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LEFT-MODULAR ELEMENTS AND EDGE LABELINGS

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LEFT-MODULAR ELEMENTS AND EDGE LABELINGS

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ABSTRACT

LEFT-MODULAR ELEMENTS AND EDGE LABELINGS

By

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This work is a study of posets and lattices in three parts.

In the first part, we give a characterization of left-modular elements and demonstrate two formulas for the characteristic polynomial of a lattice with a left-modular element. One of the formulas generalizes Stanley's Partial Factorization Theorem, and also provides an inductive proof for Blass and Sagan's Total Factorization Theorem for LL lattices. The characteristic polynomials and the Möbius functions of non-crossing partition lattices and shuffle posets are computed as examples.

The second and third parts both deal with edge labelings of posets and lattices. We construct an edge labeling for a left-modular lattice and show that it is an SL-labeling. This gives a method of labeling certain non-pure lattices, for example, the Tamari lattice.

In the third part, we study the rank-selected subposet P_S of a poset P. By constructing an induced labeling for P_S, we show that if P is an R-poset then so is P_S. Furthermore, we define the notion of a thrifty labeling and show that if a labeling is thrifty then EL- and SL-shellability are also inherited by P_S.

In memory of my father, En-Tse Liu

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INTRODUCTION

The study of partially ordered sets (posets) and lattices can be traced back to the nineteenth century. The work of G. D. Birkhoff in the 1930's began the systematic development of poset theory and lattice theory as subjects in their own right. As a fundamental invariant, the Möbius function originated in several different forms related to number theory, geometry, algebra, topology and combinatorics. The Möbius inversion formula for posets is essentially due to L. Weisner [19] in 1935 and was independently rediscovered by P. Hall [10] in 1936. However, it was not until 1964 that the work of G.-C. Rota [12] started the systematic study of the Möbius function within combinatorics. The theory of Möbius functions not only offers a very general enumerative principle, of which the inclusion-exclusion principle in set theory is a special case, but also provides a great deal of information regarding problems like determination of the Euler characteristic and counting colorings of graphs. In the first chapter of this thesis, we outline preliminaries concerning the theory of posets and lattices needed for the following chapters. Some important theorems about the Möbius function are also mentioned there.

The characteristic polynomial of a lattice was also first introduced by G. D. Birkhoff [1], and its variants have been called the Birkhoff polynomial and the Poincaré polynomial. Since it is a generating function for the Möbius function, much has been done to explore the combinatorial and algebraic properties of this polynomial. In the early 1970's, Stanley proved two factorization theorems for the characteristic polynomial. One of them states that a factor arises from a modular element in a finite geometric lattice (the Partial Factorization Theorem, [14]). The other one shows that the characteristic polynomial of a supersolvable and semimodular lattice factors over the integers (the Total Factorization Theorem, [16]) and is implied by the first result if the lattice is atomic. Various other methods for showing that this polynomial has only integer roots have been developed; see the survey article [13]. Recently, A. Blass and B. Sagan [3] generalized Stanley's Total Factorization Theorem under a weaker hypothesis requiring only that the lattice satisfy left-modular and level conditions. In Chapter 2, we give a characterization of left-modular elements and then prove a generalization of the Partial Factorization Theorem which replaces modularity with left-modularity. The latter result also provides us with an inductive proof for Blass and Sagan's theorem. So the previous three factorization theorems are unified into one. We calculate the characteristic polynomials and Möbius functions of non-crossing partition lattices and shuffle posets as examples.

R-labeling of pure (ranked) posets was introduced by R. Stanley [15]. The concept offers a combinatorial interpretation of the Möbius function for an R-labeled poset. Shortly thereafter EL- and SL-labelings, related to the concept of shelling in topology, were defined by A. Björner [4] and then generalized to CL-labeling in his joint work with M. Wachs [5]. The resulting theory allows one to compute the homology groups of the order complex of a poset. Recently, A. Björner and M. Wachs [6], [7] generalized the concept of shellability to non-pure posets. They were motivated by certain examples coming from the theory of subspace arrangements. In the first half of Chapter 3, we show that a left-modular lattice is SL-shellable by constructing a labeling which is induced by the maximal left-modular chain of the lattice. In the second half, we study the edge labeling for the rank-selected subposet P_S of a poset P. By constructing an induced labeling for P_S, we show that if P is an R-poset then so is P_S. Then we introduce the concept of a thrifty labeling which allows EL- and

SL-shellability to be inherited by rank-selected subposets.

CHAPTER 1

PRELIMINARIES

1.1 Posets and Lattices

We will use $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, and $\mathbb{R}$ for the nonnegative integers, positive integers, integers, and real numbers, respectively. For other standard notation we will follow Stanley's text [18].

A *partially ordered set*, $(P, \leq)$, or *poset* for short, is a set P together with a binary relation $\leq$ satisfying reflexivity, antisymmetry, and transitivity. Sometimes we write $\leq_P$ for $\leq$ to avoid confusion. All posets discussed in this thesis will be finite. Let P^* denote the *dual poset* of P obtained by reversing the order relation of P. Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $\eta : P \to Q$ whose inverse is also order-preserving. If x, $y \in P$ and $x \leq y$, the *closed interval* $[x, y]$ and the *open interval* (x, y) are defined by

$$\begin{aligned} [x,y] &= \{z \in P \mid x \leq z \leq y\}, \\ (x,y) &= \{z \in P \mid x < z < y\}. \end{aligned}$$

The intervals $(x, y]$ and $[x, y)$ are defined similarly. If P contains a unique minimal element (called a *bottom element*), it will be denoted by $\hat{0}$ (or $\hat{0}_P$ to avoid confusion); similarly, a unique maximal element (*top element*), if it exists, will be denoted by $\hat{1}$

(or $\hat{1}_P$). A poset is called *bounded* if it has both a $\hat{0}$ and a $\hat{1}$. In this case $[\hat{0}, \hat{1}] = P$.

We say that x *is covered by* y, and write $x \prec y$, if $x < y$ and there is no element $z \in P$ such that $x < z < y$. We also use $x \preceq y$ to mean $x \prec y$ or $x = y$. An *atom* is an element covering a minimal element of P. Let $A(P)$ denote the set of atoms in P and $A(a, b)$ denote the set of atoms in interval $[a, b]$. A *co-atom* is defined dually: it is an element covered by a maximal element.

Any subset of P will form a subposet by inheriting the order relation of P. So, in particular, intervals are subposets. A *chain* C of P is a subposet in which any two elements are comparable. We define the *length of a chain* C by $\ell(C) = |C| - 1$, where $|\cdot|$ denotes cardinality. The chain $\{x_0 < x_1 < \cdots < x_n\}$ is called *saturated* (or *unrefinable*) if $x_0 \prec x_1 \prec \cdots \prec x_n$. A *maximal chain* in P is a saturated chain from a minimal element to a maximal element of P. The *length of a poset* P is defined by

$$\ell(P) = \max\{\ell(C) \mid C \text{ is a chain of } P\}.$$

Since we only consider finite posets here, this maximum value is finite. The length of an interval $[x, y]$ of P is denoted by $\ell(x, y)$. If every maximal chain of P has the same length n, then we say P is *pure* (or *graded*) *of rank* n. In this case, there is a unique *rank function* $\rho : P \to \mathbb{N}$ such that

i. $\rho(x) = 0$ if x is a minimal element of P, and

ii. $\rho(y) = \rho(x) + 1$ if $y \succ x$ in P.

If $\rho(x) = i$, then we say that x is *of rank* i. Obviously $\ell(x, y) = \rho(y) - \rho(x)$ and $\ell(P) = \rho(z)$ for any maximal element z of P. A generalized rank function will be introduced in Section 2.2 for posets which could be non-pure.

A *lattice* L is a poset such that every pair x, $y \in L$ has a greatest lower bound and a least upper bound. We call the greatest lower bound and the least upper bound the *meet* and the *join* of x and y, respectively. The meet is denoted $x \wedge y$ (or $x \wedge_L y$)

and the join is written $x \vee y$ (or $x \vee_L y$). Clearly, every finite lattice has a $\hat{0}$ and a $\hat{1}$, namely the meet and the join of all elements in the lattice. The following two properties involving $\wedge$ and $\vee$ are easy to check and very useful.

$$x \wedge (x \vee y) = x = x \vee (x \wedge y). \tag{1.1}$$

$$x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \leq y. \tag{1.2}$$

A subset M is called a *sublattice* of L if x, $y \in M$ implies $x \vee_L y$, $x \wedge_L y \in M$. Unlike posets, not every subset of L is a sublattice. Even though a subset itself forms a lattice it might still not be a sublattice of L. An example will be given in Section 2.3 where we discuss the partition and non-crossing partition lattices. For a given subset N, the sublattice generated by N is the smallest (in cardinality) sublattice containing N.

If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets, then the *direct product* of P and Q is the poset

$$P \times Q = \{(x, y) : x \in P \text{ and } y \in Q\}$$

such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq_P x'$ and $y \leq_Q y'$. It is clear from the definition that $P \times Q$ and $Q \times P$ are isomorphic. The direct product of two lattices P, Q is still a lattice since $(x, y) \vee (x', y') = (x \vee_P x', y \vee_Q y')$ and $(x, y) \wedge (x', y') = (x \wedge_P x', y \wedge_Q y')$. The direct product of P with itself n times is denoted P^n.

The *Hasse diagram* of a poset P is a graph whose vertices are elements of P, whose edges are the cover relations, and such that if $y \succ x$ then y is drawn above x. Since a poset is completely determined by its cover relations, the Hasse diagram of a poset depicts all order relations. The following are some fundamental examples of finite lattices. The Hasse diagrams of these lattices are shown in Figures 1.1 and 1.2.

Example 1.1.1 Let $n \in \mathbb{N}$, $[n] := \{1, 2, \ldots, n\}$, and $[0, n] := \{0, 1, \ldots, n\}$.

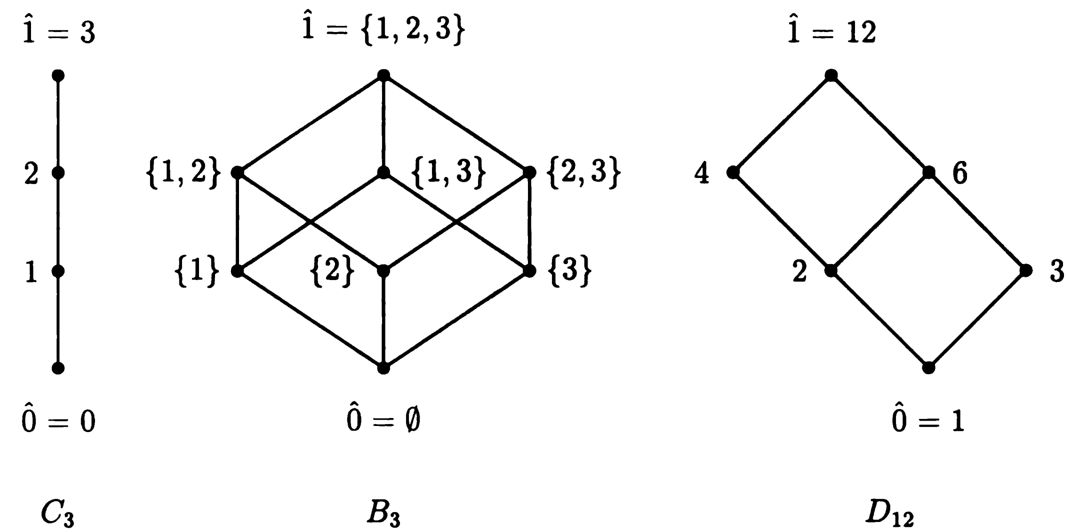


Figure 1.1. The lattices C_3, B_3 and D_{12}

1. The set $[0, n]$ ordered in the usual manner forms the *chain* C_n. For any p, $q \in [0, n]$, it is trivial that $p \wedge q = \min\{p, q\}$, $p \vee q = \max\{p, q\}$ and $\rho(p) = p$. So C_n is a lattice of rank n.

2. The *Boolean algebra* B_n consists of all subsets of $[n]$ with inclusion $\subseteq$ as the order relation. For any pair of subsets S, $T \in B_n$, we have $S \wedge T = S \cap T$ and $S \vee T = S \cup T$. The rank function is $\rho(S) = |S|$. So B_n is a lattice of rank n. B_n is isomorphic to the direct product $(C_1)^n$.

3. Let $n \in \mathbb{P}$ and $n = p_1^{m_1} p_2^{m_2} \cdots p_l^{m_l}$ with p_1, p_2, …, p_l distinct primes and m_1, m_2, …, $m_l \in \mathbb{P}$. The *divisor lattice* D_n is the set of all positive integral divisors of n ordered by divisibility, so $u \leq v$ if and only if $u|v$, i.e., u divides v. For any pair of divisors u and v of n, we have $u \wedge v = \gcd(u, v)$ and $u \vee v = \operatorname{lcm}(u, v)$. A divisor $u = p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l}$ has rank $\rho(u) = i_1 + i_2 + \cdots + i_l$. So D_n is a lattice of rank $m_1 + m_2 + \cdots + m_l$. It is easy to check that D_n is isomorphic to the direct product $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_l}$.

