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STUDENTS' ALGEBRAIC UNDERSTANDING:
A STUDY OF MIDDLE GRADES STUDENTS' ABILITY TO
SYMBOLICALLY GENERALIZE FUNCTIONS

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ANGELA S. KREBS

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STUDENTS' ALGEBRAIC UNDERSTANDING: A STUDY OF MIDDLE GRADES STUDENTS' ABILITY TO SYMBOLICALLY GENERALIZE FUNCTIONS

By

Angela S. Krebs

A DISSERTATION

Submitted to
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for the degree of

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ABSTRACT

STUDENTS' ALGEBRAIC UNDERSTANDING: A STUDY OF MIDDLE GRADES STUDENTS' ABILITY TO SYMBOLICALLY GENERALIZE FUNCTIONS

By

Angela S. Krebs

The publication of the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards* in 1989 was pivotal in mathematics reform. The National Science Foundation funded several curriculum projects to address the vision described in the *Standards*. After these materials were developed and implemented in classrooms, questions arose surrounding students' learning and understanding. This study investigates students' learning in a reform curriculum. Specifically, "What do eighth grade students know about writing symbolic generalizations from patterns which can be represented with functions, after three years in the *Connected Mathematics Project* curriculum?"

The content, the curriculum, the data, and the site chosen define the study. Initially, the study surrounded students' algebraic understanding, but I focused it to investigate students' ability to symbolically generalize functions. Although this selection is a particular slice of algebra it represents a significant piece of the discipline.

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I selected the *Connected Mathematics Project* (*CMP*) as the curriculum. I supported the authors' philosophy that the teaching and learning of algebra is an ongoing activity woven through the entire curriculum, rather than being parceled into a single grade level.

The data surrounded the solutions of four performance tasks, completed by five pairs of students. These tasks were posed for students to investigate linear, quadratic, and exponential situations. I collected and analyzed students' written responses, video recordings of the pairs' work, and follow-up interviews.

The fourth choice determined the site. I invited Heartland Middle School, a pilot site of the *CMP* to participate in this study. I approached a successful teacher, Evelyn Howard, who allowed her students to participate. Together, we selected ten students who were typical students in her classroom to participate in this study.

In conclusion, I present two major findings of this study surrounding students' understanding of algebra. First, students who had three years in the *Connected Mathematics Project* curriculum demonstrated deep understanding of a significant piece of algebra. And second, teachers can learn much more about students' understanding in algebra by drawing on multiple sources of evidence, and not relying solely on students' written work.

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I wish to first thank Dr. Sandra K. Wilcox. There is so much for which to be grateful to you that I can not begin to list everything here. Over the years, you freely gave your time, intellectual perspectives and emotional support. Thank you to Dr. Glenda T. Lappan for pushing me to think differently about mathematics education. It was through your influence that I began this process. I also thank Dr. Dan Chazan and Dr. John P. Smith III for the guidance you offered and Yvonne Grant for your assistance.

I would like to thank my family and friends. To my husband, Todd, I appreciate your support and patience during this long process. To my son Jacob, although you may have added a year to the completion, I would not have wished it any other way. To my parents, thank you for the inspiration over the years.

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CHAPTER 1

Introduction—Some Problems with Algebra

Introduction

The discipline of mathematics is in a state of rapid change and growth, but the commonly held view of school mathematics has not evolved in parallel or in conjunction with these advances. One aspect of mathematics is the discipline of algebra. Many people still consider algebra as the study of letters. I recall the reaction from one student after we found a solution of $y = \frac{3}{4}$ for an equation. She threw up her hands in disgust, and said that she thought y was always two. To her, learning algebra meant simply learning a correspondence between the letters of the alphabet and the numbers. To others learning algebra involves manipulating letters.

The NCTM *Curriculum and Evaluation Standards* (1989) and other reforms recommend a move away from the traditional algebra curriculum and teaching practice towards a discipline and instruction which is more inviting and meaningful to students. Traditional algebra is often seen as a gatekeeper. Without successfully completing an algebra course, many students are denied access to certain careers. Standard school practices are rooted in traditions several centuries old and cannot prepare students for their math

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their mathematical needs of the 21st century (Steen, 1990). Supporters of the reform believe that algebra can be a discipline where all students have access.

One response by the government was to fund several curriculum projects. With money being given to these curricula, the question becomes what can a curriculum based on these reforms contribute to a students understanding of algebra.

In this chapter I begin by describing a traditional algebra classroom. It is standard in both content and teaching. This raises concerns regarding access to algebra that has ignited a movement to encompass algebra for all students. Next, I consider aspects of a curriculum that develop algebra supporting the vision presented in the *NCTM Curriculum and Evaluation Standards* (1989). In conclusion, I pose the question surrounding the algebraic understanding of students in a curriculum such as *CMP*.

Traditional School Algebra

School algebra has not changed much in the past fifty years (Thorpe, 1989). The modal algebra classroom looks the same today—in both curriculum and teaching—as it did many years ago. The content in an Algebra I course is fairly standard and can be inferred by examining a typical text. Most algebra books consist of chapters divided into sections and sections into lessons

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into lessons. Each lesson typically spreads over two facing pages and is expected to be covered in one class meeting. The typical topics or chapters included are operations with positive and negative numbers; solving linear equations, linear inequalities, and proportions; age, digit, d = rt, work, and mixture word problems; operations on polynomials and powers; factoring of trinomials, monomial facts, and special factors; simplification and operations with rational expressions; graphs and properties of graphs; linear systems of two equations with two variables; simplification and operation with square roots; and solving quadratics equations by factoring and completing the square (Usiskin, 1987). A lesson typically offers a handful of examples, followed by a number of exercises for students to practice the new skill or procedure demonstrated in the examples. There is much repetition in this approach. Students do many exercises which follow a similar format, although the exercises may get progressively more complex as students proceed down the page.

The curriculum also has commonalties in differing sites. This traditional school algebra follows a "layer-cake" approach (Kaput, 1993; and Davis, 1993). The curriculum typically consists of some form of Algebra I in the ninth grade or earlier,[1] followed by Algebra II. Unfortunately, with this traditional approach most of a student's algebra learning is confined to at most two years of school. By keeping students' learning in separate distinct

[1] Algebra I is being taken by more and more eighth grade students to allow them the time to take the advanced mathematics courses as juniors and seniors.

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layers, or courses, the development of ideas across multiple areas of mathematics is hampered (Kaput, 1993; and Steen, 1990).

Teachers in these traditional classrooms often follow a format of instruction that parallels the typical text. The teacher might begin the class with a review of questions from the previous day's homework. After these concerns have been addressed, the class moves on to the next lesson. The teacher demonstrates a procedure by completing several examples for the class to see. This might be done interactively with input from the whole class, but the teacher typically has a predetermined agenda to show students how to progress through the problem using her method. After the teacher completes her select examples she might assign some seatwork, a kind of supervised practice, to monitor whether her students understand the procedure. Once the students appear to be on the right track, the teacher allows the rest of the class time for students to complete their assignment of exercises from the text. These assignments typically consist of more exercises similar to the examples demonstrated. A word of caution might be offered to the students that the later problems might contain some tricky or challenging aspects.

The NCTM Board of Directors (1994) summarize three major flaws in the traditional algebra courses. First they note that the focus on pencil and paper manipulations is often divorced from any meaningful context. In fact, these skills that are developed are not necessarily what students who are employment-bound or college-bound need in a technology world. The directors add that

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add that the traditional curriculum does not encourage an informal understanding of algebraic ideas in grades K-8 that could prepare students for future investigations. Finally, they acknowledge that the concepts and methods of algebra are isolated from other strands of school mathematics: statistics, geometry, and discrete mathematics. Students do not have the opportunity to integrate their learning of mathematics.

The traditional curriculum and teaching of algebra do not foster success for most students; neither is it clear how some of the content is meaningful or worthwhile to students. Instead of drawing in students, traditional algebra is viewed as a gatekeeper that effectively excludes certain students from future studies and/or careers. Traditionally minority groups and women are the ones who are most often filtered out. There is a commitment by many policy makers, professional organizations, and individual educators to help all students develop algebraic competence.

Gatekeeping and Equity

American culture places a high value on algebra. The course "algebra" listed on a student's transcript sends a message about this student's mathematical experiences and perceived competence to future employers, admission counselors, and others. In 1990 the United States Department of Labor reported that the number of mathematics courses taken during high school is the strongest predictor of earning nine years after graduation. In schools, algebra is typically a prerequisite for geometry. Nearly all students who plan

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who plan on attending college take geometry in high school regardless of their race or ethnicity (Pelavin and Kane, 1988). When students do not take algebra, they greatly limit the number of future choices and close some doors. Students who are successful with algebra leave high school with many of these doors open.

Moses (1993) called algebra the new gatekeeper of the twenty-first century. Students who have passed algebra are afforded many opportunities that would not otherwise be available. Davis (1993) noted that the United States is becoming more of a bi-modal society relative to income. Although education is not solely responsible for this distinction, it plays an important role. According to Davis, one aspect of this separation is the segregation of "those who know algebra versus those who do not." Students who find success in algebra have more opportunities available than those who do not. Algebra is often seen as a filter that only allows a select number of students to pass through. Those who are blocked are often prevented from achieving specific goals.

In the NCTM's Board of Directors (1994) statement, the authors summarized: "First year algebra in its present form is not algebra for everyone. In fact, it is not the algebra for most high school graduates today." If algebra is used as a gatekeeper, then we need to consider who is being excluded from the advantages algebra affords. Unfortunately it is most often minority students and women who are likely filtered out (Moses, 1993). NAEP data showed that less than half of the students from impoverished urban scho

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urban schools takes more than one year of algebra. As many as one in five do not take any algebra at all (Silver 1997). Moses (1993) stated that we need to develop a consensus around the right to learn algebra. He noted further in Jetter (1993) that access to algebra is an important issue for a new civil-rights movement for minorities. Moses argued that students, who do not take algebra, left high school disadvantaged, and this is a situation that can no longer be tolerated. Algebra for all students should be a top priority in education.

Algebra for All

It can no longer be tolerated that some students are restricted in their future by lack of success in an algebra course. Algebraic reasoning is a very powerful tool for students to develop not only for the leverage it brings them in society, but because it is one of the most powerful intellectual tools civilization has developed. To make the tools of algebraic understanding available to students, "algebra for all" is becoming a matter of educational policy in many states. (Olson, 1994).

NCTM commissioned several documents to represent a new vision of teaching algebra to all students. In one of their landmark publications, the authors of the NCTM *Curriculum and Evaluation Standards* (1989) envision a curriculum where algebra is expected for all students. The *Standards* establish as a goal that students become mathematically powerful problem solvers: pa

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solvers; part of this power lies in the accessibility of algebraic understanding to help students reason. In 1994, NCTM created the *Algebra Working Group*. Their charge was to clarify the vision of "algebra for all" through the K-12 setting advocated in earlier pieces of writing. The group did this by illustrating and elaborating this goal with examples, practical ideas, and promising practices to help educators raise questions about changing algebra instruction. They published "A Framework for Constructing a Vision of Algebra" in 1997 that summarizes the group's work. In the document, the authors put forth differing perspectives on algebra, argue for algebra as more than a course, and illustrate their view of algebra in a K-12 setting.

In another NCTM publication, the authors of *Algebra for Everyone* (1990) argue for the need to teach algebra to all students beginning in the elementary curriculum, continuing through the middle grades, and expanding in high school. These writers recognize that mandating the traditional algebra described earlier in this chapter in ninth or even eighth grade is unlikely to be successful at achieving mathematical equal opportunity (Silver, 1997). In fact, many argue that requirements of this sort might actually have an opposite effect and push more students away from algebra. Worse yet, forcing students into an inappropriate traditional algebra course might reinforce widely held destructive notions that algebra is only for a select few students (Silver, 1997). Mandates to require traditional algebra might have the ill-effect of accelerating students out of mathematics, rather than ope

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than opening gates (Prevost, 1985). The National Research Council (1989) stresses in *Everybody Counts* that since algebra is required for opportunities after school, all students should study a "meaningful algebra." Rather than serve as a filter, *Everybody Counts* argues that algebra should be a pump in the American education pipeline and help students pass through the gate.

It is not only access to algebra that reformers suggest needs revision, but the approaches towards algebra that is offered to students needs a very careful reexamination.

A *Standards*-Based Algebra

Most advocates of "algebra for all" do not assume the traditional symbol manipulation algebra taught in the standard ways (Fouche, 1997; Kaput, 1993; Steen, 1992; Silver, 1997; and Chazan, 1994). When educators accept the challenge of algebra for all, they also support a change in the focus of both the teaching and content of algebra (Steen, 1992; Wheeler, 1989). Chambers (1994) nicely summarizes this: "Algebra for all is the right goal at the right time. We just need to get the right algebra." (page 85).

Educators are trying to find this "right algebra." One idea that cuts across many interpretations is to consider a "strands" approach to the learning of algebra (Kaput, 1993; and Steen, 1990).

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> another to create a rich fabric, but one that has a direction, a natural flow, from the wide watershed concrete experience to generalizations and abstractions, from informal and language-based representations to more formal representations. Filters are usually built using layers, while strands provide a natural flow, gradually drawing into mathematics even more diverse experiences. (Kaput, 1993, page 34).

A strands approach assumes that students continually develop algebraic ideas and algebraic reasoning across different grades and multiple courses. The content can be developed in much more depth by continuing to build on ideas introduced earlier. Students can also benefit from the increased relevance of applications as algebra is used in more meaningful integrated contexts.

Davis (1993) adds that less risk for students might be an additional benefit to this K-12 approach. Rather than having only one or two opportunities to learn algebra, the ideas and reasoning should span over several years. Instead of filtering students out of algebra, strands allow for multiple opportunities for students to have access, drawing in more students. This approach affords diversity in student learning over the entire K-12 curriculum where each student weaves a unique tapestry.

This longitudinal K-12 approach to algebra does not simply allow more opportunity to cover the traditional curriculum. It requires careful consideration of what really makes meaningful content to be taught across the K-12 span. Every topic placed in the curriculum should have significant value to students' learning of mathematics (Thorpe, 1989). The American mathematics curriculum is often criticized as being an "inch deep and a mile wide." Cri

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wide." Critics imply that so many topics must be covered in the curriculum that each only receives a cursory mention rather than a deep exploration. The entire algebra and preparatory algebra curriculum needs to be reevaluated for meaningful content starting in kindergarten and continuing through twelfth grade. One problem with the current curriculum is that it is overcrowded (Usiskin, 1995). Fewer topics should be covered more in depth (Barbeau, 1991; and Steen, 1993). In the following section I discuss how a curriculum based on the NCTM *Curriculum and Evaluation Standards* could respond to the charge of integrating algebra for all students.

The NCTM *Curriculum and Evaluation Standards* are not intended to be curriculum materials implemented directly in a classroom. Rather, the document represents a vision of teaching meaningful mathematics to all students. An algebra curriculum in response to the NCTM *Standards* would look quite different than the traditional Algebra I or eighth grade algebra course. It would vary in *who* is taught, *what* is taught, *where* it is taught, *when* it is taught, and *how* it is taught.

First, consider *who* is taught. Algebra for all students is a primary goal. A curriculum supporting the *Standards* suggests heterogeneously grouped classes for all students. It would not promote removing the top performers or excluding those having difficulties. Rather, with the inclusion of all students, they could suggest extensions for students who are prepared and offer additional support to others as needed.

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The next concern: *what* is included in the curriculum. The standard fare of numerous symbol manipulations without a context would be minimized. The focus would shift towards understanding algebraic ideas and multiple representations.

The *when* and *where* algebra is taught could be taken together. Algebra is not a specific course or a single chapter in a text. It represents a way of thinking and reasoning. The *Standards* would support a vision of algebra integrated throughout the entire K-12 curriculum.

Finally *how* algebra is taught in a *Standards*-based curriculum would also change. The teacher is no longer the sole deliverer of knowledge. Her role is to pose challenging and engaging problems for the students to work and investigate.

Summary

There certainly seems to be a need to reconsider the content and methods used to teach traditional algebra. There is little evidence to support a claim that all students would develop solid algebraic reasoning by following the traditional approach to both content and teaching. Neither is there yet evidence to suggest that all students engaged in a curriculum based on the NCTM *Curriculum and Evaluation Standards* would be successful. If these reform ideas are to be accepted, then research must address this issue.

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This study begins to do this. The initial question that frames the study is: *What do students learn about algebra in a Standards-based curriculum?*

In Chapter 1, I discuss some of the issues surrounding the discipline of algebra and its implementation as a curriculum. Mathematics is in a state of rapid change and as a result what traditionally constitutes algebraic understanding is coming under fire. Educators are re-thinking wheat is in the curriculum and offer suggestions to afford access to all students.

I address the question of what is algebra in Chapter 2. Since there is not a consensus by educators, researchers, or mathematicians, I do not attempt to resolve the issue. Rather, I offer four organizing themes—functions and relations, modeling, structure, and language and representations—presented by the *Algebra Working Group* to cluster some of the different perspectives.

The methodology of this study is rendered in Chapter 3. I describe the reasons I made some of the choices surrounding the content—patterns which represent functions and generalizing with symbols from patterns of data, the curriculum—the *Connected Mathematics Project*, the site—Heartland Middle School, and the data—students' investigations on performance tasks, recordings while they worked, and interviews after completing the tasks.

The data is presented in Chapters 4 and 5. In the first of the two chapters I carefully consider what students have done with each task. I describe their solutions by considering the three sources of data, written responses, recordings while they worked, and interviews after they completed each ta

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each task. In the next chapter I look across students and tasks to recognize common strategies and interesting aspects in their investigations.

In the final chapter, Chapter 6, I suggest some of the implications and limitations of this study. I summarize that the students who participated have demonstrated a very solid understanding of a very important piece of algebra.

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CHAPTER 2

What is Algebra?

"A mathematician like a painter or a poet, is a maker of patterns."
--Godfrey Harold Hardy

Introduction

In the previous chapter I raise the question about what it means for students to understand algebra. Before students' understanding of algebra is considered, I need to further explore the question "What is algebra?" "Solving equations with variables" is a typical response to the question. Even among educators, researchers, and mathematicians there is not a consensus to this question. In this chapter I describe some of the differing perspectives in the literature surrounding algebra and briefly discussing some of the implications for classrooms. I use as a framework the organizing themes presented by the *Algebra Working Group*—algebra as the study of (a) functions and relations, (b) modeling, (c) structure, and (d) language and representations.

After this overview of the discipline I narrow my discussion of algebra to describe the aspect which is the focus of this study—symbolically generalizing from patterns of data. In chapter three I describe the tasks used in the st

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in the study and how they related to this area of algebra. I continue to discuss the implications for school mathematics from this perspective of algebra. Finally, I describe a sorting scheme to classify the patterns that students in this study encountered.

Some Perspectives on Algebra

Several researchers and groups offer varying perspectives of algebra that emphasize different aspects. Kaput (1995) explains that even though many use the term "algebra" there is not an encompassing discipline, no monolith which describes it. He provides a slightly different review on algebraic thinking. He categorizes five aspects of algebra:

- as generalizing and formalizing;
- as manipulations of formal objects;
- as the study of structure;
- as the study of functions; and
- as the study of languages.

Kaput is careful not to classify these as five disjoint categories, but rather as "loosely spun and richly interwoven." They represent guidelines to start thinking of the different perspectives of algebra.

Kaput's aspects are very similar, but not a direct match, to the themes offered by the *Algebra Working Group* (Burrill, 1995; and Phillips, 1995). The

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Algebra Working Group (1997) organizes the various perspectives around four key concepts:

- functions and relations—where functions underlie all big ideas in algebra,
- modeling—where finding ways to represent situations with mathematical relations or models is key to algebra,
- structure—where algebra is conceptualized as generalized arithmetic, and
- language and representation—where communicating ideas through the syntax of the representations is the focus of algebra.

The dominance of each theme varies with researcher. Some support the function approach to algebra (Kieran, Boileau, & Garancon, 1996, Heid, 1996; Fey and Good, 1985; Schwartz and Yerulshalmy, 1991; and Chazan, 1993), while others seem to suggest a structure approach (Kieran, 1989). Just as the emphasis changes by researcher, the focus has shifted with time. In the 60's, the "new math" movement was based primarily on structure, while the "back to basics" movement that followed relied on thinking of algebra as a language. The more current movement with the integration of graphing calculators has shifted the focus to functions and relations, and models (*Algebra Working Group*, 1997). Although the trend has shifted through time, researchers continue to advocate particular themes.

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Although there is much overlap between classification themes suggested by Kaput (1995) and the *Algebra Working Group* (1997), there are also differences. Their organizations overlap with the function and relation, structure, and language themes. While the *Algebra Working Group* seems to sort out modeling, Kaput does not. Instead he teases out generalizing and formalizing, and the manipulation of formal objects as two separate aspects.

In the following sections I elaborate on the *Algebra Working Group's* organizing themes and consider curricular issues surrounding each.

Functions and Relations

Both the *Algebra Working Group* (1997) and Kaput (1995) consider the basic study of functions as one of the primary perspectives on algebra. Researchers whose work falls into the function and relations theme view the function as the central object of study (Chazan, 1993; Fey and Good 1985; Yerulshalmy and Schwartz, 1991; Thorpe, 1989; and Confrey, 1994). Fey (1989) defines functions as relations where output variables depend on input variables. Some would emphasize the rate of change between the variables in the functions. An example of this dependence is how at a given rate the time (input) it takes for a trip determines the distance traveled (output).

The NCTM *Curriculum and Evaluation Standards* (1989) advocate an approach to algebra that focuses on functions and function-related ideas. With the increasing access to computers and the technology of calculators, the co

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the concentration on functions in the algebra curriculum has become a more reasonable part of study for students. Functions can be introduced to students using different representations such as tables and graphs (Confrey, 1994). Schwartz and Yerulshalmy (1992) suggest one such approach where the concept of a function is introduced much earlier than in the traditional secondary school curriculum. Students explore specially designed software that relates both the symbolic and graphical representation of functions. Both representations are important to help students understand the concepts of functions and variables (Schwartz and Yerulshalmy, ????). Kaput (1989) reminds that students need actually experiences with three different representations—table, graphs, and symbols.

Chazan (1993) also suggests a curriculum where the function is the core object of study. He clarifies his view when he describes an equation as a comparison of two functions. For example, the equation $3x - 2 = x + 5$ is identified as a question about two functions, f(x) and g(x), where f(x) = $3x - 2$ and g(x) = $x + 5$. The equation asks: when is f(x) equal to g(x)? or, for what values of x will this produce the same output in both functions? (Chazan, in press)

Chazan asserts several advantages to using this functional approach over the traditional algebra approach. Students are offered an alternative from traditional symbolic manipulation to solve equations. Rather than being limited to a single method in their solutions, students could potentially have three

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three reasonable strategies to solve equations. They might apply operations to create equivalent equations. Using the equation above:

$3x - 2 = x + 5$	Start with f(x) = g(x),
$2x - 2 = 5$	Subtract x from both f(x) and g(x),
$2x = 7$	Add 2 to both of the new functions,
$x = \frac{7}{2}$	Finally, divide both by 2.

Or students could try a 'guess and test' strategy to solve the equation and find that $x = \frac{7}{2}$ is a solution. Or they might consider a graphical representation to find solutions. In this graphical approach, students could plot each function, f(x) and g(x), on a coordinate graph and search for a point of intersection. See Figure 1 where both lines meet at the point where $x = 3\frac{1}{2}$ and is a solution to the equation.

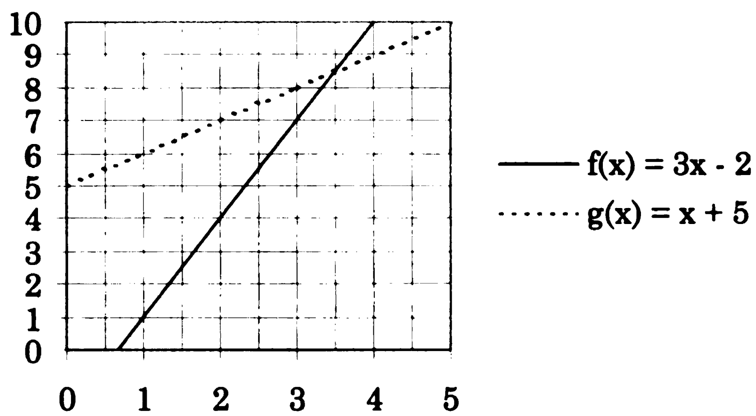

Figure 1: Graphical Representation to Find Solutions

Each of these strategies could be considered an approach following this functional view.

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Modeling

There are common ideas between the functions and relations theme and the modeling theme. Kaput (1995) notes that most models have functions as their core. Frudenthal (1983) would argue that modeling is the primary reason to study algebra. In this version of algebra, students start with some situation and their goal is to mathematize it—find a mathematical relation that models the phenomena. These mathematical models are represented with equations, graphs, or tables (Kaput, 1995). The modeling perspective is grounded more in data. It is finding mathematical relations that adequately fit the data and that can be used to make reasonable predictions. An example illustrates finding a model that will yield the weight of an object if you know how much a spring has been stretched. Students could collect the data represented in Table 1 below.

Table 1: Spring Data

Length (cm)	Weight (g)
6.5	0
7.3	100
7.5	200
8.5	500
12.2	1000

Then use it to find the weight of a rock that stretches the spring a length of 10 cm (see Figure 2).

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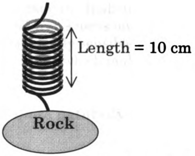

Figure 2: Spring Illustration

Part of their solution could involve making a graph, fitting a line on the graph, and making a prediction. The graph in Figure 3 can be used to estimate the weight of this rock to be about 700 grams. (I need to find a way to fit the line on the graph).

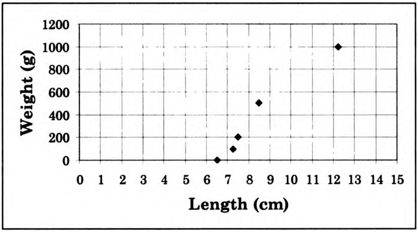

Figure 3: Spring Graph

Structure

Another way to consider algebra is to characterize the structure of the system of algebra (*Algebra Working Group*, 1997; and Usiskin, 1988). An example of this is algebra considered as generalized arithmetic (Thompson and Thompson, 1993; Kieran, 1991; Peck and Jencks, 1988; and Sfard, 1995). Peck and Jencks (1988) describe this as follows.

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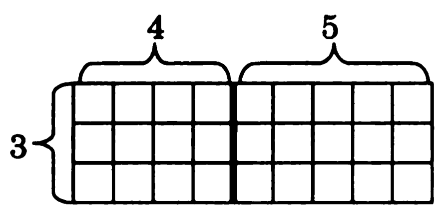
Algebra is a generalization of arithmetic. It should arise logically and naturally as a consequence of children's decisions about how arithmetic works. If arithmetic becomes completely sensible to children and becomes a tool for their thinking, the decisions which make algebra sensible flow naturally from it. (page 85).

They infer that algebra can arise quite naturally from a solid understanding of arithmetic.

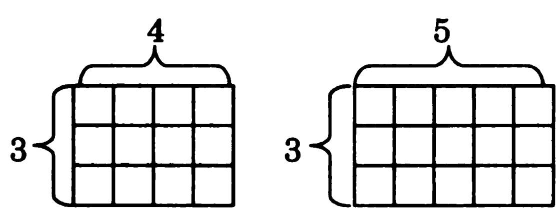
Educators whose work falls into this class support the view that algebra is learned through studying and generalizing the properties of the real number system. Algebraic assertions can be made about the numbers based on what was generalized from arithmetic. This can be illustrated with the distributive property. Before studying the general case, consider this rectangular array representation example of 3 × (4 + 5) in Figure 4.



$3 \times (4 + 5) = 3 \times (9) = 27$

Figure 4: Distributive Property of 3 × (4 + 5)

These 27 squares can be represented in another, equivalent way. This can also be shown with (3 × 4) + (3 × 5) in Figure 5.



$(3 \times 4) + (3 \times 5) = (12) + (15) = 27$

Figure 5: Distributive Property of (3 × 4) + (3 × 5)

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Studying the structure of the distributive property involves moving from the specific case of $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$ to the general case of $a \times (b + c) = (a \times b) + (a \times c)$.

Kieran (1985) asserts that students are able to learn algebra when they make the transition from the arithmetic approach, finding the values of the unknowns using simple operations, to the algebraic approach, making use of the structure of the system when the equation is the key object of study. Employing the algebraic approach means more than surface and structural operations. It is about being able to compare expressions without evaluating them directly. This can be illustrated with the equation $\mathbf{x} - 10 = 27$. Students using simple operators might think, "What minus ten is twenty-seven? Thirty minus ten is twenty, forty minus ten is thirty, thirty-seven minus ten is twenty-seven. So, the answer is thirty-seven." Whereas, students making use of the structure might think, "Since I subtract ten from X, I know that I need ten more than twenty-seven, or thirty-seven."

The first approach is case dependent. If students tried to solve $x - 17 = 41$ they would likely start this problem over, "What minus seventeen is forty-one? Fifty minus seventeen is thirty-three, sixty minus seventeen is forty-three, fifty-eight minus seventeen is forty-one. So, the answer is fifty-eight." Students who used the structure could apply the same strategy in this second equation, "Since I subtract seventeen from X, I know that I need seventeen more than forty-one, or fifty-eight."

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Kieran (1989) claims that this understanding of the structure is where many students confront difficulty learning algebra. She illustrates this with a case of students in a typical classroom. The equality between the left- and right-hand sides of an equation is a foundation in traditional algebra instruction. Students are often taught to solve equations by doing the same thing to both sides of the equal sign. When students do not see the equation as this balance, but instead see the right-hand side as the answer, they encounter problems. These students do not fully see the structure of the equations and often have difficulty solving equations in this manner. Filloy and Rojano (1989) write that these students who do not make the transition have an arithmetic notion of equality. Kieran suggests that confronting these issues that deal with structure earlier in their algebra classrooms will help students gain a better understanding of algebra.

Language and Representation

In the final theme presented by the *Algebra Working Group* (1997) the important characteristic of algebra is the language that is used to communicate. Algebra is sometimes considered the "language of arithmetic" (Kline, 1972; and Usiskin, 1988). Educators whose work falls into this theme write that algebra can be thought of as a language with syntax to be learned that communicates mathematical ideas (Booth, 1989; and Bell and Malone, 1993).

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In algebra there are many types of representations that are used in the language to communicate ideas. Many people's first thoughts of algebra relate to only the symbols present in a typical ninth grade algebra course. This is very limiting. Considering only symbols in algebra would be like restricting one's writing to include only nouns. In both instances a much more complete picture can be obtained by using multiple representations, or parts of speech. Just as a more informative story can be told when including additional parts of speech (e.g., nouns, verbs, adverbs, adjectives, and prepositions) a more complete algebraic solution might include additional representations (e.g., symbols, tables, words, graphs, and diagrams). One aspect of learning algebra means using one or more of these representations to communicate algebraic thoughts.

Learning this language is more complex than students think (Bell and Malone, 1993). Students typically possess a very simplistic view of the language of algebra (Booth, 1989). They may not recognize the intricate connections between different representations. Stacey and MacGregor, (1997) caution that under this approach there is a new grammatical structure for students to learn; the rules in the language of algebra are not the same as the rules in ordinary language. Some words that have multiple meanings in the English language have very precise mathematical definitions, such as the words "product", "and", and "or". Some operations that seem similar are not.

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For example, $2(x + y) = 2x + 2y$ is true, but many algebra teachers have been frustrated when students incorrectly conclude that $(x + y)^2 = x^2 + y^2$.

Arcavi (1994) warns that learning algebra is more than just learning what the symbols mean. Rather, obtaining "symbol sense" is at the heart of what it means to really know algebra. To him, "symbol sense" means having a "feel" for symbols, or an "accurate appreciation, understanding, or instinct regarding symbols." (pg. 28). This involves knowing when symbols should be used, when to use a different representation with symbols, when to abandon symbols, how to manipulate and read symbolic expressions, and how different symbols play different roles in different contexts. Some researchers acknowledge that symbols should be introduced earlier in the curriculum so that students can more fully appreciate their power and learn the language (Hershkowitz and Arcavi, 1990).

Patterns and the Study of Algebra

Studying patterns and finding generalizations cuts across all of the themes discussed in the previous section. Students might emphasize functions when they study the rate of change in a table to find a function to represent the pattern. They could focus on the data collected in the search for a model to predict additional values. They might study the data and genera

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generalize from the structure of the pattern. Or, they could use several different representations to help generalize.

Patterns comprise a vital component of the discipline of mathematics. Interesting patterns arise in all areas of mathematics. Mathematicians can use patterns of a sequence of growing shapes in geometry to describe characteristics of the n^{th} figure, patterns of simpler cases in probability to explain a more complex probability, or patterns of numeric data in a table in algebra to generalize the n^{th} term in a sequence, or even patterns from a continuous graph on a coordinate grid. The primary emphasis comes from the function, relation, and modeling approach when students write generalizations of functions. They often use the structure of algebra to help write these generalizations, and of course, they need knowledge of the language to do all of this.

Some researchers describe the discipline of mathematics as the "science of patterns" (Hoffman, 1989; Steen 1988; American Association for the Advancement of Science, 1989; and Schoenfeld, 1989). In Land and Becher, (1997) Van de Walle states:

> The world is full of order and pattern: in nature, in art, in buildings, and in music. Pattern and order are found in commerce, science, medicine, manufacturing, and sociology. Mathematics discovers this order and uses it in a multitude of fascinating ways... (page 301).

Steen (1988) writes,

> Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of

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> pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to 'explain' and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns. In this way mathematics follows its own logic, beginning with patterns from science and completing the portrait by adding all patterns that derive from the initial ones. (page 616).

In the next sections I further illustrate the influence on mathematics education of this view of patterns in mathematics. First, I consider the implications for school mathematics, and then I describe a scheme to classify the patterns students might study.

Patterns in School Mathematics

In school mathematics it is essential for students' experiences to include the exploration of patterns. The ability to recognize and describe patterns lies at the foundation of mathematical science (Smist and Barkman, 1996). One way this is useful is to illustrate an idea; mathematical conjectures often become more clear to observers by examining patterns (Toumasis, 1994). Some properties of positive integers can be more apparent through the exploration of patterns. There is much that can be inferred by studying the way the numbers increase. An example is illustrated with the pattern of square numbers in Figure 6.

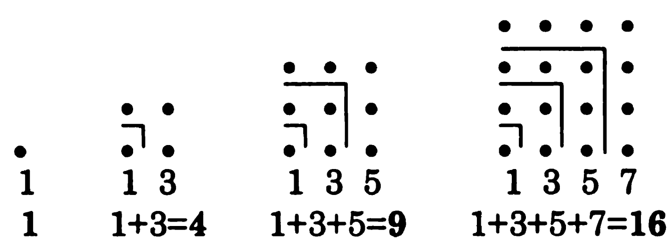

Figure 6: Square Number Pattern

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By studying the dot representations of square numbers, it becomes clearer that all square numbers can be written as the sum of consecutive odd numbers, $4 = 1 + 3$, $9 = 1 + 3 + 5$, $16 = 1 + 3 + 5 + 7$, etc.

In addition to being a general foundation in mathematics, using patterns to investigate relationships can specifically help students develop their algebraic thinking (NCTM, 1989; 1994; Silver, 1997; Ferrini-Mundy, et al., 1997; Phillips, et al., 1991; and Phillips, 1993). The authors of NCTM's *Curriculum and Evaluation Standards* (1989) and Kieran (1994) both support the view that the inductive study of patterns should represent the ground work for students' initial learning of algebra. Inductive thinking involves students studying a small number of specific terms from a pattern and then making more general statements based on their explorations. These preliminary experiences with generalizations arise through investigating patterns (Curcio and Schwartz, 1997). Students might eventually use symbolic notation to write their generalizations. Although those who do consider verbal generalizations but do not formalize their expressions with symbols are still involved in algebraic thinking (Lins, 1990).

There are other essential components to algebraic thinking based on investigating patterns besides generalizations. These include exploring and formalizing patterns, conjecturing about the patterns identified, verbalizing relationships between and among elements in patterns, extending patterns, and eventually representing the relationships using symbolic notation

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(Silver, 1997; Kenney and Silver, 1997; and Sierpinska, 1992). Some examples of these activities can be illustrated using the dot and sum representations of square numbers in Figure 6 above. Students might verbally describe or conjecture about several patterns. Students could *formalize* the pattern by noticing that the square numbers can be written as the sum of odd numbers. Students might *conjecture* that all square numbers could be written as the sum of consecutive odd numbers. This relationship might be further clarified *between elements* when students recognize that to find the next square number, one can add the next odd number to the previous square numbers. This idea is *extended* in Table 2 below to illustrate finding the first ten square numbers.

Table 2: Table of Square Numbers

N	Sequence of Odd Numbers ($2N - 1$)	Sum of Previous Square Number and Next Odd Number	N^{th} Square Number
1	1	1	1
2	3	1+3=4	4
3	5	4+5=9	9
4	7	9+7=16	16
5	9	16+9=25	25
6	11	25+11=36	36
7	13	36+13=49	49
8	15	49+15=64	64
9	17	64+17=81	81
10	19	81+19=100	100

Finally, students could *symbolize* their generalization by writing the hypothesis that to find the n^{th} square number add the first n odd numbers: $1 + 3 + 5 + \ldots + (2n\text{-}1)$ or simplify this with n^2.

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Patterns that are appropriate for middle grade students to study have regularity and predictability. These two characteristics of patterns make explicit some assumptions about the behavior of the patterns advocated in the reforms (Heaton, 1994). A general purpose of searching for patterns is to use the information gained through the investigation to make predictions about later terms in the sequence. Making predictions is only relevant when a pattern maintains aspects of regularity since it is the regularity that leads to the predictability of the pattern's behavior. Students need some reason to believe that this regularity will be maintained. The only way this occurs with reasonable certainty is in some context. (The issue of context is discussed later in this chapter.) There is not a guarantee that a pattern continues based solely on the table of numbers. Heaton (1994) adds further clarification. "...Identifying a pattern allows you to manipulate one variable and predict what will happen with the other. A relationship between two variables with this kind of regularity and predictability is a function" (p. 149).

Investigating patterns is one foundation to learning functions for students. Functions are an important concept for students to learn in mathematics; yet they often have difficulty understanding the principles of functions such as the notion of independent and dependent variable (Artigue, 1992; Eisenberg, 1992; and Sierpinska, 1992). In the *NCTM Curriculum and Evaluation Standards* (1989) the authors emphasize the significance of studyi

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studying patterns to support the learning of functions through the entire K-12 curriculum. In the section for the middle grades, the authors write:

> The theme of patterns and functions is woven throughout the 5-8 standards. It begins in K-4, is extended and made more central in 5-8, and reaches maturity with a natural extension to symbolic representation and supporting concepts, such as domain and range, in grades 9-12 (page 98).

In the elementary standards these goals are formalized in *Standard 13: Patterns and Relations*, extended for the middle grades in *Standard 8: Pattern and Functions* and *Standard 9: Algebra*, and pushed further at the secondary level in *Standard 5: Algebra* and *Standard 6: Functions*.

Several educators suggest ways to implement this approach to functions through studying patterns at both the elementary and the middle grades levels (Curcio, 1997; Austin and Thompson, 1997; and Herbert and Brown, 1997). Chappell (1997) suggests some pre-symbol experiences for students that relate to algebra at the elementary levels where students verbally describe patterns in a "guess my rule" game. She reminds that it is algebraic thinking and *not* formal algebra that should receive the emphasis at this level. Pegg and Redden (1990) describe a seventh grade course in South Wales, Australia where algebraic ideas are introduced through studying numbers patterns in data without introducing the early use of the manipulation of symbols.

A goal at the middle grades is to describe a generalization. However, the transformation to symbolic notation is not necessary for a student to initially recognize and generalize the pattern (Lee and Wheeler, 1987). Students may

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verbally describe a pattern, but not be able to write a symbolic rule from the numeric pattern in the data (Lee, 1987). This ability to generalize with words represents an important initial part of algebraic reasoning. An example of this can be illustrated in a problem involving phone charges. Suppose a phone company charges a $1.50 connection fee for a phone call and an additional $0.25 for each minute, or a fraction of a minute. Students might represent the pattern in Table 3 as follows.

