = \exp\left\{-in\frac{\pi}{2}\right\}(igb)\left\{K_\mu\{gb\}\frac{\partial s_{n,\mu}\{igb\}}{\partial gb} - s_{n,\mu}\{igb\}\frac{\partial K_\mu\{gs\}}{\partial gb}\right\} \quad (64)$$

But

$$s_{n,\mu}\{iz\} \approx \frac{(iz)^{n+1}}{(n+1)^2-\mu^2}\left(1+0\left\{\frac{1}{\mu^2}\right\}\right) \quad \text{as } |\mu| \to \infty \quad (65)$$

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This asymptotic series may be differentiated, so

$$\int_{gb}^{\infty} z^n \; K_\mu\{z\} \, dz = \text{const.} \; \frac{K_\mu\{gb\}}{\mu^2} + \text{const.} \; \frac{\partial K_\mu\{gb\}}{\mu^2 \, \partial gb} \quad (66)$$

But $$\frac{\partial K_\mu\{gb\}}{\partial \, gb} \approx \mu \, K_\mu\{gb\} \qquad \text{as} \quad |\mu| \rightarrow \infty \quad (67)$$

Therefore $$\int_{gb}^{\infty} z^n \, K_\mu\{z\} \; dz \approx \frac{K_\mu\{gb\}}{\mu} \qquad \text{as} \quad |\mu| \rightarrow \infty \quad (68)$$

From which it can be seen that

$$C\{\mu\} \approx \frac{K_\mu\{gb\}}{\mu} \qquad \text{as} \quad |\mu| \rightarrow \infty \quad (69)$$

C. Closing the contour for the first integral equations

We have already shown that

$$0 = \int_L C\{\mu\} \; I_\mu\{gr\} \; \mu \; d\mu \qquad \underline{r<b} \quad (48)$$

and that

$$C\{\mu\} \approx \frac{K_\mu\{gb\}}{\mu} \qquad \text{as} \quad |\mu| \rightarrow \infty \quad (69)$$

It can also be shown that

$$I_\mu\{gr\} \frac{K_\mu\{gb\}}{\mu} \approx \frac{1}{\mu^2} \left(\frac{b}{r}\right)^{-\mu} \qquad \text{as} \quad |\mu| \rightarrow \infty, \; \operatorname{Re} \mu > 0 \quad (70)$$

But for r<b this expression decays exponentially, and therefore the contour can be closed by an infinite semicircle enclosing the right half-plane. Therefore the function $C\{\mu\}$ is a suitable function for the application of the Wiener-Hopf technique.

Similar considerations apply to $D\{\mu\}$ [see equation (49)].

IX. The Second Equation and the Function $A\{\mu\}$

If, in equation (22), we change μ to $-\mu$ and change the direction of integration, we get

$$0 = \int_{L^*} F_1\{-\mu\} \frac{\sin\mu\pi}{\cos\mu\{\pi-\beta\}} K_\mu\{gr\}\,\mu^2\,d\mu \qquad \underline{r>b} \qquad (71)$$

But the only singularity present between 1/2 and -1/2 in the integrand is the simple pole of F_1 at zero, which is canceled by the μ^2 in the numerator. So if the original contour lies between these limits, the contour L^* may be deformed so as to coincide with the original contour L. Then we may subtract equation (33) from the resulting equation, a course suggested by the fact that the even part of F_1 obviously contributes nothing to the integral. We conclude that

$$0 = \int_L \left[F_1\{-\mu\} - F_1\{\mu\}\right] \frac{\sin\mu\pi}{\cos\mu\{\pi-\beta\}} K_\mu\{gr\}\,\mu^2\,d\mu \qquad (72)$$

$$\underline{r>b}$$

Unfortunately, the integrand of this equation exhibits an infinite number of singularities on both half-planes. We must, therefore, so modify the function that these disappear on at least one side. Let

$$\frac{F_1\{-\mu\} - F_1\{\mu\}}{\cos\mu\,(\pi-\beta)} = \frac{A\{\mu\} - A\{-\mu\}}{\mu\sin\mu\pi} \qquad (73)$$

where we would like $A\{\mu\}$ to be free of poles on the left half-plane. Let $\Delta\{\beta\}$ be the discontinuity of $\frac{\partial U}{\partial\phi}$ across β. Then from the manner in which equation (33) was obtained, it can be seen that

$$\Delta\{\beta\} - \Delta\{-\beta\} = -2\int_L F_1\{\mu\} \frac{\sin\mu\pi}{\cos\mu\,(\pi-\beta)} K_\mu\{gr\}\,\mu^2\,d\mu \qquad (74)$$

THES

Splitting the $K_\mu\{gr\}$ in the usual manner,

$$\Delta\{\beta\} - \Delta\{-\beta\} = -2\int_L (F_1\{-\mu\} - F_1\{\mu\})\,\frac{\pi}{2}\,\frac{I_\mu\{gr\}}{\cos\mu\,(\pi-\beta)}\,\mu^2\,d\mu \qquad (75)$$

or, in terms of $A\{\mu\}$

$$\Delta\{\beta\} - \Delta\{-\beta\} = -\pi\int_L \frac{A\{\mu\} - A\{-\mu\}}{\sin\mu\pi}\; I_\mu\{gr\}\,\mu\,d\mu \qquad (76)$$

It follows from our treatment of the behaviour of the solution near a corner that $\Delta \rightarrow 0$ as $r \rightarrow 0$. We may apply the inversion formula (Theorem I)

$$-i\,\pi^2\,\frac{A\{\mu\} - A\{-\mu\}}{\sin\mu\pi} = \int_0^b (\Delta\{\beta\} - \Delta\{-\beta\})\,K_\mu\{gr\}\,\frac{dr}{r} \qquad (77)$$

