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A SEPARATION PRINCIPLE FOR THE CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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ABSTRACT

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By

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It is shown that the performance of a globally bounded partial state feedback control of a certain class of nonlinear systems can be recovered by a sufficiently fast high-gain observer. The performance recovery includes recovery of asymptotic stability, the region of attraction, and trajectories. We deal with stability with respect to an equilibrium point and stability with respect to a compact, positively invariant set.

High-gain observers have been used in the design of output feedback controllers due to their ability to robustly estimate the unmeasured states while asymptotically attenuating disturbances. The available techniques for the design of high-gain observers can be classified into three groups: pole-placement algorithms, Riccati equation-based algorithms, and Lyapunov equation-based algorithms. In this work, we show that the abovementioned separation results hold for all these observer design techniques.

To my mother, Hanaa, whose life and death are an inspiration

To my father, Fawaz, who showed me the way

To my uncle, Wa'el, my mathematics professor

To my grandmother, Thurayya, the mother of all

To my wife, Aivy, whose love gives my life a meaning

To Jeanne and Mary who repainted my memory

To myself for never giving up

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CHAPTER 1

Introduction

A separation principle is the property of a control system that allows the design of a dynamic feedback controller in two separate steps. In the first step we design a state feedback controller that achieves some desired properties such as asymptotic convergence to an equilibrium point or asymptotic tracking by certain outputs of some reference signals. This design assumes that all the state variables are available for measurement. Then in the second step we design a state estimator (observer) using measurements of some outputs (functions of the state variables). The state is then replaced by its estimate in the state feedback controller to produce the dynamic output feedback controller. The separation property facilitates the design of feedback controllers in case we can only use measurements of outputs.

The literature provides several separation principles formulated for different classes of systems. Hereafter we give a quick survey of the main ones.

For the case of linear time-invariant control systems

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

if the system is stabilizable and detectable then a dynamical output feedback control can be designed such that the closed-loop system has the origin as a globally asymptotically stable equilibrium point. The control input takes the form $u = -K\hat{x}$ where K is designed to stabilize the closed-loop system under state feedback and $\hat{x}$ is the estimate of the state x provided by the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x})$$

where H is the observer gain. This observer (Luenberger observer) replicates the dynamics of the system and is driven by the output estimation error. The state estimation error $x(t) - \hat{x}(t)$ approaches zero asymptotically. This separation principle means that the poles of the system and those of the observer can be assigned separately and that the poles of the closed-loop system are the set of the state poles and the observer poles. Thus, if global stabilization is achieved by some linear state feedback $u = -Kx$, the corresponding output feedback $u = -K\hat{x}$ will globally recover this property.

We understand from the above discussion that the statement of a separation principle is a statement of recovery by an output feedback controller of performance achieved by a state feedback controller. In the case of linear systems the recovery includes recovery of asymptotic stability to an equilibrium point and asymptotic tracking of a reference. But we don't achieve closeness, in some sense, of trajectories under output feedback to trajectories under state feedback. This point will be illustrated later on.

In what follows we always assume that the origin is an equilibrium point of the system considered and all the vector fields and state functions mentioned are at least continuous in some appropriate regions of interest. We also abbreviate asymptotic stability (or asymptotically stable) by AS, local or locally by L and global or globally by G, so we may have for example LAS or GAS.

Vidyasagar in [53] formulated a separation principle for a general nonlinear MIMO (multiple-input multiple-output) control system

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ y &= r(t,x) \end{aligned}$$

This principle states that if $f(t,.,.)$ is uniformly locally Lipschitz and if the system is stabilizable and weakly detectable, then the closed-loop system composed of the stabilizing state feedback and a weak detector has the origin as a LAS equilibrium point. Stabilizability implies the existence of a state feedback law $u = h(t,x)$ that renders the origin of the system LAS. On the other hand, weak detectability implies the existence of a dynamical system (weak detector) driven by the input u and the output y and whose state is not very far from the state of the original system for initial conditions close to the origin.

This result is local, i.e. valid only in a neighborhood of the origin, and not constructive since it does not suggest a construction of the detector. The author actually stated a global version of his separation principle for the case of exponential stabilizability. This global version required, in addition to the above conditions, that the state estimation error decays exponentially and that the vector fields involved be globally Lipschitz.

It is noteworthy that the formulation of a separation principle requires the system to have some stabilizability and detectability properties. In the examples hereafter we highlight this idea. It is useful for the rest to define a feedback linearizable system. It is well known [20] that if the nonlinear system

$$\dot{\chi} = f(\chi) + g(\chi)u$$

$$y = h(\chi)$$

has a vector relative degree $(r_1, ..., r_p)$, then it can be transformed into the form

$$\begin{aligned}
\dot{\xi} &= A\xi + B[f_1(\xi, z) + g_1(\xi, z)u] \\
\dot{z} &= f_2(\xi, z, u) \\
y &= C\xi
\end{aligned}$$

where $g_1(.,.)$ is invertible in the domain of interest and y is the only measured output. If the sum of the relative degrees equals the dimension of the system it is called fully linearizable, otherwise it is called input-output linearizable. The first part of the system is a chain of integrators driven by a nonlinear function of the states and the inputs. The second part is called the zero dynamics.

Khalil and Esfandiari in [11] used a separation approach for a fully linearizable MIMO system with a high-gain observer that estimates the output and its derivatives. This observer is a chain of integrators of the form [1]

$$\begin{aligned}
\dot{\hat{y}}_1 &= \hat{y}_2 + \frac{\alpha_1}{\epsilon}(y_1 - \hat{y}_1) \\
\dot{\hat{y}}_2 &= \hat{y}_3 + \frac{\alpha_2}{\epsilon^2}(y_1 - \hat{y}_1) \\
&\vdots \\
\dot{\hat{y}}_n &= \frac{\alpha_n}{\epsilon^n}(y_1 - \hat{y}_1) + f_0(\xi, z) + g_0(\xi, z)u
\end{aligned}$$

driven by the output estimation error and by nonlinearities that reflect our partial knowledge of the system at hand. The observer gain can be regulated through the parameter ϵ in such a way that the observer is fast enough that the state under

[1] For convenience, we give the observer equations for SISO systems.

output feedback stays close to the state under state feedback and thus the stabilizing property of the controller is not lost. This observer introduces peaking into the state variables, see [11], which requires the disabling (saturation) of the controller outside some region of interest. The disabling period is made short by an appropriate choice of the observer gain. Notice that the stabilizability and detectability properties are immediately satisfied due to the particular structure of the system studied.

Tornambe in [52] proposed a local separation principle for a class of input-output linearizable non-minimum phase (unstable zeros dynamics) SISO nonlinear control systems. The system should be observable in the region of interest meaning that the state can be written as a function of the input, the output, and their derivatives. Considering the derivatives of the input and the output as state variables transforms the system into a double chain of integrators that can be linearized by state feedback. The fact that some of the states of the transformed system form a chain of integrators composed of an output and its derivatives allows the use of a high-gain observer similar to the one mentioned above to estimate these states. Two differences exist between the two observers, one is that Tornambe's observer did not use any knowledge of the nonlinearities of the system, and the other is that it did not deal with the peaking issue which makes the region of attraction of the system shrinks as the observer gain grows.

Teel and Praly in [49] proposed an interesting and constructive global separation principle for a wide class of autonomous SISO (single-input single-output) nonlinear control systems. It is a combination and extension of the ideas used in Esfandiari and Khalil [11] and Tornambe [52]. They stated that a globally stabilizable and uniformly completely observable (UCO) system can be semi-globally stabilizable by dynamic output feedback. The UCO property implies that the state of the system can be written as a function of the input, the output, and a finite (not necessarily equal) number of their derivatives. This function is called the observability map. The semi-global stabilizability means that given any compact set in the state space, then a dynamic output feedback controller can be designed such that this compact set is contained in the region of attraction of the closed loop-system. The observer used is the high-gain observer proposed by Esfandiari and Khalil in [11]. One major limitation of this separation principle is that the exact knowledge of the observability map implies a lack of robustness of the controller.

The last three results dealt with systems that can be transformed into a linear system using a function of the input, the output and their derivatives and an appropriate state feedback. For the sake of completeness we mention two more separation results that deal with systems that are affine in the inputs and whose unforced dynamics (for zero inputs) are Lyapunov stable (i.e. dissipative). The first [14] is a global separation principle for a class of SISO bilinear systems of the form

$$\begin{aligned} \dot{x} &= Ax + u(Bx + b) \\ y &= Cx \end{aligned}$$

that are stabilizable and whose unforced dynamics are observable (the unforced dynamics are linear). Structural conditions were provided to check the stabilizability and observability properties and two globally stabilizing feedback controllers were proposed. The two observers replicate the dynamics of the system and are driven by the output estimation error; one requires the input to be small and the other requires the input to be persistent.

The second [33] is a global separation result for a class of MIMO systems of the form

$$
\begin{aligned}
\dot{x} &= Ax + \sum_{i=1}^{m} g_i(x)u_i \\
y &= Cx
\end{aligned}
$$

that are stabilizable by bounded inputs (a control input was suggested) and whose unforced dynamics are observable (linear dynamics). Also, structural conditions were given to check the stabilizability of the system. The observer replicates the dynamics of the systems and is driven by the output estimation error. This result was shown to generalize the one for the bilinear system.

The above separation results were mainly concerned with AS to an equilibrium point. The following separation results are for cases of adaptive control and servomechanism where not all of the states approach the origin and the system depends on time-varying external signals (reference signals, disturbances, etc.).

In [27, 2, 1] Khalil and Aloliwi considered a SISO minimum phase nonlinear system which can be globally represented by an I/O model of the form

$$
y^{(n)} = f_0(.) + \sum_{i=1}^{p} f_i(.)\theta_i + (g_0(.) + \sum_{i=1}^{p} g_i(.)\theta_i)u^{(m)} + d(t, \theta, .)
$$

where the different nonlinearities depend on the input, the output, and their derivatives, θ is an unknown vector of parameters that belong to a compact set, and $d(.)$ is a locally Lipschitz disturbance. Adaptive state feedback controllers were designed so that the output can asymptotically track a bounded reference signal with bounded derivatives. Later, a linear high-gain observer was used to semi-globally recover what was achieved under state feedback. In some of the scenarios studied the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$ and some of the states were only proven to be bounded. In other scenarios the tracking error as well as the parameter estimation error were only ultimately bounded or were small in the mean-square sense.

In [26] Khalil used the internal model principle to regionally (in a region of interest) solve the servomechanism problem for a class of SISO uncertain systems that can be transformed into a normal form with no zero dynamics. The proposed dynamic state feedback controller achieved asymptotic convergence of the tracking error but the overall state approached an invariant zero-error manifold. A similar result was developed in [38] for a class of SISO systems represented by an I/O equation which depends nonlinearly on a vector of time varying disturbances.

In [21] Isidori has shown that his previously proposed solution for the general structurally stable regulation problem, see [22], can be coupled with the idea of high-gain observer suggested by Khalil in [11] to solve a problem of robust semiglobal output regulation in the presence of parameter uncertainties ranging over compact sets. The class of systems considered is of the form

$$\begin{aligned}
\dot{z} &= Z(\mu)z + p_0(x_1, w, \mu) \\
\dot{x} &= Fx + Gu + P(z, x, w, \mu) \\
e &= Hx - q(w, \mu)
\end{aligned}$$

where (F, G, H) is a chain of integrators, $P(.)$ is a lower triangular vector of nonlinearities, $Z(\mu)$ is Hurwitz in a compact set, and w is generated by a linear neutrally stable exosystem.

It is noteworthy that other separation results can be formulated for a specific application as is the case for polymerization reactors discussed in [54].

The purpose of this survey of literature is to show the classes of systems covered by different separation results and the observers used to achieve the goal of successful output feedback. Having done that we can exactly situate our version among the others and point out its merits compared to them. In a concise way we can say the following about our work:

(1) class of systems: it includes I/O linearizable systems, fully linearizable systems, and observable systems (that can be represented by a higher order ODE in the input and the output);

(2) observer: the robust high-gain observer of [11];

(3) state feedback: any globally bounded state feedback that stabilizes the system with respect to an equilibrium point or a compact positively invariant set. No Lyapunov function associated with the state feedback is needed;

(4) recovery properties: it recovers the AS property, trajectories, as well as the region of attraction (as opposed to local or global);

(5) it unifies and generalizes the cases of [49, 52, 11, 2, 1, 27, 26, 37, 38] which encompass a wide class of systems and a wide class of control techniques and objectives.

The goal of this work is to formulate, in a generic way, a separation principle for a wide class of nonlinear systems. This principle is based on the idea of fast estimation of the outputs and their derivatives cristalized in the high-gain observer concept. The resulting output feedback controller will be shown to recover a wide range of performance measures achieved by the state feedback controller. It is meant by "a generic way" that the statement of the separation principle does not depend on a specific state feedback nor on the knowledge of a Lyapunov function associated with this state feedback.

This work can be divided into three major sections. The first formulates a separation principle for the case where the state feedback controller achieves asymptotic stability to an equilibrium point and is given in Chapter 2. The second formulates a separation principle for the case of asymptotic stability to a compact, positively invariant set and is given in Chapters 3 and 4. The third shows that different observer design techniques yield separation results similar to those of Chapters 2 and 3, and is given in Chapter 5. Chapter 6 gives converse Lyapunov results for stability with respect to sets.

CHAPTER 2

A Separation Principle for the Stabilization of a Class of Nonlinear Systems

2.1 Introduction

A few years ago, Esfandiari and Khalil introduced in [11] a new technique in the design of robust output feedback control for input-output linearizable systems [11]. The basic ingredients of this technique are

(1) A high-gain observer that robustly estimates the derivatives of the output;

(2) A globally bounded state feedback control, usually obtained by saturating a continuous state feedback function outside a compact region of interest, that meets the design objectives. The global boundedness of the control protects the state of the plant from peaking when the high-gain observer estimates are used instead of the true states.

This technique has been the impetus for several results we have obtained over the past few years. It was used in [11] and [30] to achieve stabilization and semiglobal stabilization of fully-linearizable systems, in [26] to design robust servomechanisms for fully linearizable systems, in [37] and [38] to extend the results of [26] to systems having nontrivial zero dynamics. It was used also in adaptive control [27], variable structure control [41], and speed control of induction motors [31].

As the results of [11] became known, other researchers adopted its technique in their work. Teel and Praly [49, 50] and Lin and Saberi [35] used it in a few papers to achieve semiglobal stabilization. Jankovic [23] used it in an adaptive control problem. Isidori [21] used it to unify his pioneering work on servomechanisms [22] with Khalil's work [26]. Jiang, Hill, and Guo [24] used a reduced-order high-gain observer to achieve semiglobal stabilization for a nonlinear benchmark example.

In most of these papers the controller is designed in two steps. First, a globally bounded state feedback control is designed to meet the design objective. Second, a high-gain observer, designed to be fast enough, recovers the performance achieved under state feedback. This recovery is shown using asymptotic analysis of a singularly perturbed closed-loop system. Our goal in the current chapter is to develop this recovery property in a generic form that can be applied to any globally bounded stabilizing state feedback control. In particular, we want a separation theorem that is independent of the state feedback design and is derived under the least restrictive assumptions. To increase the utility of such a theorem, we want to allow for model uncertainty. Finally, we want to demonstrate that the performance recovery achieved with the high-gain observer is more than just asymptotic stability recovery. It includes recovery of the region of attraction and trajectories achieved under state feedback. These features of the theorem distinguish our work from the interesting separation theorem proved by Teel and Praly [49], where it is shown that global stabilizability by state feedback and observability imply semiglobal stabilizability by output feedback. The result of [49] assumes perfect knowledge of the model and shows only recovery of asymptotic stability and the semiglobal stabilization property.

The rest of the chapter is organized as follows. Section 2.2 introduces the class of systems with which the chapter is concerned. Section 2.3 states the requirements on the state feedback control. Section 2.4 introduces the high-gain observer used to estimate the states. Section 2.5 discusses and proves the recovery of performance by output feedback. Section 2.6 illustrates the previous results through simulations.

2.2 The Class of Systems

We consider a multivariable nonlinear system represented by

$$\dot{x} = Ax + B\phi(x, z, u) \tag{2.1}$$

$$\dot{z} = \psi(x, z, u) \tag{2.2}$$

$$y = Cx \tag{2.3}$$

$$\zeta = q(x, z) \tag{2.4}$$

where $u \in \mathcal{U} \subseteq R^m$ is the control input, $y \in \mathcal{Y} \subseteq R^p$ and $\zeta \in R^s$ are measured outputs, and $x \in \mathcal{X} \subseteq R^r$ and $z \in \mathcal{Z} \subseteq R^\ell$ constitute the state vector. The $r \times r$ matrix A, the $r \times p$ matrix B, and the $p \times r$ matrix C, given by

$$A = \text{block diag}[A_1, \ldots, A_p], \quad A_i = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}_{r_i \times r_i}$$

$$B = \text{block diag}[B_1, \ldots, B_p], \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r_i \times 1}$$

$$C = \text{block diag}[C_1, \ldots, C_p], \quad C_i = \begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \end{bmatrix}_{1 \times r_i}$$

where $1 \leq i \leq p$ and $r = r_1 + \ldots + r_p$ represent p chains of integrators. This system satisfies the following assumption:

Assumption 2.1 *The functions $\phi : \mathcal{X} \times \mathcal{Z} \times \mathcal{U} \rightarrow R^p$ and $\psi : \mathcal{X} \times \mathcal{Z} \times \mathcal{U} \rightarrow R^\ell$ are locally Lipschitz in their arguments over the domain of interest. In addition, $\phi(0,0,0) = 0$, $\psi(0,0,0) = 0$, and $q(0,0) = 0$.*

Assumption 2.1 guarantees that the origin is an equilibrium point of the open-loop system.

The main source of the system (2.1)–(2.4) is the normal form of a nonlinear system having a vector relative degree $(r_1, ..., r_p)$. It is well known [20] that if the nonlinear system

$$\begin{aligned} \dot{\chi} &= f(\chi) + g(\chi)u \\ y &= h(\chi) \end{aligned}$$

has a vector relative degree $(r_1, ..., r_p)$, then it can be transformed into the form

$$\begin{aligned} \dot{\xi} &= A\xi + B[f_1(\xi, z) + g_1(\xi, z)u] \\ \dot{z} &= f_2(\xi, z, u) \end{aligned}$$

$$y = C\xi$$

In this case $g_1(.,.)$ is nonsingular in the domain of interest, y is the only measured output, and equation (2.4) is dropped.

Another source, where equation (2.4) is relevant, arises when the dynamics are extended by augmenting a series of integrators at the input side [27, 49, 52]. Reference [27] considers a single-input single-output system modeled by the nth–order differential equation

$$y^{(n)} = f_0(.) + g_0(.)\mu^{(n-\rho)}$$

where μ is the input, y is the output, f_0 and g_0 are functions of y, $y^{(1)}$, ..., $y^{(n-1)}$, μ, ..., $\mu^{(n-\rho-1)}$. Augmenting $(n-\rho)$ integrators at the input side, denoting their states by $z_i = \mu^{(i-1)}$, setting $u = \mu^{(n-\rho)}$ as the control input of the augmented system, and taking $x_i = y^{(i-1)}$, results in a system of the form (2.1)–(2.4) with $r = n$ and $\ell = n - \rho$. In this case all the components of z are measured; hence $q(x,z) = z$ in (2.4). Another example of the use of extended dynamics can be found in [49]. Reference [49] considers a single-input single-output nonlinear system where complete uniform observability guarantees that the state χ can be expressed as $\chi = h(y, ..., y^{(ny)}, \mu, ..., \mu^{(nu)})$ where μ is the input, y is the output, and $h(.)$ is a known function. Furthermore, $y^{(ny+1)} = \alpha(\chi, u, ..., u^{(mu)})$ where α is a known function. The dynamics are extended by adding $l_u = \max\{n_u, m_u\}$ integrators at the input side. Taking $x_i = y^{(i)}$, for $1 \leq i \leq n_y$, $z_i = \mu^{(i)}$, for $1 \leq i \leq l_u$, and $u = \mu^{(l_u+1)}$, the system can be represented as

$$\begin{aligned} \dot{x} &= Ax + B\alpha(h(x,z),z) \\ \dot{z} &= A_0 z + B_0 u \end{aligned}$$

$$y = Cx$$

where (A, B, C) and (A_0, B_0) represent chains of n_y and l_u integrators, respectively. In this case ϕ is independent of u and all the components of z are measured; hence $q(x, z) = z$ in (2.4).

The model (2.1)–(2.4) may also arise in models of mechanical and electromechanical systems where displacement variables are measured while their derivatives (velocities, accelerations, etc.) are not measured. Examples of such models can be found in [31, 15, 36, 19, 51, 24]. A model of induction motor [31] can be represented in the form (2.1)–(2.4) with $x = [\delta, \dot{\delta}, \ddot{\delta}]^T$, where $\delta = \theta - \theta_{ref}$ is the rotor position error, and z constitutes the rotor flux and stator current. The measured variables y and ζ are the rotor position error and stator current, respectively. Examples of models that can be put in the normal form are the models given in [15] and [36] for the inverted pendulum-on-a-cart system. These models, taking the cart displacement as the measured output, have a relative degree two but are non–minimum phase. In [19] and [51], the models given of the ball and beam system fit in the form of (2.1)–(2.4). These systems can not be represented in the normal form because, taking the ball's position as one of the measured outputs, the relative degree is not well defined. A last example of systems fitting the model (2.1)–(2.4) is the model of the benchmark rotational/translational actuator given in [24] where the system has a well defined relative degree with respect to the cart's position but only locally. The design of the globally stabilizing state feedback controller of [24] does not transform the system into the normal form.

2.3 Partial State Feedback Control

Our goal is to design a feedback control to stabilize the origin of the closed-loop system using only the measured outputs y and ζ. We follow a two-step approach to this design problem. We first design a partial state feedback control that uses measurements of x and ζ. Then we use a high-gain observer to estimate x from y. We allow the state feedback control to be dynamic, which is the case, for example, in the adaptive control of [27], the speed control of induction motors of [31] which uses a flux observer, and the stabilizing control of [11] which includes a zero-dynamics observer. The state feedback control is assumed to be in the form

$$\dot{\vartheta} = \Gamma(\vartheta, x, \zeta) \tag{2.5}$$

$$u = \gamma(\vartheta, x, \zeta) \tag{2.6}$$

A non-dynamic state feedback control $u = \gamma(x, \zeta)$ will be viewed as a special case of (2.5)–(2.6) by dropping equation (2.5).

We allow any state feedback design that holds the following three properties:

Assumption 2.2

(1) Γ *and* γ *are locally Lipschitz functions in their arguments over the domain of interest,* $\Gamma(0,0,0) = 0$ *and* $\gamma(0,0,0) = 0$*;*

(2) Γ *and* γ *are globally bounded functions of* x*;*

(3) The origin $(x = 0,\ z = 0,\ \vartheta = 0)$ *is an asymptotically stable equilibrium point of the closed-loop system.*

The state feedback control may have additional properties like conformity to certain performance measures on the trajectories, and/or robustness to a certain class of uncertainties. The global boundedness requirement is typically achieved by saturation of $\Gamma(.)$ and $\gamma(.)$, or saturation of their x-input, outside a compact region of interest. The boundedness of $\gamma(.)$ may also follow from a design under control constraints.

2.4 High-Gain Observer

To implement the control (2.5)–(2.6) we use

$$\begin{aligned} \dot{\vartheta} &= \Gamma(\vartheta, \hat{x}, \zeta) && (2.7) \\ u &= \gamma(\vartheta, \hat{x}, \zeta) && (2.8) \end{aligned}$$

where the state estimate $\hat{x}$ is generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, u) + H(y - C\hat{x}) \tag{2.9}$$

The observer gain H is chosen as

$$H = \text{block diag}[H_1, \ldots, H_p], \quad H_i = \begin{bmatrix} \alpha_1^i/\epsilon \\ \alpha_2^i/\epsilon^2 \\ \vdots \\ \alpha_{r_i-1}^i/\epsilon^{r_i-1} \\ \alpha_{r_i}^i/\epsilon^{r_i} \end{bmatrix}_{r_i \times 1} \tag{2.10}$$

where ϵ is a positive constant to be specified and the positive constants α_j^i are chosen such that the roots of

$$s^{r_i} + \alpha_1^i s^{r_i-1} + \cdots + \alpha_{r_i-1}^i s^1 + \alpha_{r_i}^i = 0$$

are in the open left-half plane, for all $i = 1, \ldots, p$. The function $\phi_0(x, \zeta, u)$ is a nominal model of $\phi(x, z, u)$. If ϕ is a known function of x, ζ, and u, we can take $\phi_0 = \phi$. On the other hand, if such nominal model is not available, we can take $\phi_0 = 0$, which results in a linear observer. The function ϕ_0 is required to satisfy the following assumption:

Assumption 2.3 $\phi_0(x, \zeta, \gamma(\vartheta, x, \zeta))$ *is locally Lipschitz in its arguments over the domain of interest and globally bounded in* x [1]. *Moreover,* $\phi_0(0, 0, 0) = 0$.

The high-gain observer suggested above is a full-order one. It is possible to design a reduced-order high-gain observer of order $(r - p)$ [45]. We start by partitioning the state vector x as $x = \begin{bmatrix} y \\ v \end{bmatrix}$, where $y = [y_1, \ldots, y_p]^T$, and rewriting the system (2.1)–(2.2) as

$$\begin{aligned} \dot{y} &= A_{12} v \\ \dot{v} &= A_{22} v + B_{22} \phi(y, v, z, u) \end{aligned}$$

where, A_{22}, B_{22}, A_{12} have the same structure as A, B, C, respectively, but with all the r_i's replaced by $(r_i - 1)$'s. Then, the reduced-order observer will be

$$\begin{aligned} \dot{w} &= (A_{22} - L A_{12})(w + Ly) + B_{22} \phi_0(y, \hat{v}, \zeta, u) \\ \hat{v} &= w + Ly \end{aligned}$$

[1] *The need for this global boundedness property will be made clear in footnote 3 in Section 2.5.1. Moreover, global boundedness of* ϕ_0 *can always be achieved by saturation outside a compact set of interest in the subspace of* x.

where

$$L = \text{block diag}[L_1, \ldots, L_p], \quad L_i = \begin{bmatrix} \beta_1^i/\epsilon \\ \beta_2^i/\epsilon^2 \\ \vdots \\ \beta_{r_i - 2}^i/\epsilon^{r_i - 2} \\ \beta_{r_i - 1}^i/\epsilon^{r_i - 1} \end{bmatrix}_{(r_i - 1) \times 1}$$

The positive constants β_j^i are chosen such that the roots of

$$s^{r_i - 1} + \beta_1^i s^{r_i - 2} + \cdots + \beta_{r_i - 2}^i s^1 + \beta_{r_i - 1}^i = 0$$

are in the open left-half plane, for all $i = 1, \ldots, p$.

Remark 2.1

(1) In the case where one or more of the p channels that compose the system (2.1)–(2.3) have relative degrees equal to one, then, there is no need to estimate their states; they can be used as they are in the output feedback controller. This will not modify the forthcoming analysis.

(2) We can use the measured y in the output feedback controller instead of its estimate $\hat{y} = C\hat{x}$ even when we construct a full-order high-gain observer. This, also, will not change the forthcoming analysis.

2.5 Performance Recovery

In this section we show that the output feedback controller (2.7)–(2.9) recovers the performance of the state feedback controller (2.5)–(2.6) for sufficiently small ϵ. The performance recovery manifests itself in three points. First, the origin ($x = 0$, $z =$

0, $\vartheta = 0$, $\hat{x} = 0$) of the closed-loop system under output feedback is asymptotically stable. Second, The output feedback controller recovers the region of attraction of the state feedback controller in the sense that if $\mathcal{R}$ is the region of attraction under state feedback, then for any compact set $\mathcal{S}$ in the interior of $\mathcal{R}$ and any compact set $\mathcal{Q} \subseteq R^r$, the set $\mathcal{S} \times \mathcal{Q}$ is included in the region of attraction under output feedback control. Third, the trajectory of (x, z, ϑ) under output feedback approaches the trajectory under state feedback as $\epsilon \to 0$.

For the clarity of the proof we will follow the way used in [49] which establishes asymptotic stability of the origin in three steps. The first step is to show boundedness of trajectories, second, ultimate boundedness of these trajectories, and third, local asymptotic stability. This allows us to deal with asymptotic stability as a local property that will require some additional assumptions on the nonlinearities of the closed–loop system, stated as a condition on the modeling error, which is not the case for boundedness and ultimate boundedness of trajectories. Thus, the proof will be divided into four sections. First we prove recovery of boundedness of trajectories; second, we prove recovery of ultimate boundedness of trajectories; third, we prove trajectory convergence; and fourth, we prove recovery of asymptotic stability of the origin. In the latter case, we distinguish between asymptotic and exponential stability in order to impose less conservative restrictions on the modeling error.

2.5.1 Boundedness

Let us first, for the purpose of analysis, replace the observer dynamics by the equivalent dynamics of the scaled estimation error

$$\eta_{ij} = \frac{x_{ij} - \hat{x}_{ij}}{\epsilon^{r_i - j}}$$

for $1 \le i \le p$ and $1 \le j \le r_i$. Hence, we have $\hat{x} = x - D(\epsilon)\eta$ where

$$
\begin{aligned}
\eta &= [\eta_{11}, \ldots, \eta_{1r_1}, \ldots, \eta_{p1}, \ldots, \eta_{pr_p}]^T \\
D(\epsilon) &= \text{block diag}[D_1, \ldots, D_p] \\
D_i &= \text{diag}[\epsilon^{r_i - 1}, \ldots, 1]_{r_i \times r_i}
\end{aligned}
$$

The closed-loop system can be represented by

$$\dot{x} = Ax + B\phi(x, z, \gamma(\vartheta, x - D(\epsilon)\eta, \zeta)) \tag{2.11}$$

$$\dot{z} = \psi(x, z, \gamma(\vartheta, x - D(\epsilon)\eta, \zeta)) \tag{2.12}$$

$$\dot{\vartheta} = \Gamma(\vartheta, x - D(\epsilon)\eta, \zeta) \tag{2.13}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon B g(x, z, \vartheta, D(\epsilon)\eta) \tag{2.14}$$

where

$$g(x, z, \vartheta, D(\epsilon)\eta) = \phi(x, z, \gamma(\vartheta, \hat{x}, \zeta)) - \phi_0(\hat{x}, \zeta, \gamma(\vartheta, \hat{x}, \zeta))$$

and $\frac{1}{\epsilon}A_0 = D^{-1}(\epsilon)(A - HC)D(\epsilon)$ is an $r \times r$ Hurwitz matrix. The initial states are $(x(0), z(0), \vartheta(0)) = (x_0, z_0, \vartheta_0) \in \mathcal{S}$, and $\hat{x}(0) = \hat{x}_0 \in \mathcal{Q}$, where $\mathcal{S}$ is any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ is any compact subset of R^r; thus, we have $\eta(0) = D^{-1}(\epsilon)(x_0 - \hat{x}_0) = \eta_0$.

Remark 2.2 *In the case of a reduced-order high-gain observer, the scaled error will be*

$$\eta_{ij} = \frac{v_{ij} - \hat{v}_{ij}}{\epsilon^{r_i - 1 - j}}$$

for $1 \le i \le p$ *and* $1 \le j \le (r_i - 1)$*. We also get the same structure of (2.11)–(2.14) but with* r_i *replaced by* $(r_i - 1)$*,* x *replaced by* (y, v)*, and* $\hat{x}$ *replaced by* $(y, \hat{v})$ *where* $\hat{v} = v - D(\epsilon)\eta$*. The results of this paper will be the same for this reduced-order* *observer.*

The system (2.11)–(2.14) is a standard singularly perturbed one, and $\eta = 0$ is the unique solution of (2.14) when $\epsilon = 0$. The reduced system, obtained by substituting $\eta = 0$ in (2.11)–(2.14), is nothing but the closed-loop system under state feedback. For simplicity we write the system (2.11)–(2.13) as

$$\dot{\chi} = f_r(\chi, D(\epsilon)\eta) \tag{2.15}$$

where $\chi = [x^T, z^T, \vartheta^T]^T$ and $\chi(0) = [x_0^T, z_0^T, \vartheta_0^T]^T$. Then, the reduced system is given by

$$\dot{\chi} = f_r(\chi, 0) \tag{2.16}$$

The boundary-layer system, obtained by applying to (2.14) the change of time variable $\tau = \frac{t}{\epsilon}$ then setting $\epsilon = 0$, is given by

$$\frac{d\eta}{d\tau} = A_0\eta \tag{2.17}$$

To fix the notation, let $(\chi(t, \epsilon), \eta(t, \epsilon))$ denote the trajectory of the system (2.11)–(2.14) starting from $(\chi(0), \eta(0))$. The recovery of boundedness of trajectories is summarized in the following theorem:

Theorem 2.1 *Let Assumptions 2.1-2.3 hold; then, there exists* $\epsilon_1^\star > 0$ *such that for every* $0 < \epsilon \leq \epsilon_1^\star$, *the trajectories* (χ, η) *of the system (2.11)–(2.14) starting in* $\mathcal{S} \times \mathcal{Q}$ *are bounded for all* $t \geq 0$.

Proof: The recovery of boundedness can be shown in two steps. First, we show the positive invariance of an appropriately chosen set Λ. This set is arbitrarily small in the direction of the error variable η. Second, we show that, any closed-loop trajectory, starting in the compact set $\mathcal{S} \times \mathcal{Q}$, enters the positively invariant set Λ in finite time.

We know that the origin of (2.16) is asymptotically stable with a region of attraction $\mathcal{R}$. Then, the converse Lyapunov theorem of Kurzweil [32, Theorem 7] [2] assures the existence of a C^1 Lyapunov function $V(\chi)$ and three positive definite functions $U_1(\chi)$, $U_2(\chi)$, and $U_3(\chi)$, all defined and continuous on $\mathcal{R}$, such that:

$$U_1(\chi) \leq V(\chi) \leq U_2(\chi) \tag{2.18}$$

$$\lim_{\chi \to \partial\mathcal{R}} U_1(\chi) = \infty \tag{2.19}$$

$$\frac{\partial V}{\partial \chi} f_r(\chi, 0) \leq -U_3(\chi) \tag{2.20}$$

for all $\chi \in \mathcal{R}$. The properness of $V(\chi)$ in $\mathcal{R}$ guarantees that with any finite $c > \max_{\chi \in \mathcal{S}} V(\chi)$, the set $\Omega = \{\chi \in \mathcal{R} : V(\chi) \leq c\}$ is a compact subset of $\mathcal{R}$ and $\mathcal{S}$ is in the interior of Ω.

For the boundary-layer system we define the Lyapunov function $W(\eta) = \eta^T P_0 \eta$, where P_0 is the positive definite solution of the Lyapunov equation $P_0 A_0 + A_0^T P_0 = -I$. This function satisfies

$$\lambda_{min}(P_0)\|\eta\|^2 \leq W(\eta) \leq \lambda_{max}(P_0)\|\eta\|^2 \tag{2.21}$$

$$\frac{\partial W}{\partial \eta} A_0 \eta \leq -\|\eta\|^2 \tag{2.22}$$

Let $\Lambda = \Omega \times \{W(\eta) \leq \rho\epsilon^2\}$. Due to Assumptions 2.1, 2.2, and 2.3 (i.e., continuity of the nonlinearities and global boundedness of $\Gamma(.)$ and $\gamma(.)$ in x) we have, for all $\chi \in \Omega$ (continuous functions are bounded on compact sets) and all $\eta \in R^r$ (global

[2] This theorem is built around a stability notion called *Strong Stability* in an open set. The proof of Theorem 12 of [32] shows that asymptotic stability implies strong stability in the region of attraction which is an open invariant set. Thus, we can apply Theorem 7.

boundedness of the controller),

$$\|f_r(\chi, D(\epsilon)\eta)\| \leq k_1 \tag{2.23}$$

$$\|g(\chi, D(\epsilon)\eta)\| \leq k_2 \tag{2.24}$$

where k_1 and k_2 are positive constants independent of ϵ. Moreover, for any $0 < \tilde{\epsilon} < 1$, there is L_1, independent of ϵ, such that, for all $(\chi, \eta) \in \Lambda$ and every $0 < \epsilon \leq \tilde{\epsilon}$, we have

$$\|f_r(\chi, D(\epsilon)\eta) - f_r(\chi, 0)\| \leq L_1\|\eta\| \tag{2.25}$$

In the rest of the paper we always consider $\epsilon \leq \tilde{\epsilon}$. We start by showing that there exist positive constants ρ and ϵ_1 (dependent on ρ) such that the compact set Λ is positively invariant for every $0 < \epsilon \leq \epsilon_1$. This can be done by verifying that

$$\dot{V} \leq \frac{\partial V}{\partial \chi} f_r(\chi, 0) + \epsilon k_3 \tag{2.26}$$

for all $(\chi, \eta) \in \{V(\chi) = c\} \times \{W(\eta) \leq \rho\epsilon^2\}$, and

$$\dot{W} \leq -\frac{1}{\epsilon}\|\eta\|^2 + 2\|\eta\|\|P_0\|\|B\|k_2 \tag{2.27}$$

for all $(\chi, \eta) \in \Omega \times \{W(\eta) = \rho\epsilon^2\}$, where $k_3 = L_1 L_2\sqrt{\rho/\lambda_{min}(P_0)}$, $\|P_0\| = \lambda_{max}(P_0)$, and L_2 is an upper bound for $\|\frac{\partial V}{\partial \chi}\|$ over Ω. Taking $\rho = 16k_2^2\|P_0\|^3$ and $\epsilon_1 = \beta/k_3$, where $\beta = \min_{\chi \in \partial\Omega} U_3(\chi)$, it can be shown that, for every $0 < \epsilon \leq \epsilon_1$, we have

$$\dot{V} \leq 0 \tag{2.28}$$

for all $(\chi, \eta) \in \{V(\chi) = c\} \times \{W(\eta) \leq \rho\epsilon^2\}$, and

$$\dot{W} \leq 0 \tag{2.29}$$

for all $(\chi, \eta) \in \{V(\chi) \leq c\} \times \{W(\eta) = \rho\epsilon^2\}$. From (2.28) and (2.29) we conclude that the set Λ is positively invariant.

Now, we consider the initial state $(\chi(0), \hat{x}(0)) \in \mathcal{S} \times \mathcal{Q}$. It can be verified that the corresponding initial error $\eta(0)$ satisfies $\|\eta(0)\| \leq k/\epsilon^{(r_{max} - 1)}$ for some non-negative constant k dependent on $\mathcal{S}$ and $\mathcal{Q}$, where $r_{max} = \max\{r_1, ..., r_p\}$. Since the vector field $f_r(.,.)$ is continuous, we can write

$$\chi(t, \epsilon) - \chi(0) = \int_0^t f_r(\chi(\tau, \epsilon), D(\epsilon)\eta(\tau, \epsilon))d\tau \tag{2.30}$$

Then, using (2.23) and the fact that $\chi(0)$ is in the interior of Ω, we have

$$\|\chi(t, \epsilon) - \chi(0)\| \leq k_1 t \tag{2.31}$$

as long as $\chi(t, \epsilon) \in \Omega$. Thus, there exists a finite time T_0, independent of ϵ, such that $\chi(t, \epsilon) \in \Omega$ for all $t \in [0, T_0]$. During this time interval we have [3]

$$\dot{W} \leq -\frac{1}{2\epsilon}\|\eta\|^2, \text{ for } W(\eta) \geq \rho\epsilon^2$$

Therefore,

$$W(\eta(t, \epsilon)) \leq \frac{\sigma_2}{\epsilon^{2(r_{max} - 1)}} \exp(-\sigma_1 t/\epsilon) \tag{2.32}$$

where $\sigma_1 = 1/2\|P_0\|$ and $\sigma_2 = k^2\|P_0\|$. Choose $\epsilon_2 > 0$ small enough that

$$T(\epsilon) \stackrel{\text{def}}{=} \frac{\epsilon}{\sigma_1} \ln\left(\frac{\sigma_2}{\rho\epsilon^{2r_{max}}}\right) \leq \frac{1}{2}T_0 \tag{2.33}$$

for all $0 < \epsilon \leq \epsilon_2$. We note that ϵ_2 exists since the left-hand side of the preceding inequality tends to zero as ϵ tends to zero. It follows that $W(\eta(T(\epsilon), \epsilon)) \leq \rho\epsilon^2$, for

[3] Here we use an inequality similar to (2.27), obtained using (2.24), which is valid for $(\chi, \eta) \in \Omega \times R^r$. Inequality (2.24) requires global boundedness of ϕ_0 in x.

every $0 < \epsilon \leq \epsilon_2$. Taking $\epsilon_1^\star = \min(\tilde{\epsilon}, \epsilon_1, \epsilon_2)$ guarantees that, for every $0 < \epsilon \leq \epsilon_1^\star$, the trajectory $(\chi(t,\epsilon), \eta(t,\epsilon))$ enters Λ during the interval $[0, T(\epsilon)]$ and remains there for all $t \geq T(\epsilon)$. Thus, the trajectory is bounded for all $t \geq T(\epsilon)$. On the other hand, for $t \in [0, T(\epsilon)]$, the trajectory is bounded by virtue of inequalities (2.31) and (2.32).$\triangleleft$

Remark 2.3 *The constant* $\epsilon_1^\star$ *depends on the sets* $\mathcal{S}$ *and* $\mathcal{Q}$.

2.5.2 Ultimate Boundedness

Next, we show that trajectories of the system (2.11)–(2.14), starting in $\mathcal{S} \times \mathcal{Q}$, come arbitrarily close to the origin as time progresses. This is summarized in the following Theorem:

Theorem 2.2 *Under the conditions of Theorem 2.1, given any* $\xi > 0$, *there exist* $\epsilon_2^\star = \epsilon_2^\star(\xi) > 0$ *and* $T_1 = T_1(\xi)$ *such that, for every* $0 < \epsilon \leq \epsilon_2^\star$, *we have*

$$\|\chi(t,\epsilon)\| + \|\eta(t,\epsilon)\| \leq \xi, \ \forall t \geq T_1 \tag{2.34}$$

Proof: From the proof of Theorem 2.1 we know that, for every $0 < \epsilon \leq \epsilon_1^\star$, the trajectory of the closed-loop system, starting from $(\chi(0), \hat{x}(0)) \in \mathcal{S} \times \mathcal{Q}$, is inside the set Λ for all $t \geq T(\epsilon)$, where Λ is $O(\epsilon)$ in the direction of the variable η. Take $\epsilon_3 = \min\{\epsilon_1^\star, \frac{\xi}{2}\sqrt{\lambda_{min}(P_0/\rho)}\}$. Then, $\epsilon_3 = \epsilon_3(\xi) \leq \epsilon_1^\star$ and for every $0 < \epsilon \leq \epsilon_3$ we have

$$\|\eta(t,\epsilon)\| \leq \xi/2, \ \forall t \geq T(\epsilon_3) \stackrel{\text{def}}{=} \bar{T}(\xi) \tag{2.35}$$

In what follows we continue working with the Lyapunov function defined in the proof of Theorem 2.1. It can be shown that, for all $(\chi, \eta) \in \Lambda$, we have

$$\dot{V} \leq -U_3(\chi) + k_3\epsilon \tag{2.36}$$

where $k_3 = L_1\sqrt{\rho/\lambda_{min}(P_0)}\ \max_{\chi\ \in\ \Omega}(\|\frac{\partial V}{\partial \chi}\|)$. Thus, we conclude that

$$\dot{V} \leq -\frac{1}{2}U_3(\chi),\ \text{for}\ \chi \notin \{\chi : U_3(\chi) \leq 2k_3\epsilon \stackrel{\text{def}}{=} \mu(\epsilon)\} \tag{2.37}$$

Since $U_3(\chi)$ is positive definite and continuous, the set $\{\chi : U_3(\chi) \leq \mu(\epsilon)\}$ is a compact set for sufficiently small ϵ. Let $c_0(\epsilon) = \max_{U_3(\chi)\ \leq\ \mu(\epsilon)}\{V(\chi)\}$; $c_0(\epsilon)$ is nondecreasing and $\lim_{\epsilon\ \to\ 0} c_0(\epsilon) = 0$. Consider the compact set $\{\chi : V(\chi) \leq c_0(\epsilon)\}$. We have $\{\chi : U_3(\chi) \leq \mu(\epsilon)\} \subset \{\chi : V(\chi) \leq c_0(\epsilon)\}$. Choose $\epsilon_4 = \epsilon_4(\xi) \leq \epsilon_1^\star$ small enough such that, for all $\epsilon \leq \epsilon_4$, the set $\{\chi : U_3(\chi) \leq \mu(\epsilon)\}$ is compact, the set $\{\chi : V(\chi) \leq c_0(\epsilon)\}$ is in the interior of Ω, and

$$\{\chi : V(\chi) \leq c_0(\epsilon)\} \subset \{\chi : \|\chi\| \leq \xi/2\} \tag{2.38}$$

Then, for all $\chi \in \Omega$ but $\chi \notin \{\chi : V(\chi) \leq c_0(\epsilon)\}$, we have an inequality similar to (2.37).

Thus, we conclude that the set $\{\chi : V(\chi) \leq c_0(\epsilon)\} \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ is positively invariant and every trajectory in $\Omega \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ reaches $\{\chi : V(\chi) \leq c_0(\epsilon)\} \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ in finite time. In other words, given (2.38), there exists a finite time $\tilde{T} = \tilde{T}(\xi)$ such that, for every $0 < \epsilon \leq \epsilon_4$

$$\|\chi(t,\epsilon)\| \leq \xi/2,\ \forall t \geq \tilde{T} \tag{2.39}$$

Take $\epsilon_2^\star = \epsilon_2^\star(\xi) = \min(\epsilon_3, \epsilon_4)$ and $T_1 = T_1(\xi) = \max(\bar{T}, \tilde{T})$, then (2.34) follows from (2.35) and (2.39).$\triangleleft$

In what follows we use the results of Theorem 2.2. Although it is understood that different values of ξ give different values of $\epsilon_2^\star$, we use the same notation for simplicity.

