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ABSTRACT

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Let M be a maximal subgroup of a finite group G such that no nontrivial characteristic subgroup of M is normal in G. We will consider the problem of determining the structure of G, in particular its action on the largest normal subgroup R of G that is contained in M. This is related to a problem about graphs with vertex transitive automorphism groups that has been considered by V.I. Trovimov and R.M. Weiss. The problem under consideration is similar to the 'classical' pushing-up problem, where the assumption that M is maximal in G is replaced by the assumption that M is a p-subgroup for some prime p. Using the amalgam method, which has also been used to solve classical pushing-up problems, we will solve this problem under some additional assumptions, mainly that the components of G/R are perfect but not isomorphic to $\mathrm{PSL}_n(q)$ $(n \in \mathbb{N}$, q a power of a prime$)$.

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Contents

Introduction

By a pushing-up problem we mean the following: Given a finite group G and a subgroup M of G such that no nontrivial characteristic subgroup of M is normal in G, determine the action of G on $R := \bigcap_{g \in G} M^g$, the largest normal subgroup of G that is contained in M. Let us explain why this is called a 'pushing-up' problem. The classical pushing-up situation is as follows. Given a p-local subgroup G of a finite group X and a p-subgroup M of G with $O_p(G) \leq M$ where p is a prime, does $N_X(M) \not\leq G$ imply that G is not a maximal p-local subgroup of X? (I.e. can G be 'pushed up' to a larger p-local subgroup?) Note that if C is a nontrivial characteristic subgroup of M that is normal in G, then G is properly contained in the p-local subgroup $N_X(C)$ of X. The usual way to treat this pushing-up problem is by determining as much of the structure of the counterexamples (i.e. G is a maximal p-local, but $N_X(M) \not\leq G$) as possible, in particular the action of G on $O_p(G)$, by using the property that

(∗) no nontrivial characteristic subgroup of M is normal in G.

Note that (∗) depends only on G and M, but not on X. Also, (∗) still makes sense if M is not necessarily a p-subgroup of G.

Here we will consider pushing-up problems in which M is a maximal subgroup of G, but not necessarily a p-group. This is then related to the following graph-theoretic problem, treated by V.I. Trofimov and R.M. Weiss (cf. [17]): Assume that H is a group of automorphisms of a connected graph Γ with the following properties:

1. H acts transitively on the vertices of Γ.

2. The vertex-stabilizers H_γ in H are finite and act primitively on the set of neighbors of the vertex γ.

Then determine the structure of the vertex-stabilizers. Under these assumptions, if (γ, δ) is an edge of Γ, then $H_\gamma \cap H_\delta$ is a maximal subgroup of H_γ. Moreover, there exists an element h in the edge-stabilizer $H_{\{\gamma,\delta\}}$ that switches γ and δ. Since H acts faithfully on Γ, it follows that no nontrivial subgroup of $H_\gamma \cap H_\delta$ is normal in both H_γ and $H_{\{\gamma,\delta\}}$. In particular, no nontrivial characteristic subgroup of $H_\gamma \cap H_\delta$ is normal in H_γ. In group-theoretic terms this is equivalent to the following.

(A) G (corresponding to H_γ) is a finite group,

(B) M (corresponding to $H_\gamma \cap H_\delta$) is a maximal subgroup of G, and

(C) there exists an outer automorphism t of M such that $t^2 \in \mathrm{Inn}(M)$ and no nontrivial subgroup of M is invariant under both t and G.

Again consider the situation of the first paragraph with M a maximal of G. Let L be a normal subgroup of G that is minimal subject to $LR/R = \mathrm{Soc}(G/R)$ where $\mathrm{Soc}(G/R)$ is the socle of G/R) (i.e. the product of all minimal normal subgroups of G/R). Then $G = ML$. Notice that if A is any finite group, then $G \wr A$ and $M \wr A$ satisfy the same assumptions as G and M, respectively. In particular, there is no hope to get the action of M on R under control. Hence we will consider the problem of determining the action of L on R. In the following, assume that $[R, L] \neq 1$. Note that in contrast to the classical pushing-up problem, there is a priory no prime p involved in our pushing-up situation. But using $[R, L] \neq 1$ it can be shown that $F^*(G)$, the generalized Fitting subgroup of G, is a p-group for some prime p. We will call p the characteristic of the pushing-up problem (G, M).

By the O'Nan-Scott-Theorem (cf. [12], e.g.) we have some information about G/R. Indeed we will show that under our assumptions not all of the cases listed in this theorem can occur. In particular, $\mathrm{Soc}(G/R)$ is a minimal normal subgroup of G/R. If $\mathrm{Soc}(G/R)$ is solvable, then $\mathrm{Soc}(G/R)$ is an elementary abelian q-group and $\{p,q\} = \{2,3\}$. We will restrict to the more interesting case that $\mathrm{Soc}(G/R)$ is perfect.