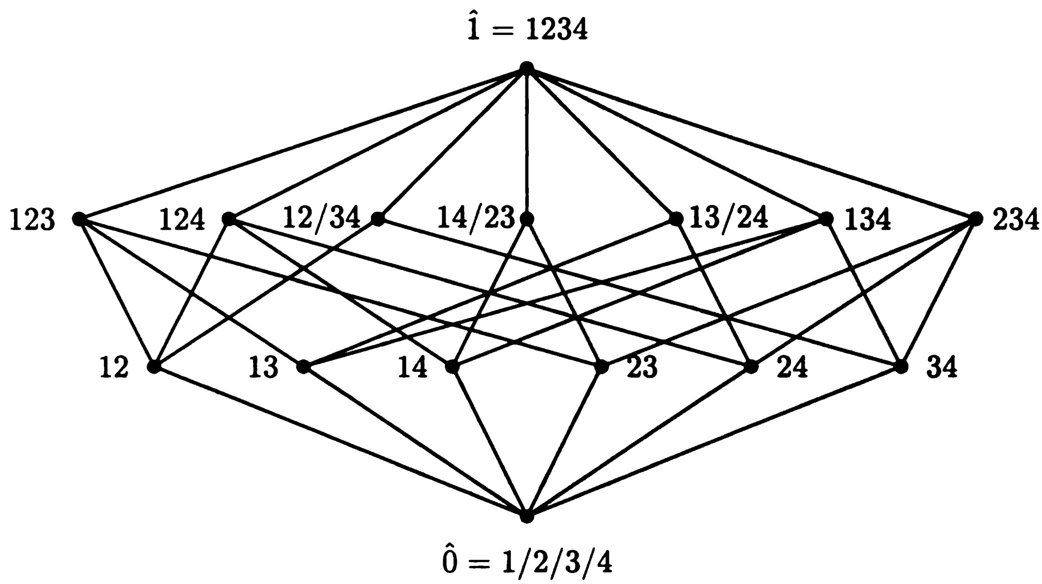


Figure 1.2. The partition lattice Π_4

4. Let $n \in \mathbb{P}$. If π is a partition of $[n]$ into k non-empty subsets, B_i, called *blocks*, then we write $\pi = B_1/B_2/\ldots/B_k \vdash [n]$. When it will cause no confusion, we will not explicitly write out any blocks that are singletons. The *partition lattice* Π_n consists of all partitions of $[n]$ with partial order $\pi \leq \sigma$ if every block of π is contained in a block of σ. We denote by $\equiv_\pi$ the equivalence relation associated with the partition π, i.e., $p \equiv_\pi q$ if and only if p and q belong to the same block of π. It is easy to check that the equivalence relation $\equiv_{\pi\vee\sigma}$ is the transitive closure of $\equiv_\pi$ and $\equiv_\sigma$; while $\equiv_{\pi\wedge\sigma}$ is their intersection. Suppose $\pi = B_1/B_2/\ldots/B_k$ then $\rho(\pi) = \sum_{i=1}^{k}(|B_i| - 1) = n -$ (number of blocks of π). Therefore Π_n is a lattice of rank $n - 1$.

1.2 Möbius Functions

One of the fundamental invariants of a poset P is its *Möbius function*, $\mu : P \times P \to \mathbb{Z}$, defined recursively by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \le z < y} \mu(x,z) & \text{else.} \end{cases}$$

Clearly $\mu(x,y) = 0$ if $x \not\le y$. Note that μ is uniquely determined by the equation

$$\sum_{x \le z \le y} \mu(x,z) = \delta_{xy}$$

where δ_{xy} is the Kronecker delta. If there is possible ambiguity, we will use μ_P to denote the Möbius function of P. In case P has a $\hat{0}$, for brevity we let $\mu(x) = \mu(\hat{0}, x)$. If in addition P has a $\hat{1}$ then we write $\mu(P) = \mu(\hat{0}, \hat{1})$.

There are some general techniques for the evaluation of μ. We begin with the simplest one. For a proof, see [18, p. 118].

Proposition 1.2.1 (The Product Formula) *Let P and Q be two finite posets. Then*

$$\mu_{P\times Q}((x,y),(x',y')) = \mu_P(x,y)\mu_Q(x',y'). \ \blacksquare$$

Example 1.2.2 The Möbius functions of the first three lattices in Example 1.1.1 are easy to find.

1. If p, $q \in C_n$, then directly from the definition

$$\mu(p,q) = \begin{cases} 1 & \text{if } q = p, \\ -1 & \text{if } q = p+1, \\ 0 & \text{else.} \end{cases}$$

2. We have B_n isomorphic to $(C_1)^n$ by identifying $S \subseteq [n]$ with a vector

$$(i_1, i_2, \ldots, i_n), \text{ where } i_k = \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Möbius function of the chain $C_1 = \{0, 1\}$ is given by $\mu(0,0) = \mu(1,1) = 1$ and $\mu(0,1) = -1$, we conclude from the product formula that if $S \leq T$ ($S \subseteq T$) in B_n then

$$\mu(S,T) = (-1)^{|T-S|} = (-1)^{\ell(S,T)}.$$

In particular $\mu(S) = \mu(\emptyset, S) = (-1)^{|S|}$.

3. Recall that $D_n \cong C_{m_1} \times C_{m_2} \times \cdots \times C_{m_l}$, where $n = p_1^{m_1} p_2^{m_2} \dots p_l^{m_l}$ with $p_1, p_2, \dots, p_l$ distinct primes and $m_1, m_2, \dots, m_l \in \mathbb{P}$. Similarly to the previous example, if $u \leq v$ in D_n and $v/u = p_1^{i_1} p_2^{i_2} \dots p_l^{i_l}$ then

$$\mu(u,v) = \mu(1, v/u) = \begin{cases} (-1)^{i_1+i_2+\cdots+i_l} & \text{if } 0 \leq i_k \leq 1 \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

The value $\mu(n) = \mu_{D_n}(1, n)$ is the classical Möbius function in number theory.

The following two results are fundamental in the theory of Möbius function. A proof of each can be found in [18, pp. 116, 117].

Theorem 1.2.3 (Möbius Inversion Formula) *Let P be a finite poset and f, $g : P \to \mathbb{R}$ be any two functions. Then*

$$g(x) = \textstyle\sum_{y \leq x} f(y) \text{ for all } x \in P, \text{ if and only if}$$
$$f(x) = \textstyle\sum_{y \leq x} \mu(y,x) g(y) \text{ for all } x \in P. \blacksquare$$

Theorem 1.2.4 (Möbius Inversion Formula, dual form) *Let P be a finite poset and f, $g : P \to \mathbb{R}$ be any two functions. Then*

$$g(x) = \textstyle\sum_{y \geq x} f(y) \text{ for all } x \in P, \text{ if and only if}$$
$$f(x) = \textstyle\sum_{y \geq x} \mu(x,y) g(y) \text{ for all } x \in P. \blacksquare$$

As an important application of Theorem 1.2.4, the Principle of Inclusion-Exclusion is just Möbius inversion over a Boolean algebra. We will show it in the following example.

Example 1.2.5 Let X be a set and let U denote a collection of properties which elements of X either have or don't have. For $S \subseteq U$, let $g(S)$ denote the number of elements in X having all the properties in S, and let $f(S)$ denote the number of elements in X having all properties in S but no others. It is clear that $g(S) = \sum_{S \subseteq T} f(T)$. So by dual form of Möbius inversion formula

$$f(S) = \sum_{S \subseteq T} \mu_{B_n}(S,T) g(T) = \sum_{S \subseteq T} (-1)^{|T-S|} g(T)$$

where $n = |U|$. In the sum we use the Möbius function of B_n because the collection of subsets of U forms a Boolean algebra. If we let $S = \emptyset$, then

$$f(\emptyset) = \sum_{T \subseteq U} (-1)^{|T|} g(T)$$

which is the well-known sieve formula for the number of elements in X having none of the properties in U.

An element of a lattice is called *atomic* if it is a join of atoms. The bottom element $\hat{0}$ is the join of the empty set, so it is atomic. A lattice is called *atomic* if all its elements are atomic. We denote by $J(L)$ the set of atomic elements of lattice L. In general, the set $J(L)$ is not a sublattice of L because the meet of two atomic elements is not necessarily atomic; however their join is atomic. An element $p \neq \hat{0}$ is called *join-irreducible* (or just *irreducible*) if for every pair x, $y \in L$

$$p = x \vee y \qquad \text{implies} \qquad p = x \text{ or } p = y.$$

It follows from the definition that an element is irreducible if and only if it covers a single element in L. So atoms are irreducible. Irreducibles which are not atoms are called *singular elements*. We can express any element $a \in L$ as $a = p_1 \vee p_2 \vee \cdots \vee p_k$ where p_i are irreducible elements. This expression is called a *decomposition* of a. The remark after the definition of irreducible implies that the (single argument) Möbius function of a singular element is zero. In fact, every non-atomic element has a Möbius function value of zero. To prove this we define a map $\delta : L \to J(L)$ by

$$\delta(x) = \bigvee\{a \in A(L) \mid a \leq x\}.$$

Equivalently, $\delta(x)$ is the maximum atomic element in $[\hat{0}, x]$. If x is a non-atomic element, then

$$\mu(x) = -\left[\sum_{y \leq \delta(x)} \mu(y) + \sum_{y \in [\hat{0},x)\backslash[\hat{0},\delta(x)]} \mu(y)\right] = -\sum_{y \in [\hat{0},x)\backslash[\hat{0},\delta(x)]} \mu(y).$$

All $y \in [\hat{0}, x)\backslash[\hat{0}, \delta(x)]$ are non-atomic and the minimal elements of $[\hat{0}, x)\backslash[\hat{0}, \delta(x)]$ are all singular. So by induction $\mu(x) = 0$. (Rota's NBC Theory [12] is another way to explain this result.) Therefore, for any atomic element $x \in L$, we have $\mu(x) = \mu_{J(L)}(x)$.

1.3 Characteristic Polynomials

Let P be a bounded pure poset of rank n and let t be an indeterminate. The *characteristic polynomial* of P is then

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{n-\rho(x)}.$$

One uses the co-rank of x rather than the rank as the exponent on t so that the polynomial will be monic. Since χ is a generating function for μ, it is of fundamental importance. The following corollary is derived directly from Proposition 1.2.1.

Corollary 1.3.1 *Let P and Q be two bounded pure posets. Then*

$$\chi(P \times Q, t) = \chi(P, t)\chi(Q, t). \blacksquare$$

Example 1.3.2 Continuing Example 1.2.2, it follows from the definition that the characteristic polynomial of C_n is

$$\chi(C_n, t) = t^{n-1}(t - 1).$$

By Corollary 1.3.1, the characteristic polynomials of B_n and D_n are

$$\chi(B_n, t) = (t-1)^n,$$

$$\chi(D_n, t) = t^{m-l}(t-1)^l.$$

where $m = m_1 + m_2 + \cdots + m_l$.

CHAPTER 2

LEFT-MODULAR ELEMENTS

2.1 Modular and Left-modular Elements

Throughout this chapter L will be a finite lattice. Given x, y, $z \in L$ with $z < y$, it is easy to check that the following inequality (the *modular inequality*)

$$z \vee (x \wedge y) \leq (z \vee x) \wedge y \tag{2.1}$$

is always true and equality holds whenever y or z is comparable to x. We say that x, y form a *modular pair* (x, y) if

$$z \vee (x \wedge y) = (z \vee x) \wedge y \tag{2.2}$$

for any $z < y$. Note that this relation is not symmetric in general. We now define two of the central concepts of this chapter.

Definition 2.1.1 *1. An element x is called a* left-modular element *if (x, y) is a modular pair for every $y \in L$.*

2. An element x is called a modular element *if both (x, y) and (y, x) are modular pairs for every $y \in L$.*

From the definition, if x is left-modular in L then it is also left-modular in the dual lattice L^*. However, this property is not true in general for modularity.

A finite lattice L is called (*upper*) *semimodular* if $x \wedge y \prec x$ implies $y \prec x \vee y$ for any x, $y \in L$. A lattice L whose dual L^* is semimodular is called *lower semimodular*. An equivalent definition of semimodularity is given in the following proposition. A proof can be found in [18, p. 103].

Proposition 2.1.2 *A finite lattice L is semimodular if and only if L is pure and the rank function ρ of L satisfies*

$$\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y)$$

for all x, $y \in L$. ■

In a semimodular lattice, the pair (x, y) is modular if and only if

$$\rho(x \wedge y) + \rho(x \vee y) = \rho(x) + \rho(y). \tag{2.3}$$

For a proof, see [2, p. 83]. So in this case the relation of being a modular pair is symmetric, and then there is no difference between modularity and left-modularity in a semimodular lattice. However, there are examples such as the non-crossing partition lattices (see Sec. 2.3) and the Tamari lattices (see Sec. 3.2) where the two concepts do not coincide.

We say a finite lattice is *geometric* if it is semimodular and atomic. In early work of R. Stanley [14], he showed that $\chi([\hat{0}, x], t)$ is a factor of $\chi(L, t)$ if x is a modular element in a geometric lattice L. One of our goals is to generalize Stanley's result by replacing x with a left-modular element and relaxing the condition that the lattice be geometric. Stanley's theorem and its generalization will be dealt with in next section. Here we would like to examine some general properties of modular elements and left-modular elements.

We say that y is a *complement* of x if $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. The following theorem that was given in [14] provides a characterization of modular elements.

Theorem 2.1.3 (Stanley [14]) *In a geometric lattice, an element x is modular if and only if no two complements of x are comparable.* ■

The analog of Theorem 2.1.3 for left-modular elements is as follows.

Theorem 2.1.4 *Let x be an element of any lattice L. The following statements are equivalent:*

i. The element x is left-modular.

ii. For any y, $z \in L$ with $z < y$, we have $x \wedge z \neq x \wedge y$ or $x \vee z \neq x \vee y$.

iii. For any y, $z \in L$ with $z \prec y$, we have $x \wedge z \neq x \wedge y$ or $x \vee z \neq x \vee y$ (but not both).

iv. For every interval $[a, b]$ containing x, no two complements of x with respect to the sublattice $[a, b]$ are comparable.

Proof. We will prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The proof of (ii) $\Leftrightarrow$ (iv) is trivial.

First we make some preliminary observations. Suppose $z < y$. We claim that $x \vee y = x \vee z$ if and only if $y = (z \vee x) \wedge y$. The forward direction is trivial by equation (1.1). Also we have $y = (z \vee x) \wedge y$ implies $y \leq x \vee z$ by (1.2). Now $z < y \leq x \vee z$, and joining all sides with x, gives $x \vee y = x \vee z$. Similarly we can show that $x \wedge y = x \wedge z$ if and only if $z = z \vee (x \wedge y)$.

For any $z < y$ the inequalities

$$z \leq z \vee (x \wedge y) \leq (z \vee x) \wedge y \leq y \tag{2.4}$$

are true by the modular inequality (2.1). Since $z \neq y$, at least one of the $\leq$'s in (2.4) should be $<$. Therefore (i) $\Rightarrow$ (ii). If $z \prec y$, then exactly two of the $\leq$'s should be $=$ and the remaining one must be $\prec$. Thus (ii) $\Rightarrow$ (iii).

