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THE KAZHDAN PROPERTY OF THE MAPPING CLASS GROUP OF CLOSED SURFACES AND THE FIRST COHOMOLOGY GROUP OF THEIR COFINITE SUBGROUPS

By

Feraydoun Taherkhani

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ABSTRACT

THE KAZHDAN PROPERTY OF THE MAPPING CLASS GROUP OF CLOSED SURFACES AND THE FIRST COHOMOLOGY GROUP OF THEIR COFINITE SUBGROUPS

By

Feraydoun Taherkhani

This dissertation studies the mapping class group of closed surfaces $\mathcal{M}_g$ with respect to the property T or the Kazhdan property. The motivation for this work finds its root in [Iv4]. *N. Ivanov* asks in this preprint if every cofinite subgroup in $\mathcal{M}_g$ has a vanishing first cohomology group. In the same preprint, he addresses the more general question of whether $\mathcal{M}_g$ is a Kazhdan group or not. If a group has cofinite subgroups having a non-trivial first cohomology group, then it cannot be a Kazhdan group. In the following we first consider $\mathcal{M}_2$, the mapping class group of a closed surface of genus 2, and show that it does not satisfy the Kazhdan property by constructing subgroups of finite index having a non-vanishing first cohomology group. As a matter of fact there are a lot of subgroups with this property (see 3.3). We also tried to follow the same approach for a genus 3 surface, which turned out to be hopeless (see section 3.1). We adopted a new method that was originally introduced by *R. Gilman* to study the automorphism group of a free group, and after minor changes we were able to apply it to $\mathcal{M}_g$. In this way we were able to construct some subgroups of finite index in the mapping class group of a genus 3 surface. We also managed to calculate their first cohomology groups, which all turned out to be trivial. Unfortunately, most of the subgroups we obtained this way contain the Torelli subgroup and are of no use to our problem (see section 3.2). In only one case did we find a cofinite subgroup that did not contain the Torelli subgroup, and it turned out to be of a trivial first cohomology group. We also used the same method to construct some more cofinite subgroups for $\mathcal{M}_2$, and all of them turned out to have

a non-trivial first cohomology group (see 3.3). The subgroups constructed for $\mathcal{M}_3$ could neither respond to Ivanov's question in the negative, nor could they deliver any substantiated evidence in favor of the mapping class group being a Kazhdan group. The current capacity of our computers have set the boundary for further subgroups to be calculated. Maybe if we manage to construct a variety of more subgroups in the future using better and more advanced machines, we would be able to shed some more light upon the questions studied in this dissertation. We have done some of the calculations in this paper by hand, but the bulk of the calculations have been carried out by the aid of a computer using the programming language GAP (see [Sc]). [1]

[1] GAP *Groups and Algorithms* is a software package developed at *RWTH-Aachen*, that primarily deals with algorithms concerning finite groups, finitely presented groups and their presentations. There are several versions of GAP available. We have used version 3 release 4.4 on April 18^{th} 1997. GAP is free and can be obtained by an anonymous ftp to the following server:
samson.math.rwth-aachen.de

To Sofia Fermina Taherkhani (my lovely daughter)

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Chapter 1

Introduction

Let $\mathcal{S}_g$ be a closed surface of genus g, $\pi_g = \pi_1(\mathcal{S}_g)$ its fundamental group and $\mathcal{M}_g$ the corresponding mapping class group, i.e. the group of the isotopy classes of orientation preserving diffeomorphisms of $\mathcal{S}_g$. The connection between $\mathcal{M}_g$ and combinatorial group theory was established by *Nielsen* towards the end of the twenties (see [Ni]). If we drop the assumption of orientation preserving and consider the group of isotopy classes of all diffeomorphisms, the so-called extended Mapping Class Group $\mathcal{M}_g^*$, then from a pure algebraic point of view $\mathcal{M}_g^*$ is isomorphic to $Out(\pi_g)$, the outer automorphism group of π_g. Hence $\mathcal{M}_g$, as a subgroup of index two in $\mathcal{M}_g^*$, can be identified with a subgroup of index two in $Out(\pi_g)$ that we denote by $Out^+(\pi_g)$. The group $\mathcal{M}_g$ is generated by special types of diffeomorphisms called *Dehn twists* around simple closed curves (see [De]). Unaware of Dehn's result, *Lickorish* finds a generating set of $3g-1$ elements (see [Li1] and [Li2]). The minimal number of Dehn twists generating $\mathcal{M}_g$ is $2g+1$ and was determined by *Humphries* (see [Hu1]). The Dehn twists around the simple closed curves $\alpha_1, \beta_1, \alpha_2, \beta_2, .., \alpha_g, \beta_g, \delta$ (see Figure 1) will be our choice for a generating set and we will denote them by $D_{\alpha_1}, D_{\beta_1}, D_{\alpha_2}, D_{\beta_2}, .., D_{\alpha_g}, D_{\beta_g}, D_\delta$.

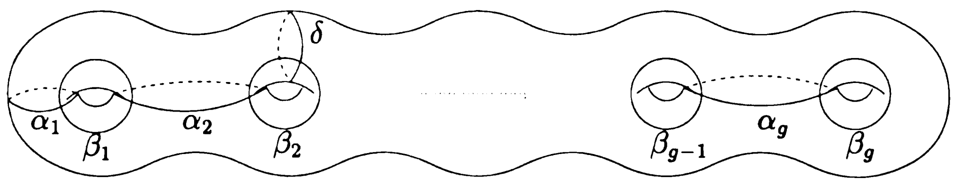

Figure 1.1: Dehn twists generating $\mathcal{M}_g$

The question of the existence of a finite presentation was settled by *Birman* and *Hilden* for the case of $g = 2$ (see [BH]) and by *McCool* (see [Mc]) for $g \geq 3$. A simple presentation was determined by *Wajnryb* in 1983 that was slightly corrected in 1994 (see [Wj1] and [BW]) based on results obtained by *Hatcher* and *Thurston* [HT]. Directly from the presentation we can establish the well-known result that $\mathcal{M}_g$ is a perfect group for $g \geq 3$ (see [P]):

$$H_1(\mathcal{M}_g, \mathbb{Z}) = \begin{cases} \mathbb{Z}_{10} & \text{if } g = 2 \\ 0 & \text{if } g \geq 3. \end{cases}$$

Another property of $\mathcal{M}_g$ is its *residual finiteness*, which was proved by *Grossman* [Gr]. Residual finiteness means that the intersection of all normal subgroups of finite index in $\mathcal{M}_g$ is trivial (see [KMS] page 116). In the mid-eighties the analogy between $\mathcal{M}_g$ and the *arithmetic groups* was established (see [Iv1], [Iv2] and [Iv3]). For the definition of an arithmetic group we refer to [Huj], however for our purposes the only arithmetic group of interest would be the symplectic group over the integers, $Sp_{2g}(\mathbb{Z})$, for $g \geq 2$. In the following theorem we will list some of its properties:

Theorem 1 *Let $\Gamma_g(m)$ be the kernel of the canonical epimorphism*

$$\Phi_g(m) : Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}_m)$$

a) *Every cofinite subgroup of $Sp_{2g}(\mathbb{Z})$ (a subgroup of finite index) contains one of the congruence subgroups $\Gamma_g(m)$.*

b) *Every non-trivial normal subgroup in $Sp_{2g}(\mathbb{Z})$ different from the center contains a congruence subgroup and hence is of finite index.*

c) *Every cofinite subgroup U of $Sp_{2g}(\mathbb{Z})$ has a vanishing first cohomology group, i.e. $H^1(U) = 0$. Since $Sp_{2g}(\mathbb{Z})$ is finitely presented, this property is equivalent to saying that U/U' is finite.*

Proof: Parts *a* and *b* of the theorem are proved by *Mennicke* for the symplectic group (see [Me]) and generalized by *Bass, Milnor* and *Serre* in [BMS]. Part *c* of the theorem is a consequence of part *b* and the following proposition:

Proposition 1 *Let $H \leq G$ be cofinite. Then if $[H : H']$ is finite, so is $[G : G']$.*

Proof : Since $H' \leq G'$, therefore $H' \leq G' \cap H$ and consequently $G' \cap H$ will be cofinite in G, but $G' \cap H \leq G'$, which means that G' is cofinite in G.

q.e.d.