Now if we identify

$$A\{\mu\} = \frac{-1}{2\pi i}\int_0^b (\Delta\{\beta\} - \Delta\{-\beta\})\,I_{-\mu}\{gr\}\,\frac{dr}{r} \qquad (78)$$

then $A\{\mu\}$ has no singularities on the entire left half-plane. Moreover, by a treatment similar to that accorded the function $C\{\mu\}$, it can be shown that

$$A\{\mu\} \approx \frac{I_{-\mu}\{gb\}}{\mu} \qquad \text{as} \quad |\mu| \rightarrow \infty \qquad (79)$$

But

$$\frac{K_\mu\{gr\}\,I_{-\mu}\{gb\}}{\mu}\,\mu \approx \frac{1}{\mu}\left(\frac{r}{b}\right)^{-|\mu|} \quad \text{as Re}\,\mu \rightarrow -\infty \qquad (80)$$

so if we split the integral (72) after substituting relation (73) we get

$$0 = \int_L A\{\mu\}\,K_\mu\{gr\}\,\mu\,d\mu - \int_L A\{-\mu\}\,K_\mu\{gr\}\,\mu\,d\mu \qquad (81)$$

Changing μ to $-\mu$ in the second integral and changing the direction of integration, we get

THES

$$0 = \int_L A\{\mu\} K_\mu\{gr\} \mu d\mu + \int_{L^*} A\{\mu\} K_\mu\{gr\} \mu d\mu \tag{82}$$

But $A\{\mu\}$ is analytic in some finite strip containing the imaginary axis and L, so we may deform L^* into L.

$$0 = \int_L A\{\mu\} K_\mu\{gr\} \mu d\mu \qquad \underline{r>b} \tag{83}$$

But in view of relation (80), this integrand decays exponentially on the left half-plane (because r>b), so we may close the contour by the addition of an infinite semicircle enclosing the left half-plane. Therefore the function $A\{\mu\}$ is suitable for the application of the Wiener-Hopf technique.

Similar considerations apply to the function defined by the equations

$$\frac{B\{-\mu\} - B\{\mu\}}{\mu \sin \mu \pi} = \frac{F_2\{\mu\} + F_2\{-\mu\}}{\sin \mu (\pi - \beta)} \tag{84}$$

and

$$B\{\mu\} = \frac{-1}{2\pi i} \int_0^b (\Delta\{\beta\} + \Delta\{-\beta\}) I_{-\mu}\{gr\} \frac{dr}{r} \tag{85}$$

Therefore

$$0 = \int_L B\{\mu\} K_\mu\{gr\} \mu \, d\mu \qquad \underline{r>b} \tag{86}$$

in which $B\{\mu\}$ is analytic on the left half plane and possesses such decay as to make it possible to close the contour on the left. $B\{\mu\}$ is a suitable function for the application of the Wiener-Hopf technique. The proof of these assertions follows exactly the same lines as that for $A\{\mu\}$.

THESI

X. The Wiener-Hopf Technique Applied to $F_1\{\mu\}$

Corresponding to F_1 we have the equations

$$\int_L A\{\mu\} K_\mu\{gr\} \mu \, d\mu \qquad \underline{r>b} \qquad (83)$$

and

$$\int_L C\{\mu\} I_\mu\{gr\} \mu \, d\mu \qquad \underline{r<b} \qquad (48)$$

of which the former may be closed on the left, the latter on the right. Then equations (47) and (24) combine to give

$$C\{\mu\} = \frac{\pi}{2 \sin \mu \pi} \Big\{ (F_1\{-\mu\} - F_1\{\mu\}) \cos \mu \beta - \frac{4}{i\pi \mu} K_\mu\{gr_o\} \cos\mu \phi_o \cos\mu \{ \pi - \beta\}\Big\} \qquad (87)$$

Substituting (73) into the above

$$C\{\mu\} = \frac{\pi}{2 \sin \mu \pi} \Big\{ (A\{\mu\} - A\{-\mu\}) \frac{\cos\mu\beta \cos\mu(\pi-\beta)}{\mu \sin \mu \pi} - \frac{4}{i \pi\mu} K_\mu\{gr_o\} \cos\mu \phi_o \cos\mu(\pi - \beta)\Big\} \qquad (88)$$

We must now separate this equation into functions which are analytic and possess at least algebraic decay on the right half-plane [plus-functions] and functions which are analytic and possess at least algebraic decay on the left half-plane [minus-functions]. It can be seen at once by equation (69) that the function $C\{\mu\}$ does not approach zero at infinity on either half-plane. We must therefore divide the equation (88) by

$$\left(\frac{gb}{2}\right)^{-\mu} \Gamma\{\mu\}$$

in order to make the quotient decay at least algebraically at infinity on the right half-plane.

THESI

For convenience in the subsequent work we define:

$$\epsilon = \frac{gb}{2} \tag{89}$$

$$P\{\mu\} = \frac{\pi}{2\mu} \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} \frac{\cos\mu\beta \cos\mu(\pi-\beta)}{(\sin\mu\pi)^2} \tag{90}$$

$$P_n = \operatorname*{Res}_{\mu \to -n} \left(P\{\mu\} \right) = -\frac{1}{2\pi} (-\epsilon)^{-n} \cos n\beta \quad \cos n(\pi-\beta) \Gamma\{n\}, \tag{91}$$

$$R\{\mu\} = \frac{2}{i\mu} \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} K_{\mu}\{gr_o\} \frac{\cos\mu\phi_o \cos\mu(\pi-\beta)}{\sin\mu\pi} \tag{92}$$

$$R_n = \operatorname*{Res}_{\mu \to n} R\{\mu\} = \frac{2(-\epsilon)^n}{i\pi n!} K_n\{gr_o\} \cos n\phi_o \cos n(\pi-\beta) \tag{93}$$