2.5.3 Trajectory Convergence

Let $\chi_r(t)$ be the solution of (2.16) starting from $\chi(0)$. The following theorem shows that $\chi(t, \epsilon)$ converges to $\chi_r(t)$ as $\epsilon \to 0$, uniformly in t, for all $t \geq 0$.

Theorem 2.3 *Under the conditions of Theorem 2.1, given any $\xi > 0$, there exists $\epsilon_3^\star > 0$ such that, for every $0 < \epsilon \leq \epsilon_3^\star$ we have*

$$\|\chi(t, \epsilon) - \chi_r(t)\| \leq \xi, \ \forall t \geq 0 \tag{2.40}$$

Proof: We divide the interval $[0, \infty)$ into three intervals $[0, T(\epsilon)]$, $[T(\epsilon), T_2]$, and $[T_2, \infty)$, where both $T(\epsilon)$ and T_2 are to be determined later, and show (2.40) for each interval. This approach gives more insight into the factors that come into play in each of these intervals.

• From Theorem 2.2 we know that there exists a finite time $\tilde{T}_2 \geq T(\epsilon)$, independent of ϵ, such that, for every $0 < \epsilon \leq \epsilon_2^\star$, we have

$$\|\chi(t, \epsilon)\| \leq \xi/2, \ \forall t \geq \tilde{T}_2 \tag{2.41}$$

From the asymptotic stability of the origin of the reduced system we know that there exists a finite time $\bar{T}_2$, independent of ϵ, such that

$$\|\chi_r(t)\| \leq \xi/2, \ \forall t \geq \bar{T}_2 \tag{2.42}$$

Take $T_2 = \max\{\tilde{T}_2, \bar{T}_2\}$. Then, using the triangular inequality along with (2.41) and (2.42), we conclude that, for every $0 < \epsilon \leq \epsilon_2^\star$, we have

$$\|\chi(t, \epsilon) - \chi_r(t)\| \leq \xi, \ \forall t \geq T_2 \tag{2.43}$$

- From the proof of Theorem 2.1 we know that

$$\|\chi(t,\epsilon) - \chi(0)\| \le k_1 t$$

during the interval $[0, T(\epsilon)]$. Similarly, it can be shown that

$$\|\chi_r(t) - \chi(0)\| \le k_1 t$$

during the same interval. Hence,

$$\|\chi(t,\epsilon) - \chi_r(t)\| \le 2k_1 T(\epsilon), \; \forall t \in [0, T(\epsilon)] \tag{2.44}$$

Since $T(\epsilon) \to 0$ as $\epsilon \to 0$, there exists $0 < \epsilon_5 \le \epsilon_2^\star$ such that, for every $0 < \epsilon \le \epsilon_5$, we have

$$\|\chi(t,\epsilon) - \chi_r(t)\| \le \xi, \; \forall t \in [0, T(\epsilon)] \tag{2.45}$$

- Over the interval $[T(\epsilon), T_2]$, the trajectory $\chi(t,\epsilon)$ satisfies

$$\dot{\chi} = f_r(\chi, D(\epsilon)\eta(t,\epsilon)), \text{ with initial condition } \chi(T(\epsilon),\epsilon)$$

Over the same time interval, the trajectory $\chi_r(t)$ satisfies

$$\dot{\chi} = f_r(\chi, 0), \text{ with initial condition } \chi_r(T(\epsilon))$$

From (2.44), we know that

$$\|\chi(T(\epsilon),\epsilon) - \chi_r(T(\epsilon))\| \le 2k_1 T(\epsilon) \stackrel{\text{def}}{=} \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0^+$. By continuous dependence of the solutions of differential equations on parameters over compact time intervals [28, Theorem 2.5], we conclude that

$$\begin{aligned}\|\chi(t,\epsilon) - \chi_r(t)\| &\leq \delta(\epsilon)\exp[L(T_2 - T(\epsilon))] \\ &\quad + \frac{L_1}{L}\sqrt{\rho/\lambda_{min}(P_0)}\epsilon\{\exp[L(T_2 - T(\epsilon))] - 1\} \\ &\leq [\delta(\epsilon) + \frac{L_1}{L}\sqrt{\rho/\lambda_{min}(P_0)}\epsilon]\exp[L(T_2 - T(\epsilon))] \end{aligned} \tag{2.46}$$

where L is the Lipschitz constant of $f_r(.,0)$ on Ω. Thus, given (2.46), there exists $0 < \epsilon_6 \leq \epsilon_2^\star$ such that for every $0 < \epsilon \leq \epsilon_6$ we have

$$\|\chi(t,\epsilon) - \chi_r(t)\| \leq \xi, \ \forall t \in [T(\epsilon), T_2] \tag{2.47}$$

Take $\epsilon_3^\star = \min(\epsilon_5, \epsilon_6)$, then, using (2.43), (2.45), and (2.47) we conclude (2.40).$\triangleleft$

2.5.4 Recovery of asymptotic stability of the origin

We treat first the case when there is no modeling error; then we proceed to the more general case when modeling error is present. In order to avoid very restrictive conditions on the modeling error, we separate the case where the origin of (2.16) is asymptotically stable from the case where it is exponentially stable. At this stage of the chapter we place ourselves in a small ball of radius $\xi > 0$ around the origin $(\chi, \eta) = (0, 0)$; the value of ξ will be determined later on. Theorem 2.2 guarantees that trajectories of the system (2.11)–(2.14), starting in $\mathcal{S} \times \mathcal{Q}$, enter this ball after a finite time and stay thereafter.

Case 1: We deal with the case where the origin of (2.16) is asymptotically stable, and there is no modeling error; i.e., we perfectly know ϕ in (2.1)–(2.4) and use it as

ϕ_0. We know [32, Theorem 7] that there exists a C^1 Lyapunov function V and a positive definite function U_3, both defined on a ball $B(0, r_1) \subseteq \mathcal{R}$ for some $r_1 > 0$, such that for all $\chi \in B(0, r_1)$

$$\frac{\partial V}{\partial \chi} f_r(\chi, 0) \leq -U_3(\chi) \tag{2.48}$$

Choose $\xi < r_1$; then given Assumptions 2.1, 2.2, and 2.3, we can show that, for all $(\chi, \eta) \in B(0, \xi) \times \{\|\eta\| \leq \xi\} = \Lambda_1$, we have

$$\|g(\chi, D(\epsilon)\eta)\| \leq L_4 \|\eta\| \tag{2.49}$$

Consider the composite function $\tilde{V}(\chi, \eta) = V(\chi) + (W(\eta))^{1/2}$ and choose $0 < \epsilon_4^\star \leq \epsilon_2^\star$ such that $1/(4\epsilon_4^\star \sqrt{\|P_0\|}) - \tilde{L}_2 L_3 - \|P_0\| L_4 / \sqrt{\lambda_{min}(P_0)} > 0$, where $\tilde{L}_2$ is an upper bound for $\left\| \frac{\partial V}{\partial \chi} \right\|$ over Λ_1 and L_3 is a Lipschitz constant of $f_r(.,.)$ in η over the same set. Then, we conclude that, for every $0 < \epsilon \leq \epsilon_4^\star$ and for all $(\chi, \eta) \in \Lambda_1$, we have

$$\dot{\tilde{V}} \leq -U_3(\chi) - \frac{1}{4\epsilon\sqrt{\|P_0\|}} \|\eta\| \tag{2.50}$$

Thus, the origin of system (2.11)–(2.14) is asymptotically stable. We summarize the above conclusion in the following theorem:

Theorem 2.4 *Let Assumptions 2.1–2.3 hold and assume that $\phi_0 = \phi$. Then, there exists $\epsilon_4^\star > 0$ such that, for every $0 < \epsilon \leq \epsilon_4^\star$, the origin of the system (2.11)–(2.14) is asymptotically stable.*

Remark 2.4 *Theorem 2.4 covers the results obtained by Teel and Praly in [49]. It also gives a nonlinear generalization of the linear separation principle.*

During the proof of cases 2 and 3, we will use the following fact which is a special case of *Young's Inequality* [18, Theorem 156]:

Fact: $\forall x, y \in R_+$, $\forall p > 1$, $\forall \epsilon_0 > 0$ we have

$$xy \leq \frac{1}{\epsilon_0} x^p + (\epsilon_0)^{p_0} y^{\frac{p}{p-1}}$$

where $p_0 = \frac{1}{p-1}$.

Case 2: We deal with the case where the origin of (2.16) is exponentially stable, whether or not we know ϕ.

In this case, there exists a C^1 Lyapunov function $V_2(\chi)$ [28, Theorem 3.13] defined over $B(0, r_2) \subseteq \mathcal{R}$, for some $r_2 > 0$, and four positive constants a_1, a_2, a_3, and a_4 such that, for all $\chi \in B(0, r_2)$ we have

$$\begin{aligned} a_1 \|\chi\|^2 \ \leq \ V_2(\chi) \ &\leq \ a_2 \|\chi\|^2 && (2.51) \\ \frac{\partial V_2}{\partial \chi} f_r(\chi, 0) \ &\leq \ -a_3 \|\chi\|^2 && (2.52) \\ \left\| \frac{\partial V_2}{\partial \chi} \right\| \ &\leq \ a_4 \|\chi\| && (2.53) \end{aligned}$$

Let us consider $\bar{V}(\chi, \eta) = V_2(\chi) + \beta W(\eta)$, where $\beta > 0$ is to be determined, as a Lyapunov function candidate for the system (2.11)–(2.14). Choose $\xi < r_2$; then, given Assumptions 2.1, 2.2, and 2.3, we have, for all $(\chi, \eta) \in B(0, \xi) \times \{\|\eta\| \leq \xi\} = \Lambda_2$,

$$\|g(\chi, D(\epsilon)\eta)\| \leq L_5 \|\chi\| + L_6 \|\eta\| \qquad (2.54)$$

Using (2.52), (2.53), (2.54), and Young's inequality, it can be shown that for all $(\chi, \eta) \in \Lambda_2$, we have

$$\dot{\bar{V}} \leq -\frac{a_3}{2} \|\chi\|^2 - \frac{\beta}{2\epsilon} \|\eta\|^2 - b_1 \|\chi\|^2 - b_2 \|\eta\|^2 \qquad (2.55)$$

where $b_1 = a_3/2 - a_4 L_7/\epsilon_0 - \beta L_5 \|P_0\|$, $b_2 = \beta/(2\epsilon) - a_4 L_7 \epsilon_0 - (4L_5 + 2L_6)\beta\|P_0\|$, L_7 is a Lipschitz constant of $f_r(.,.)$ in η over Λ_2, and $\epsilon_0 > 0$. Now, choose β small enough and ϵ_0 large enough such that $b_1 > 0$, then, it can be shown that there exists $0 < \epsilon_5^\star \leq \epsilon_2^\star$ such that, for every $0 < \epsilon \leq \epsilon_5^\star$, we have

$$\dot{V} \leq -\min\left(a_3/2, \beta/(2\epsilon)\right)\left[\|\chi\|^2 + \|\eta\|^2\right] \tag{2.56}$$

Thus, we can conclude that the origin of (2.11)–(2.14) is exponentially stable.

The foregoing result is summarized in the following theorem:

Theorem 2.5 *Let Assumptions 2.1–2.3 hold and suppose the vector field* $f_r(\chi, 0)$ *is continuously differentiable around the origin. Moreover, assume the origin of the closed-loop system under state feedback is exponentially stable. Then, there exists* $\epsilon_5^\star > 0$ *such that, for every* $0 < \epsilon \leq \epsilon_5^\star$, *the origin of the system (2.11)–(2.14) is exponentially stable.*

Case 3: We deal with the case where the origin of (2.16) is asymptotically, but not exponentially, stable. We start with an example that shows the need for some conditions on the modeling error in order for the output feedback controller to recover asymptotic stability of the origin. In addition, it gives an idea about how to formulate these conditions.

Example: Consider the system

$$\dot{x}_1 = x_2 \tag{2.57}$$

$$\dot{x}_2 = f(x) + u \tag{2.58}$$

$$y = x_1 \tag{2.59}$$

Suppose we know a nominal model $f_0(x) = -x_1$ of $f(x)$, and that the state feedback $u = -x_2$ globally stabilizes system (2.57)–(2.59) for the actual function f.

To implement the output feedback controller we use the high-gain observer

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2}{\epsilon}(x_1 - \hat{x}_1), \\
\dot{\hat{x}}_2 &= -\hat{x}_1 + u + \frac{1}{\epsilon^2}(x_1 - \hat{x}_1)
\end{aligned}
$$

By passing to the error coordinates and applying the output feedback $u = -\hat{x}_2$, we get the closed-loop system

$$
\begin{aligned}
\dot{x}_1 &= x_2 && (2.60) \\
\dot{x}_2 &= f(x) - x_2 + \eta_2 && (2.61) \\
\dot{\eta}_1 &= -\frac{2}{\epsilon}\eta_1 + \frac{1}{\epsilon}\eta_2 && (2.62) \\
\dot{\eta}_2 &= f(x) + x_1 - (\frac{1}{\epsilon} + \epsilon)\eta_1 && (2.63)
\end{aligned}
$$

Suppose now that the actual nonlinearity $f(x) = -x_1^3$. By linearization of the system (2.60)–(2.63) around the origin, we notice that, for any $\epsilon \in (0,1)$, the linearized system has a positive eigenvalue, thus, the origin of the output feedback system is unstable.

Clearly the Lyapunov analysis we performed in the exponentially stable case fails in this example. To see the source of the problem, let us note that the state feedback control $u = -x_2$ stabilizes the origin of (2.57)–(2.59) when $f = -x_1^3$, and the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 + \frac{1}{2}x_1^4$ satisfies

$$\dot{V} = -x_1^4 - x_2^2 \tag{2.64}$$

Now, notice that, around the origin, we have $|f(x) - f_0(x)| \sim |x_1|$. This modeling error is bigger than the absolute value of the derivative of the Lyapunov function $V(x)$ along trajectories of (2.57)–(2.59). This observation motivates the upper bound on the modeling error which will be stated in Assumption 2.4.

Realizing that the modeling error does not cause a problem in the exponentially stable case, we want to focus our attention on the part of the dynamics that is asymptotically, but not exponentially, stable. Towards that end we use the center manifold Theorem [28]. We need the vector field $f_r(\chi, 0)$ in (2.16) to be twice continuously differentiable.

Let us write the system (2.11)–(2.14) in the form

$$\dot{\chi} = f_r(\chi, 0) + \Delta_1(\chi, \eta) \tag{2.65}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon B\Delta_2(\chi, \eta) + \epsilon B\delta(\chi) \tag{2.66}$$

where

$$\begin{aligned}
\Delta_1(\chi, \eta) &= f_r(\chi, D(\epsilon)\eta) - f_r(\chi, 0) \\
\Delta_2(\chi, \eta) &= [\phi_0(x, \zeta, \gamma(\vartheta, x, \zeta)) - \phi_0(\hat{x}, \zeta, \gamma(\vartheta, \hat{x}, \zeta))] + \\
&\quad [\phi(x, z, \gamma(\vartheta, \hat{x}, \zeta)) - \phi(x, z, \gamma(\vartheta, x, \zeta))] \\
\delta(\chi) &= \phi(x, z, \gamma(\vartheta, x, \zeta)) - \phi_0(x, \zeta, \gamma(\vartheta, x, \zeta))
\end{aligned}$$

Since the origin of (2.16) is asymptotically, but not exponentially, stable, Theorem 3.13 of [28] shows that $\frac{\partial f_r}{\partial \chi}(0, 0)$ has all its eigenvalues with either zero or negative real parts. Thus, there is a change of variables

$$[\bar{x}^T, \bar{z}^T]^T = T\chi \tag{2.67}$$

such that (2.16) can be written as

$$\begin{aligned}\dot{\bar{x}} &= A_1\bar{x} + g_1(\bar{x},\bar{z})\\ \dot{\bar{z}} &= A_2\bar{z} + g_2(\bar{x},\bar{z})\end{aligned}$$

where A_1 has all its eigenvalues with zero real parts and A_2 is Hurwitz. In addition, Theorem 4.1 of [28] guarantees the existence of a continuously differentiable center manifold $\bar{z} = h(\bar{x})$, for all $\|\bar{x}\| \leq d_1$, for some $d_1 > 0$. Shifting the center manifold to the origin via the change of variable

$$\omega = \bar{z} - h(\bar{x}) \tag{2.68}$$

puts (2.16) in the form

$$\dot{\bar{x}} = A_1\bar{x} + g_1(\bar{x}, h(\bar{x})) + N_1(\bar{x},\omega) \tag{2.69}$$

$$\dot{\omega} = A_2\omega + N_2(\bar{x},\omega) \tag{2.70}$$

where

$$\|N_i(\bar{x},\omega)\| \leq \lambda_i\|\omega\|, \ \forall\|(\bar{x},\omega)\| \leq d_2 \tag{2.71}$$

for some $d_2 > 0$. The positive constants λ_i, $i = 1, 2$, can be made arbitrarily small by choosing d_2 small enough. By inserting all these changes into the system (2.11)–(2.14) we end up with

$$\dot{\bar{x}} = g_0(\bar{x}) + N_1(\bar{x},\omega) + M_1(\bar{x},\omega,\eta) \tag{2.72}$$

$$\dot{\omega} = A_2\omega + N_2(\bar{x},\omega) + M_2(\bar{x},\omega,\eta) \tag{2.73}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon B M_3(\bar{x},\omega,\eta) + \epsilon B\delta_1(\bar{x}) + \epsilon B\delta_2(\bar{x},\omega) \tag{2.74}$$

where $\delta_1(\bar{x}) \stackrel{\text{def}}{=} \delta(\chi)|_{\omega=0}$ is the projection of the modeling error $\delta(.)$ onto the center manifold, $\delta_2(\bar{x},\omega) \stackrel{\text{def}}{=} \delta(\chi) - \delta(\chi)|_{\omega=0}$, $[M_1^T, M_2^T]^T = T\Delta_1$, $M_3 = \Delta_2$, and $g_0(\bar{x}) = A_1\bar{x} + g_1(\bar{x}, h(\bar{x}))$.

Corollary 4.2 of [28] shows that the origin of the reduced system

$$\dot{\bar{x}} = g_0(\bar{x}) \tag{2.75}$$

is asymptotically stable. The condition on the modeling error, needed to establish the asymptotic stability of the origin of (2.11)–(2.14), can be stated as follows:

Assumption 2.4 *There exists a C^1 function $V_3(\bar{x})$ defined on $B_{\bar{x}}(0, r_3)$, a ball around $\bar{x} = 0$ contained in the projection of Ω onto the subspace of $\bar{x}$, that satisfies, for all $\bar{x} \in B_{\bar{x}}(0, r_3)$,*

$$\frac{\partial V_3}{\partial \bar{x}} g_0(\bar{x}) \leq -\alpha_5(\|\bar{x}\|) \tag{2.76}$$

$$\|\delta_1(\bar{x})\| \leq c_0\, \alpha_5^a(\|\bar{x}\|) \tag{2.77}$$

$$\left\|\frac{\partial V_3}{\partial \bar{x}}\right\| \leq c_1\, \alpha_5^b(\|\bar{x}\|) \tag{2.78}$$

for some positive constants a, $b < 1$ such that $a + b = 1$, where α_5 is a class $\mathcal{K}$ function defined on $[0, r_3]$, $c_0 \geq 0$, and $c_1 > 0$.

The existence of a Lyapunov function satisfying (2.76) is guaranteed by the converse Lyapunov theorem [28, Theorem 3.14], but what we need here is for (2.77) and (2.78) to be satisfied as well.

Remark 2.5 *When the reduced system (2.75) is one-dimensional, we can take $V_3(\bar{x}) = -\int_0^{\bar{x}} g_0(y)dy$. Then $\frac{\partial V_3}{\partial \bar{x}} = -g_0(\bar{x})$ and Assumption 2.4 is satisfied, with $a = b = 1/2$, if $|\delta_1(\bar{x})| \leq d_3\, |g_0(\bar{x})|$ for some $d_3 > 0$; i.e., $|\delta_1(\bar{x})|$ cannot approach the origin faster than $|g_0(\bar{x})|$.*

The recovery of asymptotic stability can now be stated as follows:

Theorem 2.6 *Let Assumptions 2.1–2.4 hold. Let the origin of the closed-loop system under state feedback be asymptotically, but not exponentially, stable, and let the vector field $f_r(\chi, 0)$ be twice continuously differentiable around the origin. Then, there exists $\epsilon_6^\star > 0$ such that, for all $0 < \epsilon \leq \epsilon_6^\star$, the origin of (2.11)–(2.14) is asymptotically stable.*

Proof: Consider the Lyapunov function candidate $\mathcal{V}(\bar{x}, \omega, \eta) = V_3(\bar{x}) + (\omega^T P_2 \omega)^\sigma + (W(\eta))^\sigma$, where P_2 is the positive definite solution of the Lyapunov equation $P_2 A_2 + A_2^T P_2 = -I$, and $\sigma = 1/(2a) > 1/2$.

Let $\xi < \min(d_1, d_2, r_3)$ and let $\epsilon \leq \epsilon_2^\star$, then according to Theorem 2.2 there exists a finite time T_3 after which we have $\|\chi(t,\epsilon)\| + \|\eta(t,\epsilon)\| \leq \min(\xi/(2\|T\|(1+L)), \xi)$ where L is a Lipschitz constant of $h(.)$ over $B_{\bar{x}}(0, r_3)$. Then, using (2.67) and (2.68) we can show that

$$\begin{aligned}
\|\bar{x}(t,\epsilon)\| + \|\omega(t,\epsilon)\| &\leq (1+L)\|\bar{x}(t,\epsilon)\| + \|\bar{z}(t,\epsilon)\| \\
&\leq 2(1+L)\|(\bar{x}^T(t,\epsilon), \bar{z}^T(t,\epsilon))^T\| \\
&\leq 2(1+L)\|T\|\|\chi(t,\epsilon)\| \\
&\leq \xi, \ \ \forall t \geq T_3
\end{aligned}$$

Due to Assumptions 2.1, 2.2 and 2.3 we have, for all $(\bar{x}, \omega, \eta) \in B_{\bar{x}}(0,\xi) \times B_\omega(0,\xi) \times \{\|\eta\| \leq \xi\} = \Lambda_3$,

$$\begin{aligned}
\|\delta_2(\bar{x}, \omega)\| &\leq L_8\|\omega\| && (2.79) \\
\left\| \begin{matrix} M_1(\bar{x}, \omega, \eta) \\ M_2(\bar{x}, \omega, \eta) \end{matrix} \right\| &\leq L_9\|T\|\|\eta\| && (2.80) \\
\|M_3(\bar{x}, \omega, \eta)\| &\leq L_{10}\|\eta\| && (2.81)
\end{aligned}$$

Using (2.71), (2.76)–(2.78), and (2.79)–(2.81), we can show that, for all $(\bar{x}, \omega, \eta) \in \Lambda_3$, we have

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\alpha_5(\|\bar{x}\|) + \rho_0\lambda_1\|\omega\|\alpha_5^b(\|\bar{x}\|) + \rho_1\|\eta\|\alpha_5^b(\|\bar{x}\|) - \rho_2\|\omega\|^{2\sigma} + \\ & \rho_3\lambda_2\|\omega\|^{2\sigma} + \rho_4\|\eta\|\|\omega\|^{2\sigma-1} - \tfrac{\rho_5}{\epsilon}\|\eta\|^{2\sigma} + \rho_6\|\eta\|^{2\sigma} + \\ & \rho_7\alpha_5^a(\|\bar{x}\|)\|\eta\|^{2\sigma-1} + \rho_8\|\omega\|\|\eta\|^{2\sigma-1} \end{aligned} \tag{2.82}$$

where

$$\begin{aligned}\rho_0 &= c_1,\ \rho_1 = c_1\|T\|L_9,\ \rho_2 = \sigma\gamma_1 \\ \rho_3 &= 2\sigma\gamma_2\|P_2\|,\ \rho_4 = 2\sigma\gamma_2\|P_2\|\|T\|L_9,\ \rho_5 = \sigma\gamma_3 \\ \rho_6 &= 2\sigma\gamma_4\|P_0\|L_{10},\ \rho_7 = 2\sigma\gamma_4\|P_0\|c_0,\ \rho_8 = 2\sigma\gamma_4\|P_0\|L_8\end{aligned}$$

are positive constants, and where

$$\left.\begin{aligned}\gamma_1 &= (\lambda_{min}(P_2))^{\sigma-1} \\ \gamma_2 &= \|P_2\|^{\sigma-1} \\ \gamma_3 &= (\lambda_{min}(P_0))^{\sigma-1} \\ \gamma_4 &= \|P_0\|^{\sigma-1}\end{aligned}\right\} \text{ if } \sigma \geq 1 \qquad \left.\begin{aligned}\gamma_1 &= \|P_2\|^{\sigma-1} \\ \gamma_2 &= (\lambda_{min}(P_2))^{\sigma-1} \\ \gamma_3 &= \|P_0\|^{\sigma-1} \\ \gamma_4 &= (\lambda_{min}(P_0))^{\sigma-1}\end{aligned}\right\} \text{ if } 1/2 \leq \sigma < 1$$

The positive constants λ_1 and λ_2, defined by (2.71), can be made arbitrarily small by choosing ξ small enough.

Next, we separate the five cross-product terms in (2.82) by repeatedly using Young's Inequality with $p = 2\sigma$ and $\epsilon_0 = q_1$, q_2, q_3, q_4, and q_5, respectively. Consequently, given that $\sigma = 1/(2a)$ and $a + b = 1$, we have all the terms in $\|\eta\|$ or in $\|\omega\|$ with the power 2σ and all the terms in $\alpha_5(\|\bar{x}\|)$ with the power 1.

Now, choose λ_1, λ_2 and q_1, q_2, q_3, q_4, and q_5 such that $-1/2 + \rho_0\lambda_1(q_1)^{p_1} + \rho_1(q_2)^{p_1} + \rho_7/q_4 < 0$ and $-\rho_2/2 + \rho_0\lambda_1/q_1 + \rho_3\lambda_2 + \rho_4(q_3)^{p_1} + \rho_8/q_5 < 0$, where

$p_1 = 1/(2\sigma - 1)$. Then, it can be shown that there exists $0 < \epsilon_6^\star \leq \epsilon_2^\star$ such that, for every $0 < \epsilon \leq \epsilon_6^\star$ we have $-\rho_5/(2\epsilon) + \rho_1/q_3 + \rho_6 + \rho_7(q_4)^{p_1} + \rho_8(q_5)^{p_1} < 0$ which implies that for every $0 < \epsilon \leq \epsilon_6^\star$ and for all $(\bar{x}, \omega, \eta) \in \Lambda_3$, we have

$$\dot{\mathcal{V}} \leq -\frac{1}{2}\alpha_5(\|\bar{x}\|) - \frac{\rho_2}{2}\|\omega\|^{2\sigma} - \frac{\rho_5}{2\epsilon}\|\eta\|^{2\sigma} \tag{2.83}$$

Thus, the origin of the system (2.11)–(2.14) is asymptotically stable.◁

Remark 2.6 *Theorems 2.4, 2.5 and 2.6, along with Theorems 2.1 and 2.2, show the recovery of the region of attraction.*

2.6 Examples

We apply our technique to different systems in order to illustrate the theoretical results and go beyond the theory to demonstrate some reasonable intuitions. We also take advantage of these examples to show that the results obtained in this chapter apply not only to input-output linearizable systems as in the previous work [11, 27, 26, 30, 31, 38, 37, 41] but to any kind of system that fits the model (2.1)–(2.4).

2.6.1 Example 1

We consider a second order system having an exponentially unstable mode, together with a bounded linear controller that achieves a finite region of attraction. The system is

$$\dot{x}_1 = x_2 \tag{2.84}$$

$$\dot{x}_2 = 2x_1 + 10\tanh(u) \tag{2.85}$$

where the control is $u = -x_1 - x_2$.

We consider a full-order high-gain linear observer (i.e., $\phi_0 = 0$) with $\alpha_1 = \alpha_2 = 1$. In this example we show how the output feedback controller recovers the region of attraction achieved under state feedback.

Figure 2.1 shows the region of attraction under state feedback control, in addition to three compact subsets that are recovered using the high-gain observer. In each case the compact subset is specified, then a design parameter $\epsilon^\star$ is found through multiple simulations at different points of the subset such that for every $\epsilon \leq \epsilon^\star$ the output feedback controller is able to recover the given subset; i.e., it is a part of the region of attraction of the new closed-loop system. The bound $\epsilon^\star$ is tight in the sense that for $\epsilon > \epsilon^\star$ there is a part of the given set that is not included in the region of attraction. The bounds $\epsilon^\star$'s for these subsets are 0.082, 0.057, and 0.007, respectively, starting from the smallest subset. Notice that the bigger the subset the smaller the bound $\epsilon^\star$. In all cases we take $\hat{x}(0) = 0$.

2.6.2 Example 2 - Inverted Pendulum

We consider the inverted pendulum-on-a-cart problem given in [15]. The system consists of an inverted pendulum mounted on a cart free to move on a horizontal plane. The equations of motion are given by:

$$\begin{aligned}
\dot{x}_1 &= x_2 && (2.86)\\
\dot{x}_2 &= \frac{1}{M + m\sin^2(x_3)}[-mg\sin(x_3)\cos(x_3) + mlx_4^2\sin(x_3) - \\
&\quad bx_2 + u] && (2.87)\\
\dot{x}_3 &= x_4 && (2.88)\\
\dot{x}_4 &= \frac{1}{l(M + m\sin^2(x_3))}[(M+m)g\sin(x_3) - mlx_4^2\sin(x_3)\cos(x_3) + \\
&\quad bx_2\cos(x_3) - u\cos(x_3)] && (2.89)
\end{aligned}$$

where M is the mass of the cart, m is the mass of the ball attached to the free end of the pendulum, l is the length of the pendulum, g is the gravitational acceleration, b is the coefficient of viscous friction opposing the cart's motion, x_1 is the cart's displacement, x_2 is the cart's velocity, x_3 is the the angle that the pendulum makes with the vertical, and x_4 is the pendulum's angular velocity. The values of the different parameters of the model are $M = 1.378$Kg, $m = 0.051$Kg, $g = 9.81\text{m}/sec^2$, $l = 0.325$m, and $b = 12.98$Kg/sec. The nominal value of b is b_0. It is shown in [15] that the state feedback control

$$\begin{aligned} u &= [mg\sin(x_3)\cos(x_3) - mlx_4^2\sin(x_3) + b_0x_2 + \\ &\quad (M + m\sin^2(x_3))v] && (2.90) \\ v &= -900x_2 + 900[3.22x_1 + 12x_3 + 7.44(lx_4 + x_2\cos(x_3))] && (2.91) \end{aligned}$$

stabilizes the origin[4].

Let the measured outputs be (x_1, x_3). We use a full order high-gain observer to estimate all the state variables, and use these estimates in the stabilizing control. In order to avoid peaking in the state variables, induced by peaking of the observer variables, we saturate the control input such that our region of interest is included in the system's region of attraction; we use $u = 100\tanh(./100)$. The nominal function ϕ_0 is made globally bounded by saturating x_2 and x_4 at 15 and 20, respectively. We design a full-order high-gain observer with multiple poles at $-1/\epsilon$.

Figure 2.2 shows how the velocity estimate peaks due to the difference in the initial conditions between the position and its estimate. This is done for a linear

[4]This is a special case of the tracking control of [15] when the desired trajectory is taken to be zero.

observer and the following choice of initial conditions and design parameters:

$$x(0) = (1, 0, -0.5, 0),\ \hat{x}(0) = 0,\ b_0 = 12.98,\ \epsilon = 0.001$$

Figure 2.3 shows how our output feedback controller recovers the trajectories achieved under state feedback. We use a linear high-gain observer with three values of ϵ. This is done for the following choice of initial conditions and design parameters:

$$\begin{aligned} x(0) &= (1, -4, 0.7, -9),\ \hat{x}(0) = (0, 0, 0, 0),\ b_0 = 12.98, \\ \epsilon &= 0.0015,\ 0.001,\ 0.0001 \end{aligned}$$

Intuitively we expect that a nonlinear observer that includes a model of the system's nonlinearities would outperform a linear one when the model is accurate. Figures 2.4 and 2.5 show that this intuition is justified. In Figure 2.4 we compare between a linear observer and a nonlinear one with no modeling error; i.e., the nonlinearity is perfectly known; thus $b_0 = b$. In Figure 2.5 we do it with modeling error induced by a nominal value $b_0 = 15.58$ (20% error). In these two simulations we use the following initial conditions and design parameters:

$$x(0) = (1, 0, 0.8, 0),\ \hat{x}(0) = (0, 0, 0, 0),\ \epsilon = 0.01$$

It is clear that when $b_0 = b$ the nonlinear observer outperforms the linear one. But as the model uncertainty increases, the performance of the nonlinear observer degrades towards that of the linear observer.

2.6.3 Example 3 - VTOL aircraft

We consider the simplified PVTOL (Planar Vertical Take off and Landing) aircraft modeled in [39] by

$$
\begin{aligned}
\dot{x}_1 &= x_2, \ \dot{x}_2 = -u_1 \sin(x_5) + \mu u_2 \cos(x_5) \\
\dot{x}_3 &= x_4, \ \dot{x}_4 = u_1 \cos(x_5) + \mu u_2 \sin(x_5) - g \\
\dot{x}_5 &= x_6, \ \dot{x}_6 = \lambda u_2
\end{aligned}
$$

where x_1, x_3, and x_5 are the horizontal coordinate, the vertical coordinate, and the inclination of the aircraft, respectively. We also consider, like [39], the linearizing dynamic feedback

$$
\begin{aligned}
\dot{x}_7 &= x_8, \ \dot{x}_8 = -\nu_1 \sin(x_5) + \nu_2 \cos(x_5) + x_7 x_6^2 \\
u_1 &= x_7 + \frac{\mu}{\lambda} x_6^2 \\
u_2 &= \frac{1}{\lambda x_7}(-\nu_1 \cos(x_5) - \nu_2 \sin(x_5) - 2 x_8 x_6)
\end{aligned}
$$

The equilibrium point of the closed-loop system under state feedback is $\bar{x} = (\bar{x}_1, 0, \bar{x}_3, 0, 0, 0, g, 0)$, $\bar{u} = (g, 0)$, and $\bar{\nu} = (0, 0)$. The linearizing effect of this dynamic controller can be seen by applying the change of variables

$$
\begin{aligned}
\chi_1 &= x_1 - \frac{\mu}{\lambda}\sin(x_5), \quad \chi_2 = x_2 - \frac{\mu}{\lambda} x_6 \cos(x_5) \\
\chi_3 &= -x_7 \sin(x_5), \quad \chi_4 = -x_8 \sin(x_5) - x_7 x_6 \cos(x_5) \\
\chi_5 &= x_3 + \frac{\mu}{\lambda}\cos(x_5), \quad \chi_6 = x_4 - \frac{\mu}{\lambda} x_6 \sin(x_5) \\
\chi_7 &= x_7 \cos(x_5) - g, \quad \chi_8 = x_8 \cos(x_5) - x_7 x_6 \sin(x_5)
\end{aligned}
$$

to obtain $\dot{\chi}_4 = \nu_1$ and $\dot{\chi}_8 = \nu_2$. In the new coordinates the equilibrium point is $\bar{\chi} = (\bar{x}_1, 0, \bar{x}_3 + \mu/\lambda, 0, 0, 0, 0, 0)$. We choose the equilibrium point to be at $x_{eq} =$

$(2, 0, 2, 0, 0, 0, g, 0)$. To stabilize the aircraft at this equilibrium point, or to make it track this constant trajectory, we define the change of variables $\tilde{\chi}_1 = \chi_1 - 2$, $\tilde{\chi}_3 = x_3 - 2 - \mu/\lambda$, with the rest of the χ variables unchanged. Then we take the control

$$\nu_1 = k_1\tilde{\chi}_1 + k_2\tilde{\chi}_2 + k_3\tilde{\chi}_3 + k_4\tilde{\chi}_4 \tag{2.92}$$

$$\nu_2 = k_1\tilde{\chi}_1 + k_2\tilde{\chi}_2 + k_3\tilde{\chi}_3 + k_4\tilde{\chi}_4 \tag{2.93}$$

where k_1 to k_4 are chosen to stabilize the origin $\tilde{\chi} = 0$. For the purpose of simulations we take g, μ and λ to be 1, 1, and 0.5, and k_1, k_2, k_3, and k_4 to be -24, -50, -35, and -10

Now suppose we only measure the position variables x_1, x_3, and x_5, and set $y_1 = x_1$, $y_2 = x_3$, and $y_3 = x_5$. We want to design an observer to estimate the velocity variables x_2, x_4, and x_6. Noting that the nonlinear functions $\sin(x_5)$ and $\cos(x_5)$ depend only on the measured output variable x_5, the system takes the form

$$\dot{x} = Ax + B\phi(u, y), \quad y = Cx$$

If the nonlinear function $\phi(u, y)$ is exactly known, we can design an observer that yields linear error dynamics [20, Section 4.9]. In particular, the full-order observer

$$\dot{\hat{x}} = A\hat{x} + B\phi(u, y) + L(y - C\hat{x})$$

results in the error equation

$$\dot{e} = (A - LC)e$$

where $e = x - \hat{x}$. Such observer design does not need the high-gain observer theory presented in this paper. For the purpose of comparison with our method, we design a reduced-order observer that yields linear error dynamics with eigenvalues located at -1, assuming perfect knowledge of the parameters μ and λ. We will refer to this observer as the nominal reduced-order observer. On the other hand, when there is uncertainty in modeling the nonlinearity $\phi(u, y)$, the error dynamics will no longer be linear. This case is covered by our high-gain observer which is designed to be robust with respect to uncertainties in ϕ. We design a reduced-order high-gain observer with eigenvalues located at $-1/\epsilon$. We saturate the dynamic feedback controller as follows: $\dot{x}_7 = 20\tanh(./20)$, $\dot{x}_8 = 200\tanh(./200)$, $u_1 = 40\tanh(./40)$, and $u_2 = 200\tanh(./200)$. These bounds have been figured out from extensive simulations done to see the maximal values that the state trajectories would take when the initial state is in a region of interest around x_{eq}. The model nonlinearities are naturally globally bounded so we do not need to saturate the nominal nonlinearities in the observer.

To illustrate the capability of our output feedback controller to recover the state trajectories we perform simulations with different values of the design parameter ϵ. Figure 2.6 shows that as ϵ approaches 0 the trajectories under output feedback approach the trajectories under state feedback.

To illustrate the robustness of the output feedback controller we perform simulations with a 15% error in λ. Figure 2.7 shows that our design does its job in recovering stability and trajectories for ϵ small enough. The above simulations are done with the following choice of initial conditions and parameters:

$$x(0) = (1, 0, 1, 0, 0, 0, 1, 0),\ \hat{x}(0) = (0, 0, 0),\ \epsilon = 0.02,\ 0.008,\ 0.002$$

Figures 2.8 and 2.9 show that our observer outperforms the nominal observer in terms of convergence rate and trajectory recovery. One may think that by increasing the gain of the nominal observer we can have as good of a performance as the high-gain observer. This is not true, and Figure 2.10 shows it. In the nominal observer, the input and the nonlinearity are not globally bounded. Therefore, peaking in the estimate passes to the state through the input. For this reason we saturate our dynamic feedback. The above simulations are done with the following choice of initial conditions and parameters:

$$x(0) = (1,0,1,0,0,0,1,0),\ \hat{x}(0) = (0,0,0),\ \epsilon = 0.01,\ 0.002,$$
$$nominal\ observer\ gain = 100$$

Remark 2.7 *Of course, the trajectories recovered are the entire state x_1 to x_8 but Figures 2.6 to 2.10 only show a few of them.*

2.7 Conclusion

We presented and proved a separation principle for a certain class of nonlinear systems. An output feedback controller using a sufficiently fast high-gain observer recovers the performance achieved under a state feedback controller. This includes boundedness, ultimate boundedness, convergence of trajectories, and exponential stability of the origin. We also found that we can recover asymptotic stability of the origin when the modeling error is zero, but, when this error is not zero, we need to impose some additional conditions. It is worthwhile to note that our results can only show semiglobal stabilization under output feedback even when the state feedback control achieves global stabilization. Global separation results are more challenging as the discussions of [13] reveal. We performed simulations on various types of systems and controllers to illustrate the results obtained in this paper. These simulations showed the advantage of including a model of the system's nonlinearity in the observer, when a good model is available. Furthermore, they demonstrated the effectiveness of the combination of high-gain observers and saturation in recovering asymptotic stability and trajectories achieved under state feedback.