To treat the pushing-up problem under consideration, we will use the following strategy, which has been successfully applied to classical pushing-up problems (cf. [14], e.g.):

(1) Reduction to the case that $\mathrm{Soc}(G/R)$ is simple.

(2) Solve the problem under the assumption that $\mathrm{Soc}(G/R)$ is simple, using the so called amalgam method.

Note that step (1) is a partial converse of the extension of G to $G \wr A$. In a pushing-up problem with $M \in \mathrm{Syl}_p(G)$ step (1) is staightforward. In the graph-theoretic problem described above this step seems to be much harder. Our situation (M maximal in G, satisfying $(*)$) is somewhere in between. We will reduce this problem to a pushing up problem in which $\mathrm{Soc}(G/R)$ is simple and M is 'almost' maximal in G, which then can be treated by the amalgam method. (Indeed, almost always M will still be maximal in G.)

In the amalgam method (as introduced in [7]) there is a distinguished prime involved, which will be the characteristic p of the pushing-up problem in our case. Moreover, one needs some information about

- the structure of G/R, and
- FF-modules in characteristic p for G/R.

The O'Nan-Scott-Theorem gives the necessary information about G/R. Some information about FF-modules in characteristic p for the minimal normal subgroups of $\mathrm{Soc}(G/R)$ is already needed for step (1). In order to be able to make use of the classifications of FF-modules in the literature (cf. [14](1.2), [6], and [16]) we shall restrict to the case that the minimal normal subgroups of $\mathrm{Soc}(G/R)$ belong to the class $\mathcal{L}_p$ defined as follows: $\mathcal{L}_2$ consists of all finite simple groups of Lie type in characteristic 2, and of all finite alternating groups of degree at least 5. If p is odd, then $\mathcal{L}_p$ consists of all finite simple groups of Lie type in characteristic p.

In step (2) we will embed G and the semidirect product $M\mathrm{Aut}(M)$ into the free amalgamated product $G*_M(M\mathrm{Aut}(M))$ (therefore the name amalgam method). Then we will often meet the following situation: X is a normal p-subgroup of G and $h \in G *_M (M\mathrm{Aut}(M))$ such that X and X^h act on each other. To draw conclusions from situations like this, eventually leading to a description of the action of L on R, a close relation between the commutator $[X, X^h]$ and the centralizer $C_X(X^h)$ is very useful. Therefore, we shall further restrict to the case that the minimal normal subgroups of G/R belong to the class $\tilde{\mathcal{L}}_p$ consisting of all members of $\mathcal{L}$ that are not isomorphic to $\mathrm{PSL}_n(q)$ ($n \in \mathbb{N}$, q a power of a prime).

Before we can state the result to be proven, let us explain some terminology. By a p-component of a finite group X we mean a perfect subnormal subgroup Y of X such that $YO_p(X)/O_p(X)$ is a component of $X/O_p(X)$. The numbering of the nodes in a Dynkin diagram is always chosen as in [9]. We will prove the following result:

Theorem 1 *Let G be a finite group, M a maximal subgroup of G, R the largest normal subgroup of G that is contained in G, and L a normal subgroup of G that is minimal subject to $LR/R = Soc(G/R)$. Assume that no nontrivial characteristic subgroup of M is normal in G, $[R, L] \neq 1$, and each minimal normal subgroup of $Soc(G/R)$ belongs to $\tilde{\mathcal{L}}_p$ where p is the characteristic of the pushing-up problem (G, M). Then L is the central product of the p-components $L_1, \dots, L_m$ of G that are not contained in R, and M permutes these p-components transitively. Moreover, if $i \in \{1, \dots, m\}$ then one of the following holds:*

1. *$L_iR/R \cong PSp'_{2n}(q)$ ($q \in \{2, 4\}$), $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 1 in L_iR/R, and $[R, L_i]/C_{[R,L_i]}(L_i)$ is a natural $Sp_{2n}(q)$-module for L_i.*

2. *$L_iR/R \cong PSp'_{2n}(q)$ $(q \in \{2,4\})$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 2 in L_iR/R, and $[R, L_i]$ is a natural $Sp_{2n}(q)$-module for L_i.*

3. *$L_iR/R \cong PSp_{2n}(2)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 3 in L_iR/R, and $[R, L_i]$ is a natural $Sp_{2n}(q)$-module for L_i.*

4. *$L_iR/R \cong Sp_6(q)$ $(q = 2^k)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 1 in L_iR/R, $Z([R, L_i])$ is a natural $O_7(q)$-module for L_i, and $[R, L_i]/Z([R, L_i])$ is an $O_7(q)$-spin module for L_i.*

5. *$L_iR/R \cong \Omega_8^+(2)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype $\{3, 4\}$ in L_iR/R, and $[R, L_i]$ is a natural $O_8^+(2)$-module for L_i.*