$$Q_n = \lim_{\mu \to n} \frac{(\mu-n)^2 P\{\mu\}}{\pi^2} = \frac{\epsilon^n \cos n\beta \cos n(\pi-\beta)}{2\pi n!} \tag{94}$$

$$S\{\mu\} = \frac{-2}{i\mu} \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} I_{-\mu}\{gr_o\} \frac{\cos\mu\phi_o \cos\mu(\pi-\beta)}{(\sin\mu\pi)^2} \tag{95}$$

$$S_n = \frac{2\epsilon^{-n}}{i\pi^2 n} \cos n\phi_o \cos n\beta \quad I_n\{gr_o\} \tag{96}$$

$$M_n\{\mu\} = \frac{\partial}{\partial\mu} \left\{ \frac{\pi}{2\mu} \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} \cos\mu\beta \cos\mu(\pi-\beta) A\{-\mu\} \Big|_n \frac{\mu-n}{\sin^2\mu\pi} \right. \tag{97}$$

$$M_n = \operatorname*{Res}_{\mu \to n} M_n\{\mu\}$$

$$N\{\mu\} = \frac{-2}{i\mu} \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} I_{\mu}\{gr_o\} \frac{\cos\mu\phi_o \cos\mu(\pi-\beta)}{(\sin\mu\pi)^2} \tag{98}$$

where n is a positive integer.

Rearranging equation (88) and using the above definitions we have

$$P\{\mu\} A\{\mu\} = \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} C\{\mu\} + P\{\mu\} A\{-\mu\} + R\{\mu\} \tag{99}$$

THESI

in which we have put the prospective "minus-functions" on the left hand side and prospective "plus-functions" on the right hand side.

Due to the singularities of $P\{\mu\}$, the function on the left is not yet a minus-function. It can easily be shown that the function $P\{\mu\} A\{\mu\}$ has the proper decay on the left half-plane, and also in the strip

$$0 < \mathrm{Re}\ \mu < 1$$

All the functions in equations (88) and (99) are analytic in this strip, so if we can make them all plus- or minus-functions simultaneously, we may set each side of the equation equal to zero.

In the case of the function $P\{\mu\} A\{\mu\}$ this may be done by subtracting a Mittag-Leffler series including all of the singularities of $P\{\mu\} A\{\mu\}$ on the left half-plane. We subtract

$$\sum_{n=1}^{\infty} \frac{P_n A\{-n\}}{\mu - n} + \frac{Q_o\ A\{0\}}{\mu^2} + \frac{M\{0\}}{\mu} \qquad (100)$$

from both sides of the equation. The two terms on the right are contributed by the double pole of $P\{\mu\}$ at the origin, and

$$Q_o = \frac{1}{2\pi}, \qquad (101)$$

$$M(0) = \frac{1}{2\pi}\left\{ \left.\frac{\partial A}{\partial \mu}\right|_o + \log_e \epsilon\ A\{0\} - \psi\{1\} A\{0\}\right\} \qquad (102)$$

The Mittag-Leffler series in (100) contains the unknowns $A\{-n\}$, so its convergence can not be definitely established without completely solving the problem. We know that $A\{-n\}$ decays like a gamma-function for values of the argument approaching minus infinity, but P_n approaches infinity factorially at the same time, so the convergence is not so simple to establish. We shall assume convergence.

The function $\dfrac{\epsilon^{-\mu}}{\Gamma\{\mu\}}\ C\{\mu\}$ is entire, and decays algebraically in the strip and on the right half plane. It is, therefore, a plus-function already.

THES

The function $P\{\mu\} A\{-\mu\}$ decays algebraically in the strip and like the inverse square of a gamma function on the right half-plane. However, it has double poles at the positive integers, which must be removed by subtracting

$$\sum_{n=1}^{\infty} \frac{Q_n A\{-n\}}{(\mu - n)^2} + \sum_{n=1}^{\infty} \frac{M_n}{\mu - n} \tag{103}$$

from both sides of the equation.

We must now calculate M_n explicitly.

$$\frac{\partial}{\partial \mu}\left\{\frac{\epsilon^{\mu}}{(\mu+1)} \cos\mu\beta \cos\mu(\pi-\beta) A\{-\mu\}\right\} =$$

$$\frac{-\epsilon^{\mu}}{\Gamma\{\mu+1\}} \cos\mu\beta \cos\mu(\pi-\beta) \frac{\partial A\{-\mu\}}{\partial \mu}$$

$$+ \frac{\epsilon^{\mu}\log_e\epsilon}{\Gamma\{\mu+1\}} \cos\mu\beta \cos\mu(\pi-\beta) A\{-\mu\}$$

$$\frac{-\epsilon^{\mu}\psi\{\mu+1\}}{\Gamma\{\mu+1\}} \cos\mu\beta \cos\mu(\pi-\beta) A\{-\mu\}$$

$$\frac{-\beta\,\epsilon^{\mu}}{\Gamma\{\mu+1\}} \sin\mu\beta \cos\mu(\pi-\beta) A\{-\mu\}$$

$$\frac{-(\pi-\beta)\,\epsilon^{\mu}}{\Gamma\{\mu+1\}} \cos\mu\beta \sin\mu(\pi-\beta) A\{-\mu\} \tag{104}$$

Therefore

$$M_n = \frac{-(-\epsilon)^n}{2\pi n!} \cos^2 n\beta\, A\{-n\}\left\{\frac{A'\{-n\}}{A\{-n\}} - \log_e\epsilon + \psi\{n+1\}\right.$$

$$\left. + (2\beta-\pi)\tan n\beta\right\} \tag{105}$$

The function $R\{\mu\}$ must be treated differently depending on whether $b > r_o$ or $b < r_o$. In the former case it will be found necessary to modify the equation (99).