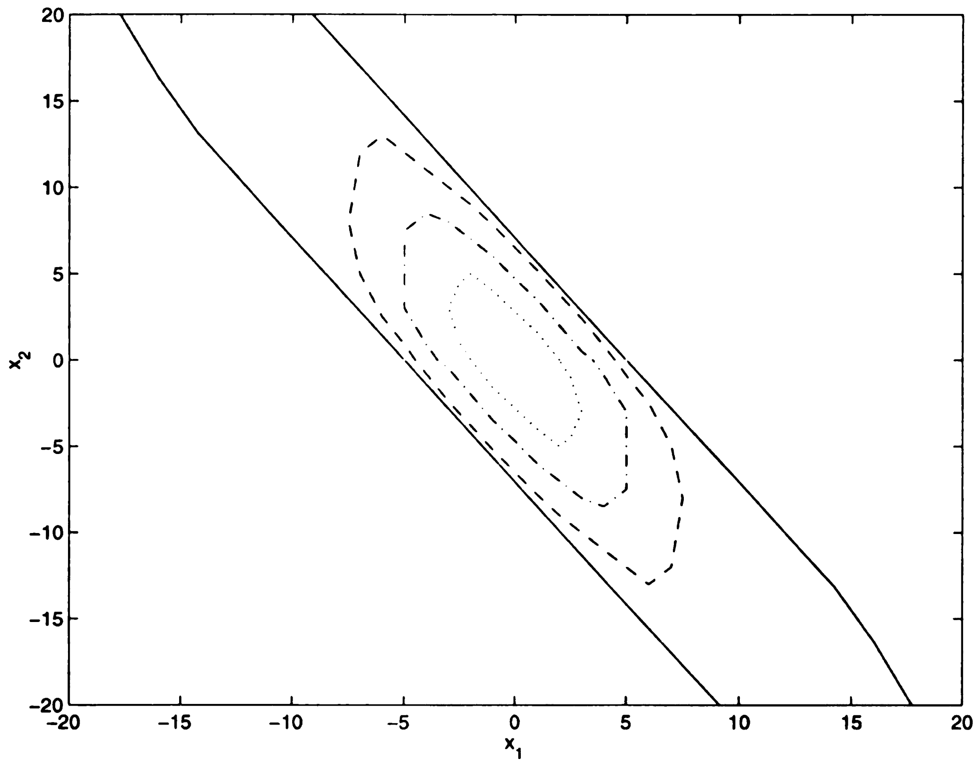

Figure 2.1. Recovery of region of attraction: $\epsilon^{\star} = 0$ (solid), $\epsilon^{\star} = 0.007$ (dashed), $\epsilon^{\star} = 0.057$ (dash-dotted), and $\epsilon^{\star} = 0.082$ (dotted)

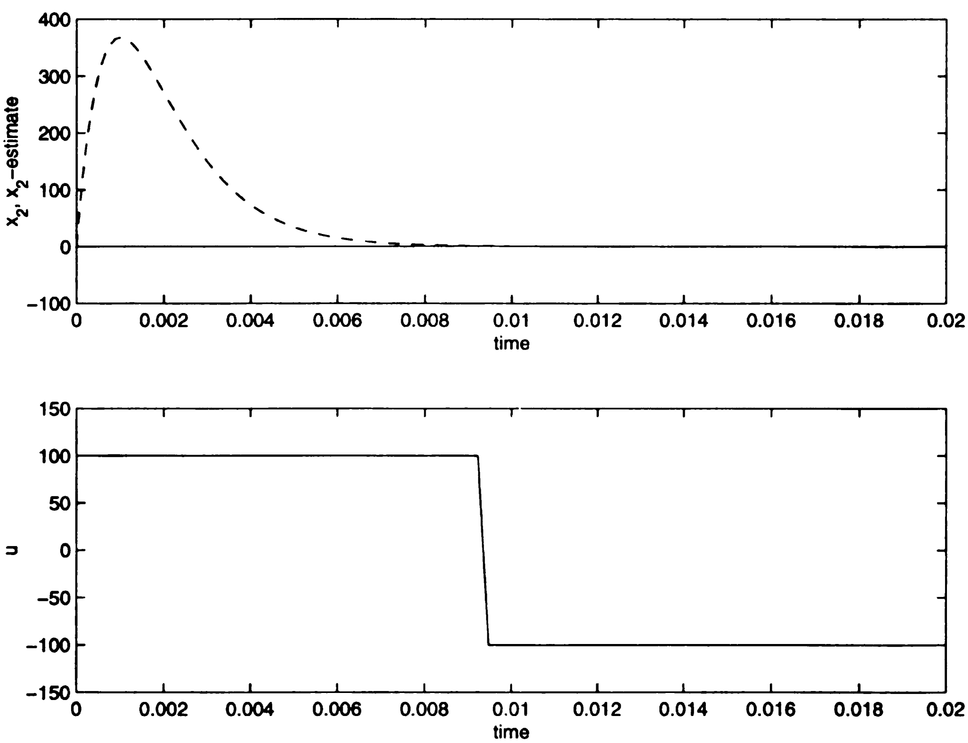

Figure 2.2. Peaking of the velocity: x_2 (solid), $\hat{x}_2$ (dashed)

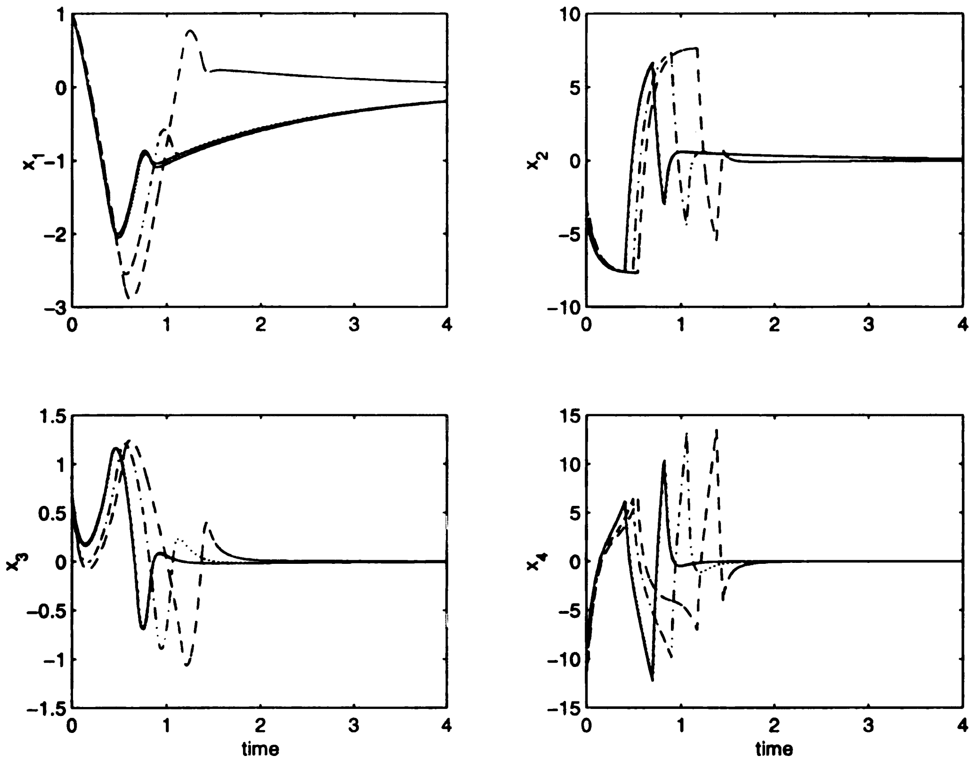

Figure 2.3. Trajectory convergence: state feedback (solid); output feedback with $\epsilon = 0.0015$ (dashed), $\epsilon = 0.001$ (dash-dotted), and $\epsilon = 0.0001$ (dotted)

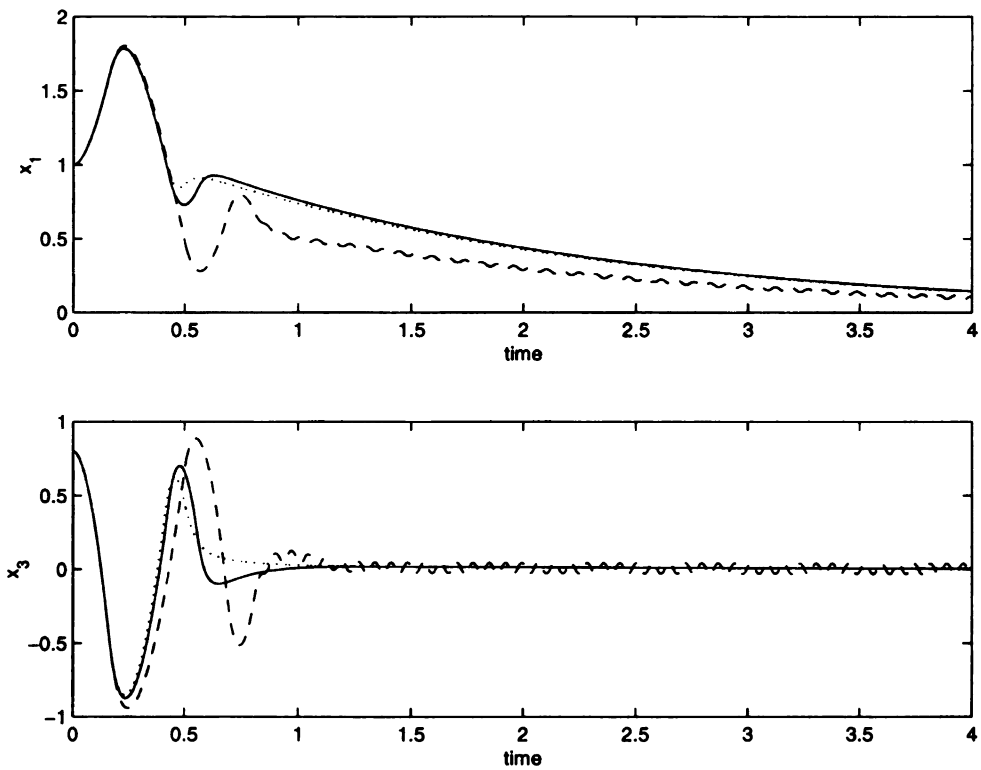

Figure 2.4. Effect of nonlinearity in the observer - without uncertainty: state feedback (solid), with linear observer (dashed), with nonlinear observer (dotted)

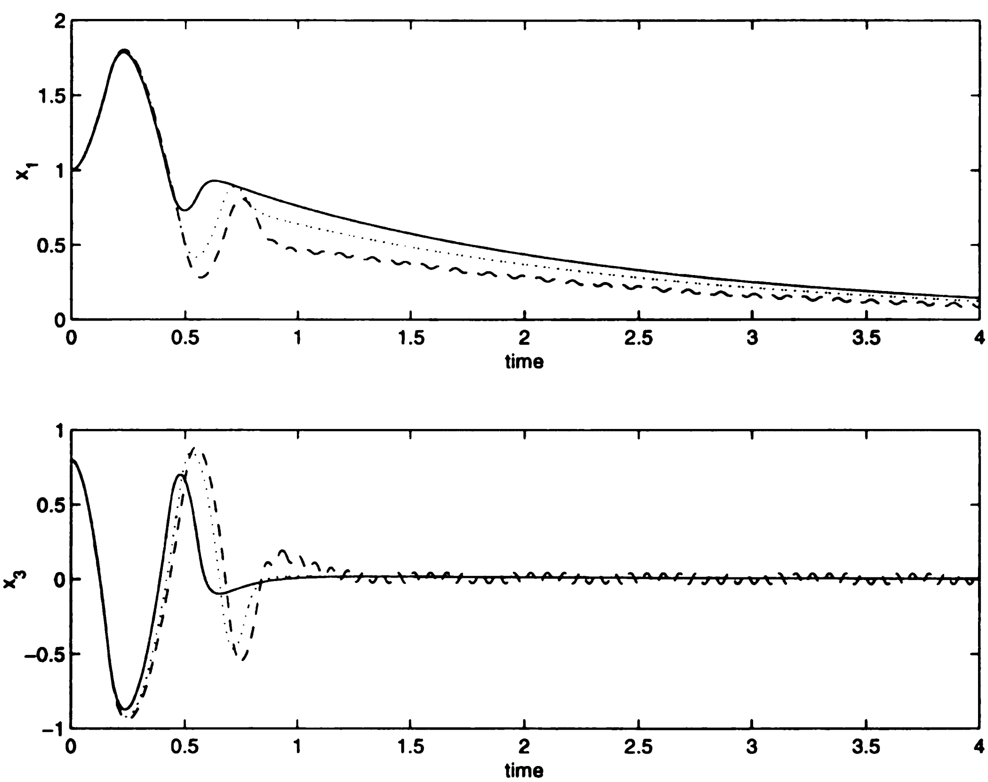

Figure 2.5. Effect of nonlinearity in the observer - with uncertainty: state feedback (solid), with linear observer (dashed), with nonlinear observer (dotted)

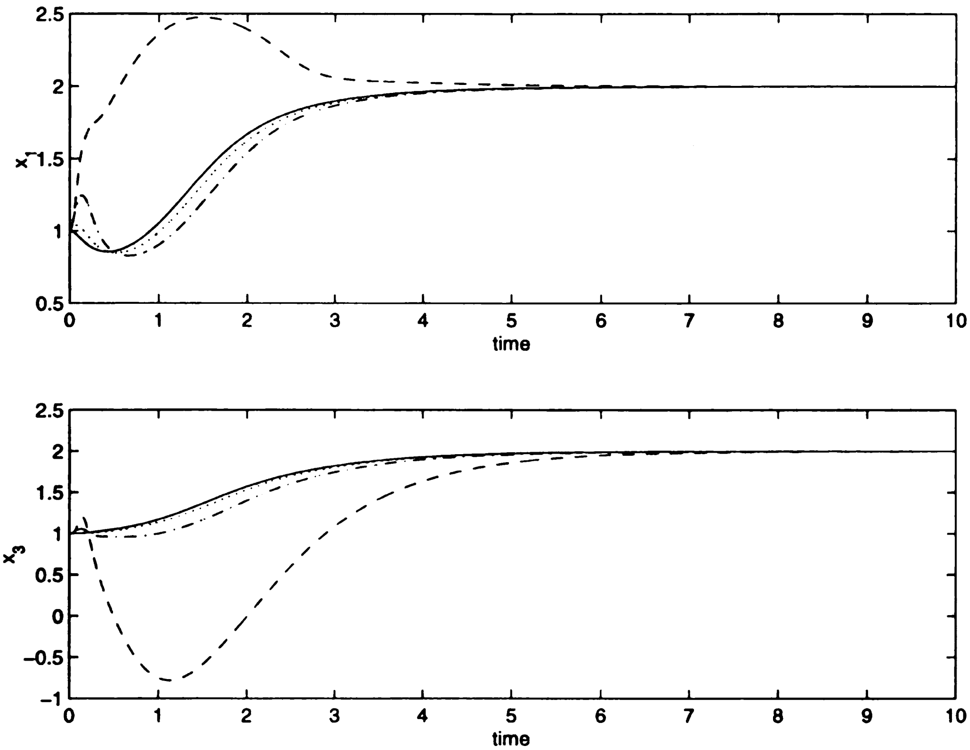

Figure 2.6. Trajectory convergence - without uncertainty: state feedback (solid); output feedback with $\epsilon = 0.02$ (dashed), $\epsilon = 0.008$ (dash-dotted), and $\epsilon = 0.002$ (dotted)

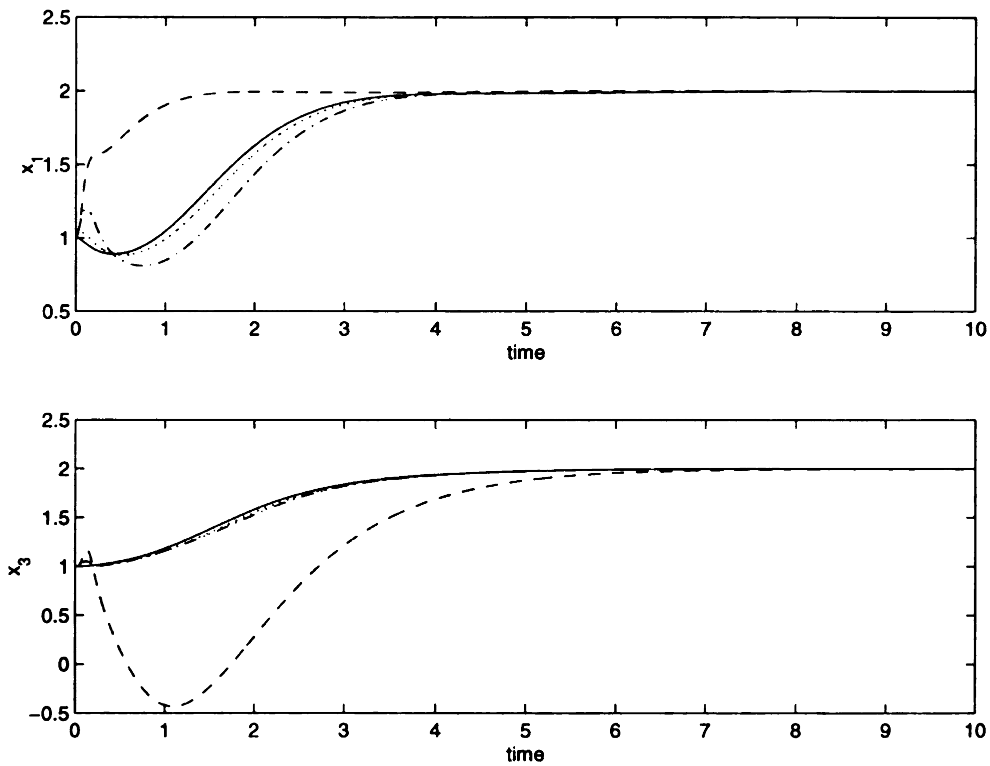

Figure 2.7. Trajectory convergence - with uncertainty: state feedback (solid); output feedback with $\epsilon = 0.02$ (dashed), $\epsilon = 0.008$ (dash-dotted), and $\epsilon = 0.002$ (dotted)

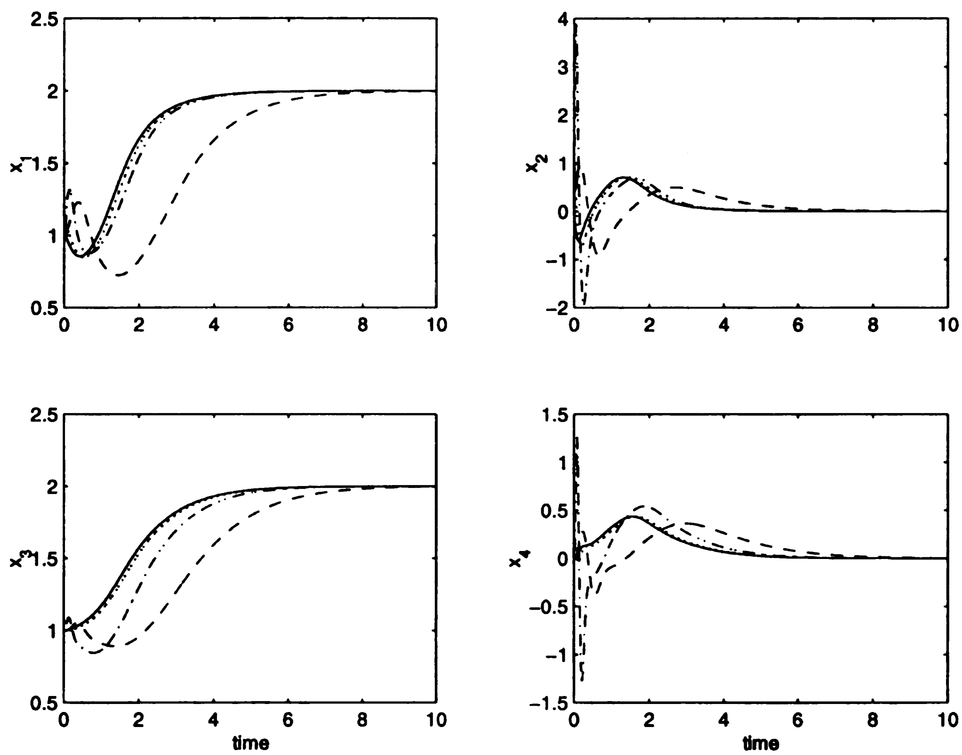

Figure 2.8. High-gain vs. nominal observer - without uncertainty: state feedback (solid), nominal observer (dashed); high-gain observer with $\epsilon = 0.01$ (dash-dotted) and $\epsilon = 0.002$ (dotted)

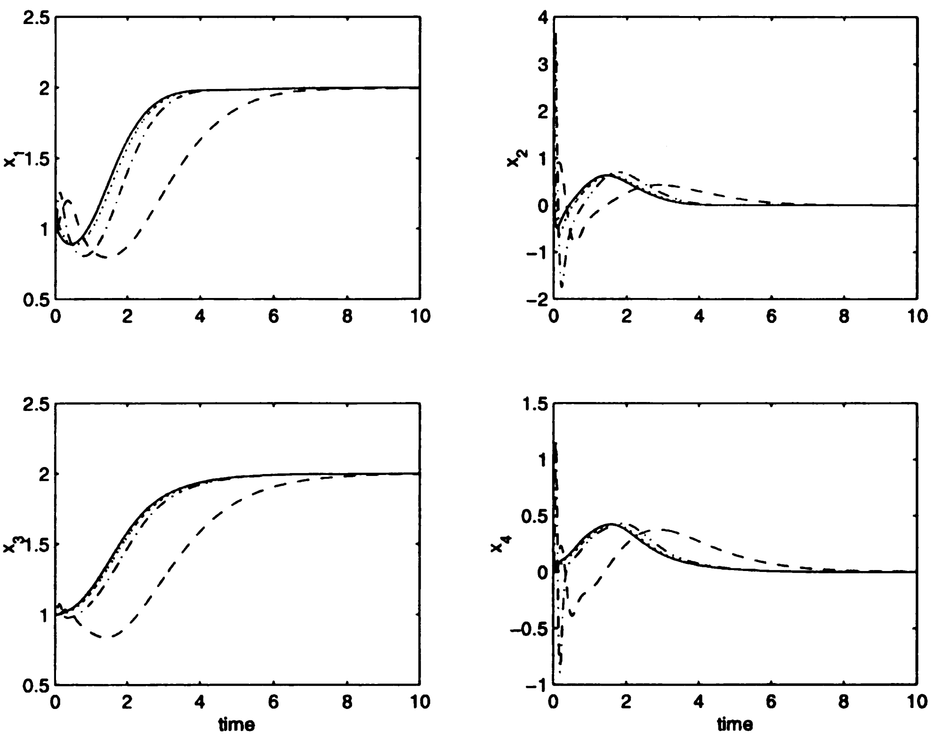

Figure 2.9. High-gain vs. nominal observer - with uncertainty: state feedback (solid), nominal observer (dashed); high-gain observer with $\epsilon = 0.01$ (dash-dotted) and $\epsilon = 0.002$ (dotted)

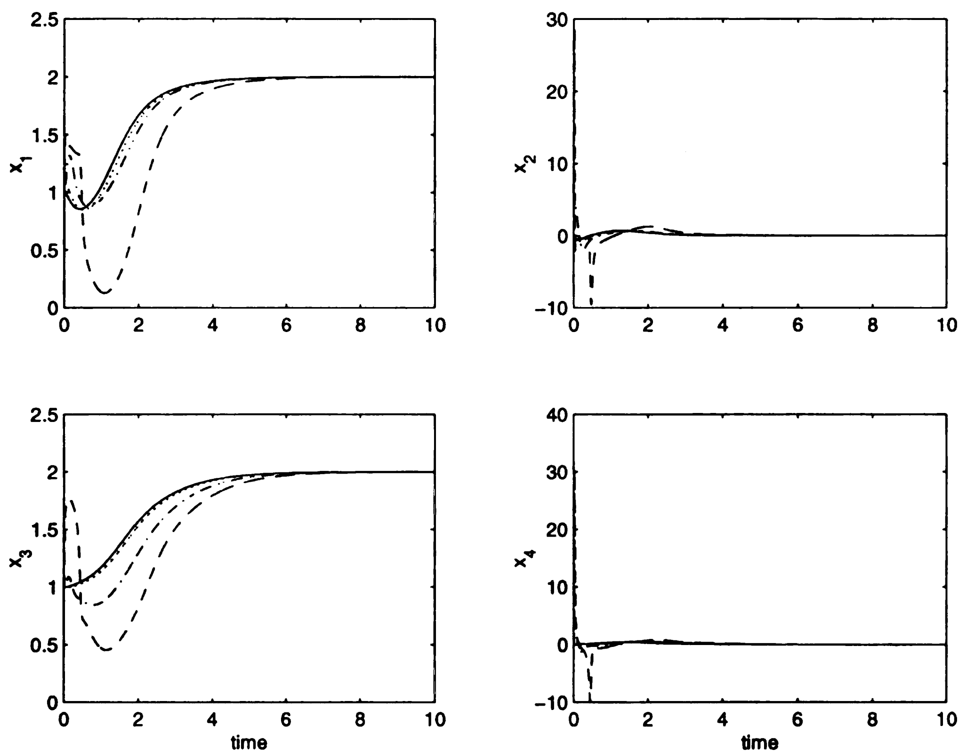

Figure 2.10. High-gain vs. nominal observer - importance of saturation: state feedback (solid), nominal observer (dashed); high-gain observer with $\epsilon = 0.01$ (dash-dotted) and $\epsilon = 0.002$ (dotted)

CHAPTER 3

A Separation Principle for the Control of a Class of Nonlinear Systems

3.1 Introduction

In Chapter 2 we introduce separation results for the stabilization of a class of systems having a chain or more of integrators in their structure. Therein, we consider state feedback controllers that make the origin of the closed-loop system an asymptotically stable equilibrium point. In this chapter we are interested in the output feedback implementation of controllers that achieve boundedness of trajectories under state feedback control but not necessarily with convergence to an equilibrium point. Such a situation can be encountered in adaptive tracking and regulation [2, 27] where only the tracking error or both the tracking error and the parameter error converge to zero. Another example is the convergence to a zero-error manifold as in the servomechanism problem discussed in [26, 37, 38, 21]. Additional examples can be found in stabilization problems in the presence of disturbances as in [50, 10] where only finite-time convergence to a set can be achieved. In all these cases, it can be shown that the trajectories approach an attractive, positively invariant, compact set.

In this chapter, we consider a class of systems similar to the one considered in Chapter 2 and characterize the performance of the state feedback controller as rendering a certain compact set positively invariant and asymptotically attractive. Furthermore, as in Chapter 2, we require the control law to be globally bounded and implement it using a high-gain observer. Then, we recover the same set of performance measures that was recovered in Chapter 2. It includes recovery of the region of asymptotic stability of the attractive set (i.e., recovery of arbitrary compact subsets of this region), as well as the convergence of trajectories under output feedback control to those under state feedback control as the observer gain approaches infinity.

We start with semiglobal separation results using results form [34]. Then, we give similar separation results for a possibly finite region of attraction. For this task we adapt the results of [34] to this case because [34] deals only with global convergence to a set. In order to illustrate the theory developed hereafter, we present in the next chapter several examples taken form [10, 50, 38, 21, 2].

This chapter is organized as follows: Section 3.2 states some definitions and recalls results from [34], Section 3.3 formulates the problem. Section 3.4 discusses the set of performance measures to be recovered by output feedback and proves this recovery in a semiglobal setting. Section 3.5 adapts the results of [34] and Section 3.4 to the finite region of attraction case.

3.2 Definitions and Converse Lyapunov Results

Consider the system

$$\dot{x}(t) = f(x(t), d(t)) \tag{3.1}$$

where for each $t \in R$, $x(t) \in R^n$ and $d(t) \in \mathcal{D}$, and where $\mathcal{D}$ is a compact subset of R^d. The map $f : R^n \times \mathcal{D} \to R^n$ is assumed to satisfy the following properties:

- f is continuous in its arguments.
- f is locally Lipschitz in x uniformly in d. This means that for each compact subset K of R^n there is some constant c such that

$$\|f(x, \mathbf{d}) - f(z, \mathbf{d})\| \leq c\|x - z\|$$

for all $x, z \in K$ and all $\mathbf{d} \in \mathcal{D}$.

Let $\mathcal{M}_{\mathcal{D}}$ be the set of all piecewise continuous functions from R to $\mathcal{D}$. For each $d \in \mathcal{M}_{\mathcal{D}}$, we denote by $x(t, x_0; d)$ the solution at time t of (3.1) with $x(0) = x_0$. This solution exists and is defined on some maximal interval $(T^-_{x_0,d}, T^+_{x_0,d})$ with $-\infty \leq T^-_{x_0,d} < 0 < T^+_{x_0,d} \leq +\infty$.

We say that a set $\mathcal{A}$ is a *positively invariant* set for (3.1) if

$$\forall x_0 \in \mathcal{A},\ \forall d \in \mathcal{M}_{\mathcal{D}},\ T^+_{x_0,d} = +\infty \text{ and } x(t, x_0; d) \in \mathcal{A},\ \forall t \geq 0$$

Let $\mathcal{A}$ be a closed, non-empty subset of R^n. The distance of $\xi \in R^n$ with respect to $\mathcal{A}$ is defined as

$$|\xi|_{\mathcal{A}} = \inf_{\eta \in \mathcal{A}} \|\xi - \eta\|$$

In the sequel we use the notation

$$L_{f_{\mathbf{d}}} V(\xi) = \frac{\partial V(\xi)}{\partial \xi} . f_{\mathbf{d}}(\xi)$$

where, for each $\mathbf{d} \in \mathcal{D}$, $f_{\mathbf{d}}(.)$ is the vector field defined by $f(., \mathbf{d})$. By "smooth" we always mean infinitely differentiable.

We define uniform asymptotic stability with respect to a set in the spirit of [4, Definitions 4.1, 4.12] and [57, Section 1.10, Definition 1].

Definition 3.1 *The system (3.1) is Uniformly Asymptotically Stable (UAS) with respect to the compact positively invariant set $\mathcal{A}$ if the following two properties hold:*

1. *Uniform Stability: for any $\epsilon > 0$, there exists a constant $\delta_1 = \delta_1(\epsilon)$ such that*

$$|x(t, x_0, d)|_{\mathcal{A}} \leq \epsilon \text{ for all } d \in \mathcal{M}_{\mathcal{D}}, \text{ whenever } |x_0|_{\mathcal{A}} \leq \delta_1(\epsilon) \text{ and } t \geq 0 \quad (3.2)$$

2. *Uniform Attraction: there is an $\alpha > 0$ and, for any $\epsilon > 0$, there is $T = T(\epsilon) > 0$, such that for every $d \in \mathcal{M}_{\mathcal{D}}$,*

$$|x(t, x_0, d)|_{\mathcal{A}} < \epsilon \text{ whenever } |x_0|_{\mathcal{A}} < \alpha \text{ and } t \geq T \quad (3.3)$$

Moreover, the system (3.1) is Uniformly Globally Asymptotically Stable (UGAS) with respect to $\mathcal{A}$, if (3.2) holds with a class $\mathcal{K}_{\infty}$ function $\delta(\epsilon)$ and (3.3) holds for any $r > 0$ with $T = T(\epsilon, r)$. Finally, the system (3.1) is Uniformly Exponentially Stable (UES) with respect to $\mathcal{A}$, if there are three positive constants r, k, and γ such that the solution $x(t, \xi, t_0, d)$, starting from ξ at time t_0 under the input $d(t)$, exists for all $t \geq t_0$ and satisfies

$$|x(t, x_0, t_0, d)|_{\mathcal{A}} \leq k e^{\gamma(t - t_0)} |x_0|_{\mathcal{A}}, \ \forall t \geq t_0$$

for all $x_0 \in \{\xi : |\xi|_{\mathcal{A}} \leq r\} \stackrel{\text{def}}{=} \Omega_0$ and all $d \in \mathcal{M}_{\mathcal{D}}$.

Remark 3.1 *Proposition 2.5 of [34] shows the equivalence between UGAS and bounding $|x(t, x_0; d)|_{\mathcal{A}}$ with a class $\mathcal{KL}$ function.*

Herein, we state the definition of a Lyapunov function for (3.1) with respect to the compact, positively invariant set $\mathcal{A}$.

Definition 3.2 *A Lyapunov function for the system (3.1) in the open set $\mathcal{R}$ with respect to a compact, positively invariant set $\mathcal{A} \subseteq \mathcal{R}$ is a function $V : \mathcal{R} \to R_{\geq 0}$ such that V is smooth on $\mathcal{R}/\mathcal{A}$ and satisfies the following properties:*

1. *There exist two class $\mathcal{K}$ functions α_1 and α_2 such that for any $\xi \in \mathcal{R}$,*

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}) \tag{3.4}$$

2. *There exists a continuous, positive definite function α_3 such that for any $\xi \in \mathcal{R}/\mathcal{A}$, and any $d \in \mathcal{M}_{\mathcal{D}}$,*

$$L_{f_{\mathbf{d}}} V(\xi) \leq -\alpha_3(|\xi|_{\mathcal{A}}) \tag{3.5}$$

A smooth Lyapunov function is one which is smooth on all of $\mathcal{R}$.

In the case $\mathcal{R} = R^n$, we require α_1 and α_2 to be class $\mathcal{K}_\infty$.

In some situations the Lyapunov function candidate is time-dependent. In this case we need a Lyapunov stability theorem for invariant sets where the Lyapunov function could depend on time.

Theorem 3.1 *Let $\mathcal{A} \subseteq \mathcal{R}$ be a compact, positively invariant subset of R^n for the system (3.1). Then, (3.1) is UAS with respect to $\mathcal{A}$ if there exist a C^1 positive definite, with respect to $\mathcal{A}$, function $V(t,\xi) : [0,\infty) \times U \to R$, where U is a neighborhood of $\mathcal{A}$, two $\mathcal{K}$-functions α_1, α_2, and a continuous positive definite function α_3 such* *that, for all $t \geq 0$, we have*

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(t,\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}) \tag{3.6}$$

for any $\xi \in U$, and

$$\frac{\partial V}{\partial t} + L_{f_{\mathbf{d}}} V(\xi) \leq -\alpha_3(|\xi|_{\mathcal{A}}) \tag{3.7}$$

for any $\xi \in U/\mathcal{A}$ and any $\mathbf{d} \in \mathcal{D}$. In addition, if $U = R^n$ and α_1 is class $\mathcal{K}_\infty$, then (3.1) is uniformly globally asymptotically stable with respect to $\mathcal{A}$.

Proof: see Appendix C.1.$\triangleleft$

Remark 3.2 *Theorem 3.1 along with [34, Proposition 2.5] constitute a powerful machinery for showing UGAS. For example, we can show UGAS using two Lyapunov functions, the first one shows ultimate boundedness and the second one shows local UAS.*

For GUAS, we state the converse Lyapunov theorem given in [34] as Theorem 2.9.

Theorem 3.2 *Let $\mathcal{A} \subset R^n$ be a compact, positively invariant set for the system (3.1). Then, (3.1) is uniformly globally asymptotically stable with respect to $\mathcal{A}$ if and only if there exists a smooth Lyapunov function $V(x)$ with respect to $\mathcal{A}$.*

Moreover, we give a converse Lyapunov theorem for UES.

Theorem 3.3 *Assume that the system (3.1) is uniformly exponentially stable with respect to the compact, positively invariant set $\mathcal{A}$. Then, there exists a function $V(t,x)$, defined and continuous on $R_{\geq 0} \times \Omega_0$, where $\Omega_0 = \{|x|_{\mathcal{A}} \leq r_0,\ r_0 > 0\}$*

and contains $\mathcal{A}$, such that

$$
\begin{aligned}
|x|_{\mathcal{A}} &\leq V(t,x) \leq k|x|_{\mathcal{A}} && (3.8)\\
|V(t,x) - V(t,\bar{x})| &\leq L\|x-\bar{x}\| && (3.9)\\
\lim_{h\to 0}\frac{V(t+h,\tilde{x}) - V(t,x)}{h} &\leq -\lambda V(t,x), \ \ \forall d \in \mathcal{M}_{\mathcal{D}} && (3.10)
\end{aligned}
$$

for all $x, \bar{x} \in \Omega_0$, and all $t \geq t_0$, where L and $\lambda < \gamma$ are positive constants and $\tilde{x} = x(t+h, x, t_0; d)$.

Proof: see Appendix C.2.◁

3.3 Problem Formulation

In many state feedback controller designs, the trajectories of the closed-loop system do not converge to an equilibrium point. For example, in the servomechanism cases discussed in [26, 38] the objective is to make the tracking error converge to zero while keeping the states of the zero dynamics bounded. The same objective is achieved in [2, 27] using an adaptive controller. Furthermore, in robust control, as in [50, 10], we often achieve ultimate boundedness. In this section we formulate these design objectives as steering the trajectories towards a compact, positively invariant set. Of course, there are problems which can neither be cast as stabilization of an equilibrium point nor as stabilization of a compact, positively invariant set. These problems, such as [2] (when we only have partial persistence of excitation) and [29], will be the subject of future research.

The class of systems considered can be represented by the multi-input multi-output nonlinear model

$$\dot{x} = Ax + B\phi(x, z, d(t), u) \tag{3.11}$$

$$\dot{z} = \psi(x, z, d(t), u) \tag{3.12}$$

$$y = Cx \tag{3.13}$$

$$\zeta = q(x, z, d(t)) \tag{3.14}$$

where $u \in R^m$ is the control input, $\zeta \in R^s$ and $y \in R^p$ are measured outputs, $x \in R^r$ and $z \in R^\ell$ constitute the state vector, and $d(t) \in R^d$ is a vector of signals that belongs to $\mathcal{M}_{\mathcal{D}}$. The matrices A, B, and C, represent p chains of integrators as in Section 2.2.

The state feedback control is assumed to be in the form

$$\dot{\vartheta} = \Gamma(\vartheta, x, \zeta, d(t)) \tag{3.15}$$

$$u = \gamma(\vartheta, x, \zeta, d(t)) \tag{3.16}$$

We allow any state feedback design that satisfies:

Assumption 3.1 *(1) Γ and γ are locally Lipschitz functions in ϑ, x, and ζ uniformly in d over the domain of interest;*
(2) Γ and γ are globally bounded functions of x;
(3) The closed-loop system is uniformly globally asymptotically stable with respect to the compact positively invariant set $\mathcal{A}$.

The system (3.11)–(3.14) with the controller (3.15)–(3.16) satisfies:

Assumption 3.2 *The functions q, ϕ and ψ are locally Lipschitz in x, z, and u uniformly in d over the domain of interest. Moreover, $\phi(x, z, d, \gamma(\vartheta, x, \zeta, d))$ is zero in $\mathcal{A}$ uniformly in d.*

To implement the control (3.15)–(3.16) we use the state estimates, $\hat{x}$, generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, d(t), u) + H(y - C\hat{x}) \tag{3.17}$$

The observer gain H is chosen as in 2.10.

The function $\phi_0(x, \zeta, d(t), u)$ is a nominal model of $\phi(x, z, d(t), u)$ which is required to satisfy the following assumption:

Assumption 3.3 ϕ_0 *is a locally Lipschitz function in* x, ζ*, and* u *uniformly in* d *over the domain of interest. Furthermore, it is globally bounded in* x *and zero in* $\mathcal{A}$*, uniformly in* d*.*

3.4 Performance Recovery - Semiglobal Separation Results

The main objective of this section is to show that the suggested output feedback implementation of the control law recovers the performance of the state feedback controller (3.15)–(3.16) for sufficiently small ϵ. The performance recovery manifests itself in three points. First, the compact set $\mathcal{A} \times \{x - \hat{x} = 0\}$ is a positively invariant set of the closed-loop system under output feedback and the closed-loop system is asymptotically stable with respect to $\mathcal{A} \times \{x - \hat{x} = 0\}$. Second, the output feedback controller achieves semiglobal stabilization; that is, for any compact set $\mathcal{S}$ which contains $\mathcal{A}$, and any compact set $\mathcal{Q} \subseteq R^r$, the set $\mathcal{S} \times \mathcal{Q}$ is included in the region of attraction under output feedback control. Third, the trajectory of (x, z, ϑ) under output feedback approaches the trajectory under state feedback as $\epsilon \to 0$.

The analysis is done in three steps. First, we show the recovery of boundedness of trajectories. Second, we show the recovery of ultimate boundedness of these trajectories. Third, we show the recovery of local asymptotic stability with respect to $\mathcal{A}$. This allows us to deal with asymptotic stability as a local property that could require some additional assumptions on the modeling error.

Let us first, for the purpose of analysis, replace the observer dynamics by the equivalent dynamics of the scaled estimation error (the scaling is similar to that of Section 2.5.1). Then, as in Chapter 2, we have $\hat{x} = x - D(\epsilon)\eta$. Hence, the closed-loop system can be represented by

$$\dot{x} = Ax + B\phi(x, z, d(t), \gamma(.)) \tag{3.18}$$

$$\dot{z} = \psi(x, z, d(t), \gamma(\vartheta, x - D(\epsilon)\eta, \zeta, d(t))) \tag{3.19}$$

$$\dot{\vartheta} = \Gamma(\vartheta, x - D(\epsilon)\eta, \zeta, d(t)) \tag{3.20}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon Bg(x, z, \vartheta, D(\epsilon)\eta, d(t)) \tag{3.21}$$

where $g(.) = \phi(x, z, d(t), \gamma(.)) - \phi_0(\hat{x}, \zeta, d(t), \gamma(.))$ and A_0 is a constant Hurwitz matrix.

The system (3.18)–(3.21) is a standard singularly perturbed one, and $\eta = 0$ is the unique solution of (3.21) when $\epsilon = 0$. Similar to Section 2.5.1, the reduced system is the closed-loop system under state feedback. For simplicity we write the system (3.18)–(3.20) as

$$\dot{\chi} = f_r(\chi, d(t), D(\epsilon)\eta) \tag{3.22}$$

where $\chi = [x^T, z^T, \vartheta^T]^T$ and $\chi(0) = [x_0^T, z_0^T, \vartheta_0^T]^T$. Then, the reduced system is given by

$$\dot{\chi} = f_r(\chi, d(t), 0) \tag{3.23}$$

The boundary-layer system is

$$\frac{d\eta}{d\tau} = A_0\eta \tag{3.24}$$

where $\tau = \frac{t}{\epsilon}$. Let $(\chi(t,\epsilon), \eta(t,\epsilon))$ denote the trajectory of the system (3.18)–(3.21) starting from $(\chi(0), \eta(0))$.

We know that (3.23) is uniformly globally asymptotically stable with respect to the compact positively invariant set $\mathcal{A}$. Then, Theorem 3.2 ensures the existence of a smooth Lyapunov function $V(\chi)$ in addition to two class $\mathcal{K}_\infty$ functions α_1, α_2 and a continuous, positive definite function α_3 such that, for all $\chi \in R^n$, where $n = r + \ell$, we have:

$$V(\chi) = 0 \Longleftrightarrow \chi \in \mathcal{A} \tag{3.25}$$

$$\alpha_1(|\chi|_{\mathcal{A}}) \leq V(\chi) \leq \alpha_2(|\chi|_{\mathcal{A}}) \tag{3.26}$$

$$\frac{\partial V}{\partial \chi} f_r(\chi, \mathbf{d}, 0) \leq -\alpha_3(|\chi|_{\mathcal{A}}), \ \forall \mathbf{d} \in \mathcal{D} \tag{3.27}$$

For the boundary-layer system we define the Lyapunov function $W(\eta) = \eta^T P_0 \eta$, where P_0 is the positive definite solution of the Lyapunov equation $P_0 A_0 + A_0^T P_0 = -I$. This function satisfies inequalities similar to (2.21)–(2.22).

3.4.1 Boundedness

Let the initial states $(x_0, z_0, \vartheta_0) \in \mathcal{S}$, and $\hat{x}_0 \in \mathcal{Q}$, where $\mathcal{S}$ is any compact set in R^n which contains $\mathcal{A}$ and $\mathcal{Q}$ is any compact subset of R^r. The recovery of boundedness of trajectories is given in the following theorem:

Theorem 3.4 *Let Assumptions 3.1–3.3 hold; then, there exists $\epsilon_1^\star > 0$ such that for every $0 < \epsilon \leq \epsilon_1^\star$, the trajectories $(\chi(t,\epsilon), \eta(t,\epsilon))$ of the system (3.18)–(3.21) starting in $\mathcal{S} \times \mathcal{Q}$ are bounded for all $t \geq 0$ and all $d \in \mathcal{M}_{\mathcal{D}}$.*

Proof: As in Section 2.5.1, we show the positive invariance of an appropriately chosen set Λ, then we show that, any closed-loop trajectory, starting in the compact set $\mathcal{S} \times \mathcal{Q}$, enters the positively invariant set Λ in finite time.

The properness of $V(\chi)$ guarantees that the set $\Omega = \{\chi \in R^n : V(\chi) \leq c\}$, for some $c > 0$, is a compact set and that $\mathcal{S}$ is in the interior of Ω.

Let $\Lambda = \Omega \times \{W(\eta) \leq \rho\epsilon^2\}$. Due to Assumptions 3.1–3.3 we have, for all $\chi \in \Omega$, all $\mathbf{d} \in \mathcal{D}$, and all $\eta \in R^r$,

$$\begin{aligned} \|f_r(\chi, \mathbf{d}, D(\epsilon)\eta)\| &\leq k_1 &\quad (3.28)\\ \|g(\chi, \mathbf{d}, D(\epsilon)\eta)\| &\leq k_2 &\quad (3.29)\end{aligned}$$

where k_1 and k_2 are positive constants independent of ϵ. Moreover, for any $0 < \tilde{\epsilon} < 1$, there is L_1, independent of ϵ, such that, for all $(\chi, \eta) \in \Lambda$, all $\mathbf{d} \in \mathcal{D}$, and every $0 < \epsilon \leq \tilde{\epsilon}$, we have

$$\|f_r(\chi, \mathbf{d}, D(\epsilon)\eta) - f_r(\chi, \mathbf{d}, 0)\| \leq L_1\|\eta\| \quad (3.30)$$

In the rest of the chapter we always consider $\epsilon \leq \tilde{\epsilon}$. It can be shown that

$$\dot{V} \leq \frac{\partial V}{\partial \chi} f_r(\chi, \mathbf{d}, 0) + \epsilon k_3, \ \ \forall \mathbf{d} \in \mathcal{D} \quad (3.31)$$

for all $(\chi, \eta) \in \{V(\chi) = c\} \times \{W(\eta) \leq \rho\epsilon^2\}$, and

$$\dot{W} \leq -\frac{1}{\epsilon}\|\eta\|^2 + 2\|\eta\|\|P_0\|\|B\|k_2, \ \ \forall \mathbf{d} \in \mathcal{D} \quad (3.32)$$

for all $(\chi, \eta) \in \Omega \times \{W(\eta) = \rho\epsilon^2\}$, where $k_3 = L_1 L_2 \sqrt{\rho/\lambda_{min}(P_0)}$, $\|P_0\| = \lambda_{max}(P_0)$, and L_2 is an upper bound for $\|\frac{\partial V}{\partial \chi}\|$ over Ω. As in Section 2.5.1, taking $\rho = 16k_2^2\|P_0\|^3$ and $\epsilon_1 = \beta/k_3$, where $\beta = \min_{\chi \in \partial\Omega} U_3(\chi)$, it can be shown that, for every $0 < \epsilon \leq \epsilon_1$, the derivatives of V and W along the trajectories are nonpositive on the boundaries of Λ. Thus, we conclude that the set Λ is positively invariant.

Now, we consider the initial state $(\chi(0), \hat{x}(0)) \in \mathcal{S} \times \mathcal{Q}$. Using (3.28) and the fact that $\chi(0)$ is in the interior of Ω, it can be shown that

$$\|\chi(t,\epsilon) - \chi(0)\| \leq k_1 t \tag{3.33}$$

as long as $\chi(t,\epsilon) \in \Omega$. Thus, there exists a finite time T_0, independent of ϵ, such that $\chi(t,\epsilon) \in \Omega$ for all $t \in [0, T_0]$. As in Section 2.5.1, we can show that there exists $\epsilon_2 > 0$ small enough and a time $T(\epsilon)$ such that $W(\eta(T(\epsilon),\epsilon)) \leq \rho\epsilon^2$ for every $0 < \epsilon \leq \epsilon_2$. Taking $\epsilon_1^\star = \min(\tilde{\epsilon}, \epsilon_1, \epsilon_2)$ guarantees that, for every $0 < \epsilon \leq \epsilon_1^\star$, the trajectory $(\chi(t,\epsilon), \eta(t,\epsilon))$ enters Λ during the interval $[0, T(\epsilon)]$ and remains there for all $t \geq T(\epsilon)$. Thus, the trajectory is bounded for all $t \geq T(\epsilon)$. On the other hand, for $t \in [0, T(\epsilon)]$, the trajectory is bounded by virtue of inequality (3.33) and an inequality similar to (2.32).$\triangleleft$

3.4.2 Ultimate Boundedness

Hereafter, we show that trajectories of the system (3.18)–(3.21), starting in $\mathcal{S} \times \mathcal{Q}$, come arbitrarily close to the set $\mathcal{A} \times \{\eta = 0\}$ as time progresses. This is summarized in the following theorem:

Theorem 3.5 *Under the conditions of Theorem 3.4, given any $\xi > 0$, there exist $\epsilon_2^\star = \epsilon_2^\star(\xi) > 0$ and $T_1 = T_1(\xi)$ such that, for every $0 < \epsilon \leq \epsilon_2^\star$, we have*

$$|\chi(t,\epsilon)|_{\mathcal{A}} + \|\eta(t,\epsilon)\| \leq \xi, \ \forall t \geq T_1 \tag{3.34}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$.

Proof: From the proof of Theorem 3.4 we know that, for every $0 < \epsilon \leq \epsilon_1^\star$, the trajectory of the closed–loop system, starting from $(\chi(0), \hat{x}(0)) \in \mathcal{S} \times \mathcal{Q}$, is inside the set Λ for all $t \geq T(\epsilon)$, where Λ is $O(\epsilon)$ in the direction of the variable η. Thus, we can find $\epsilon_3 = \epsilon_3(\xi) \leq \epsilon_1^\star$ such that for every $0 < \epsilon \leq \epsilon_3$ we have

$$\|\eta(t,\epsilon)\| \leq \xi/2, \ \forall t \geq T(\epsilon_3) \stackrel{\text{def}}{=} \bar{T}(\xi) \tag{3.35}$$

for all $d \in \mathcal{M}_\mathcal{D}$. In what follows we continue working with the Lyapunov function defined in the the proof of Theorem 3.4. It can be shown that, for all $(\chi, \eta) \in \Lambda$, we have

$$\dot{V} \leq -\alpha_3(|\chi|_\mathcal{A}) + k_3\epsilon, \ \ \forall \mathbf{d} \in \mathcal{D}$$

Without loss of generality α_3 can be chosen to be class $\mathcal{K}_\infty$. Thus, we conclude that

$$\dot{V} \leq -\frac{1}{2}\alpha_3(|\chi|_\mathcal{A}), \ \text{for}\ |\chi|_\mathcal{A} \geq \alpha_3^{-1}(2k_3\epsilon) \stackrel{\text{def}}{=} \mu(\epsilon), \ \ \forall \mathbf{d} \in \mathcal{D}$$

Choose $\epsilon_4 = \epsilon_4(\xi) \leq \epsilon_1^\star$ such that $\mu(\epsilon_4) < \alpha_2^{-1}(c)$ and $\alpha_1^{-1}(\alpha_2(\mu(\epsilon_4))) < \xi/2$. Then, by a proof similar to that of [28, Theorem 5.1, Corollary 5.1], we conclude that there exists a finite time $\tilde{T} = \tilde{T}(\xi)$ such that, for every $0 < \epsilon \leq \epsilon_4$

$$|\chi(t,\epsilon)|_\mathcal{A} \leq \xi/2, \ \forall t \geq \tilde{T} \tag{3.36}$$

for all $d \in \mathcal{M}_\mathcal{D}$. Take $\epsilon_2^\star = \epsilon_2^\star(\xi) = \min(\epsilon_3, \epsilon_4)$ and $T_1 = T_1(\xi) = \max(\bar{T}, \tilde{T})$, then (3.34) follows from (3.35) and (3.36).$\triangleleft$

3.4.3 Trajectory Convergence

Let $\chi_r(t)$ be the solution of (3.23) starting from $\chi(0)$. The following theorem shows that $\chi(t,\epsilon)$ converges to $\chi_r(t)$ as $\epsilon \to 0$, uniformly in t, for all

Theorem 3.6 *Under the conditions of Theorem 3.4, given any $\xi > 0$, there exists $\epsilon_3^\star > 0$ such that, for every $0 < \epsilon \leq \epsilon_3^\star$ we have*

$$\|\chi(t,\epsilon) - \chi r(t)\| \leq \xi, \ \forall t \geq 0 \tag{3.37}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$.