6. *$L_iR/R \cong \Omega_{2n}^+(2)$ $(n \geq 5)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 3 in L_iR/R, and $[R, L_i]$ is a natural $O_{2n}^+(2)$-module for L_i.*

7. *$L_iR/R \cong \Omega_{2n}^-(2)$ $(n \geq 3)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 3 in L_iR/R, and $[R, L_i]$ is a natural $O_{2n}^-(2)$-module for L_i.*

8. *$L_iR/R \cong \Omega_{2n}^+(2)$ $(n \geq 4)$, $(M \cap L_i)R/R$ is the centralizer in L_iR/R of a graph automorphism that switches the nodes $n-1$ and n of the Dynkin diagram, and $[R, L_i]$ is a natural $O_{2n}^+(2)$-module for L_i.*

9. *$L_iR/R \cong \Omega_{10}^+(q)$ $(q = p^k)$, $(M \cap L_i)R/R$ is a parabolic subgroup of cotype 1 in L_iR/R, and $[R, L_i]$ is an $O_{10}^+(q)$-half spin module for L_i.*

10. *$L_iR/R \cong \mathsf{G}_2(2)'$ and $[R, L_i]/C_{[R,L_i]}(L_i)$ is an irreducible FF-module for L_i with $|[R, L_i]| = 2^6$.*

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Note that (D.2.3) and (D.2.4) show that in the cases 4 and 9 of Theorem 1 no further restrictions on the field of definition of L_iR/R are possible.

In chapters 1 and 2 we derive some basic results that are needed to get the amalgam mathod started. In particular, chapter 1 contains the proof that there exists a prime p such that $F^*(G) = O_p(G)$, and the reduction to the case that $\mathrm{Soc}(G/R)$ is simple is done in chapter 2. One of the fundamental observations in the amalgam method is that $Z := \langle \Omega_1(Z(O_p(M)))^G \rangle$ is an FF-module for G. Still in chapter 2 we show that under the assumption that $\mathrm{Soc}(G/R)$ belongs to $\tilde{\mathcal{L}}_p$ Z contains exactly one noncentral L-chief factor.

In chapters 3–5 we then determine the action of L on R, distinguishing cases by the isomorphism types of LR/R, of $(M \cap L)R/R$, and of the noncentral L-chief factor in Z. Chapter 3 handles the majority of the cases where LR/R is an orthogonal, symplectic, or unitary group, $(M \cap L)R/R$ is a parabolic subgroup of LR/R, and Z contains a natural module, except the case that LR/R is symplectic and $(M \cap L)R/R$ is of cotype 1, which is done in chapter 4. Chapter 5 treats the remaining cases, i.e. 'exceptional' FF-modules and the case that $(M \cap L)R/R$ is not a parabolic subgroup of LR/R.

In chapter 6 we summarize the results from chapters 1–5 to prove Theorem 1.

Appendix A contains some general lemmas. In Appendix B we make some concrete calculations in FF-modules for groups of Lie type, needed in chapters 3–5. Appendix C lists some properties of FF-modules for alternating groups. Finally, Appendix D contains some examples for the pushing-up problem under consideration.

Chapter 1

Basic observations, Part 1

Let G be a finite group, M a subgroup of G, R the largest normal subgroup of G which is contained in M, and L a normal subgroup of G which is minimal with respect to $LR/R = \mathrm{Soc}(G/R)$. Let H be a subgroup of $\mathrm{Aut}(G)$, and consider G and H to be embedded in the semidirect product of G and H. Assume that the following hold:

(I) M and every minimal normal subgroup of G/R are H-invariant.

(II) MH is a maximal subgroup of GH.

(III) No nontrivial characteristic subgroup of M is normal in G.

(IV) $[R, L] \neq 1$.

Note that R is H-invariant, since both G and M are H-invariant.

1.1

(1.1.1) One of the following holds:

(a) i. LR/R is an elementary abelian q-group for some prime q.

ii. M/R acts faithfully and irreducibly on LR/R.

iii. $O_q(G) = O_q(M) = 1$.

(b) i. L is perfect.

ii. $L = \bigcap \{N \mid N \trianglelefteq G,\ NR/R = \mathrm{Soc}(G/R)\}$.

Proof. If X is any MH-invariant subgroup of LR, then by (II) either $XM = G$ or $X \leq M$. In particular, $L'M = G$ or $L' \leq M$, since $L'R$ is MH-invariant by (I).

First assume that $L'M = G$. This implies that $\mathrm{Soc}(G/R)$ is perfect, and then by minimality of L also L is perfect.

Let N be a normal subgroup of G with $NR/R = \mathrm{Soc}(G/R)$. Then $[L, N]R/R = \mathrm{Soc}(G/R)$ and $[L, N] \leq L \cap N$. By minimality of L, we get $L = [L, N]$ and, hence, $L \leq N$.