THES

The case $r_o > b$

The function $R\{\mu\}$ behaves properly in the strip, but at infinity on the right half plane, it approaches

$$R\{\mu\} \approx \frac{1}{\mu}\left(\frac{b}{r_o}\right)^{|\mu|} \quad \text{as Re } \mu \longrightarrow \infty \tag{106}$$

from which it can be seen that $R\{\mu\}$ decays only if $r_o > b$. Subject to this condition, however, we can treat it as we have the previous functions. $R\{\mu\}$ has simple poles at the positive integers, which we remove by subtracting

$$\sum_{n=1}^{\infty} \frac{R_n}{\mu - n} \tag{107}$$

from both sides of the equation.

The case $b > r_o$

Since the difficulty arises from the undesirable behavior of the MacDonald function, we split it using equations (7), (95) and (98)

$$R\{\mu\} = N\{\mu\} - S\{\mu\} \tag{108}$$

where $N\{\mu\}$ will become a plus-function and $S\{\mu\}$ will become a minus function. In the strip, $S\{\mu\}$ decays exponentially because we have assumed $\phi_o < \beta$. On the left half plane, $S\{\mu\}$ exhibits the behaviour

$$S\{\mu\} \quad \frac{1}{\mu} \quad \left(\frac{b}{r_o}\right)^{-|\mu|} \qquad \underline{\text{Re } \mu < 0} \tag{109}$$

from which it can be seen that $S\{\mu\}$ decays exponentially when Re $\mu \rightarrow -\infty$. $S\{\mu\}$ has simple poles at the negative integers and a double pole at zero. We subtract

THES

$$\sum_{n=1}^{\infty} \frac{S_n}{\mu + n} + \frac{N_o}{\mu^2} + \frac{N'_o}{\mu} \tag{110}$$

from each side of the equation. In the above

$$N_o = \frac{-2}{i\pi^2} \quad I_o\{gr_o\} \tag{111}$$

and

$$N'_o = \frac{2}{i\pi^2} \; I_o\{gr_o\}\left[\frac{I'_o\{gr_o\}}{I_o\{gr_o\}} - \log_e \epsilon + \psi\{1\}\right] \tag{112}$$

In the strip, $N\{\mu\}$ behaves the same as $S\{\mu\}$, while on the right half-plane it decays like the inverse square of a gamma-function. However, $N\{\mu\}$ has double poles at all the positive integers. They are removed by subtracting

$$\sum_{n=1}^{\infty} \left\{ \frac{N_n}{(\mu - n)^2} + \frac{N'_n}{\mu - n} \right\} \tag{113}$$

from both sides of the equation. In the above

$$N'_n = \frac{-2}{i\pi^2} \; \frac{(-\epsilon)^n}{n!} \cos n\,\phi_o \cos n\,\beta \; I_n\{gr_o\}\left\{ \frac{I'_n\{gr_o\}}{I_n\{gr_o\}} \right. \tag{114}$$

$$\left. + \log_e \epsilon - \psi\{n+1\} - \phi_o \tan n\,\phi_o + (\pi - \beta) \tan n\,\beta \right\}$$

and

$$N_n = \frac{-2(-\epsilon)^n}{i\pi^2 n!} \; I_n\{gr_o\} \cos n\,\phi_o \cos n\,\beta \tag{115}$$

Then for the case $b > r_o$ we have the modified equation

$$P\{\mu\}A\{\mu\} + S\{\mu\} = \frac{\epsilon^{\mu}}{\Gamma\{\mu\}} C\{\mu\} + P\{\mu\}A\{-\mu\} + N\{\mu\} \tag{116}$$

THES

We now have two equations; (99) for the case of $b < r_o$; (116) for the case of $b > r_o$. After subtracting the appropriate Mittag-Leffler series from both sides in each of these equations, we find that in each case we have a minus-function on the left, a plus-function on the right. Since these functions are analytic in the same strip $0 < \operatorname{Re} \mu < 1$, they both define the function zero. Hence

$$P\{\mu\} A\{\mu\} = \sum_{n=1}^{\infty} \frac{P_n A\{-n\}}{\mu + n} + \frac{Q_o A\{0\}}{\mu^2} + \frac{M\{0\}}{\mu} + \sum_{n=1}^{\infty} \frac{Q_n A\{-n\}}{(\mu - n)^2} + \sum_{n=1}^{\infty} \frac{M_n}{\mu - n} + \sum_{n=1}^{\infty} \frac{R_n}{\mu - n} \tag{117}$$

<u>for $b<r_o$</u>

and

$$P\{\mu\} A\{\mu\} + S\{\mu\} = \sum_{n=1}^{\infty} \frac{P_n A\{-n\}}{\mu + n} + \frac{Q_o A\{0\}}{\mu^2} + \frac{M\{0\}}{\mu} + \sum_{n=1}^{\infty} \frac{Q_n A\{-n\}}{(\mu - n)^2} + \sum_{n=1}^{\infty} \frac{M_n}{\mu - n} + \sum_{n=1}^{\infty} \frac{S_n}{\mu + n} + \frac{N_o}{\mu^2} + \frac{N'_o}{\mu} + \sum_{n=1}^{\infty} \frac{N_n}{(\mu - n)^2} + \sum_{n=1}^{\infty} \frac{N'_n}{\mu - n} \tag{118}$$

<u>for $b > r_o$</u>

THES

XI. The Infinite System of Equations for $A\{-m\}$

If, in equation (117) we let $\mu = -m$, where m may be any positive integer, we obtain the infinite system of equations

$$\sum_{\substack{n=1 \\ n\neq n}}^{\infty} \frac{P_n A\{-n\}}{n-m} + \sum_{n=1}^{\infty} \frac{Q_n A\{-n\}}{(n+m)^2} - \sum_{n=1}^{\infty} \frac{M_n}{n+m} - \sum_{n\neq 1}^{\infty} \frac{R_n}{n+m} \qquad (119)$$

$$+ \frac{Q_o A\{0\}}{m^2} - \frac{M\{0\}}{m} = \frac{(-1)^{m+1}}{2\pi} \frac{\partial}{\partial \mu}$$

$$\left\{ \epsilon^{\mu} \Gamma\{-\mu\} \cos \mu \beta \cos \mu(\pi-\beta) A\{\mu\} \right\} \Big|_{-m}$$

for $b < r_o$

in which it must be remembered that M_n contains a linear combination of $A\{-n\}$ and $A'\{-n\}$.