Proof: We divide the interval $[0,\infty)$ into three intervals $[0,T(\epsilon)]$, $[T(\epsilon),T_2]$, and $[T_2,\infty)$, where both $T(\epsilon)$ and T_2 are to be determined later, and show (3.37) for each interval.

• From Theorem 3.5 and the uniform asymptotic stability with respect to $\mathcal{A}$ of the reduced system we conclude that there exists a finite time $T_2 \geq T(\epsilon)$, independent of ϵ, such that, for every $0 < \epsilon \leq \epsilon_2^\star$, we have

$$|\chi r(t)|_{\mathcal{A}} \leq \xi/2, \ \forall t \geq T_2 \tag{3.38}$$

$$|\chi(t,\epsilon)|_{\mathcal{A}} \leq \xi/2, \ \forall t \geq T_2 \tag{3.39}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$. Then, we can write

$$\|\chi(t,\epsilon) - \chi r(t)\| \leq \|\chi(t,\epsilon) - x\| + \|\chi r(t) - x\|, \ \forall t \geq T_2 \tag{3.40}$$

for all $x \in \mathcal{A}$ and all $d \in \mathcal{M}_{\mathcal{D}}$. By taking the infimum of (3.40) over $\mathcal{A}$ and using (3.38)–(3.39), we have

$$\begin{aligned} \|\chi(t,\epsilon) - \chi r(t)\| &\leq |\chi(t,\epsilon)|_{\mathcal{A}} + |\chi r(t)|_{\mathcal{A}} \\ &\leq \xi, \ \forall t \geq T_2, \ \forall d \in \mathcal{M}_{\mathcal{D}} \end{aligned} \tag{3.41}$$

- As in Section 2.5.2, we can show that

$$\|\chi(t,\epsilon) - \chi_r(t)\| \leq 2k_1 T(\epsilon), \ \forall t \in [0, T(\epsilon)]$$

Since $T(\epsilon) \to 0$ as $\epsilon \to 0$, there exists $0 < \epsilon_5 \leq \epsilon_2^\star$ such that, for every $0 < \epsilon \leq \epsilon_5$, we have

$$\|\chi(t,\epsilon) - \chi_r(t)\| \leq \xi, \ \forall t \in [0, T(\epsilon)] \tag{3.42}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$.

- Over the interval $[T(\epsilon), T_2]$, the trajectory $\chi(t,\epsilon)$ satisfies, for all $d \in \mathcal{M}_{\mathcal{D}}$,

$$\dot{\chi} = f_r(\chi, d(t), D(\epsilon)\eta(t,\epsilon)), \text{ with } \|\chi(T(\epsilon),\epsilon) - \chi_r(T(\epsilon))\| \leq \delta(\epsilon)$$

where $D(\epsilon)\eta$ is $O(\epsilon)$ and $\delta(\epsilon) \to 0$ as $\epsilon \to 0^+$. Thus, as in Section 2.5.2, we conclude that there exists $0 < \epsilon_6 \leq \epsilon_2^\star$ such that, for every $0 < \epsilon \leq \epsilon_6$, we have

$$\|\chi(t,\epsilon) - \chi_r(t)\| \leq \xi, \ \forall t \in [T(\epsilon), T_2] \tag{3.43}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$. Take $\epsilon_3^\star = \min(\epsilon_5, \epsilon_6)$, then, using (3.41), (3.42), and (3.43) we conclude (3.37).$\triangleleft$

3.4.4 Uniform Asymptotic Stability

In this section we deal with local uniform asymptotic stability with respect to a compact positively invariant set in the absence or presence of modeling errors. We assume that the trajectory belongs to the ball $B(\mathcal{A}, \xi) = \{\chi : |\chi|_{\mathcal{A}} \leq \xi\}$. This is justified given the result of Theorem 3.5.

Case 1: Herein, we deal with the case where the system (3.23) is uniformly asymptotically stable with respect to the set $\mathcal{A}$. Moreover, there is no modeling error. We have the following theorem:

Theorem 3.7 *Let Assumptions 3.1–3.3 hold and assume that $\phi_0 = \phi$. Then, there exists $\epsilon_4^\star > 0$ such that, for every $0 < \epsilon \leq \epsilon_4^\star$, the system (3.18)–(3.21) is uniformly asymptotically stable with respect to the compact positively invariant set $\mathcal{A} \times \{\eta = 0\}$.*

Proof: We know from Theorem 3.2 that there exists a C^1 Lyapunov function V and a class $\mathcal{K}_\infty$ function α_3, both defined on a ball $B(\mathcal{A}, r_1) \subseteq \Omega$ for some $r_1 > 0$, such that for all $\chi \in B(\mathcal{A}, r_1)$

$$\frac{\partial V}{\partial \chi} f_r(\chi, \mathbf{d}, 0) \leq -\alpha_3(|\chi|_{\mathcal{A}}), \ \ \mathbf{d} \in \mathcal{D} \tag{3.44}$$

Choose $\xi < r_1$; then given Assumptions 3.1–3.3 we can show that, for all $(\chi, \eta) \in B(\mathcal{A}, \xi) \times \{\|\eta\| \leq \xi\} = \Lambda_1$ and all $\mathbf{d}$, we have

$$\|g(\chi, \mathbf{d}, D(\epsilon)\eta)\| \leq L_4 \|\eta\|, \ \ \mathbf{d} \in \mathcal{D} \tag{3.45}$$

Consider the composite function $\tilde{V}(\chi, \eta) = V(\chi) + (W(\eta))^{1/2}$ and choose $0 < \epsilon_4^\star \leq \epsilon_2^\star$ such that $1/(4\epsilon_4^\star \sqrt{\|P_0\|}) - \tilde{L}_2 L_3 - \|P_0\| L_4 / \sqrt{\lambda_{min}(P_0)} > 0$, where $\tilde{L}_2$ is an upper bound for $\left\| \frac{\partial V}{\partial \chi} \right\|$ over Λ_1 and L_3 is a Lipschitz constant of $f_r(.,.,.)$ in η over the same set. Then, we conclude that, for every $0 < \epsilon \leq \epsilon_4^\star$ and for all $(\chi, \eta) \in \Lambda_1$, we have

$$\dot{\tilde{V}} \leq -\alpha_3(|\chi|_{\mathcal{A}}) - \frac{1}{4\epsilon\sqrt{\|P_0\|}} \|\eta\|, \ \ \mathbf{d} \in \mathcal{D} \tag{3.46}$$

Thus, according to Theorem 3.1, the closed-loop system (3.18)–(3.21) is uniformly asymptotically stable with respect to $\mathcal{A} \times \{\eta = 0\}$. ◁

Case 2: We deal with the case where the system (3.23) is uniformly exponentially stable, whether or not we know ϕ.

The result is summarized in the following theorem:

Theorem 3.8 *Let Assumptions 3.1–3.3 hold and assume that the closed-loop system (3.23) is uniformly exponentially stable with respect to the set $\mathcal{A}$. Then, there exists $\epsilon_5^\star > 0$ such that, for every $0 < \epsilon \leq \epsilon_5^\star$, the system (3.18)–(3.21) is uniformly exponentially stable with respect to the set $\mathcal{A} \times \{\eta = 0\}$.*

Proof: Let $\chi_2(t)$ be the solution of (3.23) that starts from χ at time $t = 0$. In this case, according to Theorem 3.3, there exists a Lyapunov function $V_2(t, \chi)$ defined over $B(\mathcal{A}, r_2) \subseteq \Omega$, for some $r_2 > 0$, and three positive constants a_1, a_2, and a_3 such that, for all $\chi \in B(\mathcal{A}, r_2)$ we have

$$|\chi|_{\mathcal{A}} \leq V_2(t,\chi) \leq a_1 |\chi|_{\mathcal{A}} \tag{3.47}$$

$$\lim_{h \to 0} \frac{V(t+h, \chi_2(t+h)) - V(t,\chi)}{h} \leq -a_2 V(t,\chi), \ \forall d \in \mathcal{M}_{\mathcal{D}} \tag{3.48}$$

$$\|V_2(t,\chi_1) - V_2(t,\chi_2)\| \leq a_3 \|\chi_1 - \chi_2\|, \forall \chi_1, \chi_2 \in B(\mathcal{A}, r_2) \tag{3.49}$$

Let us consider $\bar{V}(t, \chi, \eta) = V_2(t, \chi) + \beta\sqrt{W(\eta)}$, where $\beta > 0$ is to be determined, as a Lyapunov function candidate for the system (3.18)–(3.21). Choose $\xi < r_2$; then we have

Claim: Given Assumptions 3.1–3.3, we have, for all $(\chi, \eta) \in B(\mathcal{A}, \xi) \times \{\|\eta\| \leq \xi\} = \Lambda_2$,

$$\|g(\chi, d, D(\epsilon)\eta)\| \leq L_5 |\chi|_{\mathcal{A}} + L_6 \|\eta\| \tag{3.50}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$.

Proof: It suffices to show that

$$\|\phi(\chi, d) - \phi_0(\chi, d)\| \leq L_5 |\chi|_{\mathcal{A}} \tag{3.51}$$

for all $d \in \mathcal{M}_{\mathcal{D}}$. Since both ϕ and ϕ_0 are zero in $\mathcal{A}$, we can write

$$\|\phi(\chi, d) - \phi_0(\chi, d)\| \leq \|\phi(\chi, d) - \phi(\chi_a, d)\| + \|\phi_0(\chi, d) - \phi_0(\chi_a, d)\| \tag{3.52}$$

for all $\chi_a \in \mathcal{A}$ and all $d \in \mathcal{M}_{\mathcal{D}}$. Using the local Lipschitz property in (3.52) yields

$$\|\phi(\chi, d) - \phi_0(\chi, d)\| \leq L_5 \|\chi - \chi_a\| \tag{3.53}$$

for some $L_5 > 0$. Taking the infimum over $\mathcal{A}$ yields (3.51).◁

Let $\chi_3(t)$ be the solution of (3.22) that starts from χ at time $t = 0$. Then, using (3.48), (3.49), and (3.50), it can be shown that, for all $(\chi, \eta) \in \Lambda_2$, we have

$$\begin{aligned}\lim_{h \to 0} \frac{\bar{V}(t+h, \chi_3(t+h), \eta(t+h, \epsilon)) - \bar{V}(t, \chi, \eta)}{h} &\leq -a_2 V_2(t, \chi) + a_3 L_7 \|\eta\| \\ - \frac{\beta}{2\epsilon\sqrt{\|P_0\|}}\|\eta\| + \frac{\beta}{\sqrt{\lambda_{min}(P_0)}}\|P_0\|(L_5|\chi|_{\mathcal{A}} + L_6\|\eta\|) \end{aligned} \tag{3.54}$$

where L_7 is a Lipschitz constant of $f_r(.,.,.)$ in η over Λ_2. Then, there exist positive constants β such that $-(a_2/2) + \beta\|P_0\|L_5/\sqrt{\lambda_{min}(P_0)}) < 0$, and $0 < \epsilon_5^\star \leq \epsilon_2^\star$ such that, for every $0 < \epsilon \leq \epsilon_5^\star$, we have $-\beta/(4\epsilon\sqrt{\|P_0\|}) + a_3 L_7 + \beta\|P_0\|L_6/\sqrt{\lambda_{min}(P_0)}) < 0$. This implies that, for every $0 < \epsilon \leq \epsilon_5^\star$, we have

$$\lim_{h \to 0} \frac{\bar{V}(t+h, \chi_3(t+h), \eta(t, \epsilon)) - \bar{V}(t, \chi, \eta)}{h}$$

$$
\begin{aligned}
&\leq -\frac{1}{2}a_2 V_2(t,\chi) - \frac{\beta}{4\epsilon\sqrt{\|P_0\|}}\|\eta\| \\
&\leq -\min\{\frac{1}{2}a_2, \frac{1}{4\epsilon\|P_0\|}\}\bar{V}(t,\chi,\eta) \qquad (3.55)
\end{aligned}
$$

Thus, there exist positive constants b_2 and b_3 such that

$$
|(\chi(t,\epsilon),\eta(t,\epsilon))|_{\mathcal{A}\times\{0\}} \leq b_2 e^{-b_3 t}|(\chi(0),\eta(0))|_{\mathcal{A}\times\{0\}}, \quad \forall d \in \mathcal{M}_{\mathcal{D}} \qquad (3.56)
$$

Thus, according to the definition of UES in Definition 3.1, the closed-loop system (3.18)–(3.21) is uniformly exponentially stable with respect to $\mathcal{A}\times\{\eta = 0\}$. $\triangleleft$

During the proof of case 3, we will use *Young's Inequality* given in Section 2.5.4.

Case 3: We deal with the case where the system (3.23) is uniformly asymptotically stable with respect to the compact, positively invariant set $\mathcal{A}$. A condition on the modeling error has to be imposed and is stated as

Assumption 3.4 *There exists a C^1 function $V_3(t,\chi)$ defined on $[0,\infty)\times U$, where $U = \{\chi : |\chi|_{\mathcal{A}} \leq r_3,\ r_3 > 0\}$ is a neighborhood of $\mathcal{A}$ in Ω, three functions ψ_1 and ψ_2, and ψ_3 defined and continuous on U which are positive definite with respect to $\mathcal{A}$ (i.e., positive everywhere and zero only in $\mathcal{A}$), such that, for all $t \geq 0$, we have*

$$
\psi_1(\chi) \leq V_3(t,\chi) \leq \psi_2(\chi) \qquad (3.57)
$$

$$
\frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial \chi} f_r(\chi,\mathbf{d},0) \leq -\psi_3(\chi) \qquad (3.58)
$$

$$
\|\phi(x,z,\mathbf{d},\gamma(\vartheta,x,\zeta,\mathbf{d})) - \phi_0(x,\zeta,\mathbf{d},\gamma(\vartheta,x,\zeta,\mathbf{d}))\| \leq c_0\, \psi_3^a(\chi) \qquad (3.59)
$$

$$
\left\|\frac{\partial V_3}{\partial \chi}(t,\chi)\right\| \leq c_1\, \psi_3^b(\chi) \qquad (3.60)
$$

for all $\chi \in U$ *and all* $\mathbf{d} \in \mathcal{D}$, *for some positive constants* $c_0 \geq 0$, $c_1 > 0$ *and* a, $b < 1$, *such that* $a + b = 1$.

The existence of a Lyapunov function satisfying (3.57)–(3.58) is guaranteed by Theorem 3.2, but what we need here is for (3.59) and (3.60) to be satisfied as well.

Remark 3.3 *Assumption 3.4 is similar to Assumption 2.4 of Chapter 2 in the sense that it relates the modeling error magnitude and the rate of convergence of trajectories near the attractor (which is a set in the case at hand). It can also be viewed as an extension of Assumption 2.4 to the case of asymptotic stability with respect to a set. However, in Assumption 2.4 we used the center manifold decomposition to reduce the size of the set over which the assumption is satisfied and to reduce the structural complexity of the modeling error by projecting it on the center manifold.*

From different examples, presented later on, it seems reasonable to allow the Lyapunov function candidate V_3 *to depend on time.*

The recovery of asymptotic stability can now be stated as follows:

Theorem 3.9 *Let Assumptions 3.1-3.4 hold. Then, there exists* $\epsilon_6^\star > 0$ *such that, for all* $0 < \epsilon \leq \epsilon_6^\star$, *the system (3.18)-(3.21) is uniformly asymptotically stable with respect to the compact positively invariant set* $\mathcal{A} \times \{\eta = 0\}$.

Proof. Consider the Lyapunov function candidate $\mathcal{V}(t, \chi, \eta) = V_3(t, \chi) + (W(\eta))^\sigma$ with $\sigma = 1/(2a) > 1/2$.

Let $\xi < r_3$ and let $\epsilon \leq \epsilon_2^\star$, then according to Theorem 3.5 there exists a finite time $T_3 > 0$ after which we have $|\chi(t, \epsilon)|_{\mathcal{A}} + \|\eta(t, \epsilon)\| \leq \xi$ for all $t \geq T_3$.

Using Assumptions 3.1–3.4, we can show that, for all $(\chi, \eta) \in \Lambda_3 = B(\mathcal{A}, \xi) \times \{\|\eta\| \leq \xi\}$, we have

$$\|f_r(\chi, \mathbf{d}, D(\epsilon)\eta) - f_r(\chi, \mathbf{d}, 0)\| \leq L_8\|\eta\| \tag{3.61}$$

$$\|g(\chi, \mathbf{d}, D(\epsilon)\eta)\| \leq c_0\psi_3(\chi) + L_9\|\eta\| \tag{3.62}$$

for all $\mathbf{d} \in \mathcal{D}$, and

$$\dot{\mathcal{V}} \le -\psi_3(\chi) + \rho_1\|\eta\|\psi_3^b(\chi) - \frac{\rho_2}{\epsilon}\|\eta\|^{2\sigma} + \rho_3\|\eta\|^{2\sigma} + \rho_4\psi_3^a(\chi)\|\eta\|^{2\sigma - 1} \quad (3.63)$$

where

$$\begin{aligned} \rho_1 &= c_1 L_8, \ \rho_2 = \sigma\gamma_1 \\ \rho_3 &= 2\sigma\gamma_2\|P_0\|L_9, \ \rho_4 = 2\sigma\gamma_2\|P_0\|c_0 \end{aligned}$$

are positive constants and

$$\left.\begin{aligned} \gamma_1 &= (\lambda_{min}(P_0))^{\sigma - 1} \\ \gamma_2 &= \|P_0\|^{\sigma - 1} \end{aligned}\right\} \text{ if } \sigma \ge 1 \qquad \left.\begin{aligned} \gamma_1 &= \|P_0\|^{\sigma - 1} \\ \gamma_2 &= (\lambda_{min}(P_0))^{\sigma - 1} \end{aligned}\right\} \text{ if } 1/2 \le \sigma < 1$$

Next, we separate the two cross-product terms in (3.63) by repeatedly using Young's Inequality (see Section 2.5.4) with $p = 2\sigma$ and $\epsilon_0 = q_1, q_2$, respectively. Consequently, given that $\sigma = 1/(2a)$ and $a + b = 1$, we have all the terms in $\|\eta\|$ with the power 2σ and all the terms in $\psi_3(\chi)$ with the power 1.

Now, choose q_1, q_2, such that $-1/2+\rho_1(q_1)^{p_1}+\rho_4/q_2 < 0$, where $p_1 = 1/(2\sigma-1)$. Then, it can be shown that there exists $0 < \epsilon_6^\star \le \epsilon_2^\star$ such that, for every $0 < \epsilon \le \epsilon_6^\star$ we have $-\rho_2/(2\epsilon) + \rho_3 + \rho_4(q_2)^{p_1} < 0$ which implies that, for every $0 < \epsilon \le \epsilon_6^\star$ and for all $(\chi, \eta) \in \Lambda_3$, we have

$$\dot{\mathcal{V}} \le -\frac{1}{2}\psi_3(\chi) - \frac{\rho_2}{2\epsilon}\|\eta\|^{2\sigma} \quad (3.64)$$

for all $\mathbf{d} \in \mathcal{D}$. Thus, according to Theorem 3.1, the closed-loop system (3.18)–(3.21) is uniformly asymptotically stable with respect to $\mathcal{A} \times \{0\}$. $\triangleleft$

Remark 3.4 *Theorems 3.7, 3.8 and 3.9, along with Theorems 3.4 and 3.5, show semiglobal stabilization.*

3.5 Regional Separation results

In many cases, asymptotic stability with respect to a set, achieved under the state feedback controller, is not global and the region of attraction is a subset of the state space. This subset may be finite (bounded) or infinite. Examples of such cases can be found in adaptive control [2, 27], and robust control [50].

In order to extend the previous separation results, we need, as we did in the previous chapter, a converse Lyapunov theorem that yields a Lyapunov function which goes to infinity at the boundary of an estimate of the region of attraction. This can be done by extending the converse Lyapunov results of [34] to an estimate of the region of attraction. In order to perform this task, we need to restrict the set of time-varying parameters to the set, denoted by $\mathcal{M}'_{\mathcal{D}}$, of all continuously differentiable functions from R to $\mathcal{D}$ where the derivative $d'(t)$ of $d(t)$ belongs to a compact set $\mathcal{D}_1 \subset R^d$.

First, we give a definition of the region of asymptotic stability of a closed set.

Definition 3.3 *The region of uniform asymptotic stability of the system (3.1) with respect to the closed positively invariant set $\mathcal{A}$ is the set of all points $x_0 \in R^n$ such that*

$$|x(t, x_0, d)|_{\mathcal{A}} \to 0 \text{ as } t \to +\infty$$

uniformly in $d \in \mathcal{M}'_{\mathcal{D}}$.

This region is not empty because it contains $\mathcal{A}$.

Second, we give the needed converse Lyapunov theorem.

Theorem 3.10 *Let $d \in \mathcal{M}'_{\mathcal{D}}$. Let $\mathcal{A} \subset R^n$ be a compact, positively invariant set for the system (3.23). Assume that the system (3.23) is UAS with respect to $\mathcal{A}$. Let $\mathcal{R}$ be an open and connected subset of the region of attraction that contains $\mathcal{A}$. Then, there exists a smooth Lyapunov function V in $\mathcal{R}$ and three positive definite, with respect to $\mathcal{A}$, functions U_1, U_2, and U_3, all defined on $\mathcal{R}$, such that*

$$
\begin{aligned}
V(\chi) &= 0 \Longleftrightarrow \chi \in \mathcal{A} && (3.65)\\
U_1(\chi) \le V(\chi) &\le U_2(\chi) && (3.66)\\
\lim_{\chi \to \partial\mathcal{R}} U_1(\chi) &= \infty && (3.67)\\
\frac{\partial V}{\partial \chi} f_r(\chi, \mathbf{d}, 0) &\le -U_3(\chi),\ \forall \mathbf{d} \in \mathcal{D} && (3.68)
\end{aligned}
$$

Proof: see the proofs of Theorem 6.1 and Corollary 6.1.◁

Remark 3.5 *The set $\mathcal{R}$ can be the region of attraction when the system is autonomous.*

In this case we can recover the same set of performance as in the previous section. First, let us replace Assumption 3.1 with the following assumption:

Assumption 3.5 *The time-varying parameter belongs to the set $\mathcal{M}'_{\mathcal{D}}$. Items 1 and 2 of Assumption 3.1 hold. The closed-loop system under the state feedback controller is uniformly asymptotically stable.*

The separation result is summarized in the following theorem:

Theorem 3.11 *Let Assumptions 3.2–3.4, and 3.5 hold. Let $\mathcal{R}$ be an open, connected subset of the region of attraction that contains $\mathcal{A}$ and S be any compact subset of $\mathcal{R}$ which contains $\mathcal{A}$. Then, the conclusions of Theorems 3.4, 3.5, 3.6, 3.7, 3.8, and 3.9 hold.*

Proof: Using Theorem 3.10, the proof of boundedness is similar to that of Theorem 3.4. The proof of ultimate boundedness is similar to that of Theorem 2.2. We show that

$$\dot{V} \leq -\frac{1}{2}U_3(\chi), \text{ for } \chi \notin \{\chi : U_3(\chi) \leq 2k_3\epsilon \stackrel{\text{def}}{=} \mu(\epsilon)\} \tag{3.69}$$

We define $c_0(\epsilon) = \max_{U_3(\chi) \leq \mu(\epsilon)}\{V(\chi)\}$. Then, we have $\{\chi : U_3(\chi) \leq \mu(\epsilon)\} \subset \{\chi : V(\chi) \leq c_0(\epsilon)\}$.

Finally, we choose $\epsilon_4 = \epsilon_4(\xi) \leq \epsilon_1^\star$ small enough such that, for all $\epsilon \leq \epsilon_4$, the set $\{\chi : U_3(\chi) \leq \mu(\epsilon)\}$ is compact, the set $\{\chi : V(\chi) \leq c_0(\epsilon)\}$ is in the interior of Ω, and

$$\{\chi : V(\chi) \leq c_0(\epsilon)\} \subset \{\chi : |\chi|_{\mathcal{A}} \leq \xi/2\} \tag{3.70}$$

Then, we conclude that the set $\{\chi : V(\chi) \leq c_0(\epsilon)\} \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ is positively invariant and every trajectory in $\Omega \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ reaches $\{\chi : V(\chi) \leq c_0(\epsilon)\} \times \{\eta : W(\eta) \leq \rho\epsilon^2\}$ in finite time.

The proofs of trajectory convergence and local uniform exponential stability are similar to those of Theorems 3.6 and 3.8, respectively.

The proofs of local uniform asymptotic stability (with or without modeling errors) are similar to those of Theorems 3.7 and 3.9. In this case we locally replace, using Lemma A.3, the functions U_i, $i = 1, 2, 3$, with class $\mathcal{K}$ functions α_i, $i = 1, 2, 3$, such that $\alpha_1(|\chi|_{\mathcal{A}}) \leq U_1(\chi)$, $\alpha_2(|\chi|_{\mathcal{A}}) \geq U_2(\chi)$, and $\alpha_3(|\chi|_{\mathcal{A}}) \leq U_3(\chi)$ on a certain ball around $\mathcal{A}$.$\triangleleft$

3.6 Conclusion

We presented a separation principle for a class of nonlinear systems in cases where trajectories do not necessarily go to the origin, namely, in cases where trajectories go to a compact, positively invariant set $\mathcal{A}$. An output feedback controller using a sufficiently fast high-gain observer recovers the performance achieved under a state feedback controller. This includes boundedness, ultimate boundedness, convergence of trajectories, and exponential stability with respect $\mathcal{A}$. We also found that we can recover asymptotic stability with respect to $\mathcal{A}$ when the modeling error is zero, but, when this error is not zero, we need to impose some additional conditions on it. It is noteworthy that our results can only show semiglobal stabilization under output feedback even when the state feedback control achieves global stabilization. As for the case where the region of attraction is not the whole space we can only recover compact subsets of an estimate of the region of attraction.

CHAPTER 4

A Separation Principle for the Control of a Class of Nonlinear Systems - Examples

4.1 Introduction

In order to illustrate the theory developed in the previous chapter, we present several examples taken form [10, 50, 38, 21, 2]. We present these examples as state feedback control designs and apply the separation results of the previous chapter to arrive at the output feedback control results proven in the respective references. Moreover, we go beyond the results of those references and conclude the trajectory convergence property mentioned earlier. This shows how we can use the results of the previous chapter as a framework for the design of output feedback controllers.

Hereafter, we discuss several output feedback control problems that fit in the generic form we discussed in Section 3.3. In each example we give the system considered and the control problem solved then we show that the system fits into the model (3.11)–(3.14) and that the separation results previously obtained apply to the problem at hand.

The forthcoming examples have been treated in several papers to be referenced later on. In our presentation of these examples we conserve the notation used by the respective papers for easy reference. The notation pertains only to the example at hand and does not refer to any previous analysis unless it is explicitly mentioned. These design cases, as presented in their corresponding references, have not been treated as separation results; i.e., they were treated as output feedback design cases. Thus, in order to apply our separation results, we reworked them as state feedback design cases which did not require considerable changes to the original analysis given in the corresponding references. Moreover, the high-gain observers suggested are the ones used in those references.

Each example is divided into three main parts. The first part introduces the system considered as well as any necessary technical assumptions. The second part states the design objective and details the design process. The third part explains how we can apply the separation results developed in the previous chapter to the example at hand. The application of the separation results follows the following points:

- We give the system to which the theory applies.
- We give the globally bounded state feedback controller.
- We prove that the system with the controller is asymptotically stable with respect to a certain compact positively invariant set that we specify. We also give an estimate of the region of attraction.
- We propose a high-gain observer for the output feedback implementation of the controller.

- We state the conclusions that can be made using the separation results and show, if needed, that the conditions are satisfied.

We start with examples related to robust control. Then, we move to examples related to tracking and servomechanism. Finally, we treat an example from adaptive control.

Remark 4.1 *To prepare the controller for output feedback implementation, we saturate the control law outside a region of interest. It is implicit in this case that the region of attraction recovered is the one achieved with the bounded controller.*

In case the state feedback controller achieves global asymptotic stability, the output feedback controller achieves semiglobal stabilization (i.e., the region of attraction contains arbitrary compact sets fixed a priori *by the designer). It is implicit that the saturation level imposed on the control law depends on the compact set that we wish to recover.*

Remark 4.2 *The design and analysis pertaining to each of the forthcoming examples are extracted from their corresponding references. But for the simplicity of the presentation we used the pronoun "we" to indicate the designers who are not necessarily "us".*

Remark 4.3 *The assumptions and claims stated in each example are related only to the example at hand and so are their numbers.*

4.2 Robust Control I - Finite Time Convergence to a Set [10]

Consider the perturbed linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + E\delta(x, u, t) \tag{4.1}$$

$$y(t) = Cx(t) \tag{4.2}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and $\delta(.) \in R^\ell$. The following assumption imposes growth and structural conditions on the nonlinearity:

Assumption 1: The disturbance vector $\delta(.)$ satisfies the following inequalities:

$$\begin{aligned} \|\delta(x,u,t)\| &\leq \delta_x\|x\| + \delta_u\|u\| + \delta_c \\ \|\delta(x,u,t) - \delta(x,\hat{u},t)\| &\leq k_1\|u - \hat{u}\| \end{aligned}$$

where $\delta_x, \delta_u, \delta_c$, and k_1 are nonnegative constants and $\delta_u < 1$.

The objective is to design a dynamic state feedback control that ensures ultimate boundedness of the state trajectories. The following assumption gives a stabilizing static state feedback controller:

Assumption 2: There exists a state feedback control $u = -f(x)$ and a Lyapunov function $W(x)$ such that

$$\begin{aligned} \frac{\partial W}{\partial x}[Ax - Bf(x) + E\delta(x,-f(x),t)] &\leq -\beta_4\|x\|^2 + (\delta_c\beta_3 + \beta_2)\|x\| + \delta_c\beta_1 \\ \underline{\eta}_\omega\|x\|^2 &\leq W(x) \leq \bar{\eta}_\omega\|x\|^2 \\ \left\|\frac{\partial W}{\partial x}\right\| &\leq k_2\|x\| \\ \|f(x) - f(\hat{x})\| &\leq k_3\|x - \hat{x}\| \\ \|f(x)\| &\leq k_4\|x\| + k_5 \end{aligned}$$

for all x and $\hat{x}$, where $\beta_4, \underline{\eta}_\omega, \bar{\eta}_\omega, k_2, k_3$ are positive constants, while $\beta_3, \beta_2, \beta_1, k_4$, and k_5 are nonnegative constants.

The first inequality ensures uniform ultimate boundedness of the closed-loop system with respect to the set

$$\Omega = \{x \in R^n : W(x) < \bar{\eta}_\omega c_\star^2\} \tag{4.3}$$

where

$$c_\star \stackrel{\text{def}}{=} \frac{\delta_c\beta_3 + \beta_2}{2\beta_4} + \left[\frac{(\delta_c\beta_3 + \beta_2)^2}{4\beta_4^2} + \frac{\delta_c\beta_1}{\beta_4}\right]^{\frac{1}{2}} \geq 0$$

If the transfer function $C(sI - A)^{-1}E$ is minimum-phase and left-invertible, then there exist nonsingular transformations $\Gamma_1, \Gamma_2, \Gamma_3$, integer K, and integer indices q_i, r_i, $i = 1, \cdots, K$ such that

$$\begin{aligned} x &= \Gamma_1\tilde{x} = [\tilde{x}_a^T, \tilde{x}_b^T, \tilde{x}_f^T]^T = [\tilde{x}_s^T, \tilde{x}_f^T]^T \\ y &= \Gamma_2[\tilde{y}_f^T, \tilde{y}_s^T]^T, \ u = \Gamma_3\tilde{u}, \ \delta(.) = \Gamma_3\tilde{\delta}(.) \end{aligned}$$

and such that the system (4.1)–(4.2) can be written as (all tildes will be dropped for notational simplicity)

$$\begin{aligned} \dot{x}_a &= A_{aa}x_a + A_{af}y_f + A_{as}y_s + B_a u \\ \dot{x}_b &= A_{bb}x_b + A_{bf}y_f + B_b u \\ \dot{x}_f &= A_f x_f + M_f y_f + B_f u + E_f[D_a x_a + D_b x_b + D_f x_f + \delta(.)] \\ y_f &= C_f x_f \\ y_s &= C_s x_b \end{aligned}$$

where (A_f, B_f) represents chains of integrators. The structure of the above matrices is explained in [10] and the references therein.

The state component x_f can be estimated using a high-gain observer. As for the components x_a and x_b, they are estimated using the observer

$$\begin{aligned}\dot{\hat{x}}_a &= A_{aa}\hat{x}_a + A_{af}y_f + A_{as}y_s + B_a u \\ \dot{\hat{x}}_b &= A_{bb}\hat{x}_b + A_{bf}y_f + L_b(y_s - C_s\hat{x}_b) + B_b u\end{aligned}$$

where L_b is to be chosen. Thus, the implemented state feedback controller is $u = -f(\Gamma_1 \hat{\bar{x}}) \stackrel{\text{def}}{=} f_1(\hat{x}_s, x_f)$. The estimation error dynamics are

$$\dot{e}_a = A_{aa}e_a \tag{4.4}$$

$$\dot{e}_b = (A_{bb} - L_b C_s)e_b \tag{4.5}$$

Choose L_b such that $(A_{bb} - L_b C_s)$ is Hurwitz.

Set $e_s = [e_a^T, e_b^T]^T$, $A_s = \text{diag}[A_{aa}, A_{bb} - L_b C_s]$, and rewrite (4.4)–(4.5) as

$$\dot{e}_s = A_s e_s \tag{4.6}$$

Let P_s be the positive definite solution of the Lyapunov equation $P_s A_s + A_s^T P_s = -I$. The function $V_s = e_s^T P_s e_s$ satisfies

$$\underline{\eta}_s \|e_s\|^2 \leq V_s \leq \bar{\eta}_s \|e_s\|^2 \tag{4.7}$$

$$\dot{V}_s \leq -\bar{\alpha}_s V_s \tag{4.8}$$

for some positive constants $\underline{\eta}_s, \bar{\eta}_s, \bar{\alpha}_s$.

Next, we show that if Assumptions 1 and 2 hold globally, then the closed-loop system (4.1), (4.6) with $u = -f_1(\hat{x}_s, x_f)$ is globally uniformly ultimately bounded with respect to the compact set

$$\Phi = \{(x^T, e_s^T)^T : V_s \leq c_s^2,\ W(x) \leq \bar{\eta}_\omega c_1\} \tag{4.9}$$

for some c_s which can be made arbitrarily small and some $c_1 > c_\star^2$ which depends on c_s.

The derivative of W along the trajectories of the closed-loop system is

$$\begin{aligned} \dot{W} &= \frac{\partial W}{\partial x}[Ax - Bf(x) + E\delta(x, -f(x), t)] + \frac{\partial W}{\partial x}B[f_1(x_s, x_f) - f_1(\hat{x}_s, x_f)] \\ &\quad + \frac{\partial W}{\partial x}E[\delta(x, -f_1(\hat{x}_s, x_f), t) - \delta(x, -f_1(x_s, x_f), t)] \end{aligned} \tag{4.10}$$

Using Assumptions 1 and 2 in (4.10) yields

$$\begin{aligned} \dot{W} &\leq -\beta_4\|x\|^2 + (\delta_c\beta_3 + \beta_2)\|x\| + \delta_c\beta_1 \\ &\quad + \left[\left\|\frac{\partial W}{\partial x}B\right\| k_3\|\Gamma_1\| + \left\|\frac{\partial W}{\partial x}E\right\| k_1k_3\|\Gamma_1\|\right] \|x_s - \hat{x}_s\| \end{aligned} \tag{4.11}$$

Using the bounds on $\partial W/\partial x$ and V_s into (4.11) yields

$$\dot{W} \leq -\left\{\beta_4\|x\|^2 - \left[(\delta_c\beta_3 + \beta_2) + 2\gamma_1\bar{\gamma}_h\left(\frac{V_s}{\underline{\eta}_s}\right)^{\frac{1}{2}}\right]\|x\| - \delta_c\beta_1\right\} \tag{4.12}$$

where $2\gamma_1 = k_2(\|B\| + \|E\|k_1)k_3$ and $\bar{\gamma}_h = \|\Gamma_1\|$.

Let

$$\begin{aligned} C_x\left(\sqrt{V_s}\right) &\stackrel{\text{def}}{=} \frac{1}{2\beta_4}\left[(\delta_c\beta_3 + \beta_2) + 2\gamma_1\bar{\gamma}_h\left(\frac{V_s}{\underline{\eta}_s}\right)^{\frac{1}{2}}\right] \\ &\quad + \frac{1}{2\beta_4}\left\{\left[(\delta_c\beta_3 + \beta_2) + 2\gamma_1\bar{\gamma}_h\left(\frac{V_s}{\underline{\eta}_s}\right)^{\frac{1}{2}}\right]^2 + 4\beta_4\delta_c\beta_1\right\}^{\frac{1}{2}} \end{aligned}$$

Notice that $C_x(0) = c_\star$. It is straightforward to see that

$$\dot{W} < 0 \text{ when } \|x\| > C_x\left(\sqrt{V_s}\right) \tag{4.13}$$

or

$$\dot{W} < 0 \text{ when } W(x) > \bar{\eta}_\omega C_x^2\left(\sqrt{V_s}\right) \tag{4.14}$$

Using (4.14) and (4.8), it can be seen that the closed-loop system is globally uniformly ultimately bounded with respect to the compact, positively invariant set Φ defined in (4.9) where $c_1 > C_x^2(c_s)$.

The separation results apply to the system

$$\begin{aligned}
\dot{x}_a &= A_{aa}x_a + A_{af}y_f + A_{as}y_s + B_a u \\
\dot{x}_b &= A_{bb}x_b + A_{bf}y_f + B_b u \\
\dot{x}_f &= A_f x_f + M_f y_f + B_f u + E_f[D_a x_a + D_b x_b + D_f x_f + \delta(x,u,t)] \\
y_f &= C_f x_f \\
y_s &= C_s x_b
\end{aligned}$$

This system fits into the model (3.11)–(3.14) with the vector of bounded disturbances being $\delta(0,0,t)$ (bounded by Assumption 1), the x state component being x_f, the z state component being (x_a, x_b), and the ζ output being y_s.

The state feedback controller considered is

$$\begin{aligned}
\dot{\hat{x}}_a &= A_{aa}\hat{x}_a + A_{af}y_f + A_{as}y_s + B_a u \\
\dot{\hat{x}}_b &= A_{bb}\hat{x}_b + A_{bf}y_f + L_b(y_s - C_s\hat{x}_b) + B_b u
\end{aligned}$$

$$u = -f_1(\hat{x}_a, \hat{x}_b, x_f)$$

Global boundedness is achieved by saturation outside a region of interest.

We have shown that the closed-loop system under this controller is globally asymptotically stable with respect to the compact positively invariant set Φ.

To implement the controller we use the high-gain observer

$$\dot{\hat{x}}_f = A_f \hat{x}_f + M_f y_f + B_f u + E_f[D_a \hat{x}_a + D_b \hat{x}_b + D_f \hat{x}_f] + L_d(y_f - C_f \hat{x}_f)$$

The structure of the gain L_d is given in [10, Appendix B]. This structure is exactly the one suggested in Section 3.3 with the difference that in [10] all channels of equal relative degree are grouped together.

According to Theorems 3.4 and 3.5, trajectories of the closed-loop system under output feedback come arbitrarily close to the set $\Phi \times \{e_f = 0\}$ in finite time, where e_f is the estimation error. Moreover, Theorem 3.6 ensures convergence of trajectories under the output feedback controller to those under the state feedback controller as the observer gain approaches infinity.

4.3 Robust Control II - Finite-time Convergence to a Set [50, Section 3]

Consider the control system

$$\dot{z} = A(z, u, d(t)) \tag{4.15}$$

$$y = C(z, d(t)) \tag{4.16}$$

where the state $z \in R^n$ and the input $u \in R$. The functions $A(.)$ and $C(.)$ are smooth in their arguments, and $d(t)$ is a time-varying smooth disturbance signal contained in a compact set $D \subset R^d$. For simplicity we denote $d(t)$ and its time derivatives $\dot{d}(t),\ \ddot{d}(t), \cdots$ by the same symbol d, i.e., $d = (d, \dot{d}, \ddot{d}, \cdots)$.

We start with the following two definitions:

Definition 1: [50, Definition 2] (Uniform Complete Observability). Consider the dynamical system

$$\dot{\zeta} = A(\zeta, u),\ y = C(\zeta)$$

The above dynamical system is UCO if there exist two integers n_y and n_u and a C^1 function Ψ such that, for each solution of

$$\dot{\zeta} = A(\zeta, u_0),\ \dot{u}_0 = u_1, \cdots, \dot{u}_{n_u} = v$$

we have, for all t where the solution makes sense,

$$\zeta(t) = \Psi(y(t), \cdots, y^{(n_y)}(t), u_0(t), \cdots, u_{n_u}(t)).$$

Remark 4.4 *In [50] the authors used the more general notion of UCO of a function $\bar{u}(\zeta)$ with respect to a dynamical system. In such a definition we have*

$$\bar{u}(\zeta(t)) = \Psi(y(t), \cdots, y^{(n_y)}(t), u_0(t), \cdots, u_{n_u}(t)).$$

In our case we need the function $\bar{u}$ to be equal to ζ because, later on, we need to *express the state variable z as function of the input, the output, and some of their derivatives.*

Definition 2: [50, Definition 4] (Semiglobal Practical Stabilizability). A point $z = 0$ is said to be semiglobally practically stabilizable by dynamic state feedback (respectively, output) feedback if, for each pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{z\ell})$, neighborhoods of 0 with $\mathcal{K}_{zs} \subset \mathcal{K}_{z\ell}$, there exist a locally Lipschitz dynamic state (respectively, output) feedback $u = \bar{u}(z,\zeta)$, $\dot{\zeta} = \theta(z,\zeta)$ (respectively, $u = \bar{u}(y,\zeta)$, $\dot{\zeta} = \theta(y,\zeta)$) and a pair of compact sets $(\mathcal{K}_{\zeta s}, \mathcal{K}_{\zeta\ell})$ such that all solutions of the closed-loop system, with initial condition in $\mathcal{K}_{z\ell} \times \mathcal{K}_{\zeta\ell}$, enter the set $\mathcal{K}_{zs} \times \mathcal{K}_{\zeta s}$ in finite time.

The objective is to design a state feedback controller that achieves semiglobal practical stabilization of the point $z = 0$.

We assume that the point $z = 0$ is semiglobally practically stabilizable by a static state feedback, as in the following assumption:

Assumption 1: [50, Assumption S-GPS]: There exists two integers N_y and N_u so that the system (4.15)-(4.16) is UCO with $n_y \leq N_y$ and $n_u \leq N_u$. For each pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{z\ell})$, neighborhoods of 0 and with $\mathcal{K}_{zs} \subset \mathcal{K}_{z\ell}$, we can find

1. a positive C^1 function V, zero at 0, which is defined on Ω, an open set containing $\mathcal{K}_{z\ell}$, and such that there exist three positive real numbers ϑ_ℓ, c_s, and c_ℓ satisfying

$$c_s < c_\ell, \ \ \{z, V(z) \leq \vartheta_\ell\} \subset \mathcal{K}_{zs}, \ \ \mathcal{K}_{z\ell} \subset \{z : V(z) \leq c_s\} \tag{4.17}$$

and so that the set $\{z : V(z) \leq c_\ell\}$ is compact and contained in Ω.

2. a C^2 function $\bar{u}(z)$ which is zero at 0, defined on Ω such that

$$\frac{\partial V}{\partial z} A(z, \bar{u}(z), d(t)) \leq -\Phi(z) \tag{4.18}$$

where $\Phi(z)$ is continuous on Ω and positive definite on $\{z : \vartheta_s \leq V(z) \leq c_\ell\}$ for some positive real number $\vartheta_s < \vartheta_\ell$.

In order to use the state feedback $\bar{u}(z)$ we need to know the output and n_y of its derivatives in addition to the input and n_u of its derivatives. Thereafter, we need to prepare to estimate the output and some of its derivatives, and design an input whose derivatives (a number of them) are known.

It can be shown that there exist $n_y + 1$ smooth functions C_i and an integer $m_u \leq n_y$ such that, for each solution of

$$\begin{aligned} \dot{z} &= A(z, u_0, d(t)) \\ \dot{u}_0 &= u_1 \\ &\vdots \\ \dot{u}_{m_u - 1} &= u_{m_u} \end{aligned} \tag{4.19}$$

we have, for all t where the solution makes sense,

$$y^{(i)} = C_i(z(t), u_0(t), \cdots, u_{m_u}(t), d(t)), \ i = 1, \cdots, n_y + 1 \tag{4.20}$$

Let $y_0 = y$ and $y_i = y^i$, $i = 1, \cdots, n_y$. Then, if we write the system as

$$\begin{aligned} \dot{y}_0 &= y_1 \\ \dot{y}_1 &= y_2 \\ &\vdots \\ \dot{y}_{n_y} &= C_{n_y + 1}(z, u_0, \cdots, u_{m_u}, d(t)) \end{aligned} \tag{4.21}$$

we use a high-gain observer with the nonlinearity $C_{n_y + 1}(0, u_0, \cdots, u_{m_u}, 0)$ to estimate the output and its derivatives that are needed in $\bar{u}(z)$.