Now assume that $L' \leq M$. Then $L' \leq R$. Let q be a prime that divides $|\mathrm{Soc}(G/R)|$. Let X be the subgroup of LR such that $R \leq X$ and $X/R = O_q(LR/R)$. Note that $X \not\leq M$, since $X \trianglelefteq G$ and $X/R \neq 1$. Since X is MH-invariant, it follows that $XM = G$, i.e., $X = LR$. Hence $\mathrm{Soc}(G/R)$ is an abelian q-group. A similar argument (with $X/R = \Phi(LR/R)$ instead of $O_q(LR/R)$) shows that $\mathrm{Soc}(G/R)$ is elementary abelian.

The irreducible M-submodules of LR/R are precisely the minimal normal subgroups of G/R, and by (I) these are also H-invariant. Hence again the same argument as above shows that M acts irreducibly on LR/R. By definition of R, M/R acts also faithfully on LR/R.

Since $C_{LR/R}(O_q(M/R))$ is MH-invariant, we get similarly to the above $O_q(M/R) = 1$ and thus $O_q(M) = O_q(R)$. But then (III) implies

$$O_q(M) = O_q(R) = 1.$$

If $O_q(G) \neq 1$, then $L = O_q(G)$ and $[L, R] \leq O_q(R) = 1$, a contradiction to (IV). ■

(1.1.2) (a) If N is a normal subgroup of G with $N \cap M \not\leq R$, then $L \leq N$.

(b) LR/R is the unique minimal normal subgroup of G/R.

(c) $F^*(G) = F^*(R) = O_p(G)$ and $O_p(M) \not\leq R$ for some prime p.

(d) $[R, L] \leq O_p(G)$.

(e) $O_p(M) \cap RL = O_p(M) \cap O_p(G)L$.

Proof. (a) This can be proven as Lemma 1.2 in [15]. For the convenience of the reader we repeat the argument here. Pick a subgroup T of G containing R such that T/R is a minimal normal subgroup of G/R. Then $T = (T \cap L)R$, since $R \leq T \leq LR$. If $T \leq MH$, then $T \leq MH \cap G = M$, contrary to $T \trianglelefteq G$ and $T \not\leq R$. Hence, by (II), $(T \cap L)MH = (T \cap L)RMH = TMH = GH$. Therefore,

$$(*) \quad G = GH \cap G = (T \cap L)MH \cap G = (T \cap L)M(H \cap G) = (T \cap L)M.$$

If $[T \cap L, N \cap M] \leq R$, then $(*)$ implies that $(N \cap M)R \trianglelefteq G$, a contradiction to $N \cap M \not\leq R$. Thus

$$(**) \quad [T \cap L, N \cap M] \not\leq R.$$

Since $[T \cap L, N \cap M]$ is contained in T and normalized by $T \cap L$ and M, it follows from $(*)$ and $(**)$ that

$$T = [T \cap L, N \cap M]R.$$

Since T/R is an arbitrary minimal normal subgroup or G/R, we get

$$\mathrm{Soc}(G/R) = [L, N]R/R.$$

By minimality of L, this implies $L = [L, N] \leq N$.

(b),(c) If M has a component K which is not contained in R, then $\langle K^G \rangle \leq C_G(R)$ and, by (a), $L \leq \langle K^G \rangle$, a contradiction to (IV). Hence $E(M) = E(R)$. Now (III) implies

$$E(M) = E(R) = 1$$

and

$$F(M) \neq F(R).$$

Suppose G/R has more than one minimal normal subgroup. Then, by (A.1.2)(b,c), also $GH/\tilde{R}$ has more than one minimal normal subgroup, where $\tilde{R} := \bigcap_{g \in GH}(MH)^g$. Then it follows from [12] that $GH/\tilde{R}$ has exactly two minimal normal subgroups $X/\tilde{R}$ and $Y/\tilde{R}$, so in particular

$$\mathrm{Soc}(GH/\tilde{R}) = L\tilde{R}/\tilde{R}.$$

Moreover, again by [12], $MH/\tilde{R} \cap L\tilde{R}/\tilde{R}$ is a diagonal between $X/\tilde{R}$ and $Y/\tilde{R}$, and $MH/\tilde{R} \cap L\tilde{R}/\tilde{R}$ ($\cong X/\tilde{R}$) is perfect. Hence $F(MH)\tilde{R}/\tilde{R}$ centralizes $MH/\tilde{R} \cap L\tilde{R}/\tilde{R}$. But then $F(MH)\tilde{R}/\tilde{R}$ centralizes $L\tilde{R}/\tilde{R}$ and, hence, is normal in $GH/\tilde{R}$. Since the only minimal normal subgroups of $GH/\tilde{R}$ are perfect, it follows that $F(MH) \leq \tilde{R}$ and therefore

$$F(M) \leq F(MH) \cap G \leq \tilde{R} \cap G = R,$$

a contradiction to $F(M) \neq F(R)$. Hence (b) holds.