Now we can prove part *c* of the theorem. We may assume that U is a normal subgroup of $G = Sp_{2g}(\mathbb{Z})$. Otherwise we can pass to $Core_G(U) = \bigcap_{g\in G} g^{-1}Ug$, which is a cofinite normal subgroup in G contained in U. Using the proposition, it suffices to prove the statement for $Core_G(U)$. Since U is a normal subgroup of G, therefore U' as a characteristic subgroup of U will be a normal subgroup of G. Using part *b* of the theorem, U' is cofinite in G hence cofinite in U.

q.e.d.

Let $N \trianglelefteq \pi_g$ be a cofinite characteristic subgroup of π_g. Then the canonical map

$$Aut(\pi_g) \to Aut(\pi_g/N)$$

factors through the outer automorphism group and after the restriction to $\mathcal{M}_g$ (recall that $\mathcal{M}_g \simeq Out^+(\pi_g)$) we obtain a homomorphism

$$\Psi_{g,N} : \mathcal{M}_g \to Out(\pi_g/N)$$

whose kernel $\Lambda_{g,N}$ is a cofinite normal subgroup in $\mathcal{M}_g$. We call these subgroups the congruence subgroups, in analogy with arithmetic groups. *N. Ivanov* asks the following questions about $\mathcal{M}_g$ (see [Iv4]).

1. **Congruence Subgroup Problem for $\mathcal{M}_g$:**
   Does any cofinite subgroup $U \leq \mathcal{M}_g$ contain one of the $\Lambda_{g,N}$?

2. Does every cofinite subgroup U of $\mathcal{M}_g$ have a vanishing first cohomology group?

3. Does $\mathcal{M}_g$ satisfy the *Kazhdan property*?

For an introduction to the Kazhdan property (or property T) we refer to [Kz], [HV] and [Lu]. The only important fact we need to know is that a T-group (a group satisfying the property T) has no cofinite subgroups having a non-vanishing first cohomology group. This means a negative response to the second question would answer the third question negatively as well.

Theorem 2 *$\mathcal{M}_g$ the mapping class group of a closed surface of genus $g = 2$ is not a Kazhdan group.*

In the following we prove Theorem 2 by constructing examples of subgroups having a non-trivial first cohomology group.

Chapter 2

Constructing Examples For $g = 2$

$\mathcal{M}_g$ acts on the first homology groups $H_1(\mathcal{S}_g, \mathbb{Z})$ and $H_1(\mathcal{S}_g, \mathbb{Z}_m)$ for every $m \in \mathbb{Z}$. This action preserves the symplectic form and gives rise to the homomorphisms

$$\Theta_g : \mathcal{M}_g \to Sp_{2g}(\mathbb{Z})$$

and

$$\Theta_g(m) : \mathcal{M}_g \to Sp_{2g}(\mathbb{Z}_m),$$

which are known to be surjective. Let $\mathcal{T}_g = Kernel(\Theta_g)$ be the *Torelli-subgroup* of $\mathcal{M}_g$ and $\mathcal{T}_g(m) = Kernel(\Theta_g(m))$ the preimage of the congruence subgroup $\Gamma_g(m)$ in $\mathcal{M}_g$. In this way we obtain a lot of cofinite normal subgroups of $\mathcal{M}_g$ all containing $\mathcal{T}_g$. In particular, in the case of $g = 2$ and $m = 2$ we get

$$\Theta_2(2) : \mathcal{M}_2 \to Sp_4(\mathbb{Z}_2) \simeq S_6.$$

$Sp_4(\mathbb{Z}_2)$ is isomorphic to S_6, the symmetric group on six elements, which has order 720, therefore $\mathcal{T}_2(2) \trianglelefteq \mathcal{M}_2$ will be a normal subgroup of index 720 in $\mathcal{M}_2$. The normal subgroup $\mathcal{T}_2(2)$ is generated by the squares of the Dehn twists around the simple closed curves and normally generated by the square of only one of the Dehn twists such as a_1, the first generator of $\mathcal{M}_2$ (see [Hu2]). Using the *Schreier-Reidemeister* method (see [Jl1] and [Jl2]) we can calculate a presentation for $\mathcal{T}_2(2)$ using GAP. The simplest presentation we can construct after all the possible reductions using *Tietze* transformations contains 14 generators and 388 relations of total length 8622. As a subgroup, $\mathcal{T}_2(2)$ is generated by the following 14 elements:

$$a_1^{-2}, b_1^{-2}, a_2^{-2}, b_2^{-2}, d^{-2},$$

$$a_1 b_1^{-2} a_1^{-1}, b_1 a_2^{-2} b_1^{-1}, a_2 b_2^{-2} a_2^{-1},$$

$$b_2 d^{-2} b_2^{-1}, a_1 b_1 a_2^{-2} b_1^{-1} a_1^{-1},$$

$$a_2 b_2 d^{-2} b_2^{-1} a_2^{-1}, a_1 b_1 a_2 b_2^{-2} a_2^{-1} b_1^{-1} a_1^{-1},$$

$$b_1 a_2 b_2^{-2} a_2^{-1} b_1^{-1}, b_1 a_2 b_2 d^{-2} b_2^{-1} a_2^{-1} b_1^{-1}.$$

By writing the 388 relations in an additive form we obtain a matrix of 388 rows and 14 columns that we refer to as the *relation matrix* of the presentation. From the presentation we can compute $H_1(\mathcal{T}_2(2))$ (the commutator factor group of $\mathcal{T}_2(2)$) by applying the *Gauß*-algorithm to this matrix to evaluate its invariant divisors. The divisors are

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 4,$$

meaning

$$H_1(\mathcal{T}_2(2)) = 9\mathbb{Z} \oplus 4\mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

As a byproduct, we see that 14 is the lowest cardinality for a generating set of this group (since none of the invariant divisors is 1). This example takes care of the genus $g = 2$.

In addition, using an algorithm called *"Low Index Subgroups"* in GAP we can calculate a complete list of the conjugacy classes of all subgroups of finite index bounded by a given number p. (The algorithm is efficient only for small indices. For example, $p = 20$ is already a huge index for $\mathcal{M}_2$). We have tabulated all the conjugacy classes of subgroups of $\mathcal{M}_2$ for $p = 10$ together with their commutator factor group in Table 2.1.

Actually, H_6 is the commutator subgroup of $\mathcal{M}_2$, which is a perfect group, and H_7 corresponds to the only subgroup (up to conjugation) of order 72 in S_6. Hereby we have found the smallest index subgroup with non-trivial first cohomology group.

Index	H	H/H'
1	H_1	$\mathbb{Z}_{10}$
2	H_2	$\mathbb{Z}_5$
3	-	-
4	-	-
5	H_3	$\mathbb{Z}_2$
6	H_4	$\mathbb{Z}_{10}$
6	H_5	$\mathbb{Z}_{80}$
7	-	-
8	-	-
9	-	-
10	H_6	0
10	H_7	$\mathbb{Z} \oplus \mathbb{Z}_2$

Table 2.1: Subgroups of low index in $\mathcal{M}_2$

Using the same method, we can find a generating set together with a presentation for H_7:

$$H_7 = \langle b_1, a_2^{-2}, b_2, a_2 b_1 a_1 b_2 a_2 d^{-1} b_2^{-1} b_1^{-1} a_2^{-1} \rangle.$$

Since H_7 has a relatively small index and is generated by only four elements, the procedure of finding a presentation can even be done by hand. The simplest presentation we can construct for H_7 consists of 4 generators and 25 relations of total length 564. In the following, we write down this presentation explicitly and give the number of occurrences of each generator in the 25 relations.

Generators:

1) g_1 197 occurrences
2) g_2 81 occurrences
3) g_3 174 occurrences
4) g_4 112 occurrences.