Table 3: Cost of Phone Call

Number of Minutes	Cost in Dollars
0	1.50
1	1.75
2	2.00
3	2.25

While completing the table they could use words to describe how the cost of the phone call is increasing with time, but not be able to write a generalization using symbolic notation.

Coming up with a symbolic representation to generalize a pattern often proves challenging for students. It is not a trivial transition for students to move from recognizing a pattern to writing an algebraic rule (MacGregor and Stacey, 1993; Pegg and Redden, 1990; and English, 1995). Lee and Wheeler, (1987) report that students can often formulate appropriate generalizations without using algebraic symbolic notation. MacGregor and Stacey (1993) identify four critical steps students must cross in order to move from reco

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recognizing a pattern in a function table to writing an algebraic rule with x as the independent and y as the dependent variables. Students should to be able to:

- articulate the relationship to find numerical values,
- look beyond a recurrence pattern to find one that links the two variables,
- know the syntax of algebra, and
- know what can and can not be said with algebra (page 187).

While MacGregor and Stacey did not necessarily present these steps as a hierarchy, I consider the order presented above. Prior to writing a symbolic rule, students should recognize a pattern; this could be with a verbal description, or by extending the pattern.

In many cases there are a number of different symbolic representations that students can write. In this study, I am interested in how students write symbolic generalizations where the dependent variable is expressed as a function of the independent variable. Although this is not the only way to represent functions, it is useful. To write a representation in this form, students must look beyond a recurrence pattern.

To write symbolic rules students need to know the syntax. If they do not know this language, writing a generalization can be nearly impossible. But, not knowing the syntax is different than the fourth of MacGregor and Stacey's steps. Knowing what can be said with algebra might be considered a more sophisticated step than the other three. To know this students would need a more complete knowledge of the discipline. Students might confuse and b

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and believe that something can not be said, when actually they just do not know the syntax to write a representation.

A Classification Scheme

There are two important, yet different, aspects in a classification of problems involving patterns. Phillips, et al. (1991) along with other research helps to distinguish these (Steen, 1990; 1988; Golos, 1981; Boles, 1990; Heaton, 1994; and *Algebra Working Group*, 1997). The first is the *content* or function type of the specific pattern; the second is the *context* in which the problem is situated.

Content Class.

The first classification is based on the mathematical content, or function type, of the pattern. Four functions suggested for a Standards-based middle grades reform curriculum are linear, exponential, polynomial, and inverse functions (NCTM, 1989). Although this list is not intended to be representative of functions, it represents some of the typical functions covered in the middle grades.

Linear functions are those which have a constant rate of change. They typically are one of the first patterns students learn to recognize. Students might observe these patterns by studying the linear graphs, or by recognizing a constant difference in a table of data. Students might encounter these as eithe

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either continuous or discrete functions. In some instances they might notice that a pattern increases (or decreases) in a table by the same constant number. For example, a problem might ask students to study the number of perimeter dots of the following dot representations of squares with length greater than or equal to two. The first four figures are illustrated in Figure 7 below.

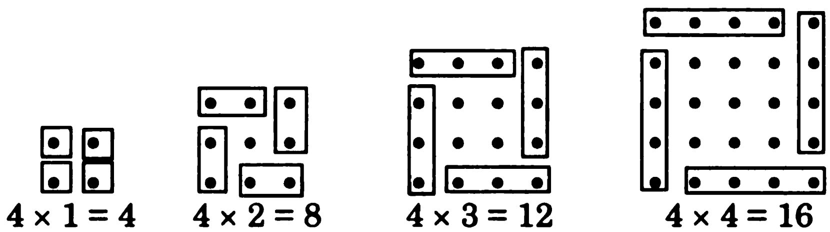

Figure 7: Perimeter Dots on Squares

Students might recognize that this is a linear function by studying the figures, tables, or a graph. In Figure 7 above, the perimeter dots for each figure of the pattern are grouped into four boxes. For each figure in the pattern, each box has one less dot than the length of the side. The number of perimeter dots can then be represented as $4 \times (n-1)$ where n is the length of the side. This form of the rule often tells students that this is a linear function. The linearity of this pattern could also be observed by studying Table 4.

Table 4: Perimeter Dots on Squares

Length of Side	Perimeter Dots	Rate of Change
2	4	
3	8	$8 - 4 = 4$
4	12	$12 - 8 = 4$
5	16	$16 - 12 = 4$

The constant rate of change can be found by subtracting subsequent terms: $8 - 4 = 4$, $12 - 8 = 4$, etc. Since these differences are all 4, this must be a linea

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linear function. Finally, a plot of the data on a graph supplies information about the pattern because the data forms a straight line, it is a linear function (See Figure 8).

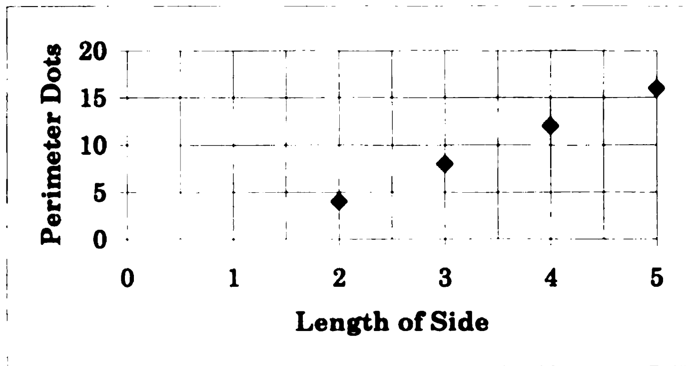

Figure 8: Linear Graph

A second function appropriate to the middle grades is the exponential pattern. Students typically experience exponential functions as either growth or decay patterns. One way to recognize this pattern is to note the multiplicative growth (or decay) factor in a numeric table of data (See Table 5).

Table 5: Exponential Table

x	$y = 4 \cdot 3^{(x-1)}$	Growth Factor
1	4	
2	12	$12 \div 4 = 3$
3	36	$36 \div 12 = 3$
4	108	$108 \div 36 = 3$

This could be presented to students as a problem where students find their allowance after 10 weeks under the following plan. They receive one cent the first week and then on subsequent weeks, they double their previous week's allowance. They would receive one cent the first week, two cents the second weel

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week, four cents the third week, eight cents the fourth week, and so on. The data can be organized in Table 6 below.

Table 6: Allowance Table

Week Number	Allowance	Growth Factor
1	$0.01	
2	0.02	$0.02 \div 0.01 = 2$
3	0.04	$0.04 \div 0.02 = 2$
4	0.08	$0.08 \div 0.04 = 2$
5	0.16	$0.16 \div 0.08 = 2$
6	0.32	$0.32 \div 0.16 = 2$
7	0.64	$0.64 \div 0.32 = 2$
8	1.28	$1.28 \div 0.64 = 2$
9	2.56	$2.56 \div 1.28 = 2$
10	5.12	$5.12 \div 2.56 = 2$

They could determine that this problem was exponential since the next term could be found by multiplying by the growth factor of 2. The pattern can be represented with a rule where A is the allowance for the w^{th} week: $A = 0.01 \times 2^{(w-1)}$. The graph of an exponential also takes a different form as is illustrated in Figure 9 below.

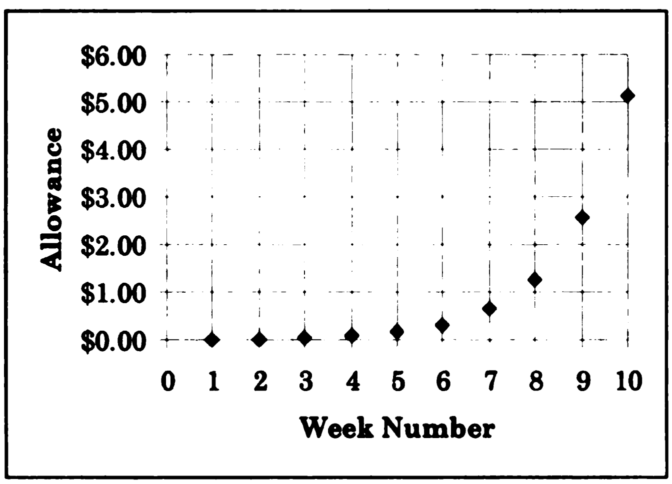

Figure 9: Allowance Graph

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In addition, students could study polynomials. They often first experience polynomial patterns as the less complex power functions: $y = x^2$, $y = x^3$, $y = x^4$, and so on. But, more generally these patterns include polynomial functions that can all be recognized by finding constant differences in numeric tables of data. In Tables 7 and 8 below, the constant differences are found for $y = x^2$ and $y = x^3$ respectively.

Table 7: $y = x^2$ Constant Differences[1]

x	$y = x^2$	1st Difference	2nd Difference
0	0		
1	1	1 – 0 = 1	
2	4	4 – 1 = 3	3 – 1 = 2
3	9	9 – 4 = 5	5 – 3 = 2
4	16	16 – 9 = 7	7 – 5 = 2
5	25	25 – 16 = 9	9 – 7 = 2
6	36	36 – 25 = 11	11 – 9 = 2

Table 8: $y = x^3$ Constant Differences

x	$y = x^3$	1st Difference	2nd Difference	3rd Difference
0	0			
1	1	1 – 0 = 1		
2	8	8 – 1 = 7	7 – 1 = 6	
3	27	27 – 8 = 19	19 – 7 = 12	12 – 6 = 6
4	64	64 – 27 = 37	37 – 19 = 18	18 – 12 = 6
5	125	125 – 64 = 61	61 – 37 = 24	24 – 18 = 6
6	216	216 – 125 = 91	91 – 61 = 30	30 – 24 = 6

In each case the first difference is not constant. In all quadratics the second difference is constant. In all cubics the third difference is constant. This can

[1] Recall that the differences are related to the derivatives.

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be extended to show that the fourth difference is constant in a quartic, but middle school students do not typically explore polynomials to greater degrees. Functions of this general polynomial type are typically more difficult for students to generalize using symbols when they are other than the power functions. In a typical problem, they might be asked to find the total number of squares in the pattern illustrated in Figure 10.

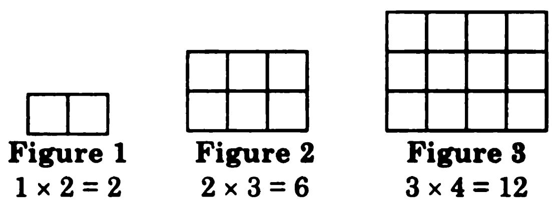

Figure 10: Polynomial Pattern of Figures

The number of squares follows a quadratic pattern. This could be recognized in either the rule or the table. A symbolic rule for the total number of squares can be written as $T = n(n+1)$ where n is the figure number. This could be recognized as a quadratic since it is a quantity of n multiplied by a quantity of n. The pattern can also be extended in Table 9 to find that the second difference is constant, which also signifies that it is a quadratic pattern.

Table 9: y = Constant Differences in Total Squares Table

Figure Number	Total Squares	1st Difference	2nd Difference
1	2		
2	6	6 – 2 = 4	
3	12	12 – 6 = 6	6 – 4 = 2
4	20	20 – 12 = 8	8 – 6 = 2
5	30	30 – 20 = 10	10 – 8 = 2

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A fourth function that students might study in the middle grades is the inverse function. These are often in the form of $\frac{1}{n}$. Students may first encounter this function when they study distance-rate-time problems, $d = rt$, and solve for the rate or time. These functions then would be of the form $r = \frac{d}{t}$ or $t = \frac{d}{r}$. Students could explore how changing the rate to walk 8 miles affects the time walked as represented in a table, rule, or graph. The rule for this case would be $t = \frac{8}{r}$, while the table and graph are represented below as Table 10 and Figure 11.

Table 10: Walking Times Table

Walking Rate (mph)	Time (in hours)
1	8
2	4
4	2
6	1.33
8	1
10	0.8
12	0.67
14	0.57
16	0.5

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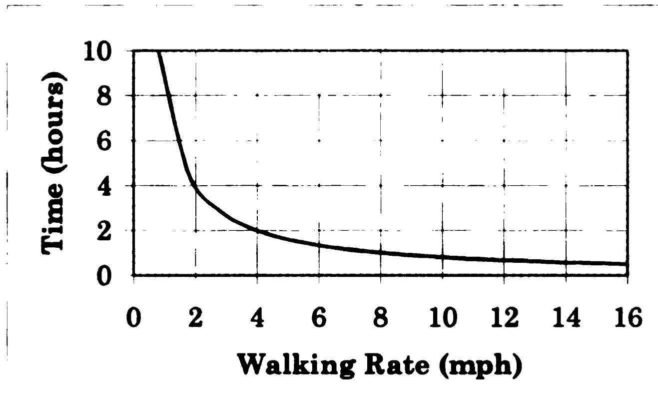

Figure 11: Walking Times Graph

Context Class.

The content is not the only important aspect of a pattern. The *Algebra Working Group* (1997) reminds that any mathematics must be *about something*. Algebra cannot be learned without some kind of context. "Students build concepts and develop ways to think in pursuit of activities that engage them in different contextual settings; such settings help students make sense of the algebra they are studying (*Algebra Working Group*, page 9)." The group identifies five contextual settings: growth and change, size and shape, data and uncertainty, number, and patterns. These are based on the settings Steen (1990) presents: dimension, quantity, shape, uncertainty, and change. Problems that involve studying expanding populations involve *growth and change*. They might include linear or exponential growth. A key idea in these type of problems is the relationship between how the change in one variable affects the other. *Size and shape* problems are geometric in nature. Students could study polygons and investigate which shapes could be used to tile a surface. *Data and uncertainty* can be considered as data and char

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chance, the ideas in statistics and probability. Initially students can delve into ways to explore the representations of data and the probability of certain outcomes. *Numbers*, or *quantities*, are fundamental to school mathematics. The problems students can perform with the availability of powerful, inexpensive calculators are no longer routine calculations. They need to be able to reason about the numbers and quantities. *Patterns* are the main context for the focus of this study. Problems with numeric patterns can be found in tables of data, or they can be recognized in a series of shapes.

Any content can be placed in a different context. For example, the exponential problem that asks students to find their allowance after ten weeks maintains the same mathematical content as a problem regarding a certain bacteria that doubles every hour. Strong problem solvers are able to investigate problems in all different contexts.

Summary

Educators, policy makers, researchers, and mathematicians have not reached a consensus on what should dominate the study of algebra. Nor do they agree on a main focus. The *Algebra Working Group* (1997) suggests a vision of algebra with multiple organizing themes: functions and relations, modeling, structure, and language and representations. In the study of functions and relations the rate of change between variables is emphasized.

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The modeling theme focuses on writing mathematical models to represent data. Supporters of the structure theme believe that the understanding of the structure of the number system lies at the root of understanding algebra. While the language and representations theme seems to suggest that understanding and communicating algebra ideas are key.

The *Algebra Working Group* and Kaput (1995) both remind that any one of these themes or aspects to the exclusion of the others is not sufficient to represent school algebra. Instead, aspects of each are important to develop a rich knowledge and understanding of algebra.

The study of generalizations of mathematical patterns is one area in the discipline where four of these themes can be emphasized at different times. Investigating patterns is suggested in the reforms as valuable mathematics and worthwhile for students to study.

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CHAPTER 3

The Study

Introduction

What is algebra? Although, I did not resolve it in the last chapter, I presented some different perspectives in this discipline. These views helped to define the study and influenced some of the choices I made to investigate the question of what students know and understand about algebra after three years in a *Standards*-based curriculum. These decisions focused my exploration of the research question. This chapter represents an opportunity for the reader to both learn how I investigated the question, and why I made specific choices. There were four initial selections I made that shaped this study into its current form.

1. The content—patterns which represent functions and generalizing with symbols from patterns of data.
2. The curriculum—the *Connected Mathematics Project.*
3. The site—Heartland Middle School.
4. The data—students' responses to performance tasks, recordings while students worked, and interviews after they completed each task.

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Having made these choices, with these parameters, the question evolved to, "What do eighth grade students know about writing symbolic generalizations from patterns which can be represented with functions, after three years in the *Connected Mathematics Project* curriculum?"

The Content

I demonstrated in the previous chapter that studying patterns renders one fundamental aspect of algebra. Although this represented the initial slice I made to study the algebraic thinking of middle grade students, the content was not sufficiently defined for this study. I looked further at my interests and the foundational aspects of algebraic understanding. Once students studied and recognized patterns, I wondered what tools they had to help them represent the patterns. The study evolved to investigate the content of how students represented patterns with symbols. I saw this as one of the fundamental aspects of algebraic understanding.

The Curriculum

A second decision I made involved the curriculum. I selected the *Connected Mathematics Project* (*CMP*) to study for six main reasons.[1] The

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[1] The *CMP* was funded by the National Science Foundation (NSF) to develop complete curriculum materials for grades 6-8. After the *NCTM Curriculum and Evaluation Standards*

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first was the stance taken towards algebra. The *CMP* takes an approach towards algebra different from traditional algebra curriculum. In *CMP*, algebra is a strand woven throughout the curriculum. It was important to me that the curriculum I studied took this stance since I was curious about the strands approach.

The second was the position towards instruction. The developers of the curriculum acknowledge that it is not possible to separate *what* is taught from *how* it is taught. Both are important to the students' understanding of mathematics. The authors of *CMP* support an investigative approach to the teaching of mathematics.[2] I saw this approach as different than traditional

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(1989) were published NSF supported several projects with 5-year grants to create materials at the elementary, middle, and secondary levels that would support the teaching and learning envisioned in the *NCTM Standards*. The goal of the *CMP* was "to develop a complete mathematics curriculum with teacher support materials for the Middle Grades, 6, 7, and 8." (page v, GTK *CMP*). Glenda Lappan, William Fitzgerald, and Elizabeth Phillips of Michigan State University, James Fey of the University of Maryland, and Susan Friel of the University of North Carolina are the principal investigators of the project.

[2] Extensive teaching materials are available to support the implementation of *CMP*. The *CMP* materials for teachers are organized around an instructional model that supports problem-centered teaching. The model considers three phases of instruction: *launch*, *explore*, and *summarize*.

Initially, during the *launch*, the teacher sets the context of the problem. She launches the problem for the class to begin their investigation. This is the time the teacher could introduce new ideas, clarify definitions, review old concepts, or connect the problem to previous work done by the students.

The *exploration* phase allows time for students to "dig in" and investigate the problem. They typically might do this with a partner or in a small group, but at times work individually or even as a whole class, depending on the problem. The teacher's role during this phase of instruction is to observe students while they work, offer prompts to students who are not making progress, redirect students as needed, and suggest extensions as students complete solutions.

The final phase of instruction calls on the teacher to aid students *summarize* their work. This is often done as a whole class when some students could share their strategies of the investigation. The teacher helps students to deepen their understanding of the mathematics in the problem by noting similarities or differences in students work or pushing students to consider an extension.

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algebra instruction and was interested how students could learn in this environment.

The third was the use of rich problems in the curriculum. The *CMP* is organized around interesting problem settings. Students are presented with worthwhile mathematics tasks to explore.[3] This approach to problems supports my use of tasks to evaluate student understanding. I will discuss the tasks in further detail in the section surrounding the data later in this chapter.

The fourth was a belief that all students can learn algebra. The authors maintain a commitment towards teaching all children mathematics; as a result they acknowledge that all students *can* learn a meaningful algebra. The *CMP* takes the stand that all students can thrive in a heterogeneously grouped classroom. The top performing students' mathematical understanding is deepened when they consider ways to justify their solutions. The teacher may also choose to suggest an extension to the problem that continues to challenge the child. Students who are low performers are in an environment where they are expected to learn valuable mathematics. The materials meet the daunting task of engaging while challenging all students.

[3] The NCTM *Professional Teaching Standards* (1991) describe the importance of posing worthwhile mathematical tasks for students to solve. They argue that the problems should be based on sound mathematics, perceptions of students experiences, and knowledge of diverse ways that students learn mathematics. The authors of *CMP* recognize the influence of the tasks chosen by the teachers on students learning of mathematics. "There is no other decision that teachers make that has a greater impact on students' opportunity to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages the students in studying mathematics." (Lappan, G., et. al., 1996, page 40).

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The authors state, "The *Connected Mathematics Project* assumes that when all students are held to the same high expectations and given a chance to explore rich problems, all students can succeed in mathematics" (Lappan, G., et. al., 1996, p. 80). This vision supports a societal goal of educational opportunities for all, as stated in the NCTM *Curriculum and Evaluation Standards*.

The fifth was the importance of connections. The curriculum is called the ***Connected** Mathematics Project*; this title alone is evidence that the authors feel connections are vital for students to make. *CMP* is based on a foundation that supports all students' learning of mathematics by connecting it with other areas of students' learning and interests. Connections in the curriculum are made between what students are learning in mathematics, different areas of mathematics, ideas from other school subjects, and the world outside of their mathematics classroom. The problem settings for each investigation where students explore mathematics present many of the opportunities for the connections. I felt it was important that for students to see the overall picture of mathematics they should recognize how mathematical ideas are related.

The sixth was my familiarity with the curriculum. I worked with the authors of the *CMP* during the development phase. I helped with revisions to both teacher and student editions in primarily the algebra strand. As a result, I had the advantage of knowing the curriculum thoroughly. This ins

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insight helped me see that the *CMP* represented a curriculum that paralleled many of my ideas regarding the teaching and learning of algebra.

CMP Algebra

In the section above I describe my interest in *CMP*'s stance towards algebra for all students. Recall in Chapter 1, I reference Chambers (1994) when she states, "Algebra for all is the right goal at the right time. We just need to get the right algebra." (p. 85). While I do not pursue CMP as necessarily the "right algebra" in this study, I do see the curriculum as an approach to algebra and a set of goals for the learners of algebra that grows out of the NCTM *Curriculum and Evaluation Standards*. In this section I describe this curriculum's approach to algebra.

Algebra Goals

The developers of *CMP* believe that every child can learn mathematics, and specifically every child can gain a meaningful understanding of algebra (Lappan, G., et. al., 1996). This means that the authors do not expect a portion of students to be skimmed off and placed in an advanced mathematics track. Rather, they believe all students together can be challenged and be successful in this curriculum. One of the first things recognized as different from the typical pre-Algebra or Algebra I in 8^{th} grade course is that the algebra is not isolated in one course or grade level. Although there are specific units focused on algebra, algebraic ideas are woven throughout the entire curriculum. This is in parallel with Kaput's suggestion for a "strands" approach to algebra.

The primary goal of the authors of *CMP* is to help students "reason and communicate proficiently in mathematics" (Connected Mathematics Project). The strand specific algebra goals for students who complete three years in the *CMP* are that most students should be able to:

- Recognize situations in which important problems and decisions involve relations among quantitative variables-one variable changing over time or several variables changing in response to each other.
- Use numerical tables, graphs, symbolic expressions, and verbal descriptions to describe and predict the patterns of change in variables.
- Recognize (in various representational forms) the patterns of change associated with linear, exponential, and quadratic functions.
- Use numeric, graphic, and symbolic strategies to solve common problems involving linear, exponential, and quadratic functions. (Lappan., G. T., et. al, 1996, pg. 22)

This list of goals looks quite different than what students in a traditional Algebra I course would be able to do. Some of the areas that are emphasized in traditional curricula that are not part of the *CMP* are: an emphasis on multiplying and factoring polynomials, operating on algebraic fractions, simplifying radicals, operating on non-linear polynomials, and completing the square (Lappan, G., et. al., 1996, p. 28). By eliminating these types of exercises, more time can be spent developing a solid underpinning in algebraic reasoning. Some of the ideas included in *CMP* that traditional curricula do not include are:

- Emphasis on variables and the representations of the relation between variables in words, numeric tables, graphs, and symbolic statements.
- Focus, on the rate of change between two variables, not only linear.
- Development of functional point of view and applications.
- Emphasis on modeling
- Earlier introduction of exponential growth and decay
- Development of alternative strategies for answering questions about algebraic expressions and equations (Lappan, G., et. al., 1996, p. 28).

It is apparent that students following this curriculum will have different experiences than students taking a traditional Algebra I course. There is less emphasis on manipulating symbols and more of a focus on understanding the relationship between variables.

Organization of the Algebra Strand

The curriculum consists of eight units at each of the three grade levels.[4] Each unit has a primary strand (content goal) as the focus of mathematical content, but all units make connections to the other strands throughout.[5] (See Appendix A for a complete list of units.)

The six units with algebra as a primary strand are listed in Table 11 by suggested grade level.

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[4] Each unit of the *CMP* curriculum is divided into four to seven investigations that are built on big mathematical problems that students solve. Some problem situations are real; some are whimsical, while others are pure mathematical investigations.

[5] The materials were developed to be used in the order suggested in Appendix A, although other paths through the curriculum may reasonably be followed, based on local circumstances. The authors caution, that some adaptations may need to be made when

Table 11: *CMP* Algebra Units

Grade Seven Algebra Units	Grade Eight Algebra Units
Variables and Patterns: *Introducing Algebra*	Thinking with Mathematical Models: *Representing Mathematical Relationships*
Moving Straight Ahead: *Linear Relationships*	Growing, Growing, Growing...: *Exponential Relationships*
	Frogs, Fleas, and Painted Cubes: *Quadratic Relationships*
	Say It with Symbols: *Algebraic Reasoning*

The first unit with a central focus of algebra, *Variables and Patterns,* is designed to start the seventh grade year. This unit builds on student's prior experiences and introduces them informally at first to the notion of variables and representations of relationships. These ideas of representations are revisited with a concentration on linear relationships in the next algebra unit, *Moving Straight Ahead.* In the following year, four of the eight units at grade eight have algebra as their primary mathematical strand. The first, *Thinking with Mathematical Models*, introduces students more formally to functions and modeling. *Growing, Growing, Growing...*, examines exponential growth and decay in tables, graphs, and simple symbolic forms. *Frogs, Fleas, and Painted Cubes*, focus on quadratic growth and functions. The last algebra unit, *Say It with Symbols*, students investigate equivalent symbolic expressions and solving linear equations symbolically.

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following alternate routes since most units build on student understanding developed in prior units.

The six units in the algebra strand do not represent the full extent of algebra in *CMP*. Consistent with the philosophy of connecting and weaving the mathematical strands, ideas from algebraic reasoning are present in all 24 of the units. Prior to working more formally with algebraic symbols students thoroughly investigate relationships in verbal, tabular, and sometimes graphic forms. When algebraic symbols are introduced, they are presented as a natural extension and representation of the ideas explored. By the end of grade eight, it is expected that students will have a deep understanding of the meaning of symbols and the relationships in tabular, graphic, and verbal forms.

Recall that both the *Algebra Working Group* (1997) and Kaput (1995) remind that a complete algebra experience for students involves all of the organizing themes—functions and relations, modeling, structure, and language and representations. This is *CMP*'s intent. Although the overall focus is on the functions and relations theme throughout the algebra strand, the other themes are present in various algebra units. A big idea in **functions and relations** is rate of change. Students study rate of change in linear patterns in *Moving Straight Ahead*, in exponential patterns in *Growing, Growing, Growing…*, and in quadratics and cubics in *Frogs, Fleas, and Painted Cubes*. Students explore **models** in *Thinking with Mathematical Models*. In *Say It with Symbols* they study the **structure** of algebra. And, in

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Variables and Patterns they spend a lot of time studying different **representations** of relations.

The authors of *CMP* acknowledge the importance of studying patterns. They state that "(o)bservations of patterns and relationships lie at the heart of acquiring deep understanding in mathematics...Students solve problems and in so doing they observe patterns and relationships; they conjecture, test, discuss, verbalize, and generalize these patterns and relationships." (Lappan, G., et. al., 1996, p. 1). Studying data to determine a pattern is a big part of the algebra curriculum of *CMP*. It is also something reasonable for middle grade students to do in their classes as a basis for algebraic understanding (NCTM, 1989). This is expressed in two of the *CMP* algebra goals: use numerical tables, graphs, symbolic expressions, and verbal descriptions to describe and predict the patterns of change in variables, and recognize (in various representational forms) the patterns of change associated with linear, exponential, and quadratic functions. Students are expected to search for patterns and relationships and then express their conclusions verbally and eventually symbolically.

The Site

Another choice I made determined the site to conduct the study. Since *CMP* was still in development when I conducted the study, I sought a location that was piloting the materials, was committed to the implementation of *CMP*, and would allow my research. I selected Heartland. This site interested me because they had demonstrated a record of their commitment to the *CMP* curriculum and because of their proximity to the university.

Heartland

Heartland is a small, rural community with a population of around 4,000 residents. A strong German Catholic heritage is often credited for the hard work ethics of the community. This ideal is often carried over into the schools. Community members respect the teachers and it is expected that children follow their parents' lead in this regard. The town had been a fairly closed community with many small family farms and blue collar automotive workers, but it is changing to more of a bedroom community as residents commute to jobs in other nearby urban areas.

Heartland Middle School, the one public middle school in the community, averages around 450 students in grades six through eight. There is a K-12 parochial school that has about 40 students per grade in the community. These schools together account for all of the students in the middle grades. Heartland Middle School has been involved in mathematics staff development for a number of years. Heartland is only about 30 minutes away from Michigan State University; this proximity to the university helped the school actively participate in the early reform movement.

Heartland Middle School has achieved academic awards in different areas. Lamar Alexander, at the time United States Secretary of Education, granted the school's mathematics department the "A+ for Breaking the Mold" award for the 1992-93 academic year. The social studies department also excels. They won many awards in the National American Express Geography Competition.

Commitment to Reform

There are a number of sources to credit for the achievement in this school: willing teachers, supportive administrators, quality materials, and an informed mathematics teacher leader. Of primary importance is the notion that this reform evolved over a number of years; it has not happened with a "quick fix."

Table 12 charts the reform projects at Heartland Middle School.

Table 12: Heartland Middle School Reform Table

Time	Reform Project
1983 – 1985	Piloted *MGMP*
1987 – 1990	Participated in a "County Project"
1991 – 1996	Piloted *CMP*

In 1983, Heartland was chosen as a pilot site for the *Middle Grades Mathematics Project (MGMP)*.[6] The middle school teachers from Heartland

[6] *MGMP* is a set of five separate stand-alone units on equivalent fractions, factors and multiples, perimeter and area, probability, and spatial visualization. They were written by

participated in workshops given by the authors of *MGMP* as part of the piloting agreement. In 1987, Emily Clark, a local leader in the school, initiated a countywide project.[7] This began a four-year project to help the mathematics teachers become more reflective about their practice. In 1991 *CMP* selected Heartland Middle School as a pilot site. Heartland seemed a reasonable choice due to its closeness to the university, the staff development that had already occurred with the teachers, and the disposition towards reform by the administrators. The school agreed to teach the curriculum to all of the students in heterogeneously grouped middle grade classes. When the piloting of *CMP* began, teachers participated in regular workshops during both the summer months and the school years to familiarize them with the *CMP* philosophy, instructional model, and materials.

Evaluating the Reform Efforts: MEAP

Schools are evaluated in Michigan based on their students' performance on the Michigan Educational Assessment of Progress (MEAP) test. Students score within one of three ranges: satisfactory, moderate, or low. In 1991, early in the reform story, only 44% of the students in Heartland scored in the satisfactory range. (See Figure 12 for the data from 1991 to 1995). Only four years later, 78.8% of the students scored in the satisfactory range. This is a

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several of the same authors as *CMP*. They were not intended to represent an entire curriculum, but rather as replacement units for these topics.

huge increase! But this does not just show that students are moving up from the moderate range, they are moving out of the low range. The number of students scoring low decreased from 20% in 1991 down to only 6.8% in 1995. Teachers and administrators were pleased. They felt this demonstrated that students enrolled in a curriculum that focuses on problem solving continue to do well on a more traditional standardized test while learning more powerful mathematics.

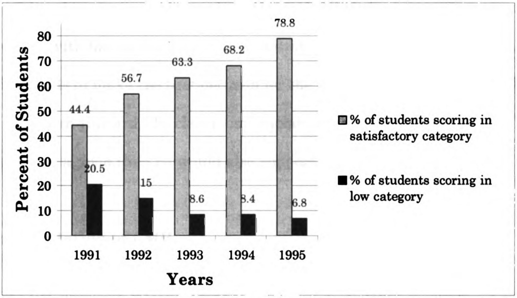

Figure 12: Heartland MEAP Data

I credit two major factors for the improvement in the middle school mathematics performance at Heartland. First was the availability of quality materials, and second was the support from the administration.[8]

The closeness to the university afford teachers the opportunity to have access to resources—both people and quality materials. The materials that

[7] Emily convinced her superintendent along with the other five school districts in the county to collectively pool their money to provide staff development for the 25 middle school

were used, first *MGMP* and later *CMP*, represented a model for the teachers of what it means to teach differently.

The final key aspect that was vital to the reform equation was the role of the administration. John Roberts, the middle school principal, has demonstrated openness, vision, and trust. He was willing to listen and allow Emily to pursue her goals for the district. He readily approved the time for the teachers to spend in their coaching sessions. I recall our conversation when I sought permission to use Heartland Middle School as a site to conduct my study; with his approval he added that he trusted his teachers as professionals to make the best decision for the students. He further stated that if Evelyn Howard, the classroom teacher whose students I wanted to participate, agreed then he would also support my choice.

Implementation of *CMP*

The *CMP* was implemented in the entire sixth grade at Heartland Middle School in the fall of 1992. The seventh grade was added the following year in 1993 as the materials were developed. Finally in the fall of 1994 the eighth grade materials were available and the entire curriculum was implemented in all three middle grades.

As the *CMP* was implemented, all students had this curriculum in their mathematics classes. Students at Heartland are not pulled out for gifted and

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mathematics teachers of the county rather than just the four teachers in their school.
[8] Conversation with Emily Clark on November 24, 1997.

talented sessions, eighth grade algebra classes, or special education classes. The classes are heterogeneously grouped. The school typically has between 10 and 15% of their students identified as special education students. All of these are mainstreamed into regular classrooms. This percentage has risen to as high as 18% at times. Evelyn Howard, the classroom teacher, has had as many as one third of her students in one class labeled as special education. There is a very minute minority population in this predominantly white community. There are typically only five minority students in Heartland Middle School in any given year.

The Students

All of the students participating were in their third year of *CMP* mathematics. They represented the second cohort of students at Heartland Middle School to have the entire *CMP* curriculum, sixth grade in 1993-94, seventh grade in 1994-1995, and eighth grade in 1995-1996. They all had Evelyn Howard as their eighth grade teacher, but came from two different classes. In the seventh grade students had one of two mathematics teachers, Evelyn or Jim Johnson. As sixth graders, these students had one of two different teachers.

I introduced this study to two of Evelyn's eighth grade mathematics classes and described my interest in understanding how students in a *CMP* classroom investigated problems with algebra. I sought permission from all students in the class to participate in this study and explained the time commitment. I wanted to study the performance of students who found varying degrees of success with the program, not just the top performers. I asked Evelyn to identify students who would give me a range of achievement, but would be able to participate in conversations with me about their sense making. Of the volunteers whose parents had signed permission slips, Evelyn and I together selected ten students to participate.

I decided that students would work with a partner because some of the tasks were challenging and I felt that the students would be more successful if they worked in pairs. I decided to videotape each pair while they worked on their tasks, I saw the additional conversation occurring between partners an additional benefit. Students worked with a partner from their class on selected problems. They worked with the same partner for all four tasks. I assigned pairs randomly with the intention of having a diverse blend of groups. I had two groups of male pairs, one group of female partners, and two groups with a male and a female student working together. The partners are listed in Table 13 below.

Table 13: List of Students

<u>Students with Partners</u>
Zachary and Todd
Ben and Joe
Anna and Katrina
Julie and Dan
Sara and Ryan

The Data

Once I had determined the content, the curriculum, and setting, my final decision involved the data to collect. I had spent a number of years studying assessment with the *Balanced Assessment* Project (*BA*).[9] While at *BA* I had the opportunity to explore what represents a quality task, and to analyze students' written responses to tasks. This helped me recognize that often some of what students did while working on a task did not make its way to the written record. This understanding helped me to determine the data that I collected in this study. I determined that assessment tasks would be part of my data. I developed and revised four performance tasks based on the algebra content described earlier for the students to complete.

I collected three sources of data in this study.

- Students' written responses to the performance tasks,
- Video recordings of the small group work, and
- Follow-up interviews.

Data Collection

I felt that each of the three sources of data could offer different evidence about these students' understanding.

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[9] Balanced Assessment was funded by the National Science Foundation in 1991. Their goal was to develop a comprehensive range of performance assessment tasks in mathematics, and assemble them into balanced packages at grades 4, 8, 10, and 12.

Written responses. This is valuable data since typically classroom teachers make decisions about student understanding and instruction based on this written record. I was interested in how these students made sense of these tasks. I felt that some of this would be revealed to me in their written responses.

Tape-recordings of group work. I also thought that a lot of what students did could potentially be lost if I limited my data to their written responses alone. So while students worked with their partners, I either videotaped or audiotaped them. (Most were videotaped, only one pair was audiotaped for two of the tasks). I had hoped that these recordings would allow me insight into the students' conversations while they worked. I could see a little more of what these students did in their solutions, when they were stuck on a problem, or changed strategies.

Follow-up interviews. I wanted the opportunity to ask the students questions about their solutions. I conducted follow-up interviews with each pair of students after they completed each task, all of which were video-recorded. I did not review the recording of the students' work prior to the interview, but I returned the students' written responses and allowed them some time to review their work before we began the interview. When they were ready I used a protocol that I wrote while designing the study for some of my initial questions. The general protocol I used to conduct each interview is included in Appendix B. I realized during the interview that additional questions were needed to clarify a response or probe a little further. For example, after Zachary and Todd replied that they made tables in all of the problems, I asked them to further describe their strategies for constructing them. Some of the additional questions I asked: When did you use the table? Okay, what does that tell you? How did you construct tables? How many of these did you have to draw before you could actually continued the table? What does that tell you that the second difference is one? Okay, what does that mean?

The data was collected over a three week period. Several pairs of students left their mathematics class each day to help me with this study. The students completed a task and then participated in the interview before completing the next task. The students spent an average of 23 minutes to complete each task and 19 minutes for the interviews. The approximate times each pair spent investigating the tasks and participating in the interviews are given in Appendix C.

Although all students completed the tasks in the same order—*Borders*, *Cutting*, *Dominoes*, and then *Toothpicks*—they were not all investigating the same problem at the same time because I had only two video-cameras to record their work. Not all interviews were done immediately after students completed their work. At times several days passed after a student completed a task before I conducted the interview. Since I had two pairs working at a time, I sometimes had to have one pair wait to complete their follow-up interviews. On some days, students were absent so I did not ask one student from the pair participate.

Task Descriptions

The four tasks in this study were chosen based on their mathematical content, context, and the students' experiences in their mathematics classrooms. Students were allowed as much time as needed to complete the tasks and had access to calculators. (See Appendix C to see how long each pair worked on each task.) If the students did not complete a problem during their class time, then I allowed them additional time in our next meeting to finish. The tasks are similar in that they all ask students to study some regularity, make predictions for future values, and then generalize about what they have found out. The tasks are dissimilar in the patterns they represent—linear, quadratic, and exponential—and the context in which each of them are set. A copy of each task is included in Appendix D.

Borders

First I asked the students to complete *Borders*. They were asked to write generalizations for both the number of blue tiles in the center and the white tiles on the outside border for any figure n in the pattern given in Figure 13.

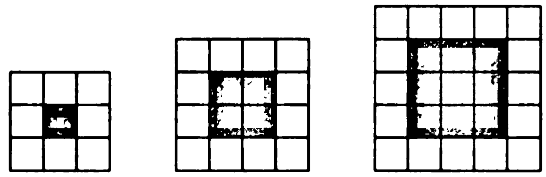


Figure 13: Borders Graphic

I thought that the two patterns—one quadratic and one linear—would be straight forward for the students to investigate. The number of blue center tiles increased in the basic quadratic pattern of n^2, while the number of white border tiles grew in the linear pattern of $4n + 4$.

I perceived this as a good initial task since I felt that all of the students could have some point of entry into the problem and therefore some success with it. I thought that most students would complete the task and find a symbolic generalization since the patterns were fairly basic and it was similar to the types of problems they saw in the curriculum—collect and organize data, and then write symbolic generalizations.