Let $\ell_u = \max\{n_u, m_u\} + 1$. Thus, by adding ℓ_u integrators to the system (4.15), we can have u and its required derivatives (n_u for $\bar{u}$ and m_u for the C_{n_y+1}) as measured states of the system

$$\begin{aligned}
\dot{z} &= A(z, u_0, d(t)) \\
\dot{u}_0 &= u_1 \\
&\vdots \\
\dot{u}_{\ell_u - 1} &= v
\end{aligned} \tag{4.22}$$

For the moment, we are left with the task of designing a state feedback controller for the system (4.22). Let $\xi_1 = u_0 - \bar{u}(z)$ and $\xi_i = \frac{u_{i-1}}{K^{i-1}}$ for $i = 1, \cdots, \ell_u$, with K being a positive real number to be specified later on. Thus, the system (4.22) can be written as

$$\begin{aligned}
\dot{z} &= A(z, \bar{u}(z) + \xi_1, d(t)) \stackrel{\text{def}}{=} h(z, \xi_1, d(t)) \\
\dot{\xi}_1 &= K\xi_2 - \frac{\partial \bar{u}}{\partial z}(z) A(z, \bar{u}(z) + \xi_1, d(t)) \\
\dot{\xi}_2 &= K\xi_3 \\
&\vdots \\
\dot{\xi}_{\ell_u} &= K^{1-\ell_u} v
\end{aligned} \tag{4.23}$$

This system satisfies, with minor adjustments, the conditions of of Lemma 2.3 of [50] which provides the desired state feedback controller. For the sake of completeness we state the main condition of this lemma.

Assumption 2 [50, Assumption ULP]: For the system

$$\dot{z} = h(z, 0, d(t))$$

there exists an open set $\Omega_1 \subseteq R^m$*, a nonnegative real number* $\vartheta < 1$*, a real number* $c \geq 1$*, and a* C^1 *function* $V : \Omega_1 \to R_{\geq 0}$ *such that the set* $\{z : V(z) \leq c + 1\}$ *is a compact subset of* Ω_1*, and we have*

$$\frac{\partial V}{\partial z} h(z, 0, d(t)) \leq -\Phi_1(z)$$

where $\Phi_1(z)$ *is continuous on* Ω_1 *and positive definite on the set* $\{z : \vartheta < V(z) \leq c + 1\}$.

Claim: There exists a C^1 *function* $V_1(z)$ *that satisfies Assumption 2.*

Proof: In order for the system (4.23) to satisfy Assumption 2 we need to adjust the function $V(z)$ and the various coefficients given is Assumption 1 as follows:
Pick ϑ_1 as an arbitrary number in $(0, 1/8)$. Let κ be a C^1 class $\mathcal{K}_\infty$ function such that

$$\kappa(\vartheta_s) = \vartheta_1, \ \kappa(\vartheta_\ell) \geq 8\vartheta_1, \ \kappa(c_s) \geq 1, \ \kappa(c_\ell) > 1 + \kappa(c_s) \tag{4.24}$$

This function exists since Assumption 1 states that $0 < \vartheta_s < \vartheta_\ell \leq c_s < c_\ell$. Now, let $V_1(z) = \kappa(V(z))$ and $c_1 = \kappa(c_s) \geq 1$. Thus, we have

$$\begin{aligned} &\{z : V_1 \leq 8\vartheta_1\} \subset \{z : V(z) \leq \vartheta_\ell\} \subset \mathcal{K}_{zs} \\ &\mathcal{K}_{z\ell} \subset \{z : V(z) \leq c_s\} \subset \{z : V_1(z) \leq c_1\} \end{aligned} \tag{4.25}$$

The set $\{z : V_1 \leq c_1 + 1\}$ is a compact set and is contained in the set $\{z : V(z) \leq$

$c_\ell\}$.

Finally, we have

$$\frac{\partial V_1}{\partial z} h(z, 0, d(t)) \leq -\Phi_1(z)$$

where $\Phi_1(z) = \frac{d\kappa}{ds}(V_z)\Phi(z)$ is continuous on Ω and positive definite on the set $\{z : \vartheta_1 \leq V_1(z) \leq c_1 + 1\} \subset \{\vartheta_s \leq V(z) \leq c_\ell\}$.

Hence, from Assumption 1 and the above adjustments, we conclude that the system (4.23) satisfies Assumption 2 with the C^1 function $V_1(z)$ defined on the open set $\Omega_1 = \Omega$ and with $\vartheta = \vartheta_1 < 1$ and $c = c_1 \geq 1$.$\triangleleft$

Hereafter, we apply Lemma 2.3 of [50] to the system (4.23). Let the polynomial

$$p(s) = s^{\ell_u + 1} + a_{\ell_u} s^{\ell_u} + \cdots + a_1$$

be Hurwitz and let A_c be the companion form matrix corresponding to $p(s)$. Let P_c be the positive definite solution of $A_c^T P_c + P_c A_c = -I$. Let $\mathcal{K}_{\xi\ell}$ be an arbitrary compact set where we choose to initialize ξ. Lemma 2.3 of [50] suggests the following controller

$$\begin{aligned} v &= -K^{\ell_u}(a_1\xi_1 + \cdots + a_{\ell_u}\xi_{\ell_u}) \\ &= -K^{\ell_u}(a_1[u_0 - \bar{u}(z)] + \cdots + a_{\ell_u}\xi_{\ell_u}) \end{aligned} \tag{4.26}$$

and provides the Lyapunov function

$$W_1(z, \xi) = \frac{c_1 V_1(z)}{c_1 + 1 - V_1} + \frac{\mu_1 \xi^T P_c \xi}{\mu_1 + 1 - \xi^T P_c \xi}$$

where

$$\mu_1 = \max\left\{1, \max_{\xi \in \mathcal{K}_{\xi\ell}} \{\xi^T P_c \xi\}\right\} \geq 1$$

Furthermore, we have

$$\mathcal{K}_{z\ell} \times \mathcal{K}_{\xi\ell} \subset \{(z,\xi) : W_1(z,\xi) \leq c_1^2 + \mu_1^2\}$$

and, by letting $\rho = \frac{\vartheta_1}{2}$, there exists $K_\star \geq 1$ such that, for all $K \geq K_\star$, the derivative of W_1 along trajectories of the closed-loop system (4.23), under the controller (4.26), satisfies

$$\dot{W}_1 \leq -\Phi_2(z,\xi) \tag{4.27}$$

where $\Phi_2(z,\xi)$ is a positive definite function on $\{(z,\xi) : \vartheta_1 + \rho \leq W_1(z,\xi) \leq c_1^2 + \mu_1^2 + 1\}$.

Thus, we have proved the following:

For any pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{z\ell})$, neighborhoods of 0 with $\mathcal{K}_{zs} \subset \mathcal{K}_{z\ell}$, we can find compact sets $\mathcal{K}_{\xi s}$ and $\mathcal{K}_{\xi\ell}$, gains a_i's, a bound $K_\star$, integers ℓ_u and n_y, and functions $\bar{u}$ and Ψ such that for each $K \geq K_\star$ the dynamic state feedback

$$\begin{aligned}
\dot{u} &= K\xi_2 \\
\dot{\xi}_2 &= K\xi_3 \\
&\vdots \\
\dot{\xi}_{\ell_u} &= K^{1-\ell_u} v \\
v &= -K^{\ell_u}(a_1[u_0 - \bar{u}(z)] + \cdots + a_{\ell_u}\xi_{\ell_u})
\end{aligned} \tag{4.28}$$

in closed-loop with the system (4.15)–(4.16) makes all solutions starting in $\mathcal{K}_{z\ell} \times \mathcal{K}_{\xi\ell}$ enter the set $\mathcal{K}_{zs} \times \mathcal{K}_{\xi s}$ in finite time.

The theory developed in the previous chapter applies to the system

$$\left.\begin{array}{rcl} \dot{y}_0 & = & y_1 \\ \dot{y}_1 & = & y_2 \\ \vdots & \vdots & \vdots \\ \dot{y}_{ny} & = & C_{ny+1}(z, u, K\xi_2, \cdots, K^{mu}\xi_{mu+1}, d(t)) \end{array}\right\} \tag{4.29}$$

$$\left.\begin{array}{rcl} \dot{u} & = & K\xi_2 \\ \dot{\xi}_2 & = & K\xi_3 \\ \vdots & \vdots & \vdots \\ \dot{\xi}_{\ell_u} & = & K^{1-\ell_u}v \end{array}\right\} \tag{4.30}$$

$$y = y_0 \tag{4.31}$$

where z can be expressed as function of the state variables given that the original system is UCO. This system fits into the model (3.11)–(3.14) with (4.29) being the x-dynamics and (4.30) being the z-dynamics. Furthermore, the vector of bounded disturbance consists of $d(t)$ and some of its derivatives, all denoted here by $d(t)$. The additional output ζ is represented here by the vector $(u, \xi_1, \cdots, \xi_{\ell_u})$. Since we are dealing with a regional result we need to restrict the time-varying parameter d to belong to $\mathcal{M}'_{\mathcal{D}}$.

We consider the state feedback controller.

$$\begin{aligned} v &= -K^{\ell_u}(a_1\xi_1 + \cdots + a_{\ell_u}\xi_{\ell_u}) \\ &= -K^{\ell_u}(a_1[u_0 - \bar{u}(z)] + \cdots + a_{\ell_u}\xi_{\ell_u}) \end{aligned}$$

where

$$z(t) = \Psi(y(t), \cdots, y^{(ny)}(t), u_0(t), \cdots, u^{(nu)}(t))$$

Global boundedness is achieved by saturating the $\bar{u}(z)$ outside the set $\{z : V(z) \leq c_\ell\}$.

We have proved that trajectories of the closed-loop system under the state feedback controller, starting in the compact set $\mathcal{K}_{z\ell} \times \mathcal{K}_{\xi\ell}$ (arbitrarily chosen), enter the compact positively invariant set $\{(z,\xi) : W_1(z,\xi) \leq \vartheta_1 + \rho\}$ in finite time (positive invariance is clear from (4.27)). An estimate of the region of attraction is the set $\Omega_0 = \{(z,\xi) : W_1(z,\xi) \leq c_1^2 + \mu_1^2 + 1\}$.

To implement the controller we use a nonlinear high-gain observer with the nonlinearity $C_{ny+1}(0, u_0, \cdots, u_{mu}, 0)$ as a nominal model for the nonlinearity $C_{ny+1}(z, u_0, \cdots, u_{mu}, d(t))$ (this is the same observer used in [50]).

Theorem 3.11 guarantees that the trajectories of the closed-loop system under output feedback controller starting in $\mathcal{S} \times \mathcal{Q}$, where $\mathcal{S}$ is a compact subset of Ω_0 and $\mathcal{Q}$ is a compact set in R^{ny}, are bounded and will come arbitrarily close to the set

$$\{(z,\xi) : W_1(z,\xi) \leq \vartheta_1 + \rho\} \times \{e = 0\}$$

where e is the estimation error. Moreover, Theorem 3.11 shows that the trajectories under output feedback control converge to those under state feedback control as the observer gain approaches infinity.

4.4 Tracking [28]

We show that a tracking problem can be viewed as asymptotic stability with respect to a positively invariant and compact set. Then we give an example where the separation results of the previous chapter apply.

Consider the globally defined single-input single-output system

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, x) \\
\dot{x} &= A_c x + B_c \frac{1}{b(x)}[u - a(x)] \\
y &= x_1
\end{aligned} \tag{4.32}
$$

with $\eta \in R^{n-r}$, $b(x) \neq 0$ for all $x \in R^r$, and (A_c, B_c) defines a chain of integrators.

Remark 4.5 *The above system is not the most general input-output linearizable system because the nonlinearities $a(.)$ and $b(.)$ depend only on a part of the state, namely x, and does not depend on η.*

Let (4.32) satisfy the following input-to-state stability assumption:

Assumption 1: There exists a C^1 function $V_1(\eta)$ such that

$$\alpha_1(\|\eta\|) \leq V_1(\eta) \leq \alpha_2(\|\eta\|) \tag{4.33}$$

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, x) \leq -\alpha_3(\|\eta\|) + \alpha_4(\|x\|) \tag{4.34}$$

for all $\eta \in R^{n-r}$ and all $x \in R^r$, where α_i, $i = 1, \cdots, 4$ are class $\mathcal{K}_\infty$ functions.

We need the output y to asymptotically track the reference signal y_R. We assume that y_R, $y_R^{(1)}, \cdots, y_R^{(r-1)}$ are bounded and continuous and $y_R^{(r)}$ is bounded and piecewise continuous.

Set $Y_R = (y_R, y_R^{(1)}, \cdots, y_R^{(r-1)})$ and $e = x - Y_R$. The system (4.32) can be written in the error coordinates as

$$\dot{\eta} = f_0(\eta, e + Y_R)$$

$$\begin{aligned} \dot{e} &= A_c e + B_c \left\{ \frac{1}{b(x)}[u - a(x)] - y_R^{(r)} \right\} \stackrel{\text{def}}{=} g(e + Y_R, u, t) \\ e_1 &= y - y_R \end{aligned} \tag{4.35}$$

The objective is to make e approach zero asymptotically. Consider the state feedback $u^\star = a(e + Y_R) + y_R^{(r)} b(e + Y_R) + Ke$ where K is such that $A_c + B_c K$ is Hurwitz. Then, using Theorem 5.2 of [28], we conclude that

$$\|e(t)\| \leq \gamma_1 e^{-\gamma_2 t} \|e(0)\| \tag{4.36}$$

for all $t \geq 0$, for some positive constants γ_1 and γ_2.

Let $r_0 = \sup_{t \geq 0} \|Y_R(t)\|$, $\rho_0(.) = \alpha_3^{-1}(2\alpha_4(.))$, and $c = \alpha_2(\rho_0(2r_0))$. Notice that ρ_0 thus defined is a class $\mathcal{K}_\infty$ function. Since $V_1(\eta)$ is continuous and radially unbounded, the set

$$\bar{A} = \{\eta : V_1(\eta) \leq c\}$$

is a compact subset of R^{n-r}. Thus, we conclude that the set $A = \{e = 0\} \times \bar{A}$ is a compact subset of R^n.

We recall from Exercise 3.33 of [28] that a class $\mathcal{K}$ function α, defined on $[a, 0)$, satisfies

$$\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2).$$

Given this result and (4.34), we can write

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, e + Y_R) \leq -\frac{1}{2}\alpha_3(\|\eta\|) + \alpha_4(2\|e\|), \text{ for } V_1(\eta) \geq c \tag{4.37}$$

for all $t \geq 0$. Inequality (4.37) can be written as

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, e + Y_R) \leq -\frac{1}{2}\alpha_3(\|\eta\|) + \alpha_4(2\|e\|), \ \forall\ \eta \notin \bar{A} \tag{4.38}$$

Now, define the function

$$\lambda(s) = \begin{cases} [\ln(1+s-c)]^2 & \text{for} \quad s \geq c \\ 0 & \text{for} \quad s \in [0, c] \end{cases} \tag{4.39}$$

The function $\lambda(.)$ is a C^1 function on $[0, \infty)$. It is also nonnegative and strictly increasing for $s \geq c$. Furthermore, using L'Hopital's rule of differentiation (to show $\lambda'(s) \to 0$ as $s \to \infty$), we have

$$\lambda'(s) \leq k, \ k > 0 \tag{4.40}$$

for all $s \geq 0$.

Consider the Lyapunov function candidate $W(\eta) = \lambda(V_1(\eta))$. The following is a technical result needed in the forthcoming analysis:

Claim 1: The function $W(\eta)$ is C^1 and radially unbounded. Furthermore, there exist three class $\mathcal{K}_\infty$ functions δ_1, δ_2, and δ_3 such that $W(\eta)$ and $\frac{1}{2}\frac{d\lambda}{ds}(V_1(\eta))\alpha_3(\|\eta\|)$ can be bounded as follows

$$\begin{aligned} \delta_1(|\eta|_{\bar{A}}) \ &\leq\ W(\eta)\ \leq\ \delta_2(|\eta|_{\bar{A}}) \qquad (4.41) \\ \frac{1}{2}\frac{d\lambda}{ds}(V_1(\eta))\alpha_3(\|\eta\|) \ &\geq\ \delta_3(|\eta|_{\bar{A}}) \qquad (4.42) \end{aligned}$$

for all $\eta \in R^{n-r}$.

Proof: The continuous differentiability of W follows from that of V_1 and λ. As for

(4.41) and (4.42), they follow from Lemma A.3.◁

Given the results of Claim 1, we can write

$$\frac{\partial W}{\partial \eta} f_0(\eta, e + Y_R) \leq -\delta_3(|\eta|_{\bar{A}}) + k\alpha_4(2\|e\|), \ \forall \ \eta \notin \bar{A} \tag{4.43}$$

where k is defined by (4.40). Let $\rho_1(s) = \delta_2(\delta_3^{-1}(2k\alpha_4(2s)))$. Then, we can write

$$\frac{\partial W}{\partial \eta} f_0(\eta, e + Y_R) \leq -\frac{1}{2}\delta_3(|\eta|_{\bar{A}}), \ \forall \ \eta \notin \bar{A}, \text{ and } W(\eta) \geq \rho_1(\|e\|) \tag{4.44}$$

for all $t \geq 0$. Now, according to [48, Lemma 2.6], there exist a class $\mathcal{KL}$ function β_1 and a class $\mathcal{K}$ function γ such that, for every $\eta(0) \in R^{n-r}$, we have

$$|\eta(t)|_{\bar{A}} \leq \beta_1(|\eta(0)|_{\bar{A}}, t) + \gamma(\sup_{0 \leq \tau \leq t} \|e(\tau)\|) \tag{4.45}$$

By Lemma A.2 we conclude that there exists a class $\mathcal{KL}$ function $\beta(.,.)$ such that

$$|(e(t), \eta(t))|_A \leq \beta(|(e(0), \eta(0))|_A, t), \ \forall t \geq 0 \tag{4.46}$$

Thus, using Proposition 2.5 of [34], we conclude that the closed-loop system is asymptotically stable with respect to the compact set A.

The separation results apply to the system

$$\begin{aligned} \dot{\eta} &= f_0(\eta, e + Y_R) \\ \dot{e} &= A_c e + B_c \left\{ \frac{1}{b(x)}[u - a(x)] - y_R^{(r)} \right\} \stackrel{\text{def}}{=} g(e + Y_R, u, t) \\ e_1 &= y - y_R \end{aligned} \tag{4.47}$$

This systems fits into the model (3.11)–(3.14) with the vector of bounded exogenous signals being $(Y_R(t), y_R^{(r)}(t))$.

The state feedback controller considered is $u = u^\star$. It achieves global asymptotic tracking uniformly in η. Global boundedness of the control law is achieved by saturation outside a region of interest.

We showed that the system (4.47) with the controller $u = u^\star$ is globally asymptotically stable with respect to the compact positively invariant set A.

To implement the controller we use a nonlinear high-gain observer of the form (3.17). Since we know the nonlinearities $a(x)$ and $b(x)$, we use them as the nominal models.

Theorems 3.4 and 3.5 guarantee that the trajectories of the closed-loop system under output feedback controller starting in $\mathcal{S} \times \mathcal{Q}$, where $\mathcal{S}$ is a compact subset of R^n and $\mathcal{Q}$ is a compact set in R^r, are bounded and will come arbitrarily close to the set $A \times \{e_f = 0\}$ where e_f is the estimation error. Furthermore, Theorem 3.7 (no modeling error) guarantees local asymptotic stability of the compact positively invariant set $A \times \{e = 0\}$.

Finally, Theorem 3.6 shows that the trajectories under output feedback control converge to those under state feedback control as the observer gain approaches infinity.

4.5 Servomechanism [26, 37, 38]

The forthcoming example presents regional results on servomechanism. It is based mainly on [38] which is a generalization of [37]. Both of these papers are continuation of the work that was started in [26].

We consider a SISO system represented by the n-th order differential equation

$$
\begin{aligned}
y^{(n)} &= \bar{f}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(m-1)}, \omega(t)) \\
&+ \bar{g}(y, \cdots, y^{(n-1)}, u, \cdots, u^{(m-1)}, \omega(t)) u^{(m)}
\end{aligned}
\tag{4.48}
$$

where u is the control input, y is the measured output, $u^{(i)}$ and $y^{(i)}$ denote the i-th derivatives of u and y, respectively, and $\omega(t)$ is a continuous time-varying disturbance signal which is assumed to belong to a compact set $D \subset R^p$. The functions $\bar{f}$ and $\bar{g}$ are (sufficiently) smooth nonlinearities with $\bar{g}(.) \neq 0$ for all values of its arguments in the domain of interest $U \times D \subseteq R^{n+m} \times R^p$. Let $r(t)$ be a time-varying reference signal and assume that $\nu(t) = \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}$ is generated by the p_1-dimensional exosystem

$$
\dot{\nu}(t) = S_0 \nu(t) \tag{4.49}
$$

where S_0 has distinct eigenvalues on the imaginary axis. Clearly, $\nu(t)$ belongs to a compact set $D_1 \subset R^{p_1}$, $\forall t \geq 0$.

We augment the system with a series of integrators at the input side and view $v = u^{(m-1)}$ as the control input of the extended system. Taking

$$
\begin{aligned}
x_i &= y^{(i-1)} - r^{(i-1)}, \ i = 1, \cdots, n \\
\zeta_i &= u^{(i-1)}, \ i = 1, \cdots, m
\end{aligned}
$$

as state variables, the extended system can be represented as

$$
\begin{aligned}
\dot{x} &= Ax + B[f(x, \zeta, \nu(t)) + g(x, \zeta, \nu(t))v] \\
\dot{\zeta} &= A_2 \zeta + B_2 v \\
e &= Cx
\end{aligned}
\tag{4.50}
$$

where $e = y - r$ is the tracking error, and (A, B, C) and (A_2, B_2) are chains of integrators.

The following assumption ensures the existence of a change of variables such that (4.50) can be written in the normal form

$$\begin{aligned} \dot{x} &= Ax + B[f(x,\zeta,\nu) + g(x,\zeta,\nu)v] \\ \dot{z} &= \psi_0(x,z,\nu) \\ e &= Cx \end{aligned} \tag{4.51}$$

Set $Y = (y, y^{(1)}, \cdots, y^{(n-1)})$.

Assumption 1 [38, Assumption 1]: There exists a diffeomorphism

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x \\ T_1(x,\zeta,\nu) \end{bmatrix} \overset{\text{def}}{=} T(x,\zeta,\nu) = \Psi_\nu(Y,\zeta) \tag{4.52}$$

that maps (x,ζ) into (x,z) for all $\nu \in D_1$ and transforms the last m state equations of (4.50) into

$$\dot{z} = \psi_0(x,z,\nu(t))$$

which, together with the first n state equations of (4.50), define a normal form which we assume to be defined in the domain $\mathcal{D} \overset{\text{def}}{=} T(U \times D) \subseteq R^{n+m}$.

To specify the domains of validity of the above transformation, we assume ([38, Assumption 3]) that there exists a domain $N = N_1 \times N_2 \subseteq R^n \times R^m$, that contains the origin, such that $\Psi_\nu^{-1}(N) \subseteq U$ for all $\nu \in D_1$.

The objective is to design a state feedback controller that makes the output y asymptotically track the reference signal r. We start by identifying the internal model of the system, then we design a dynamic state feedback controller that incorporates this model.

Of particular interest to us are the equations of motion on the zero-error manifold (where $e = 0$ which implies $x = 0$). They are

$$f(0, \zeta(t), \nu(t)) + g(0, \zeta(t), \nu(t))v = 0 \tag{4.53}$$

$$\dot{\zeta} = A_2\zeta + B_2 \left[\frac{-f(0, \zeta, \nu(t))}{g(0, \zeta, \nu(t))} \right] \tag{4.54}$$

In the forthcoming development we will use a state feedback controller of the form

$$v = g_0^{-1}(x, \zeta, \nu(t))[-f_0(x, \zeta, \nu(t)) + \tilde{v}]$$

where $g_0(.)$ and $f_0(.)$ are nominal models of $f(.)$ and $g(.)$, respectively.

The following assumption identifies the internal model:

Assumption 2 [38, Assumption 2, Remark 3]: (1) There exists a unique mapping $\zeta = \lambda_0(\nu)$ *which solves the partial differential equation*

$$\frac{\partial \lambda_0}{\partial \nu} S_0 \nu = A_2 \lambda_0(\nu) + B_2 \left[\frac{-f(0, \zeta, \nu(t))}{g(0, \zeta, \nu(t))} \right] \tag{4.55}$$

(2) There exist q constants $d_1, \cdots, d_q$ such that, on the zero-error manifold, the control component $\tilde{v}$ is

$$c(\nu) \stackrel{\text{def}}{=} -g_0(0, \lambda_0(\nu), \nu) g^{-1}(0, \lambda_0(\nu), \nu) f(0, \lambda_0(\nu), \nu) + f_0(0, \lambda_0(\nu), \nu)$$

and satisfies

$$L_S^q c(\nu) = d_1 c(\nu) + d_2 L_S c(\nu) + \cdots + d_q L_S^{q-1} c(\nu) \tag{4.56}$$

for all $\nu \in D_1$, *where* $L_S c = \frac{\partial c}{\partial \nu} S_0 \nu$. *Moreover, the polynomial equation*

$$s^q - d_q s^{q-1} - \cdots - d_2 s - d_1 = 0$$

has distinct roots on the imaginary axis.

A routine calculation shows that there exist a $q \times q$ matrix S, $q \geq p_1$, and a $1 \times q$ constant matrix Γ such that

$$\frac{\partial \mathcal{V}(\nu)}{\partial \nu} = S\mathcal{V}(\nu) \tag{4.57}$$

$$c(\nu) = \Gamma \mathcal{V}(\nu) \tag{4.58}$$

for all $\nu \in D_1$. In fact this happens for

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ d_1 & d_2 & d_3 & \cdots & d_q \end{pmatrix}$$

$$\mathcal{V}(\nu) = \begin{pmatrix} c(\nu) \\ L_S c(\nu) \\ \cdots \\ L_S^{q-2} c(\nu) \\ L_S^{q-1} c(\nu) \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Assumption 2 means that S has distinct eigenvalues on the imaginary axis.

Remark 4.6 *If on the zero-error manifold we have* $\zeta = \lambda_0(\nu)$*, then* $\lambda(\nu) \stackrel{\text{def}}{=} T_1(0, \lambda_0, \nu)$ *is a well defined function of* ν*.*

Now, we proceed with the design of the dynamic state feedback controller. First, we need a minimum-phase assumption. Before we state such an assumption, we give the dynamics satisfied by x and $\tilde{z} = z - \lambda(\nu)$:

$$\begin{aligned} \dot{x} &= Ax + B[f(x, \zeta, \nu) + g(x, \zeta, \nu)v] \\ \dot{\tilde{z}} &= \phi_0(x, \tilde{z}, \nu) \end{aligned} \tag{4.59}$$

where $\phi_0(.) = \psi(x, \tilde{z} + \lambda(\nu), \nu) - \frac{\partial \lambda}{\partial \nu} S_0 \nu$.

Assumption 3 [38, Assumption 4]: There exists a C^1 *function* W *defined on* R^m*, and four class* $\mathcal{K}$ *functions* $\alpha_1, \alpha_2, \phi_1$*, and* α*, such that*

$$\begin{aligned} \alpha_1(\|\tilde{z}\|) &\leq W(\tilde{z}) \leq \alpha_2(\|x\|) \\ \frac{\partial W}{\partial \tilde{z}} \phi_0(x, \tilde{z}, \nu) &\leq -\phi_1(\|\tilde{z}\|), \ \forall \|\tilde{z}\| \geq \alpha(\|\tilde{z}\|) \end{aligned}$$

for all $(x, z, \nu) \in N_1 \times N_2 \times D_1$.

Set $\Omega_{a_2} \stackrel{\text{def}}{=} \{z : W(\tilde{z}) \leq a_2\}$, $a_2 > 0$.

Next, we consider a dynamic state feedback of the form

$$\dot{\sigma} = S\sigma + Je \tag{4.60}$$

$$v = \varphi(x, \sigma, \zeta, \nu) \tag{4.61}$$

where $\varphi(x, \sigma, \zeta, \nu)$ is locally Lipschitz in (x, σ) uniformly in (ζ, ν). Then, [38, Assumptions 6,7,8, Lemma 1] ensure that the controller achieves convergence to the zero-error manifold $\mathcal{A} = \{x = 0\} \times \{\sigma = L\mathcal{V}(\nu), z = \lambda(\nu)\}$ for some constant matrix L. This manifold is compact, invariant, and asymptotically attractive uniformly in $\nu \in D_1$. Hereafter, for the sake of completeness, we state these assumptions and lemma.

The control strategy consists of two steps. First, the controller ensures ultimate boundedness of trajectories. Then, it achieves asymptotic convergence to an equilibrium point.

To state the next assumption let $\xi = \begin{bmatrix} \sigma \\ x \end{bmatrix}$ and $\dot{\xi} = h_1(\xi, \zeta, \nu)$, i.e., part of the dynamics of the closed-loop system.

Assumption 4 [38, Assumption 6]: There exists a C^1 function $V : R^{n+q} \to R_+$ which satisfies

$$\begin{aligned} \beta_1(\|\xi\|) &\leq V(\xi) \leq \beta_2(\|\xi\|) \\ \frac{\partial V}{\partial \xi} h_1(\xi, \zeta, \nu) &\leq -\phi_2(\|\xi\|), \ \forall V(\xi) \geq \beta(\mu) \end{aligned}$$

for all $\xi \in X_1$, uniformly in (ζ, ν) for all $(\zeta, \nu) \in U_2 \times D_1$, where β, β_1, β_2 and ϕ_2 are class $\mathcal{K}$ functions and $\mu > 0$ is a design parameter.

Define $X_1 = R^q \times N_1$ and $\Omega_{a_1} = \{\xi : V(\xi) \leq a_1\}$ where $a_1 > 0$ is chosen such that $\Omega_{a_1} \subseteq X_1$. It can be shown that the set $\Omega_{a_1} \times \Omega_{a_2}$ is positively invariant for some $a_2 > 0$. Assumption 4 along with Assumption 2 imply that the trajectory $(\xi(t), z(t))$, starting in $\Omega_{a_1} \times \Omega_{a_2}$ will, for all $\nu \in D_1$, eventually enter the positively invariant set $\mathcal{R}_\mu \stackrel{\text{def}}{=} \Lambda_\mu \times \Gamma_\mu$ where $\Lambda_\mu \stackrel{\text{def}}{=} \{\xi \in X_1 : V(\xi) \leq \beta(\mu)\}$ and $\Gamma_\mu \stackrel{\text{def}}{=} \{z : W(\tilde{z}) \leq \gamma(\mu)\}$ for some class $\mathcal{K}$ function γ. The set $\mathcal{R}_\mu$ is a neighborhood of $(\xi, z) = 0$ whose size can be made arbitrarily small by choosing μ small enough.

Assumption 5 [38, Assumption 7]: There exists a compact, positively invariant set $S_\mu \subseteq \Lambda_\mu$ such that $\xi(t)$ enters S_μ in finite time. Furthermore, inside S_μ, the control component $\tilde{v}$ takes the form

$$\tilde{v} = K_0\xi + f_2(\zeta, \nu)$$

where f_2 satisfies $f_2(\lambda_0(\nu), \nu) = L_2\mathcal{V}(\nu)$.

Lemma 4.1 *[38, Lemma 1]: Suppose that Assumptions 2 and 5 hold. Then, there exists a $q \times q$ matrix L such*

$$\Gamma = -(K_{01}L + L_2), \quad LS = SL$$

Furthermore, the set $\{\sigma = L\mathcal{V}(\nu),\ x = 0,\ \zeta = \lambda_0(\nu)\}$ is an integral manifold of the closed-loop system (4.50) and (4.60)–(4.61).

Now, we need to design a state feedback controller (choose K_0 and f_2) such that the zero-error manifold $\{\sigma = L\mathcal{V}(\nu),\ x = 0,\ \zeta = \lambda_0(\nu)\}$ is regionally attractive. To study this attractiveness, we set $\tilde{\sigma} = \sigma - L\mathcal{V}(\nu)$ and $\eta = \begin{bmatrix} \tilde{\sigma} \\ x \\ \tilde{z} \end{bmatrix}$. With this change of variables, the zero-error manifold reduces to the origin $\eta = 0$ which is an equilibrium point of the closed-loop system (4.59) and (4.60)–(4.61). The next assumption is needed to show attractiveness of the origin $\eta = 0$:

Assumption 6 [38, Assumption 8]: The origin of the closed-loop system $\dot{\eta} = \tilde{h}_1(\eta, \nu)$ is locally asymptotically stable.

The theory developed in the previous chapter can be applied to the autonomous system

$$\begin{aligned}
\dot{x} &= Ax + B[f(x, \zeta, \nu) + g(x, \zeta, \nu)v] \\
\dot{\zeta} &= A_2\zeta + B_2 v \\
\dot{\nu} &= S_0\nu \\
e &= Cx
\end{aligned} \tag{4.62}$$

This system fits the model (3.11)–(3.14) with ζ_i's, being the additional measured outputs ζ. The smoothness of f and g inherited from $\bar{f}$ and $\bar{g}$ implies a local Lipschitz property in (x, ζ) uniformly in $\nu \in D_1$.

We consider the state feedback controller

$$\begin{aligned}
\dot{\sigma} &= S\sigma + Je \\
v &= \varphi(x, \sigma, \zeta, \nu)
\end{aligned}$$

Global boundedness of the control law is achieved by saturation outside a region of interest.

We have shown that the system with this controller is asymptotically stable with respect to the zero error manifold $\mathcal{A} \stackrel{\text{def}}{=} \{\sigma = L\mathcal{V}(\nu),\ x = 0,\ \zeta = \lambda_0(\nu)\}$ with $\Omega_{a_1} \times \Omega_{a_2}$ being an estimate of the region of attraction. Lemma 4.1 shows that $\mathcal{A}$ is a center manifold for the system (4.62) with the controller. Since $\nu \in D_1$, then $\mathcal{A}$ is a compact subset of R^{n+2q+m}.

To implement the controller we use a linear high-gain observer to estimate the tracking error and its derivatives (let the estimate be $\hat{x}$). Let χ be the scaled estimation error. Then, the closed-loop system under the output feedback controller can be written compactly as

$$
\begin{aligned}
\dot{\eta} &= \tilde{h}_3(\eta, \chi, \mathcal{V}) \\
\epsilon\dot{\chi} &= A_1\chi + \epsilon B[f_1(x, \tilde{z}, \nu) + g_1(x, \tilde{z}, \nu)\varphi(\sigma, \hat{x}, \tilde{z}, \nu)] \\
&= A_1\chi + \epsilon\tilde{h}_2(\eta, \chi, \mathcal{V}) \qquad (4.63)
\end{aligned}
$$

Theorem 3.11 guarantees that the trajectories of the closed-loop system under output feedback control (starting in the compact set $\Omega_{a_1} \times \Omega_{a_2} \times \mathcal{Q}$, where $\mathcal{Q}$ is a compact subset of R^n) are bounded and come arbitrarily close to the set $\mathcal{A} \times \{x - \hat{x} = 0\}$. Moreover, Theorem 3.11 shows that the trajectories under output feedback control converge to those under state feedback control as the observer gain approaches infinity.

To establish asymptotic convergence of the closed-loop system (4.63) to the equilibrium point $(\eta, \chi) = (0, 0)$ some conditions on the nonlinearities have to be imposed. These conditions are given in the following assumption:

Assumption 7 [38, Assumption 9]: There exists a C^1 function $\tilde{V}(\eta)$: $R^{n+q+m} \to R_+$ and a continuous positive definite function ϕ_3 such that

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial \eta}\tilde{h}_3(\eta, 0, \nu) &\leq -q_0\phi_3(\eta), \; q_0 > 0 \qquad (4.64) \\
\|\tilde{h}_2(\eta, 0, \mathcal{V})\| &\leq q_1\phi_3^a(\eta), \; q_1 \geq 0 \qquad (4.65) \\
\frac{\partial \tilde{V}}{\partial \eta}[\tilde{h}_3(\eta, \chi, \nu) - \tilde{h}_3(\eta, 0, \nu)] &\leq q_2\phi_3^b(\eta)\|\chi\|^c \qquad (4.66)
\end{aligned}
$$

$0 < a \leq 1/2$, $0 < b < 1$, $c = \frac{1-b}{a}$ *and* $q_2 \geq 0$, *for all* $(\xi, z, \chi, \nu) \in S_\mu \times \Gamma_\mu \times \Omega_\epsilon \times D_1$.

Note that inequality (4.64) of Assumption 7 follows from Assumption 6. Assumption 7 provides a Lyapunov function for which Assumption 3.4 is satisfied. Both $\tilde{V}$ and ϕ_3 are positive definite with respect to $\mathcal{A}$. Thus, by Lemma A.3 the bound (3.57) is satisfied. Inequalities (4.64)–(4.65) are similar to (3.58)–(3.59). Finally, inequality (4.66) is a variation of (3.60) and yields the same results. Then, according to Theorem 3.11, the system (4.63) is asymptotically stable with respect to the set $\mathcal{A} \times \{x - \hat{x} = 0\}$.

Remark 4.7 *Notice from (4.65) that the system's nonlinearity $\tilde{h}_2(\eta, 0, \mathcal{V})$ has to be zero on the zero-error manifold. Therefore, it is very convenient to choose the nominal value of $\tilde{h}_2(\eta, 0, \mathcal{V})$ to be zero (which results in a linear observer), because otherwise we have to know the manifold and we have to choose a nominal value which is zero on this manifold.*

4.6 Servomechanism [21]

The servomechanism example considered herein deals with a system that has a triangular structure as opposed to the system considered in the previous example which has a double chain of integrators. Moreover, no assumption of complete observability (as defined in Section 4.3) is given, thus we can not proceed, by repeated differentiation of the output, to find the input-output model and continue the design process as in the previous example (complete observability is needed for the existence of an input-output model as shown in Section 4.3). Even if the system is completely observable, the design presented in this section will, in general, be different from the one presented in the previous section because the nonlinearities in the input-output model could be different from the nonlinearities in the state model due to partial differentiation while deriving the input-output model.

Consider the system

$$\begin{aligned}\dot{z} &= Z(\mu)z + p_0(x_1, \omega, \mu)\\ \dot{x} &= Fx + Gu + P(z, x, \omega, \mu)\\ e &= Hx \end{aligned} \tag{4.67}$$

where

$$\begin{aligned} F &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; G = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \\ P(z, x, \omega, \mu) &= \begin{pmatrix} p_1(z, x_1, \omega, \mu) \\ p_2(z, x_1, x_2, \omega, \mu) \\ \cdots \\ p_{r-1}(z, x_1, x_2, \cdots, x_{r-1}, \omega, \mu) \\ p_r(z, x_1, x_2, \cdots, x_r, \omega, \mu) \end{pmatrix} \end{aligned}$$

with state $x \in R^n$, control input $u \in R^m$, and regulated output $e \in R^m$. The system (4.67) is subject to an exogenous input $\omega \in R^d$ in which $\mu \in \mathcal{P} \in R^p$ is a vector of unknown parameters and $\mathcal{P}$ is compact set, $p_0(.)$ and $P(.)$ are C^k functions of their arguments (for some large k), and $p_0(0, 0, \mu) = 0$, $P(0, 0, 0, \mu) = 0$. Without loss of generality we assume $0 \in \text{int}(\mathcal{P})$. The exosystem

$$\dot{\omega} = S\omega \tag{4.68}$$

is assumed to be neutrally stable (the matrix S has distinct eigenvalues on the imaginary axis).

Assumption A: The eigenvalues of $Z(\mu)$ have negative real part, for all $\mu \in \mathcal{P}$. Moreover, the equation

$$\frac{\partial\zeta(\omega,\mu)}{\partial\omega}S\omega = Z(\mu)\zeta(\omega,\mu) + p_0(0,\omega,\mu) \tag{4.69}$$

has a solution $\zeta(\omega,\mu)$ defined for all ω,μ.

Given Assumption A and the structure of F, G, H and $P(.)$, a routine calculation shows that the system

$$\begin{aligned}
\frac{\partial(\zeta,\pi^a)(\omega,\mu)}{\partial\omega}S\omega &= \begin{pmatrix} Z(\mu)\zeta(\omega,\mu) + p_0(H\pi^a(\omega,\mu),\omega,\mu) \\ F\pi^a(\omega,\mu) + Gc^a(\omega,\mu) + P(\zeta(\omega,\mu),\pi^a(\omega,\mu),\omega,\mu) \end{pmatrix} \\
0 &= H\pi^a(\omega,\mu)
\end{aligned} \tag{4.70}$$

has a unique and globally defined solution $\pi^a(\omega,\mu)$, $c^a(\omega,\mu)$ such that $\pi^a(0,\mu) = 0$ and $c^a(0,\mu) = 0$ for all μ. Hereafter, it is assumed that the function $c^a(\omega,\mu)$ thus determined satisfies the following:

Assumption B: For some set of real numbers $a_0, a_1, \cdots, a_{q-1}$, the identity

$$L_S^q c^a(\omega,\mu) = a_0 c^a(\omega,\mu) + a_1 L_S c^a(\omega,\mu) + \cdots + a_{q-1} L_S^{q-1} c^a(\omega,\mu) \tag{4.71}$$

holds for all ω, μ, where $L_S = \frac{\partial}{\partial\omega}S\omega$. Moreover, the polynomial equation

$$s^q - a_{q-1}s^{q-1} - \cdots - a_1 s - a_0 = 0$$

has distinct roots on the imaginary axis.

Simple routine calculations show that, under Assumption B, there exist a $q \times q$ matrix Φ, a $1 \times q$ row vector Γ, and a globally defined mapping $\tau^a(\omega, \mu)$ such that

$$\begin{aligned} \frac{\partial \tau^a(\omega, \mu)}{\partial \omega} &= \Phi \tau^a(\omega, \mu) \\ c^a(\omega, \mu) &= \Gamma \tau^a(\omega, \mu) \end{aligned} \tag{4.72}$$

In fact, this happens for

$$\begin{aligned} \Phi &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{q-1} \end{pmatrix} \\ \tau^a(\omega, \mu) &= \begin{pmatrix} c^a(\omega, \mu) \\ L_s c^a(\omega, \mu) \\ \cdots \\ L_s^{q-2} c^a(\omega, \mu) \\ L_s^{q-1} c^a(\omega, \mu) \end{pmatrix} \\ \Gamma &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

By Theorem 1 in [21] and using Assumptions A and B we conclude that the problem of structurally stable output regulation for the system (4.67) is solvable.

Hereafter, we propose a feedback law for which we prove the existence of an attractive zero-error invariant manifold. Furthermore, this manifold can be made semiglobally attractive.

Set $\tilde{z} = z - \zeta(\omega, \mu)$, $\tilde{x} = x - \pi^a(\omega, \mu)$, and

$$\eta = \begin{pmatrix} e \\ e^{(1)} \\ \vdots \\ e^{(r-1)} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 + \tilde{p}_1(\tilde{z}, \tilde{x}_1, \omega, \mu) \\ \vdots \\ \tilde{x}_r + \tilde{p}_{r-1}(\tilde{z}, \tilde{x}_1, \cdots, \tilde{x}_{r-1}, \omega, \mu) \end{pmatrix} \tag{4.73}$$

Consider now a feedback law of the form

$$\begin{aligned} \dot{\xi}_1 &= \Phi\xi_1 + Ne \\ u &= M\eta + T\xi_1 \end{aligned} \tag{4.74}$$

Then, it is possible to prove the following property:

Proposition 1 : Suppose Assumptions A and B hold. Suppose that (4.74) asymptotically stabilizes the linear approximation of (4.67) at the equilibrium point $(\xi_1, z, x) = (0, 0, 0)$, $(\omega, \mu) = (0, 0)$. *Then, there exists a* $q \times q$ *matrix* Π *satisfying*

$$\Phi\Pi = \Pi\Phi, \; T\Pi = \Gamma \tag{4.75}$$

where Φ *and* Γ *are defined as in (4.72). As a consequence, the closed-loop system*

$$\begin{aligned} \dot{\xi}_1 &= \Phi\xi_1 + NHx \\ \dot{z} &= Z(\mu)z + p_0(x_1, \omega, \mu) \\ \dot{x} &= Fx + G(M\eta + T\xi_1) + P(z, x, \omega, \mu) \\ \dot{\omega} &= S\omega \end{aligned} \tag{4.76}$$

has a globally defined center manifold

$$\mathcal{M}_C = \{(\xi_1, z, x, \omega) : \xi_1 = \Pi\tau^a(\omega, \mu),\ z = \zeta(\omega, \mu),\ x = \pi^a(\omega, \mu)\} \tag{4.77}$$

at $(\xi_1, z, x, \omega) = (0, 0, 0, 0)$.

Proof: similar to that of Proposition 1 in [21], which applies to the output feedback case.◁

Now, let us design the state feedback controller that makes $\mathcal{M}_C$ semiglobally attractive. The issue here is to choose N, T and M such that this goal is achieved. Let $\tilde{\xi_1} = \xi_1 - \Pi\tau^a(\omega, \mu)$. Then, in the coordinates $(\tilde{z}, \tilde{x}, \tilde{\xi_1})$, the closed-loop system becomes

$$\begin{aligned}
\dot{\tilde{\xi_1}} &= \Phi\tilde{\xi_1} + NH\tilde{x} \\
\dot{\tilde{z}} &= Z(\mu)\tilde{z} + \tilde{p}_0(H\tilde{x}, \exp(St)\omega^0, \mu) \\
\dot{\tilde{x}} &= F\tilde{x} + G(M\eta + T\tilde{\xi_1}) + \tilde{P}(z, x, \exp(St)\omega^0, \mu)
\end{aligned} \tag{4.78}$$

where ω^0 represents the value at time $t = 0$ of the state of the exosystem. System (4.78) is an uncertain system because the actual values of μ and ω^0 are unknown. We assume that the initial value ω^0 belongs to an *a priori* known compact set $\mathcal{W} \in R^d$. The invariant manifold reduces to the origin $(\tilde{\xi_1}, \tilde{z}, \tilde{x}) = (0, 0, 0)$ where the regulation error $e = \tilde{x}_1$ is zero. Thus, output regulation is achieved if the origin is attractive.