Relations:

1) $g_3^{-1}g_4^{-1}g_3g_4$,

2) $g_4^{-1}g_2g_4g_2^{-1}$,

3) $g_2g_1g_2g_1^{-1}g_2^{-1}g_1^{-1}$,

4) $g_3^{-1}g_2g_3g_1g_3^{-1}g_2^{-1}g_3g_1^{-1}$,

5) $g_4g_1g_4^{-1}g_1g_4^{-1}g_1^{-1}g_4g_1^{-1}$,

6) $g_3^{-1}g_2g_3g_2g_3^{-1}g_2^{-1}g_3g_2^{-1}$,

7) $g_4g_1^{-1}g_3^{-1}g_4^{-1}g_3^{-1}g_1g_3g_1g_4^{-1}g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3$,

8) $g_3g_1g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3^{-1}g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}$,

9) $g_4^{-1}g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}g_4g_1g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}$,

10) $g_3g_4g_3g_1g_3g_1g_3^{-1}g_1^{-1}g_3^{-2}g_4^{-1}g_1g_4g_3g_1^{-1}g_4^{-1}g_3^{-1}g_1^{-1}$,

11) $g_3g_4g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}g_3^{-2}g_4^{-1}g_3^{-1}g_1^{-1}g_3g_4g_3g_1g_3g_1$,

12) $g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3g_1^{-1}g_3^{-1}g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-2}$
$g_4^{-1}g_1g_3g_4g_3g_1$,

13) $g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_2g_1^{-1}g_4g_1g_2^{-1}g_3g_1g_4g_1^{-1}$
$g_3^{-1}g_2g_1^{-1}g_4^{-1}g_1g_2^{-1}$,

14) $g_1^{-1}g_3g_1g_3g_1^2g_4g_1^{-2}g_3^{-1}g_1^{-1}g_3^{-1}g_1^2g_3$
$g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}$,

15) $g_2g_3^{-1}g_2^{-1}g_3g_1g_2g_4^{-1}g_1g_2g_3g_1g_4^{-1}g_3^{-1}g_1g_3$
$g_1g_2^{-1}g_4^{-1}g_1^{-1}g_4g_1^{-1}g_4^{-1}g_3^{-1}$,

16) $g_2g_1g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_2^{-1}g_4^{-1}g_3^{-1}g_2g_1g_3$
$g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}g_3^{-2}g_2^{-1}g_3$,

17) $g_2^{-1}g_1^{-1}g_4g_1^{-1}g_3^{-1}g_2g_4g_3g_2^{-1}g_3g_1g_4^{-1}g_2^{-1}$
$g_1g_2g_3^{-1}g_1^{-1}g_4g_3g_1^{-1}g_3^{-1}g_1^{-1}g_4$,

18) $g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_2g_1^{-1}g_4^{-1}g_3^{-2}g_1^{-1}g_4g_3^2g_1$
$g_2^{-1}g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_2$,

19) $g_1^{-1}g_4g_3^{-1}g_2g_3g_2^{-2}g_1^{-1}g_2^{-1}g_3g_4g_1g_4^{-1}g_1$
$g_2g_1^{-1}g_3^{-1}g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3g_2^{-1}g_4$,

20) $g_1^{-1}g_3g_4g_3g_2^{-1}g_1^{-1}g_2g_3^{-2}g_4^{-1}g_2g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-2}$
$g_2g_3^2g_1g_4g_1^{-1}g_3^{-1}g_2g_1^{-1}g_3^{-1}g_2^{-1}g_4^{-1}g_3^{-1}$,

21) $g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_4^{-1}g_2^{-1}g_3^{-1}g_1g_3g_1g_4^{-1}g_1^{-1}$
$g_3^{-1}g_2g_1g_2^{-1}g_3g_1g_4^{-1}g_1^{-1}g_4^{-1}g_3^{-1}g_1g_2g_3^{-1}$
$g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3g_4^2g_1^{-1}g_3^{-1}g_1^{-1}g_3g_4g_2^{-1}g_1^{-1}g_4$,

22) $g_2g_4g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_2^{-1}g_1^{-1}g_2^{-2}g_4g_1^{-1}$
$g_4g_2^{-1}g_3^{-1}g_2g_3^2g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_2^{-1}g_3^{-1}g_1^{-1}g_4g_2$
$g_1^{-1}g_3^{-2}g_4^{-1}g_1g_3^2g_4g_1g_2^{-1}g_4^{-1}g_1g_3$,

23) $g_3^{-1}g_4g_2g_3g_2^{-1}g_1^{-1}g_2^{-1}g_1^{-1}g_2^{-1}g_4g_1^{-1}g_3^{-1}g_1^{-1}$
$g_3g_2^{-1}g_1g_4g_1^{-1}g_2g_4g_3g_4g_1^{-1}g_3^{-1}g_4^{-1}g_3^{-1}g_2g_3^2g_4g_1$
$g_4^{-1}g_3^{-1}g_4^{-1}g_1g_4^{-1}g_1^{-1}g_2g_3^{-1}g_1g_3g_2^{-1}g_1g_4g_1^{-1}$,

24) $g_2g_4^{-1}g_1g_4g_1^{-1}g_3^{-1}g_1^{-1}g_3g_4^2g_1^{-1}g_3^{-1}g_1^{-1}g_3g_1g_4g_1^{-1}$
$g_3^{-1}g_2^{-1}g_1^{-1}g_4g_3g_1g_4g_1^{-1}g_3^{-1}g_2g_1^{-1}g_2^{-1}g_3g_1g_4^{-1}$
$g_1^{-1}g_3^{-1}g_1^{-1}g_3^{-1}g_2g_1g_3g_1g_4^{-1}g_1^{-1}g_3^{-1}g_2^{-1}g_4^{-1}g_1$,

25) $g_3g_1g_4g_1^{-1}g_3^{-1}g_1^{-2}g_4g_1^{-1}g_3^{-2}g_2g_3^2g_1g_4^{-1}g_1^2g_3g_1$
$g_4^{-1}g_1^{-1}g_3^{-2}g_2g_3^2g_1g_4^{-1}g_1^{-1}g_3^{-1}g_1^{-1}g_3^{-1}g_2^{-2}g_4g_1^{-1}$
$g_4g_1^{-1}g_3^{-1}g_2g_3g_2^{-1}g_1^{-1}g_4g_1^{-1}g_2g_3g_1g_3g_1g_4g_1^{-1}g_3^{-2}g_2^{-1}g_3$.

At the end we calculate the commutator factor group of H_7. The invariant divisors of the relation matrix are

$$1, 1, 2, 0,$$

meaning

$$H_1(H_7) = \mathbb{Z}_2 \oplus \mathbb{Z},$$

or

$$H^1(H_7) = \mathbb{Z}.$$

This proves Theorem 2.

Chapter 3

Constructing Examples For $g = 3$

The result for genus $g = 2$ is not very surprising, because of the exceptional status of $g = 2$. There are many properties that all surfaces of $g \geq 3$ share, but a surface of $g = 2$ does not (see [Jd1]). Therefore, the interesting examples would be for surfaces of higher genera.

The case $g = 3$ is not only much more difficult to handle but also quite different in nature. The first reason is the following theorem proved by *J. McCarthy* (see [Ma]).

Theorem 3 *Let $\Gamma \leq \mathcal{M}_g$ be a subgroup of finite index containing the Torelli subgroup $\mathcal{T}_g$, then $H^1(\Gamma) = 0$.*

The proof uses a result of *D. Johnson* in [Jd2] and the fact that the image of Γ in $Sp_{2g}(\mathbb{Z})$ contains some congruence subgroup. According to this theorem, if a cofinite subgroup of $\mathcal{M}_g$ with non-trivial first cohomology group exists, it has to be found among those that do not contain $\mathcal{T}_g$. The residual finiteness of $\mathcal{M}_g$ assures us the existence of subgroups not containing the Torelli subgroup. The main problems we encounter for the construction of these subgroups are the following three:

1. How to find cofinite subgroups of $\mathcal{M}_g$?

2. How to check whether they contain $\mathcal{T}_g$ or not?

3. How to calculate the first cohomology groups of these subgroups?

Problem 1: In order to construct a whole series of cofinite subgroups of $\mathcal{M}_g$ we have adopted a method that was originally introduced by *R. Gilman* (see [Gi]) to study the automorphism groups of free groups. We have modified this method and have applied it to $\mathcal{M}_g$ as follows:

Let Q be an arbitrary finite group, and G an arbitrary finitely presented group. Two epimorphisms ϕ_1 and ϕ_2 from G onto Q have the same kernel if and only if they differ by an automorphism of Q, i.e $\phi_1 = \psi\phi_2$, where $\psi \in Aut(Q)$. If we denote the set of all epimorphisms of G onto Q by $\mathcal{E}pim(G,Q)$, then we have the following bijection:

$$\mathcal{N}_Q = \{N \trianglelefteq G | G/N \simeq Q\} \longleftrightarrow \mathcal{E}pim(G,Q)/Aut(Q).$$

Now let $G = \pi_g$. The automorphism group of π_g acts on $\mathcal{N}_Q$ as a permutation group. Let k be the cardinality of $\mathcal{N}_Q$ and S_k the symmetric group on k elements. We obtain a homomorphism

$$\Phi_{g,Q} : Aut(\pi_g) \to S_k.$$

The inner automorphisms act trivially on $\mathcal{N}_Q$, hence $\Phi_{g,Q}$ factors through the outer automorphism group of π_g. After restriction to $\mathcal{M}_g$, we obtain a homomorphism

$$\Phi_{g,Q} : \mathcal{M}_g \to S_k.$$

$\Sigma_{g,Q}$, the kernel of $\Phi_{g,Q}$, will be a cofinite normal subgroup of $\mathcal{M}_g$. In this way we obtain a lot of different normal subgroups of $\mathcal{M}_g$ for different choices of Q.