Cutting

After the students completed *Borders*, I asked them to explore *Cutting*. Like *Borders*, *Cutting* also asked students to search for two generalizations, but these patterns were not as familiar to the students. The first pattern followed an exponential growth pattern, 2^n, while the second represented exponential decay, $\frac{32}{2^n}$. Although, I administered the task with the expectation that most students would not find symbolic generalizations for these patterns, I still considered this a worthwhile problem for the students to explore for three main reasons. First, I was interested to find if the students used any language that would allow them to verbally describe a generalization with a recursive pattern. For example, they might have stated that the number of sheets was always double the number of sheets for the cut before. Second, I wondered what strategies they would rely on to solve unfamiliar problems—would they call upon a comfortable process or search for something new? Third, I was curious to see how students from a *Standards*-based curriculum would struggle with a generalization for a rule that was unfamiliar to them—for what length of time would they pursue the problem, at what point they became frustrated with the problem, how they handled their frustration, and how they coped with an unfamiliar problem.

Since this exponential pattern has a base of two, I felt students could conduct a meaningful exploration of the problem prior to writing a symbolic rule. I predicted that they could work with the doubling in the problem prior to their formal introduction of exponential symbolism. I discovered after the pairs of students finished the task that some did recall doing similar types of problems in earlier *CMP* units or their seventh grade science class. I found it interesting that some students were able to make this connection.

I felt that the generalizations for the rules in this task would prove to be more challenging for these students. Since, these students had yet to complete the *CMP* unit on exponentials, *Growing, Growing, and Growing...*, I saw this task as more challenging. They were scheduled to begin this unit during their final weeks of school, shortly after the completion of this study.

Dominoes

The third problem students completed was *Dominoes*. The final question asked students to find the total number of domino faces possible with from zero to n dots.

I predicted that the students might observe and extend the pattern in the data for a specific case, but would probably have difficulty writing a rule for the n^{th} case, $\frac{1}{2}(n + 2)(n + 1)$.

I saw this as a problem that would challenge all of the students, but for different reasons than with *Cutting*. Unlike the exponential unit, *Growing, Growing, and Growing...*, all of the students in my study had completed the *CMP* unit on quadratics, *Frogs, Fleas, and Painted Cubes*. The students had some experiences in their mathematics class classifying patterns as quadratics, but they did not have a systematic way to write generalizations based on data for a quadratic relationship. This was not a basic quadratic pattern that I felt the students would find easily.

Toothpicks

Next, the students explored *Toothpicks* which involved two number patterns based on stair-step shape figures in Figure 14 below.

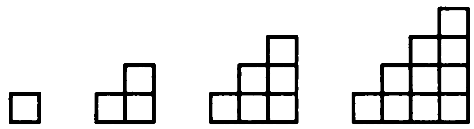


Figure 14: *Toothpicks* Figures

Students explored both the number of toothpicks in the perimeter, and the total number of toothpicks in the figures. This task possesses a similar structure to *Borders*; one pattern is linear and one pattern is quadratic in both tasks.

The pattern of the perimeters of the *Toothpick* shapes is linear, $4n$, while the total number of toothpicks is quadratic, $n(n + 3)$. I saw the pattern for the quadratic total number of toothpicks, as a little more difficult for the students than the quadratic pattern in *Borders*, n^2, but not as challenging as the rule in *Dominoes*, $\frac{1}{2}(n + 2)(n + 1)$. I was interested to see what tools the students would use to write a rule for this pattern.

I felt that most students could find these generalizations, unlike *Dominoes*. The linear perimeter pattern was more basic and the students spent time in the curriculum studying factored.

Task Solutions

One of the strengths of the four tasks the students completed was the potential for multiple solution strategies. This opportunity for a variety of approaches is one aspect that makes a worthwhile mathematical task according to the NCTM *Professional Teaching Standards* (1991). One possible solution is explored in Appendix E for each of the tasks. This section is not intended to be representative of the students' work, rather offer one possible solution to each task for the reader.

In my solutions I make assumptions that these patterns do continue infinitely in either the linear, quadratic, or exponential pattern I represented. I realize that given any finite amount of data these patterns could reasonably be extended in an infinite number of ways. The students also seem to make this assumption, but it is important to note that these regular extensions of the patterns are not the only possible completions.

Data Analysis

In my initial analysis I used only students' written responses. I grouped the responses by student pairs. I first studied all tasks completed by one pair of students and looked for whether they used common strategies across several tasks. I repeated this process with the other four pairs focusing on each pair and searching for similarities across their solutions. After I collected notes regarding what the students did by pairs, I looked across the five pairs, looking for common approaches. I formulated tentative hunches of what I saw in the students' work. One of the strategies fairly common across tasks and across pairs was the construction of a table and then the search for a pattern in the table. I noted after studying the written work that:

- Tables seemed to be a starting point for students when they encountered unfamiliar problem situations.
- Tables seemed to serve as a means to systematically generate and organize data.
- Students were able to answer questions about specific cases.
- Students used tables to study the data in a search for patterns by looking at constant differences.

Students' written work was not sufficient to inform me about the students' understanding. It did two things for me. First, the written responses offered confirming evidence for some of my hunches, and second, this work raised some questions for me about students' understanding. I still had questions about how the students used their tables, and I was puzzled in the instances where students did not construct tables.

I looked at other sources of data to support my assertions. In the videotapes while students worked I saw further evidence that confirmed that constructing tables was a reliable strategy for these students. When several pairs independently voiced their uncertainty, one of the pair usually suggested making a table. With this lead, the all made progress with the tasks. During the interview the students elaborated further that the tables gave them a lot of information and usually worked in their problems. This represents an example of how I found confirming evidence and triangulated the data (Bogdan and Biklen, 1982).

Summary

This chapter represents a narration of the choices that shaped this study to investigate the question of what eighth grade students know after three years in the *Connected Mathematics Project* about writing symbolic generalizations for patterns which can be represented with functions.

The next two chapters offer data and some analysis for the reader to see how the question was addressed. First, in chapter four, I thoroughly describe what each pair of students have done in their investigation for each of the tasks. All three sources of data—written responses, recordings while working and recordings of the interviews—are used to support this. Next, in chapter five, I step back and look across students and across tasks to describe two common strategies that were used in most of the solutions, making tables and studying the shapes.

CHAPTER 4

Digging Deep—

A Careful Look at Student Investigations

Introduction

In this chapter, I describe students' solutions in four performance tasks. I rely on three different sources of data—students' written responses, recordings while students worked, and interview recordings after students completed each task—to help me understand what the students have done. I describe each pairs' investigation separately in each task.

Most students classified the four tasks I administered to them as primarily algebra because they searched for symbolic generalizations. A few classified some of the tasks as mostly geometric because of the shapes that were involved.

The students' interpretations of the problems, algebraic or geometric, influenced their solution strategy. When the students considered a problem algebraic, they constructed a table of numeric data to study the pattern. In some of the less complex, linear cases, the students recognized a pattern in the numeric data prior to organizing it in a table. When they saw a problem as geometric, they studied the changing shapes to describe the pattern.

The dominant strategy used across students and tasks was to construct a table of data. Students who made tables followed some or all of these steps. They used tables to (a) record or re-present data, (b) extend given data, (c) find specific cases, (d) study patterns, (e) write rules, and (f) verify rules.

Students followed an alternative strategy when they saw tasks as geometric. They did not make a numeric table of data, instead they focused on the changing shapes and found generalizations based on how the figures grew. In a small number of cases, students made pictorial representations to study the patterns. These students often found generalizations based on sketches.

Regardless of the strategy used in the investigation, the students demonstrated understanding about the patterns in all cases, even when they did not write a symbolic generalization. The students demonstrated this at times by describing the patterns verbally. In other cases some made connections between the patterns of numeric data in a table to the shapes of the graphical representations.

Zachary and Todd

Students often made and studied tables to help them solve situations involving patterns. Zachary and Todd constructed tables for all four tasks. The pair successfully found rules for all patterns.

Borders

These students described *Borders* as an algebra problem since it dealt primarily with "equations and variables." This interpretation led them to solve the problem by creating and studying tables as discussed during our interview.

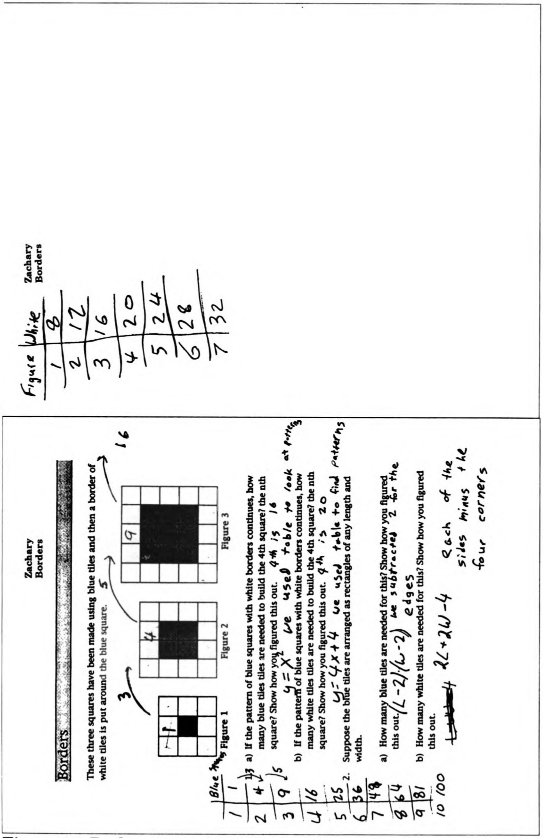
Todd: We could have just looked at pictures. First, I did and we saw that it got taller and wider, but it was kind of hard to make an equation based just on that.

Zachary: We just tried to find patterns in the table to see how much it changes by.

Todd: Yeah, it was easier seeing the numbers.

Using a Table

Zachary and Todd began working on *Borders* by counting the tiles in the drawings and making marks on the figures as they counted. Then, they made tables to record their data. Next, they generated new data by observing the pattern of change in the numbers they had recorded in their table. In the case of the blue tiles, they extended the table to Figure 10; in the case of the white tiles they extended the table to Figure 8. After they had extended their tables for both blue and white tiles, they used the data in their table to read the number of tiles needed for the specific case of the fourth figure.


Zachary
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 4th is 16
$y = x^2$ we used table to look at patterns

b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 4th is 20
$y = 4x + 4$ we used table to find patterns

2. Suppose the blue tiles are arranged as rectangles of any length and width.

a) How many blue tiles are needed for this? Show how you figured this out. $(L-2)\cdot(W-2)$ we subtracted 2 for the edges

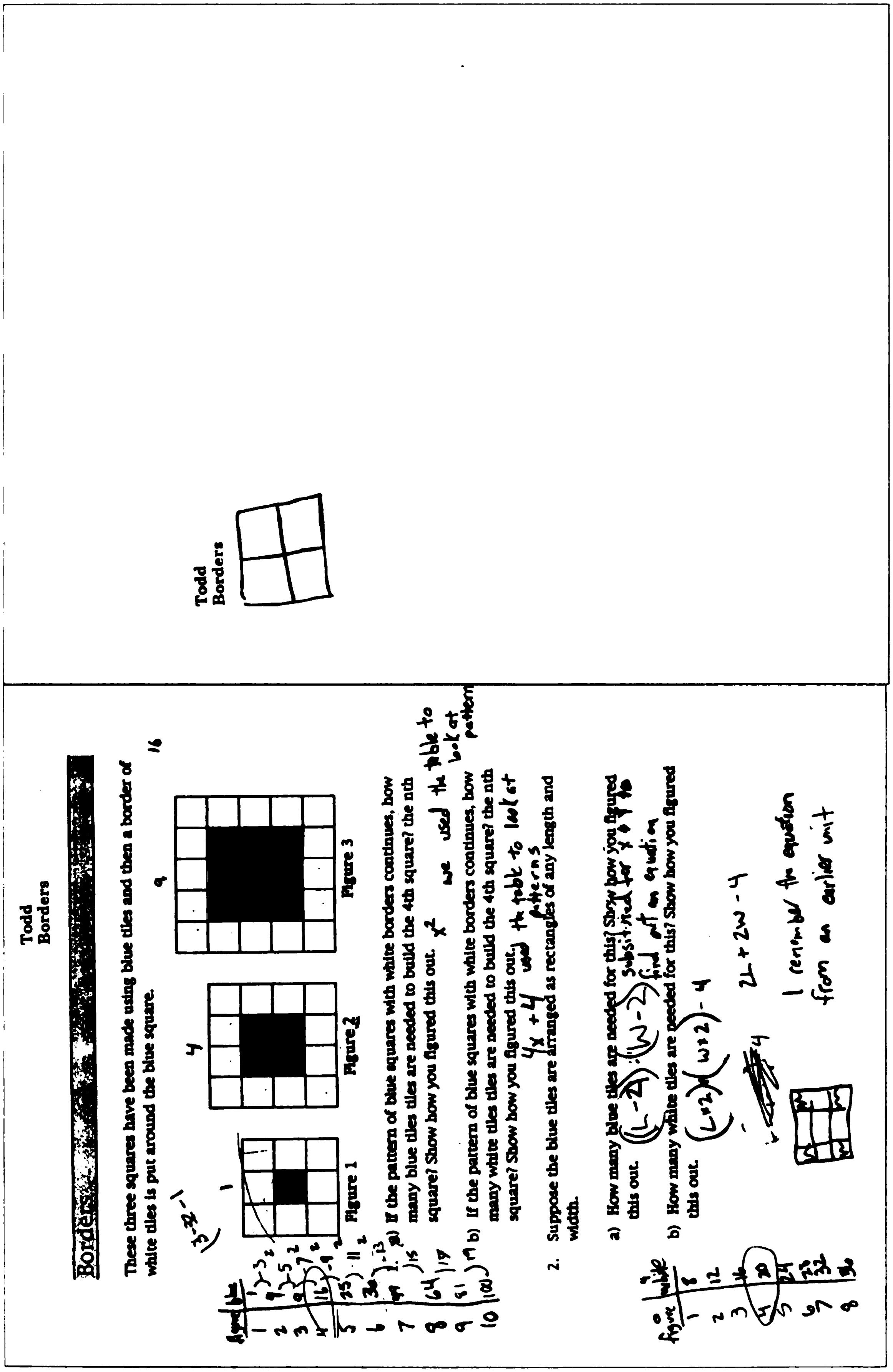
b) How many white tiles are needed for this? Show how you figured this out. $2L + 2W - 4$ each of the sides minus the four corners

	Blue Squares
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Zachary
Borders

Figure	White
1	8
2	12
3	16
4	20
5	24
6	28
7	32

Figure 15: Zachary's work on *Borders*



Todd
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1) a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. x^2 we used the table to look at pattern

b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. $4x+4$ used the table to look at patterns

2. Suppose the blue tiles are arranged as rectangles of any length and width.

a) How many blue tiles are needed for this? Show how you figured this out. $(L-2)(W-2)$ substituted for x & y to find out an equation

b) How many white tiles are needed for this? Show how you figured this out. $(L+2)(W+2)-4$

$2L+2W-4$

I remember the equation from an earlier unit

Todd
Borders

Figure 16: Todd's work on *Borders*

The pair studied the patterns in the data table to extend their table and to determine if a relationship was linear or non-linear. They quickly noted the constant increase in the number of white tiles while working on the task.

Todd: After the first one is eight. Wait, we can start at zero, the y-intercept is four.

Zachary: It increases by four each time.

Todd: So, for each blue[1] square, it's four times...

Once they recognized this change of four they attempted to write a generalization.

The pattern for the blue tiles did not result in a linear relationship, so the pair examined the data to see if and when they would get a constant difference. Zachary noticed, "The increase increases by two each time, so three, five, seven, nine. One times one is one, two times two is four, three times three is nine." In an interview I asked them about this.

AK: When did you stop looking for the pattern?

Todd: When the differences were the same.

AK: When was that?

Zachary: The increase increases by two each time.

Todd: I stopped about here [points to the table around four].

Using this they found the rule of x^2 for the number of blue tiles.

---

[1] Todd says "blue" but he refers to the pattern in the white tiles.

Zachary and Todd found the linear rule for the white tiles by using the constant rate of change they calculated from the table and extending the pattern back to find the number of white tiles for figure 0.

Todd: It would be what?

Zachary: Four X plus four?

Todd: That [four] would be the intercept, even though we don't have zero [in our table].

In an interview they explained further.

AK: What did you get for your equations?

Todd: Four X plus four.

AK: How is this four [the X-coefficient], how is how it changes shown in your work?

Todd: Right here [pointing to the increase in the table].

AK: What does that last four show?

Todd: That's the y-intercept and it means...

Zachary: ...where it starts.

Todd: Yeah. This figure over here [pointing to the left of figure 1] wouldn't have any blue squares only four white squares.

Zachary: For the four corners.

Todd: That would be figure zero.

In this complete explanation they elaborated on what each four in their rule of $4x + 4$ represented, describing how it related back to the geometric context in which the problem was posed.

Cutting

Although Zachary and Todd recalled completing a problem similar to this in their seventh grade science class, they did not remember the process they used to solve the problem. They were initially unsure how to start this problem, but decided to make a table again, influenced by the demand of the task to find an algebraic rule.

Using a Table

The pair began by immediately writing the number of sheets in each stack above the figures on their paper and completed the pattern up to 64 sheets in a stack after six cuts (see Zachary's calculations above and to the right of the four figures). They soon recognized that a table would be useful as evidenced in their conversation while they worked.

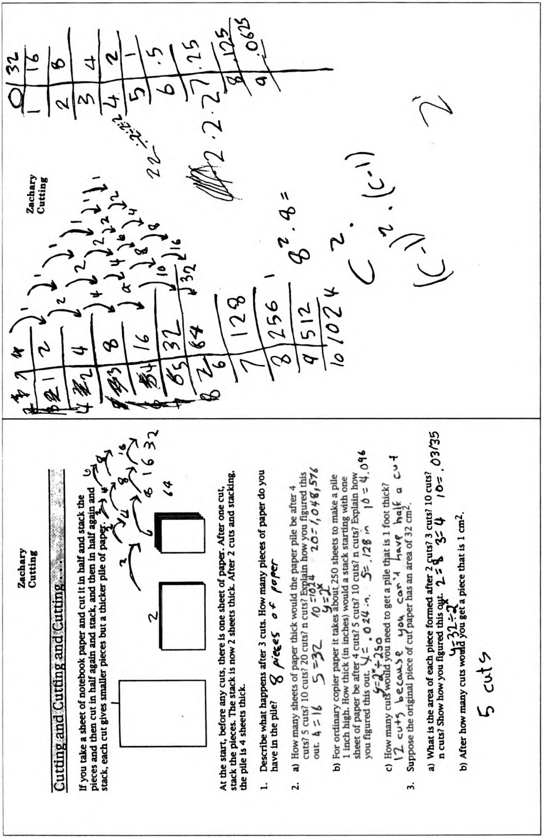
Zachary: So, one, two, four...

Todd: Squared?

Zachary: ...times two, so eight.

Todd: Then, sixteen, then thirty-two? [pause] I don't know.

Zachary: Do you want to try to make a table?



Zachary
Cutting

Cutting and Cutting and Cutting...

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? 8 pieces of paper

2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out. 4 = 16 5 = 32 10 = 1024 20 = 1,048,576 $y = 2^x$

b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out. 4 = .024 in. 5 = .128 in 10 = 4.096 $y = 2^x \div 250$

c) How many cuts would you need to get a pile that is 1 foot thick? 12 cuts because you can't have half a cut

3. Suppose the original piece of cut paper has an area of 32 cm².

a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out. 2 = 8 3 = 4 10 = .03135 $y = 32 \div 2^x$

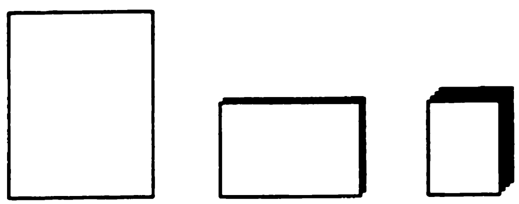
b) After how many cuts would you get a piece that is 1 cm²? 5 cuts

Figure 17: Zachary's work on *Cutting*

Todd
Cutting

Cutting and Cutting and Cutting...

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.



At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? $4 \times 2 = 8$

2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain ~~how you~~ figured this out. 2^x
   4 cuts = 16 5 cuts = 32 10 cuts = 1024 20 = 1,048,576

   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out. 4 = .024 in. 5 = .128 10 = 4.096 in
   $2^x \div 250$

   c) How many cuts ~~would you need~~ to get a pile that is 1 foot thick?
   12 cuts

3. Suppose the original piece of cut paper has an area of 32 cm^2.

   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out.
   $32 \div 2^x$ We reversed 2 cuts = 16 3 = 8 10 = .046875

   b) After how many cuts would you get a piece that is 1 cm^2.
   5 cuts

Todd
Cutting

cuts	paper
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

2 +2 +2
4 +4 +4
8 +8 +4
16 +16
32 +32
64 +64
128 +128
256 +256
512

0	32
1	16
2	8
3	4
4	2
5	1
6	.5
7	.25
8	.125
9	.0625
11	.0312

$1 \times x^2$

2^x

$c - 1$

$8^2 + 8 \rightarrow 9$

c^{2x}

$(c-x)^2 \times (c-1)$

Figure 18: Todd's work on *Cutting*

Each student proceeded to flip his paper over and recorded the data in a table for one to 10 cuts. Before they generalized their pattern, they used their calculators to continue doubling the pattern and found 1,048,576 sheets in a stack for the specific case of twenty cuts.

This exponential pattern was puzzling to the pair. They readily found the recursive doubling pattern to find the number of sheets after each successive cut by studying the pattern, but they had difficulty writing a co-variational rule for the pattern. Part of the difficulty was that this situation did not produce a constant difference. They continued to search out to the sixth difference. In an interview I asked them about what they had tried.

AK: How did you fill out your table?

Zachary: Times two, times two, times two.

Todd: It was easy filling out the table.

AK: Why did it take so much longer to answer the question after you filled in the table?

Todd: Not a real obvious problem. I tried finding the first difference, second difference, and third difference and there really wasn't any other pattern besides it doubling.

AK: Why did you find the differences?

Zachary: To try to find the equation, that works for the one in the last problem [*Borders*].

They knew that constant differences helped them write algebraic rules, but this problem did not lend itself to this strategy.

However, they recalled doing a similar problem in their science class the year before. They explained during our interview.

Todd: We were trying to find the time for bacteria to touch the moon. How many times it duplicated. Looking at the equation, it was almost the same.

AK: Did that help you to solve this problem?

Todd: I wouldn't have known how to use the y-x key.[2]

Even with this recognition they struggled for some time before they finally came up with the rule. They followed a "guess and adjust" strategy recording $(C - 1)$, C^2, and $(C - 1)^2 \times (C - 1)$. After about twenty minutes, Zachary recognized something that led to the rule.

Zachary: It's whatever number that is, the number of two times. I'm not sure though how to write that in an equation.

Todd: It's that right there.

Zachary: Two X [and writes 2^x].

Once the students had written their rule of $y = 2^x$, they tested it by trying a specific case. In our interview, they "proved" that their rule was appropriate by demonstrating how it worked for the values in their table.

---

[2] They had everyday access to the *TI-30 Challenger* in both their seventh and eighth grade mathematics classrooms. These calculators used a y^x to represent the exponential function or (Footnote continued on next page.)

This problem proved to be challenging for Zachary and Todd. What was interesting was they way in which they persevered in their investigation. Even when I told them they could stop and we could discuss the rule together, the pair seemed to believe that they should be able to find an equation given any problem. They continued their investigation until they had completed the problem and found a symbolic generalization.

Dominoes

Zachary and Todd made tables again in *Dominoes*. But, before they did, they sketched representations of the dominoes to generate their data. First, they drew the dominoes with zero to two white spots. Todd drew nine possibilities (0-0, 0-1, 0-2, 1-0, 1-1, 1-2, 2-0, 2-1, and 2-2), decided that the 0-1 and 1-0 dominoes were the same, and crossed off the doubles. Rather than re-draw the dominoes with up to two spots again in the set of dominoes with up to three spots, Zachary drew the four additional dominoes to arrive at a total of ten dominoes with up to three white spots. Zachary then started to sketch the dominoes with up to six spots, but quickly abandoned that strategy to try an alternative approach.

"power key". Students also had access to the *TI-82* graphing calculators in their classroom. I supplied both calculators for the students to use as needed while they completed the tasks.

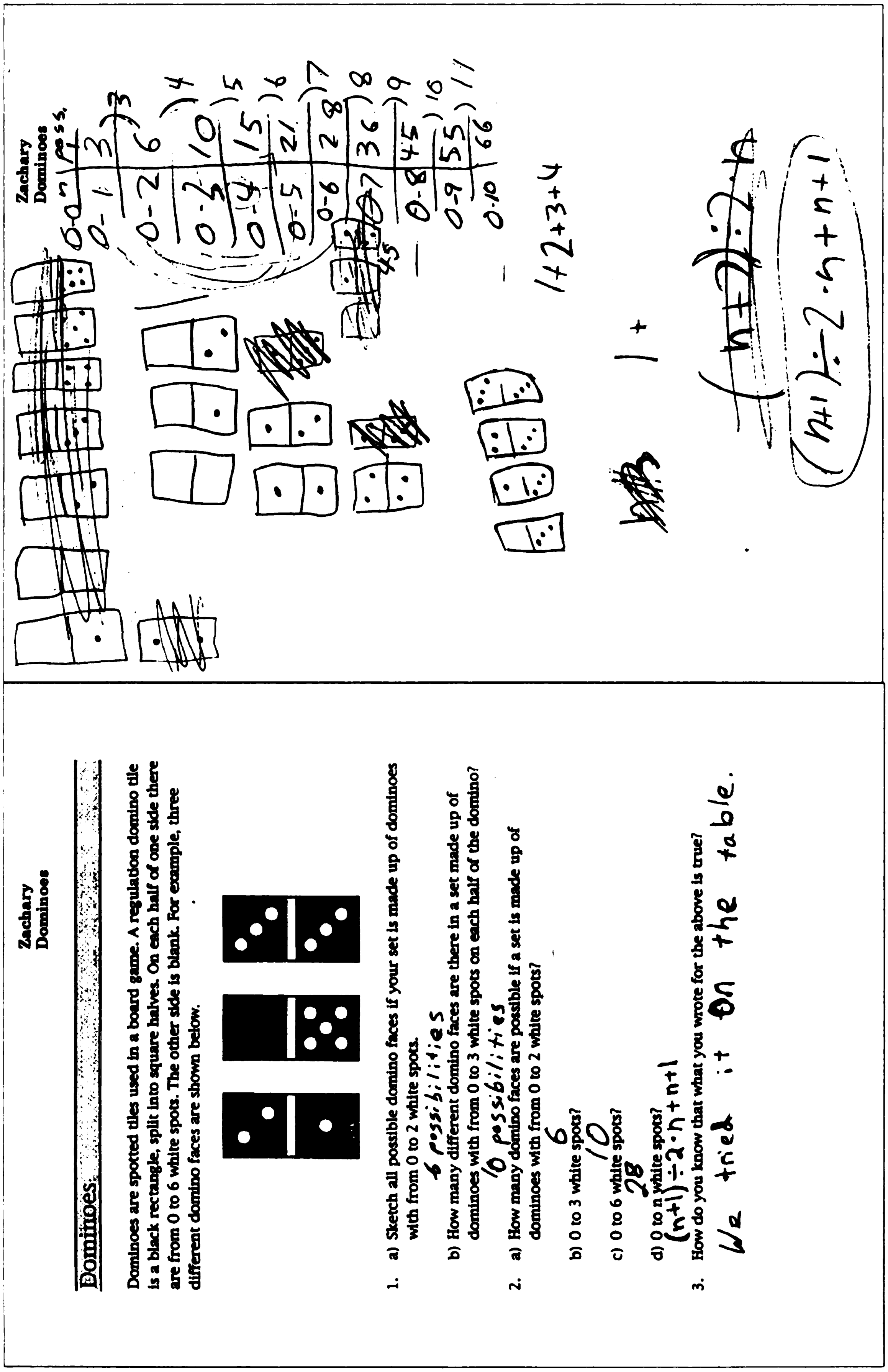


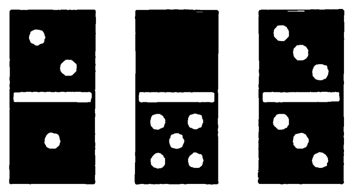
Figure 19: Zachary's work on *Dominoes*

Todd
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

0 – 0, 1, 2
1 – 0, 1, 2
2 – 0, 1, 2



1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots. back

   b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?
   10

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?
   6

   b) 0 to 3 white spots?
   10

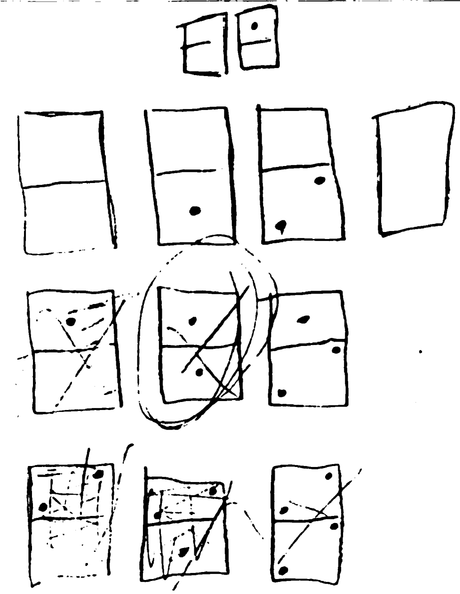
   c) 0 to 6 white spots?
   28

   d) 0 to n white spots?
   $(n+1) \div 2 \cdot N + N+1$

3. How do you know that what you wrote for the above is true?

We tested it out on the table

Todd
Dominoes



0 - 0	1
0 - 1	3
0 - 2	6
0 - 3	10
0 - 4	15
0 - 5	21
0 - 6	28
0 - 7	36
0 - 8	45
0 - 9	55
0 - 10	66

+1

$(n+2) \div (2 \cdot N)$

$(n-1)^2 + ((n-1)-2)$

$(n+1) \div 2 \cdot n + n+1$ $\quad$ $3n \div 2 + 1$

Figure 20: Todd's work on *Dominoes*

Using a Table

With the preliminary sketches made, they recorded and organized the data into a table to complete the task. They studied the pattern of the data in the table and then extended their table up to the number of dominoes possible with one to 10 white spots.

Zachary first recognized that once he had the dominoes with from zero to two white spots, he could find the ones with up to three spots by adding four additional dominoes (0-3, 1-3, 2-3, and 3-3). He saw that the pattern would continue; to find the dominoes with zero to four spots he needed to add five more dominoes. They repeated this pattern to find the specific case that there were 28 dominoes with from zero to six white spots.

This iterative pattern did not help them write the symbolic rule. It was when they organized the data into a table and studied constant differences that they found something in the pattern that could help with the rule. This was noticed while they completed the task.

> **Zachary:** There is kind of a pattern, like that changes, three, then four. Remember? So that would be fifteen, the next would be twenty-one. Is that what we got?

They used the 28 dominoes they found earlier for zero to six to confirm the pattern of differences.

Zachary and Todd collected a lot of information about the pattern prior to writing their symbolic rule. During the interview they explained that since the first di

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the first difference increased by a constant amount of one, they knew the pattern was quadratic.

AK: How many of these did you have to draw before you could actually continue the table?

Todd: Probably the first three, after it went up by three, then four, then five.

AK: Then that was enough for you to say [what]?

Todd: The second difference was one.

AK: What does that tell you, that the second difference is one?

Todd: That, uhh, it's a quadratic.

AK: Okay, what does that mean?

Todd: Umm, the graph makes a parabola and X is multiplied by itself or a quantity of X is multiplied by itself.

They found one as the constant term of the rule by extending their table back to find that there was one domino possible with up to zero spots. They seemed to try to fit their understanding of constant terms in linear situations to constant terms in quadratic situations. During their investigation they tried potential rules that were quadratic and had a constant term of one.

Zachary: The starting point would be one, so zero zero would be one.

Todd: So, its got to be something N plus one. N squared divided by two plus one? N squared divided by three plus one?

Zachary: Maybe minus one? N [pause] N minus two times N minus?

Todd: You have to add one.

These trials did not result in an appropriate symbolic rule.

The pair considered that they might need to divide by two because of the way they counted the dominoes and crossed off the doubles. They explained during our interview.

AK: Okay, what was your strategy for this one?

Todd: First we tried to make a chart to find all possible ways...

AK: Okay, and then what did you do?

Todd: Took out doubles.

AK: How did that help you with the problem?

Todd: Kind of found out that in the equation, that we will probably needed to divide our n by two and then add one.

They used all of this information to make some trial guesses for a rule. Zachary started to consider the consecutive integers when he noticed for the 10 dominoes with zero to three white spots, or "1 + 2 + 3 + 4." After some calculations, he arrived at $(n + 2) \div 2 \times n$, but realized his answers were off, so he adjusted that expression and wrote his final rule.

Todd: Did you get it?

Zachary: Yea, I think I might, [n plus two] divided by two times N. That does not work.

Todd: That work?

Zachary: It doesn't work for that, but it gets the answer before it, so [pause] you have to add N plus one. [pause] Let's try that out.

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Todd: It works for ten.

When the adjusted rule satisfied the case for 10 spots, the pair recorded their rule of $(n + 1) \div 2 \times n + n + 1$.

Toothpicks

Zachary and Todd also saw this as an algebraic task. They continued with their numeric strategy of making and studying tables to address the questions in this task.

Using a Table

The pair began the problem by counting the toothpicks and recording these numbers above the shapes. They extended the pattern by drawing a sketch of figure five. They recorded the perimeter toothpicks they counted in the table. The pair used pattern of the increase of four they studied in the table to complete it and find 24 toothpicks for the perimeter of the specific case for figure 6.[3]

Even though their strategy of looking at differences was not apparent in their written work for *Toothpicks* they demonstrated that they recognized when the first difference was constant by referring to the increase when

[3] Question 2 asked about the total number of toothpicks of figure six, while Zachary and Todd recorded 24, the number of toothpicks in the perimeter of figure 6. I believe that this pair would have likely been able to find the total number for figure 6 based on the table they made and the pattern they noted for the total number of toothpicks.

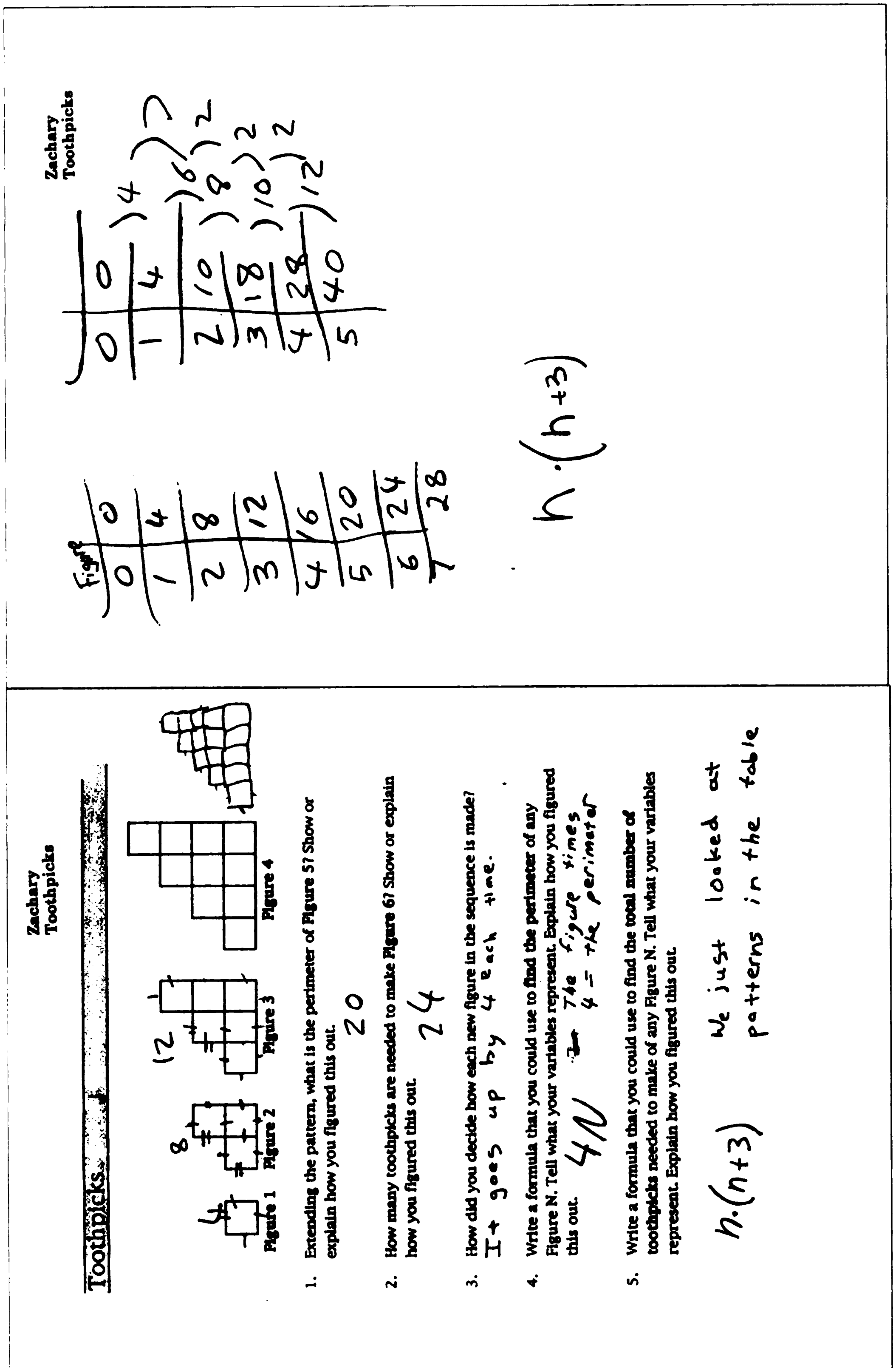

Figure 21: Zachary's work on *Toothpicks*

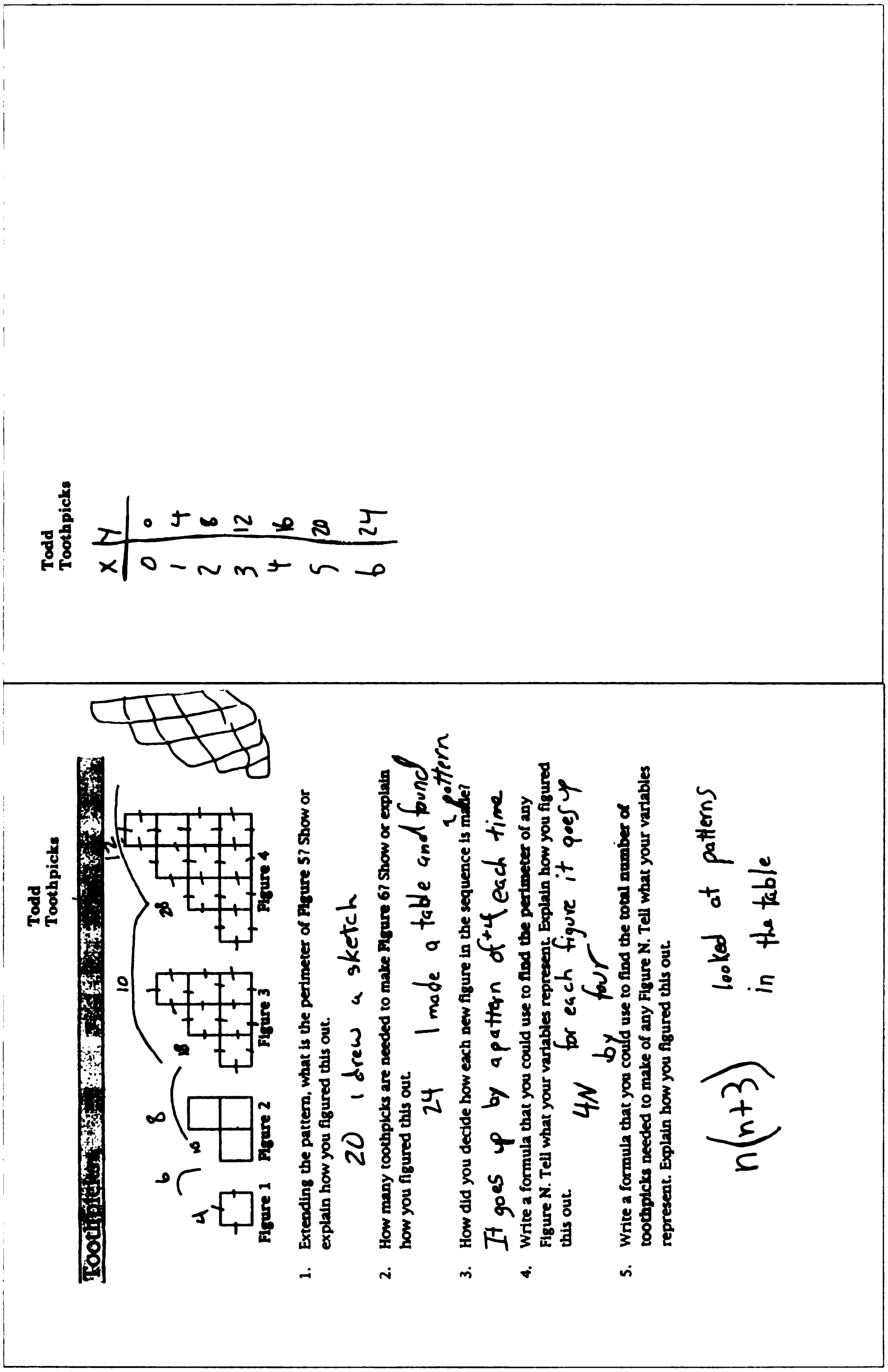

Figure 22: Todd's work on *Toothpicks*

counting the perimeter. Todd noted, "It goes up by four," while studying the numbers he recorded above the figures.