Suppose G has a component K which is not contained in R. Then, by (b),

$$(*) \quad L \leq \langle K^G \rangle R \leq C_G(R)R.$$

If L is perfect, then (1.1.1)(b)(ii) and $(*)$ imply $L \leq C_G(R)$, a contradiction to (IV). If L is not perfect, then (1.1.1)(a)(i) and $(*)$ imply that $LR \cap C_G(R)$ is a nilpotent normal subgroup of G. Since q (as in (1.1.1)(a)) divides the order of $LR \cap C_G(R)$, we get a contradiction to (1.1.1)(a)(iii). Hence

$$E(G) = E(R) = 1.$$

Choose a prime p such that $O_p(R) \neq 1$. Then $O_p(M) \neq O_p(R)$ by (III). Hence, by (a), $L \leq \langle O_p(M)^G \rangle \leq C_G(O_{p'}(R))$. Suppose $O_r(R) \neq 1$ for some prime $r \neq p$. Then $L \leq C_G(O_{p'}(R)O_{r'}(R)) = C_G(F^*(R))$ and thus

$$(**) \quad [R, L, L] \leq [R \cap L, L] \leq [C_R(F^*(R)), L] \leq [F^*(R), L] = 1.$$

If L is perfect, the Three-Subgroup Lemma and $(**)$ imply $[R, L] = 1$, a contradiction to (IV). If L is not perfect, then (1.1.1)(a) and $(**)$ imply $[L, L, L, L] = 1$, i.e., L is nilpotent. Since q (as in (1.1.1)(a)) divides the order of L, we get a contradiction to (1.1.1)(a)(iii).

Hence $F^*(R) = O_p(R)$ and $F^*(G) = F(G)$. Suppose $F(G) \not\leq R$. Then (1.1.1) implies that L is not perfect and $F(G)R/R$ is a q-group (q as in (1.1.1)(a)). But then $O_q(G) \neq 1$, a contradiction to (1.1.1)(a)(iii).

(d) Since $O_p(G)$ is not a characteristic subgroup of M there exists $t \in \mathrm{Aut}(M)$ such that $O_p(G)^t \not\leq R$. Then $[R, O_p(G)^t] \leq R \cap O_p(M) = O_p(R) = O_p(G)$ and $L \leq \left\langle (O_p(G)^t)^G \right\rangle$.

(e) Put $X := R \cap O_p(M)L$. Let $S \in \mathrm{Syl}_p(X)$. Note that $X/(R \cap O_p(G)L)$ is a p-group, since

$$X/(R \cap O_p(G)L) = X/(X \cap O_p(G)L) \cong XO_p(G)L/O_p(G)L \leq$$

$$O_p(M)L/O_p(G)L.$$

Together with $R \cap O_p(G)L = O_p(G)(R \cap L)$ this implies

$$X = S(R \cap O_p(G)L) = SO_p(G)(R \cap L) = S(R \cap L).$$

From (d) and (c) it follows that

$$[X, R \cap L] \leq [R, R \cap L] \cap [O_p(M)L, R \cap L] \leq O_p(G) \cap [O_p(M), R] \leq O_p(G).$$

Therefore $S \trianglelefteq X$, whence $S \trianglelefteq R$. Thus $S = O_p(G)$ by (c), and therefore

$$X = S(R \cap L) = O_p(G)(R \cap L).$$

Let $x \in R$ and $y \in L$ with $xy \in O_p(M)$. Then $x \in R \cap O_p(M)L = O_p(G)(R \cap L)$ and hence $xy \in O_p(G)(R \cap L)y \leq O_p(G)L$. ∎

In the following, p denotes the prime defined in (1.1.2)(c).

1.2

Let $G *_M (M\mathrm{Aut}(M))$ be the free amalgamated product of G and the semidirect product $M\mathrm{Aut}(M)$ over M. Let Γ be the coset graph of $G *_M (M\mathrm{Aut}(M))$ with respect to G and $M\mathrm{Aut}(M)$, where we identify G and $M\mathrm{Aut}(M)$ with their images in $G *_M (M\mathrm{Aut}(M))$. For vertices $\gamma, \lambda \in \Gamma$ let $d(\gamma, \lambda)$ be the distance between γ and λ in Γ, and write $\gamma \sim \lambda$ if γ and λ are in the same orbit under the action of $G *_M (M\mathrm{Aut}(M))$ on Γ by right multiplication. Put

$$\Delta^{(k)}(\gamma) := \{\lambda \in \Gamma \mid d(\gamma, \lambda) = k\} \quad \text{for all } \gamma \in \Gamma \text{ and } k \in \mathbb{N} \cup \{0\}, \text{ and}$$

$$\Delta(\gamma) := \Delta^{(1)}(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