Remark 1 *Let $N \in \mathcal{N}_Q$, then $H := \bigcap_{\phi \in Aut(\pi_g)} \phi(N)$ will be a cofinite characteristic subgroup of π_g, and $\Lambda_{g,H} \leq \Sigma_{g,Q}$, i.e each of the subgroups $\Sigma_{g,Q}$ contains at least one congruence subgroup.*

Problem 2: Based on Johnsons' results on the Torelli subgroup [Jd1] we will give a method to determine whether $\mathcal{T}_g \leq \Sigma_{g,Q}$ or not. This will be discussed later, when we construct some subgroups (see section 3.1).

Problem 3: This is the most difficult part of the calculation to deal with. In order to determine $H^1(U)$, a presentation of U has to be given. The algorithms we use to determine a presentation for a subgroup of a finitely presented group are based on the following classical theorem from combinatorial group theory:

Theorem 4 (Schreier-Reidemeister) *Let G be a finitely presented group with n generators and m relations. In addition, let $H \leq G$ be a subgroup of finite index with $[G : H] = p$. Then H is also finitely presented, and there is an algorithm to construct a presentation for H on $pn - p + 1$ generators and at most mp relations.*

There are generally two different methods to construct such a presentation: *Schreier-Reidemeister* and *Todd-Coxeter* (see [Jl1] and [Jl2]). A modified version of both algorithms has been implemented on GAP. As we notice from the theorem, the complexity of the presentation increases with the index p. In addition, the length of the defining relations for H depends on p as well and can become eventually very large. (See the presentation that was constructed for $\mathcal{T}_2(2)$ in the previous chapter. For instance, the first relation is a word of length 4, while the length of the last relation is 50). Therefore, constructing presentations for subgroups of huge indices can become an infeasible task (for instance $p = 1000$ is already huge in the case of $\mathcal{M}_2(2)$).

In the following sections we consider some examples for different choices of Q. At first we show that abelian Q's are of no help for our purposes.

Example 1 $Q = \mathbb{Z}_2$.

$Aut(\mathbb{Z}_2)$ is trivial, hence there are as many subgroups of index 2 in $\pi_3 = \pi_1(\mathcal{S}_3)$ as different epimorphisms from π_3 onto $\mathbb{Z}_2$. Therefore there are exactly $2^6 - 1 = 63$ subgroups of index 2 in π_3. Consequently we obtain the following homomorphism:

$$\Phi_{2,\mathbb{Z}_2} : \mathcal{M}_3 \to S_{63}.$$

The action of $\mathcal{M}_3$ on these subgroups coincides with the symplectic action of $\mathcal{M}_3$ on the subspaces of codimension 1 in $(\mathbb{Z}_2)^6 \simeq H_1(\mathcal{S}_3, \mathbb{Z}_2)$. Therefore $\Phi_{2,\mathbb{Z}_2}$ factors through $Sp_6(\mathbb{Z}_2)$ and gives us a faithful permutation representation of $Sp_6(\mathbb{Z}_2)$,

which is also transitive (see [Hup] page 221). This implies that $\Sigma_{3,\mathbb{Z}_2} = \mathcal{T}_3(2)$ is a normal subgroup of index $|Sp_6(\mathbb{Z}_2)|$=1451520 that contains $\mathcal{T}_g$, and consequently $H^1(\Sigma_{3,\mathbb{Z}_2}) = 0$.

Theorem 5 *For every Abelian Q, $H^1(\Sigma_{g,Q}) = 0$.*

Proof: $\mathcal{T}_g$ acts trivially on $H_1(\mathcal{S}_g, \mathbb{Z})$. Since Q is Abelian, every $N \trianglelefteq \pi_g$ with $\pi_g/N \simeq Q$ contains π'_g. As a result $\mathcal{T}_g$ acts trivially on $\{N \ |\pi_g/N \simeq Q\}$, i.e $\mathcal{T}_g \leq \Sigma_{g,Q}$.

q.e.d.

In the following section, Q will be always a non-abelian group, and we will describe the procedure to construct the homomorphism $\Phi_{g,Q}$ and its kernel $\Sigma_{g,Q}$.

3.1 The Procedure to Construct $\Phi_{g,Q}$

We proceed as follows:

1. We need to know $Aut(Q)$. In general there is no algorithm to calculate the automorphism group of a finite group, unless the group belongs to a certain category such as P-groups or more generally nilpotent groups (see [Sc]). But for small Q this might be done by hand.

2. To find the set $\mathcal{E}pim(\pi_g, Q)$, we consider all tuples $q_1, q_2, .., q_{2g} \in Q$ generating Q and satisfying the only defining relation of π_g:

$$r_0 = [q_1, q_2][q_3, q_4]...[q_{2g-1}, q_{2g}] = 1.$$

   $Aut(Q)$ acts on the set of all these tuples. Let us choose an orbit representative for this action. This yields us the set $\mathcal{E}pim(\pi_g, Q)/Aut(Q)$.

3. For every $\phi \in \mathcal{E}pim(\pi_g, Q)$, we determine $N_\phi = Kernel(\phi)$ as the normal closure of a finite set of elements of π_g in the following manner:
   We construct a presentation for Q on generators $q_1, q_2, .., q_{2g}$. Besides r_0, the only relation of π_g, this presentation of Q will satisfy some more relations such as $r_1, r_2, .., r_k$. If we rewrite $r_1, r_2, .., r_k$ as words in generators of π_g, we will obtain a normally generating set for N.

4. $\mathcal{M}_g \simeq Out^+(\pi_g)$ acts on the conjugacy classes of π_g. Via a calculation carried out by hand we determine the action of $D_{\alpha_1}, D_{\beta_1}, .., D_{\alpha_g}, D_{\beta_g}, D_\delta$ (see figure 3.2) on the conjugacy classes of the generators of π_g (see figure 3.1). We choose a fixed presentation of π_g as follows:

$$\pi_g = \langle\ a_1, b_1, ..a_g, b_g \ \mid\ [a_1, b_1][a_2, b_2]..[a_g, b_g] = 1\rangle.$$

In figure 3.1 we have drawn $2g$ loops, with the appropriate orientation, whose isotopy classes represent the $2g$ generators of π_g satisfying the only relation of π_g. Hereby $[a, b] = aba^{-1}b^{-1}$ is the commutator of a and b, and we multiply curves from left to right, i.e. the curve ab is obtained by traversing first curve a then curve b.