In order to be able to use the separation results of Chapter 3, Assumption C of [21] has to be modified as follows:

Assumption C: There exists a positive definite smooth function $V(\tilde{z})$ satisfying

$$\alpha_1\|\tilde{z}\|^2 \leq V(\tilde{z}) \leq \alpha_2\|\tilde{z}\|^2 \tag{4.79}$$

$$\frac{\partial V}{\partial \tilde{z}}(Z(\mu)\tilde{z} + \tilde{p}_0(H\tilde{x}, \exp(St)\omega^0, \mu)) \leq -\alpha_3\|\tilde{z}\|^2 + c|H\tilde{x}|^2 \tag{4.80}$$

for all $\tilde{z}$, $\tilde{x}, t$ and all $(\omega^0, \mu) \in \mathcal{W} \times \mathcal{P}$, where α_i's are positive constants and $c > 0$.

For N choose any matrix such that the pair (Φ, N) is controllable. Then, given any compact set $\mathcal{S}$ of initial conditions $(\tilde{\xi}_1(0), \tilde{z}(0), \tilde{x}(0)) \in R^q \times R^{n-r} \times R^r$, find (via backstepping methods and high-gain feedback, for example) a pair of matrices M and T such that the origin is locally exponentially stable with a basin of attraction that includes the set $\mathcal{S}$.

In order to apply our separation results, we consider the system

$$\begin{aligned}
\dot{z} &= Z(\mu)z + p_0(x_1, \omega, \mu) \\
\dot{\eta} &= A\eta + G(u + \tilde{p}_r(z, \eta, \omega, \mu)) \\
\dot{\omega} &= S\omega \\
e &= H\eta
\end{aligned}$$

This system fits the model (3.11)–(3.14) with μ being the vector of bounded disturbances (constant in this case, thus it belongs to $\mathcal{M}'_{\mathcal{D}}$).

We consider the state feedback controller.

$$\begin{aligned}
\dot{\xi}_1 &= \Phi\xi_1 + Ne \\
u &= M\eta + T\xi_1
\end{aligned}$$

This controller achieves semiglobal tracking uniformly in ω and μ. Global boundedness is achieved by saturation outside a region of interest.

We showed, by construction, that the system with the controller is exponentially stable with respect to the compact positively invariant zero-error manifold $\mathcal{M}_C$ with $\mathcal{S}$ being an estimate of the region of attraction.

To implement the controller we use a linear high-gain observer. Boundedness, ultimate boundedness, and convergence of trajectories under the output feedback controller (starting in $\mathcal{S} \times \mathcal{Q}$, where $\mathcal{Q}$ is a compact subset of R^n) are guaranteed by Theorem 3.11. Moreover, Theorem 3.11 guarantees exponential stability with respect to the compact positively invariant set $\mathcal{M}_C \times \{\eta - \xi_0 = 0\}$, where ξ_0 is the estimate of η.

4.7 Adaptive Control [27, 2, 1]

We consider the system represented globally by the n-th order differential equation

$$y^{(n)} = f_0(\cdot) + \Sigma_{i=1}^{p} f_i(\cdot)\theta_i + [g_0(\cdot) + \Sigma_{i=1}^{p} g_i(\cdot)\theta_i]u^{(m)} \tag{4.81}$$

where u is the control input, y is the measured output, $y^{(i)}$ denotes the i-th derivative of y, and $m < n$. The functions f_i and g_i are known smooth nonlinearities which may depend on the output, the input, and their derivatives up to the $(n-1)$-th order and $(m-1)$-th order, respectively. The constant parameters are unknown, but the vector $\theta = [\theta_1, \cdots, \theta_p]^T$ belongs to Ω, a known compact convex subset of R^p.

The objective is to design a state feedback controller that renders all the signals bounded and makes the output asymptotically track the bounded reference signal $y_r(t)$. We assume that all derivatives of y_r up to the n-th order are bounded and that $y_r^{(n)}(t)$ is piecewise continuous.

Let $x_i = y^{(i-1)}$, $e_i = x_i - y_r^{(i-1)}$ for $i = 1, \cdots, p$; $z_i = u^{(i-1)}$, $i = 1, \cdots, m$, and $v = u^{(m-1)}$. Let $\mathcal{Y}_r = [y_r(t), \cdots, y_r^{(n-1)}(t)]^T$ and $\mathcal{Y}_R = [\mathcal{Y}_r, y^{(n)}(t)]^T$. To ensure that the system is input/output linearizable we make the following assumption:

Assumption 1: $|g_0(x, z) + \theta^T g(x, z)| \geq k > 0$, $\forall x \in R^n$, $z \in R^m$ *and* $\theta \in \Omega_1$, *where* Ω_1 *is a compact set that contains* Ω *in its interior.*

We assume that there is a global change of variables $\zeta = T_1(y, \cdots, y^{(n-1)}, u, \cdots, u^{(m-1)})$ such that the system (4.81) can be written

$$\begin{aligned} \dot{e} &= A_m e + b\{Ke + f_0(e + \mathcal{Y}_r, z) + \theta^T f(e + \mathcal{Y}_r, z) \\ &\quad + [g_0(e + \mathcal{Y}_r, z) + \theta^T g(e + \mathcal{Y}_r, z)]v - y_r^{(n)}\} \qquad (4.82) \\ \dot{\zeta} &= F(\zeta, e + \mathcal{Y}_r, \theta) \qquad (4.83) \end{aligned}$$

where $A_m = A - bK$, (A, b) is a chain of integrators, and K is such that A_m is Hurwitz.

Before designing the controller, we have to make a crucial minimum-phase assumption:

Assumption 2: The system $\dot{\zeta} = F(\zeta, \mathcal{Y}_r, \theta)$ *has a unique bounded steady-state solution* $\bar{\zeta}$. *Moreover, with* $\tilde{\zeta} = \zeta - \bar{\zeta}$ *the system*

$$\begin{aligned} \dot{\tilde{\zeta}} &= F(\bar{\zeta} + \tilde{\zeta}, e + \mathcal{Y}_r, \theta) - F(\zeta, \mathcal{Y}, \theta) \\ &= F_2(\tilde{\zeta}, e + \mathcal{Y}_r, \theta, \bar{\zeta}) \end{aligned}$$

has a continuously differentiable function $V_1(t, \tilde{\zeta})$, *possibly dependent on* θ, *that satisfies*

$$\begin{aligned}
\eta_1 \|\tilde{\zeta}\|^2 &\leq V_1(t, \tilde{\zeta}) \leq \eta_2 \|\tilde{\zeta}\|^2 \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial \tilde{\zeta}} F_2(.) &\leq -\eta_3 \|\tilde{\zeta}\|^2 + \eta_4 \|\tilde{\zeta}\| \|e\|
\end{aligned}$$

where $\eta_1, \eta_2, \eta_3 > 0$, *and* $\eta_4 \geq 0$ *are independent of* $\mathcal{Y}_r$, $\bar{\zeta}$, *and* θ.

The state feedback controller is designed to adaptively cancel the nonlinearities and stabilize the system (4.82). It consists of the dynamic controller

$$\dot{\tilde{\theta}} = \Gamma_p(\tilde{\theta} + \theta, e, z, \mathcal{Y}_R) \tag{4.84}$$

$$v = \psi(\tilde{\theta} + \theta, e, z, \mathcal{Y}_R) \tag{4.85}$$

where $\tilde{\theta} = \hat{\theta} - \theta$ ($\hat{\theta}$ is the estimate of θ) and

$$\psi(.) = \frac{-Ke + y_r^{(n)} - f_0(e + \mathcal{Y}_r, z) - \hat{\theta}^T f(e + \mathcal{Y}_r, z)}{g_0(e + \mathcal{Y}_r, z) + \hat{\theta}^T g(e + \mathcal{Y}_r, z)} \tag{4.86}$$

The functions Γ_p and ψ are locally Lipschitz in their arguments uniformly in $\mathcal{Y}_R$ and θ. The adaptive law is a projection-type law (for more details see [27]).

Consider the Lyapunov function candidate $V = e^T Pe + \frac{1}{2}\tilde{\theta}\Gamma^{-1}\tilde{\theta}$, where P is the positive definite solution of the Lyapunov equation $PA_m + A_m^T P = -Q$, where $Q = Q^T > 0$ and Γ^{-1} is a positive definite matrix. The derivative of V along the trajectories of the closed-loop system (4.82)–(4.86) is

$$\dot{V} = -e^T Qe + \tilde{\theta}^T \Gamma^{-1} [\dot{\hat{\theta}} - \Gamma\phi] \tag{4.87}$$

where

$$\phi(e, z, \mathcal{Y}_R, \hat{\theta}) = 2e^T Pb[f(e + \mathcal{Y}_r, z) + g(e + \mathcal{Y}_r, z)\psi(e, z, \mathcal{Y}_R, \hat{\theta})]$$

The adaptive law is chosen to ensure that

$$\tilde{\theta}^T \Gamma^{-1}[\dot{\hat{\theta}} - \Gamma\phi] \leq 0 \tag{4.88}$$

and $\hat{\theta}(t) \in \Omega_\delta$ for all $t \geq 0$ and all $\hat{\theta}(0) \in \Omega$, where Ω_δ is a compact set chosen such that $\Omega \subset \Omega_\delta \subset \Omega_1$. Inequality (4.88) ensures that $\dot{V} \leq -e^T Qe$. Therefore, all signals are bounded for all $t \geq 0$. Since $\mathcal{Y}_r$ is bounded, we conclude that $x(t)$ is bounded, which implies, in view of assumption 2, that $z(t)$ is bounded. By using [28, Theorem 4.4] we can conclude that

$$e(t) \to 0 \text{ as } t \to \infty \tag{4.89}$$

To discuss parameter convergence let us define the regressor vector $\omega_r(t)$ as

$$\omega_r(t) = f(\mathcal{Y}_r, \bar{z}) + g(\mathcal{Y}_r, \bar{z})\psi(0, \bar{z}, \mathcal{Y}_R, \theta) \tag{4.90}$$

where $\bar{z}$ is the steady-state solution of the zero dynamics, determined uniquely from $\bar{\zeta} = T_1(\mathcal{Y}_r, \bar{z})$.

Assumption 3: The regressor vector is persistently exciting.

Let

$$\omega(t) = f(e + \mathcal{Y}_r, z) + g(e + \mathcal{Y}_r, z)\psi(e, z, \mathcal{Y}_R, \hat{\theta})$$

Then, the $\dot{e}$- and $\dot{\tilde{\theta}}$-dynamics can be written as

$$
\begin{aligned}
\dot{e} &= A_m e - b\tilde{\theta}^T \omega_r + b\tilde{\theta}^T(\omega_r - \omega) \\
\dot{\tilde{\theta}} &= \Gamma_p(.)
\end{aligned}
$$

Now let us work on $(\omega_r - \omega)$.

Claim: We can write

$$
\omega_r - \omega = (\bar{f} - f) + (\bar{g}\tilde{\psi} - g\psi) + \bar{g}\frac{\tilde{\theta}^T}{\bar{g}_0 + \hat{\theta}^T \bar{g}}\omega_r \tag{4.91}
$$

where

$$
\begin{aligned}
&\bar{f}(.) = f(\mathcal{Y}_r, \bar{z}), \quad \bar{g}(.) = g(\mathcal{Y}_r, \bar{z}) \\
&\bar{f}_0 = f_0(\mathcal{Y}_r, \bar{z}), \quad \bar{g}_0(.) = g_0(\mathcal{Y}_r, \bar{z}), \quad \tilde{\psi}(.) = \psi(0, \bar{z}, \mathcal{Y}_R, \hat{\theta})
\end{aligned}
$$

Proof: Let $\bar{\psi}(.) = \psi(0, \bar{z}, \mathcal{Y}_R, \theta)$. Then, we have

$$
\begin{aligned}
\omega_r - \omega &= (\bar{f} + \bar{g}\bar{\psi}) - (f + g\psi) \\
&= (\bar{f} - f) + (\bar{g}\tilde{\psi} - g\psi) + \bar{g}(\bar{\psi} - \tilde{\psi})
\end{aligned}
$$

The claim is proved if we show that

$$
(\bar{\psi} - \tilde{\psi}) = \frac{\tilde{\theta}^T}{\bar{g}_0 + \hat{\theta}^T \bar{g}}\omega_r
$$

We have

$$
\begin{aligned}
\bar{\psi} - \tilde{\psi} &= \frac{y_r^{(n)} - \bar{f}_0 - \theta^T \bar{f}}{\bar{g}_0 + \theta^T \bar{g}} - \frac{y_r^{(n)} - \bar{f}_0 - \hat{\theta}^T \bar{f}}{\bar{g}_0 + \hat{\theta}^T \bar{g}} \\
&= \frac{\tilde{\theta}^T \bar{g}(y_r^{(n)} - \bar{f}_0 - \theta^T \bar{f}) + \tilde{\theta}^T \bar{f}(\bar{g}_0 + \theta^T \bar{g})}{(\bar{g}_0 + \theta^T \bar{g})(\bar{g}_0 + \hat{\theta}^T \bar{g})}
\end{aligned}
$$

Using the expression of $\bar{\psi}$ and ω_r we can write

$$\begin{aligned}
\bar{\psi} - \tilde{\psi} &= \frac{\tilde{\theta}^T \bar{g} \bar{\psi}}{\bar{g}_0 + \hat{\theta}^T \bar{g}} + \frac{\tilde{\theta}^T \bar{f}}{\bar{g}_0 + \hat{\theta}^T \bar{g}} \\
&= \frac{\tilde{\theta}^T}{\bar{g}_0 + \hat{\theta}^T \bar{g}} (\bar{f} + \bar{g}\bar{\psi}) \\
&= \frac{\tilde{\theta}^T}{\bar{g}_0 + \hat{\theta}^T \bar{g}} \omega_r \quad \triangleleft
\end{aligned}$$

By using the above claim, the $\dot{e}$- and $\dot{\tilde{\theta}}$-dynamics can be written as

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_m & -b\mathcal{G}\omega_r^T \\ 2\Gamma \mathcal{G} \omega_r b^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} \tilde{\Lambda}_s(.) \\ \tilde{\Lambda}_e(.) \end{bmatrix} \tag{4.92}$$

where

$$\begin{aligned}
K_{\mathcal{G}1} > \mathcal{G} &= \frac{g_0(\mathcal{Y}_r, \bar{z}) + \theta^T g(\mathcal{Y}_r, \bar{z})}{g_0(\mathcal{Y}_r, \bar{z}) + \hat{\theta}^T g(\mathcal{Y}_r, \bar{z})} > K_{\mathcal{G}2} \\
\tilde{\Lambda}_s(.) &= b\tilde{\theta}^T [(\bar{f} - f) + (\bar{g}\bar{\psi} - g\psi)] \\
\tilde{\Lambda}_e(.) &= \Gamma_p(.) - 2\Gamma \mathcal{G} \omega_r b^T P e
\end{aligned}$$

Since f, g_0, g, and ψ are Lipschitz functions in their arguments uniformly in $\mathcal{Y}_r$ and θ and since $\tilde{\theta}$ is bounded, we have

$$\|\tilde{\Lambda}_s(.)\| \leq \delta_1 \|e\| + \delta_2 \|\tilde{\zeta}\| \tag{4.93}$$

$$\|\tilde{\Lambda}_e(.)\| \leq \delta_3 \|e\| \tag{4.94}$$

for some $\delta_i \geq 0,\ i = 1, \cdots, 3$. Inequality (4.94) becomes clearer from the explicit form of $\Gamma_p(.)$ (see [27]).

From well known results in adaptive control theory (see for example [28, Section 13.4.2]) and the fact that ω_r is persistently exciting and $\mathcal{G}$ is bounded, it can be shown that the system

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_m & -b\mathcal{G}\omega_r^T \\ 2\Gamma\mathcal{G}\omega_r b^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix} \tag{4.95}$$

is exponentially stable. Then, from the converse Lyapunov theorem, there exists a Lyapunov function $V_2(t, e, \tilde{\theta})$ whose derivative along (4.92) satisfies

$$\lambda_1 \|(e, \tilde{\theta}_1)\|^2 \leq V_2(t, e, \tilde{\theta}) \leq \lambda_2 \|(e, \tilde{\theta})\|^2 \tag{4.96}$$

$$\begin{aligned} \dot{V}_2 \leq & -\delta_4\|e\|^2 - \delta_5\|\tilde{\theta}\|^2 + \delta_6\|e\|^2 + \delta_7\|e\|\|\tilde{\theta}\| \\ & + \delta_8\|e\|\|\tilde{\zeta}\| + \delta_9\|\tilde{\theta}\|\|\tilde{\zeta}\| \end{aligned} \tag{4.97}$$

(4.98)

$$\left\| \frac{\partial V_2}{\partial(e, \tilde{\theta})} \right\| \leq \lambda_3 \|(e, \tilde{\theta})\|^2 \tag{4.99}$$

for some positive constants δ_4, δ_5, and λ_i, $i = 1, 2, 3$, and some non-negative constants δ_i, $i = 7, \cdots, 9$.

The derivative of V along the trajectories of the closed-loop system (4.82)–(4.86) is

$$\dot{V} \leq -e^T Q e \leq -k_1\|e\|^2 \tag{4.100}$$

where $k_1 > 0$. Consider the Lyapunov function candidate $W(t, e, \tilde{\zeta}, \tilde{\theta}) = \alpha V(e, \tilde{\theta}) + \beta V_1(t, \tilde{\zeta}) + V_2(t, e, \tilde{\theta})$. Then, using (4.92)–(4.94), (4.97), (4.100), and Assumption 2 it can be shown that the derivative of W along trajectories of the closed-loop system

(4.82)–(4.86) satisfies

$$\dot{W} \leq - \begin{bmatrix} \|e\| \\ \|\tilde{\theta}\| \\ \|\tilde{\zeta}\| \end{bmatrix}^T M \begin{bmatrix} \|e\| \\ \|\tilde{\theta}\| \\ \|\tilde{\zeta}\| \end{bmatrix} \tag{4.101}$$

where M is given by

$$M = \begin{bmatrix} \alpha k_1 + \delta_4 - \delta_6 & -\frac{\delta_7}{2} & -\frac{\beta\eta_4 + \delta_8}{2} \\ -\frac{\delta_7}{2} & \delta_5 & -\frac{\delta_9}{2} \\ -\frac{\beta\eta_3 + \delta_8}{2} & -\frac{\delta_9}{2} & \beta\eta_3 \end{bmatrix}$$

Choose β large enough to make

$$\begin{bmatrix} \delta_5 & -\frac{\delta_9}{2} \\ -\frac{\delta_9}{2} & \beta\eta_3 \end{bmatrix}$$

positive definite; then choose α large enough to make M positive definite. Therefore, we conclude that $(e = 0, \tilde{\theta} = 0, \tilde{\zeta} = 0)$ is exponentially stable.

We consider the system

$$\begin{aligned} \dot{e} &= Ae + b\{f_0(e + \mathcal{Y}_r, z) + \theta^T f(e + \mathcal{Y}_r, z) \\ &\quad + [g_0(e + \mathcal{Y}_r, z) + \theta^T g(e + \mathcal{Y}_r, z)]v - y_r^{(n)}\} \\ \dot{\tilde{\zeta}} &= F_2(\tilde{\zeta}, \mathcal{Y}_r, \theta, \bar{\zeta}) \end{aligned}$$

This system fits the model (3.11)–(3.14) with the vector of time-varying bounded perturbations being $d(t) = (\mathcal{Y}_R(t), \theta, \bar{\zeta}(t))$ and the additional outputs ζ being the input and $(m-1)$ of its derivatives; i.e., z. Since we are dealing with a regional result and in order to apply the separation results of Chapter 3, we need to restrict the time-varying parameter d to belong to $\mathcal{M}'_{\mathcal{D}}$. This can be guaranteed by assuming that $y_r^{(n+1)}(t)$ exists and is continuous and bounded for all $t \geq 0$ (notice that $\bar{\zeta}(t) \in \mathcal{M}'_{\mathcal{D}}$ by Assumption 2).

We consider the state feedback controller.

$$\begin{aligned} \dot{\tilde{\theta}} &= \Gamma_p(\tilde{\theta} + \theta, e, z, \mathcal{Y}_R) \\ v &= \psi(\tilde{\theta} + \theta, e, z, \mathcal{Y}_R) \end{aligned}$$

This controller achieves exponential tracking of the reference signal with $R^n \times R^m \times \Omega_\delta$ being an estimate of the region of attraction. Global boundedness is achieved by saturation of $\psi(.)$ and $\Gamma_p(.)$ outside a region of interest. [1]

In the case at hand the compact positively invariant set A reduces to the origin $(e, \tilde{\theta}, \tilde{\zeta}) = 0$ which is exponentially stable. Thus, we are dealing with a stabilization problem of a time-varying system.

In [2] a case where only a part of the regressor vector is persistently exciting is considered. This case does not fit into our formulation because we we do not have stability with respect to a compact, positively invariant set.

Now, in order to implement the controller (4.84)–(4.85) and recover the performance achieved under it, we use a linear high-gain observer to estimate the tracking error and its derivatives.

Boundedness, ultimate boundednes, convergence of trajectories under output feedback control, as well as local UES, are guaranteed by Theorem 3.11.

Remark 4.8 *As in Remark 4.7 we choose the nominal value of the system's nonlinearity ϕ to be zero (which results in a linear observer), because otherwise we have*

[1] Actually, the saturation of $\Gamma_p(.)$ may not be needed since the projection-type adaptive law guarantees boundedness of $\hat{\theta}$.

to know the steady state solution $\bar{z}$ and we have to choose a nominal value which is zero at the steady state.

4.8 Conclusion

In this chapter we presented separation results for design cases where the trajectories converge to a compact, positively invariant set. For each of these cases, we showed how the state feedback controller performance can be cast as convergence to a compact, positively invariant set. Then, after suggesting an output feedback implementation of the control law, we applied the previous chapter's results and listed the set of performance measures that can be recovered.

CHAPTER 5

Separation Results for the Control of Nonlinear Systems Using Different High-gain Observer Designs

5.1 Introduction

In this chapter we are concerned with the separation approach to the design of stabilizing output feedback control using high-gain observers. In the separation approach the design is pursued in two steps. The first step focuses on the design of a stabilizing state feedback controller, while the second step is concerned with the design of a high-gain observer that successfully provides state estimates such that the overall closed-loop system is asymptotically stable.

High-gain observers are attractive because of their robustness, namely their ability to estimate the unmeasured states while rejecting the effect of disturbances. The available techniques for the design of high-gain observers can be classified into three groups. First, pole-placement algorithms which lead to either a two-time scale structure as in [11] or a multiple time-scale structure as in [46]. Second, Riccati equation-based algorithms which lead to either an H_2 algebraic Riccati equation (ARE) as in [9] and [43, Section 4.4.1] or to an H_∞ ARE as in [42] and [43, Section 4.4.2]. Third, Lyapunov equation-based algorithm as in [17].

For linear time-invariant systems the recovery of asymptotic stability through the use of high-gain observers is shown in [43], and references therein, in a Loop Transfer Recover (LTR) context and in [42] in an H_∞ context. As for nonlinear systems, Esfandiari and Khalil in [11] and [30] use pole placement/singular perturbation to design a one-parameter observer gain and recover the robustness properties of a controller designed to stabilize a fully linearizable system. This design results in a standard two-time scale singularly perturbed system. In [46] Saberi and Sannuti design a multiple-parameter observer to recover the global stabilizability of an uncertain linear system. This design results in a standard multiple time-scale singularly perturbed system. Tornambe in [52] uses a similar pole placement technique to recover local asymptotic stability for a class of input-output linearizable systems. Nicosia and Tornambe in [40] use singular perturbations to recover local asymptotic stability for the case of a robot with elastic joints. Teel and Praly [49] combine ideas from [11] and [52] to achieve semiglobal stabilization for a wide class of nonlinear systems. Isidori in [21] shows that his previously proposed solution for the general structurally stable regulation problem [22] can be coupled with ideas from [11] to solve a problem of robust semiglobal output regulation. In [7] Busawon *et al* present local and global separation results for a class of nonlinear systems using a high-gain observer designed with a Lyapunov equation-based algorithm.

A characteristic of a high-gain observer is the peaking phenomenon of its transient response. This phenomenon is examined carefully in [11]. Peaking occurs in the observer variables and propagates to the state variables through the control law. This peaking could be destabilizing in the case of a finite region of attraction, which only allows recovery of local asymptotic stability as in [52], [40], and [7]. However, despite peaking, global asymptotic stability results can be obtained at the expense of imposing restrictive global Lipschitz conditions, as is the case of [46] and [7], and tolerating a clearly unacceptable transient response. To remedy this problem the idea of saturating the control law outside a region of interest was introduced in [11]. It is this saturation feature that leads to the semiglobal results of [11, 30, 49, 21] and the regional results of Chapter 2 and 3.

Aside from some special cases like the case of relative degree-one systems, all the different ideas to design high-gain observers boil down to various asymptotic methods to approximate the derivatives of the outputs. In [25] Khalil illustrates this observation through an example and in this chapter we prove it rigorously.

In this chapter we show that separation results, similar to those of Chapters 2 and 3, can be obtained if the other available algorithms for observer gain design are applied in combination with global boundedness of the control law. Section 5.2 describes the different algorithms that can be used to design the gain of a high-gain observer and shows that in all of them the gain asymptotically matches the gain structure used in Chapters 2 and 3. Section 5.3 argues the separation results of Chapters 2 and 3 when we use alternative observer gain designs.

5.2 High-gain Observers - A Comparative Study

We consider the system represented by (2.1)–(2.4). The estimation error dynamics for the subsystem (2.1) can be written as

$$\dot{e} = (A - HC)e + B\delta(x, \hat{x}, z, u) \tag{5.1}$$

where $\delta(.) = \phi(x, z, u) - \phi_0(\hat{x}, \zeta, u)$. We need to design an observer gain H that stabilizes $(A - HC)$ while rejecting the effect of the disturbance $\delta(.)$. This is achieved if we could design H such that the transfer function between the disturbance input and the estimation error $(sI - A + HC)^{-1}B$ is identically zero or arbitrarily close to zero. We design H as a function of a parameter ϵ or a set of parameters $\epsilon_i, i = 1, \cdots, p$ such that this transfer function approaches zero as ϵ or ϵ_i's tend to zero. We present three different approaches to the design of H and show that in all three cases the gain H has approximately the structure (2.10).

5.2.1 Pole Placement/Time-structure Assignment

In pole placement we assign one or multiple time-scale eigenstructures to the observer matrix and ultimately make the closed-loop system under output feedback transformable into a standard singularly perturbed system. This is the approach used in Chapter 2. A detailed exposé of this technique as applied to full or reduced order observers can be found in [43, Section 4.3], [11], and Chapter 2.

5.2.2 Riccati Equation-Based Algorithms

Optimization-based techniques to design the high-gain observer can be reached through two paths: Loop Transfer Recovery and disturbance attenuation. Both can be applied to general stabilizable and detectable linear time-invariant systems

$$\dot{x} = Ax + Bu \tag{5.2}$$

$$y = Cx \tag{5.3}$$

The LTR algorithm consists of two steps. First, design a state feedback controller $u = Fx$ to shape the loop transfer function as desired. For example, by breaking the feedback loop at the input side the loop transfer function is given by $T(s) =$

$F(sI - A)^{-1}B$. Second, design a Luenberger observer to estimate the state x by $\hat{x}$ and implement the feedback control $u = F\hat{x}$.

The error E between the target loop transfer function and the realized one is given by $E(s) = M(s)[I + M(s)]^{-1}(I + F(sI - A)^{-1}B)$, see [43, Lemma 2.2.1], where $M(s) = F(sI - A + HC)^{-1}B$. It is also shown that, for all $0 \leq |\omega| < \infty$, $E(j\omega) = 0$ if and only if $M(j\omega) = 0$. The observer gain can be designed to exactly make $E(s) = 0$ or to depend on a small positive parameter μ; i.e. $H = H(\mu)$, such that the loop transfer function under output feedback control will approach that under state feedback control asymptotically as μ tends to zero; i.e. to asymptotically make $E(s)$ tend to zero.

The observer design for asymptotic LTR can be achieved by pole placement or optimization-based methods. Pole placement is discussed in the previous section. In optimization-based algorithms the objective is to find a gain $H(\mu)$ that asymptotically minimizes either the H_2 or the H_∞ norm of $M(s)$. In other words, let $M(s, \mu)$ denote the matrix $M(s)$ with H substituted by the designed $H(\mu)$, then it can be shown, [43, Theorem 4.4.2], that $\|M(s, \mu)\| \to \inf \|M(s)\|$ as μ tends to zero. A historical survey as well as clear explanation of this approach can be found in [43, Section 4.4]. Basically, the idea is to solve a standard H_2 or H_∞ control problem for the following auxiliary system

$$\dot{x} = A^T x + C^T u + F^T w \tag{5.4}$$

$$y = x \tag{5.5}$$

$$z = B^T x \tag{5.6}$$

The standard H_2 or H_∞ control consists of determining the control gain H^T to minimize the H_2 or H_∞ norm of the transfer function between the disturbance input w and the controlled output z over the set of all possible gains while rendering the matrix $(A - HC)$ asymptotically stable. This transfer function is equal to $M^T(s)$. The infinimum of its H_2 or H_∞ norm over all possible gains is equal to zero for a minimum-phase left-invertible system for any loop gain F. It is worth noting that these LTR techniques work for general stabilizable and detectable systems; i.e, not necessarily minimum-phase nor left-invertible, but $\|M(s,\mu)\|$ is only guaranteed to converge to some finite value, except for some cases where F satisfies certain conditions.

It is shown in [43, Section 4.4.1] and references therein that the H_2 optimization-based technique yields, for a minimum-phase left-invertible system, a standard algebraic Riccati equation of the form

$$AP + PA^T - \frac{1}{\mu^2} PC^T CP + BB^T = 0 \tag{5.7}$$

The observer gain is $H_2(\mu) = (1/\mu^2)P_2(\mu)C^T$ where $P_2(\mu)$ is the unique positive definite solution of (5.7) that makes the matrix $[A - H_2(\mu)C]$ asymptotically stable.

The H_∞ optimization yields an H_∞ algebraic Riccati equation of the form

$$AP + PA^T - \frac{1}{\mu^2} PC^T CP + \frac{1}{\gamma^2} PF^T FP + BB^T + \mu^2 I = 0 \tag{5.8}$$

The observer gain is $H_\infty(\mu) = (1/\mu^2)P_\infty(\mu)C^T$ where $P_\infty(\mu)$ is the unique positive definite solution of (5.8) that makes the matrix $(A - H_\infty(\mu)C)$ asymptotically stable. According to [56], this solution exists for an appropriately large γ and sufficiently small μ (when the system (A, B, C) is stabilizable and detectable). Let $\gamma^\star$ denote the infinimum of $\|M(s)\|_\infty$ over all possible gains. Then, equation (5.8) is actually solvable for all $\gamma > \gamma^\star$ and every $0 < \mu < \mu^\star$ where $\mu^\star$ depends on γ. Moreover, it can be shown, see [56], that $\|M(s,\gamma)\|_\infty < \gamma$ for $\gamma > \gamma^\star$ (in our case $\gamma^\star = 0$).

The disturbance attenuation algorithm for observer design is detailed in [42], where it is applied to a minimum-phase left-invertible linear system. It is based on a parameterized H_∞ algebraic Riccati equation for a dual system. In our case it can be applied to the system

$$
\begin{aligned}
\dot{x} &= Ax + B\phi_0(x, \zeta, u) + B\delta(x, \hat{x}, z, u) && (5.9)\\
y &= Cx && (5.10)\\
z &= x && (5.11)
\end{aligned}
$$

considering ϕ_0 as the control input, $\delta(.)$ as the disturbance input, and z as the controlled output. This methods yields, after some scaling, an H_∞ Algebraic Riccati Equation similar to (5.8).

Now, let us go back to our problem of designing the gain H and apply the optimization-based techniques suggested by the LTR theory. The observer design problem formulated in (5.1), can fit into a Loop Transfer Recovery scheme if we consider $\phi(.)$ to be the control input and $F = I$ to be the controller gain. Thus, in this case we have $M(s) = (sI - A + HC)^{-1}B$. The problem is solvable since (A, B) and (A, C) are controllable and observable pairs, respectively (left invertibility and minimum-phase are implied by the structure of (A, B, C)).

The structure of the stabilizing solution of a general algebraic Riccati equation as well as the eigenstructure of the observer matrix have been investigated in [47]. Here, we apply these results to our case of interest; namely, when the triplet (A, B, C) represents a set of chains of integrators and $F = I$. In Appendix B we establish similar results for the H_∞ algebraic Riccati equation (5.8). Basically the same structure in (B.18)–(B.19), obtained in Appendix B, applies to both types of equations. The observer gain is given by $H = \frac{1}{\mu^2} P_\infty C^T$ = block diag$[H1, \cdots, Hp]$. It can be shown that, for $i = 1, \cdots, p$, we have

$$H_i = \begin{bmatrix} (a_1^i + O(\epsilon_i))/\epsilon_i \\ (a_2^i + O(\epsilon_i))/\epsilon_i^2 \\ \vdots \\ (a_{r_i}^i + O(\epsilon_i))/\epsilon_i^{ri} \end{bmatrix} \tag{5.12}$$

To apply the analysis of Chapter 2 we have to transform the closed-loop system into a standard singularly perturbed form similar to (2.11)–(2.14) by scaling the estimation error. The scaling turns out to be

$$\begin{aligned} \hat{x} &= x - S(\epsilon)\eta, \text{ where} \\ S &= \text{diag}[S_1, \cdots, S_K], \; S_i = \text{diag}[\epsilon_i^{r_i - 1}, \cdots, \epsilon_i, 1] \end{aligned} \tag{5.13}$$

This scaling, when applied to the observer (2.9) with the gain H just calculated, yields

$$S^{-1}(A - HC)S = \mathcal{E}^{-1}\Gamma + \mathcal{E}^{-1}\Delta.R(\mathcal{E}) \tag{5.14}$$

where

$$\begin{aligned} \mathcal{E} &= \text{diag}[\epsilon_1 I_{r_1}, \cdots, \epsilon_{rp} I_{rp}] \\ R(\mathcal{E}) &= \text{diag}[O(\epsilon_1)I_{r_1}, \cdots, O(\epsilon_p)I_{rp}) \end{aligned} \tag{5.15}$$

$$\Gamma = \text{block diag}[\Gamma_1, \cdots, \Gamma_p], \; \Gamma_i = \begin{bmatrix} -a_1^i & 1 & \dots & \dots & 0 \\ -a_2^i & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ -a_{r_i-1}^i & \dots & \dots & 0 & 1 \\ -a_{r_i}^i & \dots & \dots & \dots & 0 \end{bmatrix}_{r_i \times r_i}$$

$$\Delta = \text{block diag}[\Delta_1, \cdots, \Delta_p], \; \Delta_i = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 1 & \dots & \dots & 0 & 0 \\ 1 & \dots & \dots & \dots & 0 \end{bmatrix}_{r_i \times r_i}$$

and Γ is nonsingular. This result yields a multiple time-scale singularly perturbed closed loop system similar to (2.11)–(2.14) with the estimation error equation given by

$$\mathcal{E}\dot{\eta} = \Gamma\eta + \mathcal{E}B\delta(x, z, \vartheta, D(\mathcal{E})\eta) + \Delta.R(\mathcal{E})\eta \tag{5.16}$$

In Section 5.4 we show how the results of Chapter 2 can be extended to the closed-loop system formed of (2.11)–(2.13) and (5.16).

5.2.3 Lyapunov Equation-Based Algorithm

In [17] Gauthier, Hammouri, and Othman presented a class of single-input–single-output nonlinear systems transformable into the triangular form

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ \varphi(x) \end{bmatrix} + \begin{bmatrix} g_1(x_1) \\ g_2(x_1, x_2) \\ \vdots \\ g_{n-1}(x_1, \cdots, x_{n-1}) \\ g_n(x) \end{bmatrix} u = F(x) + G(x)u \qquad (5.17)$$

$$y = x_1 = Cx \qquad (5.18)$$

To implement an output feedback control scheme, the following observer was suggested

$$\dot{\hat{x}} = F(\hat{x}) + G(\hat{x})u + S_\infty^{-1} C^T (y - C\hat{x}) \qquad (5.19)$$

where the gain S_∞ is the solution of the following Lyapunov equation

$$(A + \frac{1}{2\epsilon} I)^T S_\infty + S_\infty (A + \frac{1}{2\epsilon} I) - C^T C = 0 \qquad (5.20)$$

and A is an $n \times n$ matrix representing a chain of integrators. This solution exists for sufficiently small ϵ. Through a Lyapunov analysis coupled with global Lipschitz conditions, global exponential stability of the estimation error is established. These results were extended to the single-input–multi-output case in [5].

The system (2.1)–(2.4), with the z dynamics dropped, becomes a special case of (5.17)–(5.19) when $g_i(.) = 0$ for $i = 1, \cdots, n-1$. Equation (5.20) is written for a single chain of integrators. Let us examine its solution for a multiple chain of integrators. Multiply (5.20) by S_∞^{-1} from the left and from the right to obtain

$$AS_\infty^{-1} + S_\infty^{-1}A^T - S_\infty^{-1}C^T CS_\infty^{-1} + \frac{1}{\epsilon}S_\infty^{-1} = 0 \tag{5.21}$$

It can be shown that the positive definite solution of (5.21) is a block-diagonal matrix and is given by

$$\begin{aligned} S_\infty^{-1}(\epsilon) &= \text{block diag}[S_{\infty,1}^{-1}(\epsilon), \cdots, S_{\infty,p}^{-1}(\epsilon)] \\ S_{\infty,i}^{-1}(\epsilon) &= [(S_{\infty,i}^{-1}(\epsilon))_{\ell j}]_{r_i \times r_i} = [(S_{\infty,i}^{-1}(1))_{\ell j} \frac{1}{\epsilon^{\ell + j - 1}}] \end{aligned} \tag{5.22}$$

Thus, the structure of the observer gain $S_\infty^{-1}C^T$ is exactly the structure (2.10). Consequently, if the scaling of Section 2.5.1 is used for the estimation error we obtain a two time-scale singularly perturbed system similar to (2.11)–(2.14). Thus, if global boundedness of the control law is implemented, we obtain the results of Chapter 2. It is noteworthy that in [17] all nonlinearities are required to be known which undermines the robustness of the observer. However in the case where $g_i(.) = 0$ for $i = 1, \cdots, n-1$, the results of Chapter 2 show that imperfect knowledge of the two remaining nonlinearities $g_n(x)$ and $\varphi(x)$ can be allowed.

5.3 More Separation Results

In Chapter 2 we decompose the closed-loop system into a reduced model corresponding to the closed-loop system under state feedback controller and a boundary-layer model corresponding to the Hurwitz matrix A_0 of the error dynamics. Then, we reduce the parameter ϵ to make the observer fast enough such that it brings the state estimate close enough to its real value in short time and restores the stabilizing powers of the feedback controller. This is carried out using two Lyapunov functions: one, $V(x, z, \vartheta)$, for the reduced model and another, $W(\eta) = \eta^T P_0 \eta$ for the boundary-layer model. It establishes boundedness of trajectories by proving that trajectories enter, in a short time, the positively invariant set $\Lambda = \{V(x, z, \vartheta) \leq c\} \times \{W(\eta) \leq \rho\epsilon^2\}$ where c and ρ are positive constants.

We notice that the observer gain of Chapter 2 depends only on one parameter ϵ in all of the p channels which results in a closed-loop system with a two time-scale structure. Alternatively, one may have different parameters in the different channels as we have seen. This eventually, after scaling, results in a closed-loop system with a multiple time-scale structure.

For the case, where the observer gain is designed using one of the above-mentioned algorithms, we can apply a multiple time-scale analysis to (5.16) and show that we have p decoupled boundary-layer models. The i-th fast subsystem is an exponentially stable linear time-invariant system with the Hurwitz matrix Γ_i. Thus we can repeat the same steps of the previous analysis with the Lyapunov functions $V(x, z, \vartheta)$, $W_1(\eta_1), \cdots, W_p(\eta_p)$ where $W_i(\eta_i) = \eta_i^T P_{0i} \eta_i$ with P_{0i} being the unique positive definite solution of the Lyapunov equation $P\Gamma_i + \Gamma_i^T P = -I$. In this case the positively invariant set is $\Lambda = \{V(x, z, \vartheta) \leq c\} \times \{W_1(\eta_1) \leq \rho_1\epsilon_1^2\} \times \cdots \times \{W_p(\eta_p) \leq \rho_p\epsilon_p^2\}$. All of the parameters ϵ_i's should be simultaneously reduced to make the different boundary-layer models fast enough to bring the state estimate close to its real value in short time. Ultimate boundedness and convergence of trajectories as well as asymptotic stability analysis can be performed as in Chapter 2 using the above-given Lyapunov functions. The foregoing analysis and discussion are summarized in the following theorem

Theorem 5.1 *Let the observer gain be designed using one of the above-given algorithms, leading to the gain structure (5.12). Suppose that:*

- *The vector fields of the system (2.1)-(2.4), the dynamic controller (2.5)-(2.5), and the observer (2.9) be locally Lipschitz in their arguments over the domain of interest and vanish at the origin* $(x, z, \vartheta) = \chi = 0$.

- *The functions* $\Gamma(\vartheta, x, \zeta)$, $\gamma(\vartheta, x, \zeta)$, *and* $\phi_0(x, \zeta, u)$ *are globally bounded in* x.
- *The origin* $\chi = 0$ *is an exponentially stable equilibrium point of the closed-loop system under state feedback, with* $\mathcal{R}$ *as its region of attraction.*

Let the initial state $\chi(0)$ *be in a compact subset* $\mathcal{S}$ *of* $\mathcal{R}$ *and the initial observer state* $\hat{x}(0)$ *be in a compact set* $\mathcal{Q}$ *of* R^r.

Then, for sufficiently small ϵ_i, $i = 1, \cdots, p$, *the origin* $(\chi, \hat{x}) = (0, 0)$ *is an exponentially stable equilibrium point of the closed-loop system under output feedback. Moreover, for any compact subsets* $\mathcal{S} \subset \mathcal{R}$ *and* $\mathcal{Q} \subset R^r$, *the set* $\mathcal{S} \times \mathcal{Q}$ *is a subset of the region of attraction. Furthermore, as the parameters* ϵ_i*'s tend to zero the trajectory* $\chi(t, \mathcal{E})$ *under output feedback approaches the trajectory* $\chi_r(t)$ *under state feedback control, uniformly in* t, *for* $t \geq 0$.

Remark 5.1 *For simplicity we stated the separation results only for the case where the origin* $\chi = 0$ *is exponentially stable. Similar results can be easily proved for the case of asymptotic stability along the lines of Chapter 2, but we have to require additional conditions on the local growth of the modeling errors.*

Remark 5.2 *Separation results similar to those of Theorem 5.1 and Remark 5.1 can be stated for the case of stability with respect to a compact, positively invariant set along the line of Chapter 3.*

5.4 Conclusion

In this chapter we discussed the problem of output feedback control for a wide class of nonlinear systems using high-gain observers designed using different algorithms. These algorithms involved either pole placement, algebraic Riccati equation-based techniques, or Lyapunov equation-based technique. We basically showed that all these designs yield observer gains that asymptotically have the structure of the observer gain used in Chapters 2 and 3. Consequently, if the idea of global boundedness of the control law is implemented, the separation results of Chapters 2 and 3 apply to the cases where these alternative techniques are used to design the observer gain.

CHAPTER 6

A Smooth Converse Lyapunov Theorem for Robust Stability

6.1 Introduction

In [34] Lin, Sontag, and Wang present converse Lyapunov function theorems for stability with respect to sets. Their work allows arbitrary bounded time-varying parameters in the system description, results in smooth Lyapunov functions, applies to stability with respect to not necessarily compact sets, and deals with global asymptotic stability. In addition to the global results of [34], we need a converse Lyapunov theorem that yields a smooth function defined on a possibly finite open set and that approaches infinity at the boundary of this set. Thus our objective here is to modify the results of [34] to make it suitable for our purposes.

The tools used to modify the results of [34] are inspired by Kurzweil [32]. It consists of replacing the distance (with respect to a set in this case) by a continuous positive definite function that has all the important properties of the distance (except, may be, the triangular inequality) and that approaches infinity at the boundary of the open set of interest.

In Section 6.2 we give basic definitions and the main results. In Sections 6.3 we prove the converse Lyapunov theorem for uniform asymptotic stability.

The proof of the converse Lyapunov theorem closely follow that of [34] except in places where some special discussions have to be carried. We will point out these matters whenever they arise.

6.2 Definitions and the Main Results

Consider the system (3.1) (we repeat it here for easy reference)

$$\dot{x}(t) = f(x(t), d(t)) \tag{6.1}$$

Let $\mathcal{M}_{\mathcal{D}}$ be as in Chapter 3. Let $\mathcal{R}$ be an open subset of R^n. The system is said to be *forward complete* in $\mathcal{R}$ if $T^+_{x_0, d} = +\infty$ for all $x_0 \in \mathcal{R}$ and all $d \in \mathcal{M}_{\mathcal{D}}$. It is *backward complete* in $\mathcal{R}$ if $T^-_{x_0, d} = -\infty$ for all $x_0 \in \mathcal{R}$ and all $d \in \mathcal{M}_{\mathcal{D}}$, and it is *complete* in $\mathcal{R}$ if it is both forward and backward complete in $\mathcal{R}$. In this chapter we use notation and definitions given in Section 3.2.

6.2.1 Strong Stability

Let $\mathcal{A}$ be a compact subset of the open connected set $\mathcal{R}$. Define the function $\omega_{\mathcal{A}}$: $R^n \to R_{\geq 0} \cup \{+\infty\}$ by

$$\omega_{\mathcal{A}}(\xi) = \begin{cases} \max(|\xi|_{\mathcal{A}}, \frac{1}{|\xi|_F} - \frac{2}{|F|_{\mathcal{A}}}) & \text{if } x \in \mathcal{R} \\ +\infty & \text{if } x \notin \mathcal{R} \end{cases} \tag{6.2}$$

where F is the complement of $\mathcal{R}$ in R^n, and $|F|_{\mathcal{A}} = \inf_{(\eta, \mu) \in \mathcal{A} \times F} \|\eta - \mu\|$.