To find the total number of toothpicks, a quadratic pattern, the pair repeated their process of counting the toothpicks, recording the data, observing the increase, and extending the table. They stopped searching for patterns early in their solutions and continued the pattern they saw. Todd explained this during our interview.

AK: When do you know to stop looking for a pattern and start to generalize [to find the total number of toothpicks]?

Todd: We filled out five and six [in the table].

After Zachary and Todd determined that the perimeter of the toothpicks was a linear pattern, they extended their tables back to zero. In their tables they found corresponding values for Figure 0, I was curious how they thought about this, so I asked them during the interview.

AK: How did you find zero?

Zachary: If it goes up by four, then it goes down by four.

AK: Does that make sense with the figures?

Zachary: Yes.

Although it was not clear how this made sense based on the pattern or figures, in the pattern of the data (0,0) was reasonable. They used the y-intercept of zero and the increase of four to write a rule of $4n$ for the total number of toothpicks.

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The rule for the total number of toothpicks, $n{\cdot}(n{+}3)$, came fairly quickly to Zachary. He studied the table and wrote a rule that seemed to work. Zachary explained to Todd what he saw.

Zachary: So, N times N plus three.

Todd: Yeah. So how did you get the answer?

Zachary: I was looking to see what it was. It is kind of a coincidence that I found it. Like five times two equals ten, so then five is three more than two. Then I did the next one, six times three.

Todd: So you looked at the pattern in the table?

Zachary: Yeah.

Although they do not directly state it, Zachary began his trials with a rule in the factored form of "n times itself or a quantity of itself." They knew that the pattern was quadratic since they found a constant second difference. This helped them to make their initial guess that worked for the pattern of data.

Ben and Joe

During each interview I asked Ben and Joe what they felt was the mathematics in the problem. The pair saw the four tasks as algebra, or a blend of both algebra and geometry. They claimed that all of the tasks involved algebra because they were asked to find rules from patterns in the table

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tables. They added that two of the tasks also included geometry since they involved changing shapes. The strategy they used to study the patterns—numeric or geometric—seemed to determine their view of the mathematics in the problem.

The pair constructed tables in three out of the four tasks. In the problem where they did not make a table, they used a geometric approach to investigate the problem. In all situations, Ben and Joe successfully wrote symbolic rules.

Borders

In *Borders*, Ben described the task during our interview as "algebra a lot, maybe a little geometry. They made tables to investigate the problem, but verified their solution using a geometric approach.

Using a Table

Ben and Joe first counted the white border tiles and then recorded their data into a table. They constructed tables with data for figures 0 through 10. The pattern they observed helped them to extend the table to find the specific cases.

AK: When do you know to stop looking for a pattern and start to generalize?

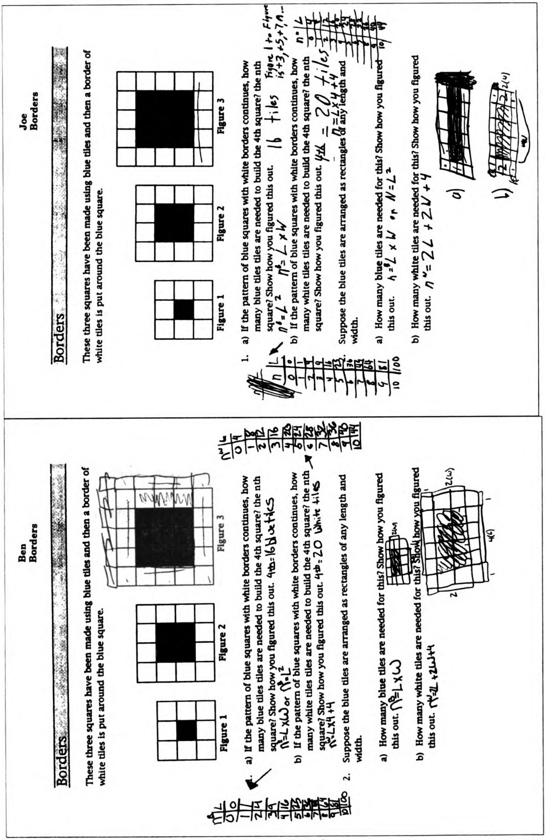
Ben: For one b, [we] just saw a pattern once we got to four.

From the table, they formalized the pattern.

Figur

Ben
Borders

Joe
Borders


Ben
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 4th=16 blue tiles
$n^B = L \times W$ or $n^B = L^2$
b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 4th=20 White tiles
$n^W = L \times 4 + 4$

n^B	L
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

n^W	L
0	4
1	8
2	12
3	16
4	20
5	24
6	28
7	32
8	36
9	40
10	44

2. Suppose the blue tiles are arranged as rectangles of any length and width.

a) How many blue tiles are needed for this? Show how you figured this out. $n^B = L \times W$

b) How many white tiles are needed for this? Show how you figured this out. $n^W = 2L + 2W + 4$

2(W) 2 4(L) 2(W)

Joe
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 16 tiles Figure 1 to Figure is +3, +5, +7, n...
$n^B = L^2$ $n^B = L \times W$
b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. 4th = 20 tiles
$n = L \times 4 + 4$

n	L
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

n^W	L
0	4
1	8
2	12
3	16
4	20
5	24
6	28
7	32
8	36
9	40
10	44

2. Suppose the blue tiles are arranged as rectangles of any length and width.

a) How many blue tiles are needed for this? Show how you figured this out. $n^B = L \times W$ or $N = L^2$

b) How many white tiles are needed for this? Show how you figured this out. $n^W = 2L + 2W + 4$

a)

b) 2(W)

Figure 23: Ben and Joe's work on *Borders*

AK: What did you use the table for?

Joe: To find the pattern. Pattern made it easier to find the rule.

AK: Did you find the table before the rules?

Ben: For one a, we found the equation first. For one b we found table first. L times four plus four.[4]

Joe described the constant increase of the first difference in his response to 1 a, when he wrote, "Figure 1 to Figure 2 is +3, +5, +7, +9..." (See Joe's work under 1 a).

In this problem the pair wrote the quadratic rule, L^2, for the blue interior tiles quickly. The linear white tile pattern also proved straightforward to generalize for the pair of students. They wrote $L \times 4 + 4$. The increase of 4 in the data was represented as the coefficient of L. They extended the tables back to find that 4 corresponded to a value of 0 for n and used this 4 as the constant term.

Geometric Approach

Although this pair used a numeric approach to investigate this problem, when asked to justify their results, they switched to a geometric approach.

AK: How would you justify your solution?

Joe: Sides are always the same length so it is squared.

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[4] 1. a) If the pattern of blue squares with white borders continues, how many **blue** tiles are needed to build the 4th square? the nth square? Show how you figured this out.
(Footnote continued on next page.)

AK: In one b, what is L?

Ben: It looks like the length of blue tiles was L. That way there are four on each side of the blue tiles, so that is L times four plus the four corners.

They pointed to figure 3 and traced lines to illustrate their thinking as illustrated in Figure 24.

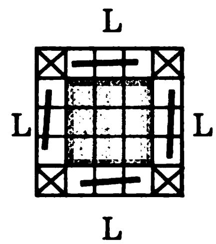

Figure 24: Ben's *Borders* diagram

Cutting

This was an unfamiliar problem for Ben and Joe since, "we had not worked on anything with an X power before." When I conducted my study the class had not yet completed the *CMP* unit on exponential growth, but during our interview these two students recalled a similar problem from the previous year.

AK: Have you ever done this problem before?

Joe: In science class last year when we stacked things up.

Ben: In biology class we were multiplying things.

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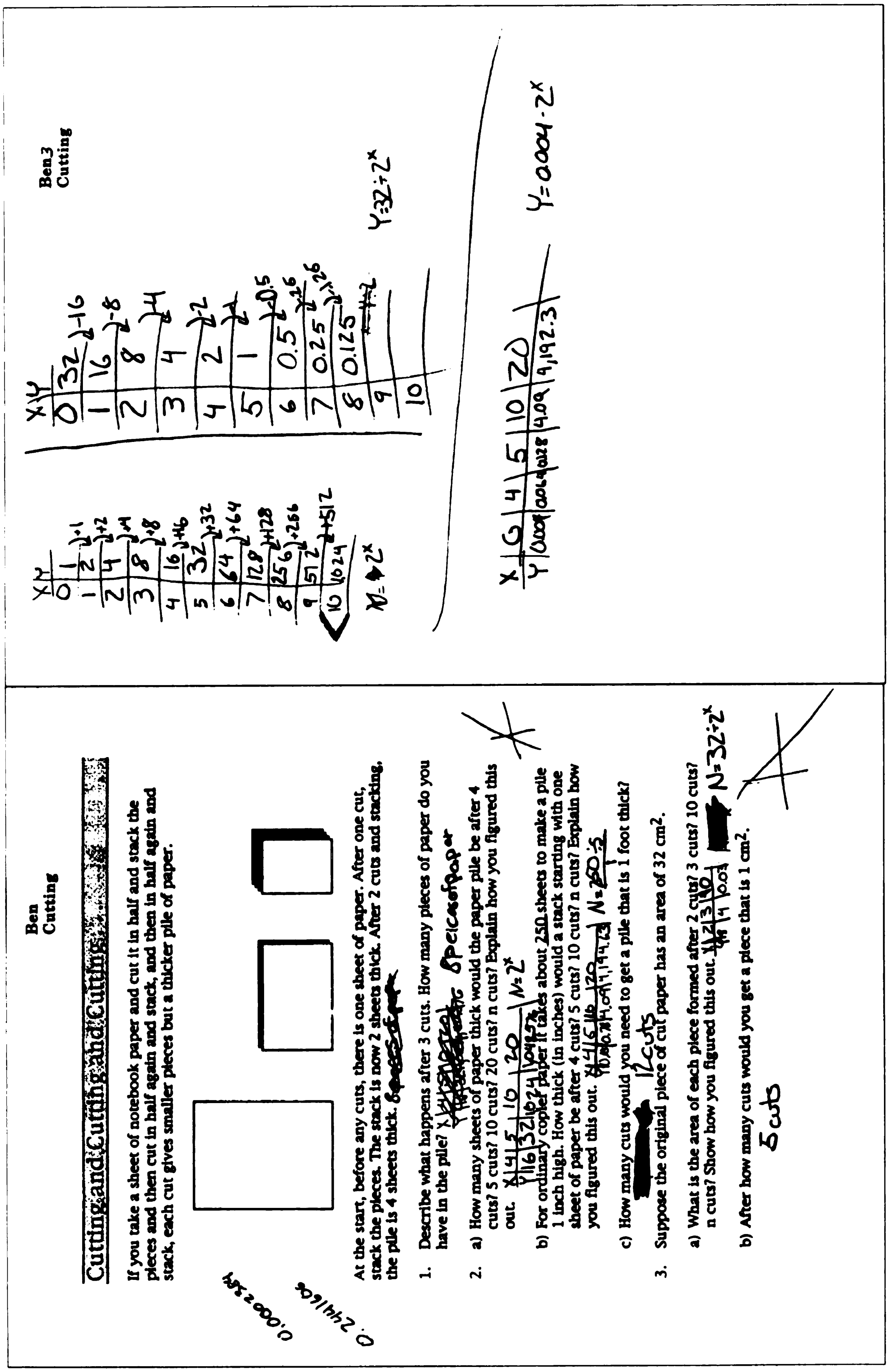
b) If the pattern of blue squares with white borders continues, how many **white** tiles are needed to build the 4th square? the nth square? Show how you figured this out.



Ben
Cutting

Cutting and Cutting and Cutting

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

0.00002384

0.2441606

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? 8 peices of paper

2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out.

X	4	5	10	20
Y	16	32	1024	1048576

$N = 2^x$

b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.

$N = 250 \div 5$

c) How many cuts would you need to get a pile that is 1 foot thick?

[redacted] 12 cuts

3. Suppose the original piece of cut paper has an area of 32 cm^2.

a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out.

X	2	3	10
Y	8	4	0.03

[redacted] $N = 32 \div 2^x$

b) After how many cuts would you get a piece that is 1 cm^2.

5 cuts

Ben3
Cutting

X	Y
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

+1, +2, +4, +8, +16, +32, +64, +128, +256, +512

$N = 2^x$

X	Y
0	32
1	16
2	8
3	4
4	2
5	1
6	0.5
7	0.25
8	0.125
9	
10	

−16, −8, −4, −2, −1, −0.5, −.25, −.125

$Y = 32 \div 2^x$

X	0	4	5	10	20
Y	0.004	0.064	0.128	4.09	4,192.3

$Y = 0.004 \cdot 2^x$

Figure 25: Ben's work on *Cutting*

Joe Cutting

Joe Cutting

Figu

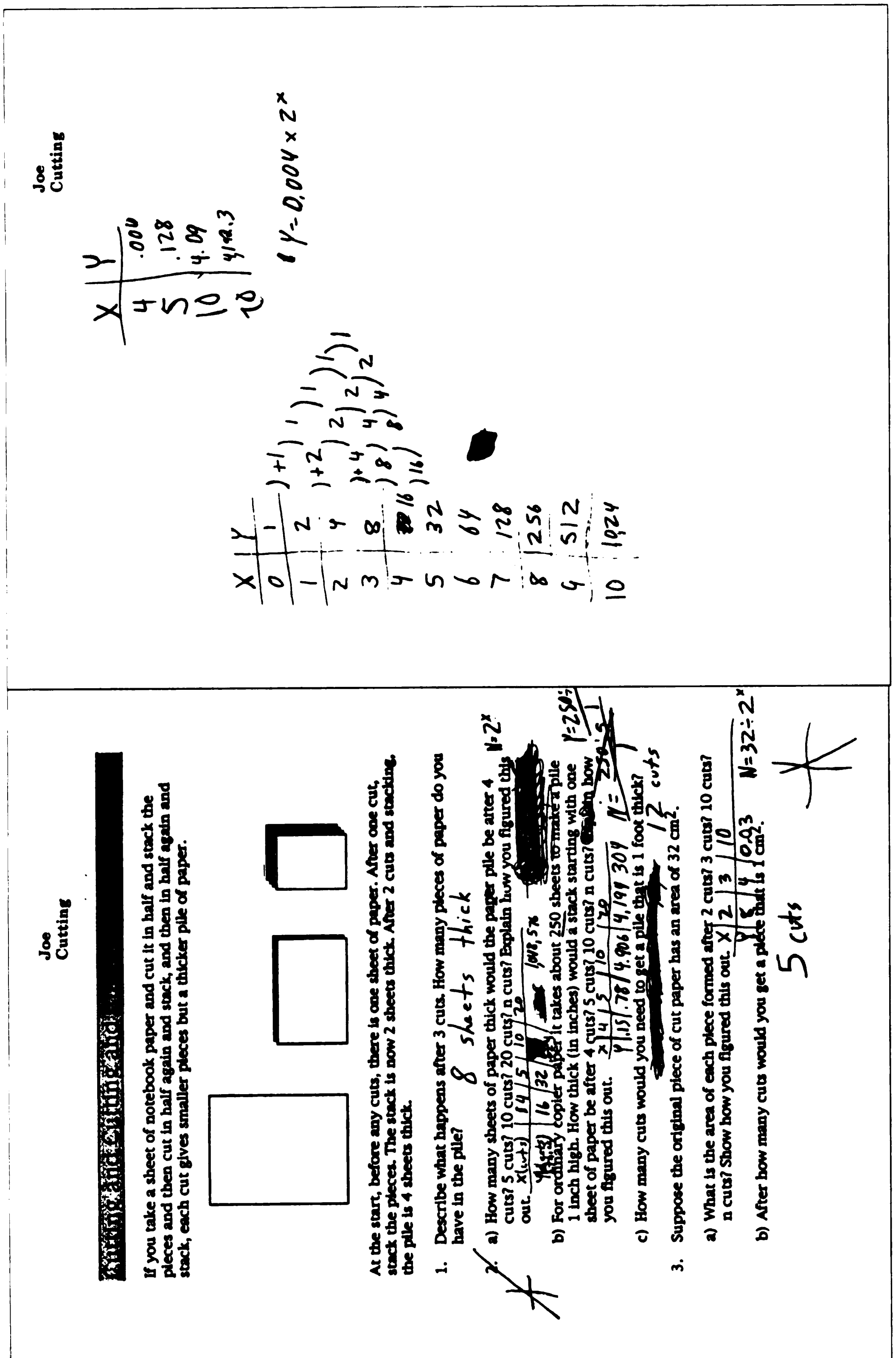


Figure 26: Joe's work on *Cutting*

Joe: We talked about cells doubling to reach the moon.

They referenced a problem from their seventh grade science class that had a similar mathematical structure of exponential growth.

They saw *Cutting* as "(a)lgebra in writing a rule and table. Just algebra. It doesn't seem to have geometry." Making and studying tables was a strategy that they eventually used.

AK: What strategy did you use for this problem?

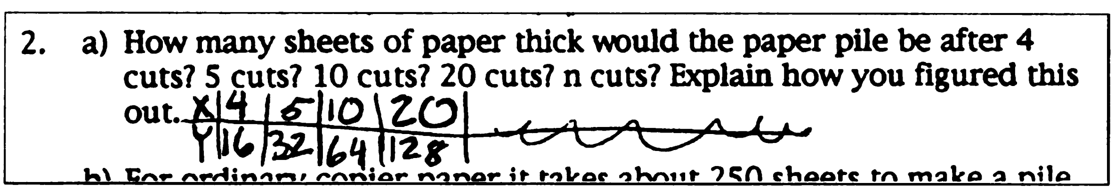
Ben: Guess and check

Joe: Made a table.

Ben: Yeah, and from the table we got the equation.

Using a Table

Initially Ben and Joe recorded their data in an incomplete table and made the table shown in Figure 27


2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out.

X	4	5	10	20
Y	16	32	64	128

b) For ordinary copier paper it takes about 250 sheets to make a pile

Figure 27: Ben's initial response to question 2 a in *Cutting*

They soon recognized their error and corrected it on a fresh sheet of paper.

Ben: It wouldn't be sixty-four, for ten. Do you have an eraser or a sheet of paper [to me]? It doubles from five to six, not five to ten. It wouldn't be sixty-four it would be something else.

Joe: Yeah, so double it.

They created new tables and recorded data that ranged from 1 to 10 cuts, including all numbers in between. They found that the number of sheets in a stack after 10 cuts was 1,024 and not the 64 they originally wrote.

The pair completed the tables based on the doubling pattern they observed in the data. They continued the pattern to correctly find the specific case of 1,048,576 sheets in the stack after 20 cuts. Next, they searched for a rule by first examining the differences. They tried several iterations of differences but did not find a constant.

The symbolic rule eluded Ben and Joe for some time. They found the doubling factor for the growth problem and the halving factor in the decay problem but had difficulty translating this into a symbolic rule.

Ben: I don't see where the two is. For the first, second cuts, its double, the number of cuts is two sheets and four is double two, for three you get eight sheets.

Once Ben and Joe described the pattern in an iterative form, they searched for a closed form of the rule. Joe began to guess.

Joe: Okay, X times Y times two?

Ben: Times? No, you can't have X and Y in the same. You are trying to find Y. This right here is Y [points to sheets thick]. You don't know these, you are trying to get these [sheets thick] from X [cuts].

Ben seemed to have a clear understanding of how the equation involved X (the number of cuts) and Y (the number of sheets thick) as independent and dependent variables. They eventually wrote $N = 2^x$. Ben credits, "I was just thinking and it popped into my head." It remains a puzzle how they found their rule. The pair did not articulate any further how they found it, nor is there evidence in any of the videotape to suggest anything else.

Dominoes

Ben and Joe saw Dominoes as an algebraic task since they were asked to find a rule. Making and studying a table was the strategy they relied on for algebraic tasks.

Using a Table

Their investigation began with the pair sketching dominoes. They counted the number sketched and recorded the data in a table. They followed the familiar strategy of recognizing a pattern, generating new data, answering specific cases, and classifying the pattern.

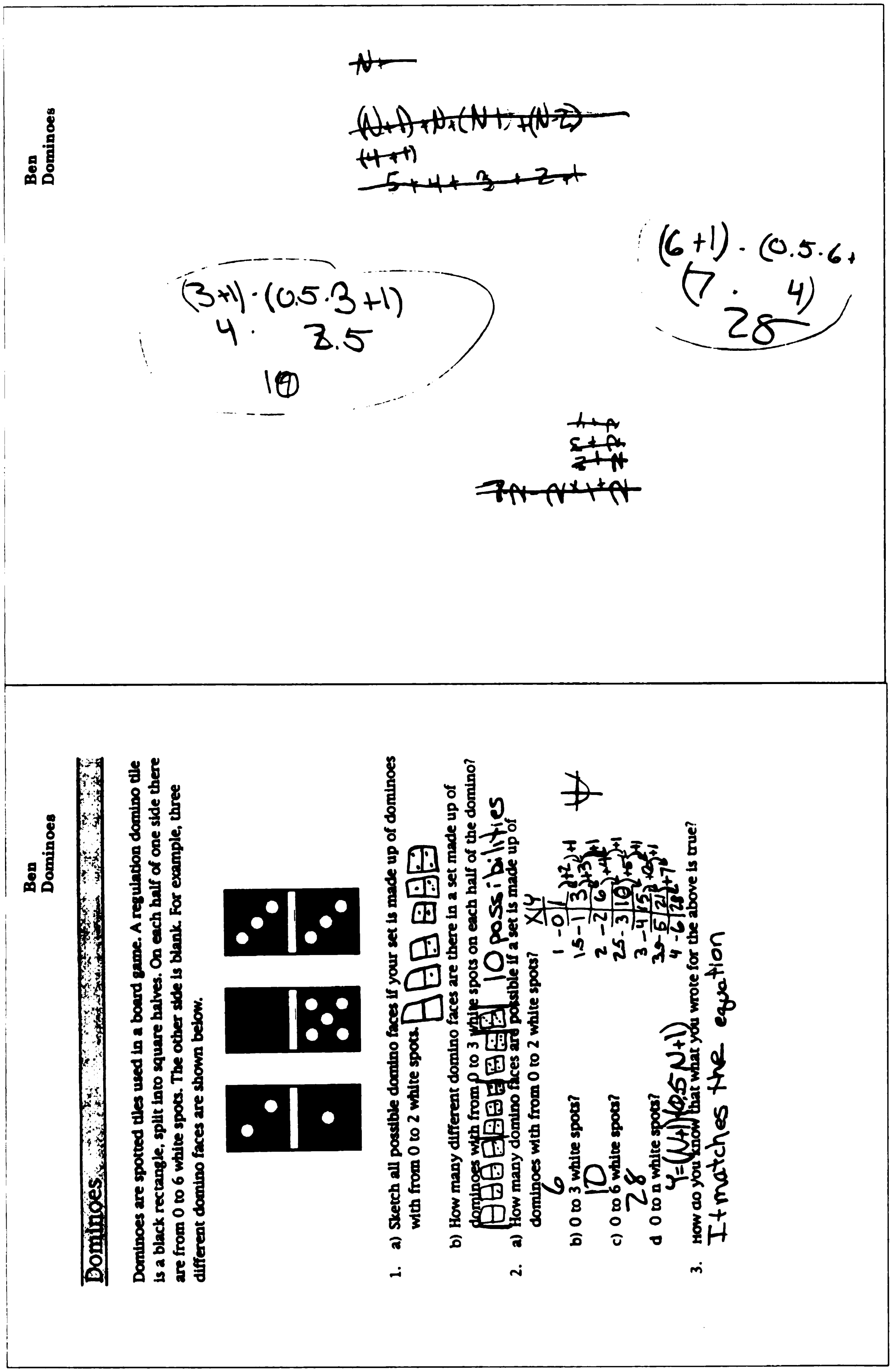
They recognized the pattern of increase in the first difference and used that to help them complete the table.

AK: When did you know when to stop looking for a pattern?

Ben: We filled in the table from one, two, and three and then looked for constant differences.

AK: For zero to six spots? How did you find that one?

Figu



Ben
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.
   b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?
   10 possibilities
2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?
   6
   b) 0 to 3 white spots?
   10
   c) 0 to 6 white spots?
   28
   d 0 to n white spots?
   $Y=(N+1)(0.5N+1)$

	X	Y
1	0	1
1.5	1	3
2	2	6
2.5	3	10
3	4	15
3.5	5	21
4	6	28

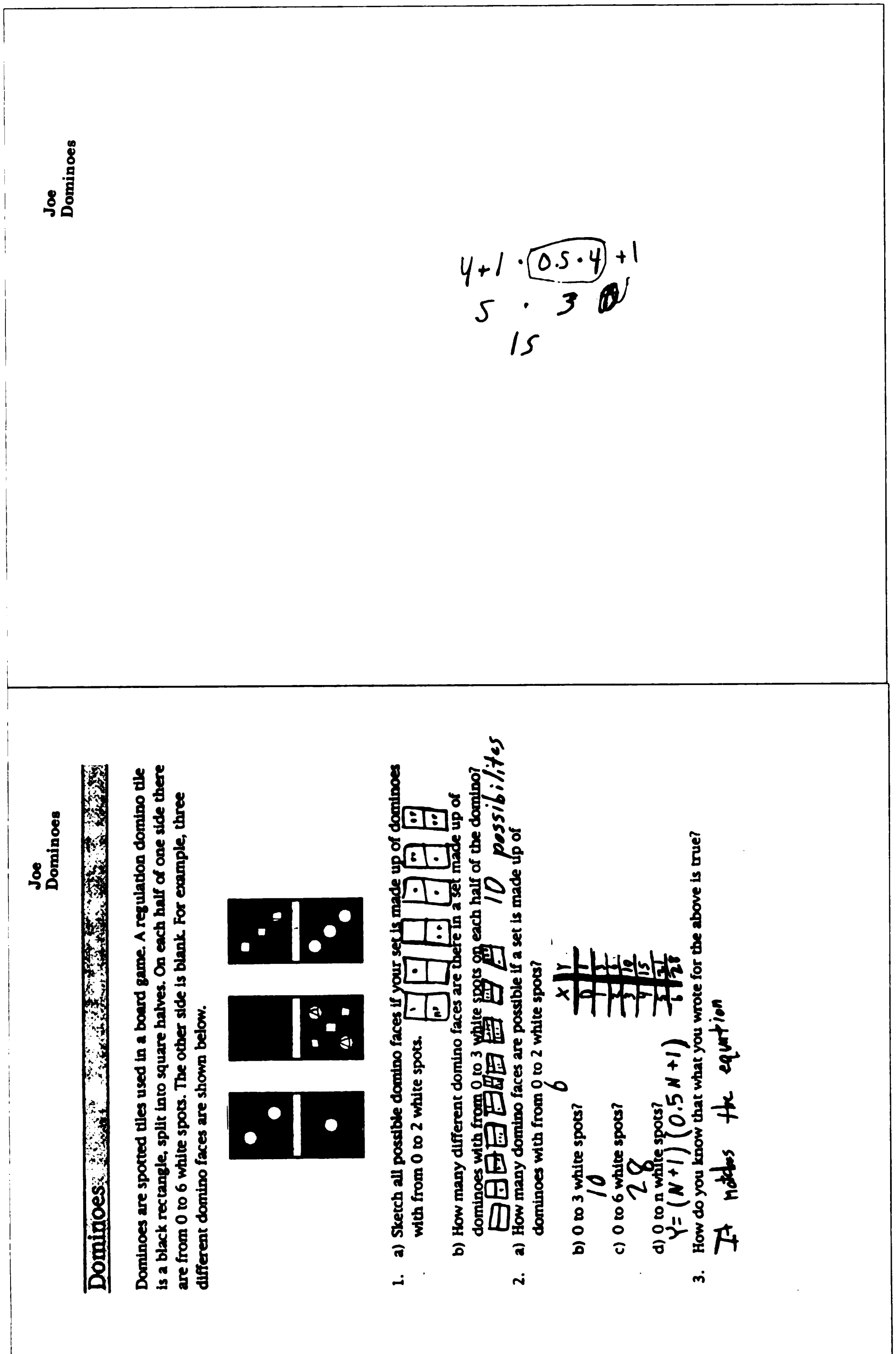
3. How do you know that what you wrote for the above is true?
   It matches the equation

Ben
Dominoes

$(3+1)\cdot(0.5\cdot 3+1)$
$4 \cdot 3.5$
10

$(6+1)\cdot(0.5\cdot 6+$
$(7 \cdot 4)$
28

Figure 28: Ben's work on *Dominoes*



Joe
Dominoes

$4+1 \cdot \boxed{0.5 \cdot 4} + 1$

$5 \cdot 3$

15

Joe
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

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2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots? 6

   b) 0 to 3 white spots? 10

   c) 0 to 6 white spots? 28

   d) 0 to n white spots? $Y=(N+1)(0.5N+1)$

X	Y
0	1
1	3
2	6
3	10
4	15
5	21
6	28

3. How do you know that what you wrote for the above is true? It matches the equation

Figure 29: Joe's work on *Dominoes*

Ben: We did the table, we saw it went up by two, three, four, five. So we put one there [extended the first difference back] and for 0 [dominoes with up to zero spots] it would be one then for the third one we put ten, then added five, for fifteen. Add one every time. So I found the second difference of plus one.

After the pair recognized the pattern, they then used that to find the 28 dominoes with from zero to six spots.

During the interview they explained how the constant difference helped them write the equation.

AK: Okay, let me ask you something different now. When you found that the second difference was one, what did that tell you about the equation?

Ben: I knew the n had to be multiplied by something.

They started to describe a factored form of a quadratic.

As an initial guess, they seemed to connect the numbers in the table to the sum of consecutive integers. They partitioned the 10 dominoes they drew for zero to three white spots into sections of four dominoes, three dominoes, two dominoes, and one domino in their written work (See Ben's work beneath 1 b). These 10 dominoes could be represented by the sum of consecutive integers, 4 + 3 + 2 + 1. They continued and represented the number of dominoes with from zero to four white spots as the sum of 5 consecutive integers (5 + 4 + 3 + 2 + 1) on the back of Ben's paper. After they wrote this sum, it prompted them to use N+1 × N + (N-1) as their initial guess. Ben adjusted this guess and tried N+2, N+3, N+4, as different variations.

It was unclear what the pair tried for their rules after these guesses, but they eventually wrote $y = (N+1)(0.5N+1)$. They verified their rule with the zero to three, zero to four, and zero to six cases by finding the 10, 15, and 28 dominoes respectively. (See the back of Ben and Joe's written responses.)

Toothpicks

Ben and Joe used a different approach in *Toothpicks*. They did not construct any tables as they did for the algebra tasks. Instead, they classified it as geometric and studied the pattern of the changing shapes.

Geometric Approach

The pair used the geometric patterns of change in the figures to help write rules. They explained their approach in our interview.

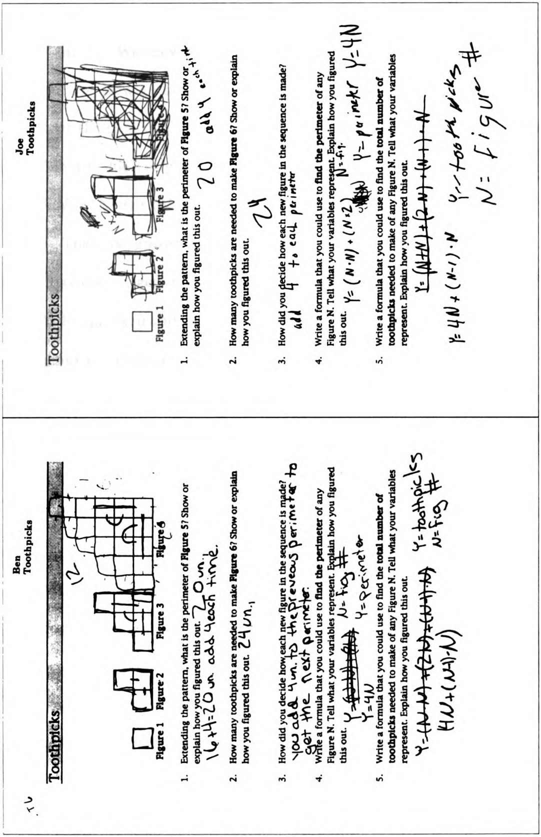
AK: Do you want to describe your strategy you used for this problem?

Ben: For one, we...

Joe: ...we drew pictures.

Ben: We added on to the first one, we drew extra steps and we did that for number two also.

Joe: We added on to the pictures.



Ben
Toothpicks

Toothpicks

Figure 1 Figure 2 Figure 3 Figure 4

1. Extending the pattern, what is the perimeter of **Figure 5**? Show or explain how you figured this out. 20 un. 16+4=20 un. add 4 each time.
2. How many toothpicks are needed to make **Figure 6**? Show or explain how you figured this out. 24 un.
3. How did you decide how each new figure in the sequence is made? you add 4 un. to the preveous perimeter to get the next perimeter.
4. Write a formula that you could use to **find the perimeter** of any Figure N. Tell what your variables represent. Explain how you figured this out. Y=4N N=fig # Y=perimeter
5. Write a formula that you could use to find the **total number of toothpicks** needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out. Y=~~(N·N)+(2N)+(N+1)·N~~ 4N+(N-1)·N Y=toothpicks N=fig #

Joe
Toothpicks

Toothpicks

Figure 1 Figure 2 Figure 3 Figure 4

1. Extending the pattern, what is the perimeter of **Figure 5**? Show or explain how you figured this out. 20 add 4 each time
2. How many toothpicks are needed to make **Figure 6**? Show or explain how you figured this out. 24
3. How did you decide how each new figure in the sequence is made? add 4 to each perimeter
4. Write a formula that you could use to **find the perimeter** of any Figure N. Tell what your variables represent. Explain how you figured this out. Y=(N·N)+(N×2) N=fig. Y=perimeter Y=4N
5. Write a formula that you could use to find the **total number of toothpicks** needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out. Y=~~(N+N)+(2N)+(N+1)·N~~ Y=4N+(N-1)·N Y=toothpicks N=figure #

Figure 30: Ben and Joe's work on *Toothpicks*

AK: What did you mean you added on?

Ben: We drew the extra steps

Ben and Joe drew additional toothpicks on the steps of figure four to represent the next two figures in the pattern (See Ben's Figure 4 in his work).

Ben and Joe's original rules of $Y = (N \cdot N) + (N \times 2)$ for the perimeter and $Y = (N \cdot N) + (N \times 2) + (N\text{-}1) \cdot N$ for the total number of toothpicks were not correct. They recognized their error during the interview, revised their rules and wrote correct generalizations.

Although their initial rules were incorrect, the reasoning they offered during our interview to find them was reasonable.

Ben: Umm, for four, we knew that each one had n times n to get the two straight sides [the base and right side], then Joe came up with this [the step portion on the left side] being two n. Because you've got n here [vertical steps] and n here [horizontal steps], so that you get two n.

AK: Do you want...?

Joe: Yeah, n times n would be this times this part of the perimeter [points to the base and right side]. Then n times two [the steps portion] plus this perimeter is two, and right here is two times two would be four. And it works over here too [See Joe's marks on Figure 3 in his work].

The pair had mistakenly used $n \times n$ (rather than $n + n$) to represent the number of toothpicks in the base and on the right hand side. After they recognized their mistake they correctly adjusted their rule.

Ben: We counted this one. Oh, for the n times n, it was this times this. For two n and there's an n and two n, three n there and four n. I think it would actually be four n.

AK: Why's that?

Ben: Because you've got one n here, two n, three n there, and four n there. It works the same here for one, two, three, so it would be Y is four n.

The pair found the total number of toothpicks by considering two separate categories: the perimeter toothpicks and the interior toothpicks. They revised and wrote $Y = 4N + (N\text{-}1) \cdot N$ for the total number of toothpicks.

It was unclear how they verified that $(N\text{-}1) \cdot N$ would always result in the interior toothpicks. Ben demonstrated on his paper how he could show 2 groups of 1 for figure 1, 3 groups of 2 for figure 2, and 4 groups of 3 for Figure 3 (See Ben's work for Figures 1-3). Even though this will be numerically true, it was unclear how to continue this pattern and mark 5 groups of 4 in Figure 4, or the general case of n+1 groups of n in Figure n, nor was it evident how this pair saw this aspect of their generalization continuing.

Julie and Dan

This pair of students used two different approaches in their investigation of the four tasks. Their strategy for each problem focused on their interpretation of the mathematical content. Although they thought that all of the tasks involved some algebra, they saw *Borders* and *Toothpicks* as primarily geometry.

When Julie and Dan felt the main mathematics of a problem was algebra, they constructed tables of numeric data to initiate their investigation. For the tasks they interpreted as geometric they used a strategy based on studying the shapes.

The pair found correct rules in *Borders* and *Toothpicks*. They found a rule in *Dominoes* for Case 1 where they counted 1-0 and 0-1 as two distinct dominoes, but were unsuccessful in Case 2 when they counted them as one unique domino. They were unable to write a rule for the exponential patterns in *Cutting*.

Borders

The pair seemed to find their rules in *Borders* by observing geometric characteristics of the figures in the task. Their solution primarily focused on geometry, but Julie also used the numeric values to search for a pattern in part of the problem.

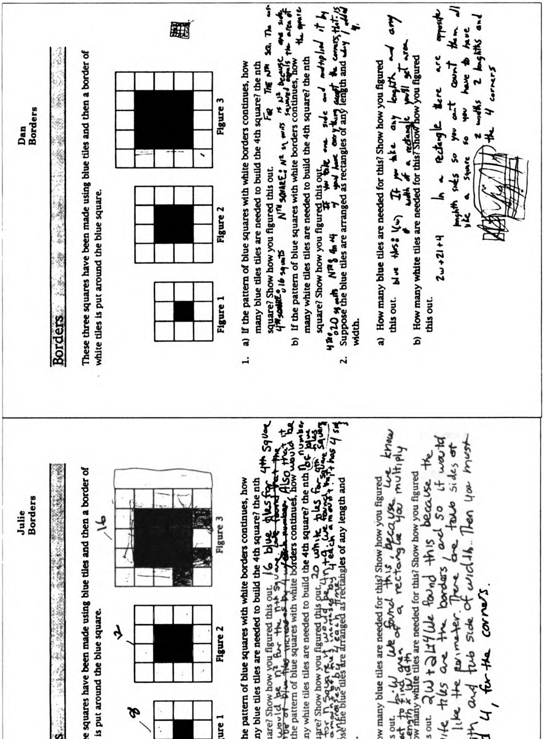


Figure 31: Julie and Dan's work on *Borders*

Geometric Approach

Julie and Dan found their rule of n^2 for the blue center tiles by studying the shape of the figures. Dan supported his rule and wrote, "For the N^{th} square the answer is N^2 because one side squared equals the area of the square." They recognized that the blue tiles formed a square with dimensions of n, where n was the figure number. The pair described their strategy during our interview.