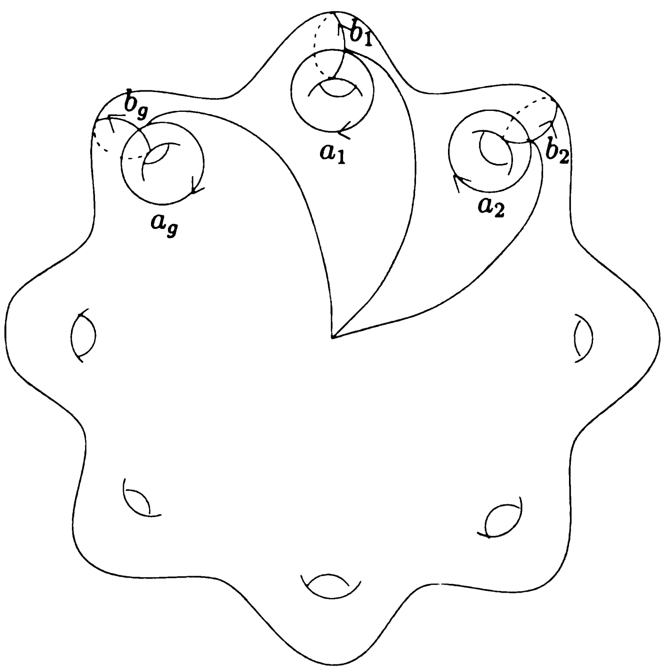

Figure 3.1: The generators of π_g

In addition to $D_{\alpha_1}, D_{\beta_1}, .., D_{\alpha_g}, D_{\beta_g}, D_\delta$ we need also to know the action of D_η (see figure 3.2). The reason is the following fact established by D. Johnson [Jd1]. The Torelli subgroup $\mathcal{T}_g$ is generated by all the elements of the form $D_\delta D_\eta^{-1}$, where δ and η represent a bounding pair, i.e. a pair of disjoint simple closed curves, representing the same non-trivial $\mathbb{Z}$-homology class. Johnson further defines the *genus* of a bounding pair to be the smaller of the genera of the two pieces of the surface cut by the two curves, and proves that $\mathcal{T}_g$ is normally generated by any genus 1 bounding pair. Our choice in figure 3.2 is of genus 1. This result will be used to settle problem 2. That is, in order to determine if a certain normal subgroup of $\mathcal{M}_g$ such as $\Sigma_{g,Q}$ contains $\mathcal{T}_g$ or not, all we have to do is to determine if $\Phi_{g,Q}(D_\delta D_\eta^{-1})$ is the trivial permutation or not, a fact that can easily be checked.

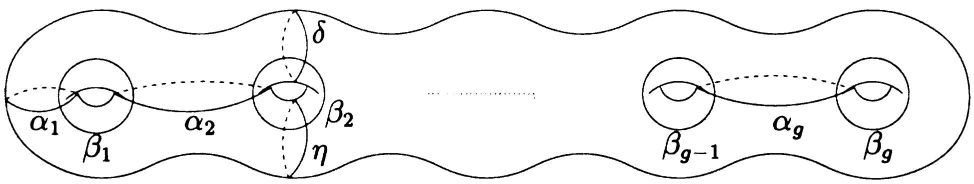

Figure 3.2: The bounding pair generating $\mathcal{T}_g$

In the following we list the action of the Dehn twists on the generators of π_g. Every Dehn twist changes only some of the generators, so we will not list those generators that remain invariant. We will need some new elements of π_g, defined as follows:

$$\alpha_1 = b_1^{-1}, \quad \alpha_i = a_i b_i^{-1} a_i^{-1} b_{i-1}, \qquad \text{for } i = 2, .., g$$

$$\beta_i = a_i, \qquad \text{for } i = 1, ..g$$

$$\delta = b_2, \quad \eta = \alpha_3^{-1} b_3^{-1}.$$

These elements of π_g, namely $(\alpha_i, \beta_i, \delta, \eta)$, are denoted by the same letters as the simple closed curves used in the presentation of $\mathcal{M}_g$ because they all happen to be simple closed curves representing the same isotopy classes. In figure 3.3 we have represented these new elements. For the right Dehn twists we obtain:

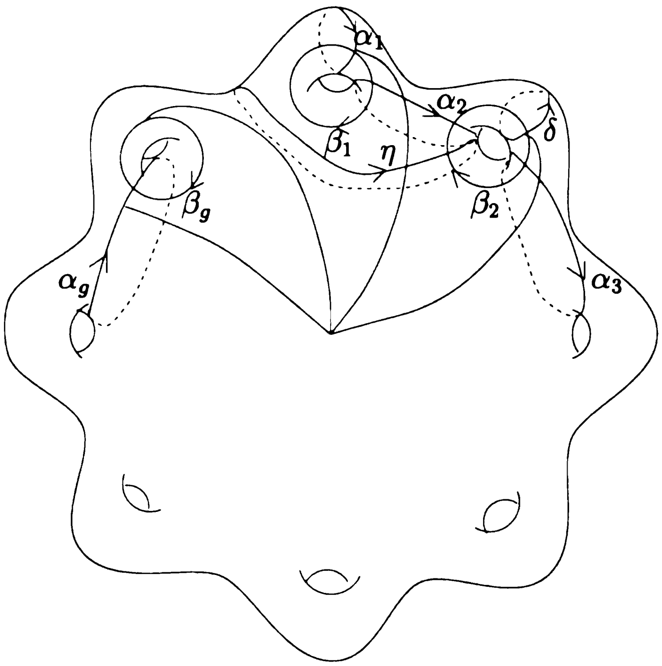

Figure 3.3: The action of Dehn twists

$$
\begin{array}{llll}
D_{\alpha_1}: & a_1 & \mapsto & a_1\alpha_1 \\
 & & & \\
D_{\alpha_i}: & a_{i-1} & \mapsto & a_{i-1}\alpha_i^{-1} \\
 & b_{i-1} & \mapsto & \alpha_i b_{i-1}\alpha_i^{-1} \qquad\qquad i=2,..,g \\
 & a_i & \mapsto & \alpha_i a_i \\
 & & & \\
D_{\beta_i}: & b_i & \mapsto & b_i\beta_i \qquad\qquad i=1,..,g \\
 & & & \\
D_\delta: & a_2 & \mapsto & a_2\delta^{-1} \\
 & & & \\
D_\eta: & a_2 & \mapsto & a_2\eta \\
 & b_2 & \mapsto & \eta^{-1}b_2\eta \\
 & a_3 & \mapsto & \eta^{-1}a_3\eta \\
 & b_3 & \mapsto & \eta^{-1}b_3\eta.
\end{array}
$$

We will also need the action of the left Dehn twists:

$$\begin{array}{llll}
D_{\alpha_1}^{-1}: & a_1 & \mapsto & a_1\alpha_1^{-1}
\end{array}$$

$$\begin{array}{lllll}
D_{\alpha_i}^{-1}: & a_{i-1} & \mapsto & a_{i-1}\alpha_i & \\
 & b_{i-1} & \mapsto & \alpha_i^{-1}b_{i-1}\alpha_i & i=2,..,g \\
 & a_i & \mapsto & \alpha_i^{-1}a_i &
\end{array}$$

$$\begin{array}{lllll}
D_{\beta_i}^{-1}: & b_i & \mapsto & b_i\beta_i^{-1} & i=1,..,g
\end{array}$$

$$\begin{array}{llll}
D_{\delta}^{-1}: & a_2 & \mapsto & a_2\delta
\end{array}$$

$$\begin{array}{llll}
D_{\eta}^{-1}: & a_2 & \mapsto & a_2\eta^{-1} \\
 & b_2 & \mapsto & \eta b_2\eta^{-1} \\
 & a_3 & \mapsto & \eta a_3\eta^{-1} \\
 & b_3 & \mapsto & \eta b_3\eta^{-1}.
\end{array}$$

5. Let $w = \omega(a_1, b_1, .., a_g, b_g)$ be some word in the generators of π_g, and let D_α be some Dehn twist. Then we define

$$D_\alpha(w) = \omega(D_\alpha(a_1), D_\alpha(b_1), .., D_\alpha(a_g), D_\alpha(b_g)).$$

Now if N is the normal closure of $n_1, n_2, .., n_l$, where n_i are some words in $a_1, b_1, .., a_g, b_g$, then $D_\alpha(N)$ will be a normal subgroup normally generated by $D_\alpha(n_1), .., D_\alpha(n_l)$. Now let ϕ be an element of $\mathcal{E}pim(\pi_g, Q)$ and $N = N_\phi$ be the corresponding kernel of ϕ. If we apply D_α to N, we will obtain another $N_{\phi'}$ for some other $\phi' \in \mathcal{E}pim(\pi_g, Q)$. Since we know all ϕ's together with their kernels, we can find the appropriate normal subgroup that N gets mapped to under the action of D_α. In this way we construct the homomorphism

$$\Phi_{g,Q}: \mathcal{M}_g \to S_k.$$

In the following example we choose Q to be the smallest non-abelian finite group S_3. First of all we need some facts from classical group theory.