Remark 6.1 *A similar function was first defined in Kurzweil [32] for the case where $\mathcal{A}$ reduces to an equilibrium point.*

We can make some useful observations concerning $\omega_{\mathcal{A}}(\xi)$; they are summarized in the following Lemma:

Lemma 6.1 *Let $\mathcal{A}$ be a compact set contained in the open connected set $\mathcal{R} \subseteq R^n$. The function $\omega_{\mathcal{A}}(\xi)$ restricted to the set $\mathcal{R}$ satisfies the following properties:*

1. *It is positive definite with respect to $\mathcal{A}$; i.e., $\omega_{\mathcal{A}}(\xi) > 0$ for all $\xi \in \mathcal{R}/\mathcal{A}$ and $\omega_{\mathcal{A}}(\xi) = 0$ for all $\xi \in \mathcal{A}$. $\omega_{\mathcal{A}}(\xi) = |\xi|_{\mathcal{A}}$ if $\mathcal{R} = R^n$;*

2. *It is continuous in $\mathcal{R}$;*

3. *It approaches infinity as ξ approaches the boundary of $\mathcal{R}$;*

4. *The set $\{\xi \in \mathcal{R} : r_1 \leq \omega_{\mathcal{A}}(\xi) \leq r_2\}$ for $r_1, r_2 \geq 0$ is compact in $\mathcal{R}$;*

5. *It is locally Lipschitz.*

Proof: See Section 6.4.1.◁

Remark 6.2 *The function $\omega_{\mathcal{A}}(.)$ shares some properties with $|.|_{\mathcal{A}}$, namely, positive definiteness with respect to $\mathcal{A}$, continuity, and local Lipschitz property. This fact makes adapting many results of [34] to our case a straightforward process. Therefore, we omit the proofs where replacing $|.|_{\mathcal{A}}$ by $\omega_{\mathcal{A}}(.)$ does not pose any challenge.*

The following is a notion of stability in an open set inspired by Definition 1 of [32] and Definition 2.2 of [34]:

Definition 6.1 *Let $\mathcal{R} \subseteq R^n$ be an open connected set that contains $\mathcal{A}$. We say that the system (6.1) is Strongly Stable with respect to the compact positively invariant set $\mathcal{A}$ if the following two properties hold:*

1. *Uniform Stability in $\mathcal{R}$: there exists a class $\mathcal{K}_\infty$ function $\delta(.)$ such that for any $\epsilon \geq 0$ and every $d \in \mathcal{M}_{\mathcal{D}}$, we have*

$$\omega_{\mathcal{A}}(x(t, x_0; d)) \leq \epsilon, \text{ whenever } \omega_{\mathcal{A}}(x_0) \leq \delta(\epsilon) \text{ and } t \geq 0 \tag{6.3}$$

2. *Uniform Attraction in $\mathcal{R}$: for any $r, \epsilon > 0$, there is $T = T(\epsilon, r) > 0$, such that for every $d \in \mathcal{M}_{\mathcal{D}}$,*

$$\omega_{\mathcal{A}}(x(t, x_0; d)) < \epsilon \text{ whenever } \omega_{\mathcal{A}}(x_0) < r \text{ and } t \geq T. \tag{6.4}$$

Remark 6.3 *The definition of Strong Stability with respect to a set is an adaptation of the notion of global UAS [34, Definition 2.2] to a possibly finite open set $\mathcal{R}$ ($\omega_{\mathcal{A}}(.) = |.|_{\mathcal{A}}$ if $\mathcal{R} = R^n$). This notion is a generalization to a compact set of the definition of strong stability given in [32].*

The following definition generalizes [34, Definition 2.6] to the case of strong stability:

Definition 6.2 *A Lyapunov function for the system (6.1) in the open set $\mathcal{R}$ with respect to a compact, positively invariant set $\mathcal{A} \subseteq \mathcal{R}$ is a function $V : \mathcal{R} \to R_{\geq 0}$ such that V is smooth on $\mathcal{R}/\mathcal{A}$ and satisfies the following properties:*

1. *There exist two class $\mathcal{K}_{\infty}$ functions α_1 and α_2 such that for any $\xi \in \mathcal{R}$,*

$$\alpha_1(\omega_{\mathcal{A}}(\xi)) \leq V(\xi) \leq \alpha_2(\omega_{\mathcal{A}}(\xi)) \tag{6.5}$$

2. *There exists a continuous, positive definite function α_3 such that for any $\xi \in \mathcal{R}/\mathcal{A}$, and any $d \in \mathcal{M}_{\mathcal{D}}$,*

$$L_{f_{\mathbf{d}}} V(\xi) \leq -\alpha_3(\omega_{\mathcal{A}}(\xi)) \tag{6.6}$$

A smooth Lyapunov function is one which is smooth on all of $\mathcal{R}$.

It follows from the previous definition that V is continuous in all of $\mathcal{R}$, zero if and only if $\xi \in \mathcal{A}$, and onto.

The first main result of this chapter is:

Theorem 6.1 *Let $\mathcal{A} \subset R^n$ be a compact, positively invariant set for the system (6.1). Let $\mathcal{R}$ be an open and connected set that contains $\mathcal{A}$. Assume that the system (6.1) is strongly stable in $\mathcal{R}$ with respect to $\mathcal{A}$. Then, there exists a smooth Lyapunov function V in $\mathcal{R}$ with respect to $\mathcal{A}$.*

Remark 6.4 *It is possible to state Theorem 6.1 as a necessary and sufficient condition. Moreover, a result similar to Proposition 2.5 of [34] for the case of strong stability in an estimate of the region of attraction is also correct. The proofs of such results follow from the corresponding ones in [34] by replacing $|.|_{\mathcal{A}}$ with $\omega_{\mathcal{A}}(.)$.*

It is useful to restate the uniform attraction property as in the following lemma.

Lemma 6.2 *The uniform attraction property defined in Definition 6.1 is equivalent to the following:*
There exists a family of mappings $\{T_r\}_{r\,>\,0}$ such that

- *for each fixed $r > 0$, $T_r : R_{>\,0} \to R_{>\,0}$ is onto, continuous, and strictly decreasing;*
- *for each fixed $\epsilon > 0$, $T_r(\epsilon)$ is strictly increasing in r and $\lim_{r\,\to\,\infty} T_r(\epsilon) = \infty$*

and, for each $d \in \mathcal{M}_{\mathcal{D}}$, we have

$$\omega_{\mathcal{A}}(x(t, x_0; d)) < \epsilon \text{ whenever } \omega_{\mathcal{A}}(x_0) < r \text{ and } t \geq T_r(\epsilon)$$

Proof: straightforward from Lemma 3.1 in [34].$\triangleleft$

6.2.2 Uniform Asymptotic Stability

Let $\mathcal{M}'_{\mathcal{D}}$ be the set of all continuously differentiable functions from R to $\mathcal{D}$ where the derivative $d'(t)$ of $d(t)$ belongs to a compact set $\mathcal{D}_1$.

The following lemma relates the notion of strong stability to that of uniform asymptotic stability given in Chapter 3:

Theorem 6.2 *Let $d \in \mathcal{M}'_{\mathcal{D}}$. Let the system (6.1) be Uniformly Asymptotically Stable (UAS) with respect to the compact positively invariant set $\mathcal{A}$. Let $\mathcal{R}$ be an open, connected subset of the region of attraction that contains $\mathcal{A}$. Then, the system (6.1) is Strongly Stable in $\mathcal{R}$ with respect to $\mathcal{A}$.*

Proof: see Section 6.4.2.$\triangleleft$

The second main result of this chapter is:

Corollary 6.1 *Let $d \in \mathcal{M}'_{\mathcal{D}}$. Let $\mathcal{A} \subset R^n$ be a compact, positively invariant set for the system (6.1). Assume that the system (6.1) is UAS with respect to $\mathcal{A}$. Let $\mathcal{R}$ be an open, connected subset of the region of attraction that contains $\mathcal{A}$. Then, there exists a smooth Lyapunov function V in $\mathcal{R}$ with respect to $\mathcal{A}$.*

Proof: if follows directly from Theorems 6.1 and 6.2.$\triangleleft$

Remark 6.5 *The open set $\mathcal{R}$ can be any time-independent estimate of the region of attraction of (6.1) with respect to $\mathcal{A}$. This region of attraction is not necessarily time-independent nor positively invariant unless the system (6.1) is autonomous (see [4, Chapter V,Proposition 4.15]). Thus, we can not take $\mathcal{R}$ to be the region of attraction as in [32, Theorem 12] unless the system (6.1) is autonomous.*

6.3 Proof of Theorem 6.1

The proof is divided into three main parts. In the first part we assume that the system (6.1) is complete and we construct a Lyapunov function which is smooth in $\mathcal{R}/\mathcal{A}$. In the second part we drop the completeness assumption and we construct an auxiliary complete system for which a Lyapunov function is available from the first part. Then, we prove that this function is also a Lyapunov function for the system (6.1). Finally, we smooth the resulting Lyapunov function in $\mathcal{R}$.

6.3.1 Case of a Complete System

Theorem 6.3 *Let the assumptions of Theorem 6.1 hold. Moreover, assume that the system (6.1) is complete in $\mathcal{R}$. Then, there exists a Lyapunov function V in $\mathcal{R}$ with respect to $\mathcal{A}$ which is smooth in $\mathcal{R}/\mathcal{A}$.*

Proof. First, we construct a useful locally Lipschitz function $g(\xi)$ and use it to construct a not necessarily smooth Lyapunov function $U(\xi)$. Then, we use a smoothing result given in [34, Theorem B.1] to show the existence of a smooth Lyapunov function $V(\xi)$.

The system (6.1) is strongly stable in $\mathcal{R}$ with respect to $\mathcal{A}$. Let δ and T_r be as in Definition 6.1 and Lemma 6.2.

Lemma 6.3 *The function $g : \mathcal{R} \to R$, defined by*

$$g(\xi) = \inf_{t \leq 0, d \in \mathcal{M}_{\mathcal{D}}} \{\omega_{\mathcal{A}}(x(t, \xi; d))\} \tag{6.7}$$

is well defined, continuous everywhere in $\mathcal{R}$, locally Lipschitz on $\mathcal{R}/\mathcal{A}$, and satisfies

$$g(x(t, \xi; d)) \leq g(\xi), \ \forall t > 0, \ \forall d \in \mathcal{M}_{\mathcal{D}} \tag{6.8}$$

$$\delta(\omega_{\mathcal{A}}(\xi)) \leq g(\xi) \leq \omega_{\mathcal{A}}(\xi) \tag{6.9}$$

for all $\xi \in \mathcal{R}$.

Proof.

Some additional steps should be added to the proof of [34]. One is the proof that

$g(.)$ is well defined on $\mathcal{R}$, and the other is the proof of the local Lipschitz property of $g(.)$. The function $g(\xi)$ is well defined because the trajectory $x(t, \xi; d)$, $t \leq 0$ starting from $\xi \in \mathcal{R}$ exists for any $d \in \mathcal{M}_{\mathcal{D}}$. Moreover, it stays in $\mathcal{R}$ for at least a finite interval $[T_e, 0]$ in the negative time. Since the function $\omega_{\mathcal{A}}$ is equal infinity on the boundary and outside of $\mathcal{R}$, then the infimum function always has a finite value for every ξ in $\mathcal{R}$. The proof of (6.8) and (6.9) is a straightforward extension of the corresponding proof in [34].

Let us characterize $g(.)$ in the set $K_{\epsilon, r} \stackrel{\text{def}}{=} \{\xi \in \mathcal{R} : \epsilon \leq \omega_{\mathcal{A}}(\xi) < r\}$ defined for any $0 < \epsilon < r$ ($K_{\epsilon, r}$ is not necessarily compact). Note that, for all ϵ and r with $0 < \epsilon < r$, there exists $q_{\epsilon, r} \leq 0$ such that:

$$\xi \in K_{\epsilon, r},\ d \in \mathcal{M}_{\mathcal{D}},\ \text{and } t < q_{\epsilon, r} \Longrightarrow \omega_{\mathcal{A}}(x(t, \xi; d)) \geq r$$

Therefore, for any $\xi \in K_{\epsilon, r}$, we have

$$g(\xi) = \inf\{\omega_{\mathcal{A}}(x(t, \xi; d)) : t \in [q_{\epsilon, r}, 0],\ d \in \mathcal{M}_{\mathcal{D}},\ \omega_{\mathcal{A}}(x(t, \xi; d)) \leq r\} \qquad (6.10)$$

Let us prove that $g(.)$ is locally Lipschitz on $\mathcal{R}/\mathcal{A}$. Fix any $\xi_0 \in \mathcal{R}/\mathcal{A}$, and let

$$s = \min\left\{\frac{|\xi_0|_{\mathcal{A}}}{2}, \frac{|\xi_0|_F}{2}\right\}$$

Let $\bar{B}(\xi_0, s)$ denote the closed ball of radius s centered at ξ_0. Then $\bar{B}(\xi_0, s) \subseteq K_{\sigma, r}$, for some $0 < \sigma < r$. Pick a constant C as in Proposition 5.5 of [34] with respect to the closed ball $\bar{B}(\xi_0, s)$ and $T = |q_{\sigma, r}|$.

Pick any $\zeta, \eta \in \bar{B}(\xi_0, s)$. Then, from (6.10) (a property of the inf function), for any

$\epsilon > 0$ there exist some $d_{\eta,\epsilon} \in \mathcal{M}_{\mathcal{D}}$ and some $t_{\eta,\epsilon} \in [q_\sigma, r, 0]$ such that

$$g(\eta) \geq \omega_{\mathcal{A}}(x(t_{\eta,\epsilon}, \eta; d_{\eta,\epsilon})) - \epsilon \tag{6.11}$$

Moreover, from (6.10), we have

$$g(\zeta) \leq \omega_{\mathcal{A}}(x(t_{\eta,\epsilon}, \zeta; d_{\eta,\epsilon})) \tag{6.12}$$

From (6.11) and (6.12), we conclude

$$g(\zeta) - g(\eta) \leq \omega_{\mathcal{A}}(x(t_{\eta,\epsilon}, \zeta; d_{\eta,\epsilon})) - \omega_{\mathcal{A}}(x(t_{\eta,\epsilon}, \eta; d_{\eta,\epsilon})) + \epsilon \tag{6.13}$$

Since $\bar{B}(\xi_0, s)$ is compact, then, by Proposition 5.1 of [34] the points $x(t_{\eta,\epsilon}, \zeta; d_{\eta,\epsilon})$ and $x(t_{\eta,\epsilon}, \eta; d_{\eta,\epsilon})$ belong to a compact set $\tilde{E}$ in R^n; moreover, by (6.10), they belong to $\{\xi : \omega_{\mathcal{A}}(\xi) \leq r\}$. Consider the set $E = \tilde{E} \bigcap \{\omega_{\mathcal{A}}(\xi) \leq r\}$. We know from Lemma 6.1 that $\{\omega_{\mathcal{A}}(\xi) \leq r\}$ is compact, then the set E is compact and is contained in $\mathcal{R}$.

Using the Lipschitz property of $\omega_{\mathcal{A}}$ in E , proved in Lemma 6.1, yields

$$g(\zeta) - g(\eta) \leq L\|x(t_{\eta,\epsilon}, \zeta; d_{\eta,\epsilon}) - x(t_{\eta,\epsilon}, \eta; d_{\eta,\epsilon})\| + \epsilon \tag{6.14}$$

where L is a Lipschitz constant of $\omega_{\mathcal{A}}$ in E. By Proposition 5.5 in [34], applied in $\bar{B}(\xi_0, s)$, we have

$$g(\zeta) - g(\eta) \leq CL\|\zeta - \eta\| + \epsilon \tag{6.15}$$

Note that (6.15) holds for all $\epsilon > 0$, then it follows that

$$g(\zeta) - g(\eta) \leq CL\|\zeta - \eta\|$$

By symmetry we have $g(\eta) - g(\zeta) \leq CL\|\zeta - \eta\|$ which proves that $g(.)$ is locally Lipschitz in $\mathcal{R}/\mathcal{A}$. The Lipschitz constant CL depends only on $\bar{B}(\xi_0, s)$ (although L depends on E, E depends only on $\bar{B}(\xi_0, s)$ if we fix the time T).

Let us show that $g(.)$ is continuous everywhere in $\mathcal{R}$. From the Lipschitz property we conclude that $g(.)$ is continuous in $\mathcal{R}/\mathcal{A}$. Since, for $\xi \in \mathcal{A}$, $\omega_{\mathcal{A}}(\xi) = g(\xi) = 0$, then, using (6.9), we have

$$|g(\eta) - g(\xi)| \leq |\omega_{\mathcal{A}}(\eta) - \omega_{\mathcal{A}}(\xi)|$$

for $\xi \in \mathcal{A}$ and $\eta \in \mathcal{R}$. Since $\omega_{\mathcal{A}}(.)$ is continuous in $\mathcal{R}$, the above inequality shows that $g(.)$ is continuous in $\mathcal{R}$.$\lhd$

Lemma 6.4 *The function $U : \mathcal{R} \to R_{\geq 0}$, defined by*

$$U(\xi) = \sup_{t \geq 0, d \in \mathcal{M}_{\mathcal{D}}} \{g(x(t, \xi; d))k(t)\} \tag{6.16}$$

is well defined given that $k : R_{\geq 0} \to R_{> 0}$ is any strictly increasing, smooth function that satisfies:

- *there are two constants $0 < c_1 < c_2 < \infty$ such that $k(t) \in [c_1, c_2]$ for all $t \geq 0$;*
- *there is a bounded, positive, decreasing, and continuous function $\tau(.)$ such that*

$$k'(t) \geq \tau(t) \text{ for all } t \geq 0$$

(For instance, $\dfrac{c_1 + c_2 t}{1 + t}$ is one example of $k(t)$).

Moreover the function U is continuous everywhere on $\mathcal{R}$, locally Lipschitz on $\mathcal{R}/\mathcal{A}$, and satisfies

$$c_1\delta(\omega_{\mathcal{A}}(\xi)) \leq U(\xi) \leq c_2\omega_{\mathcal{A}}(\xi) \tag{6.17}$$

for all $\xi \in \mathcal{R}$. Finally, the function $U(\xi)$ decreases along trajectories of the system (6.1); i.e.,

$$L_{f_{\mathbf{d}}}U(\xi) \leq -\hat{\alpha}(\omega_{\mathcal{A}}(\xi)), \ \forall d \in \mathcal{M}_{\mathcal{D}} \tag{6.18}$$

for all $\xi \in \mathcal{R}/\mathcal{A}$, where $\hat{\alpha}$ is a continuous positive definite function.

Proof:

Some additional steps should be added to the proof of [34]. One is the proof that $U(.)$ is well defined on $\mathcal{R}$, and the other is the proof of the local Lipschitz property of $U(.)$. The function $U(\xi)$ is well defined for $\xi \in \mathcal{R}$ because the value $g(x(t,\xi;d))$ exists for all $t \geq 0$ and all $d \in \mathcal{M}_{\mathcal{D}}$ given that

$$0 \leq g(x(t,\xi;d)) \leq g(\xi), \ \forall \xi \in \mathcal{R}, \ \forall t > 0$$

and $g(\xi)$ exists for $\xi \in \mathcal{R}$. The proof of (6.17) is a straightforward extension of the corresponding proof in [34].

Let us characterize $U(.)$ in any set of the form $\{\xi \in \mathcal{R} : 0 < \omega_{\mathcal{A}}(\xi) < r, r > 0\}$. Note that, for any $0 < \omega_{\mathcal{A}}(\xi) < r$,

$$U(\xi) = \sup_{0 \leq t \leq t_{\xi};\, d \in \mathcal{M}_{\mathcal{D}}} \{g(x(t,\xi;d))k(t)\}$$

where $t_\xi = T_r\left(\frac{c_1}{2c_2}\delta(\omega_{\mathcal{A}}(\xi))\right)$.

Let us prove the local Lipschitz property of $U(.)$ on $\mathcal{R}/\mathcal{A}$. For any compact set $K \subseteq \mathcal{R}/\mathcal{A}$, let

$$t_K \stackrel{\text{def}}{=} \max_{\xi \in K} t_\xi < \infty$$

Finiteness follows from the above expression of $U(\xi)$, since $K \subseteq \{\xi \in \mathcal{R} : 0 < \omega_{\mathcal{A}}(\xi) < r\}$ for some $r > 0$, and from continuity of $T_r(.), \delta(.)$, and $\omega_{\mathcal{A}}(.)$.

For $\xi_0 \notin \mathcal{A}$, pick a compact neighborhood K_0 of ξ_0 in $\mathcal{R}$ such that $K_0 \bigcap \mathcal{A} = \emptyset$. By (6.17) we have (the continuous function $\omega_{\mathcal{A}}(.)$ reaches its minimum on the compact set K_0)

$$U(\xi) \geq r_0, \ \forall \xi \in K_0 \tag{6.19}$$

for some constant $r_0 > 0$. Let $r_1 = \frac{r_0}{2c_2}$ and let

$$K_1 = K_0 \bigcap \left\{\eta : \|\eta - \xi_0\| \leq \frac{r_1}{2CL}\right\}$$

where C is a constant such that (see Proposition 5.5 of [34])

$$\|x(t,\xi;d) - x(t,\eta;d)\| \leq C\|\xi - \eta\|, \ \forall \xi, \eta \in K_0, \ 0 \leq t \leq t_{K_0}, \ d \in \mathcal{M}_{\mathcal{D}} \tag{6.20}$$

and L is to be determined later on. The set K_1 is a compact neighborhood of ξ_0 in $\mathcal{R}/\mathcal{A}$. In the following we will show that there exists some constant $\tilde{L}$ such that for any $\xi, \eta \in K_1$, we have

$$|U(\xi) - U(\eta)| \leq \tilde{L}\|\xi - \eta\| \tag{6.21}$$

We have to find a compact set in $\mathcal{R}/\mathcal{A}$ on which we can apply the Lipschitz property of $g(.)$ (this set will be K_3). We know that (it is a property of the supremum), for any $\xi \in K_1$ and any $\epsilon \in (0, r_0/2)$, there exist $t_{\xi,\epsilon} \in [0, t_{K_0}]$ and $d_{\xi,\epsilon} \in \mathcal{M}_{\mathcal{D}}$, such that

$$U(\xi) \leq g(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon}))k(t_{\xi,\epsilon}) + \epsilon$$

Furthermore, using (6.9) yields

$$U(\xi) \leq c_2 \omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) + \epsilon$$

Thus, using (6.19), we obtain

$$\omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) \geq \frac{r_0}{c_2} - \frac{\epsilon}{c_2}$$

Since $\epsilon \in (0, r_0/2)$

$$\omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) \geq r_1 \tag{6.22}$$

From Proposition 5.5 in [34] we know that there exists a compact set K_2 such that

$$x(t, \xi; d) \in K_2, \ \forall \xi \in K_1, \ \forall t \in [0, t_{K_1}], \text{ and } \forall d \in \mathcal{M}_{\mathcal{D}}$$

The set K_2 is contained in $\mathcal{R}$, given the definition of strong stability (especially the definition of stability in $\mathcal{R}$). Let L be the Lipschitz constant of $\omega_{\mathcal{A}}(.)$ in K_2. Then, for all $\eta \in K_1$, we have (we apply the Lipschitz property of $\omega_{\mathcal{A}}(.)$ in K_2)

$$\omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})) \geq \omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) - L\|x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon}) - x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})\|$$

Therefore, using (6.20) and (6.22), for all $\eta \in K_1$, we have (we apply the Lipschitz property of $x(t, \xi; d)$ in K_1))

$$\omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})) \geq r_1 - LC\|\xi - \eta\|$$

Thus, from the expression of K_1, we have

$$\omega_{\mathcal{A}}(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})) \geq \frac{r_1}{2} \tag{6.23}$$

The compact set $K_3 \overset{\text{def}}{=} K_2 \bigcap \{\xi \in \mathcal{R} : \omega_{\mathcal{A}}(\xi) \geq \frac{r_1}{2}\}$ is a subset of $\mathcal{R}/\mathcal{A}$. The inequalities (6.22) and (6.23) show that $x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})$ and $x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})$ belong to K_3. Then, we can apply the Lipschitz property of $g(.)$ on the compact K_3 and obtain

$$\begin{aligned} &|g(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) - g(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon}))| \\ &\leq C_1\|x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon}) - x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})\| \end{aligned} \tag{6.24}$$

for some $C_1 > 0$. Therefore, using the bounds on $k(t)$ and (6.24), we have

$$\begin{aligned} U(\xi) - U(\eta) &\leq g(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon}))k(t_{\xi,\epsilon}) + \epsilon - g(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon}))k(t_{\xi,\epsilon}) \\ &\leq k(t_{\xi,\epsilon})|g(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) - g(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon}))| + \epsilon \\ &\leq c_2|g(x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon})) - g(x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon}))| + \epsilon \\ &\leq C_1c_2\|x(t_{\xi,\epsilon}, \xi; d_{\xi,\epsilon}) - x(t_{\xi,\epsilon}, \eta; d_{\xi,\epsilon})\| + \epsilon \end{aligned} \tag{6.25}$$

Using Proposition 5.5 of [34], inequality (6.25) yields

$$U(\xi) - U(\eta) \leq \tilde{L}\|\xi - \eta\| + \epsilon \tag{6.26}$$

for some constant $\tilde{L} = C_1 c_2 > 0$ that only depends on the compact set K_1. Since (6.26) holds for any $\epsilon \in (0, r_0/2)$,

$$U(\xi) - U(\eta) \leq \tilde{L} \|\xi - \eta\|, \ \forall \xi, \eta \in K_1$$

Thus, by symmetry, we can prove (6.21).

Let us prove continuity of $U(.)$ everywhere in $\mathcal{R}$. From the Lipschitz property of $U(.)$ we conclude that $U(.)$ is continuous in $\mathcal{R}/\mathcal{A}$. For all $\xi \in \mathcal{A}$ we have $U(\xi) = 0$ and $\omega_{\mathcal{A}}(\xi) = 0$. Then, for all $\eta \in \mathcal{R}$, we have

$$|U(\xi) - U(\eta)| = U(\eta) \leq c_2 |\omega_{\mathcal{A}}(\xi) - \omega_{\mathcal{A}}(\eta)|$$

Since $\omega_{\mathcal{A}}(.)$ is continuous in $\mathcal{R}$, so is $U(.)$.

The proof of (6.18) is a straightforward extension of the corresponding proof in [34]. Finally, let us smooth the function $U(\xi)$ in $\mathcal{R}/\mathcal{A}$. By [34, Theorem B.1], there exists a C^{∞} function $V : \mathcal{R}/\mathcal{A} \to R_{\geq 0}$ such that for all $\xi \in \mathcal{R}/\mathcal{A}$, we have

$$\|V(\xi) - U(\xi)\| < \frac{U(\xi)}{2}$$

and

$$L_{f_{\mathbf{d}}} V(\xi) \leq -\frac{1}{2} \hat{\alpha}(\omega_{\mathcal{A}}(\xi)), \ \forall d \in \mathcal{M}_{\mathcal{D}}$$

Extend V to $\mathcal{R}$ by letting $V = 0$ on $\mathcal{A}$ and denote the extension by V. Note that V is continuous on $\mathcal{R}$ and smooth on $\mathcal{R}/\mathcal{A}$. Thus, V is the desired Lyapunov function with $\alpha_1(s) = \frac{c_1}{2} \delta(s)$, $\alpha_2(s) = \frac{3c_2}{2} s$, and $\alpha_3 = \frac{1}{2} \hat{\alpha}$. $\triangleleft$

6.3.2 Case of a Forward Complete System

Hereafter, we establish a Converse Lyapunov result without completeness of the system. In order to use the previous result, we modify the vector field of (6.1) to make it complete. Then we show that, if the system (6.1) is strongly stable, then the new system with the modified vector field is also strongly stable. Finally, by applying Theorem 6.3 we show the existence of a Lyapunov function for the new system. The same function will be the desired Lyapunov function.

Let us define a new but complete vector field.

Lemma 6.5 *Let $f : R^n \times \mathcal{D} \to R^n$ be continuous, where $\mathcal{D}$ is a compact subset of R^m. Then there exists a smooth function $a_f : R^n \to R$, with $a_f(x) \geq 1$ everywhere, such that $\|f(x, \mathbf{d})\| \leq a_f(x)$ for all $x \in R^n$ and all $d \in \mathcal{D}$.*

Proof: Let $a(x) = \max_{\mathbf{d} \in \mathcal{D}} \|f(x, \mathbf{d})\|$. This function is continuous because of the continuity of $f(., \mathbf{d})$. Choose any smooth function a_f such that $a_f(x) \geq 1 + a(x)$ for all $\xi \in R^n$. The function a_f is the desired one.◁

For any given system

$$\Sigma : \; \dot{x} = f(x, \mathbf{d}) \tag{6.27}$$

not necessarily complete, the system

$$\Sigma_b : \; \dot{x} = \frac{1}{a_f(x)} f(x, \mathbf{d}) \stackrel{\text{def}}{=} \tilde{f}(x, \mathbf{d}) \tag{6.28}$$

is complete since $\frac{\|f(x, \mathbf{d})\|}{a_f(x)} \leq 1$ for all $x \in R^n$ and all $\mathbf{d} \in \mathcal{D}$ (see [16, Theorem 2.1, page 17]).

Assume that f is continuous and locally Lipschitz in x uniformly in $\mathbf{d}$, then $\tilde{f}$ has these same two properties.

Lemma 6.6 *Assume that $\mathcal{A}$ is a compact subset of R^n. Let $\mathcal{R}$ be an open, connected set that contains $\mathcal{A}$. Suppose that the system Σ is strongly stable in $\mathcal{R}$ with respect to $\mathcal{A}$. Then, system Σ_b is strongly stable in $\mathcal{R}$ with respect to $\mathcal{A}$ as well.*

Proof: straightforward from Lemma 7.2 of [34].$\triangleleft$

Now consider the system (6.1) and let Σ_b be the corresponding complete system. By Lemma 6.6, we know that the system Σ_b is strongly stable in $\mathcal{R}$ with respect to the set $\mathcal{A}$. By applying Theorem 6.3 to the complete system Σ_b in $\mathcal{R}$, there exists a Lyapunov function V for Σ_b such that

$$\begin{aligned} \alpha_1(\omega_{\mathcal{A}}(\xi)) \ &\leq \ V(\xi) \ \leq \ \alpha_2(\omega_{\mathcal{A}}(\xi)), \ \forall \xi \in \mathcal{R} \\ L_{\tilde{f}_{\mathbf{d}}} V(\xi) \ &\leq \ -\alpha_3(\omega_{\mathcal{A}}(\xi)), \ \forall \xi \in \mathcal{R}/\mathcal{A}, \text{ and } \forall \mathbf{d} \in \mathcal{D} \end{aligned}$$

for some class $\mathcal{K}_\infty$ functions α_1, α_2 and some continuous positive definite function α_3. Since $a_f(\xi) \geq 1$ in $\mathcal{R}$, it follows that

$$L_{f_{\mathbf{d}}} V(\xi) \ \leq \ -\alpha_3(\omega_{\mathcal{A}}(\xi)), \ \forall \xi \in \mathcal{R}/\mathcal{A}, \text{ and } \forall \mathbf{d} \in \mathcal{D}$$

Thus, we conclude that V is also a Lyapunov function for Σ. $\triangleleft$

6.3.3 Smoothing of Lyapunov functions

The Lyapunov function found previously is only smooth on $\mathcal{R}/\mathcal{A}$. Hereafter we smooth this function in the domain of interest $\mathcal{R}$. To do this, we first construct an appropriate smooth function β then we prove that the function $W = \beta o V$ is the desired smooth Lyapunov function. This procedure is summarized in the following lemma and proposition:

Lemma 6.7 *Assume that* $V : \mathcal{R} \to R_{\geq 0}$ *is* C^0*, the restriction* $V|_{\mathcal{R}/\mathcal{A}}$ *is* C^∞*,* $V|_{\mathcal{A}} = 0$*, and* $V|_{\mathcal{R}/\mathcal{A}} > 0$*. Then, there exists a class* $\mathcal{K}_\infty$ *functions* β *which is smooth on* $(0, \infty)$ *such that* $\beta^{(i)}(t) \to 0$ *as* $t \to 0^+$ *for each* $i = 0, 1, \cdots$*;* $\beta' > 0$*,* $\forall t > 0$*, and* $W = \beta \circ V$ *is a* C^∞ *function over* $\mathcal{R}$*.*

Proof: see [34, Lemma 4.3] we just need to restrict the domain of interest to $\mathcal{R}$ instead of R^n.$\triangleleft$

Proposition 6.1 *If there is a Lyapunov function for (6.1) in* $\mathcal{R}$ *with respect to* $\mathcal{A}$*, then there is also a smooth Lyapunov function.*

Proof: straightforward from Proposition 4.2 in [34].$\triangleleft$

6.4 Proofs

6.4.1 Proof of Lemma 6.1

1) Positive definiteness with respect to $\mathcal{A}$: From the definition of $\omega_{\mathcal{A}}(.)$, we have $\omega_{\mathcal{A}}(\xi) \geq |\xi|_{\mathcal{A}} \geq 0$. Now, if $\omega_{\mathcal{A}}(\xi) = 0$ then $|\xi|_{\mathcal{A}} = 0$. Thus, ξ belongs to the closure of $\mathcal{A}$, then, since $\mathcal{A}$ is closed, we have $\xi \in \mathcal{A}$. Next, if $\xi \in \mathcal{A}$, then $|\xi|_{\mathcal{A}} = 0$ and $|\xi|_F \geq |F|_{\mathcal{A}}$; i.e., $\frac{1}{|\xi|_F} - \frac{2}{|F|_{\mathcal{A}}} < 0$. Thus, $\omega_{\mathcal{A}}(\xi) = |\xi|_{\mathcal{A}} = 0$, and we conclude positive definiteness of $\omega_{\mathcal{A}}(.)$ with respect to $\mathcal{A}$. Finally, if $\mathcal{R} = R^n$, then, the term $\frac{1}{|\xi|_F} - \frac{2}{|F|_{\mathcal{A}}}$ is zero and, since $|\xi|_{\mathcal{A}} \geq 0$, we have $\omega_{\mathcal{A}}(\xi) = |\xi|_{\mathcal{A}}$.

2) Continuity in $\mathcal{R}$: It follows from the fact that the maximum of two continuous functions is continuous; see [6, Exercise 4, page 136].

3) Properness in $\mathcal{R}$: As ξ approaches the boundary of $\mathcal{R}$ we have $|\xi|_F \to 0$. From the definition of $\omega_{\mathcal{A}}(.)$ we always have

$$\omega_{\mathcal{A}}(\xi) \geq \frac{1}{|\xi|_F} - \frac{2}{|F|_{\mathcal{A}}}$$

Therefore, since $\frac{2}{|F|_{\mathcal{A}}}$ is finite ($\mathcal{A}$ is compact), then $\omega_{\mathcal{A}}(\xi) \to \infty$ as ξ approaches the boundary of $\mathcal{R}$.

4) Compact Level Sets: Since $\omega_{\mathcal{A}}(.)$ is continuous in $\mathcal{R}$, then the set $\{\xi \in \mathcal{R} : r_1 \leq \omega_{\mathcal{A}}(\xi) \leq r_2\}$ is closed. Now we only need to show that this set is bounded. Note that

$$\{\xi \in \mathcal{R} : r_1 \leq \omega_{\mathcal{A}}(\xi) \leq r_2\} = \{\xi \in \mathcal{R} : r_1 \leq \omega_{\mathcal{A}}(\xi)\} \cap \{\xi \in \mathcal{R} : \omega_{\mathcal{A}}(\xi) \leq r_2\}$$

Thus, it suffices to prove that the set $\Omega \stackrel{\text{def}}{=} \{\xi \in \mathcal{R} : \omega_{\mathcal{A}}(\xi) \leq r_2\}$ is bounded.

In case $\mathcal{R}$ is bounded, the set Ω is necessarily bounded (of course this is possible because $\mathcal{A}$ is bounded). Now, consider the case where $\mathcal{R}$ is not bounded. Suppose that Ω is not bounded. Then, as $\|\xi\| \to \infty$; i.e., as ξ approaches the boundary of $\mathcal{R}$, we have that $\omega_{\mathcal{A}}(\xi) \to \infty$ which is a contradiction because, in Ω, we have $\omega_{\mathcal{A}}(\xi) \leq r_2$. The assumption that $\mathcal{A}$ be bounded is needed because otherwise the set Ω is necessarily unbounded.

5) Local Lipschitz property: Let K be a compact subset of $\mathcal{R}/\mathcal{A}$. Let us show that there exists a positive constant k such that, for all $\xi_1,\ \xi_2 \in K$, we have

$$|\omega_{\mathcal{A}}(\xi_1) - \omega_{\mathcal{A}}(\xi_2)| \leq k\|\xi_1 - \xi_2\| \tag{6.29}$$

First, consider the case where $\omega_{\mathcal{A}}(\xi_1) = |\xi_1|_{\mathcal{A}}$ and $\omega_{\mathcal{A}}(\xi_2) = |\xi_2|_{\mathcal{A}}$. In this case we have

$$|\omega_{\mathcal{A}}(\xi_1) - \omega_{\mathcal{A}}(\xi_2)| = |\, |\xi_1|_{\mathcal{A}} - |\xi_2|_{\mathcal{A}} \,| \tag{6.30}$$

Since $|.|_{\mathcal{A}}$ is locally Lipschitz, we can write

$$\left| |\xi_1|_{\mathcal{A}} - |\xi_2|_{\mathcal{A}} \right| \leq \|\xi_1 - \xi_2\| \tag{6.31}$$

Then, using (6.31), the inequality (6.29) is true.

Second, consider without loss of generality that

$$\begin{aligned} \omega_{\mathcal{A}}(\xi_1) &= |\xi_1|_{\mathcal{A}} && (6.32) \\ \omega_{\mathcal{A}}(\xi_2) &= \frac{1}{|\xi_2|_F} - \frac{2}{|F|_{\mathcal{A}}} && (6.33) \end{aligned}$$

Then, from the definition of $\omega_{\mathcal{A}}(.)$, we have

$$\begin{aligned} \omega_{\mathcal{A}}(\xi_1) &\geq \frac{1}{|\xi_1|_F} - \frac{2}{|F|_{\mathcal{A}}} && (6.34) \\ \omega_{\mathcal{A}}(\xi_2) &\geq |\xi_2|_{\mathcal{A}} && (6.35) \end{aligned}$$

Using (6.32) and (6.35) yields

$$\omega_{\mathcal{A}}(\xi_1) - \omega_{\mathcal{A}}(\xi_2) \leq |\xi_1|_{\mathcal{A}} - |\xi_2|_{\mathcal{A}} \tag{6.36}$$

Then, using (6.31) in (6.36), we have

$$\omega_{\mathcal{A}}(\xi_1) - \omega_{\mathcal{A}}(\xi_2) \leq \|\xi_1 - \xi_2\| \tag{6.37}$$

Using (6.33) and (6.34) yields

$$\begin{aligned}
\omega_{\mathcal{A}}(\xi_2) - \omega_{\mathcal{A}}(\xi_1) &\leq \frac{1}{|\xi_2|_F} - \frac{2}{|F|_{\mathcal{A}}} - \frac{1}{|\xi_1|_F} + \frac{2}{|F|_{\mathcal{A}}} \\
&\leq \frac{1}{|\xi_2|_F} - \frac{1}{|\xi_1|_F} \\
&\leq \left| \frac{1}{|\xi_2|_F} - \frac{1}{|\xi_1|_F} \right| \qquad (6.38)
\end{aligned}$$

Now, if we can prove that, in K, there exists a positive constant k_2 such that

$$\left| \frac{1}{|\xi_2|_F} - \frac{1}{|\xi_1|_F} \right| \leq k_2 \|\xi_1 - \xi_2\| \qquad (6.39)$$

then, using (6.37), (6.38), and (6.39) in addition to (6.31) proves the local Lipschitz property of $\omega_{\mathcal{A}}(.)$ with $k = \max(1, k_2)$. Hence, it suffices to prove (6.39). Since $\mathcal{R}$ is an open set, F is a closed set. Thus, with a similar reasoning as the one that lead to (6.31) we have

$$\Big| |\xi_1|_F - |\xi_2|_F \Big| \leq \|\xi_1 - \xi_2\| \qquad (6.40)$$

Since K is compact, then there exists a positive constant k_2 such that

$$\frac{1}{|\xi_1|_F |\xi_2|_F} \leq k_2 \qquad (6.41)$$

Using (6.40) and (6.41) we can write

$$\begin{aligned}
\left| \frac{1}{|\xi_1|_F} - \frac{1}{|\xi_2|_F} \right| &\leq \left| \frac{|\xi_2|_F - |\xi_1|_F}{|\xi_1|_F |\xi_2|_F} \right| \\
&\leq L_2 \|\xi_1 - \xi_2\| \qquad (6.42)
\end{aligned}$$

This concludes the proof of Lemma 6.1.◁

6.4.2 Proof of Theorem 6.2

Assume that the system (6.1) is UAS with respect to $\mathcal{A}$ with $\mathcal{R}$ as in the theorem. Then, the system is forward complete in $\mathcal{R}$ and Definition 3.1 applies. Let us show strong stability of (6.1) with respect to $\mathcal{A}$. We divide the proof into two parts. First, we show uniform stability in $\mathcal{R}$. Then, we show uniform attraction in $\mathcal{R}$. The forthcoming proof is inspired by that of Theorem 12 of [32].

Since $\mathcal{R}$ is open, then there exists an open ball $B_0 = \{\xi : |\xi|_{\mathcal{A}} < r_0,\ r_0 > 0\}$ around $\mathcal{A}$ such that B_0 is the largest ball in $\mathcal{R}$ that conatins $\mathcal{A}$ (i.e., B_0 is the first ball whose boundary intersects with the boundary of $\mathcal{R}$). It is important to notice that r_0 may be finite. Let us define the function

$$\phi(s) = \sup_{|\xi|_{\mathcal{A}} \leq s} \omega_{\mathcal{A}}(\xi), \quad \text{for } s \in [0, r_0)$$

the function $\phi(s)$ is continuous, positive definite, increasing (not necessarily strictly), and goes to infinity as $s \to r_0$. Moreover, we have

$$\omega_{\mathcal{A}}(\xi) \leq \phi(|\xi|_{\mathcal{A}})$$

for all $\xi \in B_0$. Let $\gamma(s)$ be a class $\mathcal{K}$ function such that $\gamma(s) \geq k\phi(s)$ with $k > 1$. Then, using the above inequality and the definition of $\omega_{\mathcal{A}}(.)$, we have

$$|\xi|_{\mathcal{A}} \ \leq\ \omega_{\mathcal{A}}(\xi)\ \leq\ \gamma(|\xi|_{\mathcal{A}}) \tag{6.43}$$

$$\lim_{s \to r_0} \gamma(s) \ =\ \infty \tag{6.44}$$

Part I - Uniform Stability in $\mathcal{R}$:

Fix $\epsilon > 0$. Consider the positive constant $\eta = \gamma^{-1}(\epsilon)$; γ^{-1} exists since $\gamma : [0, r_0] \to$

$[0,\infty)$ is strictly increasing and onto. From Definition 3.1 we know that there exists a positive constant $\delta_1(\eta)$ such that, for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq 0$, we have

$$|x(t, x_0; d)|_{\mathcal{A}} \leq \eta \tag{6.45}$$

whenever $|x_0|_{\mathcal{A}} \leq \delta_1(\eta)$.

Now, let $\omega_{\mathcal{A}}(x_0) \leq \delta_1(\eta)$ which implies, given the definition of $\omega_{\mathcal{A}}$, that $|x_0|_{\mathcal{A}} \leq \delta_1(\eta)$. This implies (6.45). Then, using (6.43), we have

$$\omega_{\mathcal{A}}(x(t, x_0; d)) \leq \gamma(\eta) = \epsilon$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq 0$, whenever $\omega_{\mathcal{A}}(x_0) \leq \tilde{\delta}(\epsilon) = \delta_1(\gamma^{-1}(\epsilon))$.

Now, we have to find a class $\mathcal{K}_\infty$ function δ to replace $\tilde{\delta}$ in the above statement. For a fixed ϵ, let $\bar{\delta}(\epsilon)$ be the supremum of such $\tilde{\delta}$. Then,

$$\omega_{\mathcal{A}}(x_0) < \bar{\delta}(\epsilon) \Longrightarrow \omega_{\mathcal{A}}(x(t, x_0; d)) \leq \epsilon \tag{6.46}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq 0$, and if $\delta_2 > \bar{\delta}$, then there exists at least one initial state $\tilde{x}_0$ and one function $\tilde{d} \in \mathcal{M}'_{\mathcal{D}}$ such that

$$\omega_{\mathcal{A}}(\tilde{x}_0) \leq \delta_2 \text{ and } \sup_{t \geq 0} \omega_{\mathcal{A}}(x(t, \tilde{x}_0, \tilde{d})) > \epsilon \tag{6.47}$$

Let $\hat{\delta}(\epsilon) = \frac{1}{2}\bar{\delta}(\epsilon)$. Then, (6.46) can be written as

$$\omega_{\mathcal{A}}(x_0) \leq \hat{\delta}(\epsilon) \Longrightarrow \omega_{\mathcal{A}}(x(t, x_0; d)) \leq \epsilon \tag{6.48}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq 0$. The function $\hat{\delta}(\epsilon)$ is positive and non-decreasing, but not necessarily continuous. Furthermore, we have $\lim_{\epsilon \to 0} \hat{\delta}(\epsilon) = 0$ because we can see from (6.48) that $\hat{\delta}(\epsilon) \leq \epsilon$, otherwise we would get a contradiction at $t = 0$.