Similarly,

$$\sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{P_n A\{-n\}}{m=n} - \sum_{n=1}^{\infty} \frac{Q_n A\{-n\}}{(m+n)^2} + \sum_{n=1}^{\infty} \frac{M_n}{m+n} + \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{S_n}{m-n}$$

$$- \sum_{n=1}^{\infty} \frac{N_n}{(m+n)^2} + \sum_{n=1}^{\infty} \frac{N'_n}{M+n} - \frac{Q_o A\{0\}}{m^2} + \frac{M\{0\}}{m} \qquad (120)$$

$$- \frac{N_o}{m^2} + \frac{N'_o}{m} = \frac{(-1)^m}{2\pi} \frac{\partial}{\partial \mu}$$

$$\left[\epsilon^{\mu} \Gamma\{-\mu\} \cos \mu \beta \cos \mu(\pi-\beta) A\{\mu\} \right|_{-m}$$

$$+ \frac{2(-1)^m}{i\pi} \frac{\partial}{\partial \mu} \left[\epsilon^{\mu} \Gamma\{-\mu\} \cos \mu \phi_o \cos \mu(\pi-\beta) I_{-\mu}\{g r_o\} \right|_{-m}$$

for $b > r_o$

THES

XII. The Infinite System of Equations for $B\{-m\}$

Following an analysis similar to that used for $A\{\mu\}$, we have

$$D\{\mu\} = \frac{\pi}{2\sin\mu\pi}\Big\{ -F_2\{-\mu\}\sin\mu\beta - F_2\{\mu\}\sin\mu\beta$$

$$+ \frac{4}{i\pi\mu} K_\mu\{gr_o\}\sin\mu\,\phi_o\,\sin\mu(\pi-\beta)\Big\} \qquad (121)$$

or, substituting from equation (84)

$$D\{\mu\} = \frac{\pi}{2\sin\mu\pi}\Big\{ \frac{(B\{\mu\} - B\{-\mu\})\sin\mu\beta\,\sin\mu(\pi-\beta)}{\sin\mu\pi}$$

$$+ \frac{4}{i\pi\mu} K_\mu\{gr_o\}\sin\mu\,\phi_o\,\sin\mu(\pi-\beta)\Big\} \qquad (122)$$

The application of the Wiener-Hopf technique follows exactly the same lines as that for $A\{\mu\}$, except that there are no poles at zero. In fact, it can easily be seen that $B\{-m\}$ obeys equations (117), (118), (119), and (120) with the provision that, in each case, the cosines are replaced with sines and the following changes are introduced:

$$Q_o \rightarrow 0 \qquad M\{0\} \rightarrow 0 \qquad N_o \rightarrow 0 \qquad N'_o \rightarrow 0 \qquad (123)$$

THES

XIII. The Series Expansion of the Solution

According to our original representation

$$U_s = \int_L (F_1\{\mu\}\cos\mu\phi + F_2\{\mu\}\sin\mu\phi)\, K_\mu\{gr\}\,\mu d\mu \quad (11)$$

in Region I

Changing μ to $-\mu$, and changing the direction of integration, we get

$$U_s = -\int_{L^*} (F_1\{-\mu\}\cos\mu\phi - F_2\{-\mu\}\sin\mu\phi)\, K_\mu\{gr\}\,\mu d\mu \quad (124)$$

But relation (42) shows that the integrand of equation (124) is analytic in the strip $-1 < \mathrm{Re}\,\mu < 1$ at least. Then if the original contour lies within that strip, we may now deform L^* into L, and add equations (11) and (124).

$$2U_s = \int_L \Big\{ (F_1\{\mu\} - F_1\{-\mu\})\cos\mu\phi + (F_2\{\mu\} + F_2\{-\mu\})\sin\mu\phi \Big\} \cdot K_\mu\{gr\}\,\mu d\mu \quad (125)$$

Using equations (73) and (84),

$$2U_s = \int_L \Big\{ (A\{-\mu\} - A\{\mu\}) \frac{\cos\mu\phi\cos\mu(\pi-\beta)}{\mu\sin\mu\pi} + (B\{-\mu\} + B\{\mu\}) \frac{\sin\mu\phi\sin\mu(\pi-\beta)}{u\sin\mu\pi} \Big\} K_\mu\{gr\}\,\mu d\mu \quad (126)$$

In the following, it will be assumed that L lies in the strip $-1 < \mathrm{Re}\,\mu < 1$. Let us split the integral and consider, for the moment, only the terms involving $A\{-\mu\}$ and $B\{-\mu\}$.