If $\gamma \sim M\mathrm{Aut}(M)$, i.e., $\gamma = M\mathrm{Aut}(M)x$ for some $x \in G *_M (M\mathrm{Aut}(M))$, put

$$M_\gamma := M^x,$$

$$Q_\gamma := O_p(M_\gamma) \quad \text{and}$$

$$Z_\gamma := \Omega_1(Z(Q_\gamma)).$$

If $\gamma \sim G$, i.e., $\gamma = Gx$ for some $x \in G *_M (M\mathrm{Aut}(M))$, put

$$G_\gamma := G^x,$$

$$R_\gamma := R^x,$$

$$L_\gamma := L^x,$$

$$Q_\gamma := O_p(G_\gamma),$$

$$Z_\gamma := \langle Z_\lambda \mid \lambda \in \Delta(\gamma)\rangle,$$

$$U_\gamma := [Z_\gamma, L_\gamma],$$

$$T_\gamma := C_{Z_\gamma}(L_\gamma) \quad \text{and}$$

$$C_\gamma := C_{G_\gamma}(Z_\gamma/T_\gamma).$$

For each $\gamma \in \Gamma$ put

$$V_\gamma := \langle Z_\lambda \mid \lambda \in \{\gamma\} \cup \Delta^{(2)}(\gamma)\rangle, \quad \text{and}$$

$$W_\gamma := \langle Z_\lambda \mid \lambda \in \{\gamma\} \cup \Delta^{(2)}(\gamma) \cup \Delta^{(4)}(\gamma)\rangle.$$

Let (α, α') be a critical pair (i.e., α and α' are two vertices of Γ whose distance b is minimal with respect to $Z_\alpha \not\leq Q_{\alpha'}$). For each $i \in \{0, \ldots, b\}$, we denote the element of $\Delta^{(i)}(\alpha) \cap \Delta^{(b-i)}(\alpha')$ by $\alpha + i$ and the element of $\Delta^{(i)}(\alpha') \cap \Delta^{(b-i)}(\alpha)$ by $\alpha' - i$.

(1.2.1) (a) $Q_\alpha \leq C_\alpha \leq R_\alpha$.

(b) α and α' are cosets of G.

(c) $[Z_\alpha, Z_{\alpha'}] \neq 1$. In particular, (α', α) is a critical pair.

(d) $Z_\alpha Z_{\alpha+2} \ntrianglelefteq G_{\alpha+2}$ and $U_\alpha Z_{\alpha+2} \ntrianglelefteq G_{\alpha+2}$.

(e) Assume that for each critical pair (μ, μ') there exists a critical pair (ν, ν') with $\nu \in \Delta^{(2)}(\mu)$ and $\nu' \in \Delta^{(b-2)}(\mu) \cap \Delta^{(2)}(\mu')$. Then $Z_\alpha Z_{\alpha+2} \ntrianglelefteq G_\alpha$ and $Z_\alpha U_{\alpha+2} \ntrianglelefteq G_\alpha$.

Proof. (b) If α is not a coset of G, then $Z_\alpha \leq Z_{\alpha+1} \leq Q_{\alpha'}$, a contradiction. If α' is not a coset of G, then $Z_\alpha \leq Q_{\alpha'-1} \leq Q_{\alpha'}$, again a contradiction. (a) Since (1.1.2)(c) implies $Q_\alpha \leq Q_\lambda$, for each $\lambda \in \Delta(\alpha)$, we get $Q_\alpha \leq C_\alpha$. If $C_\alpha \not\leq R_\alpha$, then $G_\alpha = C_\alpha M_{\alpha+1}$ and hence $Z_{\alpha+1} \trianglelefteq G_\alpha$, contrary to (III).
(c) Note that (1.1.2)(c) implies that $R_\gamma \cap Q_\lambda = Q_\gamma$, for all $\gamma \sim \alpha$ and $\lambda \in \Delta(\gamma)$. Hence the statements $Z_\alpha \not\leq Q_{\alpha'}$ and $[Z_\alpha, Z_{\alpha'}]$ are equivalent.
(d) Suppose that $Z_\alpha Z_{\alpha+2} \trianglelefteq G_{\alpha+2}$. In particular, $b > 2$. Pick $g \in G_{\alpha+2}$ such that $(\alpha+1)^g = \alpha+3$. Then $Z_\alpha Z_{\alpha+2} = Z_\alpha^g Z_{\alpha+2} \leq Q_{\alpha'}$, since $d(\alpha^g, \alpha') < b$, a contradiction.

Note that $U_\alpha \not\leq Q_{\alpha'}$, since $Z_\alpha = U_\alpha Z_{\alpha+1}$ and $Z_{\alpha+1} \leq Q_{\alpha'}$. Hence similarly to the above we get $U_\alpha Z_{\alpha+2} \ntrianglelefteq G_{\alpha+2}$.