Definition 1 *Let G be a group and H a permutation group on a finite set Ω with $|\Omega| = n$. The wreath product $G \wr H$ of G with H is defined to be the set*

$$G \wr H = \{(g_1, g_2, .., g_n, h) \mid g_i \in G, h \in H\}$$

with the group multiplication

$$(g_1, g_2, .., g_n, h)(g_1', g_2', .., g_n', h') = (g_1 g'_{h(1)}, g_2 g'_{h(2)}, .., g_n g'_{h(n)}, hh').$$

Theorem 6 *Let G and H be defined as in definition 1. Then $G \wr H$ has a normal subgroup isomorphic to the direct product of n copies of G and is isomophic to a semidirect product of this normal subgroup by H. Hence the order of $G \wr H$ will be*

$$|G \wr H| = |G|^n |H|.$$

See [Hup] page 95 for a proof.

Definition 2 *A permutation group G on the set Ω is imprimitive if there exists a proper subset $\Delta \subset \Omega$ such that for all $g \in G$ either $g\Delta = \Delta$ or $g\Delta \cap \Delta = \emptyset$. Δ is called an imprimitivity region of G. A transitive permutation group that is not imprimitive is called primitive.*

Theorem 7 *Let G be an imprimitive permutation group on Ω with imprimitivity region Δ, and let $H = \{g \in G \mid g\Delta = \Delta\}$ be the stabilizer of Δ. Let further R be a coset representative of G/H. Then*

a) $\Omega = \bigcup_{r \in R} \ r\Delta$.

b) If $|\Omega| = n$ is finite, then $|\Omega| = |\Delta| \ [G : H]$. Let $|\Delta| = k$ and $[G : H] = p$, then G is isomorphic to a subgroup of the wreath product of $S_k \wr S_p$.

c) H acts transitively on Δ.

The proof can be found again in [Hup] page 146.

Theorem 8 *Let $G = \langle g_1, g_2, .., g_m \mid \ r_1, r_2, ..r_k\rangle$ be a finitely presented group and*

$$\phi : G \to S_n$$

be a transitive permutation representation of G on n elements. Let us denote the image of ϕ by H. Then we can construct a presentation for H on $h_1 = \phi(g_1), .., h_m = \phi(g_n)$ (see section 3.4),

$$H = \langle h_1, h_2, ..h_m \mid r'_1, r'_2, .., r'_{k'} \rangle,$$

and N, the kernel of ϕ, is normally generated by $r'_1, r'_2, .., r'_{k'}$. (Note that $r'_1, .., r'_{k'}$ are the same words used in the presentation of H but written in the generators of G). Let further $\{s_1, s_2, ..s_t\}$ be a set of representatives of the preimages of all elements of H under ϕ. Then a generating set for N as a subgroup will be

$$N = \langle s_i^{-1} r'_j s_i \mid i = 1, .., t, \quad j = 1, .., k' \rangle.$$

If U is a subgroup of H (such as the stabilizer of one point), generated by $u_1, .., u_l$, and $w_1, .., w_l$ are their preimage representatives in G, then $\phi^{-1}(U)$ is generated by

$$\phi^{-1}(U) = \langle w_1, .., w_l, s_i^{-1} r'_j s_i \mid i = 1, .., t, \quad j = 1, ..k \rangle.$$

Proof: The proof is trivial.

Now we are ready to look at some examples.

Example 2 $Q = S_3$

$Aut(S_3)$ is isomorphic to S_3. The set $\mathcal{E}pim(\pi_g, Q)/Aut(S_3)$ contains exactly 2520 elements, i.e there are exactly 2520 normal subgroups $N \trianglelefteq \pi_1(\mathcal{S}_3)$ with $\pi_1(\mathcal{S}_3)/N \simeq S_3$. As a result we obtain the homomorphism

$$\Phi_{3,S_3} : \mathcal{M}_3 \rightarrow S_{2520}.$$

S_3 is solvable with $S'_3 \simeq \mathbb{Z}_3$. As mentioned earlier in example 1 there are exactly 63 subgroups $M_1, M_2, .., M_{63}$ in $\pi_1(\mathcal{S}_3)$ with $\mathbb{Z}_2$-quotient. Each of these normal subgroups M_i turns out to contain $N_{i,1}, N_{i,2}, .., N_{i,40}$ normal subgroups with $\mathbb{Z}_3$ as a quotient such that each $N_{i,j}$ is also normal in $\pi_1(\mathcal{S}_3)$ and $\pi_1(\mathcal{S}_3)/N_{i,j} \simeq S_3$. Let's denote the set of these subgroups by $\mathcal{N}_{S_3} = \{N_{i,j}\}$. In this way we obtain a partitioning of these 2520 normal subgroups into 63 blocks, $\Delta_1, .., \Delta_{63}$, each containing 40 normal

subgroups. The action of $\mathcal{M}_3$ on $\mathcal{N}_{S_3}$ turns out to be transitive but not primitive. The imprimitivity regions are exactly the blocks $\Delta_1, .., \Delta_{63}$. The permutation of the $\Delta_i's$ is determined by the action of $\mathcal{M}_3$ on the normal subgroups $M_1, M_2, .., M_{63}$, which gives us a faithful transitive permutation representation of $SP_6(\mathbb{Z}_2)$ on 63 elements as we saw in example 1. The stabilizer of one of the $\Delta_i's$, such as Δ_1 for instance, under the action of $\mathcal{M}_3$ acts transitively on Δ_1 (see theorem 7), therefore the restriction of this action on Δ_1 will be a transitive permutation representation on 40 elements. Let us denote the image of this representation by U. We can also compute a set of generators for U. The following six permutations generate U:

$$(5,23,32)(6,24,34)(7,25,33)(8,26,38)(9,27,40)(10,28,39)$$

$$(11,29,35)(12,30,37)(13,31,36),$$

$$(14,32,23)(15,33,24)(16,34,25)(17,35,26)(18,36,27)(19,37,28)(20,38,29)$$

$$(21,39,30)(22,40,31),$$

$$(2,3,4)(8,9,10)(11,13,12)(17,18,19)(20,22,21)(26,27,28)(29,31,30)$$

$$(35,36,37)(38,40,39),$$

$$(1,2,4)(6,11,10)(7,8,12)(15,20,19)(16,17,21)(24,29,28)(25,26,30)$$

$$(33,38,37)(34,35,39),$$

$$(2,3,4)(8,9,10)(11,13,12)(17,18,19)(20,22,21)(26,27,28)(29,31,30)$$

$$(35,36,37)(38,40,39),$$

$$(2,22,19)(3,21,17)(4,20,18)(5,25,34)(6,23,33)(7,24,32)(11,37,31)$$

$$(12,36,29)(13,35,30).$$

U turns out to be a simple group of order 25920. The only simple group of this order is $PSp_4(\mathbb{Z}_3)$, (see [AFG]). Thus we proved the following:

$$Im(\Phi_{3,S_3}) \simeq PSp_4(\mathbb{Z}_3) \wr Sp_6(\mathbb{Z}_2),$$

and the order of this image is

$$(25920)^{63} \times 1451520 \approx 10^{284}.$$

It's needless to mention that computing a presentation for $\Sigma = \Sigma_{3,S_3}$ is a hopeless task, but we are not actually interested in computing this presentation, but rather knowing if $H^1(\Sigma)$ is finite or not. We are going to look at $H = \Phi_{3,S_3}^{-1}(Stab_{\Phi_{3,S_3}(\mathcal{M}_3)}(N_{1,1}))$. As mentioned above the image of $\mathcal{M}_3$ under Φ_{3,S_3} turns out to be transitive, so the stabilizer of one of the $\{N_{i,j}\}$, such as $N_{1,1}$, will be a subgroup of index 2520, and its preimage H will be a subgroup of the same index in $\mathcal{M}_3$. What seems to be realistic to handle is computing a presentation for H. On the other hand, $\Sigma_{3,S_3} = Core_{\mathcal{M}_3}(H) = \bigcap_{g\in\mathcal{M}_3} g^{-1}Hg$, meaning that $\mathcal{T}_3 \leq \Sigma_{3,S_3}$ if and only if $\mathcal{T}_3 \leq H$.