AK: How did you decide what strategy to use?

Julie: Kind of looked at a pattern for these and that [points to Figure 2, Figure 1, and Figure 3], when two by two, one by one, three by three, so we figured four by four and got sixteen.

They observed the pattern in the shapes, extended it to answer questions about specific cases, and then finally wrote the generalization.

Julie and Dan continued with this strategy to find the generalization of the white border tiles. Dan's written work again offered evidence to support their rule. "If you take one side and multiplied it by four, youd [sic] have everything accept [sic] the corners, that's why I added four." They further explained their strategy during an interview.

AK: How would you justify your solution for the white tiles in the equation?

Julie: Yeah, four N plus four?

AK: Mm, hmm. Could you prove it to me?

Julie: See, this [traces the side of the blue square] would be N, four of these, then four corners.

Dan: You always have four corners left over.

Numeric Approach

Julie searched for a numeric pattern in at least one instance while solving *Borders*, as illustrated in her written work where she counted the number of white tiles and wrote the numbers, 8, 12, and 16, above the figures. While she worked on the task she noticed an increase of four in her data, "No, there's [pause] you need to find out how many borders. Each increases by four." She further noted the increase in her written work where she supported her rule of $4n + 4$ by claiming, "So for n square, it would be four n plus four. We found that the amount of tiles increases by four each amount." In her justification, she seemed to consider the numeric increase of four in her statement rather than the geometric change.

Cutting

Even though Julie and Dan did not find symbolic rules, they demonstrated understanding of the pattern and correctly responded to all of the specific cases.

Us

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calc

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Using a Table

The pair made and studied tables to investigate the task.

AK: How did you decide what strategy to use?

Dan: We tried to figure out simple ones, the first few. We tried to make a table on back. We tried guess and check.

Julie: We found a pattern for them, but we couldn't find the rule.

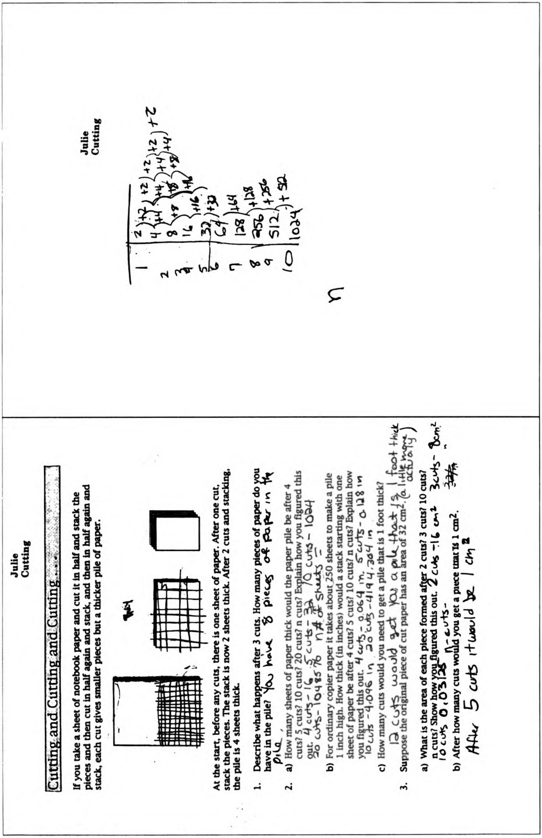
They used repeated multiplication on their calculators and recorded data in their table. As evidenced on Julie's paper, they correctly found the specific cases for the number of sheets of paper in a stack, the height of the stack in inches, and the area of the pieces of paper. As Julie passed her calculator over to Dan to see the display she said, "That's how many after ten cuts. Would you believe that?" She recorded 1,024 on her paper from the calculator display. Using the procedures they devised, Julie and Dan correctly answered all of the questions related to specific cases.

It was apparent by the marks on the tables that both students searched for constant differences while investigating the task in an attempt to describe the pattern.

Julie: What's it [the differences] going up by? Each time it increases. Maybe [pause] I have no idea how to do this. Plus two, plus sixteen, plus thirty-two. Then plus two, plus four.

Dan: That's what I mean.

Julie: It just keeps going out.


Julie
Cutting

Cutting and Cutting and Cutting . . .

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? You have 8 pieces of paper in the pile.
2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out. 4 cuts - 16 5 cuts - 32 10 cuts - 1024 20 cuts - 1048576 n # of sheets =
   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out. 4 cuts - 0.064 in. 5 cuts - 0.128 in 10 cuts - 4.096 in 20 cuts - 4194.304 in
   c) How many cuts would you need to get a pile that is 1 foot thick? 12 cuts would get you a pile that is 1 foot thick (a little more actually)
3. Suppose the original piece of cut paper has an area of 32 cm^2.
   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out. 2 cuts - 16 cm^2 3 cuts - 8 cm^2 10 cuts 0.03125 n-cuts- ~~32/n~~
   b) After how many cuts would you get a piece that is 1 cm^2. After 5 cuts it would be 1 cm^2

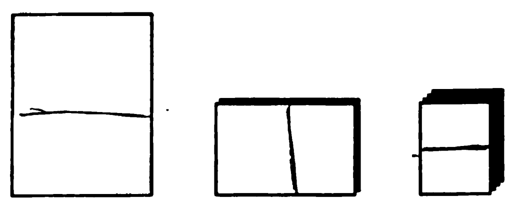
Julie
Cutting

1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

+2 +4 +8 +16 +32 +64 +128 +256 +512

n

Figure 32: Julie's work on *Cutting*



Dan
Cutting

Cutting and Cutting and Cutting.....

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? 8 The number of stacks seems to double
2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out. 4:16 5:32

   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.

   c) How many cuts would you need to get a pile that is 1 root thick?
3. Suppose the original piece of cut paper has an area of 32 cm^2.

   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out. $32/2^n$

   b) After how many cuts would you get a piece that is 1 cm^2.

Dan
Cutting

1 x2 = 2
2x2 = 4
3 = 1x2 = 8
4 = 16
5 = 1x3 = 32

1	2
2	4
3	8
4	16
5	32
6	64

+2 +2 +4 +4 +8 +16 +32

Figure 33: Dan's work on Cutting

They found up to the fifth difference, but it was not constant. Although they recognized that the column of differences kept repeating (2, 4, 8, 16, 32) this did not seem to help them write the rule. Dan hypothesized that this pattern would never produce a constant difference. Julie was not convinced of Dan's generalization.

Julie: I suppose if you make it really long, it will.

Dan: No, it won't.

During the interview the pair clarified their purpose of searching for constant differences.

Julie: We tried to find if it was a quadratic.

AK: What did you find?

Julie: It [the differences] kept going and increasing.

AK: Will it ever stop?

Julie: No.

Although they did not find a constant difference, Julie felt they did find a pattern in the data.

Julie: We found a pattern for them [the differences], but we couldn't find the rule.

AK: Okay, what was the pattern you found?

Julie: It went up by, it like increased by itself, like okay. Like it you had four, four cuts, sixteen. Five cuts would increase by sixteen.

But, this description of a pattern of increase did not help the pair write the symbolic rules.

Julie and Dan searched for a rule that would give them the number of sheets of paper in a stack from the number of cuts.

Julie: Want to get the next one?

Dan: I'm not sure. There has to be a rule...times a half? Find out what the heck six has to do with sixty-four?

Julie: That's what six has to do with sixty-four. It doesn't make any sense.

After we discussed the rule $y = 2^n$ for the number of sheets of paper in a stack after n cuts, I asked them to reconsider the other questions.

AK: Take a second to see if you can figure out a rule for three a.[5]

Julie: It would be thirty two divided by two to the n, I think.

Julie and Dan possessed a great deal of understanding about the task that was not first apparent by analyzing only their written responses. On first glance one might assume that Dan's contribution to the solution was minimal since his written work appears sparse. This was not the case; he continued to puzzle over the rule for the number of sheets in a stack, while Julie continued with other specific cases. He was determined to find the rule, even after I suggested that they could continue with the rest of the problem.

---

[5] 3. Suppose the original piece of cut paper has an area of 32 cm^2.

a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out.

Dominoes

Julie and Dan saw this task as algebra because of the rules, variables, and symbols they used to explore the problem. They held two different interpretations of *Dominoes*. In both situations it was evident that they used primarily numeric solutions to investigate the problems.

Their two interpretations differed in the way they counted the 1-0 and the 0-1 dominoes.

Table 14: Two Cases of *Dominoes*

Case 1:	Case 2:
Count the 0-1 and 1-0 as two different dominoes. Therefore four dominoes possible with from zero to one white spot (0-0, 0-1, 1-0, and 1-1).	Count the 0-1 and 1-0 dominoes as the same. Using this counting scheme, only three domino faces possible with from zero to one white spot (0-0, 0-1, and 1-1).

The labels Julie used on her tables in her written work to distinguish the two patterns were unclear. She seemed to reverse her labels when she called Case 1 (where the 0-1 and the 1-0 dominoes were classified as two distinct dominoes) as "same," and Case 2 (where the 0-1 and 1-0 dominoes are counted as one domino) "different." The pair successfully found a correct symbolic generalization for Case 1 but was unable to write a rule for Case 2.

Using a Table

Julie explained their data collection: "The tree diagrams helped us with like making our tables." Both students organized their data with tables for

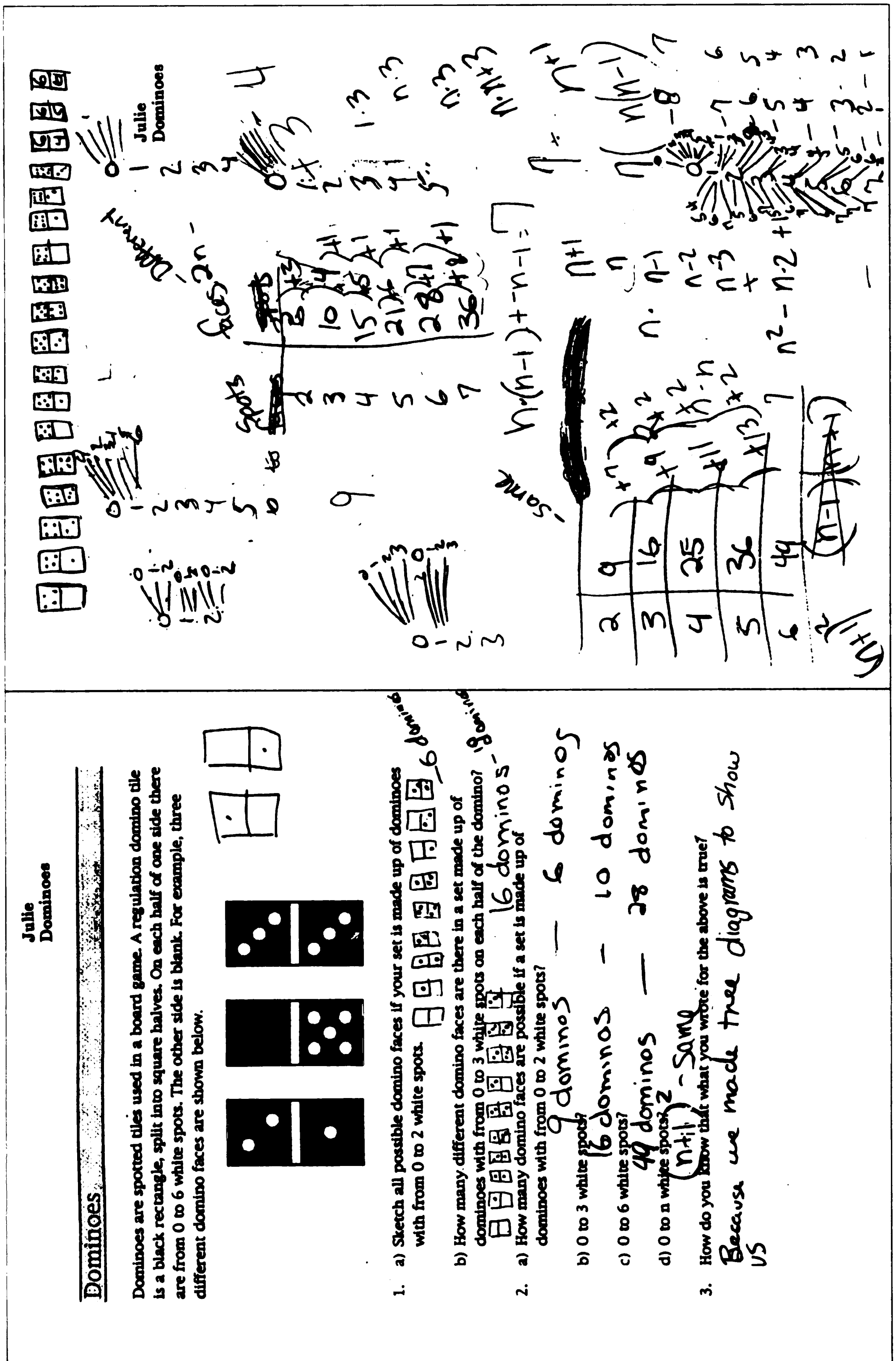
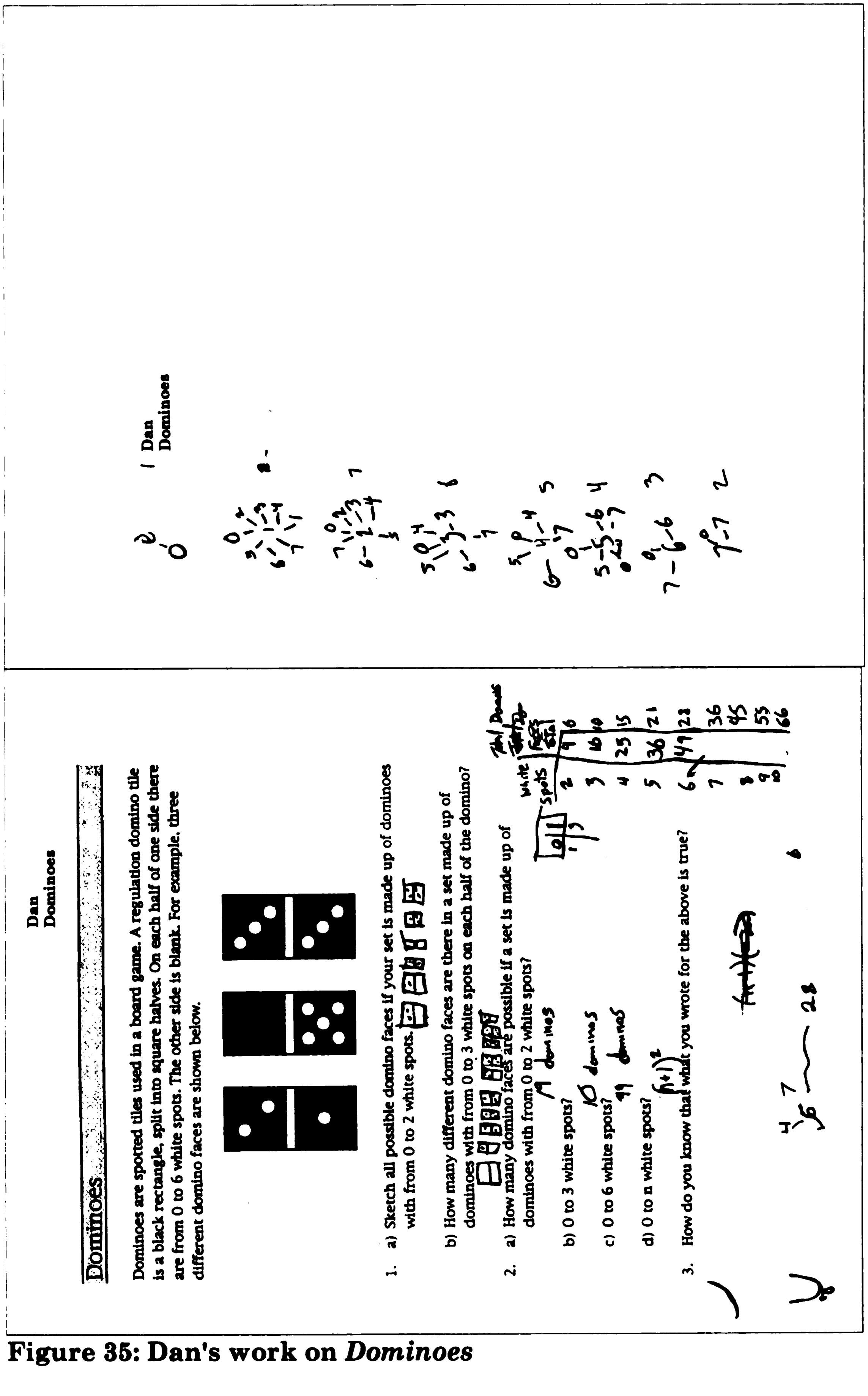


Figure 34: Julie's work on *Dominoes*



Dan
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.

b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?

b) 0 to 3 white spots?

c) 0 to 6 white spots?

d) 0 to n white spots?

3. How do you know that what you wrote for the above is true?

Dan
Dominoes

Figure 35: Dan's work on *Dominoes*

each case. In Julie's tables there was evidence that she searched for constant differences. In the interview the students clarified how the constant second differences informed them regarding the pattern.

AK: When the dominoes aren't counted as the same,[6] how did you fill in that table?

Julie: Well that we found a pattern, every single time, the first one it went up by four, and then five, six, seven, eight, nine so we found the pattern in the table, but not anything else.

AK: Does that pattern tell you anything?

Julie: It looked to me like a quadratic

AK: What does that mean that it is a quadratic?

Dan: I think that Ms. Howard told us that if something goes up by two degrees she called it that it was possibly a quadratic.

They seemed to recall from their mathematics class that a constant second difference meant that the pattern was quadratic; it was unclear what this meant to Julie and Dan.

They successfully found a quadratic rule by studying the table for Case 1, $(n + 1)^2$ where n was the maximum number of white spots.

AK: How did you find N plus one squared [rule for Case 1]?

Julie: Well, we know that four plus one is five, obviously, and then like five times five is twenty five. Five plus one is six, six

---

[6] This is using Julie's labels, but it refers to Case 2.

times six. I don't know. We just kind of saw the pattern in the table.

This situation seemed to be a straight forward rule for them to find.

Although Julie and Dan did not find a generalization for Case 2, they possessed some additional knowledge about the relationship. First, they knew that the equation was quadratic while they worked.

Julie: This is a quadratic right?

Dan: I don't know.

Julie: It is because it takes two degrees to get a constant.

Dan: Oh, yeah, but what is that going to help.

Julie: It helps with the equation. Umm, I don't know its not like one of the real obvious ones.

This seemed to inform them about a possible form of the rule. Although it was still not evident what quadratic meant to this pair, they recorded $n(n + 3)$ and $n(n - 1)$ as reasonable quadratic guesses for the pattern. It was not clear why they tried these two expressions. They might have started with them because they believed quadratics could be written in the factored form of $n(n + x)$. Besides being quadratic, they also knew that the rule had a constant term of one. They explained.

Dan: For, zero it is one so, we have to find like some way to get one from zero.

Julie: So, then you would have to add one. So, it has to be something plus one.

They combined these and tried $n^2 - 2n + 1$, but continued to be unsuccessful in their search.

Additionally, they referenced the shape of the graph. Dan noted during the interview that the graph held the shape of a parabola and sketched one on his paper (See the lower left corner of Dan's work).

Toothpicks

The pair easily found a generalization based on the numeric data for the perimeter pattern prior to making a table for the total number of toothpicks. Julie and Dan systematically collected data, searched for patterns, and wrote symbolic rules for both patterns in this task. Making tables played a partial role in their solutions. They only used tables to investigate the total number of toothpicks pattern.

Numeric Approach

In the perimeter pattern the pair counted toothpicks, recorded the data on the shapes, and readily recognized the increase of four. While they worked, Julie described how she used that pattern to find the perimeter of Figure 5, "They go up by four each time, so the perimeter of five would be twenty." Once they had the increase of four, they readily wrote, $y = 4n$.

Using a Table

While working, they easily found the pattern in the perimeter number of toothpicks.

Julie: The increase?

Dan: It's four.

Julie: Four…N.

Dan: N equals the base.

Julie: I got Y equals four N.

Dan: Okay.

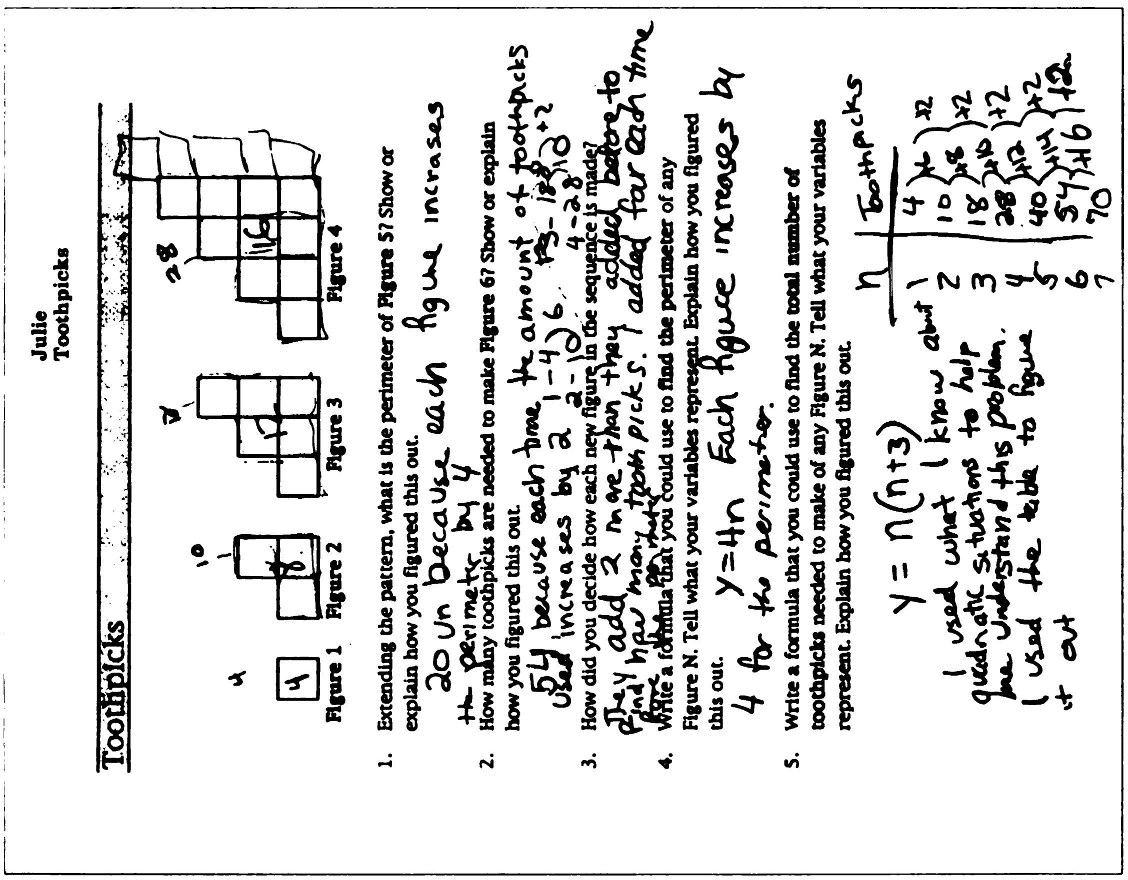
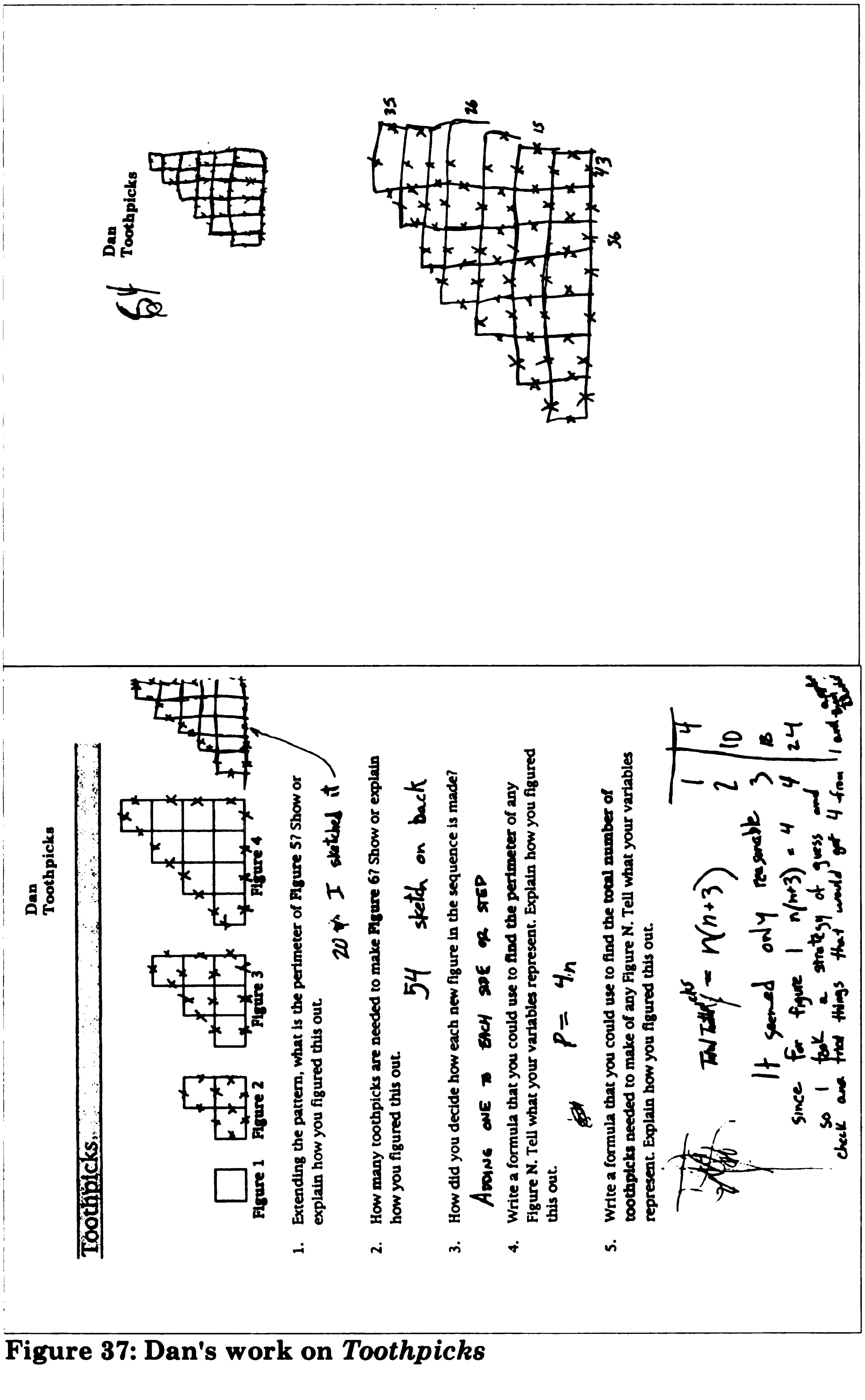


Figure 36: Julie's work on *Toothpicks*



Dan
Toothpicks

Toothpicks

Figure 1 Figure 2 Figure 3 Figure 4

1. Extending the pattern, what is the perimeter of Figure 5? Show or explain how you figured this out.

20 I sketched it

2. How many toothpicks are needed to make Figure 6? Show or explain how you figured this out.

54 sketch on back

3. How did you decide how each new figure in the sequence is made?

Adding one to each side or step

4. Write a formula that you could use to find the perimeter of any Figure N. Tell what your variables represent. Explain how you figured this out.

P = 4n

5. Write a formula that you could use to find the total number of toothpicks needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out.

Total Toothpicks = n(n+3)

It seemed only resonable

Since for figure 1 n(n+3) = 4

so I took a strategy of guess and check and tried things that would get 4 from 1 and

1	4
2	10
3	18
4	24

Dan
Toothpicks

64

35
26
15
43
36

Figure 37: Dan's work on *Toothpicks*

They further described how they found the rule in the interview.

AK: The perimeters, four, eight, twelve, and sixteen, you found that by counting?

Julie: Yeah.

AK: What was the pattern?

Julie: It increased by four each time.

AK: How does that help you write the rule?

Julie: Because I know one times four, two times four, I just figured N times four gives perimeter.

Once they collected the data for the total number of toothpicks they recorded it in a table, searched for a rule, and easily recognized that the second difference was constant.

AK: What about total number of toothpicks? You sketched and counted?

Dan: [nods]

AK: Julie?

Julie: I used my pattern. First, I made a table and found the amount of increase [second difference] by two, then I could find fifty-four.

To verify how Julie extended the pattern in the table, Dan sketched figure 6 on the back of his paper.

Once they had the table complete, the rule came quickly to Julie as she explained during our interview.

AK: What about total number of toothpicks?

Julie: I don't know, I just like looked at my table and it [the rule] just popped into my head. I guess I don't know. I looked at it [the table] and I tried it [the rule] and it was right.

AK: How did you know it was right?

Julie: I like checked with my table with different variables and it worked with all of them. It just kind of popped in here. I just kind of tried it.

AK: Just sort of tried some for this one then?

Julie: No that was the first one I got.

Julie's first trial in their guess and adjust strategy yielded an appropriate rule $n(n + 3)$. Julie and Dan verified their rule by checking it against any ordered pair in the table. When Dan saw that it yielded the appropriate value, it convinced him that their rule was reasonable.

Sara and Ryan

According to Sara and Ryan, all of the tasks were algebra because they "looked for patterns" and "made tables". They added that *Toothpicks* also involved some geometry since it involved shapes. In three out of their four solutions they studied numeric data, but in only two of these three tasks did they use tables. Additionally, in some cases they created graphs to help explore the patterns. The pair only found rules in two out of the four tasks, but they demonstrated significant understanding of the patterns in those instances where they were unable to generalize symbolically.

Borders

Sara and Ryan used a geometric approach to investigate *Borders* by studying the shapes of the figures and they way they changed. The pair made an interesting choice of variables in their generalizations. The question asked them to generalize about the n^{th} square, where n was the figure number, but Sara and Ryan considered the length and width as their variables.

Geometric Approach

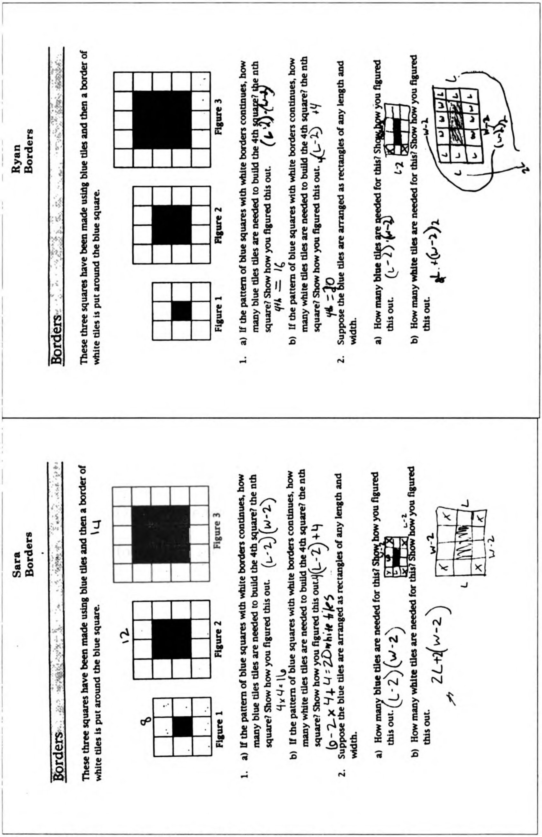
The pair used the drawings of the figures in the task to find the generalization for the number of blue interior tiles.

AK: What strategy did you use for this problem?

Ryan: What do you mean?

AK: When you sat down and I gave you this paper, what did you do?

Ryan: Umm, we started with the perimeters and know we needed to get minus two from the center area. So we kind of looked at [pause] there's two extra on every side, so we took two away,



Sara
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square. 14

8 12

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. $(L-2)(W-2)$
   $4 \times 4 = 16$
   b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. $4(L-2)+4$
   $6-2 \times 4 + 4 = 20$ white tiles
2. Suppose the blue tiles are arranged as rectangles of any length and width.
   a) How many blue tiles are needed for this? Show how you figured this out. $(L-2)(W-2)$
   b) How many white tiles are needed for this? Show how you figured this out. $2L+2(W-2)$

Ryan
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. $(L-2)\cdot(W-2)$
   4th = 16
   b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. $4(L-2)+4$
   4th = 20
2. Suppose the blue tiles are arranged as rectangles of any length and width.
   a) How many blue tiles are needed for this? Show how you figured this out. $(L-2)\cdot(W-2)$
   b) How many white tiles are needed for this? Show how you figured this out. $2L+(W-2)2$

Figure 38: Sara and Ryan's work on *Borders*

we go length minus two. To get the area of anything length times width.

AK: Did you do this by the pictures?

Ryan: Yeah.

They wrote the rule, $(L-2)\cdot(W-2)$, for the number of blue interior tiles using L and W as the dimensions of the figure. Using similar notation, Sara and Ryan wrote an accurate rule to find the number of white tiles, $4(L-2)+4$. In this expression they recognized the regularity of the squares, and used one variable L to represent the length of all four sides. During our interview they demonstrated their rule.

AK: What about one b?[7]

Ryan: L minus two times four plus four.

AK: How did you get that?

Sara: Oh, like we took away these corners, length minus two, so we took away these two corners times four, so you'd be like multiplying these ones [white tile sides minus the corners], then you add the four corners.

Ryan: Yeah. Take away these areas [the four corners], then times by four, then the four corners.

[7] Question one b refers to the white border tiles. It states:
"If the pattern of blue squares with white borders continues, how many **white** tiles are needed to build the 4th square? the nth square? Show how you figured this out."

Ryan traced the drawing for Figure 3 in the task as illustrated in Figure 39.

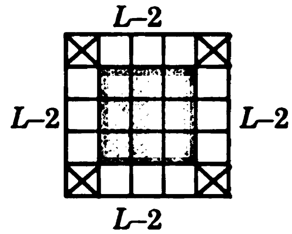


Figure 39: Ryan's tracing in *Borders*

He covered the four corners; each side had ($L - 2$) tiles remaining. Since there were four sides, they multiplied $L - 2$ by 4. To find the total number of white tiles they added the four corner tiles back and wrote $4(L - 2) + 4$ as their rule. Their geometric approach of studying characteristics of the shapes helped them write their expressions.

Much was left unclear in the students' written work about their choice of variables. In the first pattern for the blue tiles they did not state that L and W represented the dimensions of the figures, although it can be inferred from their written work. Nor, did they consider the specific case of a square where $L = W$. In the rule they wrote for the number of white tiles, they specifically used squares with length, L, for all four sides. Their rules were reasonable with the dimensions of L and W, but the pair never related these lengths back to the figure number; the students never clarified that the side of the squares, L, was two more than the figure number, n. They explained in an interview that they chose dimensions because that seemed the important characteristics of each shape to them.

Sara's written work offers some evidence of her understanding of a specific case. When she wrote, "4(L - 2) + 4,   6 - 2 × 4 + 4 = 20 white tiles" she recorded that the length of figure 4 was six (See Sara's work under 1 b). Sara successfully used their rule to support finding twenty white tiles. It still remains unclear if they could adapt this to find the length of figure n for the general case.

Cutting

Ryan and Sara approached *Cutting* by constructing tables. They explained in an interview.

AK: Okay, umm, how did you decide what strategy to use?

Ryan: Looked for patterns.

AK: How did you do that?

Sara: Made a table.

Ryan: Yeah.

...

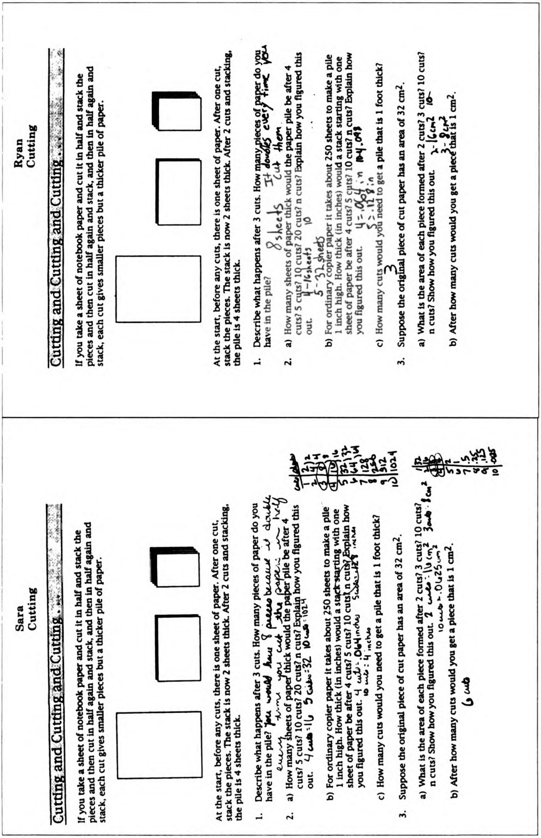
AK: What was an advantage to using the table?

Sara: To see a pattern.

AK: What do you mean by that?

Sara: Get how much each of the variables increased by.

They felt confident using this approach when they were in an unfamiliar problem context.



Sara
Cutting

Cutting and Cutting and Cutting . . .

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? you would have 8 pieces because it double every time you cut the paper in half
2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out. 4 cuts = 16 5 cuts = 32 10 cuts = 1024

   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a ~~stack starting~~ with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out. 4 cuts = .064 inches 5 cuts = .128 inches 10 cuts = 4 inches

   c) How many cuts would you need to get a pile that is 1 foot thick?

cuts	sheets
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

3. Suppose the original piece of cut paper has an area of 32 cm^2.

   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out. 2 cuts = 16 cm^2 3 cuts = 8 cm^2 10 cuts = .0625 cm^2

   b) After how many cuts would you get a piece that is 1 cm^2.
   6 cuts

1	32
2	16
3	8
4	4
5	2
6	1
7	.5
8	.25
9	.125
10	.0625

Ryan
Cutting

Cutting and Cutting and Cutting . . .

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile? 8 sheets It doubles every time you cut them.
2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out. 4 - 16 sheets 5 - 32 sheets 10

   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out. 4 = .064 in 5 = .128 in 10 - 4.098

   c) How many cuts would you need to get a pile that is 1 foot thick? 3
3. Suppose the original piece of cut paper has an area of 32 cm^2.

   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out. 2 - 16 cm^2 3 - 8 cm^2 10 -

   b) After how many cuts would you get a piece that is 1 cm^2.

Figure 40: Sara and Ryan's work on *Cutting*

Using a Table

The pair seemed to understand the doubling nature in finding the number of sheets in a stack after n cuts. They found answers for most of the specific cases requested in the problem by doubling the previous number of sheets.

Sara: Maybe it goes up by eight and then sixteen. Yeah, it does. Do you see what I am saying?

Ryan: Yeah, sort of.

Sara: It's just thirty-two. You just double it. Double that, it goes up by sixty-four. But, I don't know how to find the equation.

Sara and Ryan successfully found the number of sheets in a stack up to 10 cuts and recorded this data in a table. They used this data to find the specific cases for the thickness of the stack. The pair worked hard to make sense of the situation. When they started question 2 b[8], Ryan suggested an answer while Sara was not clear how he found this. She pushed him to clarify his process for her until she finally understood what he intended. The following transcript illustrates their discussion.

Ryan: Okay, do you have that number, like six percent, I think.

Sara: Six percent?

---

[8] 2. b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.

Ryan: Yeah, okay, umm. Cause that's how many sheets. You get that, I guess it's like a fraction.