(e) By the assumption, we can extend the path $(\alpha, \alpha+1, \ldots, \alpha')$ to a path $(\alpha - b + 2, \alpha - b + 2, \ldots, \alpha - 1, \alpha, \ldots, \alpha')$ such that $(\alpha - 2i, \alpha' - 2i)$ is a critical pair, for each $i \in \{0, \ldots, \frac{b-2}{2}\}$. Then (e) follows from (d), applied to the critical pair $(\alpha+2, \alpha-b+2)$ in place of (α, α'). ∎

By (1.2.1)(c) we can fix notation such that

$$|Z_\alpha / Z_\alpha \cap R_{\alpha'}| \leq |Z_{\alpha'} / Z_{\alpha'} \cap R_\alpha|.$$

(1.2.2) Let V be a G_α-submodule of Z_α with $[Z_\alpha, L_\alpha] \not\leq V$. Then Z_α / V is an FF-module for G_α and $Z_{\alpha'}$ acts as an offending subgroup on Z_α / V.

Proof. From (1.1.2)(c) and (1.2.1)(a) it follows that $Z_\alpha \cap R_{\alpha'} = Z_\alpha \cap Q_{\alpha'}$. Since $(Z_\alpha \cap Q_{\alpha'})V/V \leq C_{Z_\alpha/V}(Z_{\alpha'})$, we get

$$(*) \quad |(Z_\alpha/V)/C_{Z_\alpha/V}(Z_{\alpha'})| \leq |Z_\alpha/(Z_\alpha \cap Q_{\alpha'})V| \leq |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| =$$

$$|Z_\alpha/Z_\alpha \cap R_{\alpha'}|.$$

Note that (1.1.2)(b) implies

$$(**) \quad C_{G_\alpha}(Z_\alpha/V) \leq R_\alpha.$$

Now the claim follows from $(*)$, $(**)$, and the choice of α and α'. ∎

(1.2.3) Let $\lambda \in \Delta(\alpha)$. Let X be an elementary abelian normal p-subgroup of M_λ. Then the following are equivalent:

(a) $[X, L_\alpha] = 1$.

(b) Each M_λ-submodule of X is G_α-invariant.

Proof. Clearly (a) implies (b). Assume that (b) holds. In particular, X is a G_α-module. Let A/B be any G_α-composition factor of X. Since $C_{A/B}(Q_\lambda)$ is nontrivial and M_λ-invariant, (b) and the irreducibility of A/B as G_α-module imply that $[A, Q_\lambda] \leq B$. Since $L_\alpha \leq \langle Q_\lambda^{G_\alpha} \rangle$, we get $[A, L_\alpha] \leq B$. Since $L_\alpha = O^p(L_\alpha)$, it follows that $[X, L_\alpha] = 1$. ■

For all $\mu \sim \alpha$ put

$$\Xi_\mu := \{\mu' \in \Gamma \mid (\mu, \mu') \text{ is critical}\},$$

$$\Xi_{\mu,\nu} := \{\mu' \in \Xi_\mu \mid d(\mu,\nu) + d(\nu,\mu') = b\} \quad \text{for all } \nu \in \Gamma, \text{ and}$$

$$A_{\mu,\nu} := \bigcap_{\mu' \in \Xi_{\mu,\nu}} [Z_\mu, Z_{\mu'}] \quad \text{for all } \nu \in \Gamma \text{ with } \Xi_{\mu,\nu} \neq \emptyset.$$

(1.2.4) Let (μ, μ') be a critical pair, $\lambda \in \Delta(\mu)$ and $\gamma \in \Delta^{(b-2)}(\mu) \cap \Delta^{(2)}(\mu')$. Assume that the following hold:

(i) $b > 2$.

(ii) $G_\mu = \langle Z_{\mu'}, M_\lambda \rangle$.

Then $[\bigcap_{g \in M_\lambda} Z_\gamma^g, L_\mu] = 1$.

Proof. Put $D := \bigcap_{g \in M_\lambda} Z_\gamma^g$. Since $b > 2$ and $D \leq Z_\gamma$, we get $[D, Z_{\mu'}] = 1$. Hence the claim follows from $D \trianglelefteq M_\lambda$, (ii) and (1.2.3). ■

(1.2.5) Assume that the following hold:

(i) $b > 4$.

(ii) For any critical pair (μ, μ') there exists $\lambda \in \Delta(\mu)$ with $G_\mu = \langle Z_{\mu'}, M_\lambda \rangle$.