3.2 Computing a Presentation for H

Even 2520 turns out to be a huge index, and we (our computer) won't be able to compute a presentation for H. Therefore, we will break down the calculations in two steps. At first we look at $K = \Phi_{3,\mathbb{Z}_2}^{-1}(Stab_{\Phi_{3,\mathbb{Z}_2}(\mathcal{M}_3)}(M_1))$, which will be of index 63 in $\mathcal{M}_3$. Since K fixes M_1, it acts on $\{N_{1,1},..,N_{1,40}\}$. Therefore Φ_{3,S_3} can be restricted to K and we will get a homomorphism (that we denote by the same letter)

$$\Phi_{3,S_3} : K \rightarrow S_{40}.$$

In the previous example we denoted the image of K by U. Now we look at $H = \Phi_{3,S_3}^{-1}(Stab_{\Phi_{3,S_3}(K)}(N_{1,1}))$, which will be a subgroup of index 40 in K, and a subgroup of index $40 \times 63 = 2520$ in $\mathcal{M}_3$. Using Theorem 8 we can find a generating set for K. Then we construct a presentation for K on this generating set (see section 3.4). The advantage of this specific representation of K is that every element of K (written as a word in its generators) can be directly rewritten as a word in the generators of $\mathcal{M}_3$. This means that every subgroup of K can be directly realized as a subgroup of $\mathcal{M}_3$, and any homomorphism from $\mathcal{M}_3$ onto any permutation group can be restricted on K and easily evaluated. We can also find a generating set (using again Theorem 8) for H as a subgroup of K together with a presentation. Using the presentation we can evaluate its first homology group, and taking advantage of the special presentation of K we can realize H as a subgroup of $\mathcal{M}_3$. The most time consuming part of the calculations involves the computation of a presentation for H. Here it would be reasonable to check at first if H contains the Torelli subgroup or not. The smallest generating set we can find for K consists of the following 8 elements:

$$a_1, b_1, a_2^{-2}, b_2, a_3, b_3, d^{-2}, a_2 b_2 d b_2^{-1} a_2^{-1}.$$

Then we calculate a presentation for K, and apply the Tietze transformations to simplify it. The simplest presentation we get has 8 generators and 242 relations of total length 66790. H is generated by the following nine elements:

$$a_1, b_1, a_2^{-2}, b_2, b_3, d^{-2}, a_2 b_2 d b_2^{-1} a_2^{-1},$$

$$a_3 b_2^{-1} a_3, a_3 b_3 b_2^{-1} a_3^{-1},$$

and turns out not to contain the Torelli subgroup. H has a presentation on 94 generators and 9401 relations of total length 2120026. At the end we determine the commutator factor group of H. The non-trivial invariant divisors are

$$2, 6, 12.$$

Hence

$$H_1(H) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}.$$

Although H is a subgroup that does not contain the Torelli subgroup, its first cohomology group turns out to be zero.

The next attempt will be undertaken using the quaternion group Q_8 instead of S_3.

Example 3 $Q = Q_8$

Q_8 is a solvable group with $Q_8' \simeq \mathbb{Z}_2$ and $Q_8/Q_8' \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have exactly $\frac{(2^6-1)(2^6-2)}{(2^2-1)(2^2-2)} = 651$ subspaces of codimension two in $(\mathbb{Z}_2)^6$, meaning there are exactly 651 normal subgroups $M_1, .., M_{651}$ in $\pi_1(S_3)$ with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as quotient. The action of

$\mathcal{M}_3$ on these subspaces won't be transitive. There are exactly two types of subspaces of codimension two, and the action of $\mathcal{M}_3$ will be transitive on each of these families. One of them is called *isotropic*, on which the symplectic form vanishes, and the other one is called *hyperbolic*, on which the symplectic form is non-degenerate (see [Hup]). In our situation we will be only interested in the hyperbolic ones and there will be 315 of those. Each of these hyperbolic subgroups M_i's contains exactly 16 normal subgroups $\{N_{i,1},..,N_{i,16}\}$ such that their quotient in $\pi_1(S_3)$ is isomorphic to Q_8. In this way we obtain $315 \times 16 = 5040$ normal subgroups of quotient Q_8. The homomorphism

$$\Phi_{3,Q_8} : \mathcal{M}_3 \to S_{5040}$$

will have an image isomorphic to $U \wr L$, where L is a transitive permutation representation of $Sp_6(\mathbb{Z}_2)$ on 315 elements and U is a solvable permutation group of order 9216 on 16 elements. The order of this image will be

$$(9216)^{315} \times 1451520 \approx 10^{1255}.$$

Again we look at $K = \Phi^{-1}_{3,\mathbb{Z}_2\oplus\mathbb{Z}_2}(Stab_{\Phi_{3,\mathbb{Z}_2\oplus\mathbb{Z}_2}(\mathcal{M}_3)}(M_1))$ and at its subgroup $H = \Phi^{-1}_{3,Q_8}(Stab_{\Phi_{3,Q_8}(K)}(N_{1,1}))$. K is a subgroup of index 315 containing the Torelli subgroup, and H is a subgroup of index 5040 that turns out not to contain the latter group. K is generated by the following 10 elements :

$$a_1, b_1, a_2^{-2}, b_2, a_3^{-2}, b_3, d^{-2}, a_3b_3b_2^{-1}a_3^{-1},$$

$$a_2b_2db_2^{-1}a_2^{-1},$$

$$a_2b_1a_1b_2a_2b_1a_3b_2da_2^{-1}b_2^{-1}a_3^{-1}b_1^{-1}a_2^{-1}b_2^{-1}a_1^{-1}b_1^{-1}a_2^{-1},$$

and has a presentation with 10 generators and 1140 relations of total length 285918. H is generated by the following 13 elements:

$$a_1, b_1, a_2^4, b_2, a_3^4, b_3, d^4, a_3b_3b_2^{-1}a_3^{-1},$$

$$a_2b_2db_2^{-1}a_2^{-1}, a_2^{-2}b_2a_2^2, a_2^{-2}a_3b_3b_2^{-1}a_3a_2^2,$$

$$a_2b_1a_1b_2a_2b_1a_3b_2da_2^{-1}b_2^{-1}a_3^{-1}b_1^{-1}a_2^{-1}b_2^{-1}a_1^{-1}b_1^{-1}a_2^{-1}, a_2^{-2}b_1^{-2}a_2^2,$$

and has a presentation with 86 generators and 18105 relations of total length 3935640. At the end we look at the commutator factor group of H. The non-trivial invariant divisors are:

$$2, 4, 8.$$

Hence

$$H_1(H) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8.$$

Therefore $H^1(H) = 0$.
In the following we will list the result of our calculations for all the other choices for Q that we have been able to handle. Although in each case the calculations are slightly different and need to be treated separately, we will not get into the details and will just state the results.

Example 4 $Q = D_8$

There are 15120 normal subgroups of quotient D_8 (the dihedral group of order 8) in $\mathcal{M}_3$. The action of $\mathcal{M}_3$ on these 15120 subgroups is not transitive and has 2 orbits of equal length of 7560 elements. The stabilizer of one of the elements in the first orbit is a subgroup H_1 of index 7560 in $\mathcal{M}_3$ generated by 10 elements. A presentation on 126 generators and 27169 relations of total length 6069283 can be constructed for H_1. The non-trivial invariant divisors of H_1/H_1' are:

$$2, 2, 2, 4, 4, 4.$$

Hence

$$H^1(H_1) = 0.$$

As a matter of fact H_1 contains the Torelli subgroup. The stabilizer of one of the elements in the second orbit is a subgroup H_2 of index 7560 in $\mathcal{M}_3$ generated by 10 elements. A presentation on 126 generators and 27168 relations of total length 6060990 can be constructed for H_2. The non-trivial invariant divisors of H_2/H_2' are:

$$2, 2, 4, 8.$$

Hence

$$H^1(H_2) = 0.$$

It turns out that H_2 also contains the Torelli subgroup.

Example 5 $Q = D_{10}$

There are 9828 normal subgroups of quotient D_{10} (the dihedral group of order 10) in $\mathcal{M}_3$ falling into $\frac{9828}{63} = 156$ blocks. The action of $\mathcal{M}_3$ on these 9828 subgroups is an imprimitive transitive group. The stabilizer of one of its element is a subgroup H of index 9828 in $\mathcal{M}_3$ generated by 11 elements. A presentation on 259 generators and 34795 relations of total length 5206667 can be constructed for H. The non-trivial invariant divisors of H/H' are:

$$2, 4, 4.$$

Hence

$$H^1(H) = 0.$$

Here again H contains the Torelli subgroup.