Sara: Do you get a percent?

Ryan: I don't know. I don't know, I just did a bunch of stuff, okay.

Sara: So if it's six percent, what is it six percent of? Six percent of one inch?

Ryan: Help me out.

Sara: I'm doing it. [Pause] Okay, point oh six inches. Would you just take one inch divided by that [points to 250 sheets].

Ryan: I tried that, it didn't work.

Sara: One inch divided by [pause] fifteen point six [250 sheets = 1 inch, Sara calculates 250 ÷ 16 = 15.6]?

Ryan: Explain how you figured it out.

Sara: I umm, isn't there sixteen sheets [in a stack after four cuts], this [250 sheets] is one inch. [Pause] And then divide that by one inch [16 ÷ 250]?

Ryan: Yeah, that makes sense.

Sara: It does?

Ryan: Yeah.

Sara: Point oh six four [pause] inches?

Ryan: Yeah.

Not only did the pair note the doubling in the number of sheets in a stack, they further described a doubling in the thickness of the stack.

Ryan: It [the thickness after five cuts] should have doubled, right?

Sara: Doubled? So, it would be like point one two eight [double the thickness after four cuts]?

Ryan: We've got to figure out a rule.

They had clearly described both patterns with verbal recursive rules that the pattern doubles, but the symbolic expressions continued to elude them.

Graphic Connection

Although the students were not asked to create graphs in the tasks, Sara and Ryan considered the nature of the shape of the graph while searching for the rules.

While the pair worked, Ryan described the growth and decay graphs, "Whatever it is, it would take, this [the number of sheets in a stack pattern] sort of rises, going in the positive direction and this [the area of the pieces] is going in the negative direction [slopes down]." In their search for generalizations, they studied all they knew about the pattern, including the shape of the graphs. They created representations of the graphs, and tried to use this to help find the generalizations. They could still not find the rules.

Dominoes

Sara and Ryan mainly considered the algebraic nature of *Dominoes* to search for a symbolic rule. Sara described to me the path they followed during their investigation, "I got this answer by drawing pictures, then I put it in the table, and then I found a pattern."

This pair did not find a symbolic rule. But they did demonstrate a deep understanding of the pattern when they discussed the shape of the graph.

Using a Table

Prior to recording the data in a table Sara and Ryan made sketches and counted the dominoes. This strategy of drawing or representing dominoes helped them to find the number of dominoes for specific cases. They discussed how to count the 0-1 and 1-0 dominoes and completed the task under the assumption that they would count them as one unique domino. Sara continued with her strategy of sketching dominoes to find the number of dominoes with up to six white spots. She began to systematically sketch dominoes (See the top three rows on the back of Sara's work). First, she made the doubles for zero through six, next she represented the dominoes that had a one, then two, and she continued up to six. After sketching all of these she realized that she had repeated some and she used an "X" to cross them off. This sketch was incomplete since she had forgotten the six additional dominoes that could be made with zero spots (0-1, 0-2, 0-3, 0-4, 0-5, and 0-6).

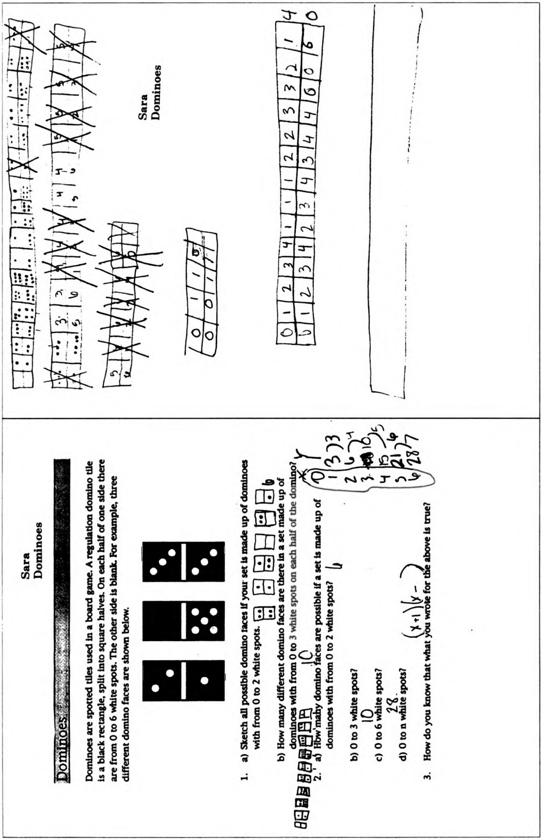


Figure 41: Sara's work on *Dominoes*

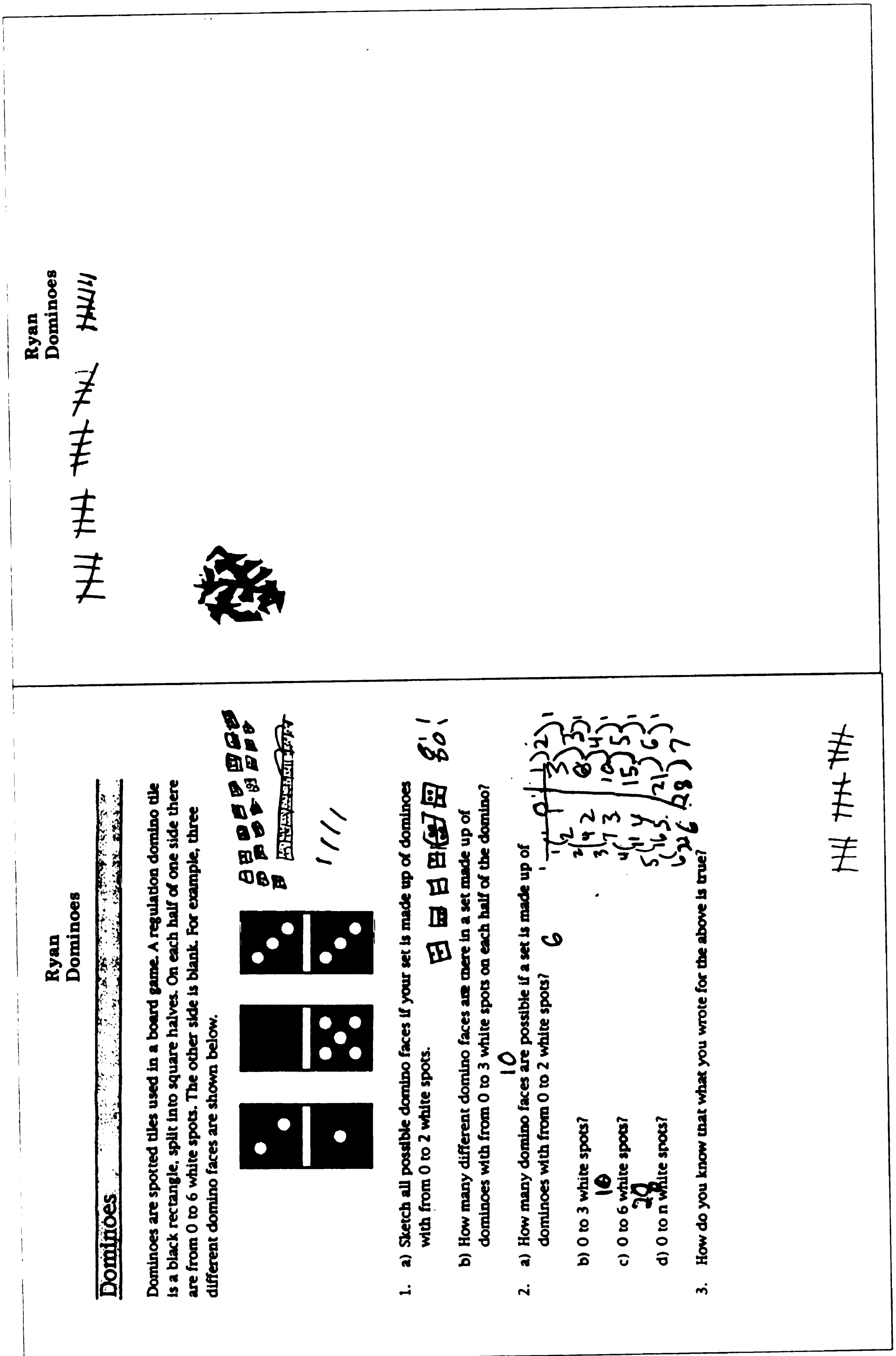


Figure 42: Ryan's work on *Dominoes*

She initially answered question 2 c, with the 22 dominoes she sketched and later revised her response when she recognized a pattern in the table.

Ryan opted for an alternative representation for his dominoes.

AK: When did you decide to try a new strategy?

Sara: I drew them, he did something different.

Ryan: I first kind of thought what they were. I thought about it and drew tally marks.

AK: So, you thought about it and had a mental picture?

Ryan: Yeah.

Ryan mentally represented the dominoes. After noting the 28 dominoes with up to six white spots, he proceeded to use his "tally marks" representations to count the 15 dominoes with up to four white spots. (See the lower right corner on the front of Ryan's work.)

After they represented the dominoes, they used a table to organize the data. They sketched additional domino representations to complete their tables. Sara drew the number of dominoes with up to zero, up to one, and then up four white spots and inserted these numbers into her table.

The pair studied the data in the table by searching for constant differences. They used the pattern of a constant second difference of one to complete their table and concluded that there were 21 dominoes with up to five white spots, and 28 with up to six white spots. Here Sara recognized that she had incorrectly represented 22 dominoes for the number of dominoes with up to six spots and changed the value in both her table and response to 28.

During the interview Sara and Ryan explained the ease with which they changed the data. The pattern they found in the table was more convincing to them than the sketches. Besides, Sara added, it was quite possible that she missed some dominoes in her sketches.

Sara and Ryan did not find a symbolic rule for the number of dominoes. They did suggest some reasonable ideas regarding the form of the equation, based on their work. After they found the second difference was constant, Sara used (X + 1) (X -   ) as a template for her initial guesses. Because the second difference was constant, they knew the rule was a quadratic (that meant that the rule would be X or a quantity of X times a quantity of X). They continued with a guess and adjust strategy but never found an appropriate symbolic generalization.

Graphic Connection

Ryan described a graph of the pattern in *Dominoes* during our interview.

Ryan: It forms a parabola in the graph.

AK: How do you know it forms a parabola in the graph?

Ryan: The way the table is written when you write it out.

AK: What about the table tells you that it is a parabola?

Ryan: It increases by more each time. See right here, it increases by three, then it increases by four. Makes it curve up more.

They analyzed an aspect of the pattern (the shape of their graph) even though the task did not pose questions about graphical representations of the situations. Ryan offered a very reasonable description of a parabola based on this analysis of the data in their table. As he said, "Makes it curve up more," his hand curved up to indicate the upward swing of the curve.

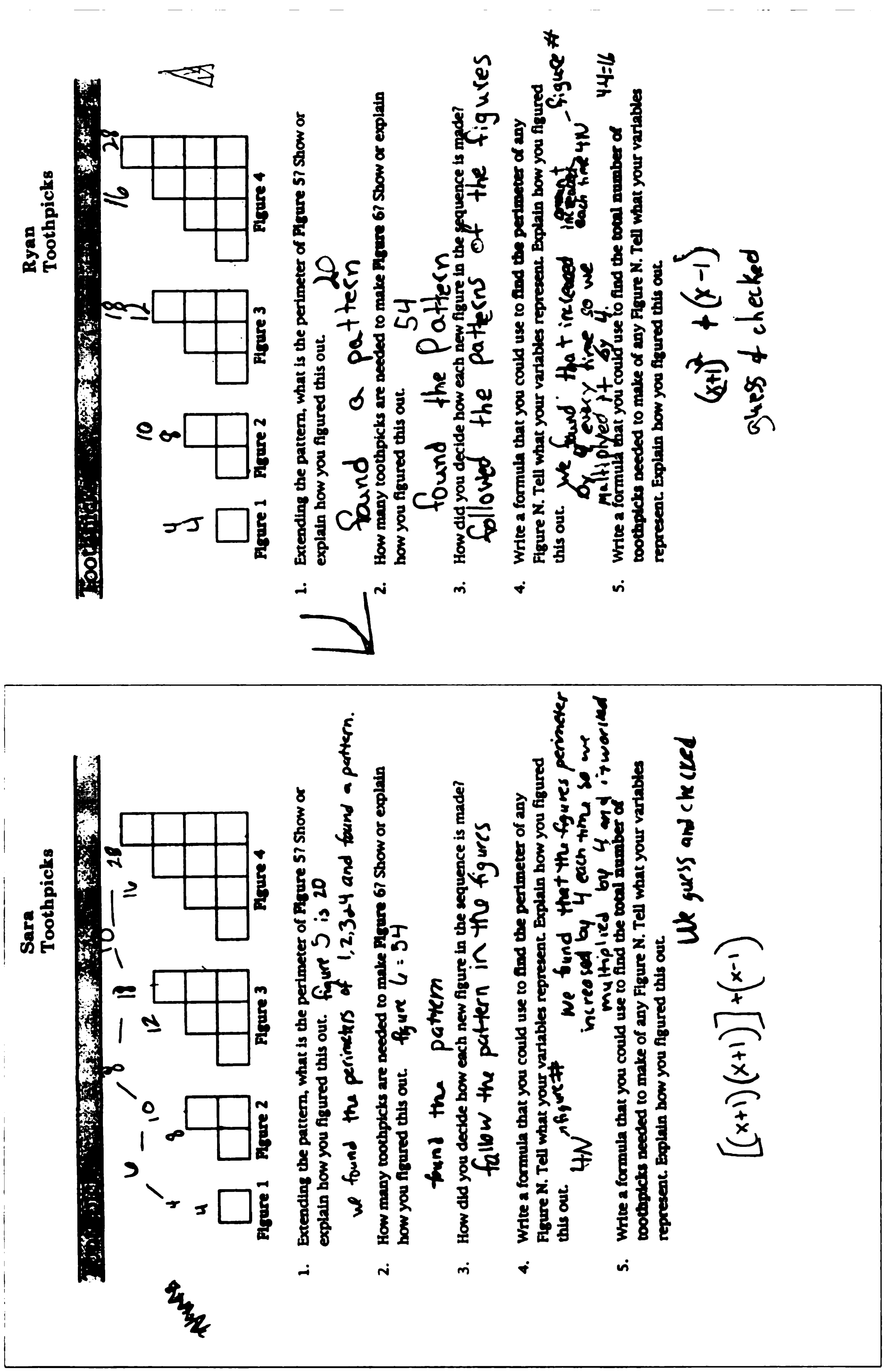
Not only did he know that the graph would be a parabola, he offered a very reasonable description of the graph based on the data in the table. They made a strong connection between two representations (tables and graphs) of quadratic functions.

Toothpicks

Although Sara and Ryan did not make a formal table in their solution to *Toothpicks* they studied the pattern in the numeric data. They followed a similar strategy of collecting data, generating new data, studying patterns, and writing the rules.

Numeric Approach

Sara and Ryan first counted the number of toothpicks in the perimeter for each of the four figures given and wrote these values (4, 8, 12, and 16) above each shape. When they began investigating the second pattern for the total number of toothpicks they repeated their process and counted the total



Sara
Toothpicks

Figure 1 Figure 2 Figure 3 Figure 4

4 — 6 — 10 — 8 — 18 — 10 — 16 — 28

1. Extending the pattern, what is the perimeter of **Figure 5**? Show or explain how you figured this out. figure 5 is 20
we found the perimeters of 1,2,3&4 and found a pattern.

2. How many toothpicks are needed to make **Figure 6**? Show or explain how you figured this out. figure 6 : 54
found the pattern

3. How did you decide how each new figure in the sequence is made?
follow the pattern in the figures

4. Write a formula that you could use to **find the perimeter** of any Figure N. Tell what your variables represent. Explain how you figured this out.
4N ^figure#
we found that the figures perimeter increased by 4 each time so we multiplied by 4 and it worked

5. Write a formula that you could use to find the **total number of toothpicks** needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out.
We guess and checked
$[(x+1)(x+1)]+(x-1)$

Ryan
Toothpicks

Figure 1 Figure 2 Figure 3 Figure 4

4 4 10 8 18 12 16 28

1. Extending the pattern, what is the perimeter of **Figure 5**? Show or explain how you figured this out. 20
found a pattern

2. How many toothpicks are needed to make **Figure 6**? Show or explain how you figured this out. 54
found the Pattern

3. How did you decide how each new figure in the sequence is made?
followed the patterns of the figures

4. Write a formula that you could use to **find the perimeter** of any Figure N. Tell what your variables represent. Explain how you figured this out.
we found that increased by 4 every time so we multiplyed it by 4. increased each time 4N - figure #
4·4=16

5. Write a formula that you could use to find the **total number of toothpicks** needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out.
$(x+1)^2+(x-1)$
guess & checked

Figure 43: Sara and Ryan's work on *Toothpicks*

number of toothpicks in the four figures and wrote these numbers (4, 10, 18, and 28) above the perimeters.

Sara and Ryan quickly saw the pattern of increase in the perimeter data.

AK: Can you describe your strategy for this problem?

Ryan: We tried finding a pattern.

AK: How did you try to find a pattern?

Sara: We figured out like the perimeter of these.

AK: You counted?

Sara: Yeah, it went up by four.

AK: Okay, did you use that to answer the question about five or did you draw five?

Sara: Five, we just counted them.

Ryan: We just found the pattern goes up by four every time.

AK: How many of these did you have to find, to find the pattern?

Ryan: The first three, but we just kind of wrote in the last one.

AK: Like a check?

Ryan: Yeah.

The total number of toothpicks pattern did not come to the pair as quickly, but they were still successful in their search. Once they had the total number of toothpicks for the four figures (4, 10, 18, and 28) they wrote out the first differences of 6, 8, and 10 above the numbers. They did not write the second difference, but demonstrated their recognition that the second difference was two when Sara verbally extended the first differences during our interview, "It increased by six, then eight, then ten, so then we added twelve and then fourteen."

Sara and Ryan accurately found rules for both the perimeter and the total number of toothpicks of the n^{th} figure. Finding the perimeter rule was straight forward for them once they saw the constant increase of four in the data. Sara supported her rule of $4 \cdot N$ in her written response with, "We found that the figures [sic] perimeter increased by 4 each time so we multiplied by 4 and it worked."

It was not as clear how they arrived at their second rule of $[(x+1)(x+1)] + (x-1)$ for the total number of toothpicks. Sara and Ryan both wrote that they "guess and checked." Their discussion during our interview suggested that they knew a form the equation might take and used that to inform their guesses.

Sara: We saw how much they increased by.

Ryan: Yeah. We just started guessing and we knew it formed a parabola.

AK: How did you know it formed a parabola?

Ryan: Because we sort of plotted it.

AK: Okay, what did that tell you when you knew it formed a parabola?

Ryan: In the equation X had to multiplied by another form of X.

The constant increase they observed in the first differences of 6, 8, 10, 12, 14 and so on, helped describe the shape of the graph to the pair. The shape of the graph determined the form of the equation. Once they knew the form of the equation, they embarked on their guess and check strategy.

AK: How did you guess and check that one?

Sara: We knew it would be a form of X times itself or a form of X, and then I don't know.

AK: Do you remember?

Ryan: It was boom. There it was. I just tried something, I just tried the plus one and it worked.

Their guess and check resulted in a rule that satisfied the data. Although they seemed to interpret their finding a rule for the pattern to be luck, their initial guesses were guided by some fundamental understanding about the pattern. They used a reasonable guess based on what they knew of the data and were then successful in finding a rule.

Anna and Katrina

This pair of students claimed that all of these tasks involved algebra since their goal in all was to write symbolic rules. They described algebra during an interview.

AK: What do you think algebra is?

Katrina: Taking math to a different level, using numbers and symbols in the rules

Anna: I think using tables and graphs to get rules using data to find patterns.

Although, they considered all of the tasks to be algebra their solutions varied. In *Cutting* and *Toothpicks* they constructed tables, in *Borders* they followed a geometric approach, and in *Dominoes* they considered a pictorial strategy. They found adequate rules for the patterns in three out of the four tasks.

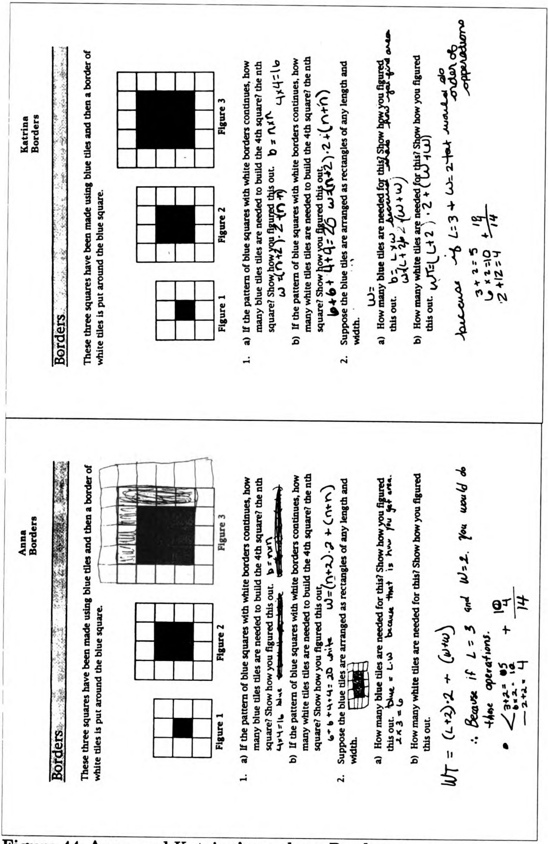
Borders

Anna and Katrina wrote their symbolic rules for the patterns in *Borders* based on the shapes of the figures, a geometric approach. They found the rules early in their solutions and used these rules to generate specific cases.

Geometric Approach

Although their written work offered minimal insight into their approach to finding the rules, the videotape provided evidence of their process. Anna and Katrina readily described the rules $b = n \times n$ and $w = (n + 2)\cdot 2 + (n + \mathrm{n})$ in words shortly after they began the task.

Anna: So, for this, the fourth square is like N times N. So, for the number of blue tiles it would be N times N. So, four times four



Anna
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. b = n×n
4×4=16 blue

b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out.
6+6+4+4=20 white W=(n+2)·2+(n+n)

2. Suppose the blue tiles are arranged as rectangles of any length and width.

a) How many blue tiles are needed for this? Show how you figured this out. blue = L·W because that is how you get area. 2×3=6

b) How many white tiles are needed for this? Show how you figured this out.

WT = (L+2)·2 + (W+W)

∴ Because if L = 3 and W=2. You would do these operations.

3+2=5, 5×2=10, 2+2=4; 10 + 4 = 14

Katrina
Borders

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many blue tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out. b = n×n 4×4=16
w = (n+2)·2+(n+n)

b) If the pattern of blue squares with white borders continues, how many white tiles tiles are needed to build the 4th square? the nth square? Show how you figured this out.
6+6+4+4=20 w=(n+2)·2+(n+n)

2. Suppose the blue tiles are arranged as rectangles of any length and width.

W=

a) How many blue tiles are needed for this? Show how you figured this out. b= L×W because that's how you find area
W=(L+2)·2+(W+W)

b) How many white tiles are needed for this? Show how you figured this out. WT=(L+2)·2+(W+W)

because if L=3 + W=2 that would do order of opperations

3+2=5, 5×2=10, 2+2=4; 10 + 4 = 14

Figure 44: Anna and Katrina's work on *Borders*

is sixteen [pause]. For, white tiles, N plus one, or N plus two, plus N plus two plus N plus N.

While Anna discussed her generalization with Katrina, she motioned along the lines sketched in Figure 44 below to the corresponding sides of figure 3 to illustrate the pieces of the rule.

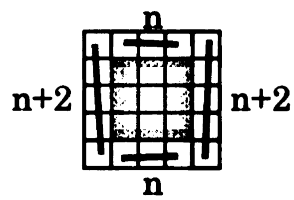


Figure 45: Anna's figure 3 in *Borders*

When pushed to describe the strategy they used to complete this problem, Katrina claimed that is was a simple problem, "we just looked for a pattern in our heads."

The pair found the number of blue and white tiles in figure 4 by using the rules they generated. Anna described how she found the 20 white tiles for figure 4 using her rule, "In this case N is three, and it goes, you add two on each end [pause]. Okay, so in this case N is four, so N plus two is six, is twelve, plus four plus four is twenty." While stating her explanation she wrote, "6 + 6 + 4 + 4 = 20 white tiles."

Once they found the 16 blue and 20 white tiles for figure 4 using the rules, they verified this for themselves by sketching figure 4 and counting tiles. Not until this confirmation were they satisfied with their work.

During the interview Anna and Katrina again justified their results with a specific case.

AK: Please justify your solution in one a[9], how did you get N times N?

Anna: N times N is the area of the blue squares, Like for figure three, three blue squares across, three here, so three times three.

AK: How about one b?[10] N plus two times two plus N plus N?

Anna: N plus two is like taking three and adding these two corners, so that whole length would be five, then you times it by two, because you have another one up here. What you have left is N, then three, then N or another three.

Although Anna specifically used figure 3 for her support, her description could be generalized for the n^{th} figure. It was unclear based on the evidence if Anna and Katrina saw this further generalization.

Cutting

This was an unfamiliar task to Anna and Katrina. They did not recall completing it or a similar problem, but because *Cutting* asked the students to write rules, they felt it was an algebra task. When faced with a new

---

[9] 1. a) If the pattern of blue squares with white borders continues, how many **blue** tiles are needed to build the 4th square? the nth square? Show how you figured this out.

[10] 1. b) If the pattern of blue squares with white borders continues, how many **white** tiles are needed to build the 4th square? the nth square? Show how you figured this out.

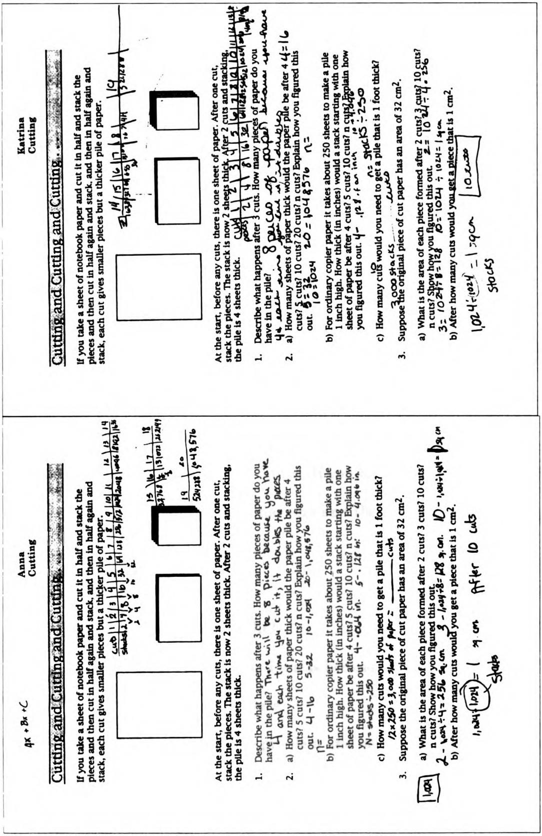


Figure 46: Anna and Katrina's work on *Cutting*

situation, Anna and Katrina made and studied tables to investigate the pattern.

Using a Table

The first step Anna and Katrina followed was to collect their data and record that in a table.

Anna: After one cut, two pieces, right?

Katrina: Yeah.

Anna: After two, you cut these pieces in half. You have four.

Katrina: Uh huh.

Anna: Then you take those four and cut them.

Katrina: I don't know its either going to be eight or six.

Anna: Okay, let's think about it. If you have four pieces of paper in your hand just regular size and then in half, it would be [pause] eight.

Katrina: Eight.

Anna: So, would it be eight?

Katrina: It would have to be. I guess because if you rip them in half it would be eight papers.

They considered the context of the problem and used that knowledge to help them collect their data.

They found the data for up to 20 cuts by using calculators. They recorded the data into a table and then used that representation to find the specific cases for which the problem asked.

Anna: For ten it would be . . . one thousand twenty four.

Katrina: All right, twenty? [Long pause]. Is that what you got? One oh four eight five seven six?

Anna: Yup.

Next, they tried to find the number of sheets in a stack after 10 cuts. They studied the table and searched for patterns. They calculated the first difference in the table, but that did not seem to help them describe the pattern.

Although Anna and Katrina did not find the symbolic rules for the exponential patterns it was evident during our interview that they recognized that they needed the rules for the number of sheets in a stack to find the thickness of a stack in a later problem.

AK: What about b?[11] What was your strategy?

Anna: We had to take the number of stacks and divide by two hundred fifty. Our table helped us, so we needed to know for four, so we took sixteen stacks and divided that two hundred fifty and we got point oh six four.

---

[11] 2. b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.

AK: All but the N?

Anna: We needed to know the equation for a, but we never figured it out.[12]

AK: Katrina?

Katrina: I don't think we did it.

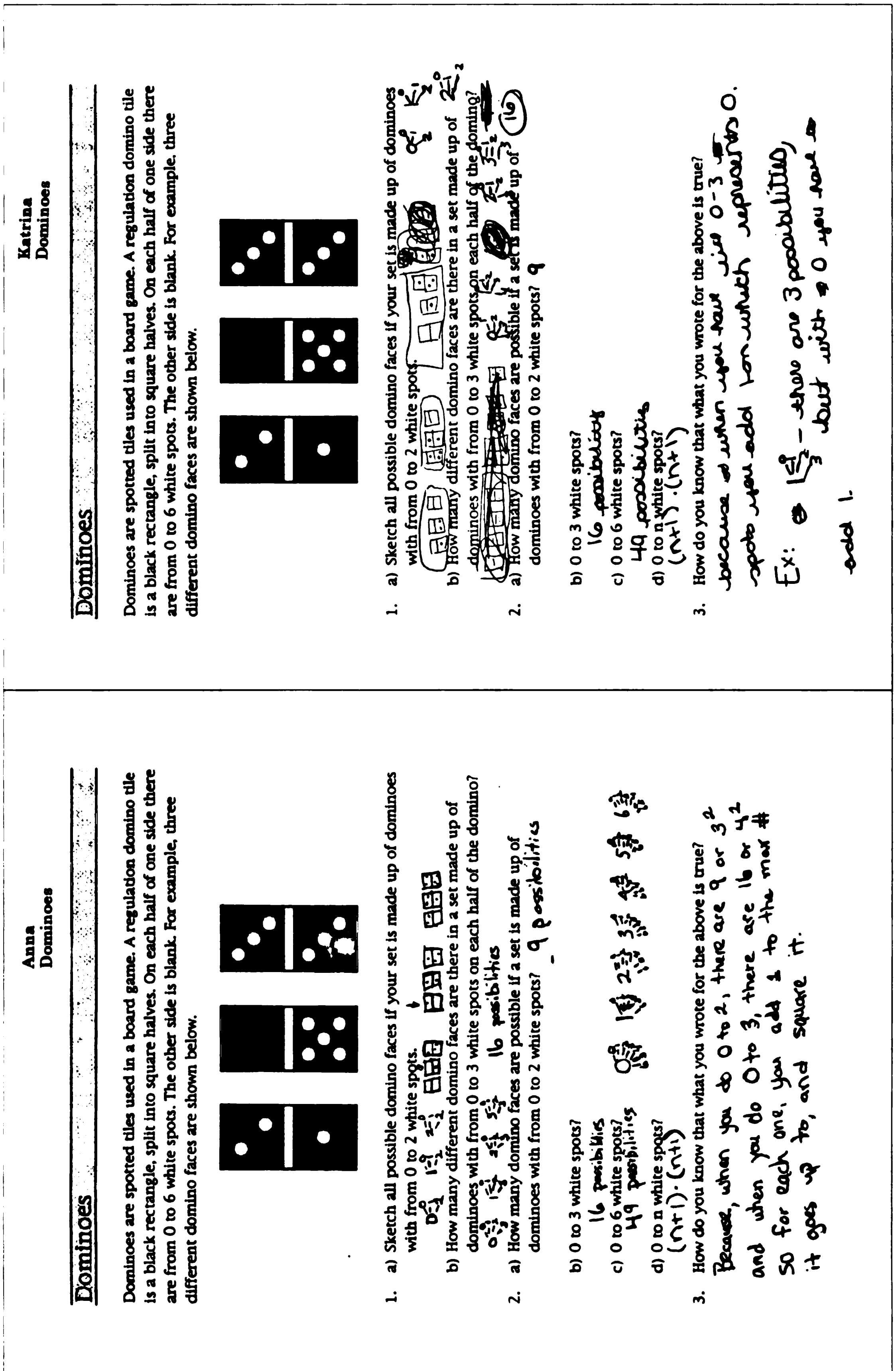
Anna: I wrote how you could do it. [Pause] Four cuts, gives you sixteen stacks of paper, so sixteen divided by two fifty, so point oh six four of an inch. So, for any number of cuts , it would be stacks divided by two fifty.

They generalized that they could find the thickness of the stack when they were given the number of sheets in a stack.

Dominoes

Anna and Katrina saw this problem as algebra and probability. Algebra, "because you need to write a rule" and probability, "to get different ways of charting out all possibilities." Their approach was to represent the dominoes with tree diagrams and find a pattern in their sketches to generalize.

[12] 2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out.



Anna
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.

   b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?
   16 possibilities

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots? 9 possibilities

   b) 0 to 3 white spots?
   16 possibilities

   c) 0 to 6 white spots?
   49 possibilities

   d) 0 to n white spots?
   $(n+1)\cdot(n+1)$

3. How do you know that what you wrote for the above is true?
Because, when you do 0 to 2, there are 9 or 3^2 and when you do 0 to 3, there are 16 or 4^2 so for each one, you add 1 to the max # it goes up to, and square it.

Katrina
Dominoes

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.

   b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots? 9

   b) 0 to 3 white spots?
   16 possibilities

   c) 0 to 6 white spots?
   49 possibilities

   d) 0 to n white spots?
   $(n+1)\cdot(n+1)$

3. How do you know that what you wrote for the above is true?
because when you have 0-3 spots you add 1 which represents 0.
Ex: – there are 3 possibilities, but with 0 you have add 1.

Figure 47: Anna and Katrina's work on *Dominoes*

Pictorial Representation

The pair had to first determine how to count the 0-1 and 1-0 dominoes. This question arose early during their investigation. They decided 0-1 and 1-0 were two distinct dominoes and completed the task with this assumption (Julie and Dan's Case 1).

These two students made tree diagrams to represent the dominoes after they sketched out the nine dominoes with up to two white spots. Making tree diagrams was a strategy they recalled from the probability they studied in their mathematics class. From the tree diagrams, they saw how to write the rules of $(n + 1)\cdot(n + 1)$.

Anna: Zero to three white spots it would be sixteen possibilities right?

Katrina: Uh huh.

Anna: All you do is you add, through, add one to the total number, so, for six it would be seven times seven, so it would be forty-nine.

Katrina: How did you get that?

Anna: You do like, when you draw the diagrams you do zero, then one, two, then three, up to three. When you include zero, its' like one more, so you count its like one two three four.

Katrina: All right, I understand.

Katrina demonstrated her understanding further in her written explanation for question three when she wrote, "because when you have like 0-3 spots you add 1 on which represents 0." She illustrated this with a diagram to clarify (See Katrina's work under question 3).

Toothpicks

This was a straight forward task for Anna and Katrina. They readily recognized the linear pattern for the perimeter toothpicks in a table. It took a little more time but they also found the quadratic pattern for the total number of toothpicks in another table.

Using a Table

The pair made tables to investigate the problem and to collect the data for the specific case in the first pattern.

Anna: Let's do a table to see if there is a pattern [in the perimeter pattern]. Be four.

Katrina: Then twelve [is the perimeter in Figure 3].

Anna: One-two-three-four-five-six-seven-eight [counts perimeter in figure 2].

When they investigated the second pattern for the total number of toothpicks, Anna again suggested a table.

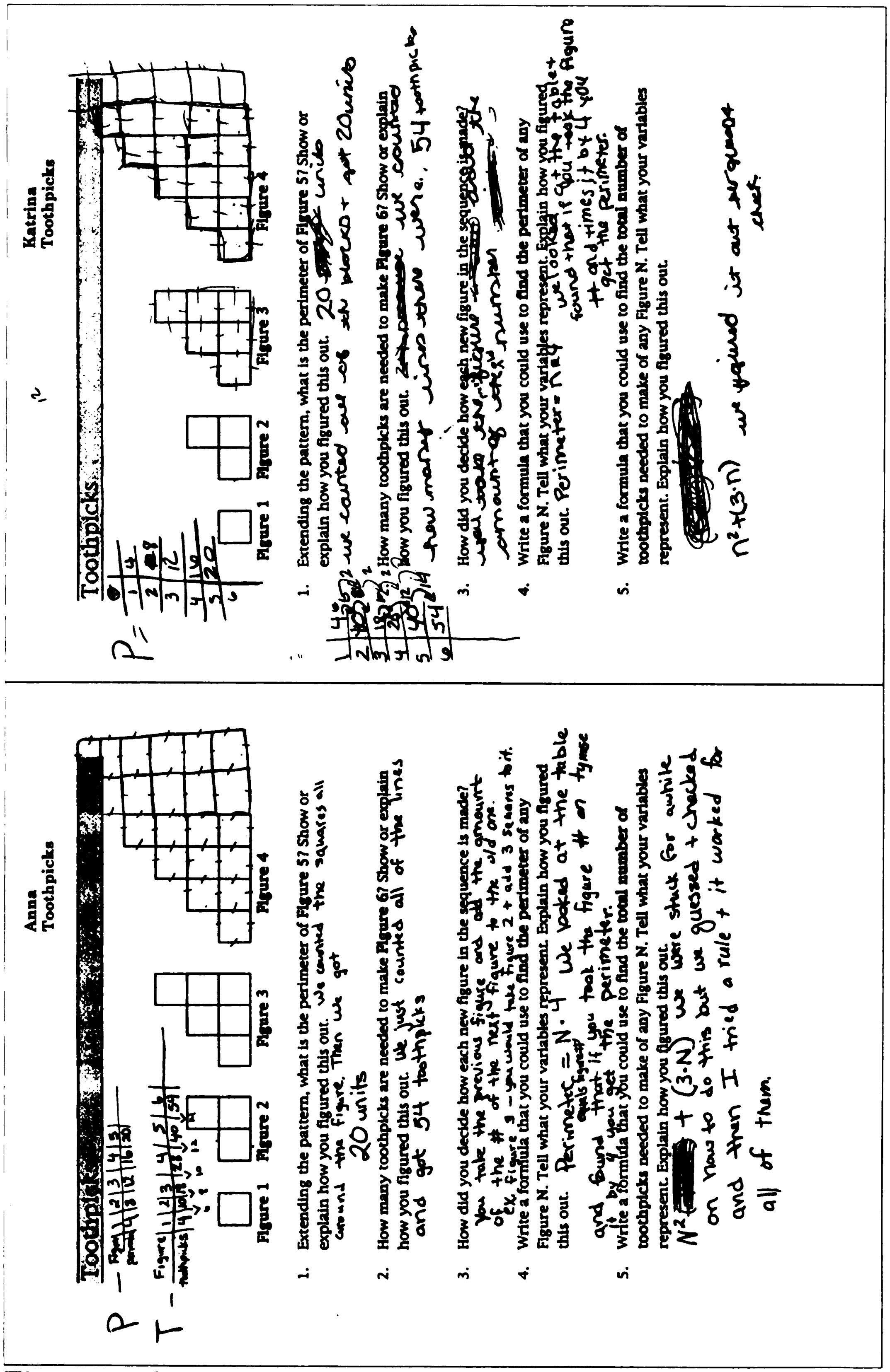


Figure 48: Anna and Katrina's work on *Toothpicks*

Anna: Okay, let's do a table for this one too [the total number of toothpicks]. Figure one there's one, two, three, four. This is for perimeter and this is for toothpicks. Okay.

Once they decided to use a table they collected data by counting the toothpicks. Anna described a shortcut for counting the total number of toothpicks.

Anna: One, two, three, four, five, six, seven, eight, nine, nine, yup eighteen [is the perimeter for figure 3]. All it is if you find all of the toothpicks that go like this way [lie in the horizontal direction] and you double 'em, because its like going one, two, nine, and then instead of counting all of the one like this [in the vertical direction] you can just turn it and it would be the ones that go across. Do you know what I am saying?