To show (iii) $\Rightarrow$ (i), let us consider the contrapositive: assume that there are u, $v \in L$ with $u < v$ such that $u \vee (x \wedge v) < (u \vee x) \wedge v$. Given any y, $z \in [u \vee (x \wedge v), (u \vee x) \wedge v]$ with $z \prec y$, we have $y \leq (u \vee x) \wedge v \leq v$ implies $u \vee (x \wedge y) \leq u \vee (x \wedge v) \leq z$, so that $x \wedge y \leq z$. It follows that $z \vee (x \wedge y) = z$ and also $(z \vee x) \wedge y = y$ similarly, i.e., $x \wedge z = x \wedge y$ and $x \vee z = x \vee y$. ■

The existence of a left-modular element in L implies that one is also present in certain sublattices as the next proposition shows.

Proposition 2.1.5 *Let x be a left-modular element in lattice L. Then for any $y \in L$*

1. *the meet $x \wedge y$ is a left-modular element in $[\hat{0}, y]$, and*

2. *the join $x \vee y$ is a left-modular element in $[y, \hat{1}]$.*

Proof. Let a, $b \in [\hat{0}, y]$ with $b < a$. By left-modularity of x, we have

$$\begin{aligned} b \vee ((x \wedge y) \wedge a) &= b \vee (x \wedge (y \wedge a)) = (b \vee x) \wedge (y \wedge a) \\ &= ((b \vee x) \wedge y) \wedge a = (b \vee (x \wedge y)) \wedge a. \end{aligned}$$

So $x \wedge y$ is a left-modular element in $[\hat{0}, y]$. The proof for join is similar and will be omitted. ■

We obtain a result from [14] as a corollary.

Corollary 2.1.6 (Stanley [14]) *Let x be a modular element in a semimodular lattice L. Then for any $y \in L$*

1. *the meet $x \wedge y$ is a modular element in $[\hat{0}, y]$, and*

2. *the join $x \vee y$ is a modular element in $[y, \hat{1}]$.* ■

2.2 Left-modular Elements and $\chi(L,t)$

First, let us say a few words in motivation. There are two important factorization theorems for characteristic polynomials given by Stanley. This factorization offers an easy way to calculate the characteristic polynomials of various lattices. The following theorem from [14] shows that if L is a geometric lattice then there is a factor of its characteristic polynomial arising from a modular element.

Theorem 2.2.1 (Partial Factorization Theorem, Stanley [14]) *Let L be a finite geometric lattice. If x is a modular element of L, then*

$$\chi(L,t) = \chi([\hat{0},x],t) \sum_{b: b\wedge x=\hat{0}} \mu(\hat{0},b) t^{\rho(\hat{1})-\rho(x)-\rho(b)}. \blacksquare$$

In another paper [16], Stanley defined a *supersolvable lattice* to be a pair (L,Δ) where L is a lattice, $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ is a maximal chain of L, and Δ together with any other chain of L generates a *distributive lattice*, i.e., one such that any elements x, y, z of it satisfy the distributive laws

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

The lattice formed by all subgroups of a supersolvable group partially ordered by inclusion is an example of a supersolvable lattice. Stanley showed that a supersolvable lattice is pure. If L has a maximal modular chain Δ, i.e., every element of Δ is modular, then (L,Δ) is supersolvable. Also, if (L,Δ) is supersolvable then the elements of Δ are left-modular. So if (L,Δ) is both supersolvable and semimodular then the elements of Δ are modular.

Theorem 2.2.2 (Total Factorization Theorem, Stanley [16]) *Let (L,Δ) be a supersolvable, semimodular lattice (SS lattice for short) of rank n, where $\Delta : \hat{0} =$*

$x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$. *Then*

$$\chi(L,t) = (t-a_1)(t-a_2)\cdots(t-a_n) \tag{2.5}$$

where a_i is the number of atoms of L that are below x_i but not below x_{i-1}. ■

If the SS lattice is also atomic, then Theorem 2.2.2 can be derived by inductively applying the Partial Factorization Theorem to the chain Δ of modular elements. In fact, we will show (Corollary 2.2.10) that the atomic restriction can be removed from Theorem 2.2.1. So induction can applied even to lattices which are not necessarily atomic such as the divisor lattice.

Recently A. Blass and B. Sagan [3] were able to generalize Theorem 2.2.2. To replace supersolvability with a weaker hypothesis, they defined a *left-modular lattice* L to be one such that

L has a maximal chain Δ, all of whose elements are left-modular.

Motivated by their result, we will generalize the Partial Factorization Theorem by replacing x with a left-modular element.

Lemma 2.2.3 *Let L be a lattice with an arbitrary function $r : L \to \mathbb{R}$ and let $n \in \mathbb{R}$. If $x \in L$ is a left-modular element, then*

$$\sum_{y \in L} \mu(y) t^{n-r(y)} = \sum_{b \wedge x = \hat{0}} \mu(b) \sum_{y \in [b, b \vee x]} \mu(b,y) t^{n-r(y)}.$$

Proof. We will mimic Stanley's proof in [14]. By Crapo's Complementation Theorem [8], for any given $a \in [\hat{0}, y]$

$$\mu(y) = \sum_{a',a''} \mu(\hat{0},a')\zeta(a',a'')\mu(a'',y),$$

where a' and a'' are complements of a in $[\hat{0}, y]$, and ζ is the zeta function defined by $\zeta(u,v) = 1$ if $u \leq v$ and $\zeta(u,v) = 0$ else. Let us choose $a = x \wedge y$. The element a is left-modular in $[\hat{0}, y]$ by Proposition 2.1.5. But no two complements of a in $[\hat{0}, y]$ are comparable by Theorem 2.1.4. Thus

$$\mu(y) = \sum_{b} \mu(\hat{0}, b)\mu(b, y), \tag{2.6}$$

where the sum is over all complements b of a in $[\hat{0}, y]$, i.e., over all b satisfying $b \leq y$, $b \wedge (x \wedge y) = \hat{0}$ and $b \vee (x \wedge y) = y$. Since x is left-modular, it is equivalent to say that the sum in (2.6) is over all $b \in L$ satisfying $b \wedge x = \hat{0}$ and $y \in [b, b \vee x]$. Thus we have

$$\begin{aligned} \sum_{y \in L} \mu(y) t^{n-r(y)} &= \sum_{y \in L} \sum_{\substack{b \wedge x = \hat{0} \\ y \in [b, b \vee x]}} \mu(\hat{0}, b)\mu(b, y) t^{n-r(y)} \\ &= \sum_{b \wedge x = \hat{0}} \sum_{y \in [b, b \vee x]} \mu(b)\mu(b, y) t^{n-r(y)}. \blacksquare \end{aligned}$$

Obviously the previous lemma is true for the rank function. To apply this result to more general functions we make the following definition.

Definition 2.2.4 *A* generalized rank function *of a lattice L is a function $\rho : \{(x, y) \in L \times L \mid x \leq y\} \to \mathbb{R}$ such that for any $a \leq b \leq c$*

$$\rho(a, c) = \rho(a, b) + \rho(b, c).$$

In this case, we say L is generalized graded *by ρ.*

For short we write $\rho(x) = \rho(\hat{0}, x)$. Conversely, if we take any function $\rho : L \to \mathbb{R}$ such that $\rho(\hat{0}) = 0$, then we can easily construct a generalized rank function, namely $\rho(x, y) = \rho(y) - \rho(x)$. So the ordinary rank function is a special case.

If L is generalized graded by ρ, we now define a generalized characteristic polynomial of L by

$$\chi(L, t) = \sum_{x \in L} \mu(x) t^{\rho(x, \hat{1})} = \sum_{x \in L} \mu(x) t^{\rho(\hat{1}) - \rho(x)}. \tag{2.7}$$

Note that χ will depend on which generalized rank function we pick. Since the restriction of a generalized rank function to an interval $[a, b]$ still satisfies Definition 2.2.4

with $L = [a,b]$, the characteristic polynomial of the interval is defined in the same manner.

Theorem 2.2.5 *Let L be generalized graded by ρ. If $x \in L$ is a left-modular element, then*

$$\chi(L,t) = \sum_{\substack{b\in J(L)\\ b\wedge x=\hat{0}}} \left[\mu(b)t^{\rho(\hat{1})-\rho(b\vee x)}\chi([b,b\vee x],t)\right]. \tag{2.8}$$

Proof. Directly from Lemma 2.2.3 and the definition of the generalized rank function, we get

$$\begin{aligned}
\chi(L,t) &= \sum_{b\wedge x=\hat{0}} \mu(b) \sum_{y\in[b,b\vee x]} \mu(b,y)t^{\rho(\hat{1})-\rho(y)} \\
&= \sum_{b\wedge x=\hat{0}} \mu(b)t^{\rho(\hat{1})-\rho(b\vee x)} \sum_{y\in[b,b\vee x]} \mu(b,y)t^{\rho(b\vee x)-\rho(y)} \\
&= \sum_{b\wedge x=\hat{0}} \mu(b)t^{\rho(\hat{1})-\rho(b\vee x)} \sum_{y\in[b,b\vee x]} \mu(b,y)t^{\rho(y,b\vee x)} \\
&= \sum_{\substack{b\in J(L)\\ b\wedge x=\hat{0}}} \left[\mu(b)t^{\rho(\hat{1})-\rho(b\vee x)}\chi([b,b\vee x],t)\right]. \blacksquare
\end{aligned}$$

In the sum (2.8), the term $\chi([b, b\vee x],t)$ depends on b. To get a factorization formula, we will remove the dependency by applying certain restrictions so that $\chi([b, b\vee x],t) = \chi([\hat{0},x],t)$ for all b in the sum.

First, we will obtain a general condition under which two lattices have the same characteristic polynomial. In the following discussion, let L and L' be two lattices and let $\tau : L \to L'$ be any map. For convenience, we also denote $\hat{0} = \hat{0}_L$, $\hat{0}' = \hat{0}_{L'}$ and similarly for $\hat{1}$, $\hat{1}'$, μ, μ', etc.

We say τ is a *join-preserving* map if

$$\tau(u\vee v) = \tau(u)\vee\tau(v)$$

for any u, $v \in L$. Note that if τ is join-preserving then it is also order-preserving since

$$x\le y \;\Rightarrow\; y = x\vee y \;\Rightarrow\; \tau(y) = \tau(x\vee y) = \tau(x)\vee\tau(y) \;\Rightarrow\; \tau(x)\le\tau(y).$$

If τ is join-preserving, then given any $x' \in \tau(L)$, we claim that the subset $\tau^{-1}(x')$ has a unique maximal element in L. Suppose that $\tau(u) = \tau(v) = x'$ for some u, $v \in L$. We have $\tau(u \vee v) = \tau(u) \vee \tau(v) = x'$. Thus $u \vee v \in \tau^{-1}(x')$ and the claim follows. In addition, if τ is also surjective then we can define a map $\sigma : L' \to L$ by

$$\sigma(x') = \text{ the maximal element of } \tau^{-1}(x'). \tag{2.9}$$

Theorem 2.2.6 *Using the previous notation, suppose that τ is surjective and join-preserving and that σ is order-preserving with $\sigma(\hat{0}') = \hat{0}$. Then for any $x' \in L'$ we have*

$$\mu'(x') = \sum_{y \in \tau^{-1}(x')} \mu(y).$$

Proof. This is trivial when $x' = \hat{0}'$, since the second hypothesis on σ implies $\tau^{-1}(\hat{0}') = \{\hat{0}\}$. Let $x = \sigma(x')$. From the assumptions on τ and σ it is easy to see that

$$[\hat{0}, x] = \biguplus_{y' \in [\hat{0}', x']} \tau^{-1}(y'). \tag{2.10}$$

Now, by surjectivity of τ and induction, we get

$$\begin{aligned} \mu'(x') &= -\sum_{y' < x'} \mu'(y') \\ &= -\sum_{\substack{y \in \tau^{-1}(y') \\ y' < x'}} \mu(y) \\ &= \sum_{y \in \tau^{-1}(x')} \mu(y). \blacksquare \end{aligned}$$

Let L and L' be generalized graded by ρ and ρ', respectively. We say an order-preserving map $\tau : L \to L'$ is *rank-preserving* on a subset $S \subseteq L$ if $\rho(x, y) = \rho'(\tau(x), \tau(y))$ for any x, $y \in S$, $x \leq y$. Also we define a *support set* of L by

$$H(L) = \{x \in L \mid \mu(x) \neq 0\}.$$

Theorem 2.2.7 *If, in addition to the hypotheses of Theorem 2.2.6, the map τ is rank-preserving on $H(L) \cup \{\hat{1}\}$ then*

$$\chi(L,t) = \chi(L',t).$$

Proof. From (2.10) in the proof of Theorem 2.2.6, we know $L = \biguplus_{x' \in L'} \tau^{-1}(x')$. Then by Theorem 2.2.6 and the rank-preserving nature of τ, we have

$$\begin{aligned} \chi(L',t) &= \sum_{x' \in L'} \mu'(x') t^{\rho'(x',\hat{1}')} \\ &= \sum_{x' \in L'} \sum_{y \in \tau^{-1}(x')} \mu(y) t^{\rho'(x',\hat{1}')} \\ &= \sum_{x' \in L'} \sum_{y \in \tau^{-1}(x')} \mu(y) t^{\rho'(\tau(y),\tau(\hat{1}))} \\ &= \sum_{y \in H(L)} \mu(y) t^{\rho(y,\hat{1})} \\ &= \chi(L,t). \blacksquare \end{aligned}$$

It is easy to generalize the previous theorem to arbitrary posets as long as the map σ is well defined. However, we know of no application of the result in this level of generality.

Returning to our factorization theorem, we still need one more tool. For any given a, b in a lattice, we define

$$\sigma_a : [b, a \vee b] \to [a \wedge b, a] \quad \text{by} \quad \sigma_a(u) = u \wedge a,$$

$$\tau_b : [a \wedge b, a] \to [b, a \vee b] \quad \text{by} \quad \tau_b(v) = v \vee b.$$

The map τ_b is the one we need to achieve $\chi([b, b \vee x], t) = \chi([\hat{0}, x], t)$. We write $H(x,y)$ for $H([x,y])$ which is the support set of the sublattice $[x,y]$.

Lemma 2.2.8 *Let L be generalized graded and let $x \in L$ be a left-modular element. If $b \in L$ is such that $b \wedge x = \hat{0}$ and τ_b is rank-preserving on $H(\hat{0}, x) \cup \{x\}$, then*

$$\chi([b, b \vee x], t) = \chi([\hat{0}, x], t).$$

Proof. We need only verify that the hypotheses of Theorem 2.2.7 are satisfied. By left-modularity of x, we have

$$\tau_b \sigma_x(y) = b \vee (x \wedge y) = (b \vee x) \wedge y = y \tag{2.11}$$

for any $y \in [b, b \vee x]$. So τ_b is surjective. And it is easy to check that τ_b is join-preserving.