THESI

Changing the variable to $-\mu$ in these expressions, and changing the direction of integration, we have

$$\int_L \frac{A\{-\mu\}\cos\mu\phi\cos\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu$$

$$= -\int_{L^*} \frac{A\ \mu\ \cos\mu\phi\cos\mu\ \pi-\beta}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu \tag{127}$$

and

$$\int_L \frac{B\{-\mu\}\sin\mu\phi\sin\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu$$

$$= -\int_{L^*} \frac{B\{\mu\}\sin\mu\phi\sin\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu \tag{128}$$

We will close these integrals on the left, so we must deform the path of integration of the other two to conform with L^*. However, the integrand of the first has a simple pole at zero, so we must include the appropriate residue.

$$\int_L \frac{B\{\mu\}\sin\mu\phi\sin\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu$$

$$= \int_{L^*} \frac{B\{\mu\}\sin\mu\phi\sin\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu \tag{129}$$

but

$$\int_L \frac{A\{\mu\}\cos\mu\phi\cos\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu$$

$$= \int_{L^*} \frac{A\{\mu\}\cos\mu\phi\cos\mu(\pi-\beta)}{\mu\sin\mu\pi} K_\mu\{gr\}\mu\, d\mu + 2iA\{0\}K_o\{gr\} \tag{130}$$

Therefore

$$U_s = -iA\{0\}K_o\{gr\} - \int_L A\{\mu\}\frac{\cos\mu\phi\cos\mu(\pi-\beta)}{\sin\mu\pi} K_\mu\{gr\}\, d\mu$$

$$- \int_L B\{\mu\}\frac{\sin\mu\phi\sin\mu(\pi-\beta)}{\sin\mu\pi} K_\mu\{gr\}\, d\mu \tag{131}$$

THESI

It can be seen that these integrals converge for $|\phi| < \beta$, which is exactly the range of validity of the representation (11). It can also be seen, from equations (79) and (80), that these integrals can be closed on the left if r > b. In that case

$$U_s = -iA\{0\} K_o\{gr\} - 2i \sum_{n=1}^{\infty} (-1)^n A\{-n\} \cos n(\pi - \beta) \cos n\phi K_n\{gr\}$$

$$- 2i \sum_{n=1}^{\infty} (-1)^n B\{-n\} \sin n(\pi - \beta) \sin n\phi K_n\{gr\}$$

$$\text{for } r > b \quad \text{and} \quad |\phi| < \beta \qquad (132)$$

Expanding $\cos n(\pi - \beta)$ and $\sin n(\pi - \beta)$

$$U_s = -iA\{0\} K_o\{gr\} - 2i \sum_{n=1}^{\infty} A\{-n\} \cos n\beta \cos n\phi K_n\{gr\}$$

$$+ 2i \sum_{n=1}^{\infty} B\{-n\} \sin n\beta \sin n\phi K_n\{gr\} \qquad (133)$$

$$\text{for } r > b$$

It can be easily shown that equation (133) is valid for all values of ϕ.

THESI

XIV. The Scattering Cross-Section for the Symmetric Case

In expression (133), let $\phi_o = 0$. Then $B\{\mu\} = 0$. Returning to real wave numbers

$$U_s = \frac{\pi}{2} A\{0\} H_o^1\{kr\} + \pi \sum_{n=1}^{\infty} (i)^n A\{-n\} \cos n\beta \; \cos n\phi \; H_n^1\{kr\} \tag{134}$$

But, for large r,

$$H_n^1\{kr\} \longrightarrow \sqrt{\frac{2}{\pi\, kr}} \; \exp\left\{i(kr - \{2n+1\} \frac{\pi}{4})\right\} \tag{135}$$

So, as $r \longrightarrow \infty$

$$U_s \rightarrow \sqrt{\frac{\pi}{2kr}} \; e^{i\left\{kr - \frac{\pi}{4}\right\}} \; A\{0\} + 2 \sum_{n=1}^{\infty} A\{-n\} \cos n\beta \cos n\phi \tag{136}$$

and, since $U = E_z$

$$H_\phi = \frac{1}{i\,\omega\,\mu_o} \; \frac{\partial U}{\partial r} \tag{137}$$

Taking $\frac{1}{2}$ Re $\{E_z \; H_\phi^*\}$, we have the outgoing energy

$$\overline{E}x\overline{H}\Big|_r = \frac{\pi}{4\,\omega\,\mu_o r} \; \left|A\{0\} + 2 \sum_{n=1}^{\infty} A\{-n\} \cos n\beta \; \cos n\phi\right|^2 \tag{138}$$

and the outgoing power

$$\frac{\partial W}{\partial t} = \frac{\pi^2}{2\omega\,\mu_o} \left\{ |A\{0\}|^2 + 2 \sum_{n=1}^{\infty} |A\{-n\}|^2 \; \cos^2 n\beta \right\} \tag{139}$$

The cross-section is this expression divided by the incident energy, which for the symmetric case is

$$\frac{2\pi}{\omega\mu_o} \arcsin\left(\frac{b \sin\beta}{r_o}\right) \tag{140}$$

So, for the symmetric case, the cross-section is

$$\frac{\pi}{4}\left[\arcsin\left(\frac{b \sin\beta}{r_o}\right)\right]^{-1}\left\{|A\{0\}|^2 + 2\sum_{n=1}^{\infty}|A\{-n\}|^2\cos^2 n\beta\right\} \tag{141}$$

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XV. Special Values of $\frac{\partial A}{\partial \mu}$

If, in equation (88), we let μ approach any positive integer $\underline{m}$, the function $C\{\mu\}$ on the left is bounded, so the same must be true of the function on the right. Moreover, since

$$I_{-m}\{gr\} = I_m\{gr\} \tag{142}$$

inserting this relation into equation (78) shows that

$$A\{-m\} = A\{m\} \tag{143}$$

Using this information, and applying L'Hospital's rule to equation (88), we have

$$A'\{m\} + A'\{-m\} = \frac{4}{i} K_m\{gr_o\} \frac{\cos m\,\phi_o}{\cos m\,\beta} (-1)^m \tag{144}$$

and, in particular,

$$A'\{0\} = \frac{2}{i} K_o\{gr_o\} \tag{145}$$

XVI. The Case of the Strip

A. The System

If we examine the special case $\beta \longrightarrow \frac{\pi}{2}$, $\phi_o = 0$ the wedge becomes an infinite strip of width $2b$. Since the arrangement is symmetric, we can disregard the $B\{\mu\}$.