Katrina: Ohhh, okay.

Anna: For four: one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, twenty-eight.

They proceeded to use this shortcut to collect the remainder of the data.

With the data recorded in the table, they studied it in search of a pattern. They quickly noticed the increase of four in the table and wrote the rule $4N$.

Anna: Sixteen? So, its...

Katrina: ... increases by four each time?

Anna: The figure times four?

Katrina: Yup.

They studied their second table, for the total number of toothpicks, and found the first differences. Katrina continued to find the constant second difference of two, but it was unclear how and if they used that information.

Anna: Okay, from here to here, six, eight, ten, twelve, fourteen. What are you doing?

Katrina: And then we can do this again, they each increase by two.

Anna: Okay, so how can we do that two times X plus six. No, that doesn't work, then umm...

Once they had this pattern the rule for the total number of toothpicks was not immediately evident.

Anna: I totally forgot how to write equations from tables. Is it something like a X squared plus b...?

Katrina: Umm. I remember that for.

Anna: Plus b X? So [pause] no wait [pause] I totally forgot how to do equations when they are like this, where you have to know the previous ones. That's why we had problems with that one [*Cutting*]. Right?

Katrina: Mm, hmm. That's paper stacks.

Anna eventually suggested that the equation be of the form x^2 + b or x^2 + bx. It was unclear what guided her to select these families of quadratics. Rather than basing her use of a quadratic on the constant second difference, she seemed to focus on the iterative nature of the problem. But she was successful finding the rule, $n^2 + 3n$, using this starting point.

Anna: Oh my gosh.

Katrina: Huh?

Anna: I got it. I got it, hold on. Three times four is twelve. Okay remember twelve.

Katrina: Right.

Anna: I got it. It's N times N plus three times N. Don't ask me how I got it. It was just.

Katrina: Guess and check?

Anna: So, three times three is nine plus three oops, times three is nine.

Katrina: Nine plus nine.

Anna: Eighteen. Okay.

Katrina: Good job.

A key to Anna finding their rule was starting with the form $x^2 + bx$. She did not articulate a reason for her selection during their work or during our interview. But, something about the pattern seemed to guide her to a quadratic that in turn helped the pair find a rule.

Summary

Making and studying tables represents a reliable strategy for these students when studying patterns in data. There were 31 instances of students investigating patterns in this study.[13] The students constructed tables in over 60% of the instances (19 out of 31). The range of values the students used for their independent variables in the tables is listed in Table 15. Of the tables that were made, half ranged from 0 to 10 or 1 to 10. Only one table extended further. The less complex generalizations are represented in the first four rows of the table (*Borders*—linear, *Toothpicks*—linear, *Borders*—quadratic, and *Dominoes*—Case 1). The more challenging problems are listed in the last three rows (*Dominoes*—Case 2, *Toothpicks*—quadratic, and *Cutting*).

Ten out of the 12 instances where students did not make and study a table were in the less complex cases. Six of these were from students studying *Borders*, a fairly basic task where students immediately recognized the patterns. These were both linear and quadratic patterns. Three of these were the linear pattern in *Toothpicks*. The only remaining situation where the students did not use a table was when Anna and Katrina completed

[13] *Borders* and *Toothpicks* each had two patterns (one linear and one quadratic). *Cutting* and *Dominoes* both had only one pattern, but Julie and Dan studied two patterns in *Dominoes*. The 5 student pairs each studied 6 patterns or 30 total, plus Julie and Dan's additional pattern from *Dominoes*, gives a final total of 31.

Table 15: Range of Values for the Independent Variable in Tables Made by Students

Function Content	Task	Students				
		Zachary and Todd	Ben and Joe	Julie and Dan	Sara and Ryan	Anna and Katrina
Linear	*Borders*	1-8	0-10	No table	No table	No table
	Toothpicks	0-7	No table	No table	No table	1-5
Quadratic	*Borders*	1-10	0-10	No table	No table	No table
	Dominoes:Case 1	–	–	0-10	–	No table
	Dominoes:Case 2	0-10	0-6	2-6	0-6	–
	Toothpicks	0-5	No table	1-7	No table	1-6
Exponential	*Cutting*	1-10	0-10	1-10	1-10	1-20

Dominoes. They solved this using the Case 1 assumption,[14] making it a fairly straightforward pattern to generalize with symbols.

In the 14 more complex instances only two did not involve tables. Students used tables in 85% of the more complex instances. In the two instances where they did not use tables students followed a geometric approach to solve *Toothpicks*. In *Cutting*, which was an unfamiliar task for the students all constructed tables.

There were only five cases where the students did not find symbolic rules. In all of these instances they constructed tables in their solutions. Table 16 lists the symbolic rules found by students.

In the following chapter I step back and look for common ideas across students and across tasks. I also consider some of the differences that make some of the solutions unique.

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[14] In the Case 1 assumption, the 1-0 and 0-1 dominoes are counted as two unique dominoes.

Table 16: Symbolic Rules Found by Students

Function Content	Task	Students				
		Zachary and Todd	Ben and Joe	Julie and Dan	Sara and Ryan	Anna and Katrina
Linear	*Borders*	$4x + 4$	$L \times 4+4$	$4n+4$	$4(L-2)+4$	$w = (n+2)\cdot 2 + (n+n)$
	Toothpicks	$4N$	$Y = 4N$	$y = 4n$	$4N$	4N
Quadratic	*Borders*	x^2	L^2	n^2	$(L-2)(W-2)$	$b = n \times n$
	Dominoes: 1	–	–	$(n+1)^2$	–	$(n+1)^2$
	Dominoes: 2	$(n+1)\div 2\cdot n+n+1$	$y = (N+1)(0.5N+1)$	No rule	No rule	–
	Toothpicks	$n\cdot(n+3)$	$y = 4N + (N-1)N$	$y = n(n+3)$	$[(x+1)(x+1)]+x-1$	$n^2 + 3n$
Exponential	*Cutting*	$y = 2^x$	$N = 2^x$	No rule	No rule	No rule

CHAPTER 5

Stepping Back—

An Analysis of What These Students Know

Introduction

In the previous chapter I describe what each of the pairs of students did with each of the individual tasks. Now I consider a different framework and review the data as a collection and describe common threads across students and across tasks.

All of the students hold some strategies to help them formalize generalizations. In the majority of situations, students constructed tables of data. They studied patterns of data and regularly sought out constant differences to help write rules. In some instances, the students used visual models and studied changing shapes in the patterns. A few students went beyond the demands of the task and considered the graphs of the data to help them think about the pattern.

Another common idea I saw across students was the ability to make connections. These students made connections between the representations of the patterns—tables, symbols, and graphs. They also made connections between the tasks and recognized similarities in the mathematical structure of the problems. Finally, some students made connections between these tasks and other mathematics that they recalled.

In the final section, I write a bit about the students' disposition towards mathematics and mathematics problems. They all seemed frustrated and surprised when they had difficulty with a problem.

Strategies to Formalize Generalizations

These students demonstrated that there was much that they knew about formalizing generalizations in specific situations. This was evident in the work they completed and in the interviews in which they participated. All student pairs made reasonable progress in each of the tasks; all were able to draw some strategies that afforded them access into the tasks.

Two aspects of the tasks determined the strategy students used to investigate it. First was the mathematics they saw in the problems, and second, the familiarity of the problems. When the students classified the problem as primarily an algebra task, they constructed a table of data to (a) record or re-present data, (b) extend given data, (c) find specific cases, (d) study patterns, (e) write rules, and (f) verify rules. If they considered it a geometry task, they used a geometric approach to investigate and studied the changing shapes. They considered a single figure and related the figure number to the measure of interest, then used the next figure to confirm their assertion. In familiar situations the students had several strategies they used in their investigations. In unfamiliar situations they studied tables as an entry point into the problem.

The students had several different strategies they used to investigate the situations. Most used tables, others followed a geometric approach, while a few considered graphic representations.

Numeric Approach: Generalizing From a Table of Data

Constructing and studying tables was the dominant strategy used by students in these four tasks. In over 60% (19 of 31) of the situations from this study, students made tables of numeric data to help with their investigations. Of the 31 situations, there were only five where the students did not successfully find symbolic generalizations. In all of these five cases, the students constructed tables to investigate the patterns.

The students saw making tables as a reliable strategy to investigate challenging problems. When the students realized that a problem was demanding, I heard many conversations where one student suggested trying to make a table after they were unsure how to proceed. In the only problem that was an unfamiliar content, *Cutting*, all five pairs of students constructed tables to analyze the data. When the students were uncertain how to approach a problem, making a table afforded them a reasonable entry point to begin their investigation. After the students constructed tables of data, they usually found the constant differences and used this information to help them write their rules.

Once students re-presented their data in a table, they usually did not relate what they found back to the context of the problem. Most studied the numeric data in the table and did not verify that their solution with the numeric data was reasonable in the setting of the problems. In a few instances when students related their solution back to the context they extended their tables back to include 0 for the independent variable and referenced figure 0 in their pattern.

Finding Constant Differences

The students were quite facile at finding constant differences. In most of the tables they wrote the differences they found. They used this information to describe and extend the pattern and then write the rules.

<u>Linear functions</u>. Writing a linear rule was fairly routine for the students in the study. When they found a constant first difference, they knew that the pattern was linear. The students often wrote linear rules in the form of $y = mx + b$. They easily recognized the rate of change in the table as the coefficient of x and proceeded to find the y-intercept. They found this by either considering Figure 0 in the pattern of shapes, or by extending the numeric pattern back in the table to include case 0. Some referred to this as

the y-intercept, while others called it the constant term. All student pairs easily found the linear rules in *Borders* and *Toothpicks*.

Quadratic Functions. Rules that corresponded to a quadratic function proved to be more difficult for the students to find. A constant second difference was a clue to the students that the pattern was quadratic.[1] A quadratic meant to most of the students that it was a rule in the form of a quantity of x (or x itself) times a quantity of x; their views represented the factored forms $(x + \mathrm{a})\cdot(x + \mathrm{b})$ and $x\cdot(x + \mathrm{b})$. To others, it led to a specialized expanded form of a quadratic $x^2 + \mathrm{b}x$ or $x^2 + \mathrm{b}$.[2] Students did not possess an algorithm to write a quadratic rule. But, they knew some things about quadratics that informed a conjecture and adjust strategy. They knew a potential form of the rule, $(x + \mathrm{a})\cdot(x + \mathrm{b})$ or $x\cdot(x + \mathrm{b})$, that helped them write an initial guess. From there they used a guess and check strategy to try to find a rule in quadratic situations. They checked to see if their rule satisfied the data; is so they were done, if not they adjusted the rule and tried again.

Some students tried to fit what they knew about linear cases to guide their guesses. The most common idea was the notion of the constant term. In the quadratic cases some students extended their tables back to include a

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[1] This is a reasonable conclusion to draw. When the x terms differ by one, the second difference is actually the second derivative. Recall that the second derivative of a quadratic is a constant (non-zero) term.

[2] These factored and expanded forms of the rules restricted students' guesses. They eliminated all quadratics with irrational roots from their trials. Some limited their guesses to only integer roots.

value of 0 for their independent variable and used the corresponding value for the dependent variable as a constant term in their rule.

Students' knowledge of quadratic situations was not as robust as it was with linear situations. However, they had enough understanding of the situation and a set of tools to conduct a reasonable investigation. They knew how to determine if a pattern was quadratic from a table of data. They knew the form of a quadratic rule. Some recognized the factored form, while others focused on an expanded form. They also knew things about the graphs of the pattern, which I describe in later sections.

Exponential functions. Quadratics were not the only non-linear patterns the students explored. I asked them to consider an exponential pattern. When the students could not find a constant difference, they were unsure what form the rule might take. Most followed a guess and check strategy hoping they might find something that worked for their situation. Even though they were in an unfamiliar situation, they had some strategies that allowed them access into the task.

Formulating Rules

After these students constructed tables, it was not trivial for them to find symbolic rules in non-linear situations. MacGregor and Stacey (1993) found that when students had difficulties writing algebraic rules from function tables there were several different steps along the way where students might lack the needed skills or knowledge. They identified four of these steps that students must cross before they can make the transition to symbolic generalizations with x as the independent variable and y as the dependent variable. Students must be able to (a) articulate the relationship to find numerical values, (b) look beyond a recurrence pattern to find one that links the two variables, (c) know the syntax of algebra, and (d) know what can and cannot be said with algebra.

Students who were successful finding symbolic rules demonstrated competence in each of the four steps outlined. Students who did not find symbolic rules stumbled in at least one of the steps. This is illustrated in the following sections.

Articulating the relationship to find numeric values. All students found numeric values for specific cases in the tasks. They generally did this in one of two ways. In some situations they extended a recursive pattern, at other times they continued the pattern of differences. Examples to illustrate these different strategies can be pulled from two tasks, *Cutting* and *Toothpicks*. In *Cutting* students were asked to find the number of sheets in the stack of paper after 20 cuts. Anna and Katrina extended their table by doubling the previous term to find the 1,048,576 sheets after 20 cuts; they used a recursive pattern to complete this. After finishing *Toothpicks*, Julie clarified during our interview how she used her understanding of the constant second difference to find the total number of toothpicks in the sixth figure. "I used my pattern.

First, I made a table and found the amount of increase by two [the second difference], then I could find fifty-four [for the sixth figure]."

The numeric patterns in the data represented very strong support for the students. When they noticed that a value they found through sketching or counting did not agree with the values predicted by the pattern, they quickly abandoned their counting strategy in favor of the pattern.

When the rules were straight forward to write, as in the linear case, there were a few instances where students wrote their rules first and used this articulated relationship to find the specific cases. This process seemed to be the exception; students only did this in simpler linear cases like *Toothpicks* where the pattern was $4n$. Most students readily saw the increase of 4 after they found the perimeter for two only cases. They noticed that the perimeters were also multiples of 4 and wrote the rule

Looking beyond a recurrence pattern. This is the step that determined for most students whether they were able to write the symbolic rules. In nearly 84% (26 out of 31) of the situations, students were successful in writing rules. Three pairs of students did not generalize with symbols for the exponential case in *Cutting* and two pairs could not find the rules for a difficult quadratic situation in *Dominoes*. In all five of these cases the students had difficulty expressing a relationship that linked the two variables.

Cutting proved to be quite a challenging task for all five pairs of students. The three pairs who did not write a rule, all described the doubling in the tables, and used that doubling pattern to find the specific cases. Yet, they could not make the transition from doubling to an exponential to link the number of cuts to the number of sheets in a stack.

It was clear that some students knew they were searching for the relationship that linked the two variables. Dan said while they worked, "There has to be a rule [pause]. Times a half? Find out what the heck six has to do with sixty four?"

Ben and Joe also discussed this link. Joe suggested "X times Y times two?" as a possible rule. Ben responded "Times? No, you can't have X and Y in the same. You are trying to find Y. This right here is Y [points to sheets thick]. You don't know these, you are trying to get these [sheets thick] from X [cuts]."

Both pairs demonstrated that they recognized the roles of the independent and dependent variables in the generalizations. Dan did this with a specific case of searching for a relationship between 6 and 64 while Ben generalized the relationship using x and y. While Ben and Joe finally wrote $N = 2^x$ to represent the number of sheets in a stack after x cuts, Julie and Dan were not successful searching for a rule. One possible factor to explain this distinction was the recognition of the structure of the problem. Julie and Dan never recalled seeing a problem like this, while Brad and Joe recalled a similar problem from their seventh grade science class. The science problem involved studying bacteria doubling. Although Brad and Joe remembered the bacteria problem they added that they did not recall using x as the exponent.

The second task where students did not write a rule was *Dominoes*. Two pairs of students were unsuccessful with this rule; they both constructed tables and used the patterns to extend the table, but neither found a rule.

The rule the students searched for was difficult. The total number of dominoes with up to n spots could be written in factored form as $\frac{1}{2}(n + 2)(n + 1)$ or $(\frac{1}{2}n + 1)(n + 1)$. Both of these are slightly different than the factored form the students used to guide their search, $(x + a)(x + b)$. Their rules all included a coefficient of 1 for the x^2 term.

All of the rules that the students wrote in all situations did correctly link independent and dependent variables in their patterns. When students had difficulty finding the rule, it was not that they did not know they needed to make the link, rather they did not know how to do this. In *Dominoes*, all students recognized that they were trying to find a rule that related the maximum number of spots (independent variable) to the total number of dominoes (dependent variable), some could just not find a rule.

Knowing the syntax. A student who could not find a symbolic rule because of the syntax might have possessed a verbal generalization, but did not know the syntax, or algebraic language, that could represent the rule with symbols. The lack of knowledge about the syntax seemed to prevent the students from writing their rules in *Cutting.* Since the students were unfamiliar with the appropriate syntax needed, they had difficulties verbalizing their generalizations in a way that could be represented with algebraic language. In the interviews after they completed the tasks I discussed the possible solutions. We talked about adding specific columns to their tables, illustrated in Table 17, to help them think about a rule.

Table 17: *Cutting* Table with Factors of 2 Represented

Number Of Cuts	Number of Sheets in a Stack		
1	2	2×1	2
2	4	2×2	2×2
3	8	2×4	2×2×2
4	16	2×8	2×2×2×2
...	...	...	...
10	1,024	2×512	2×2×2×2×2×2×2×2×2×2
...	...	...	...

The three pairs who did not write rules were not familiar with the exponential notation for functions. Although they had worked numerically with exponents, they were unsure whether they could use them in an algebraic rule. The lack of familiarity with the syntax represented at least one step where the students tripped in their solutions.

The pairs who successfully found rules recalled a similar problem from their seventh grade science class, which involved doubling bacteria. Todd stated that if had he not done the problem the previous year he would not have known how to use the "y-x" key. It seemed that the understanding of the calculator syntax from their science class helped the pair write their rule of $y = 2^x$ for the number of sheets in a stack after x cuts.

The other task where some students encountered difficulties with a rule was *Dominoes*. The students systematically observed the pattern of increase in the differences from the table. Since the rules did not fit the factored forms of some students' vision of quadratics, these students did not write rules in this situation.[3] They did not have the appropriate syntax, or language, to represent the rule. The quadratic pattern in *Toothpicks* nicely fit their assumed form, so that all pairs were successful. For the students who saw the factored form it was $n(n + 3)$. Anna and Katrina found $n^2 + 3n$ to fit their general form of $x^2 + bx$. It was unclear that Sara and Ryan saw how their solution matched the factored form. They wrote $[(x + 1)(x + 1)] + x - 1$, which is an equivalent expression. Ryan started with $(x + 1)(x + 1)$, a factored form, and recognized that he needed to adjust his rule and added $(x - 1)$.

Knowing what can and cannot be said. This step might involve higher order thinking than middle grade students' algebraic understanding. Since they do not have a complete view of the discipline they could confuse what they don't know the syntax for with something that cannot be said with algebra.

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[3] Students looked for rules in the factored forms of $x{\cdot}x$, $x(x + \text{b})$, or $(x + \text{a})(x + \text{b})$. Some used the expanded forms of $x^2 + \text{b}$ *or* $x^2 + \text{b}x$.

Geometric Approach: Generalizing about Changing Shapes

Part of being a mathematically powerful problem solver is for students to have a number of strategies at their disposal to investigate problems. The authors of the *NCTM Curriculum and Evaluation Standards* (1989) wrote that instruction should increase its focus on having students use geometry to investigate problems. A few students did successfully use geometric approaches to solve the situations presented. Using these strategies, rather than generating numeric data in a table, students observed all the patterns spatially to recognize a generalization. When the students followed a geometric approach, they were all successful seeing the geometric structure of the pattern.

Four out of the five pairs used a geometric approach in the solution to *Borders*. Julie and Dan, Sara and Ryan, and Anna and Katrina all used it to launch their investigation while Ben and Joe followed a numeric approach in their solution but verified their rule geometrically during the interview. The students also followed a geometric approach for the quadratic pattern of blue tiles in *Borders*, but they represented the pattern as n^2 or $n \times n$, without much discussion. This pattern seemed trivial to these students. The geometric interpretations of the patterns to find the white border tiles allowed different ways to count. The rules the students wrote for white tiles follow in Table 18.

Table 18: Rules for White Tiles in *Borders*

Students	Rule
Julie and Dan	$4n + 4$
Sara and Ryan	$4(L - 2) + 4$
Anna and Katrina	$w = (n + 2)\cdot 2 + (n + n)$
Ben and Joe	$L\times 4 + 4$

The first two rules, $4n + 4$ and $4(L - 2) + 4$, could represent counting the sides without including the corner tiles. The students knew that there were four of these sides (n or $(L - 2)$), and then added the four corner tiles (+ 4). Anna and Katrina's rule represented counting the tiles slightly differently. They counted the two long sides ($n + 2$), including the corners, and then added the two short sides ($n + n$).

There were only two other situations where students studied the way the pattern changed spatially to write their generalizations. Anna and Katrina used a visual model in their solution of *Dominoes*. Ben and Joe analyzed *Toothpicks* geometrically to find their rules.

Recall that Anna and Katrina explored the Case 1 (considering 1-0 and 0-1 as two distinct dominoes) interpretation of *Dominoes*. The girls relied on a visual model or tree diagram to describe how they wrote $(n+1)^2$. Katrina wrote a nice explanation to visually support their rule (See Figure 49).

3. How do you know that what you wrote for the above is true?

because when you have just 0 - 3 spots you add 1 on which represents 0.

Ex: 1 1 2 3 - there are 3 possibilities, but with 0 you have to add 1.

Figure 49: Katrina's Answer to Question 3 in *Dominoes*

Ben and Joe described how the individual pieces of the shapes in *Toothpicks* were represented in the rule. They initially had made an error for the perimeter toothpicks, but recognized their mistake and corrected it. After they rewrote their rule for the perimeter, it was easy for them to revise the total number of toothpicks based on their correction.

When students studied these problems using a geometric approach, they considered the shape of the n^{th} figure of representation. They supplied visual support of how the parts of the rule related to the pieces of figure n. This keeps their rules related to the context of the problem. The students who wrote $4n + 4$ for the number of white border tiles knew that the constant term of 4 represented the 4 corners in the shape.

Graphic Representations

Strong problem solvers often have multiple strategies to investigate problems. Some students relied on a third strategy when their others did not result in a rule. In some instances the students discussed the graphs of the patterns as a way to explore the problem. They were not asked to consider the graph, but some used this to aid in their investigation.

Sara and Ryan discussed graphs while investigating *Cutting* and *Dominoes*. They did not find rules in either of these patterns, but considered graphs in both.

Although the exponentials represented an unfamiliar task for these students, Ryan offered a reasonable description of the shape of both the growth and decay graphs, "Whatever it is, it would take, this [the number of sheets in a stack pattern] sort of rises, going in the positive direction and this [the area of the pieces] is going in the negative direction."

Ben and Joe sketched similar increasing and decreasing graphs on their papers to represent the patterns while they searched for a rule. There are several cases where the students made some reasonable sketches to help them think about the problems. Although, it is not clear how the students used this to help them write rules, this demonstrated an understanding of the pattern that may not otherwise be evident.

Connections

Making mathematical connections is an important idea for students. One of the four common standards in the *NCTM Curriculum and Evaluation*

Standards (1989) is "Mathematical Connections." A part of the middle grades standards state,

> STANDARD 4: MATHEMATICAL CONNECTIONS
> In grades 5-8, the mathematics curriculum should include the investigation of mathematical connections so that students can--
> - see mathematics as an integrated whole;
> - explore problems and describe results using graphical, numerical, physical, algebraic, and verbal mathematical models or representations;
> - use a mathematical idea to further their understanding of other mathematical ideas;
> - apply mathematical thinking and modeling to solve problems that arise in other disciplines, such as art, music, psychology, science, and business;
> - value the role of mathematics in our culture and society. (page ??).

The curriculum that the students had used was called the *Connected Mathematics Project*; this title emphasizes the importance of connections to the authors.

Students demonstrated many different connections in their solutions to these problems. There was evidence of the first three bullets as stated in the Standards. They made (a) connections in each pattern between different representations, (b) connections between the four tasks and additional problems, and (c) connections with other mathematics.

Connections between Representations

The students readily made connections between several different representations of the patterns. I have already described the links between tabular and symbolic forms in the earlier discussion surrounding constant differences. Students recognized that tables of data with constant first differences were linear and tables of data with constant second differences were quadratic. They were unsure when they could not find a constant difference about the form of the equation. Several students offered evidence connecting a third representation, connecting the graphs of the patterns.

Connections between Tasks

Some students connected the linear problems in the *Borders* and *Toothpicks* as similar. They recognized the rule, tables, and graphs as similar. Although all students did not write symbolic rules for *Dominoes* some connected it to the other quadratics either through sketches in their work (see Dan's work for *Dominoes*) or discussions we had. In an exchange during an interview Ryan explained the similarities in the graphs between patterns in several tasks.

AK: Is this [*Dominoes*] similar to any of the other problems I have asked you to do?

Ryan: The *Toothpicks* one, sort of ...

AK: How is it similar to *Toothpicks*?

Ryan: It forms a parabola in the graph.

AK: How do you know it forms a parabola in the graph?

Ryan: The way the table is written when you write it out.

AK: What about the table tells you that it is a parabola?

Ryan: It increases by more each time. See right here, it increases by three, then it increases by four. Makes it curve up more.

AK: Was that the same as *Toothpicks*?

Ryan: Yes.

AK: Were there any other parabolas?

Ryan: This one, one a in *Borders*.[4]

Since they additionally found symbolic rules in *Borders* and *Toothpicks* they extended their connections between three representations—tables, graphs, and symbols.

The students completed a fifth task, *Dot Pattern*, which I do not analyze here.[5] Sara and Ryan again made connections between the quadratic situations in *Dominoes* and *Toothpicks* with the quadratic pattern in *Dot Pattern*. They recognized the similarities in the quadratic patterns in both tabular and graphic form. When questioned regarding any similarities or

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[4] 1. a) If the pattern of blue squares with white borders continues, how many **blue** tiles are needed to build the 4th square? the nth square? Show how you figured this out.

[5] *Dot Pattern* asked the students to create their own pattern based on this first figure and then write several statements based on their pattern. Sara and Ryan completed the pattern with the triangular numbers that formed a quadratic pattern similar to the one they found in *Dominoes*. The 3, 6, 10, 15, and 18 were the total number of dots for the first five figures sketched below. The numbers between these numbers (3, 4, 5, and 6) represented the first differences. The following written represents their written work.

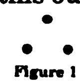


Figure 50: ***Dot Pattern*** **Graphic**

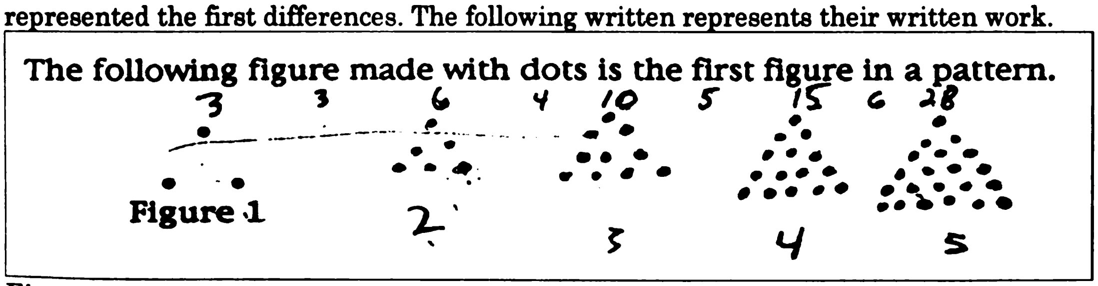


Figure 51: ***Dot Pattern*** **Written Response**

differences between the tasks, Ryan first noted that *Toothpicks* and *Dominoes* were similar. His clarification was, "Similar in the table, the way it increases. In *Dominoes*: eight, ten, twelve and fourteen. In *Toothpicks*: three, four, five, six, and seven." He recognized the constant increase in the first difference or that the second difference in both cases was a constant number.

They added to their earlier discussion of the similarity of the graphs but described that all had the shape of parabolas because of the data from the table.

AK: Were there any other patterns?

Ryan: Forms a parabola, the number of dots.

AK: How do you know that?

Ryan: The way the pattern is.

Sara: They increase by three, then four, then five, then six.

AK: How do you know that is a parabola?

Ryan: If you were to plot it. It starts out shallow and starts to get steeper and steeper until it is almost straight up and down.

AK: How do you know?

Ryan: It increases by more and more each time.

While saying this Ryan made the sketch in Figure 52 to support his claim.

Figure 52: Ryan's sketch during *Dot Pattern* interview.

In solving the problems that the pair saw as algebra, Sara and Ryan made comparisons across the three different representations—tables, graphs, and equations—of the relationship. Although they did not always find symbolic generalizations for the patterns, they demonstrated a deep understanding of the pattern.

Other students sketched graphs in their written work; some discussed it during their investigations or interviews. Although they were not specifically asked to consider graphs, they extended their solutions to include this representation. The following diagram, Figure 51, illustrates the connections that several students made between tasks, and the representations in the tasks.

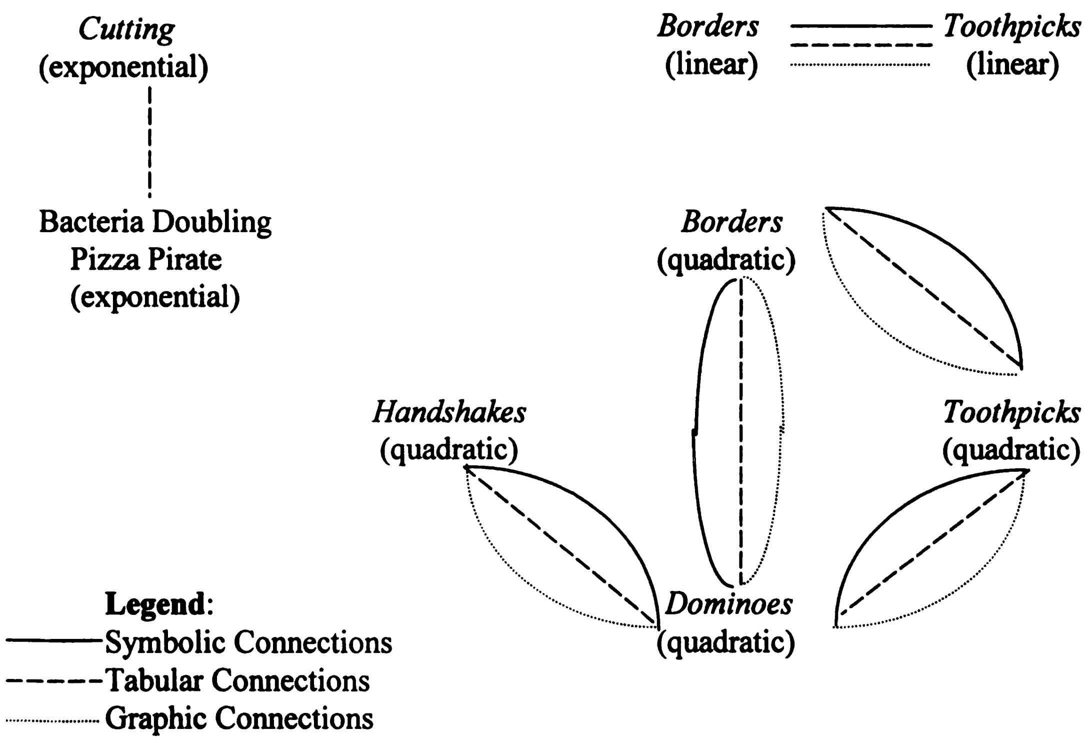


Figure 53: Connections between Tasks

In addition, some students linked the mathematical structure of the tasks with problems they had previously solved. While two pairs connected *Cutting* to their seventh grade science problem, Anna and Katrina related it to a problem from their seventh grade mathematics.

AK: What do you think this problem [*Cutting*] is about?

Anna: Umm, like taking things and cutting them in half and keep cutting them in half. There was a pizza problem that we did like this last year.[6] When they came at night and took half of the pizza and left half, and then half. They kept taking half.

AK: How was this similar?

Anna: Pretty much the whole thing, see how much you have.

The Pizza Pirate Problem the pair referenced had a discrete exponential pattern. They recognized that both patterns looked at the way data doubled and halved.

Both of these problems had similar mathematical structure. In the science class, the students did use exponential notation to discuss the time it took doubling bacteria to reach the moon. The students recalling the Pizza Pirate Problem did not use exponential notation. The problem was posed to

---

[6] This was Problem 4.3 from the *Bits and Pieces II* unit of *CMP*. A Pizza Pirate was raiding the pizza in a freezer that a class was saving for their party. On the first night he crept in and ate half of the pizza. On the second night, he ate half of what was left. Each night after that he ate half of the pizza that remained. Students investigated what fraction of the pizza was left after so much time.

consider using fractions. But, both problems were reasonable to use as comparisons.

Another comparison students made connected with *Dominoes*. Although these students did not see that task prior to my administration of it, several pairs recognized it a similar structure to some of the earlier problems they did in their mathematics class.

Both pairs, Zachary and Todd, and Anna and Katrina felt they had done similar problems prior to this in their mathematics class. Anna and Katrina felt it was somewhat like the "handslaps, high fives, or handshakes" problems they completed because of the way they charted the possibilities. Zachary and Todd noted other similarities.

AK: Have you ever done this problem before?

Todd: I don't think so.

Zachary: I don't remember.

AK: Did you do anything kind of like it?

Todd: Yeah.

AK: When was that?

Todd: It was like only about a week ago, in this book, it was how many games would a team play if there were five teams and every team played every other team

AK: How was it the same?

Todd: I remembered making a chart kind of like this, where there were certain possibilities, certain ones that were doubled so we crossed them out.

Later, during the interview, Zachary admitted that it seemed similar to another problem, "I remember a problem like this except it did not have the N plus one. In the book of *Frogs*[7], I think, with handshakes."

In both of these tasks, *Cutting* and *Dominoes*, students connected the problems to previous problems they had completed and used that information to assist with completing the tasks. The tasks may be situated in very different contexts, but the students looked beyond this to see the similarities in the structure of these tasks. This is a very important problem solving skill for these students to possess.

Connections with Other Mathematics

After each pair of students completed the tasks, I asked what mathematics was involved in the problem. One student stated the connection in the curriculum during an interview. Once they completed the final task, I pushed further to find out what they felt were some of the different areas of mathematics. Todd responded, "I don't know. In *CMP* math the teacher doesn't say right now you are doing algebra, right now you are doing geometry." Todd did not want to parse out his mathematics into separate

[7] He referred to the *Frogs, Fleas, and Painted Cubes* quadratic unit from *CMP*.

categories, he further determined that he saw mathematics as mostly related and connected.

Disposition

There is more to learning mathematics than knowing the content. Again I refer to the *NCTM Curriculum and Evaluation Standards* (1989) document as a source to consider what educators envision. The authors wrote of helping students develop a mathematical disposition,

> *Disposition* refers not simply to attitudes but to a tendency to think and to act in positive ways. Students' mathematical dispositions are manifested in the way they approach tasks--whether with confidence, willingness to explore alternatives, perseverance, and interest—and in their tendency to reflect on their own thinking (page ??).

When I initially designed this study, I had not intended to look at students' disposition. Then, some evidence unfolded that pushed me to consider their attitudes toward mathematical problem solving more carefully.

There is strong evidence that these students approached mathematical tasks with positive attitudes. They used a number of strategies to investigate the problems. When they were presented with an unfamiliar problem they had some strategies to investigate the situations. They seemed confident that they would be able to solve the problems and were disappointed when they could not.

I was also surprised at how long the students persevered on a challenging problem. It frustrated them when they could not finish a problem, but they persisted. Julie was quite animated while working. During their solution of *Cutting*, she bemoaned, "I hate not being able to do this." When they investigated *Dominoes* she exclaimed, "This problem is driving me crazy!" while she mocked pulling at her hair. She expressed her frustration, yet always continued to explore the problems.

The students all worked at *Cutting* until I reassured them that it was understandable that they might have a difficult time with the problem and we could discuss a solution together. I think that they might have worked on the problem longer had I given them the opportunity. Some students did continue their investigation, even after my warning.

These students all demonstrated confidence in their mathematics ability, willingness to explore alternatives, and perseverance. I felt they all revealed a very positive disposition towards mathematics.

Summary

When these students who had spent three years in a reform curriculum confronted a situation involving patterns in functions, they demonstrated understanding in several areas of mathematics. Each of the situations the student pairs investigated could be represented by a function—linear, quadratic, or exponential—embedded in different contexts. These students demonstrated a deep understanding about various algebraic functions. They all had a solid understanding regarding linear functions. There seemed to be a lot that they understood about quadratics, but occasionally had some difficulties in pulling it together to write a generalization. When a pattern did not fit one of these two categories they relied on several strategies to help them investigate, but were unclear what to do for the generalization.

They all had reasonable strategies at their disposal to investigate various function types. Students made connections among the tasks they solved for this study and with other problems they had previously investigated. In all cases, they held dispositions towards mathematics that encouraged them to persevere with the difficult problems.

CHAPTER 6

Implications and Limitations

Implications

There are two major findings in this study surrounding students' understanding. First, students who had three years in the *Connected Mathematics Project* curriculum demonstrated a rich understanding of a significant piece of algebra. And second, teachers can learn much more about students' understanding in algebra by drawing on multiple sources of evidence, and not relying solely on their written work.

A Rich Understanding of an Important Piece of Algebra

There is not consensus among educators regarding what constitutes algebraic thinking. Some would take issue with my view on the discipline. I do not claim that this study represents students' complete algebraic understanding, but it does demonstrate that these students have significant understanding of an important aspect of algebra.

Although this study draws on a narrow slice of algebra, it represents a very significant piece of the discipline. In Chapter Two, I write of the importance that studying patterns plays in the foundation of algebraic thinking. Studying and symbolically generalizing these patterns cut across all of the organizing themes presented by the *Algebra Working Group* (1997).

NCTM published a discussion draft of the *Principles and Standards for School Mathematics* in 1998 to build on the foundation of the three *Standards* documents published earlier. In their overview, the authors write the following algebra standard across grades K-12.

Standard 2: Patterns, Functions, and Algebra

> Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students-
> - understand various types of patterns and functional relationships;
> - use symbolic forms to represent and analyze mathematical situations and structures;
> - use mathematical models and analyze change in both real and abstract contexts (page 56).

The NCTM authors write further that,

> Patterns, functions, and algebra encompass the systematic use of symbols, algebraic characteristics of mathematical systems, modeling of phenomena, and the mathematical study of change. These notions are not only linked to one another, but also closely linked to number and operations and to geometry. They are essential to all areas of mathematics and form the basic language in which mathematics is expressed. *Ideas included within this standard compose a major component of the school curriculum* (page 56). (Emphasis added)

This study addresses all three of the main components of the Patterns, Functions, and Algebra Standard.

Studying patterns was the starting point for this study. Mathematics is sometimes considered the "science of patterns," and represents a fundamental aspect of the discipline. In studying patterns the regularity one notices can be used to predict other values. I narrowed my field of patterns to only those that held the potential for representation as a function. This kind of work can be seen as a precursor to more formal investigations with functions.

In addition to studying patterns, I asked students to generalize the patterns they recognized with symbolic representations. The use of symbolic notation is also a powerful idea in mathematics. Some say that "(s)ymbolic representation of quantitative relationships lies at the heart of algebra" (NCTM, 1998, page 58).

This study, focused on the study of patterns, also included mathematical modeling. Once the students recognized their patterns, they were asked to write symbolic generalizations that modeled the situations. "One of the most powerful uses of mathematics is the mathematical modeling of phenomena (NCTM, 1998, page 60)." They were given a situation and asked to write a mathematical model to represent it.

The students in this study demonstrated their competence in algebra in a number of ways. The *Algebra Working Group* (1997) and Kaput (1995) both recognize that solid algebraic understanding involves experiences across all organizing themes of algebra. The students demonstrated competence across three of the *Algebra Working Group's* themes: *functions and relations*, *modeling*, and *language and representations*. I did not see evidence of their understanding of the *structure* of algebra, but that was not the intent of the tasks or this study.