As for σ_x, we must check that it satisfies the definition (2.9), i.e., for any $y \in [b, b \vee x]$

$$\sigma_x(y) = \max \tau_b^{-1}(y).$$

Given $z \in \tau_b^{-1}(y)$ we have $y = \tau_b(z) = z \vee b$. So by the modular inequality (2.1) we get

$$\sigma_x(y) = y \wedge x = (z \vee b) \wedge x \geq z \vee (b \wedge x) \geq z.$$

Since this is true for any such z, we have $\sigma_x(y) \geq \max \tau_b^{-1}(y)$. But equation (2.11) implies $\sigma_x(y) \in \tau_b^{-1}(y)$, so we have equality. Now $\hat{0}_{[b,b\vee x]} = b$ so $\sigma(b) = b \wedge x = \hat{0}$ as desired. Noticing that σ_x is order-preserving, we complete the proof. ∎

We can now prove our main result.

Theorem 2.2.9 *Let L be generalized graded by ρ and let $x \in L$ be an left-modular element. If the map τ_b is rank-preserving on $H(\hat{0}, x) \cup \{x\}$ for every $b \in H(L)$ satisfying $b \wedge x = \hat{0}$. Then*

$$\begin{aligned} \chi(L,t) &= \chi([\hat{0},x],t) \sum_{\substack{b \in H(L) \\ b \wedge x = \hat{0}}} \mu(b) t^{\rho(\hat{1}) - \rho(x) - \rho(b)} \\ &= \chi([\hat{0},x],t) \sum_{b \wedge x = \hat{0}} \mu(b) t^{\rho(\hat{1}) - \rho(x) - \rho(b)}. \end{aligned}$$

Proof. By Lemma 2.2.8, we need only worry about the exponent on t in Theorem 2.2.5. But since τ_b is rank-preserving on $H(\hat{0}, x) \cup \{x\}$, we get

$$\begin{aligned} \rho(b \vee x) &= \rho(\hat{0}, b) + \rho(b, b \vee x) \\ &= \rho(\hat{0}, b) + \rho(\hat{0}, x) = \rho(b) + \rho(x). \ \blacksquare \end{aligned}$$

To apply this theorem, instead of $H(\hat{0},x)$ and $H(L)$ it is sometimes more convenient to check the hypotheses for two sets containing them. For example, they can be replaced by $J(\hat{0},x)$ and $J(L)$, or even by $[\hat{0},x]$ and L.

Here we state a corollary of the previous result which has a weaker hypothesis than Theorem 2.2.1. So Theorem 2.2.9 generalizes Stanley's Partial Factorization Theorem.

Corollary 2.2.10 *Let L be a finite semimodular lattice graded by the ordinary rank function ρ. If x is a modular element of L, then*

$$\chi(L,t) = \chi([\hat{0},x],t) \sum_{\substack{b\in H(L)\\ b\wedge x=\hat{0}}} \mu(b)t^{\rho(\hat{1})-\rho(x)-\rho(b)}.$$

Proof. To apply Theorem 2.2.9, it suffices to show that $\rho(\hat{0},z) = \rho(b, z\vee b)$ for every $z \in [\hat{0},x]$. Since (b,x) is a modular pair, we have $(z\vee b)\wedge x = z\vee(b\wedge x) = z\vee\hat{0} = z$. By Corollary 2.1.6, we know $z = (z\vee b)\wedge x$ is a modular element in $[\hat{0}, z\vee b]$, so (z,b) forms a modular pair in the same interval. Thus $\rho(z\wedge b)+\rho(z\vee b) = \rho(z)+\rho(b)$, because $[\hat{0}, z\vee b]$ is a semimodular lattice (see equation (2.3)). Since $z\wedge b = \hat{0}$ we are done. ∎

We take the divisor lattice D_n as an example. It is semimodular, but not atomic in general, so Theorem 2.2.1 does not apply. However Corollary 2.2.10 can be used for any $x \in D_n$, since all elements are modular.

We will now present two applications of the previous results in the following two sections.

2.3 Non-crossing Partition Lattices

The non-crossing partition lattice was first studied by Kreweras [11] who showed its Möbius function is related to Catalan number. By using NBB sets (see Sec. 2.5 for the

π : 123/4578/6 (non-crossing) 134/256/78 (crossing)

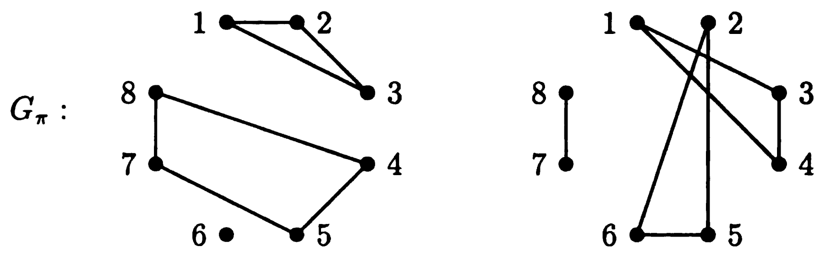


Figure 2.1. Partitions and their graphs

definition), Blass and Sagan [3] combinatorially explained this fact. In this section we will calculate the characteristic polynomial for a non-crossing partition lattice and then offer another explanation for the value of its Möbius function.

Recall the definition of the partition lattice Π_n in Section 1.1. We say that a partition $\pi \vdash [n]$ is *non-crossing* if there do not exist $i, k \in B$ and $j, l \in C$ for two distinct blocks B, C of π with $i < j < k < l$. Otherwise π is *crossing*.

Another way to view non-crossing partitions will be useful. Let $G = (V, E)$ be a graph with vertex set $V = [n]$ and edge set E. We say that G is *non-crossing* if there do not exist edges ik, $jl \in E$ with $i < j < k < l$. Equivalently, G is non-crossing if, when the vertices are arranged in their natural order clockwise around a circle and the edges are drawn as straight line segments, no two edges of G cross geometrically. Given a partition π we can form a graph G_π by representing each block $B = \{i_1 < i_2 < \ldots < i_l\}$ by a cycle with edges $i_1i_2, i_2i_3, \ldots, i_li_1$. (If $|B| = 1$ or 2 then B is represented by an isolated vertex or edge, respectively.) Then it is easy to see that π is non-crossing as a partition if and only if G_π is non-crossing as a graph. In Figure 2.1 we have displayed two partitions and their graphs.

The set of non-crossing partitions of $[n]$ forms a meet-sublattice NC_n of Π_n with the same rank function. However unlike Π_n, the non-crossing partition lattice is not semimodular in general, since if $\pi = 13$ and $\sigma = 24$ then $\pi \wedge \sigma = \hat{0}$ and $\pi \vee \sigma = 1234$. So we have

$$\rho(\pi) + \rho(\sigma) = 2 < 3 = \rho(\pi \wedge \sigma) + \rho(\pi \vee \sigma).$$

The join $\pi \vee_{\Pi_n} \sigma = 13/24$ also explains why NC_n is not a sublattice of Π_n.

Let $\pi = 12\ldots(n-1)$. We claim that π is left-modular in NC_n. It is well-known [16] that π is modular in Π_n and so left-modular there. Let a, $b \in NC_n$ with $a < b$ and both incomparable to π. It is clear that $a \vee \pi = b \vee \pi = \hat{1}$ in Π_n as well as in NC_n. By Theorem 2.1.4 we get $a \wedge \pi < b \wedge \pi$ in Π_n. Since NC_n is a meet-sublattice of Π_n, this inequality for the two meets still holds in NC_n. By Theorem 2.1.4 again, π is left-modular in NC_n. In general, π is not modular in NC_n. If $n \geq 4$, let $\sigma = 2n$ and $\phi = 1(n-1)/23\ldots(n-2)$. Clearly $\phi < \pi$, $\pi \wedge \sigma = \phi \wedge \sigma = \hat{0}$ and $\pi \vee \sigma = \phi \vee \sigma = \hat{1}$ in NC_n, so that (σ, π) is not a modular pair.

Proposition 2.3.1 *The characteristic polynomial of the non-crossing partition lattice NC_n satisfies*

$$\chi(NC_n, t) = t\ \chi(NC_{n-1}, t) - \sum_{i=1}^{n-1} \chi(NC_i, t)\chi(NC_{n-i}, t).$$

Proof. Let $\pi = 12\ldots(n-1)$. By Theorem 2.2.5 we have

$$\chi(NC_n, t) = \sum_{b \wedge \pi = \hat{0}} \left[\mu(b) t^{\rho(\hat{1}) - \rho(b \vee \pi)} \chi([b, b \vee \pi], t)\right].$$

Note that $b \wedge \pi = \hat{0}$ if and only if any two numbers of $[n-1]$ are in different blocks of b, so either $b = \hat{0}$ or $b = mn$ with $1 \leq m \leq n-1$.

If $b = \hat{0}$, then $\chi([b, b \vee \pi], t) = \chi([\hat{0}, \pi], t) = \chi(NC_{n-1}, t)$. Thus we get the first term of the formula. Now let $b = mn$. It is clear that $b \vee \pi = \hat{1}$, so we need to consider the sublattice $[b, \hat{1}]$. Given any $\omega \in [b, \hat{1}]$, the edge mn (which may not be in $E(G_\omega)$) geometrically separates the graph G_ω into two parts, $G_{\omega,1}$ and $G_{\omega,2}$, which are induced by vertex sets $\{1, 2, \ldots, m, n\}$ and $\{m, m+1, \ldots, n-1, n\}$, respectively. By contracting the vertices m and n in both $G_{\omega,1}$ and $G_{\omega,2}$, we get two non-crossing graphs $\bar{G}_{\omega,1}$ and $\bar{G}_{\omega,2}$. It is easy to check that the map $f : [b, \hat{1}] \to NC_m \times NC_{n-m}$ defined by $f(G_\omega) = (\bar{G}_{\omega,1}, \bar{G}_{\omega,2})$ is an isomorphism between these two lattices. Therefore

$$\chi([b, b \vee \pi], t) = \chi(NC_m, t)\chi(NC_{n-m}, t),$$

and the proof is complete. ∎

For any $\omega = B_1/B_2/\ldots/B_k \in NC_n$, the interval $[\hat{0}, \omega] \cong \prod_i NC_{|B_i|}$. Hence to compute the Möbius function of NC_n, it suffices to do this only for $\hat{1}$. By Proposition 2.3.1 we have the recurrence relation

$$\begin{aligned} \mu(NC_n) &= \chi(NC_n, 0) \\ &= -\sum_{i=1}^{n-1} \chi(NC_i, 0)\chi(NC_{n-i}, 0) \\ &= -\sum_{i=1}^{n-1} \mu(NC_i)\mu(NC_{n-i}) \end{aligned}$$

with the initial condition $\mu(NC_1) = 1$.

Consider a product $x_0x_1 \cdots x_n$. In how many ways can we insert parentheses such that there is no ambiguity as to the order of the multiplications? This number is called the *Catalan number* and denoted by C_n. For example, $C_0 = 1$ for x_0; $C_1 = 1$ for x_0x_1; $C_2 = 2$ for $(x_0x_1)x_2$, $x_0(x_1x_2)$; $C_3 = 5$ for $((x_0x_1)x_2)x_3$, $(x_0(x_1x_2))x_3$, $x_0((x_1x_2)x_3)$, $x_0(x_1(x_2x_3))$, $(x_0x_1)(x_2x_3)$. It is routine to check that C_n is uniquely determined by the recurrence relation

$$C_n = \sum_{i=0}^{n-1} C_iC_{n-1-i}$$

with the initial condition $C_0 = 1$. Therefore, by induction, $\mu(NC_n) = (-1)^{n-1}C_{n-1}$. An explicit expression for the Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

2.4 Shuffle Posets

The poset of shuffles was introduced by Greene [9], and he obtained a formula for its characteristic polynomial

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^{m+n} \sum_{i \geq 0} \binom{m}{i} \binom{n}{i} (\frac{1}{1-t})^i.$$

In this section we will derive an equivalent formula by using Theorem 2.2.9. Before doing this we need to recall some definitions and results which were given by Greene. Let $\mathcal{A}$ be a set, called the *alphabet of letters*. A *word* over $\mathcal{A}$ is a sequence $\mathbf{u} = u_1 u_2 \ldots u_n$ of elements of $\mathcal{A}$. All of our words will consist of distinct letters and we will sometimes also use $\mathbf{u}$ to stand for the set of letters in the word, depending upon the context. A *subword* of $\mathbf{u}$ is $\mathbf{w} = u_{i_1} \ldots u_{i_l}$ where $i_1 < \ldots < i_l$. If $\mathbf{u}$, $\mathbf{v}$ are any two words then the *restriction* of $\mathbf{u}$ to $\mathbf{v}$ is the subword $\mathbf{u_v}$ of $\mathbf{u}$ whose letters are exactly those of $\mathbf{u} \cap \mathbf{v}$. A *shuffle* of $\mathbf{u}$ and $\mathbf{v}$ is any word $\mathbf{s}$ such that $\mathbf{s} = \mathbf{u} \uplus \mathbf{v}$ as sets (disjoint union) and $\mathbf{s_u} = \mathbf{u}$, $\mathbf{s_v} = \mathbf{v}$ as words.

Given nonnegative integers m and n, Greene defined the *poset of shuffles* $\mathcal{W}_{m,n}$ as follows. Fix disjoint words $\mathbf{x} = x_1 \ldots x_m$ and $\mathbf{y} = y_1 \ldots y_n$. The elements of $\mathcal{W}_{m,n}$ are all shuffles $\mathbf{w}$ of a subword of $\mathbf{x}$ with a subword of $\mathbf{y}$. The partial order is that $\mathbf{v} \leq \mathbf{w}$ if $\mathbf{v_x} \supseteq \mathbf{w_x}$, $\mathbf{v_y} \subseteq \mathbf{w_y}$ as sets and $\mathbf{v_w} = \mathbf{w_v}$ as words. The covering relation is more intuitive: $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{w}$ can be obtained from $\mathbf{v}$ by either adding a single y_i or deleting a single x_j. It is easy to see that $\mathcal{W}_{m,n}$ has bottom element $\hat{0} = \mathbf{x}$, top element $\hat{1} = \mathbf{y}$ and is graded by the rank function

$$\rho(\mathbf{w}) = (m - |\mathbf{w_x}|) + |\mathbf{w_y}|.$$

For example, $\mathcal{W}_{2,1}$ is shown in Figure 2.2 where $\mathbf{x} = de$ and $\mathbf{y} = D$.