Example 6 $Q = D_{12}$

There are 78120 normal subgroups of quotient D_{12} (the dihedral group of order 12) in $\mathcal{M}_3$. The action of $\mathcal{M}_3$ on these subgroups is not transitive and falls into 4 orbits. We were not able to calculate a generating set much less a presentation for the stabilizer of any of these subgroups.

Example 7 $Q = D_{14}$

There are 25200 subgroups of quotient D_{14} (the dihedral group of order 14) in $\mathcal{M}_3$ falling into $\frac{25200}{63} = 400$ blocks. The action of $\mathcal{M}_3$ on theses subgroups is an imprimitive transitive group. The stabilizer of one of its element is a subgroup H of index 25200 in $\mathcal{M}_3$ generated by 11 elements. A presentation on 701 generators and 89308 relations of total length 12995940 can be constructed for H. However, we were not able to calculate the invariant divisors of H/H'. But we could establish the fact that H contains the Torelli subgroup, therefore

$$H^1(H) = 0.$$

Example 8 $Q = A_4$

For A_4, the alternating group of order 12, we were not even able to calculate a full list of normal subgroups in $\mathcal{M}_3$ having an A_4-quotient.

Example 9 $Q = T_{12}$

Besides D_{12} and A_4 there is another non-abelian group of order 12 that we name T_{12}. In the following we will give a presentation:

$$T_{12} = \langle t_1, t_2 \mid t_1^3, t_2^4, t_1 t_2^{-1} t_1 t_2 \rangle$$

and a permutation representation for T_{12}:

$$T_{12} = \langle (1,2,4)(3,6,5)(7,8,9)(10,12,11), (1,3,7,10)(2,5,8,11)(4,6,9,12) \rangle$$

There are 80640 subgroups of quotient T_{12} in $\mathcal{M}_3$ falling into $\frac{80640}{63} = 1280$ blocks. The action of $\mathcal{M}_3$ on these group is an imprimitive transitive group. The stabilizer of one of its element is a subgroup H of index 80640 in $\mathcal{M}_3$ generated by 24 elements. We were able to construct a presentation on 2336 generators and 287680 relations of total length 40604058 for H. Although we were not able to calculate the invariant divisors, we could establish that H contains the Torelli subgroup, therefore

$$H^1(H) = 0.$$

The choices for Q, considered in the examples above, were the only cases we were able to handle for $g = 3$. However, for $g = 2$ we were able to construct more subgroups.

3.3 More Examples for $g = 2$

In the following we will list some more subgroups we have been able to construct for $\mathcal{M}_2$ using the method we described in this chapter.

Example 10 $Q = S_3$

There are 60 normal subgroups of quotient S_3 in $\mathcal{M}_2$. The action of $\mathcal{M}_2$ on these 60 subgroups is a transitive permutation group. The stabilizer of one of these subgroups is a subgroup H of index 60 in $\mathcal{M}_2$ generated by 7 elements. A presentation on 11 generators and 165 relations of total length 3496 can be constructed for H. The non-trivial invariant divisors of H/H' are:

$$2, 2, 0, 0.$$

Hence

$$H^1(H) = \mathbb{Z} \oplus \mathbb{Z}.$$

Example 11 $Q = D_8$

There are 180 normal subgroups of quotient D_8 in $\mathcal{M}_2$. The action of $\mathcal{M}_2$ on these 180 subgroups has 2 orbits of equal length 90. The stabilizer of one of these subgroups in the first orbit is a subgroup H_1 of index 90 in $\mathcal{M}_2$ generated by 8 elements. A presentation on 35 generators and 391 relations of total length 6675 can be constructed for H_1. The non-trivial invariant divisors of H_1/H_1' are:

$$2, 2, 2, 0, 0, 0.$$

Hence

$$H^1(H_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

The stabilizer of one of the subgroups in the second orbit is a subgroup H_2 of index 90 in $\mathcal{M}_2$ generated by 7 elements. A presentation on 36 generators and 393 relations of total length 6682 can be constructed for H_2. The non-trivial invariant divisors of H_2/H_2' are:

$$2, 2, 8, 0, 0.$$

Hence

$$H^1(H_2) = \mathbb{Z} \oplus \mathbb{Z}.$$

Example 12 $Q = Q_8$

There are 60 normal subgroups of quotient Q_8 in $\mathcal{M}_2$. The action of $\mathcal{M}_2$ on these 60 subgroups is a transitive permutation group. The stabilizer of one of these subgroups is a subgroup H of index 60 in $\mathcal{M}_2$ generated by 11 elements. A presentation on 24 generators and 263 relations of total length 4562 can be constructed for H. The non-trivial invariant divisors of H/H' are:

$$2, 2, 4, 0.$$

Hence

$$H^1(H) = \mathbb{Z}.$$

Example 13 $Q = D_{10}$

There are 90 normal subgroups of quotient D_{10} in $\mathcal{M}_2$. The action of $\mathcal{M}_2$ on these 90 subgroups is a transitive permutation group. The stabilizer of one of these subgroups is a subgroup H of index 90 in $\mathcal{M}_2$ generated by 7 elements. A presentation on 15 generators and 245 relations of total length 5063 can be constructed for H. The non-trivial invariant divisors of H/H' are:

$$2, 2, 4, 0, 0.$$

Hence

$$H^1(H) = \mathbb{Z} \oplus \mathbb{Z}.$$

Example 14 $Q = T_{12}$

There are 480 normal subgroups of quotient T_{12} in $\mathcal{M}_2$. The action of $\mathcal{M}_2$ on these 480 subgroups is a transitive permutation group. The stabilizer of one of the these subgroups H is a subgroup of index 480 in $\mathcal{M}_2$ generated by 11 elements. A presentation on 68 generators and 1370 relations of total length 23395 can be constructed for H. The non-trivial invariant divisors of H/H' are:

$$2, 2, 2, 0, 0, 0.$$

Hence

$$H^1(H) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

3.4 Final Remarks

As mentioned in the introduction, almost all of the calculations have been carried out using the programming language GAP. All of the programs have been written by the author, except for the following two cases, where some new programs, that at the time of writing this paper were not yet implemented in GAP's library, were needed in:

1. calculating a presentation for a finite permutation group (see Theorem 8),
2. producing a presentation on a given set of generators (see section 3.2).

The author's special thanks goes to *T. Brauer* from *RWTH-Aachen*, not only for providing the programs in advance, but also for his valuable help and advice.

The first machine we have used for the purpose of our calculation has been an *Ultrasparc 1 Model 170* with 170 MGHz cpu and 128 MG RAM. The calculations for the groups S_3, Q_8 and D_8 have been carried out using this machine. The three major parts of the calculations involve

1. calculating $\mathcal{E}pim(\pi_g, Q)$ (see section 3.1),
2. evaluating $\Phi_{g,Q}$ (see section 3.1),
3. producing the presentations for H and K (see section 3.2).

Part 3 is the most time consuming. The first two parts together take only about 20 percent of the total computing time. In the following we list the time used for the first three examples discussed in the paper:

1. $Q = S_3$, 3142319 milliseconds ($\approx$ 1 hour),
2. $Q = Q_8$, 17031669 milliseconds ($\approx$ 5 hours),
3. $Q = D_8$, 24913769 milliseconds ($\approx$ 7 hours).

The other groups have been treated by a faster machine (a *Dell Power Edge, Pentium II* processor with 400 MGHz cpu and 256 MG RAM). Although we have been able

to construct only a few number of subgroups inside $\mathcal{M}_3$, all of them have turned out to have a trivial first cohomology group. Unfortunately, the fact that almost all of them (except for $Q = S_3$) contain the Torelli subgroup makes it very difficult for us to guess one way or the other, if $\mathcal{M}_3$ is a Kazhdan group. Considering that in our calculation Q has always been a solvable group, the choice of a simple group for Q would be an interesting case that deserves attention. Although the calculations for a simple Q might shed some light on this problem, even the smallest choice A_5 (the alternating group on 5 elements) of order 60 turns out to be hopeless to manage, at least at this point of time. The GAP programs used for these calculations are long (about 2300 lines) and have not been included in this paper, but they are available to the interested reader on the following website:
http://www.math.msu.edu/~ferry/

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