While the ten students of this study demonstrated understanding across the organizing themes, they met the ambitious algebra goals established by the *CMP* authors. The students showed that they had more than a procedural understanding of the algebra and were able to think deeply about the mathematics involved in this collection of problems.

Understanding Functions and Relations

When approaching algebra through the *functions and relations* theme, one of the main ideas is the focus on the rate of change. Making numeric tables of data was the dominant strategy employed by all students in this study to attend to the rate of change. After students made tables, a common tool for analysis was to search for constant differences. When they found that the first or second difference was constant this informed them about the patterns. A constant first difference meant the pattern was linear, while a constant second difference told them it was quadratic.

All five pairs of students considered the rate of change when they constructed tables and found constant differences. The students not only noted the differences, but also related the differences to graphs and described various patterns of change. They described that the linear patterns increased by the same amount while quadratic patterns, "increase by more each time."[1] These students recognized how the rates of change in the tables affect the patterns. They used this knowledge to inform their generalizations. They had reasonable general forms for the symbolic expressions of linear and quadratic patterns that they matched with the pattern.

The students in this study also demonstrated a very strong sense of changing quantities in the situations. They recognized the important quantities that changed and the dependence between the quantities. All students drew reasonable conclusions about the quantities in all tasks. There were several instances in the challenging tasks (the quadratic patterns in *Toothpicks* and *Dominoes*, and the exponential patterns in *Cutting*) that while the students searched for rules they worked hard to make sense of the variables. There was evidence while they worked of how they clarified the independent and dependent variables in the situation.

The students did not seem to struggle as much with the variables in the less complex problems (the linear patterns in *Borders* and *Toothpicks*, and the quadratic pattern in *Borders*). They easily selected variables that seemed to fit the problem. In *Borders*, one pair used different variables (length and width instead of figure number) than the others for their independent variable, but with their interpretation they found a reasonable solution for the problem.

[1] Quote from Ryan after during our interview after he completed *Dominoes*.

Understanding Modeling

The students *modeled* the situations when they sought symbolic generalizations to represent the patterns in the data. In all situations they collected, organized, and studied their data in search of a rule. Some students studied numeric data in a table, while others considered a geometric representation. Once the students recognized a pattern in their data they usually wrote their generalization. They often extended their table to find additional values and then verified the additional data generated from the table with their rule. In all instances the students looked for rules that modeled their data, and helped them to predict other values.

In some instances, the students used a graph model to describe the data. They discussed the shapes of the graphs in some tasks and used patterns in the shapes to help predict other values. These students felt that studying the graphs could be helpful to describe their patterns.

Understanding Language and Relationships

All students demonstrated competence with the *language* of several different *representations* of the patterns. All constructed tables, all found some symbolic rules, and some pairs considered graphical representations. In all instances the students used and connected different representations.

The dominant representation the students used to study the patterns was a numeric table. They recognized both linear and quadratic patterns in this tabular representation by finding constant differences. Once they knew the function type, linear or quadratic, these students drew reasonable conclusions about the symbolic representations. The tasks of this study did not pose specific questions that asked students to consider other representations, but some students additionally discussed their understanding of linear and quadratic patterns in graphic form and related that to other representations.

There was not sufficient evidence from this study to evaluate all students' understanding of the graphical representations since I did not pose questions that addressed this in any of the tasks. However, several students did demonstrate competence with this representation.

The students were not as solid in exponentials, the third pattern, that I asked them to investigate. Some pairs were unfamiliar with this pattern since they had not yet studied the *CMP* exponential unit in their mathematics class. It is important to note that although they could not classify the exponential pattern, they did not try to classify it as linear or quadratic when they observed the pattern of differences in the table. They did not use linear or quadratic rules inappropriately. The students who sketched a graph also recognized it as different than the other two patterns.

Understanding Deeply

Masingila (1998) writes about the difference between knowing some mathematical idea procedurally and knowing it conceptually. She acknowledges the importance of students' understanding mathematical concepts. If what these students learned through the *CMP* curriculum is to be classified as meaningful mathematics, they needed to learn more than just a new algorithm to generate symbolic rules. I believe that these students did have a deep understanding of symbolically generalizing patterns from data.

If these students were to have only a procedural understanding, then you could expect all of the students to arrive at similar generalizations by following the procedure. In most of the less complex cases (linear patterns in *Borders* and *Toothpicks* and the quadratic pattern in *Borders*) students did arrive at expressions that appear similar, but in the more complex quadratic cases of *Toothpicks* and *Dominoes* students found very different looking equivalent expressions.

The students quickly wrote the symbolic rules in both linear cases in *Borders* and *Toothpicks* and the quadratic case of *Borders*. It is apparent in Table 19 below that all students found the same form of a symbolic generalization for the linear case in *Toothpicks*. All pairs, except Sara and Ryan, used the generalization of x^2 or $x{\times}x$ for the *Borders* quadratic pattern; Sara and Ryan found a slightly different expression based on their choice of variables. Zachary and Todd, Ben and Joe, and Julie and Dan all found similar looking rules in the linear case of *Borders*, while Sara and Ryan and Anna and Katrina based their rules on geometric interpretations and found something that looked slightly different.

Table 19: Symbolic Rules Generated for Simple Cases

Students	Tasks		
	***Borders* Linear**	***Toothpicks* Linear**	***Borders* Quadratic**
Zachary and Todd	$4x + 4$	$4N$	x^2
Ben and Joe	$L\times4+4$	$Y = 4N$	L^2
Julie and Dan	$4n+4$	$y = 4n$	n^2
Sara and Ryan	$4(L\text{-}2)+4$	$4N$	$(L\text{-}2)(W\text{-}2)$
Anna and Katrina	$w = (n+2)\cdot2+(n+n)$	4N	$b = n\times n$

There was not much discussion between the pairs in the recordings regarding their solutions for these patterns. I suggest that these students do have a solid understanding in these cases, although their work might appear to be somewhat procedural. Their ease of working with these patterns and their connections with other representations during our discussions supports this view. Additionally, their work with the more complex quadratic suggests a solid understanding of these cases.

These students seemed to have a conceptual understanding of the more complex cases also. Table 20 below lists the four equivalent, but different expressions they wrote to represent the patterns found by the five pairs of students for the symbolic generalizations for the quadratic patterns in *Toothpicks* and *Dominoes*.[2]

Table 20: Symbolic Rules Generated for More Complex Cases

Students	Tasks	
	***Dominoes* Quadratic**	***Toothpicks* Quadratic**
Zachary and Todd	$(n+1)\div 2\cdot n+n+1$	$n\cdot(n+3)$
Ben and Joe	$y = (N+1)(0.5N+1)$	$y = 4N + (N-1)N$
Julie and Dan	No rule	$y = n(n+3)$
Sara and Ryan	No rule	$[(x+1)(x+1)]+x-1$
Anna and Katrina	–	$n^2 + 3n$

In their solutions, three pairs made tables and studied constant differences; Sara and Ryan studied the differences in numeric data not organized in a table, while Ben and Joe studied the pattern of the *Toothpicks* in the changing shapes to write the rules. Only two of the four pairs of students who worked with this interpretation of *Dominoes* arrived at symbolic rules. These two expressions looked quite different.

In all of these more complex cases the students had access to a number of tools that helped them investigate the problem in a meaningful way. Since their final rules did not look the same, in fact some appear quite different, this supports the view that these students did not follow an algorithm to

[2] I refer to the Case 2 interpretation of *Dominoes*, counting the 0-1 and 1-0 as one domino.

generate symbolic rules. They used their understanding of the situations to generate rules.

Assessment

This study also offers supporting evidence surrounding the uses of multiple forms of assessment. If teachers want to know students' sense making of algebraic ideas, then a range of forms of assessment is required. To paint a picture of students' understanding it requires administering high-quality tasks, observing students while they work, and talking with students.

High-quality tasks have some or all of the following characteristics. They must engage all students while challenging them at the same time. They should allow for multiple ways to find a solution. They should capture important mathematics. They should support student discussions. They should require higher levels of thinking. They should have the potential for students to make connections. The problems should be based on sound mathematics, perceptions of students experiences, and knowledge of diverse ways that students learn mathematics (NCTM, 1991).

It is not enough to consider only students' written responses. In this study, much of the students' thinking was not recorded in their written work. In some instances I saw that students understood much more than what they recorded on their papers. Watching the videotapes of the pairs working gave me some insight into this additional understanding. Together, written responses and careful observations still left some aspects of their understanding hidden. I learned much more by talking with students about their understanding that was not evident in the other data sources. Much of students' understanding about the graphs became apparent in our discussions. In their work with graphs, the students often connected the pattern in the problem with the pattern for other problems and connected the representations. I might not have seen what these students really understood had I not collected this additional data.

A goal of assessment is to accurately represent what students understand; good assessment strategies will aid teachers with this task. For classroom teachers, this means that they need to be aware that the written record may not tell everything about students' understanding. When evaluating students' understanding teachers need to be diligent about collecting multiple forms of assessment to more accurately represent this. They need to be careful observers while students work. It may not be practical for them to make video recordings of students working, but they can make careful notes while observing students engaged in the tasks. They should talk with students about what they have written and ask them to clarify what they have done.

Limitations

In any study it is important to acknowledge the limitations. In Chapter Three I offer support regarding the decisions I made for this study. In this section I describe three limitations based on the design of this study. First, my selection of the performance tasks I administered limited what I could learn about the students' understanding. Second, I did not study the implementation of the curriculum, and so I can not dismiss the importance of the teachers' role in the success these students had with algebra. And, third, I did not have a control group and so I am unable to draw any comparison conclusions.

Tasks

My selection of tasks limits what I could learn about these students' algebraic understanding. This study considers students' symbolic generalizations of data from specific patterns. This cut on algebra is not intended to represent the entire discipline or evaluate students' complete algebraic understanding, rather to survey students' reasoning in an aspect that represents a part of the foundation of the discipline. I described earlier in this chapter that this narrow cut of algebra is quite significant for students to learn. But, all of the tasks involved whole numbers and one could speculate that other tasks with rational numbers might have proved more challenging.

Implementation

I recognize that the nature of the implementation of any curriculum is vital to its success or failure. But, the implementation of the *CMP* curriculum was not the focus of my study. Even though this was not part of my study, I acknowledge that students coming out of the eighth grade are not just prepared because of the curriculum. Quality curriculum is not sufficient for success; it may be necessary, but not sufficient. There are a number of interesting stories to be told surrounding Heartland. Some have already been done regarding this site (Bouck, M. and Wilcox, S., 1996).

A key element to these students' achievement is that they had quite skilled teachers in grades 6 through 8. There is quite a bit of local knowledge surrounding Evelyn Howard, the teacher these students all had in grade eight, and some in grade seven. She is a very accomplished mathematics teacher who is held in high regard in the community. I feel quite certain that Evelyn's skill teaching mathematics is a contributing factor to the success for these students.

It should again be noted that this school had been seriously working on professional development for teachers for a number of years and offered considerable support for teachers. Many, including Evelyn had the opportunity to work closely with the developers of the *CMP* curriculum as part of the piloting agreement. This association helped the teachers refine their teaching and better understand the curriculum.

Comparison

This study was not designed to compare the algebraic understanding of students from the *CMP* curriculum with that of students who had more traditional experiences in mathematics. Instead, my intent was to carefully describe the potential for students with three years in the *CMP* curriculum to develop students' algebraic thinking. This study suggests that there is great promise for *CMP* students to develop a solid understanding of symbolically generalizing from patterns of data.

There have been other studies done as a comparison and found that CMP students did significantly better on challenging open-ended response tasks than non-CMP students. In addition, on a traditional, multiple-choice test, CMP students made gains comparable to the other students (Hoover, M., Zawojewski, J., and Ridgway, J., 1997). This means that the CMP students do better on the open-ended items and do not do any worse on the traditional items.

Summary

This study represents an analysis of the learning of students in one curriculum, in one site, and of one important slice of algebra. Even with these limitations, this study provides an opportunity to get a very good look into these students' reasoning about algebraic situations. It offers compelling evidence that this *Standards*-based curriculum has great potential for student learning in algebra. A curriculum that focuses on algebra as much more than symbolic manipulation, that has taken functions, modeling, and representations as key components can provide students with a solid understanding in a fundamental area of algebra.

A study such as this begins to answer some questions, but it also raises many more. There is certainly much more to be learned about students' understanding regarding other areas of algebra with these curricula. An additional set of questions important to study would look at the implementation of a standards-based curriculum. The authors of the *CMP* curriculum recognize the importance of good teaching. What does it take to successfully implement a curriculum like *CMP*? What kinds of experiences do teachers need to have to teach this curriculum? What do we know about students' knowledge prior to entering the *CMP* curriculum in the sixth grade? All of these questions are worthy of future study.

APPENDICIES

APPENDIX A

List of *CMP* Units

The units are listed in the form: **Title:** ***Subtitle,*** *Strand* in Table 17.

Table 21: Order of *CMP* Units

Grade Six Units	Grade Seven Units	Grade Eight Units
1. **Prime Time:** ***Factors and Multiples*** *Number*	1. **Variables and Patterns:** ***Introducing Algebra*** *Algebra*	1. **Thinking with Mathematical Models:** ***Representing Mathematical Relationships*** *Algebra*
2. **Data About Us:** ***Statistics*** *Prob. and Stats.*	2. **Stretching and Shrinking:** ***Similarity*** *Geom. and Meas.*	2. **Looking for Pythagoras:** ***The Pythagorean Theorem*** *Geom. and Meas., and Number*
3. **Shapes and Design:** ***2-D Geometry*** *Geom. and Meas.*	3. **Comparing and Scaling:** ***Ratio, Proportion, and Percent*** *Geom. and Meas., and Number*	3. **Growing, Growing, Growing...:** ***Exponential Relationships*** *Algebra*
4. **Bits and Pieces I:** ***Understanding Rational Numbers*** *Number*	4. **Accentuate the Negative:** ***Integers*** *Number*	4. **Frogs, Fleas, and Painted Cubes:** ***Quadratic Relationships*** *Algebra*
5. **Covering and Surrounding:** ***2-D Measurement*** *Geom. and Meas., and Number*	5. **Moving Straight Ahead:** ***Linear Relationships*** *Algebra*	5. **Say It with Symbols:** ***Algebraic Reasoning*** *Algebra*
6. **How Likely Is It:** ***Probability*** *Prob. and Stats.*	6. **Filling and Wrapping:** ***3-D Measurement*** *Geom. and Meas., and Number*	6. **Hubcaps, Kaleidoscopes, and Mirrors:** ***Symmetry and Transformations*** *Geom. and Meas., and Algebra*
7. **Bits and Pieces II:** ***Using Rational Numbers*** *Number*	7. **What Do You Expect?:** ***Probability and Expected Value*** *Prob. and Stats.*	7. **Samples and Populations:** ***Data and Probability*** *Prob. and Stats.*
8. **Ruins of Montarek:** ***Spatial Visualization*** *Geom. and Meas.*	8. **Data Around Us:** **Number Sense** *Number, and Prob. and Stats.*	8. **Clever Counting:** ***Combinatorics*** *Prob. and Stats.*

APPENDIX B

Interview Questions

1. Have you ever done this problem before?
2. Have you ever done this type of problem before? If yes, explain.
3. What do you think this problem is about?
4. What do you think you are asked to do in this problem?
5. What mathematics do you think is involved in this problem?
6. How did you decide what strategy to use? What strategies did you use?
7. When did you decide to try a new strategy (if so)?
8. How did you decide what variables to use to solve this problem?
9. When do you know to stop looking for a pattern and start to generalize?
10. How would you justify your solution?
11. How would you verify your results?
12. Is this similar to any of the other problems I have asked you to do? In what ways?
13. Did anything about this problem surprise you?

APPENDIX C

Time Spent by Students Working on Tasks and During Interviews

Table 22: Approximate Number of Minutes Spent on Each Task

Students	Tasks			
	Borders	*Cutting*	*Dominoes*	*Toothpicks*
Zachary and Todd	17	21	24	8
Ben and Joe	10	34	33	8
Julie and Dan	13	30	31	10
Sara and Ryan	15	27	41	44
Anna and Katrina	20	29	13	26

Table 23: Approximate Number of Minutes Spent on Each Interview

Students	Tasks			
	Borders	*Cutting*	*Dominoes*	*Toothpicks*
Zachary and Todd	16	13	22	20
Ben and Joe	18	29	30	17
Julie and Dan	10	24	30	25
Sara and Ryan	14	17	10	13
Anna and Katrina	5	19	22	18

APPENDIX D

Name ______________
Date ______________

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

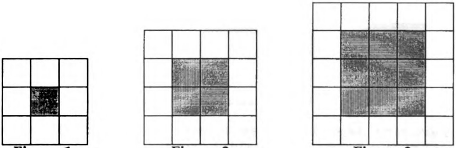


Figure 1 Figure 2 Figure 3

1. a) If the pattern of blue squares with white borders continues, how many **blue** tiles are needed to build the 4th square? the nth square? Show how you figured this out.

   b) If the pattern of blue squares with white borders continues, how many **white** tiles are needed to build the 4th square? the nth square? Show how you figured this out.

2. Suppose the blue tiles are arranged as rectangles of any length and width.

   a) How many **blue** tiles are needed for this? Show how you figured this out.

   b) How many **white** tiles are needed for this? Show how you figured this out.

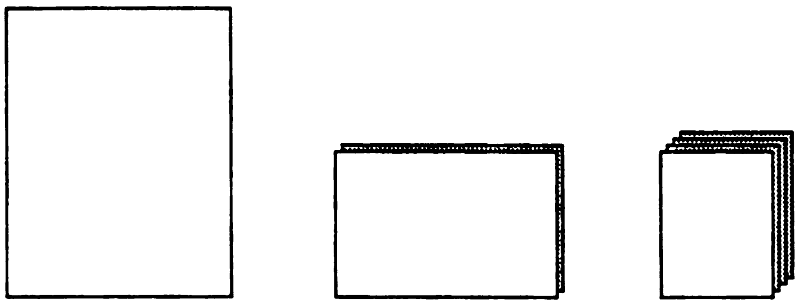
Figure 54: *Borders* Task

Name ____________________

Date ____________________

Cutting and Cutting and Cutting . . .

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.



At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. Describe what happens after 3 cuts. How many pieces of paper do you have in the pile?

2. a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out.

   b) For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.

   c) How many cuts would you need to get a pile that is 1 foot thick?

3. Suppose the original piece of cut paper has an area of 32 cm^2.

   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out.

   b) After how many cuts would you get a piece that is 1 cm^2.

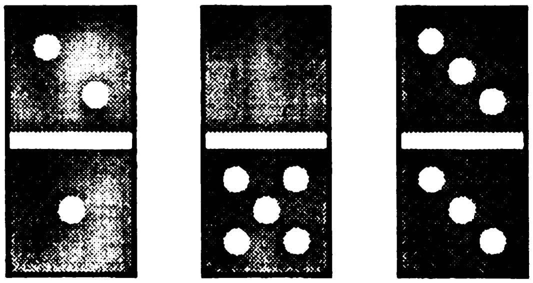
Figure 55: *Cutting* Task

Name ____________________

Date ____________________

Dominoes

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.



1. a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.

   b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?

   b) 0 to 3 white spots?

   c) 0 to 6 white spots?

   d) 0 to n white spots?

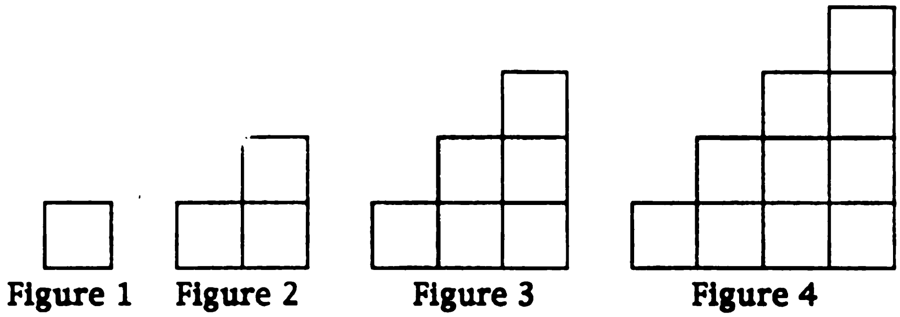
3. How do you know that what you wrote for the above is true?

Figure 56: *Dominoes* Task

Name ______________________

Date ______________________

Toothpicks



1. Extending the pattern, what is the perimeter of **Figure 5**? Show or explain how you figured this out.

2. How many toothpicks are needed to make **Figure 6**? Show or explain how you figured this out.

3. How did you decide how each new figure in the sequence is made?

4. Write a formula that you could use to **find the perimeter** of any Figure N. Tell what your variables represent. Explain how you figured this out.

5. Write a formula that you could use to find the **total number of toothpicks** needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out.

Figure 57: ***Toothpicks*** **Task**

APPENDIX E

Task Solutions

Borders

These three squares have been made using blue tiles and then a border of white tiles is put around the blue square.

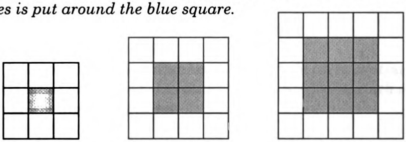


Figure 1 *Figure 2* *Figure 3*

Figure 58: *Borders* Figures

1. a) *If the pattern of blue squares with white borders continues, how many* **blue** *tiles are needed to build the 4th square? the nth square? Show how you figured this out.*

The pattern of squares can be continued to sketch figure four as in Figure 59. The blue interior tiles can be counted from that figure to find 16 blue squares.

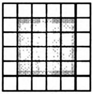


Figure 59: Figure 4 in *Borders* solution

To find the number of blue tiles in the n^{th} square, try to observe a pattern in the data. First, count the interior blue tiles from the shapes in figures 1, 2, and 3 and organize that information in Table 24.

Table 24: Figure Number and Blue Tiles in *Borders* solution

Figure Number	Blue Tiles
1	1
2	4
3	9
4	16

The number of blue tiles is the figure number squared, or

Blue tiles = n^2, where n is the figure number.

b) *If the pattern of blue squares with white borders continues, how many* ***white*** *tiles are needed to build the 4th square? the nth square? Show how you figured this out.*

To find the number of white tiles in figure four, use the sketch drawn above to count. The fourth figure has 20 white squares.

Use a similar strategy of observing a pattern to find the number of white tiles in the n^{th} square. Count the perimeter in figures 1, 2, and 3 and record that information in Table 25.

Table 25: Figure Number and White Tiles in *Borders* solution

Figure Number	White Tiles
1	8
2	12
3	16
4	20

The number of white tiles is a linear pattern since there is a constant rate of increase of four in the table. This can be used to extend the table back to find the number of white tiles in figure 0 to be four. The rule can be written as: White tiles = $4n + 4$, where n is the figure number.

2. *Suppose the blue tiles are arranged as rectangles of any length and width.*
   a) How many ***blue*** *tiles are needed for this? Show how you figured this out.*

To find the tiles in rectangular figures, consider the general rectangle with dimensions L by W as illustrated in Figure 60.

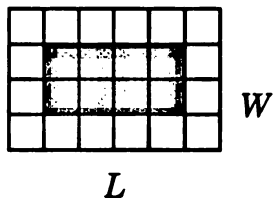


Figure 60: General Rectangle in *Borders*

The number of blue squares would be the inside area. The dimensions of the interior blue rectangle is $(L - 2)$ by $(W - 2)$. So, the number of blue tiles is:

$$\text{Blue tiles} = (L - 2)(W - 2)$$

b) How many ***white*** *tiles are needed for this? Show how you figured this out.*

If the number of white tiles were written as $2L + 2W$, the four corner pieces are double counted, so four must be subtracted:

$$\text{White tiles} = 2L + 2W - 4$$

Cutting

If you take a sheet of notebook paper and cut it in half and stack the pieces and then cut in half again and stack, and then in half again and stack, each cut gives smaller pieces but a thicker pile of paper.

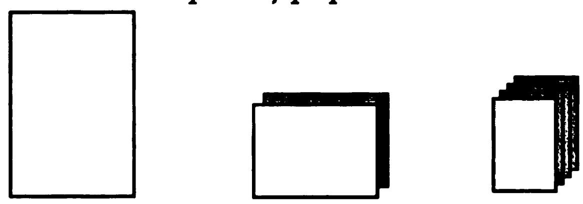


Figure 61: *Cutting* Graphic

At the start, before any cuts, there is one sheet of paper. After one cut, stack the pieces. The stack is now 2 sheets thick. After 2 cuts and stacking, the pile is 4 sheets thick.

1. *Describe what happens after 3 cuts. How many pieces of paper do you have in the pile?*

A table that gives the number of sheets of paper in a stack after each cut could be used to organize and display the data. First, the data that is given in the problem is put into Table 26.

Table 26: Cut Number and Sheets of Paper in Solution of *Cutting*

Cut Number	Sheets of Paper
0	1
1	2
2	4
3	?

It appears in the table that the data is doubling from the previous term. The number of sheets after 1 cut is double the number of sheets after 0 cuts. 2 is double the previous term of 1. Likewise, the number of sheets after 2 cuts is double the number of sheets after 1 cut, or 4 is double the previous term of 2. So, to find the number of sheets after 3 cuts, it would be double the number of sheets after 2 cuts, or 8 sheets of paper in the stack after 3 cuts.

2. *a) How many sheets of paper thick would the paper pile be after 4 cuts? 5 cuts? 10 cuts? 20 cuts? n cuts? Explain how you figured this out.*

From here the pattern can be extended to find the number of sheets after 4, 5, and 10 cuts. It seems rather cumbersome to complete the table to find 20, so first search for a pattern in Table 27 to find n.

Table 27: Cut Number and Sheets of Paper in *Cutting* solution up to 10 cuts

Cut Number	Sheets of Paper
0	1
1	2
2	4
3	8
4	16
5	32

Cut Number	Sheets of Paper
6	64
7	128
8	256
9	512
10	1024

To find each of the answers, multiply the previous term by 2, so the table can be rewritten as Table 28:

Table 28: *Cutting* solution with exponential notation

Cut Number	Sheets of Paper	Doubling Notation	Exponential Notation
0	1	1	2^0
1	2	1 x 2 = 2	$2 = 2^1$
2	4	2 x 2 = 4	$2 \times 2 = 2^2$
3	8	4 x 2 = 8	$2 \times 2 \times 2 = 2^3$
4	16	8 x 2 = 16	$2 \times 2 \times 2 \times 2 = 2^4$
5	32	16 x 2 = 32	$2 \times 2 \times 2 \times 2 \times 2 = 2^5$
6	64	32 x 2 = 64	$2 \times \ldots \times 2 = 2^6$
7	128	64 x 2 = 128	$2 \times \ldots \times 2 = 2^7$
8	256	128 x 2 = 256	$2 \times \ldots \times 2 = 2^8$
9	512	256 x 2 = 512	$2 \times \ldots \times 2 = 2^9$
10	1024	512 x 2 = 1024	$2 \times \ldots \times 2 = 2^{10}$

The answer to the questions are displayed in Table 29:

Table 29: Cut Number and Sheets of Paper in *Cutting* solution for 4, 5, 10, 20, and n cuts

Cut Number	Sheets of Paper
4	16
5	32
10	$2^{10} = 1,024$
20	$2^{20} = 1,048,576$
n	2^n

b) *For ordinary copier paper it takes about 250 sheets to make a pile 1 inch high. How thick (in inches) would a stack starting with one sheet of paper be after 4 cuts? 5 cuts? 10 cuts? n cuts? Explain how you figured this out.*

Since it takes 250 sheets to make 1 inch, after 4 cuts the paper stack would be $16 \div 250 = 0.064$ inches. The answers to the specific cases asked about in this question are displayed in Table 30.

Table 30: Cut Number and Inches Thick in *Cutting* solution

Cut Number	Inches Thick
4	$16 \div 250 = 0.064$
5	$32 \div 250 = 0.128$
10	$1{,}024 \div 250 = 4.096$
n	$2^n \div 250$

c) How many cuts would you need to get a pile that is 1 foot thick?

To have a stack one foot thick, that would be 12 inches, or $12 \times 250 = 3{,}000$ sheets thick. After 10 cuts it would be 1,024 sheets thick. By examining the table, we see that after 11 cuts, it would be double that or 2,048 sheets. 12 cuts would produce a stack 4,096 sheets thick. So it would not be until 12 cuts that the stack was at least one foot thick.

3. *Suppose the original piece of cut paper has an area of 32 cm^2.*
   a) What is the area of each piece formed after 2 cuts? 3 cuts? 10 cuts? n cuts? Show how you figured this out.

The area after 2 cuts would be $32 \div 4 = 8$ cm^2 since the sheet of paper is cut into 4 equal pieces after 2 cuts. The answers to the specific cases asked about in this question are displayed in Table 31.

Table 31: Cut Number and Area in *Cutting* solution

Cut Number	Area
2	$32 \div 4 = 8$ cm^2
3	$32 \div 8 = 4$ cm^2
10	$32 \div 1{,}024 = 0.03125$ cm^2
n	$32 \div 2^n$

b) After how many cuts would you get a piece that is 1 cm^2.

It would be when the sheet of paper was cut into 32 equal pieces. By examining the table earlier, we can see that this is after 5 cuts.

$32 \div 2^5 = 32 \div 32 = 1 \text{ cm}^2$.

Dominoes

Students interpreted this problem in two different ways. The first, CASE 1, was to count the 0-1 and 1-0 domino as one unique domino. In CASE 2, students counted 0-1 and 1-0 as two distinct dominoes. Since both solutions follow in my analysis of students' work I present both alternatives in this section.

Dominoes are spotted tiles used in a board game. A regulation domino tile is a black rectangle, split into square halves. On each half of one side there are from 0 to 6 white spots. The other side is blank. For example, three different domino faces are shown below.

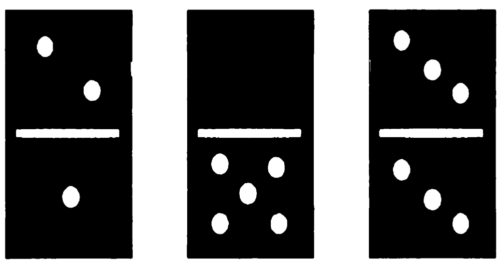


Figure 62: *Dominoes* Graphic

CASE 1: 0-1 and 1-0 as one unique domino

1. *a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.*

The sketch of the dominoes with from 0 to 2 white spots is represented in Figure 63.

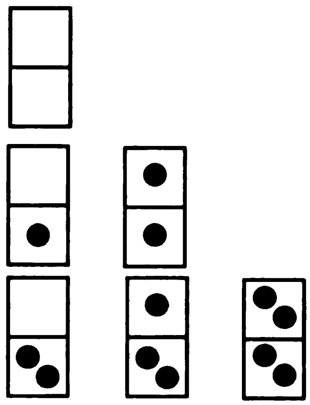


Figure 63: Sketch of Dominoes with 0-2 Spots—Case 1

b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?

The dominoes with from 0 to 3 white spots would include the six dominoes with from 0 to 2 (above) plus the following 4 dominoes in Figure 64.

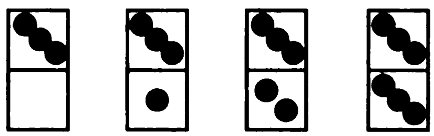


Figure 64: Additional Dominoes with 3 Spots

2. a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?

Counting the dominoes drawn in question 1 a, above, there are 6 dominoes with from 0 to 2 spots.

b) 0 to 3 white spots?

To find the dominoes with 3 spots add the four dominoes sketched in question 1 b, to the set of six dominoes with up to 2 white spots. There are 10 dominoes possible with from 0 to 3 spots.

c) 0 to 6 white spots?

The dominoes possible with from 0 to 6 white spots would be 28. For each set of dominoes add one more than the maximum number of spots to the previous set of dominoes. To find the number of dominoes with up to 3 spots add 4 for the dominoes 0-3, 1-3, 2-3, and 3-3. To find the dominoes with up to 4 spots, add 5: 0-4, 1-4, 2-4, 3-4, and 4-4 to the 10 dominoes with up to 3 spots. This is illustrated in Table 32.

Table 32: Maximum Spots and Possible Dominoes

Maximum Spots	Dominoes Possible
2	6
3	6 + 4 = 10
4	10 + 5 = 15
5	15 + 6 = 21
6	21 + 7 = 28

d) 0 to n white spots?

To find the number of dominoes with up to n white spots, consider the triangular array in Figure 65 sketched for up to 2 spots:

0-0		
0-1	1-1	
0-2	1-2	2-2

Figure 65: Triangular Array of Dominoes with up to 2 Spots

This array can be placed in the rectangle that has dimensions 3 by 4 illustrated in Figure 66.

0-0		
0-1	1-1	
0-2	1-2	2-2

Figure 66: Triangular Array in 3 by 4 rectangle

The number of dominoes is one half the area of the rectangle,

$\frac{1}{2}(3 \times 4) = \frac{1}{2}(12) = 6$ dominoes. This arrangement can be extended for any

n in a rectangle of dimensions $(n+1)$ by $(n+2)$ illustrated in Figure 67.

0-0		
0-1	1-1	
...	...	...
0-n	...	n-n

Figure 67: Triangular Array in $(n+1)$ by $(n+2)$ rectangle

The number of dominoes is still one half the area of the rectangle,

$\frac{1}{2}(n+1)(n+2)$

3. *How do you know what you wrote above is true?*

This can be verified using the diagrams sketched above.

CASE 2: 0-1 and 1-0 counted as two distinct dominoes

1. *a) Sketch all possible domino faces if your set is made up of dominoes with from 0 to 2 white spots.*

The sketch of the dominoes with from 0 to 2 white follows in Figure 68.

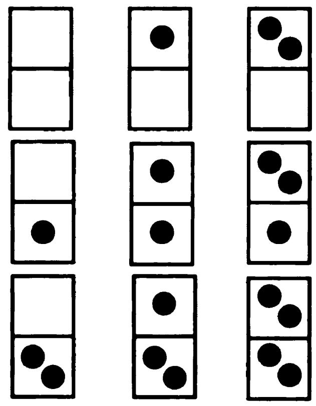


Figure 68: Sketch of Dominoes with 0-2 Spots—Case 2

b) How many different domino faces are there in a set made up of dominoes with from 0 to 3 white spots on each half of the domino?

The dominoes with from 0 to 3 white spots are sketched below in Figure 69.

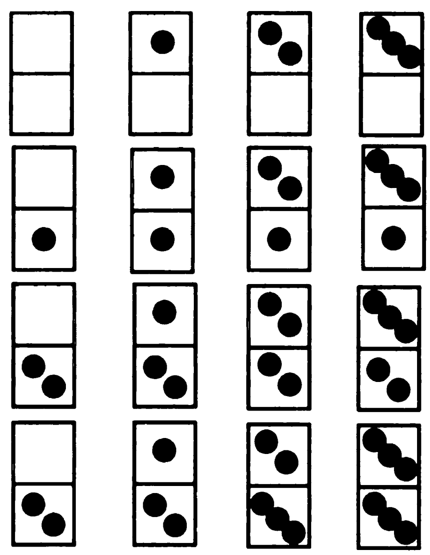


Figure 69: Sketch of Dominoes with 0-3 Spots—Case 2

2. *a) How many domino faces are possible if a set is made up of dominoes with from 0 to 2 white spots?*

Counting the dominoes drawn in question 1 a, above, there are 9 dominoes with from 0 to 2 spots.

b) 0 to 3 white spots?

Count the 16 dominoes drawn in question 1 b.

c) 0 to 6 white spots?

There are 49 dominoes possible with from 0 to 6 white spots. For each set of dominoes add one more than the maximum number of spots to include the zero. Then this number should be squared. So, for 6, you take 7-squared to get 49. This is illustrated in Table 33.

Table 33: Maximum Spots and Possible Dominoes in *Dominoes* Case 2 solution

Maximum Spots	Dominoes Possible
2	$3^2 = 9$
3	$4^2 = 16$
4	$5^2 = 25$
5	$6^2 = 36$
6	$7^2 = 49$

d) 0 to n white spots?

Following the description above, this can be generalized for n, $(n+1)^2$

3. *How do you know what you wrote above is true?*

We can show why you need to add one with the dominoes sketched with 0 to 2 white spots. You need to add one, to include the value for zero.

Toothpicks

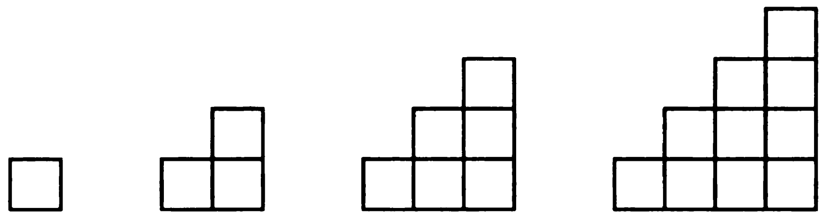


Figure 1 ***Figure 2*** ***Figure 3*** ***Figure 4***

Figure 70: *Toothpicks* Graphic

1. *Extending the pattern, what is the perimeter of* ***Figure 5****? Show or explain how you figured this out.*

   A sketch of figure 5 is shown in Figure 71, The perimeter can be counted from the sketch as 20 toothpicks.

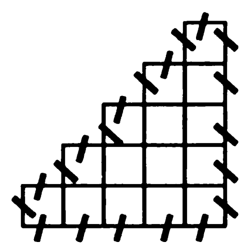


Figure 71: *Toothpicks* Figure 5

2. *How many toothpicks are needed to make* ***Figure 6****? Show or explain how you figured this out.*

   Figure 6 is sketched in Figure 72, 54 total toothpicks can be counted.

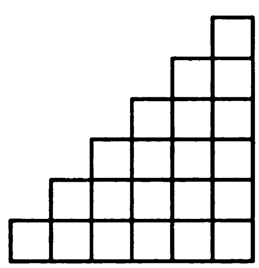


Figure 72: *Toothpicks* Figure 6

3. *Write a formula that you could use to* ***find the perimeter*** *of any Figure N. Tell what your variables represent. Explain how you figured this out.*

The perimeter toothpicks can be found from counting the first four shapes and recording that data in Table 34.

Table 34: Figure and Perimeter in *Toothpicks* solution

Figure	Perimeter
1	4
2	8
3	12
4	16
5	20

The table has a constant increase of 4, so this is a linear relationship.

The table can be extended back to find a perimeter of 0 for figure 0.

Perimeter = $4n$, where n is the figure number.

4. *Write a formula that you could use to find the* ***total number of toothpicks*** *needed to make of any Figure N. Tell what your variables represent. Explain how you figured this out.*

Table 35 shows the total number of toothpicks.

Table 35: Figure and Total *Toothpicks* in *Toothpicks* solution

Figure	Total Toothpicks
1	4
2	10
3	18
4	28
5	40
6	54

Figure three is used to demonstrate how to write the rule. First, count all of the horizontal toothpicks that are marked on the shape below in Figure 73.

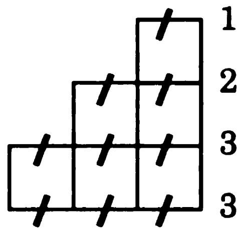


Figure 73: Horizontal Toothpicks

There are 1 + 2 + 3 + 3 toothpicks. Likewise the vertical toothpicks can be counted. They are marked in Figure 74.

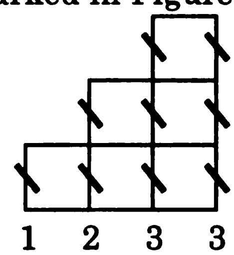


Figure 74: Vertical Toothpicks

There are 1 + 2 + 3 + 3 vertical toothpicks. The total number of toothpicks is the sum of the horizontal and vertical toothpicks or (1+2+2+3) + (1+2+3+3) which is 18 toothpicks. This can be re-written as 2(3) + 2(1+2+3) and generalized with n: $2n + 2(1+2+\ldots+n)$.

$1 + 2 + 3 + \ldots + n$ can be rewritten by pairing off the sums:

$n+1$

$n+1$

$1 + 2 + 3 + \ldots + n\text{-}1 + n$, this is $\frac{n}{2}(n+1)$

So the total number of toothpicks can be rewritten $2n + 2(\frac{n}{2}(n+1))$. In simplified terms the total number of toothpicks $= n^2 + 3n$.

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