It was shown by Greene that every shuffle poset is actually a lattice. To describe the join operation in $\mathcal{W}_{m,n}$, Greene defined crossed letters as follows. Given $\mathbf{u}$, $\mathbf{v} \in \mathcal{W}_{m,n}$ then $x \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{x}$ is *crossed* in $\mathbf{u}$ and $\mathbf{v}$ if there exist letters y_i, $y_j \in \mathbf{y}$ with

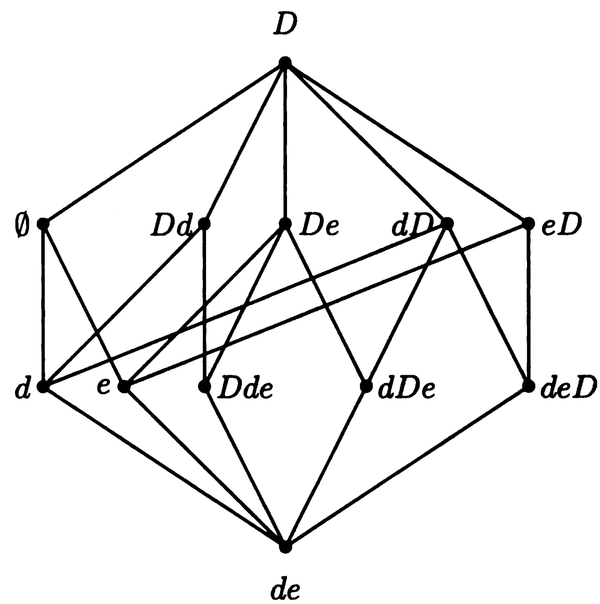


Figure 2.2. The lattice $\mathcal{W}_{2,1}$

$i \leq j$ and x appears before y_i in one of the two words but after y_j in the other. For example, let $\mathbf{x} = def$ and $\mathbf{y} = DEF$. Then in the two shuffles $\mathbf{u} = dDEe$, $\mathbf{v} = Fdef$, the only crossed letter is d. The join of $\mathbf{u}$, $\mathbf{v}$ is then the unique word $\mathbf{w}$ greater than both $\mathbf{u}$, $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{w_x} &= \{x \in \mathbf{u_x} \cap \mathbf{v_x} \mid x \text{ is not crossed}\} \\ \mathbf{w_y} &= \mathbf{u_y} \cup \mathbf{v_y}. \end{aligned} \tag{2.12}$$

In the previous example, $\mathbf{u} \vee \mathbf{v} = DEFe$. This join also shows that $\mathcal{W}_{m,n}$ is not semimodular in general, because $\rho(\mathbf{u}) + \rho(\mathbf{v}) = 3 + 1 < 5 = \rho(\mathbf{u} \vee \mathbf{v}) \leq \rho(\mathbf{u} \vee \mathbf{v}) + \rho(\mathbf{u} \wedge \mathbf{v})$. Since $(\mathcal{W}_{n,m})^* = \mathcal{W}_{m,n}$, the meet operation in $\mathcal{W}_{m,n}$ is as same as the join operation in $(\mathcal{W}_{n,m})^*$. So to find the meet in the analogous way we need to consider those letters $y \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{y}$ crossed in $\mathbf{u}$ and $\mathbf{v}$.

Greene also showed that subwords of $\mathbf{x}$ and subwords of $\mathbf{y}$ are modular elements of $\mathcal{W}_{m,n}$. In particular, the empty set $\emptyset$ is modular. It is also atomic since $[\hat{0}, \emptyset] \cong B_m$. We now give our formula for the characteristic polynomial of $\mathcal{W}_{m,n}$.

Proposition 2.4.1 *The characteristic polynomial of the shuffle poset is*

$$\chi(\mathcal{W}_{m,n},t) = (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \qquad (2.13)$$

Proof. If $\mathbf{u} \wedge \emptyset = \mathbf{w}$ in $\mathcal{W}_{m,n}$ then $\mathbf{w_x} = \mathbf{u_x} \cup \emptyset_\mathbf{x} = \mathbf{u_x}$. So the meet $\mathbf{u} \wedge \emptyset = \hat{0}$ if and only if $\mathbf{x}$ is a subword of $\mathbf{u}$, i.e., the element $\mathbf{u}$ is a shuffle of $\mathbf{x}$ with a subword of $\mathbf{y}$. Furthermore there is no crossed letter x in $\mathbf{u}$ and any $\mathbf{v} \in [\hat{0}, \emptyset]$ since $\mathbf{v_y} = \emptyset$. It follows that $(\mathbf{u} \vee \mathbf{v})_\mathbf{x} = \mathbf{u_x} \cap \mathbf{v_x} = \mathbf{v}$ and $(\mathbf{u} \vee \mathbf{v})_\mathbf{y} = \mathbf{u_y} \cup \mathbf{v_y} = \mathbf{u_y}$ as sets. Then we get

$$\begin{aligned} \rho(\mathbf{u} \vee \mathbf{v}) - \rho(\mathbf{u}) &= [(m - |\mathbf{v}|) + |\mathbf{u_y}|] - [(m-m) + |\mathbf{u_y}|] \\ &= m - |\mathbf{v}| = \rho(\mathbf{v}) - \rho(\hat{0}). \end{aligned}$$

Thus the map $\tau_\mathbf{u} : [\hat{0}, \emptyset] \to [\mathbf{u}, \emptyset \vee \mathbf{u}]$ is rank-preserving.

Since $[\hat{0}, \emptyset] \cong B_m$, by Theorem 2.2.9 we get

$$\chi(\mathcal{W}_{m,n},t) = (t-1)^m \sum_{\mathbf{u} \wedge \emptyset = \hat{0}} \mu(\mathbf{u}) t^{(m+n)-m-\rho(\mathbf{u})}.$$

It is easy to see that the interval $[\hat{0}, \mathbf{u}]$ is isomorphic to B_i where $i = |\mathbf{u_y}|$. So $\mu(\mathbf{u}) = (-1)^{|\mathbf{u_y}|} = (-1)^{\rho(\mathbf{u})}$. Now we conclude that

$$\begin{aligned} \chi(\mathcal{W}_{m,n},t) &= (t-1)^m \sum_{i=0}^{n} \begin{bmatrix} \text{the number of ways to} \\ \text{shuffle } \mathbf{x} \text{ with } i \text{ letters of } \mathbf{y} \end{bmatrix} (-1)^i t^{n-i} \\ &= (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \blacksquare \end{aligned}$$

To determine the Möbius function of $\mathcal{W}_{m,n}$, it suffices to compute $\mu(\hat{1})$ since for any $\mathbf{w} \in \mathcal{W}_{m,n}$ the interval $[\hat{0}, \mathbf{w}]$ is isomorphic to a product of $\mathcal{W}_{p,q}$'s for certain $p \leq m$ and $q \leq n$. For example, let $\mathbf{x} = defgh$ and $\mathbf{y} = DEFGHIJK$ then $\mu(DEeGhIK) = \mu(\mathcal{W}_{1,2})\mu(\mathcal{W}_{2,1})\mu(\mathcal{W}_{0,2})$ because

$$[defgh, DEeGhIK] \cong \mathcal{W}_{d,DE} \times \mathcal{W}_{fg,G} \times \mathcal{W}_{\emptyset,IK}.$$

Simply plugging $t = 0$ into formula (2.13) gives us the Möbius function $\mu(\mathcal{W}_{m,n})$.

Corollary 2.4.2 (Greene [9]) *We have*

$$\mu(\mathcal{W}_{m,n}) = (-1)^{m+n}\binom{m+n}{n}. \blacksquare$$

2.5 NBB Sets and Factorization Theorems

Applying Theorem 2.2.9 we intend to inductively prove the Total Factorization Theorem of the characteristic polynomial if L is an LL lattice. This theorem was given by Blass and Sagan [3] and generalizes Stanley's Total Factorization Theorem for SS lattices. First of all, we would like to outline their work.

Given a lattice L, recall that $A = A(L)$ is the set of atoms of L. Assign A an arbitrary partial order, which we denote $\trianglelefteq$ to distinguish it from the partial order $\leq$ in L. A nonempty set $D \subseteq A$ is *bounded below* or *BB* if, for every $d \in D$ there is an $a \in A$ such that

$$a \triangleleft d \qquad \text{and} \qquad a < \bigvee D.$$

We say that $B \subseteq A$ is *NBB* (*not bounded below*) if B does not contain any D which is bounded below. In this case we will call B an *NBB base* for the element $x = \bigvee B$. One of the main results of Blass and Sagan's paper is the following theorem which is a simultaneous generalization of both Rota's NBC and Crosscut Theorems (for the crosscut $A(L)$).

Theorem 2.5.1 (Blass and Sagan [3]) *Let L be a finite lattice and let $\trianglelefteq$ be any partial order on A. Then for all $x \in L$ we have*

$$\mu(x) = \sum_{B} (-1)^{|B|}$$

where the sum is over all NBB bases B of x. $\blacksquare$

Given an arbitrary lattice L, let $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ be a maximal chain of L. Define the i^{th} *level* of A by

$$A_i = \{a \in A \mid a \leq x_i \text{ but } a \not\leq x_{i-1}\},$$

and we partially order A by setting $a \lhd b$ if and only if $a \in A_i$ and $b \in A_j$ with $i < j$. We say a is in lower level than b or b is in higher level than a if $a \lhd b$. Note that the level A_i is an empty set if and only if x_i is a non-atomic element. Also an atom a cannot be $\leq \bigvee S$ for any set S of atoms from strictly lower levels, since there is an x_i that is $\geq$ all elements of S as well as $\bigvee S$ but not $\geq a$. If all elements of Δ are left-modular, then we say (L, Δ) (or simply L) is a *left-modular* lattice. In this case, another property involving a join of atoms is given next.

Lemma 2.5.2 (Blass and Sagan [3]) *If a and b are distinct atoms from the same level A_i in a left-modular lattice, then $a \vee b$ is above some atom $c \in A_j$ with $j < i$.* ∎

A pair (L, Δ) is said to satisfy the *level condition* if this partial order $\unlhd$ of A has the following property.

$$\text{If } a \lhd b_1 \lhd b_2 \lhd \ldots \lhd b_k \text{ then } a \not\leq \bigvee_{i=1}^{k} b_i.$$

A pair (L, Δ) is called an *LL lattice* if it is left-modular and satisfies the level condition. In this case, the characterization of NBB sets is described as follows.

Lemma 2.5.3 (Blass and Sagan [3]) *In an LL lattice, a set $B \subseteq A$ is NBB if and only if $|B \cap A_i| \leq 1$ for every i.* ∎

We define a generalized rank function $\rho : L \to \mathbb{N}$ by

$$\rho(x) = \text{ number of } A_i \text{ containing atoms less than or equal to } x.$$

Note that, for any $x \in L$, we have $\rho(x) = \rho(\delta(x))$ where $\delta(x)$ is the maximum atomic element in $[\hat{0}, x]$. So $\rho(\hat{1})$ is not necessary equal to n, the length of Δ. The following lemma states the relationship between this ρ and NBB bases.

Lemma 2.5.4 (Blass and Sagan [3]) *Let B be an NBB set in an LL lattice. Then every atom $a \leq \bigvee B$ is in the same level as some element of B. In particular, any NBB base for x has exactly $\rho(x)$ atoms.* ∎

Blass and Sagan generalized Stanley's Total Factorization Theorem to LL lattices using their theory of NBB sets. Here we present two inductive proofs for the LL factorization theorem. In the first proof we will apply Theorems 2.2.9 as well as the theory of NBB sets.

Theorem 2.5.5 (Total Factorization Theorem, Blass and Sagan [3]) *If (L, Δ) is an LL lattice then its characteristic polynomial factors as*

$$\chi(L,t) = \prod(t - |A_i|)$$

where the product is over all non-empty levels A_i.

Proof of Theorem 2.5.5 I. We will induct on n, the length of Δ. The theorem is trivial when $n \leq 1$. If $A_n = \emptyset$, then $\rho(x_n) = \rho(x_{n-1})$ and thus $\chi(L,t) = \chi([\hat{0}, x_{n-1}], t)$. So we are done by induction.

If $A_n \neq \emptyset$, consider $y \in H(L)$. Then, by Theorem 2.5.1, y must have an NBB base. So if $b \in H(L)$ and $b \vee x_{n-1}$ then, by Lemma 2.5.3, b has an NBB base of at most one atom from A_n. So $b = \hat{0}$ or $b \in A_n$ and $\rho(b) \leq 1$. Now it suffices to check that τ_b is rank-preserving on $H(\hat{0}, x_{n-1}) \cup \{x_{n-1}\}$ for every $b \in A_n$ since then we get $\chi(L,t) = \chi([\hat{0}, x_{n-1}], t)(t - |A_n|)$ by Theorem 2.2.9. Because $A_n \neq \emptyset$ and $\rho(b) = 1$, τ_b is rank-preserving on $\{x_{n-1}\}$. Given any $y \in H(\hat{0}, x_{n-1})$, suppose B be an NBB base for y. By Lemma 2.5.3, $B' = B \cup \{b\}$ is an NBB base for $\tau_b(y)$. Now $\rho(\tau_b(y)) = |B'| = |B| + 1 = \rho(y) + \rho(b)$ by Lemma 2.5.4. Hence $\rho(b, \tau_b(y)) = \rho(\tau_b(y)) - \rho(b) = \rho(y) = \rho(\hat{0}, y)$. ∎

In a similar way, Corollary 2.2.10 provides us with an inductive proof for Theorem 2.2.2. Note that the lattice in Theorem 2.2.2 is pure, so $\rho(\hat{1})$ equals the length of Δ. Therefore the product (2.5) is over all levels A_i (including empty ones).

We will use Theorem 2.2.5 for the second proof. This demonstration sidesteps the machinery of NBB sets and reveals some properties of LL lattices in the process. To prepare we need the following two lemmas.