The infinite system reduces to

$$\sum_{\substack{n=1\\n\neq m}}^{\infty} \frac{-\epsilon^{-2n}\Gamma\{2n\}\ A\{-2n\}}{2n - 2m} + \sum_{n=1}^{\infty} \frac{\epsilon^{2n}\ A\{-2n\}}{4\Gamma\{2n+1\}\ (n+m)^2}$$

$$- \sum_{n=1}^{\infty} \frac{\frac{\partial}{\partial\mu}\left[\frac{\epsilon^{\mu} A\{-\mu\}}{\Gamma\{\mu+1\}}\right]_{2n}}{2\ (n+m)} - \sum_{n=1}^{\infty} \frac{2\ \epsilon^{2n}(-1)^n K_{2n}\{gr_o\}}{i\,\Gamma\{2n+1\}\ (n+m)}$$

$$+ \frac{A\{0\}}{4\ m^2} - \frac{1}{2\ m}\left\{A'\{0\} + \log_e \epsilon\ A\{0\} - \psi\{1\}\ A\{0\}\right\}$$

$$= -\left\{\epsilon^{-2m}\log_e \epsilon\ \Gamma\{2m\}\ A\{-2m\} - \epsilon^{-2m}\Gamma\{2m\}\psi\{2m\}A\{-2m\}\right.$$

$$\left. + \epsilon^{-2m}\Gamma\{2m\}\ \frac{\partial A\{\mu\}}{\partial \mu}\Big|_{-2m}\right\} \qquad (146)$$

The solution of this infinite system is not expected to be materially simpler than for the unsymmetric case, since we still have a system involving both the unknown $A\{\mu\}$ and its derivative. We could further simplify this system by allowing r_o to approach infinity. In that event we use the source function

$$U_i = \exp\{-gr\cos\phi\} = \frac{1}{i\pi} \int_L \cos\mu\phi\ \ K_\mu\{gr\}\ d\mu$$

(147)

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instead of the Hankel function [equation (1)]. Then wherever we have the MacDonald function of r_o, we may replace it by $\frac{1}{2}$.

$$K_n \{ gr_o \} \longrightarrow \frac{1}{2} \tag{148}$$

which does not ameliorate our difficulties to any significant extent.

We can, however, separate the dependence of the unknown on ϵ from its dependence on μ, which is done in the next section.

B. Expansion in Powers of ϵ

$$A\{\mu\} = \frac{-1}{2\pi i} \int_0^b (\Delta\{\beta\} - \Delta\{-\beta\}) \; I_{-\mu}\{gr\} \frac{dr}{r} \tag{78}$$

But, by symmetry,

$$\Delta \{\beta\} = - \Delta \{-\beta\} \tag{149}$$

Treatment of the local field near the origin shows that $\frac{\Delta\{\beta\}}{r}$ may be expanded in even powers of r. The radius of convergence of this series extends to the singularity of Δ nearest the origin. This can hardly be nearer than the edge of the strip, and in fact, the analysis related to equations (54) shows that Δ is bounded there. Therefore, the series must converge at least up to $r = b$. Explicitly:

$$\frac{1}{r} \Delta\{\beta\} = \sum_{n=0}^{\infty} u_{2n} \; \eta^{2n} \tag{150}$$

where, for convenience, we are expanding in terms of

$$\eta = \frac{gr}{2} \tag{151}$$

Then, using the series expansion for the Bessel function,

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$$A\{\mu\} = \frac{-2}{i\pi g}\int_0^{\epsilon}\sum_{n=1}^{\infty} u_{2n}\,\eta^{2n}\sum_{j=0}^{\infty}\frac{\eta^{2j-\mu}}{j!\,\Gamma\{j+1-\mu\}}\,d\eta \quad (152)$$

Interchanging the order of integration and summation and integrating

$$A\{\mu\} = \frac{2i}{\pi g}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{u_{2n}\,\epsilon^{2n+2j+1-\mu}}{j!\,\Gamma\{j+1-\mu\}\,(2n+2j+1-\mu)} \quad (153)$$

where we must remember that the u_{2n} are unknown functions of ϵ. Inserting this expression into the infinite system (146), we have:

$$\sum_{n=0}^{\infty} u_{2n}\left\{\sum_{j=0}^{\infty}\left[\sum_{\substack{S=1\\ S\neq m}}^{\infty}\frac{-\Gamma(2S)\,\epsilon^{2j+2n+1}}{2(S-m)\,\Gamma(j+1)\,\Gamma(j+2S+1)\,(2j+2n+2S+1)}\right.\right.$$

$$+\sum_{S=1}^{\infty}\frac{1}{2}\,\frac{\epsilon^{4S+2j+2n+1}}{\Gamma(2S+1)\,\Gamma(j+1)\,\Gamma(2S+j+1)\,(2j+2S+2n+1)\,(S+m)}\,\cdot$$

$$\left.\left.\cdot\left(\psi(2S+1+j)+\psi(2S+1)-2\log\epsilon+\frac{1}{2(S+m)}+\frac{1}{2j+2S+2n+1}\right)\right]\right\}$$

$$+\sum_{S=1}^{\infty}\frac{\pi g\,\epsilon^{2S}\,(-1)^{S}\,K_{2S}\{gr_0\}}{(2S+1)\,(S+m)}$$

$$+\sum_{n=j}^{\infty} u_{2n}\sum_{j=0}^{\infty}\left\{\frac{\epsilon^{2j+2n+1}}{[\Gamma(j+1)]^2[2j+2n+1]\,2m}\mp\left[\frac{1}{2m}-2\log\epsilon\right.\right.$$

$$\left.+\psi(1)+\psi(j+1)+\frac{1}{2j+2n+1}\right]$$

$$-\frac{\epsilon^{2j+2n+1}\,\Gamma(2m)}{\Gamma(j+1)\,\Gamma(j+1+2m)\,[2j+2n+2m+1]}\,\cdot$$

$$\left.\left[\psi(2m)-\psi(2m+j+1)-\frac{1}{2j+2n+2m+1}\right]\right\}=0$$

(154)

an infinite system of equations for the u_{2n}.