Claim 1:

$$\lim_{\epsilon \to \infty} \hat{\delta}(\epsilon) = \infty \tag{6.49}$$

Proof. Assume that $\lim_{\epsilon \to \infty} \hat{\delta}(\epsilon) = \delta_{\infty} < \infty$. According to (6.47), for every $i \geq 1$ and for $\delta_2(i) = 2\hat{\delta}(i) + 1 = \bar{\delta}(i) + 1 > \bar{\delta}(i)$, there exists $x_{0i} \in \mathcal{R}$ and $d_i \in \mathcal{M}'_{\mathcal{D}}$ such that

$$\omega_{\mathcal{A}}(x_{0i}) \leq \delta_2(i) \text{ and } \sup_{t \geq 0} (\omega_{\mathcal{A}}(x(t, x_{0i}; d_i))) > i \tag{6.50}$$

This means that

$$\limsup_{i \to \infty} \{ \sup_{t \geq 0} (\omega_{\mathcal{A}}(x(t, x_{0i}; d_i))) \} = \infty \tag{6.51}$$

Let us find $0 < T^{\star} \leq \infty$ and a solution $x(t, x_0; d)$, where $x_0 \in \mathcal{R}$ and $\omega_{\mathcal{A}}(x_0) \leq \delta_{\infty} + 1$, such that $\lim_{t \to T^{\star}} x(t, x_0; d) = \infty$ and let us show that this constitutes a contradiction to the fact that x_0 belongs to $\mathcal{R}$, a subset of the region of attraction.

Let $x_i(t) \stackrel{\text{def}}{=} x(t, x_{0i}; d_i)$, then $\{x_i(t)\}$ is the sequence of solutions defined by (6.50). Define

$$T^{\star} \stackrel{\text{def}}{=} \sup\{T \geq 0 : \limsup_{i \to \infty} (\max_{0 \leq t \leq T} \omega_{\mathcal{A}}(x_i(t))) < \infty\} \tag{6.52}$$

Note that $0 < T^{\star} \leq \infty$. Given (6.51), the quantity $T^{\star}$ is the supremum of all $T \geq 0$ such that all of the elements of the sequence are finite for all $t \in [0, T]$.

Case 1: First, suppose that

$$T^\star < \infty$$

Consider a sequence $\{\tau_i\}$ of positive constants such that for every $i \geq 1$ we have

$$\text{if } \omega_{\mathcal{A}}(x_0) \leq i, \text{ then } \omega_{\mathcal{A}}(x(t, x_0; d)) \leq i + 1, \;\; 0 \leq t \leq 2\tau_i \tag{6.53}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$. This sequence is not empty because of the continuity of $x(t, x_0; d)$. Without loss of generality we can take $T^\star > \tau_1 > \tau_2 > \cdots$ and $\lim_{i \to \infty} \tau_i = 0$.

Let $\{x_i^1(t)\}$ be the subsequence of $\{x_i(t)\}$ such that

$$\omega_{\mathcal{A}}(x_i^1(T^\star - \tau_1)) > 1, \;\; \forall i \geq 1 \tag{6.54}$$

This subsequence is infinite. To show that, assume that it is finite and let I_1 be the set of indices such that $x_i(t)$, $i \in I_1$, does not belong to $\{x_i^1(t)\}$ (i.e., I_1 is infinite). Take $i \in I_1$, then $\omega_{\mathcal{A}}(x_i(T^\star - \tau_1)) \leq 1$, then, from the definition of τ_1, (6.53), and uniqueness of solutions, we have

$$\omega_{\mathcal{A}}(x_i(t)) \leq 2, \;\; \text{for } T^\star - \tau_1 \leq t \leq T^\star + \tau_1 \tag{6.55}$$

Then, from the definition (6.52) of $T^\star$, we have

$$\limsup_{i \in I_1} \{ \max_{0 \leq t \leq T^\star - \tau_1} (\omega_{\mathcal{A}}(x_i(t))) \} < \infty \tag{6.56}$$

Then, using (6.55) and (6.56), we can write

$$\limsup_{i \in I_1} \{ \max_{0 \leq t \leq T^\star + \tau_1} (\omega_{\mathcal{A}}(x_i(t))) \} < \infty \tag{6.57}$$

Since I_1 is infinite, then

$$\limsup_{i \to \infty} \{ \max_{0 \leq t \leq T^\star + \tau_1} (\omega_{\mathcal{A}}(x_i(t))) \} < \infty \tag{6.58}$$

which is a contradiction with (6.52). Thus, I_1 is finite and the subsequence $\{x_i^1(t)\}$ is infinite.

Moreover, let $\{x_i^2(t)\}$ be a subsequence of $\{x_i^1(t)\}$ such that

$$\omega_{\mathcal{A}}(x_i^2(T^\star - \tau_2)) > 2, \ \ \forall i \geq 1 \tag{6.59}$$

As previously, we can show that the subsequence $\{x_i^2(t)\}$ is infinite. Similarly, we construct a family of subsequences. Now, as we continue, we end up with the sequence $\{\bar{x}_j(t)\}$ such that

$$\omega_{\mathcal{A}}(\bar{x}_i(T^\star - \tau_j)) \geq j, \ \ \forall j \geq 1, \ \ \forall i \geq 1 \tag{6.60}$$

Since $\omega_{\mathcal{A}}(\bar{x}_i(0)) \leq \delta_\infty + 1 < \infty$, then, using Proposition 5.1 of [34], the set of solutions starting from $\bar{x}_i(0)$ is compact on $[0, T]$ for any $0 < T < T^\star$. Thus, the sequence of solutions $\{\bar{x}_i\}$ is uniformly bounded (uniformly in i) on compact intervals of time $[0, T]$, $0 < T < T^\star$. Let $\{\bar{d}_i\}$ be the sequence of time-varying parameters corresponding to the sequence of solutions $\{\bar{x}_i\}$. It can be shown that

$$\|\bar{x}_i(t_1) - \bar{x}_i(t_2)\| \leq \int_{t_1}^{t_2} \|f(\bar{x}_i(\tau), \bar{d}_i(\tau))\| d\tau \tag{6.61}$$

Since the sequences $\{\bar{x}_i\}$ and $\{\bar{d}_i\}$ are uniformly bounded and $f(.,.)$ is continuous, then there exists a constant c_x, independent of i, such that

$$\|\bar{x}_i(t_1) - \bar{x}_i(t_2)\| \leq c_x |t_1 - t_2| \tag{6.62}$$

for all $i \geq 1$. Then, using (6.62), for every $\epsilon > 0$, there exists a positive constant $c_t > \epsilon / c_X$ such that

$$\|\bar{x}_i(t_1) - \bar{x}_i(t_2)\| < \epsilon \tag{6.63}$$

for all t_1, t_2 such that $|t_1 - t_2| < c_t$ and all $i \geq 1$. Then, see [6, Section 4.5, page 208], the sequence $\{\bar{x}_i\}$ is equicontinuous. Thus, by Ascoli-Arzela's Theorem [6, Theorem 4.5.8; Exercises 4.5.9, Probleml], this sequence has a subsequence (indexed by i_j) that converges (as $i_j \to \infty$) for every $t \in [0, T]$, $0 < T < T^\star$. Given the compactness of $[0, T]$, the previous subsequence converges uniformly in t (see [6, Section 4.5, page 206] for the definition of uniform convergence). By [12, Theorem 2.11], we see that the limit is a continuous function in t on $[0, T]$, $0 < T < T^\star$.

Let $\{\bar{d}_{i_j}\}$ be the subsequence corresponding to the subsequence $\{\bar{x}_{i_j}\}$. By the Mean Value Theorem we can show that the sequence $\{\bar{d}_{i_j}\}$ is equicontinuous and that it has a subsequence (indexed by i_{j_k}) that converges (as $i_{j_k} \to \infty$) uniformly (in t) in every interval $[0, T]$, $0 < T < T^\star$. It can be shown that the set of functions of $\mathcal{M}'_{\mathcal{D}}$ defined on $[0, T]$ is a closed subset of the set of continuous functions defined on $[0, T]$ with values in R^d. Then, the limit of the convergent subsequence belongs to $\mathcal{M}'_{\mathcal{D}}$ for all $0 < T < T^\star$.

Let $\{\bar{x}_{0i_{j_k}}\}$ be the subsequence corresponding to the subsequence $\{\bar{x}_{i_{j_k}}\}$. Since $\omega_{\mathcal{A}}(\bar{x}_i(0)) \leq \delta_\infty + 1$, we conclude, using Lemma 6.1, that the sequence of initial conditions belongs to a compact set. Therefore, the sequence of initial states indexed by i_{j_k} has a convergent subsequence in $\{\xi : \omega_{\mathcal{A}}(\xi) \leq \delta_\infty + 1\}$. Thus, the corresponding subsequences (now indexed simply by i) of solutions $\{\bar{x}_i\}$ and time-varying parameters $\{\bar{d}_i\}$ converge uniformly (in t) in every interval $[0, T]$, $0 < T < T^\star$. Let $\bar{x}_0$, $\bar{x}(t)$, and $\bar{d}(t)$ be the respective limits.

Let us prove that $\bar{x}(t)$ is the solution of $\dot{x} = f(x, \bar{d})$ for $x(0) = \bar{x}_0$. Since $f(x, d(t))$

is continuous in x and t we can write

$$\bar{x}_i(t) = \bar{x}_{0i} + \int_0^t f(\bar{x}_i(\tau), \bar{d}_i(\tau))d\tau, \ \ 0 \le t \le T < T^\star \tag{6.64}$$

Since $f(.,.)$ is continuous in its arguments and the sequences $\{\bar{x}_i\}$ and $\{\bar{d}_i\}$ are uniformly bounded, then the sequence $\{f(x_i, d_i)\}$ is uniformly bounded (in i) and converges to $f(\bar{x}, \bar{d})$ for every $t \in [0, T]$, $0 < T < T^\star$ (see [6, Exercises 4.5.3, Problem 2]). Then, by the Lebesgue Dominated Convergence Theorem (see [12, Section 5.11, page 232]), we can exchange limit and integration. Thus, we have

$$\bar{x}(t) = \bar{x}_0 + \int_0^t f(\bar{x}(\tau), \bar{d}(\tau))d\tau, \ \ 0 \le t \le T < T^\star \tag{6.65}$$

Therefore, $\bar{x}(t)$ is the solution of $\dot{x} = f(x, \bar{d})$ for $x(0) = \bar{x}_0$; it is defined for $t \in [0, T^\star)$.

From (6.60) and the fact that $\lim_{i \to \infty} \tau_i = 0$ we conclude that

$$\lim_{t \to T^\star} \omega_{\mathcal{A}}(\bar{x}(t)) = \infty \tag{6.66}$$

This is impossible since $\bar{x}_0$ belongs to the region of attraction.

Case 2: Now, let

$$T^\star = \infty$$

Since $T^\star = \infty$, then

$$\limsup_{i \to \infty} \{ \max_{0 \le t \le T} (\omega_{\mathcal{A}}(x_i(t))) \} < \infty \tag{6.67}$$

for all finite $T > 0$. Since all x_{0i} belong to the region of attraction then, by Definition

3.3, for $\mu > 0$ there exist a finite $T_i(\mu) > 0$ such that

$$|x_i(t)|_{\mathcal{A}} < \mu, \ \ t \geq T_i \tag{6.68}$$

for all $i \geq 1$. Choose μ such that $\mu < r_0$. Then, using (6.43), we have

$$\omega_{\mathcal{A}}(x_i(t)) \leq \gamma(\mu), \ \ \text{for } t \geq T_i \tag{6.69}$$

Thus, using (6.67) and (6.69), we have

$$\limsup_{i \to \infty}\{\sup_{t \geq 0}(\omega_{\mathcal{A}}(x_i(t)))\} < \infty \tag{6.70}$$

which contradicts (6.51).

Thus, we showed that if δ_∞ is finite, we reach a contradiction. Therefore, $\delta_\infty = \infty$.$\triangleleft$

Since, by Claim 1, $\lim_{\epsilon \to \infty} \hat{\delta}(\epsilon) = \infty$, then we can choose a class $\mathcal{K}_\infty$ function $\delta(.)$ such that $\delta(r) \leq \hat{\delta}(r)$. Hence, given $\epsilon > 0$, we have

$$\omega_{\mathcal{A}}(x(t, x_0; d)) \leq \epsilon$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq 0$, whenever $\omega_{\mathcal{A}}(x_0) \leq \delta(\epsilon) \leq \hat{\delta}(\epsilon)$. Thus, we have proved property 1.

Part II - Uniform Attraction in $\mathcal{R}$:

Fix ϵ. Consider the positive constant $\eta = \gamma^{-1}(\epsilon)$. From Definition 3.1 we know that there exists $T = T(\eta) > 0$ and $\alpha > 0$ such that

$$|x(t, x_0; d)|_{\mathcal{A}} < \eta \tag{6.71}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq T$, whenever $|\xi_0|_{\mathcal{A}} < \alpha$. Since $\mathcal{R}$ is a time-independent subset of the region of attraction, without loss of generality, we can choose α such that $\{\xi : |x_0|_{\mathcal{A}} < \alpha\} \subset \mathcal{R}$.

Let $\omega_{\mathcal{A}}(x_0) < \alpha$ which implies that $|x_0|_{\mathcal{A}} < \alpha$. Let $\tilde{T}(\epsilon) = T(\gamma^{-1}(\epsilon))$. Then, using (6.71), we have

$$\omega_{\mathcal{A}}(x(t, x_0; d)) < \gamma(\eta) < \epsilon \tag{6.72}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq \tilde{T}$, whenever $\omega_{\mathcal{A}}(x_0) < \alpha$.

Fix ϵ, $r > 0$ and let us prove that there exits $T = T(\epsilon, r) > 0$ such that

$$\omega_{\mathcal{A}}(x(t, x_0; d)) < \epsilon \tag{6.73}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq T$, whenever $\omega_{\mathcal{A}}(x_0) < r$.

Assume that the above is not true. Then, $\forall T > 0$ there exists an initial state $\bar{x}_0 = \bar{x}_0(T)$ with $\omega_{\mathcal{A}}(x_0) < r$ such that $\omega_{\mathcal{A}}(x(\bar{t}, \bar{x}_0, \bar{d})) \geq \epsilon$ for some $\bar{t} \geq T$ and some $\bar{d} \in \mathcal{M}'_{\mathcal{D}}$. In other words, if we take a sequence $\{T_i\}$ of positive numbers such that $\lim_{i \to \infty} T_i = \infty$, then there exist sequences $\{t_i\} \subset R_{\geq 0}$, $\{x_{0i}\} \subset \mathcal{R}$, and $\{d_i\} \subset \mathcal{M}'_{\mathcal{D}}$ such that

$$\lim_{i \to \infty} t_i = \infty, \;\; \omega_{\mathcal{A}}(x_{0i}) < r, \;\; \omega_{\mathcal{A}}(x(t_i, x_{0i}; d_i)) \geq \epsilon \tag{6.74}$$

Using the uniform stability property (6.48) and uniqueness of solutions, we can write, for any $\tau \geq 0$,

$$\omega_{\mathcal{A}}(x(t, x_0; d)) < \epsilon \tag{6.75}$$

for all $d \in \mathcal{M}'_{\mathcal{D}}$ and all $t \geq \tau$, whenever $\omega_{\mathcal{A}}(x(\tau, x_0; d)) < \hat{\delta}(\epsilon)$ (note that ϵ is fixed). Hence,

$$\omega_{\mathcal{A}}(x(t, x_{0i}; d_i)) \geq \hat{\delta}(\epsilon), \;\; \forall\, t \in [0, t_i] \tag{6.76}$$

Since $\omega_{\mathcal{A}}(x_{0i}) < r$, using the uniform stability property (6.48), we have

$$\omega_{\mathcal{A}}(x(t, x_{0i}; d_i)) < \hat{\delta}^{-1}(r), \;\; \forall t \geq 0 \tag{6.77}$$

Thus the sequence of solutions $\{x(t, x_{0i}; d_i)\}$ is uniformly bounded (uniformly in i).

As in Part I, we can find subsequences (indexed by i) $\{\bar{x}_i\}$, $\{\bar{x}_{0i}\}$, and $\{\bar{d}_i(t)\}$ which converge on any interval $[0, T]$, $T > 0$ to $\bar{x}(t)$, $\bar{x}_0$, and $\bar{d}(t)$, respectively. Furthermore, $\bar{x}_0 \in \{\xi \in \mathcal{R} : \omega_{\mathcal{A}}(\xi) \leq r\}$, $\bar{d} \in \mathcal{M}'_{\mathcal{D}}$, for any $T > 0$. Moreover, $\bar{x}(t)$ is the solution of $\dot{x} = f(x, \bar{d})$, $x(0) = \bar{x}_0$, defined for $0 \leq 0 < \infty$.
From (6.76) and the fact that $\lim_{i \to \infty} t_i = \infty$ we conclude that

$$\omega_{\mathcal{A}}(\bar{x}(t)) \geq \hat{\delta}(\epsilon), \;\; t \in [0, \infty) \tag{6.78}$$

This is impossible since $\bar{x}_0 \in \mathcal{R}$ that is, the solution $\bar{x}(t)$ must converge to $\mathcal{A}$; i.e. $\omega_{\mathcal{A}}(x(t)) \to 0$ as $t \to \infty$. Thus $T(\epsilon, r)$ exits such that (6.73) is satisfied.$\triangleleft$

CHAPTER 7

Conclusions

In order to implement a controller using output feedback, we can rely on a separation principle which divides the design process into two steps. The first step consists of designing a state feedback controller to achieve the desired performance. The second step consists of estimating the state using an observer, and then using these estimates in the control law. This implementation should recover the performance achieved under the state feedback control, at least asymptotically. This is the task we have set to achieve in this work for a certain class of nonlinear systems. First, we gave a comprehensive formulation for the state feedback controller performance which describes the behavior of the entire state. Then, we recovered this behavior using an observer, thus proving the possibility of an output feedback implementation of the control law.

State feedback controllers achieve a wide range of performance. Sometimes we need to steer the state to a point and have it stay thereafter. Other times we need a part of the state to track a constant or a time-varying reference. We also may require the input-output response of the system to satisfy certain criteria and limitations or we may require the controller to minimize some performance index appropriate to the task at hand. We opted to a stabilization-type of performance description. We found that this formulation covers a wide range of design objectives. We found that numerous design objectives can be formulated as a stabilization problem of an equilibrium point or a compact, positively invariant set. In both cases we considered a system that contains one or several chains of integrators. In the case of an equilibrium point we considered time-invariant systems. However, in the case of invariant set we allowed a time-varying bounded parameter in the system's nonlinearities. Moreover, we used a high-gain observer to implement the control law and we required this law to be globally bounded which can be achieved by saturation outside a region of interest. It is noteworthy that this separation principle does not require any particular state feedback structure as long as the control law is globally bounded.

The high-gain observer has an adjustable gain which allowed us to estimate the state quickly enough to recover the full stabilizing power of the control law. In addition to recovering local stability, we recovered arbitrary compact subsets of the region of attraction (or sometimes an estimate of it) achieved under the state feedback controller. Moreover, we showed that trajectories under the output feedback controller approach those achieved under the state feedback controller as the observer gain approaches infinity. Finally, for each stabilization case we presented several examples to show some sources of the class of systems considered and to illustrate how we can apply the separation results previously proved.

To prove the recovery of the aforementioned set of performance measures we divided the task into several steps. First, we showed that trajectories starting in an *a priori* given compact subset of the region of attraction are bounded. For this we needed a Lyapunov function that approaches infinity at the boundary of this region. This Lyapunov function was supplied by results due to Kurzweil [32] in the case of an equilibrium point and by results due to Sontag *et al* [34] for the case of an invariant set. Second, we showed that these trajectories come arbitrarily close to the attractor (a point or a set). Before concluding asymptotic stability, we showed convergence of these trajectories to the ones achieved under the state feedback controller. As for the local property of asymptotic stability, we discussed three cases. First, we discussed the case where the modeling error is zero; i.e., we perfectly know the nonlinearities at the end of the chains of integrators and we use them in the design of the observer. Second, we discussed the case where the convergence to the attractor under the state feedback controller is of an exponential type. In this case we used a local Lyapunov function supplied by the classic Lyapunov theory for the case of an equilibrium point and by an adaptation of some results of Yoshizawa [55] to our purposes for the case of a set. Third, we discussed the case where we have asymptotic stability under the state feedback controller but where we have nonzero modeling error; i.e., we used an imperfect knowledge of the system's nonlinearities to build the observer. In this last case, some conditions on the size of the modeling error had to be imposed in order to recover local asymptotic stability. These conditions state that the modeling error should be proportional to the rate of convergence of trajectories to the attractor (raised to a power less than one) under the state feedback controller. Moreover, we note that our results can only show semiglobal stabilization under output feedback even when the state feedback control achieves global stabilization.

It is noteworthy that for the case where the attractor is a set we used some results provided in [34] when the region of attraction is the whole state space. However, when this region is a subset of the state space we had to adapt the results of [34] to our purposes which required that we restrict the class of the time-varying parameters and we only recovered an open estimate of the region of attraction.

Finally, we reviewed several techniques in designing high-gain observers. These techniques are: pole-placement algorithms, Riccati equation-based algorithms, and Lyapunov equation-based algorithms. In this work we showed that the observer gain structures provided by these different techniques are asymptotically similar to the structure used to show the separation results. Therefore, we concluded that the abovementioned separation results hold for all these observer design techniques.

The idea of this work (providing a formulation of controller performance and establishing separation results based on this formulation) is still a fertile ground for future research. Actually, we noticed that the controller performance of some design examples can be described as achieving partial stability (stability of part of the state) of the system as in [2], or as input-to-state stability (the magnitude of the trajectory is function of the magnitude of the input) as in [29]. Moreover, an optimization-type of performance can be investigated (minimization of a performance index, for example) as in [1].

APPENDICES

APPENDIX A

Technical Lemmas

Lemma A.1 *Let $A = \bar{A} \times \{e = 0\}$, where $\bar{A}$ is a compact subset of R^n. Then, the following holds:*

$$|(e,\eta)|_A = [\|e\|^2 + |\eta|_{\bar{A}}^2]^{1/2} \tag{A.1}$$

$$\|e\|^2 \leq |(e,\eta)|_A^2 \tag{A.2}$$

$$|\eta|_{\bar{A}}^2 \leq |(e,\eta)|_A^2 \tag{A.3}$$

Proof: The definition of the distance with respect to a set implies

$$\begin{aligned} |(e,\eta)|_A &= \inf_{b \in A} \|(e,\eta) - b\| = \inf_{\bar{b} \in \bar{A}} \|(e,\eta) - (0,\bar{b})\| \\ &= [\|e\|^2 + \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2]^{1/2} \end{aligned}$$

To prove (A.1) , it suffices to show that

$$|\eta|_{\bar{A}}^2 = \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2 \tag{A.4}$$

First, since $\bar{A}$ is compact, then, there exists $\bar{b}_0 \in \bar{A}$ such that

$$|\eta|_{\bar{A}} = \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\| = \|\eta - \bar{b}_0\|$$

Thus, from the definition of infimum, we have

$$\inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2 \leq \|\eta - \bar{b}_0\|^2 = |\eta|^2_{\bar{A}} \tag{A.5}$$

Next, the definition of infimum implies that, for any $\epsilon > 0$, there exists $\bar{b}_1 \in \bar{A}$ such that

$$\|\eta - \bar{b}_1\|^2 - \epsilon \leq \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2$$

Thus, since $|\eta|^2_{\bar{A}} \leq \|\eta - \bar{b}_1\|^2$, we have

$$|\eta|^2_{\bar{A}} - \epsilon \leq \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2$$

This inequality is true for an arbitrary $\epsilon > 0$; then,

$$|\eta|^2_{\bar{A}} \leq \inf_{\bar{b} \in \bar{A}} \|\eta - \bar{b}\|^2 \tag{A.6}$$

From (A.5) and (A.6) we conclude (A.4).

Finally, using (A.1) , we have

$$\begin{aligned} \|e\|^2 &\leq \|e\|^2 + |\eta|^2_{\bar{A}} = |(e, \eta)|^2_A \\ |\eta|^2_{\bar{A}} &\leq \|e\|^2 + |\eta|^2_{\bar{A}} = |(e, \eta)|^2_A \end{aligned}$$

Thus, we conclude (A.2) and (A.3).◁

Lemma A.2 *Let $A = \bar{A} \times \{e = 0\}$, where $\bar{A}$ is a compact subset of R^n. Let $(\eta(t), e(t))$ be the solution of the system (4.35) under the state feedback controller $u^\star$ and assume that*

$$|\eta(t)|_{\bar{A}} \leq \beta_1(|\eta(0)|_{\bar{A}}, t) + \gamma(\sup_{0 \leq \tau \leq t} \|e(\tau)\|), \ \forall t \geq 0 \tag{A.7}$$

$$\|e(t)\| \leq \beta_2(\|e(0)\|, t), \ \forall t \geq 0 \tag{A.8}$$

where $\beta_1(.,.)$ and $\beta_2(.,.)$ are class $\mathcal{KL}$ functions. Then, there exists a class $\mathcal{KL}$ function $\beta(.,.)$ such that

$$|(e(t), \eta(t))|_A \leq \beta(|(e(0), \eta(0))|_A, t), \ \forall t \geq 0 \tag{A.9}$$

Proof: With a proof similar to that of Lemma 5.6 of [28], we show asymptotic stability with respect to the set A.

First, using (A.7) and (A.8) we can write

$$|\eta(t)|_{\bar{A}} \leq \beta_1(|\eta(s)|_{\bar{A}}, t-s) + \gamma(\sup_{s \leq \tau \leq t} \|e(\tau)\|) \tag{A.10}$$

$$\|e(t)\| \leq \beta_2(\|e(s)\|, t-s) \tag{A.11}$$

for $0 \leq s \leq t$.

Next, let us rewrite (A.10) for $s = t/2$

$$|\eta(t)|_{\bar{A}} \leq \beta_1(|\eta(t/2)|_{\bar{A}}, t/2) + \gamma(\sup_{t/2 \leq \tau \leq t} \|e(\tau)\|) \tag{A.12}$$

In order to estimate $\eta(t/2)$ we use (A.10) with $s = 0$

$$|\eta(t/2)|_{\bar{A}} \leq \beta_1(|\eta(0)|_{\bar{A}}, t/2) + \gamma(\sup_{0 \leq \tau \leq t/2} \|e(\tau)\|) \tag{A.13}$$

Furthermore, using (A.11), we have

$$\sup_{0 \leq \tau \leq t/2} \|e(\tau)\| \leq \beta_2(\|e(0)\|, 0) \tag{A.14}$$

$$\sup_{t/2 \leq \tau \leq t} \|e(\tau)\| \leq \beta_2(\|e(0)\|, t/2) \tag{A.15}$$

Now, by substituting (A.13), (A.14) and (A.15) into (A.12) we get

$$\begin{aligned}
|\eta(t)|_{\bar{A}} &\leq \beta_1(\beta_1(|\eta(0)|_{\bar{A}}, t/2) + \gamma(\sup_{0 \leq \tau \leq t/2} \|e(\tau)\|), t/2) \\
&\quad + \gamma(\sup_{t/2 \leq \tau \leq t} \|e(\tau)\|) &\text{(A.16)} \\
|\eta(t)|_{\bar{A}} &\leq \beta_1(\beta_1(|\eta(0)|_{\bar{A}}, t/2) + \gamma(\beta_2(\|e(0)\|, 0)), t/2) \\
&\quad + \gamma(\beta_2(\|e(0)\|, t/2)) &\text{(A.17)}
\end{aligned}$$

Notice that

$$|(e, \eta)|_A = \inf_{\bar{b} \in \bar{A}} \|(e, \eta - \bar{b})\| \leq \|e\| + |\eta|_{\bar{A}} \tag{A.18}$$

Define

$$\beta(r, s) = \beta_1(\beta_1(r, s/2) + \gamma(\beta_2(r, 0)), s/2) + \gamma(\beta_2(r, s/2)) + \beta_2(r, s)$$

Notice that β is a class $\mathcal{KL}$ function. Then, using (A.18) and (A.2)–(A.3), we have

$$\begin{aligned}
|(e(t), \eta(t))|_A &\leq \beta_1(\beta_1(|(e(0), \eta(0))|_A, t/2) + \gamma(\beta_2(|(e(0), \eta(0))|_A, 0)), t/2) \\
&\quad + \gamma(\beta_2(|(e(0), \eta(0))|_A, t/2)) + \beta_2(|(e(0), \eta(0))|_A, t) &\text{(A.19)} \\
|(e(t), \eta(t))|_A &\leq \beta(|(e(0), \eta(0))|_A, t) &\text{(A.20)}
\end{aligned}$$

for all $t \geq 0$.◁

Lemma A.3 *Let $\mathcal{A}$ be a compact set contained in the domain $D = \{x : |x|_{\mathcal{A}} < r_0\} \subset R^n$. Let $V(t,x) : [0,\infty) \times D \to R$ be a continuous function such that there exist two continuous positive definite with respect to $\mathcal{A}$ functions $V_1(x)$ and $V_2(x)$, defined on D, such that*

$$V_1(x) \leq V(t,x) \leq V_2(x)$$

for all $t \geq 0$. Let $B_r = \{x : |x|_{\mathcal{A}} \leq r\} \subset D$, for some $0 < r < r_0$. Then, there exist a class $\mathcal{K}$ functions α_1 and α_2, both defined on $[0,r]$, such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(t,x) \leq \alpha_2(|x|_{\mathcal{A}})$$

for all $x \in B_r$ and all $t \geq 0$. Moreover, if $D = R^n$ and $V_1(x)$ is radially unbounded, then α_1 and α_2 can be chosen to be class $\mathcal{K}_\infty$.

Proof: similar to that of [28, Lemma 3.5].$\lhd$

APPENDIX B

Cheap Control and H_∞ Disturbance attenuation

The objective of this appendix is to find the structure of the stabilizing solution of (5.8). For this purpose we perform some of the steps of [44]. The main idea is to transfer the singularity from the Riccati equation to the system through appropriate scaling of the state variables. In so doing the singular optimal control problem is transformed into a regular optimal control design for a singularly perturbed system. For the purpose of clarity we perform the analysis directly on the auxiliary system (5.4)–(5.6), with $F = I$.

Consider the cheap control problem of minimizing the performance index

$$J = \frac{1}{2}\int_0^\infty (\mu^2 x^T x + z^T z + \mu^2 u^T u - \gamma^2 d^T d)\, dt \tag{B.1}$$

subject to the worst case of disturbance and to the dynamic constraint

$$\dot{x} = A^T x + C^T u + d \tag{B.2}$$

$$z = B^T x \tag{B.3}$$

Since (A, C) is an observable pair, then, according to [56], this disturbance attenuation problem admits, for sufficiently large γ, the unique stabilizing solution $u^\star = -\frac{1}{\mu^2} C P_\infty x$, where P_∞ is the unique positive definite solution of (5.8) that renders $(A^T - \frac{1}{\mu^2} C^T C P_\infty)$ asymptotically stable. This controller achieves a disturbance attenuation level of γ.

Now apply to both the system and the performance index the following scaling

$$
\begin{aligned}
\epsilon_i &= \mu^{1/r_i} \\
x_f &= Sx \\
\bar{u} &= \tilde{u} - z = \mu u
\end{aligned}
$$

where S is given in (5.13). We get the following problem

$$
\begin{aligned}
J &= \frac{1}{2} \int_0^\infty (\mu^2 x_f^T S^{-2} x_f + z^T z + \bar{u}^T \bar{u} - \gamma^2 d^T d)\, dt && \text{(B.4)} \\
\epsilon_i \dot{x}_{if} &= A_i^T x_{if} + C_i^T \bar{u}_i + \epsilon_i S_i d && \text{(B.5)} \\
z_i &= B_i^T x_{if},\ i = 1, \cdots, p && \text{(B.6)}
\end{aligned}
$$

This can be written in the compact form

$$
\begin{aligned}
J &= \frac{1}{2} \int_0^\infty (x_f^T M^2 x_f + z^T z + \bar{u}^T \bar{u} - \gamma^2 d^T d)\, dt && \text{(B.7)} \\
\dot{x}_f &= \mathcal{E}^{-1} A^T x_f + \mathcal{E}^{-1} C^T \bar{u} + Sd && \text{(B.8)} \\
z &= B^T x_f && \text{(B.9)} \\
M &= \text{block diag}[M_1, \cdots, M_p],\ M_i = \text{diag}[\epsilon_i, \cdots, \epsilon_i^{r_i}]
\end{aligned}
$$

where $\mathcal{E}$ is given in (5.15).

Let us consider the subproblems corresponding to the boundary layer systems.

For the i-th boundary layer system we have the following problem

$$
\begin{aligned}
J_i &= \frac{1}{2}\int_0^{\infty}(z_i^T z_i + \bar{u}_i^T \bar{u}_i)\, d\tau_i && \text{(B.10)}\\
\frac{dx_{if}}{d\tau_i} &= A_i^T x_{if} + C_i^T \bar{u}_i && \text{(B.11)}\\
z_i &= B^T x_{if},\ i = 1, \cdots, p,\ \tau_i = t/\epsilon_i && \text{(B.12)}
\end{aligned}
$$

For every $i = 1, \cdots, p$, the i-the boundary layer problem (BLP) is a standard quadratic regulator problem. Since (A_i, B_i) and (A_i, C_i) are controllable and observable pairs, respectively, then the i-th BLP has the stabilizing solution $\bar{u}_i^\star = -C_i P_i x_{if}$ where P_i is the unique positive definite solution of the algebraic Riccati equation

$$A_i P_i + P_i A_i^T - P_i C_i^T C_i P_i + B_i B_i^T = 0 \qquad \text{(B.13)}$$

that renders the matrix $A_i^T - C_i^T C_i P_i$ asymptotically stable.

We know that the scaled complete problem (B.7)–(B.9) has a stabilizing solution $\bar{u}^\star = -CKx_f$ where K is the positive definite solution of the H_∞ algebraic Riccati equation

$$\mathcal{E}^{-1}AK + KA^T\mathcal{E}^{-1} - K\mathcal{E}^{-1}C^T C\mathcal{E}^{-1}K + \frac{1}{\gamma^2}KS^2K + BB^T + M^2 = 0 \qquad \text{(B.14)}$$

By uniqueness of the solution, we can show that K is a block diagonal matrix. From [3, Theorem 8.2] we borrow the solution structure

$$K(\mathcal{E}) = \text{block diag}[\epsilon_1 K_1, \cdots, \epsilon_p K_p] \qquad \text{(B.15)}$$

and substitute in (B.14) we get

$$A_iK_i + K_iA_i^T - K_iC_i^TC_iK_i + \frac{1}{\gamma^2}\epsilon_i^2K_iS_i^2K_i + B_iB_i^T + M_i^2 = 0 \tag{B.16}$$

for $i = 1, \cdots, p$. To study the asymptotic behavior of K we let $\epsilon_i = 0,\ i = 1, \cdots, p$ in (B.16) and we get

$$A_iK_i(0) + K_i(0)A_i^T - K_i(0)C_i^TC_iK_i(0) + B_iB_i^T = 0 \tag{B.17}$$

By uniqueness of solution we conclude that $K_i(0) = P_i$, for $i = 1, \cdots, p$. Going along the lines of [8] we can show that for $i = 1, \cdots, p$, there exist $\epsilon_i^\star$ such that, for $0 \leq \epsilon_i \leq \epsilon_i^\star$ we have

$$K_i(\epsilon_i) = P_i + O(\epsilon_i) \tag{B.18}$$

The solution of the parameterized algebraic Riccati equation (5.8) of the original problem (B.1)–(B.3) is then given by

$$P_\infty = SK(\mathcal{E})S \tag{B.19}$$

APPENDIX C

Stability Results

C.1 Proof of Theorem 3.1

Pick any $d \in \mathcal{M}_{\mathcal{D}}$. The derivative of V along the trajectories of (3.1) is given by

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, d(t)) \leq -\alpha_3(|x|_{\mathcal{A}}), \ \forall t \geq 0$$

Let us establish a set of initial conditions (independent of the initial time) for which the solution of (3.1) is forward complete. Choose $r > 0$ and $\rho > 0$ such that $B_r = \{x \in R^n : |x|_{\mathcal{A}} \leq r\} \subset U$ and $\rho < \min_{|x|_{\mathcal{A}} = r} \alpha_1(|x|_{\mathcal{A}})$ (exists because $\mathcal{A}$ is compact, thus $\{x : |x|_{\mathcal{A}} = r\}$ is also compact). Then the set $\{x \in B_r : \alpha_1(|x|_{\mathcal{A}}) \leq \rho\}$ is in the interior of B_r.

Now, define the time-dependent set $\Omega_{t,\rho}$ by

$$\Omega_{t,\rho} = \{x \in B_r : V(t,x) \leq \rho\}$$

Since

$$\alpha_2(|x|_{\mathcal{A}}) \leq \rho \implies V(t,x) \leq \rho$$

the set $\Omega_{t,\rho}$ contains $\{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\}$. On the other hand, since

$$V(t,x) \leq \rho \implies \alpha_1(|x|_{\mathcal{A}}) \leq \rho$$

$\Omega_{t,\rho}$ is a subset of $\{x \in B_r : \alpha_1(|x|_{\mathcal{A}}) \leq \rho\}$.

Thus,

$$\{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\} \subset \Omega_{t,\rho} \subset \{x \in B_r : \alpha_1(|x|_{\mathcal{A}}) \leq \rho\} \subset B_r$$

for all $t \geq 0$.

Since $\dot{V}(t,x)$ is negative on $U/\mathcal{A}$, hence, $V(t,x)$ is decreasing. Therefore, for any $t_0 \geq 0$ and any $x_0 \in \{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\} \subset \Omega_{t_0,\rho}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,\rho}$, for all $t \geq t_0$. Since $\Omega_{t,\rho}$ is bounded, the solution starting at (t_0, x_0) is defined for all $t \geq t_0$ and $x(t) \in B_r$.

Now, for any $x_0 \in \{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\}$, we can write

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, d(t)) \leq -\alpha_3(|x|_{\mathcal{A}}) \leq -\alpha(V(x(t))), \ \forall t \geq 0$$

where α is the continuous positive definite function defined by

$$\alpha(.) \stackrel{\text{def}}{=} \alpha_3(\alpha_2^{-1}(.))$$

on $[0, r]$. Now, let β_α be the $\mathcal{KL}$-function defined in Lemma 4.4 of [34]. Thus, we have

$$V(t, x(t)) \leq \beta_\alpha(V(t_0, x_0), t - t_0), \ \forall V(t_0, x_0) \in [0, \rho]$$

Therefore, for any $d \in \mathcal{M}_{\mathcal{D}}$ and any $x_0 \in \{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\}$, the solution $x(t)$ of (3.1) satisfies

$$|x(t)|_{\mathcal{A}} \leq \alpha_1^{-1}(\beta_\alpha(\alpha_2(|x(t_0)|_{\mathcal{A}}), t - t_0)) \stackrel{\text{def}}{=} \beta(|x(t_0)|_{\mathcal{A}}, t - t_0)$$

The function $\beta(.,.)$ is a class $\mathcal{KL}$ function defined on $[0, \alpha_2^{-1}(\rho)] \times [0, \infty)$. This implies, by a mechanism similar to the first part of the proof of Proposition 2.5 of [34], that the system (3.1) is UAS with respect to the set $\mathcal{A}$.

If $\alpha_1(.)$ is class $\mathcal{K}_\infty$, then so is $\alpha_2(.)$. Therefore, the set $\{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\}$ is bounded for any $\rho > 0$. For any $x_0 \in R^n$, we can choose ρ large enough so that $x_0 \in \{x \in B_r : \alpha_2(|x|_{\mathcal{A}}) \leq \rho\}$. Thus, using an argument similar to the previous one, we can conclude uniform global asymptotic stability of the system (3.1).$\triangleleft$

C.2 Proof of Theorem 3.3

This proof is an extension of the proof of Theorems 19.1 and 19.2 of [55] to the case of exponential stability with respect to sets.

Define the function

$$V(t, \xi) = \sup_{\tau \geq 0, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t + \tau, \xi, t; d)|_{\mathcal{A}} e^{q\gamma\tau}\} \tag{C.1}$$

for all $(t, \xi) \in R_{\geq 0} \times \Omega_0$, where $0 < q < 1$. Clearly, for $\tau = 0$ we have

$$|\xi|_{\mathcal{A}} \leq V(t, \xi) \tag{C.2}$$

Furthermore, using the definition of UES in Definition 3.1, we have

$$\begin{aligned} V(t,\xi) &\leq \sup_{\tau \geq 0} \{ke^{-\gamma\tau}|\xi|_{\mathcal{A}}e^{q\gamma\tau}\} && \text{(C.3)} \\ &\leq k|\xi|_{\mathcal{A}} && \text{(C.4)} \end{aligned}$$

Inequalities (C.2) and (C.4) prove (3.8).

Using the local Lipschitz property of $|.|_{\mathcal{A}}$ and Proposition 5.5 of [34], we have (notice that, since $\mathcal{A}$ is compact, the set Ω_0 is compact)

$$\big|\,|x(t,\xi,t_0;d)|_{\mathcal{A}} - |x(t,\eta,t_0;d)|_{\mathcal{A}}\,\big| \leq C\|\xi-\eta\| \quad \text{(C.5)}$$

for all $\xi, \eta \in \Omega_0$, all $d \in \mathcal{M}_{\mathcal{D}}$, and all $t \in [0,T]$, for any fixed $T > 0$.

There is no loss of generality in choosing $k > 1$. Hence, we can choose $T > 0$ such that $k = e^{(1-q)\gamma T}$. Then, for $\tau \geq T$, we have

$$\begin{aligned} |x(t+\tau,\xi,t;d)|_{\mathcal{A}}e^{q\gamma\tau} &\leq ke^{-(1-q)\gamma\tau}|\xi|_{\mathcal{A}} \\ &\leq |\xi|_{\mathcal{A}} && \text{(C.6)} \end{aligned}$$

which implies that

$$V(t,\xi) = sup_{0 \leq \tau \leq T, d \in \mathcal{M}_{\mathcal{D}}}\{|x(t+\tau,\xi,t;d)|_{\mathcal{A}}e^{q\gamma\tau}\} \quad \text{(C.7)}$$

Now, using (C.5) and (C.7), we have

$$\begin{aligned} |V(t,\xi)-V(t,\eta)| &\leq sup_{0 \leq \tau \leq T, d \in \mathcal{M}_{\mathcal{D}}}\{|\,|x(t+\tau,\xi,t;d)|_{\mathcal{A}} \\ &\quad - |x(t+\tau,\eta,t;d)|_{\mathcal{A}}\,|e^{q\gamma\tau}\} \\ &\leq C\|\xi-\eta\|e^{q\gamma T} \\ &\leq L\|\xi-\eta\| && \text{(C.8)} \end{aligned}$$

for all $\xi, \eta \in \Omega_0$, where $L = Ce^{q\gamma T}$. Thus, we have proved (3.9). This local Lipschitz property in x implies continuity of $V(t,x)$ in x.

Let us prove the continuity of $V(t,x)$ in t. Take $\delta \geq 0$. Then, for $\xi \in \Omega_0$, we can write

$$\begin{aligned} |V(t+\delta,\xi) - V(t,\xi)| \;\leq\; & |V(t+\delta,\xi) - V(t+\delta, x(t+\delta,\xi,t;d))| \\ & + |V(t+\delta, x(t+\delta,\xi,t;d)) - V(t,\xi)| \qquad \text{(C.9)} \end{aligned}$$

for $d \in \mathcal{M}_{\mathcal{D}}$. For δ small enough, even if the trajectory $x(t+\delta,\xi,t;d))$ may exit from Ω_0, the function is still defined and locally Lipschitz with a constant that we denote by L. This implies

$$\begin{aligned} |V(t+\delta,\xi) - V(t,\xi)| \;\leq\; & L\|\xi - x(t+\delta,\xi,t;d)\| \\ & + |V(t+\delta, x(t+\delta,\xi,t;d)) - V(t,\xi)| \qquad \text{(C.10)} \end{aligned}$$

Let us discuss the second term of the right-hand side of (C.10). Let $\tilde{\xi} = x(t+\delta,\xi,t;d)$. By the uniqueness of solutions and a change of variable $\tau' = \tau + \delta$, we can write

$$\begin{aligned} V(t+\delta, x(t+\delta,\xi,t;d)) &= sup_{\tau \geq 0, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t+\delta+\tau,\tilde{\xi},t+\delta;d)|_{\mathcal{A}} e^{q\gamma\tau}\} \\ &= sup_{\tau \geq 0, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t+\delta+\tau,\xi,t;d)|_{\mathcal{A}} e^{q\gamma\tau}\} \\ &= sup_{\tau' \geq \delta, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t+\tau',\xi,t;d)|_{\mathcal{A}} e^{q\gamma\tau} e^{-q\gamma\delta}\} \\ &= a(\delta) e^{-q\gamma\delta} \qquad \text{(C.11)} \end{aligned}$$

where $a(\delta) = sup_{\tau \geq \delta, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t+\tau,\xi,t;d)|_{\mathcal{A}} e^{q\gamma\tau}\}$. Given (C.11), (C.10) can be written as

$$|V(t+\delta,\xi) - V(t,\xi)| \leq L\|\xi - x(t+\delta,\xi,t;d)\| + |a(\delta)e^{-q\gamma\delta} - a(0)| \qquad \text{(C.12)}$$

Given the fact that $a(\delta)$ goes to $a(0)$ as δ goes to zero and the continuity of the solutions of (3.1), we conclude from (C.12) that $V(t,\xi)$ is continuous in t.

Finally, let us prove (3.10). Let $\bar{\xi} = x(t+h,\xi,t;d)$, $h > 0$. Again, by uniqueness of solutions and a change of variable, we have

$$\begin{aligned} V(t+h,\bar{\xi}) &= sup_{\tau \geq h, d \in \mathcal{M}_{\mathcal{D}}} \{|x(t+h+\tau,\bar{\xi},t+h;d)|_{\mathcal{A}} e^{q\gamma\tau}\} e^{-q\gamma h} \\ &\leq V(t,\xi)e^{-q\gamma h} \end{aligned} \tag{C.13}$$

Using (C.13), we have

$$\begin{aligned} \lim_{h \to 0} \frac{V(t+h,\bar{\xi}) - V(t,\xi)}{h} &\leq V(t,\xi) \lim_{h \to 0} \frac{e^{-q\gamma h} - 1}{h} \\ &\leq -q\gamma V(t,\xi) \end{aligned} \tag{C.14}$$

Take $\lambda = q\gamma$.◁

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