Lemma 2.5.6 *If w is a left-modular element in L and $v \prec w$, then $v \vee u \preceq w \vee u$ for any $u \in L$.*

Proof. Suppose not and then there exists $s \in L$ such that $v \vee u < s < w \vee u$. Taking the join with w and using $v \vee w = w$, we get $w \vee (v \vee u) = w \vee s = w \vee (w \vee u)$. So we should have $w \wedge (v \vee u) < w \wedge s < w \wedge (w \vee u) = w$ by Theorem 2.1.4. Combining this with $v \leq w \wedge (v \vee u)$, we have a contradiction to $v \prec w$. ∎

Lemma 2.5.7 *If (L, Δ) is an LL lattice with $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ and $A_n \neq \emptyset$, then $([b, \hat{1}], \Delta')$ is also an LL lattice for any $b \in A_n$ where Δ' consists of the distinct elements of the multichain*

$$b = x'_0 \preceq x'_1 \preceq x'_2 \preceq \ldots \preceq x'_{n-2} \preceq x'_{n-1} = \hat{1}$$

where $x'_i = x_i \vee b, 0 \leq i \leq n-1$. Furthermore we have $|A_i| = |A'_i|$ for such i, where

$$A'_i = \{a \in A(b, \hat{1}) \mid a \leq x'_i \text{ but } a \not\leq x'_{i-1}\}.$$

Proof. By Lemma 2.5.6, the chain Δ' is indeed saturated. So Δ' is a left-modular maximal chain by Proposition 2.1.5.

Let $\tau(x) = \tau_b(x) = x \vee b$. It is surjective (see the proof of Lemma 2.2.8) and order-preserving from $[\hat{0}, x_{n-1}]$ to $[b, \hat{1}]$. Also let $A = A(\hat{0}, x_{n-1})$ and $A' = A(b, \hat{1})$. First, We prove that the map $\tau : A \to A'$ is well-defined and bijective. Suppose that there is an $a \in A_i$ such that $b \prec x < \tau(a) = a \vee b$ for some x. By the level condition and Lemma 2.5.2, in L any atom $c \leq a \vee b$ is in a level at least as high as a and only one such c is in A_i, namely a. Since $x < a \vee b$ and $a \not\leq x$, any atom $d \leq x$ is in a higher level than a. It follows that $x_i \wedge x = \hat{0}$. Now $b \vee (x_i \wedge x) = b$ and $(b \vee x_i) \wedge x \geq (b \vee a) \wedge x = x$ contradicts the left-modularity of x_i. We conclude that $\tau : A \to A'$ is well-defined. The restriction $\tau|_A$ is surjective since τ is surjective and order-preserving. To show injectivity of $\tau|_A$, let us suppose there are two distinct atoms u and v such that $\tau(u) = \tau(v)$. If u and v are from two different levels then this contradicts the level condition. If u and v are from the same level, by Lemma 2.5.2, there exists an atom c in a lower level such that $c \leq u \vee v \leq \tau(u) \vee \tau(v) = \tau(u)$, contradicting the level condition again.

Now let us prove $|A_i| = |A'_i|$. This is trivial for $i = 1$. Let $u \in A_i$ for some nonempty A_i with $2 \leq i \leq n-1$. It is clear that $u \vee b \leq x_i \vee b$. Assuming that $u \vee b \leq x_{i-1} \vee b$, we get $(b \vee x_{i-1}) \wedge (u \vee b) = u \vee b$. But $b \vee (x_{i-1} \wedge (u \vee b)) = b \vee \hat{0} = b$ contradicts the modularity of x_{i-1}. Thus $\tau(A_i) \subseteq A'_i$ and then the bijectivity of $\tau|_A$ implies that $|A_i| = |A'_i|$ for all $i \leq n-1$.

Since $\tau|_A$ is bijective and level-preserving, if $\tau(a) \leq \bigvee_{i=1}^{k} \tau(b_i)$ for some $\tau(a) \lhd \tau(b_1) \lhd \tau(b_2) \lhd \ldots \lhd \tau(b_k)$ in $[b, \hat{1}]$, then $a < a \vee b \leq (\bigvee_{i=1}^{k} b_i) \vee b$ with $a \lhd b_1 \lhd b_2 \lhd \ldots \lhd b_k \lhd b$ in L. Thus $([b, \hat{1}], \Delta')$ satisfies the level condition. ∎

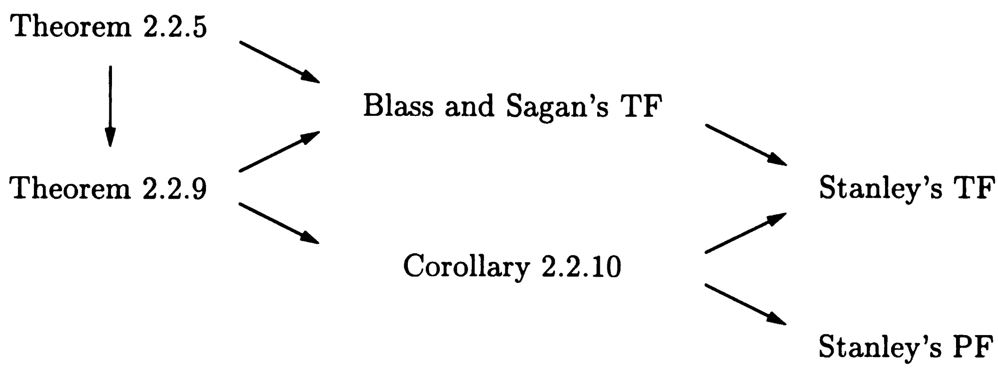
Proof of Theorem 2.5.5 II. We will induct on $n = \ell(\Delta)$. The cases $n \leq 1$ and $A_n = \emptyset$ are handled as before.

If $A_n \neq \emptyset$ and b is atomic with $b \wedge x_{n-1} = \hat{0}$, then b can only be above atoms in A_n. So by Lemma 2.5.2, b must be the join of at most one atom, i.e., either $b = \hat{0}$ or $b \in A_n$. Thus by Lemma 2.5.7 and induction we get, for any $b \in A_n$

$$\chi([b, \hat{1}], t) = \prod_{i \leq n-1} (t - |A_i|) = \chi([\hat{0}, x_{n-1}], t)$$

where the product is over non-empty A_i. Applying Theorem 2.2.5 gives $\chi(L, t) = \chi([\hat{0}, x_{n-1}], t)(t - |A_n|)$, so again we are done. ∎

To end this chapter, we depict the relationships between all factorization theorems that we have discussed. In the following chart, "TF" means "Total Factorization Theorem" and "PF" stands for "Partial Factorization Theorem."


Theorem 2.2.5
Blass and Sagan's TF
Theorem 2.2.9
Stanley's TF
Corollary 2.2.10
Stanley's PF

CHAPTER 3

EDGE LABELINGS OF POSETS

3.1 R-Labelings and Shellable Posets

This section is an introduction to the basic terminology and theorems for edge labelings of posets. The covering relation $x \prec y$ is also an edge in the Hasse diagram of P, so the set $\mathcal{E}(P) = \{xy \mid x \prec y \text{ for } x, y \in P\}$ is called the *edge set* of P. If $[u, v] \subseteq P$ we write $\mathcal{E}(u, v)$ for $\mathcal{E}([u, v])$. Sometimes, instead of using xy, we will write the edge pair as $x \prec y$. An *edge labeling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where $(\Lambda, \leq)$ is some poset (usually the integers). We will use "$\leq$" for both P and Λ if there is no confusion; otherwise we will write $\leq_\Lambda$ for Λ to distinguish from $\leq$ for P. Given $x \prec y \prec z$, if $\lambda(xy) < \lambda(yz)$ we will say λ is *strictly rising* at y in this chain; otherwise λ has a *strict descent* at y. Note that at a strict descent we do not necessary have $\lambda(xy) \geq \lambda(yz)$ since Λ is a poset. "*Weakly rising*" and "*weak descent*" are defined in the analogous way. Here we study two important concepts: R-labeling and lexicographic shelling. Both of them unveil combinatorial properties of posets.

Definition 3.1.1 *Let P be a bounded poset. An edge labeling $\lambda : \mathcal{E}(P) \to \Lambda$ is called a* strict R-labeling *of P if, for every interval $[x, y]$ of P, there is a unique saturated* *chain* $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ *satisfying*

$$\lambda(x_0x_1) < \lambda(x_1x_2) < \cdots < \lambda(x_{k-1}x_k). \tag{3.1}$$

We will call this chain the rising chain *from* x *to* y *and denote it by* $C_r(x, y)$. *A poset* P *possessing a strict R-labeling* λ *is called a* strict R-poset.

Replacing "$<$" with "$\leq$" in (3.1) defines a *weak* R-labeling and a *weak* R-poset. A strict R-labeling is not necessary a weak R-labeling, since one might have more than one weakly rising chain. An *R-labeling* is one that is either a strict or weak R-labeling. The concept of *R-poset* is defined analogously. Note that if $[x, y]$ is an interval of an R-poset P, then the restriction of λ to $\mathcal{E}(x, y)$ is still an R-labeling. So any property satisfied by an R-posets P is also satisfied by any interval of P.

Let P be an poset with a strict R-labeling λ. A *falling chain* from x to y is a saturated chain $D : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ satisfying

$$\lambda(x_{i-1}x_i) \not< \lambda(x_ix_{i+1}), \text{ for all } 1 \leq i \leq k-1.$$

Replacing "$\not<$" with "$\not\leq$" we can define a falling chain in a weak R-poset.

Let P be a bounded pure poset of rank n with the rank function ρ. If $S \subseteq [n-1]$ then we define the poset

$$P_S = \{x \in P \mid x = \hat{0}, \hat{1} \text{ or } \rho(x) \in S\},$$

called the *S-rank-selected subposet* of P. For instance, we have $P_\emptyset = C_1$ and $P_{[n-1]} = P$.

The following theorems give connections between R-labeling and the Möbius function.

Theorem 3.1.2 (Stanley [17]) *Let* P *be a pure R-poset of rank* n *and let* $S \subseteq [n-1]$. *Then* $(-1)^{|S|+1}\mu(P_S)$ *is equal to the number of maximal chains* $M : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ *of* P *for which the labeling* λ *has descents exactly at those* x_i *with* $i \in S$. ∎

Corollary 3.1.3 (Stanley [17]) *If P is a pure R-poset than for all x, $y \in P$*

$$\mu(x,y) = (-1)^{\rho(y)-\rho(x)}(\textit{number of falling } x\text{–}y \textit{ chains}). \blacksquare$$

Theorem 3.1.4 (Björner and Wachs [6]) *If P is an R-poset (not necessary a pure one), then for all x, $y \in P$*

$$\begin{aligned} \mu(x,y) &= \textit{number of even length falling chains in } [x,y] \\ &\quad - \textit{number of odd length falling chains in } [x,y]. \blacksquare \end{aligned}$$

Shellability is studied for simplicial complexes in algebraic topology and also has properties of a combinatorial nature. We will only treat this subject for posets because we are focusing on combinatorics, not topology. A poset P is said to be *shellable* if its maximal chains can be ordered M_1, M_2,..., M_t in such a way that if $1 \leq i < j \leq t$ then there exist $1 \leq k < j$ and $x \in M_j$ such that $M_i \cap M_j \subseteq M_k \cap M_j = M_j - \{x\}$. Various types of edge-labelings were introduced by Björner and Wachs that imply shellability.

For the labeling poset Λ, let Λ^∞ denote the set of all strings $(\alpha_1, \ldots, \alpha_p)$ with $\alpha_i \in \Lambda$ and variable length p. The *lexicographic order* $\leq_l$ on Λ^∞ is the one such that

$$(\alpha_1, \ldots, \alpha_p) <_l (\beta_1, \ldots, \beta_q)$$

if and only if either

1. $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, q$ and $q < p$, or

2. $\alpha_i \neq \beta_i$ for some i and $\alpha_i < \beta_i$ for the least such i.

Given a saturated chain $C : x_0 \prec x_1 \prec \ldots \prec x_n$ in an edge-labeled poset we label the chain by

$$\lambda(C) = (\lambda(x_0 x_1), \ldots, \lambda(x_{n-1} x_n)).$$

Definition 3.1.5 *An edge labeling λ of P is called an* EL-labeling (edge lexicographic labeling) *if*

1. *λ is an R-labeling, and*

2. *given any interval $[x, y]$, the unique rising chain C_r satisfies $\lambda(C_r(x, y)) <_l \lambda(C)$ for any other maximal chain C in $[x, y]$.*

Building on the work of Björner [4] for pure posets, Björner and Wachs [6] showed that if any poset has an EL-labeling then it is shellable. So a poset admitting an EL-labeling is said to be *EL-shellable.* An equivalent condition to EL-shellability is as follows.

Proposition 3.1.6 (Björner and Wachs [6]) *Labeling λ is an EL-labeling of P if and only if*

1. *λ is an R-labeling, and*

2. *given any interval $[x, y]$, if $C_r(x, y) : x = x_0 \prec x_1 \prec \ldots \prec x_n = y$ then $\lambda(xx_1) < \lambda(xz)$ for all $z \neq x_1$ such that $x \prec z \leq y$.*

Some important EL-sellable posets have following stronger property which is the dual of (2) in previous proposition.

Definition 3.1.7 *An edge labeling λ of P is called an* SL-labeling *if*

1. *λ is an EL-labeling, and*

2. *given any interval $[x, y]$, if $C_r(x, y) : x = x_0 \prec x_1 \prec \ldots \prec x_n = y$ then $\lambda(zy) < \lambda(x_{n-1}y)$ for all $z \neq x_{n-1}$ such that $x \leq z \prec y$.*

A poset admitting an SL-labeling is said to be *SL-shellable.* Depending on whether the R-labeling is strict or weak, the same will be said about the EL-labeling or SL-labeling. However, the requirement (2) in Proposition 3.1.6, as well as in Definitions 3.1.5 and 3.1.7, is always "strictly less" even for a weak EL- or SL-labeling.