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The system (154) is an improvement over (146) insofar as there is only one set of unknowns--the u_{2n}. However, all attempts to solve this system by iteration for small values of ϵ have been thwarted by the apparent logarithmic behaviour of the u_{2n}. We can, nevertheless, obtain reasonable agreement with Bouwkamp [2] by means of reasonable assumptions about the behaviour of the u_{2n} as ϵ goes to zero, as shown in the next section.

C. Approximate Treatment of the u_{2n}

Formally differentiating equations (153), and letting μ go to zero

$$A'\{0\} = \frac{-2i}{\pi g} \sum_{n=0}^{\infty} \frac{u_{2n}\, \epsilon^{2n+2j+1} \left(\log_e \epsilon - \psi(j+1) - \frac{1}{2j+2n+1}\right)}{j!\,\Gamma\{j+1\}\ (2j+2n+1)} \tag{155}$$

But, using equation (145), we have

$$\frac{2}{i} K_o\{gr_o\} = \frac{-2i}{\pi g} \sum_{n=0}^{\infty} \frac{u_{2n}\, \epsilon^{2n+2j+1} \left(\log_e \epsilon - \psi(j+1) - \frac{1}{2j+2n+1}\right)}{j!\,\Gamma\{j+1\}\ (2j+2n+1)} \tag{156}$$

which must be identically satisfied in ϵ. Since the left hand side does not contain ϵ, the right side must also be a constant. Therefore, for small ϵ, u_{2n} is bounded by

$$u_{2n} \leq \frac{g\, d_n}{\epsilon^{2n+1} \log_e \epsilon} \tag{157}$$

where the d_n are independent of μ and ϵ, and some or all may be zero.

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Inserting this expression into the formula for the cross-section, we obtain

$$\text{X-section} \leq \frac{\pi^2}{2b} \left[\frac{4\ g^2}{\pi^2\ g^2\ |\log_e \epsilon\ |^2} \right] \left| \sum_{n=0}^{\infty} \frac{d_n}{2n+1} \right|^2 \qquad (158)$$

for the symmetric-plane-wave case.

But equation (156), for the plane wave case ($K_0(gr_0) \rightarrow \frac{1}{2}$) shows that

$$\sum_{n=0}^{\infty} \frac{dn}{2n+1} = \frac{\pi}{2} \qquad (159)$$

so:

$$\text{X-section} \leq \frac{\pi^2}{2} \left(\log_\epsilon \frac{kr}{2}\right)^{-2} \qquad (160)$$

where we have returned to real k and ignored the constant added to the logarithm.

XVII. The Case of the Wedge

The expression for a wedge corresponding to equation (153) for the strip is easily obtained. For the case of the source located on the axis, it follows from (78) and (149) that

$$A\{\mu\} = \frac{i}{\pi} \int_0^b \Delta\{\beta\} I_\mu\{gr\} \frac{dr}{r} \tag{161}$$

But our treatment of the singularities at an edge shows that $\Delta\{\beta\}$ may be expressed as

$$\Delta\{\beta\} = \sum_{n=1}^{\infty} \left\{ a_n \, \eta^{\frac{n\pi}{2\beta}} + b_n \, \eta^{\frac{n\pi}{2(\pi-\beta)}} \right\} \tag{162}$$

where η is as defined in (151). Inserting this relation into equation (161) and integrating term by term

$$A\{\mu\} = \frac{i}{\pi} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \left\{ \frac{a_n \, \epsilon^{2j-\mu+\frac{n\pi}{2\beta}}}{j!\,\Gamma\{j+1-\mu\}(2j-\mu+\frac{n\pi}{2\beta})} + \frac{b_n \, \epsilon^{2j-\mu+\frac{n\pi}{2(\pi-\beta)}}}{j!\,\Gamma\{j+1-\mu\}\,(2j-\mu+\frac{n\pi}{2(\pi-\beta)})} \right\} \tag{163}$$

Differentiating this expression, and setting $\mu = 0$, it follows from (145) that

$$a_n \leq \frac{x_n}{\epsilon^{\frac{n\pi}{2\beta}} \log_e \epsilon}, \text{ and } b_n \leq \frac{y_n}{\epsilon^{\frac{n\pi}{2(\pi-\beta)}} \log_e \epsilon} \tag{164}$$

where the x_n and y_n are unknown real numbers.

Therefore

$$A\{0\} \leq 2i \frac{K_o\{gr_o\}}{\log_e \epsilon} \tag{165}$$

where we have used equation (145) as before.

It will be noticed that this expression is the same as that for the strip, so to this degree of approximation, THE RADIATED ENERGY IS THE SAME FOR ANY β. The cross-section will be different, but only because the intercepted energy is different.

A more detailed analysis suggests that $A\{0\}$ can be written

$$A\{0\} = 2i \frac{K_o\{gr_o\}}{\log_e \epsilon + X} \tag{166}$$

where X depends on β and possibly on ϵ. Comparing this with the results of Bouwkamp [2], we see that

$$X = -\log_e 2 \quad \text{(for the strip)}.$$

Unfortunately, we are not able to obtain this number even by approximation. It appears to be necessary to solve the infinite systems (119) or (154) in order to improve upon the solution (160). We have attempted to solve (154) by iteration, but the presence of the logarithm in the denominator casts considerable doubt on the validity of the procedure. We have also tried to extract more information from the definition (78), using equation (144), without success.

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