3.2 Left-modular Lattices

Unless noted otherwise in this section we assume that the lattice L is left-modular with a left-modular maximal chain $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$. We define an edge labeling λ_Δ of L by

$$\lambda_\Delta(xy) = \min\{i \mid x_i \vee x = x_i \vee y\}. \tag{3.2}$$

Clearly $\lambda_\Delta(x_{i-1}x_i) = i$ for $i = 1, 2, \ldots, n$. So λ_Δ is a natural labeling induced by Δ. Since all x_i are left-modular, by Theorem 2.1.4 we have an equivalent definition

$$\lambda_\Delta(xy) = \min\{i \mid x_i \wedge x \neq x_i \wedge y\}.$$

It is easy to check that if $x_i \vee x = x_i \vee y$ then $x_j \vee x = x_j \vee y$ for any $j > i$. Thus we have two more equivalent definitions

$$\begin{aligned} \lambda_\Delta(xy) &= \max\{i+1 \mid x_i \vee x \neq x_i \vee y\} & (3.3)\\ &= \max\{i+1 \mid x_i \wedge x = x_i \wedge y\}. & (3.4) \end{aligned}$$

Combining (3.2) and (3.4), we get following property.

Lemma 3.2.1 *The labeling $\lambda_\Delta(xy) = i$ if and only if both $x_i \vee x = x_i \vee y$ and $x_{i-1} \wedge x = x_{i-1} \wedge y$.* ■

In the following proposition we do not require that all elements of Δ are left-modular. The left-modularity of a single element x_i is sufficient.

Proposition 3.2.2 *1. If x_i is left-modular and $\lambda_\Delta(xy) = i$ for an edge $xy \in \mathcal{E}(P)$, then*

$$x_{i-1} \vee x \prec x_{i-1} \vee y = x_i \vee x = x_i \vee y, \tag{3.5}$$

$$x_{i-1} \wedge x = x_{i-1} \wedge y = x_i \wedge x \prec x_i \wedge y. \tag{3.6}$$

2. *If x_i is left-modular, then there are no u, v, x, $y \in L$ such that $u \prec v \leq x \prec y$ and $\lambda_\Delta(uv) = \lambda_\Delta(xy) = i$.*

Proof. (1) By definition we have $x_{i-1} \vee x < x_{i-i} \vee y \leq x_i \vee y = x_i \vee x$, and $x_{i-1} \vee x \preceq x_i \vee x$ from Lemma 2.5.6. So (3.5) holds. The second statement follows by a dual argument from the equivalent definition (3.4).

(2) Suppose $\lambda_\Delta(uv) = \lambda_\Delta(xy) = i$. Let $s = x_{i-1} \vee x$ and $t = x_{i-i} \vee y$. From part (1) and $v \leq x$, we get $x_i \vee v = x_{i-1} \vee v \leq s \prec t = x_i \vee y$. Now take the meet with x_i to get $x_i = x_i \wedge s = x_i \wedge t = x_i$. On the other hand, $x_i \vee s = x_i \vee x = x_i \vee y = x_i \vee t$. This is a contradiction to Theorem 2.1.4. ■

Property (2) shows that the number i cannot be a label twice on any saturated chain of L.

Now we would like to introduce another induced labeling. Let L be any lattice. Recall the definition of irreducible elements in Section 1.2 and let $I(L)$ be the set of irreducible elements of L. Given a map $\omega : I(L) \to \mathbb{P}$, it induces an edge labeling by the rule

$$\lambda_I(xy) = \min\{\omega(z) \mid z \in I(L) \text{ and } x \prec x \vee z = y\}.$$

This is well-defined, because every $x \in L$ can be written as a join of irreducibles. If λ_I is an R-labeling then ω is called an *admissible map.*

Now let (L, Δ) be a left-modular lattice, and let

$$\omega(z) = \min\{i \mid z \leq x_i\}$$

for any $z \in I(L)$. The values of ω partition $I(L)$ into n blocks (levels) $I_1, I_2, \ldots, I_n$ where $I_i = \{z \in I(L) \mid \omega(z) = i\}$. We also denote $I_x = \{z \in I(L) \mid z \leq x\}$ for any $x \in L$. Clearly $I_{x_i} = \biguplus_{j=1}^{i} I_j$ for $i = 1, 2, \ldots, n$. An equivalent definition for λ_I is

$$\lambda_I(xy) = \min\{\omega(z) \mid z \in I_y - I_x\}.$$

We will show that the map ω induced by the left-modular chain Δ is admissible. First of all, we would like to show that $\lambda_I = \lambda_\Delta$ on $\mathcal{E}(L)$. It is clear that $\lambda_I(x_{i-1}x_i) = i = \lambda_\Delta(x_{i-1}x_i)$ for any i. Also if z is an irreducible element with $y \prec z$ then

$$\lambda_I(yz) = \omega(z) = \lambda_\Delta(yz), \tag{3.7}$$

where the second equality is from Lemma 3.2.1.

Lemma 3.2.3 *We have* $\lambda_\Delta = \lambda_I$.

Proof. If $\lambda_I(xy) = i$ there is a $z \in I(L)$ such that $x \vee z = y$ and $\omega(z) = i$, so $x_i \vee x = (x_i \vee z) \vee x = x_i \vee y$. This implies $\lambda_\Delta(xy) \le i$, so $\lambda_\Delta \le \lambda_I$ on $\mathcal{E}(L)$.

We assume, towards a contradiction, that $\lambda_\Delta(xy) = j < i = \lambda_I(xy)$ for some $y \in L$ with $y \succ x$. By Proposition 3.2.2(2) no edge in $[\hat{0}, x]$ is labeled j by λ_Δ, so all irreducibles in $[\hat{0}, x]$ are labeled other than j by (3.7), i.e., $j \notin \omega(I_x)$. Also we know that $\omega(I_y - I_x) \subseteq [i, n]$, so $j \notin \omega(I_y)$ and then $I_j \cap I_y = \emptyset$. Thus $x_j \wedge y = \bigvee(I_{x_j} \cap I_y) = \bigvee(I_{x_{j-1}} \cap I_y) = x_{j-1} \wedge y$. This contradicts (3.6). ∎

Let $\Omega_x = \{\omega(z) \mid z \in I(L), z \le x\}$. A new method of labeling is defined by

$$\lambda_\Omega(xy) = \min\ (\Omega_y - \Omega_x).$$

Since $\lambda_I = \lambda_\Delta$, Proposition 3.2.2(2) gives $\lambda_\Omega = \lambda_I$. So from now on we will just use λ to represent all three labelings.

We need to state two lemmas before our main result.

Lemma 3.2.4 *Let L be left-modular with $z \in I(L)$ and $x < z \vee x$ for some $x \in L$. Then each edge in the interval $[x, z \vee x]$ is labeled by a number $\le \omega(z)$.*

Proof. Let $\omega(z) = i$ and pick any edge $u \prec v$ in $[x, z \vee x]$. It is easy to check that $z \vee u = z \vee x = z \vee v$, so we have $u \vee x_i = u \vee (z \vee x_i) = v \vee (z \vee x_i) = v \vee x_i$. Thus $\lambda(uv) \le i$. ∎

Lemma 3.2.5 *Given any $x < y$ in a left-modular lattice L. Let z, $z' \in I_y - I_x$ be two distinct irreducibles such that $\omega(z) = \omega(z') = \min \omega(I_y - I_x)$. Then we have $x \prec z \vee x = z' \vee x$.*

Proof. The covering relation is directly from the previous lemma and Proposition 3.2.2(2). In fact $\lambda(x \prec z \vee x) = \omega(z)$. If $z \vee x \neq z' \vee x$ then $z' \in I_{z' \vee (z \vee x)} - I_{z \vee x}$. By the result we have just proved, we should have $z \vee x \prec z' \vee (z \vee x)$ and $\lambda(z \vee x \prec z' \vee (z \vee x)) = \omega(z')$ but this contradicts Proposition 3.2.2(2). ∎

Theorem 3.2.6 *Left-modular lattices are SL-shellable.*

Proof. Given any interval $[x, y]$ in L. If $x \prec y$, the edge itself is the unique rising chain. If $x \not\prec y$, let $z \in I_y - I_x$ be an irreducible such that $\omega(z) = i = \min \omega(I_y - I_x)$. The chain formed by concatenating the edge $x \prec z \vee x$ and $C_r([z \vee x, y])$ (which is the unique rising chain obtained by induction) forms a rising chain from x to y. It is strict by Proposition 3.2.2(2). Since the number i must be a label on some edge of any maximal chain in $[x, y]$, any rising chain must have its first edge labeled by i. So Lemma 3.2.5 implies the uniqueness of the rising chain. By the same lemma, λ is an EL-labeling.

Applying what we have just shown to the dual lattice L^*, which has left-modular chain Δ^*, we see that L^* admits an EL-labeling λ^*. Using (3.4) we see that $\lambda^*(xy) = n + 1 - \lambda(yx)$. So EL-shellability of L^* implies SL-shellability of L. ∎

It follows from Proposition 3.2.2(2) that λ is a strict as well as a weak R-labeling. This theorem is a generalization of one for supersolvable lattices [16], but left-modular lattices are not always pure as supersolvable ones are. In the case of a non-pure lattice, we see that Δ is labeled by $[n]$ while any other maximal chain is labeled by a subset of $[n]$ without repetition. Thus we have following proposition.

Proposition 3.2.7 *Let (L, Δ) be a left-modular lattice. The length of Δ is maximum among all maximal chains in L.* ∎

This result also follows from a general theory of shellability [6]: the first maximal chain of a shelling must be of maximum length among all maximal chains.

We now look at the Tamari lattice, T_n, as an example. Consider all proper parenthesizations π of the word $x_0 x_1 \ldots x_n$. It is well known that the number of these is the Catalan number C_n as was mentioned in Section 2.3. Partially order this set by saying that σ covers π whenever

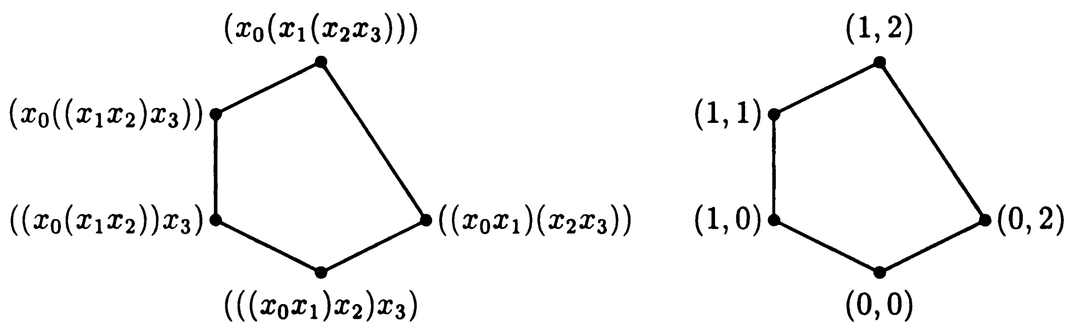
$$\pi = \ldots((AB)C)\ldots \qquad \text{and} \qquad \sigma = \ldots(A(BC))\ldots$$

for some subwords A, B, C. The corresponding poset turns out to be a lattice called the *Tamari lattice* T_n.

A *left bracket vector*, $(v_1, \ldots, v_{n-1})$, is a vector of nonnegative integers satisfying

1. $0 \leq v_i \leq i$ for all i, and
2. if $S_i = [v_i, i]$ then for any pair S_i, S_j either one set contains the other or $S_i \cap S_j = \emptyset$.

The number of left bracket vectors is also C_n. In fact given a parenthesized word π we have an associated left bracket vector $v = (v_1, \ldots, v_{n-1})$ defined as follows. To calculate v_i, start at x_i in π and move left, counting the number of x's you pass (including x_i itself) and comparing it with the number of left parentheses you pass until these two numbers are equal. Then $v_i = j$ where x_j is the last x passed before the numbers balance. It is not hard to show that this gives a bijection between parenthesizations and left bracket vectors, thus inducing a partial order on the latter. In fact this induced order is just the component-wise one. The elements covering $(v_1, \ldots, v_{n-1})$ are those $(w_1, \ldots, w_{n-1})$ such that $w_i = v_i$ for all except one value j, and w_j is the least number $> v_j$ that does not violate hypotheses (1) and (2). Figure 3.1 gives the parenthesized and bracket vector versions of T_3.



(a) Parenthesized version (b) Left bracket version

Figure 3.1. The Tamari lattice T_3

Given left bracket vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ then

$$v \vee w = (\max\{v_1, w_1\}, \ldots, \max\{v_n, w_n\}).$$

The Tamari lattice T_n is left-modular and a left-modular chain was given in [3] as follows.

$$\begin{aligned}\Delta : (0, \ldots, 0) &< (1, 0, \ldots, 0) < (1, 1, 0, \ldots, 0) \\ &< (1, 2, 0, \ldots, 0) < (1, 2, 1, 0 \ldots, 0) < \ldots < (1, 2, 3, \ldots, n-1).\end{aligned}$$

For an edge $v = (v_1, \ldots, v_j, \ldots, v_{n-1}) \prec w = (v_1, \ldots, v_{j-1}, w_j, v_{j+1}, \ldots, v_{n-1})$, we consider

$$\begin{aligned} x_k &= (1, 2, \ldots, j-1, w_j, 0, \ldots, 0), \text{ and} \\ x_{k-1} &= (1, 2, \ldots, j-1, w_j - 1, 0, \ldots, 0) \end{aligned}$$

where $k = 1 + 2 + \ldots + (j-1) + w_j$, and compute

$$\begin{aligned} v \vee x_k = (1, 2, \ldots, j-1, w_j, v_{j+1}, \ldots, v_{n-1}) = w \vee x_k, \text{ but} \\ v \vee x_{k-1} = (1, 2, \ldots, j-1, w_j - 1, v_{j+1}, \ldots, v_{n-1}) \\ \neq w \vee x_{k-1} = (1, 2, \ldots, j-1, w_j, v_{j+1}, \ldots, v_{n-1}) \end{aligned}$$

Therefore $\lambda(vw) = k$ is the explicit SL-labeling for the edge vw. Figure 3.2 shows this labeling for T_4.

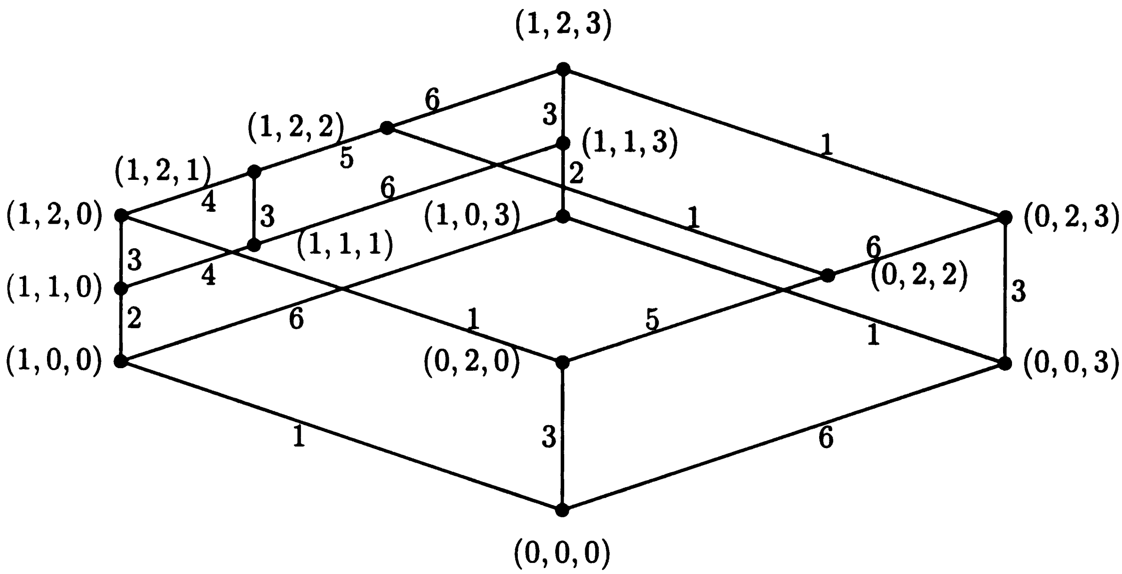


Figure 3.2. The Tamari lattice T_4 and its SL-labeling

3.3 Shellability of Rank-selected Posets

Let P be a bounded pure poset with rank function ρ and $\rho(L) = n$. Recall the definition of rank-selected poset P_S in Section 3.1. In particular, if $0 \le k+1 < l \le n$ and $S = [1,k] \cup [l, n-1]$ we have the *truncation*

$$P_k^l = P_S = \{x \in P \mid \rho(x) \in [0,k] \cup [l,n]\}.$$

If P admits an R-labeling $\lambda : \mathcal{E}(P) \to \Lambda$, we would like to construct an R-labeling for P_k^l. Let us write $\hat{\Lambda}$ for Λ with a $\hat{1}$ and $\hat{0}$ adjoined on top and bottom and order $\hat{\Lambda} \times \hat{\Lambda}$ component-wise, i.e., $(a,b) \le (c,d)$ if and only if $a \le c$ and $b \le d$ in $\hat{\Lambda}$.

Define a labeling $\hat{\lambda} : P_k^l \to \hat{\Lambda} \times \hat{\Lambda}$ by

$$\hat{\lambda}(xy) = \begin{cases} (\lambda(xy), \hat{0}) & \text{if } \rho(y) \le k, \\ (\hat{1}, \lambda(xy)) & \text{if } \rho(x) \ge l, \\ (\lambda(xx'), \lambda(y'y)) & \text{else, where } C_r(x,y) = x,\ x', \ldots,\ y',\ y. \end{cases}$$

Lemma 3.3.1 *$\hat{\lambda}$ is an R-labeling for P_k^l if P is an R-poset.*

Proof. The only interesting case is for an interval $[x, y]$ where $\rho(x) = p \leq k$ and $\rho(y) = q \geq l$. Suppose that $C = C_r(x, y) : x = x_p, x_{p+1}, \ldots, x_q = y$ in P, then $C_k^l : x_p, \ldots, x_k, x_l, \ldots, x_p$ is a rising chain in P_k^l. Given any other chain $D_k^l : x = y_p, y_{p+1}, \ldots, y_k, y_l, \ldots, y_p = y$ in P_k^l, it can be extended to be a chain $D : x = y_p, y_{p+1}, \ldots, y_{k-1}, C_r(y_k, y_l), y_{l+1}, \ldots, y_p = y$ in P. This chain D must be different from $C_r(x, y)$, so it has descents at some of $y_{p+1}, \ldots, y_k$ and $y_l, \ldots, y_{p-1}$, and then so does D_k^l. Thus the rising chain C_k^l is unique. ∎

Note that this lemma works for both strict and weak R-posets.

Theorem 3.3.2 *If P, a pure poset of rank n, has an R-labeling then so does P_S for any $S \subseteq [n-1]$.* ∎

Suppose $S \cup \{0, n\} = [l_1, k_1] \cup [l_2, k_2] \cup \ldots \cup [l_p, k_p]$ with $0 = l_1 < k_1 + 1 < l_2 < k_2 + 1 < l_3 < \ldots < l_p \leq k_p = n$. Now we merely truncate ranks in the intervals (k_i, l_{i+1}) one by one. A much simpler R-labeling for P_S is $\hat{\lambda} : \mathcal{E}(P_S) \to \hat{\Lambda}^p$ defined by

$$\hat{\lambda}(xy) = \begin{cases} (\hat{1}, \ldots, \hat{1}, \lambda(xy), \hat{0}, \ldots, \hat{0}) & \text{if } \rho(x) \in [l_i, k_i - 1], \text{ where } \lambda(xy) \\ & \text{is the } i^{th} \text{ term of the string,} \\ (\hat{1}, \ldots, \hat{1}, \lambda(xx'), \lambda(y'y), \hat{0}, \ldots, \hat{0}) & \text{if } \rho(x) = k_i, \text{ where } \lambda(xx') \text{ is the } i^{th} \\ & \text{term and } C_r(x, y) = x, x', \ldots, y', y. \end{cases}$$

Example 3.3.3 Let $B = B_n$, the Boolean algebra on $[n]$. A natural R-labeling for an edge $U \prec V$ ($U \subset V$ and $|U| + 1 = |V|$) is

$$\lambda(UV) = V - U$$

where $V - U$ represents the single number in the set. This labeling is a strict as well as weak one.

We can count the falling chains C in B_k^l to compute the Möbius function. Let U, $V \in C$ and $|U| = k$, $|V| = l$. There are unique $\emptyset - U$ and $V - [n]$ falling chains. So we need only ensure descents at both U and V where $U \subset V$. By definition of $\hat{\lambda}$, this amounts to

$$\text{(a) } \min U > \min(V - U) \quad \text{and} \quad \text{(b) } \max(V - U) > \max([n] - V).$$

Elementary set theory shows this is equivalent to

$$\text{(a}'\text{) } \min V \notin U \quad \text{and} \quad \text{(b}'\text{) } n \in V,\ [n-i+1, n] \not\subseteq U$$

where $[n - i + 1, n] = \{n - i + 1, n - i + 2, \ldots, n\}$ is the final run in V for some $1 \le i \le l$. So $|\mu(B_k^l)|$ equals

$$\begin{aligned}
&\sum_{i=1}^{l} (\#\text{ of } V \text{ with final run } [n-i+1,n])\ (\#\text{ of corresponding } U) \\
&\quad = \sum_i \binom{n-i-1}{l-i}\left[\binom{l-1}{k} - \binom{l-i-1}{k-i}\right] \\
&\quad = \binom{n-1}{l-1}\binom{l-1}{k} - \sum_i \binom{n-i-1}{l-i}\binom{l-i-1}{k-i}.
\end{aligned}$$

This sum has no closed form, but when $k = 0$ it zeros out and we obtain the same result as in [20]:

$$\mu(B_0^l) = (-1)^{n-l+1}\binom{n-1}{l-1}.$$

If P is shellable so is P_S. But whether a rank-selected poset preserves EL- or SL-shellability remains open. To inherit these two properties by using the induced labeling $\hat{\lambda}$ we need a stronger hypothesis. A *thrifty labeling* is a strict R-labeling such that

$$|\lambda(\mathcal{E}(x, y))| = \rho(x, y)$$

for any x, $y \in P$. Since $\lambda(C_r(\hat{0}, \hat{1})) = \lambda(\mathcal{E}(L))$ as sets in this case, the labeling poset Λ must be a total order.

Theorem 3.3.4 *If P admits a thrifty EL-labeling (resp. SL-labeling) then P_S is EL-shellable (resp. SL-shellable) for any $S \subseteq [n-1]$.*

Proof. It suffices to prove this for P_k^l. By Theorem 3.3.2, we need only show that $\hat{\lambda}$ satisfies Proposition 3.1.6(2) for those intervals $[x, y]$ with $\rho(x) = k$ and $\rho(y) = q > l$. In P_k^l, suppose xx' is the first edge in the $x - y$ rising chain and $x'' \in P_k^l$ with $x \prec x'' \leq y$, $x'' \neq x'$. By the thrifty condition in P, $\lambda(C_r(x, x'))$ is an increasing string consisting of the $l - k$ least numbers in $\lambda(\mathcal{E}(x, y))$ while $\lambda(C_r(x, x''))$ is also an increasing string consisting of some other $l - k$ distinct numbers in the same set. Therefore $\hat{\lambda}(xx') < \hat{\lambda}(xx'')$. SL-shellability can be prove in an analogous manner. ∎

Many lattices have edge labelings satisfying the thrifty hypothesis, for example, the natural labeling for the Boolean algebra B_n. In fact, the labeling λ_Δ for any pure left-modular lattice (L, Δ) is a thrifty SL-labeling.

For the rest of this section we will discuss a thrifty edge labeling for the non-crossing partition lattice NC_n which is not a left-modular lattice. The partition lattice Π_n is a supersolvable lattice with a left-modular maximal chain $\Delta : \hat{0} < 12 < 123 < \ldots < [n] = \hat{1}$. We will label each edge $1 \ldots (i-1) \prec 1 \ldots i$ on this chain by i instead of $i - 1$, for example $\lambda(\hat{0} \prec 12) = 2$. The induced labeling $\lambda = \lambda_\Delta$ is a thrifty SL-labeling with $\lambda(\mathcal{E}(\Pi_n)) = [2, n]$. In fact, there is an explicit formula for λ. If $\pi \prec \sigma$ with two blocks B, C in π merged into one in σ then

$$\lambda(\pi\sigma) = \max\{\min B, \min C\}.$$

Suppose $\pi < \sigma$ in Π_n, and let $\pi = A_1/B_1/C_1/\ldots/A_2/B_2/C_2/\ldots/\ldots$, $\sigma = Z_1/Z_2/\ldots$ where $Z_i = A_i \uplus B_i \uplus C_i \uplus \cdots$. We also assume that $a_i = \min A_i < b_i = \min B_i < c_i = \min C_i < \ldots$ for each i. Then $C_r(\pi, \sigma)$ in Π_n is the unique chain by only merging blocks with the same subscript such that $\lambda(C_r(\pi, \sigma))$ is the sequence gotten from $\{b_1, c_1, \ldots, b_2, c_2, \ldots\}$ by rearranging the numbers in increasing order. If both π and σ are non-crossing, it is clear that $C_r(\pi, \sigma) \subseteq NC_n$. So λ is still a thrifty SL-labeling for NC_n.

3.4 CR-labelings and CL-labelings

Let $\mathcal{M}(P)$ be the set of maximal chains of P, and $\mathcal{ME}(P)$ be the set of pairs $(M, xy) \in \mathcal{M}(P) \times \mathcal{E}(P)$ consisting of a maximal chain M and an edge xy along that chain. A *chain-edge labeling* of P is a map $\lambda : \mathcal{ME}(P) \to \Lambda$, where Λ is some poset, satisfying

> *Axiom CE:* If two maximal chain $M : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_k = \hat{1}$ and $M' : \hat{0} = x'_0 \prec x'_1 \prec \ldots \prec x'_l = \hat{1}$ coincide along their first d edges, then $\lambda(M, x_{i-1}x_i) = \lambda(M', x'_{i-1}x'_i)$ for $i = 1, \ldots, d$.

An edge labeling λ naturally induces a chain-edge labeling λ' by letting $\lambda'(m, xy) = \lambda(xy)$ for any maximal chain m containing x and y.

If $[x, y]$ is an interval and R is a saturated chain from $\hat{0}$ to x, then the pair $([x, y], R)$ will be called a *rooted interval* with root R, and will be denoted $[x, y]_R$. If M is any maximal chain of $[x, y]$, we shall also consider it as a maximal chain of the rooted interval $[x, y]_R$ and denote it by M_R. Then $R \cup M$ is maximal chain of $[\hat{0}, y]$.

Let λ be a chain-edge labeling of P and $[x, y]_R$ a rooted interval. By axiom CE, if M is a maximal chain of $[x, y]$ and M', M'' are maximal chains of P that contain $R \cup M$, then the first d entries of $\lambda(M')$ and $\lambda(M'')$ coincide where $d = \ell(R \cup M)$. Hence the labeling on M depends only on a given root R of $[x, y]$ but not on an extended maximal chain M' in P. Like an edge labeling, with each maximal chain $M : x = x_0 \prec x_1 \prec \ldots \prec x_k = y$ in a rooted interval $[x, y]_R$ we associate the ordered string

$$\lambda(M_R) = (\lambda(M', x_0x_1), \ldots, \lambda(M', x_{k-1}x_k))$$

where M' is any maximal chain containing $R \cap M$. Note that the length of the tuple $\lambda(M_R)$ depends on the length of the chain M. We will use the lexicographic order on these strings.

Definition 3.4.1 *Let $\lambda : \mathcal{ME} \to \Lambda$ be a chain-edge labeling of a bounded poset P.*

1. λ *is called a* CR-labeling *if in every rooted interval* $[x, y]_R$ *there is a unique maximal rising chain which is denoted by* $C_r([x, y]_R)$.

2. *A CR-labeling* λ *is called* CL-labeling *if for every* $[x, y]_R$ *the unique rising chain is strictly lexicographically first, i.e.,* $\lambda(C_r([x, y]_R)) <_l \lambda(N_R)$ *for any other maximal chain* N *in* $[x, y]_R$.

A poset admitting a CL-labeling (resp. CR-labeling) is called a *CL-shellable* poset (resp. *CR-poset*). The definitions of CR- and CL-labeling generalize the notion of R- and EL-labeling, respectively. Björner and Wachs [5] introduced these generalizations and proved that CL-shellable posets are shellable. They also gave the analog of the main result in Section 3.3 in this context.

Theorem 3.4.2 (Björner and Wachs [5]) *If* P *is a pure CL-shellable poset (resp. CR-poset) of rank* n*, then* P_S *is still a CL-shellable poset (resp. CR-poset) for all* $S \subseteq [n-1]$. ■

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