(iii) If (μ, μ') is any critical pair and $\lambda \in \Delta(\mu)$ with $G_\mu = \langle Z_{\mu'}, M_\lambda \rangle$, then

$$\Xi_{\nu'} \cap \Delta(\lambda) = \{\nu \in \Delta(\lambda) \mid Z_\nu Z_\mu \ntrianglelefteq G_\mu\},$$

where $\nu' \in \Delta^{(b-2)}(\mu) \cap \Delta^{(2)}(\mu')$.

Then $[\bigcap_{g \in M_{\alpha+1}} Z_\nu^g, L_\alpha] = 1$, for some $\nu \in \Xi_{\alpha,\alpha+2}$. In particular, $A_{\alpha,\alpha+2} \leq T_\alpha$.

Proof. Note that $\{\nu \in \Delta(\lambda) \mid Z_\nu Z_\mu \ntrianglelefteq G_\mu\} \neq \emptyset$ for all $\mu \sim \alpha$ and $\lambda \in \Delta(\mu)$, since V_λ is a nontrivial characteristic subgroup of M_λ. Hence (iii) and (1.2.1)(e) imply that

$$(*) \quad Z_\mu Z_\nu \ntrianglelefteq G_\mu \quad \text{for all } \mu \sim \alpha \text{ and } \nu \in \Delta^{(2)}(\mu) \text{ with } \Xi_{\mu,\nu} \neq \emptyset.$$

Pick $x \in L_\alpha$ such that $G_\alpha = \langle Z_{\alpha'}, M_{(\alpha+1)^x}\rangle$ and put $\alpha - 2 := (\alpha+2)^x$. By $(*)$ and (iii), $(\alpha-2, \alpha'-2)$ is a critical pair. Pick $y \in L_{\alpha-2}$ such that $G_{\alpha-2} = \langle Z_{\alpha'-2}, M_{(\alpha+1)^{xy}}\rangle$ and put $\alpha - 4 := \alpha^y$ and $\nu := (\alpha'-4)^{y^{-1}x^{-1}}$. Again by $(*)$ and (iii), $(\alpha-4, \alpha'-4)$ is a critical pair. Therefore,

$$\nu \in \Xi_{\alpha-4,\alpha-2}^{y^{-1}x^{-1}} = \Xi_{\alpha,\alpha+2}.$$

Define

$$D := \bigcap_{g \in M_{\alpha+1}} Z_\nu^g.$$

Since $D^{xy} \leq Z_\nu^{xy} = Z_{\alpha'-4}$, (i) implies that

$$(**) \quad [D^{xy}, Z_{\alpha'}] = 1.$$

From (1.2.4) (with $\alpha-2$, $\alpha'-2$, and $(\alpha+1)^{xy}$ in place of μ, μ', and λ, respectively) it follows that $L_{\alpha-2}$ centralizes D^{xy}. Hence

$$(***) \quad D^{xy} = D^x \trianglelefteq M_{(\alpha+1)^x}.$$

Now $(**)$, $(***)$ and (1.2.3) imply that L_α centralizes D^x. In particular, $D = D^x$ and hence $[D, L_\alpha] = 1$. ■

(1.2.6) Let (μ, μ') be a critical pair, $\lambda \in \Delta(\mu)$, $\nu \in \Delta(\lambda)$ and $\gamma \in \Delta^{(b-2)}(\mu) \cap \Delta^{(2)}(\mu')$. Assume that the following hold:

(i) $G_\mu = \langle Z_{\mu'}, M_\lambda\rangle$.

(ii) $Z_\nu Z_\mu \ntrianglelefteq G_\mu$.

(iii) $[Z_\mu, Z_{\mu'}] = [C_{Q_\gamma}(Z_\mu), Z_{\mu'}]$.

Then (ν, γ) is a critical pair.

Proof. Suppose that (ν, γ) is not a critical pair. Then $Z_\nu \leq Q_\gamma \leq G_{\mu'}$. If $b > 2$, then $Z_\nu \in C_{Q_\gamma}(Z_\mu)$, since $d(\nu, \mu) = 2 < b$. If $b = 2$, then $\gamma = \mu$ and again $Z_\nu \in Q_\gamma = C_{Q_\gamma}(Z_\mu)$, since $Q_\gamma = Q_\mu$. Now (iii) implies

$$[Z_\nu, Z_{\mu'}] \leq [C_{Q_\gamma}(Z_\mu), Z_{\mu'}] = [Z_\mu, Z_{\mu'}] \leq Z_\mu.$$

Hence $Z_{\mu'}$ normalizes $Z_\nu Z_\mu$, contrary to (i) and (ii). ∎

Chapter 2

Basic observations, Part 2

In addition to (I)–(IV), we assume

(V) The minimal normal subgroups of $\mathrm{Soc}(G/R)$ belong to the class $\mathcal{L}_p$, as defined in the introduction. In particular, L is perfect.

2.1

For $\gamma \sim \alpha$ define

$$U_\gamma^i := [Z_\gamma, L_\gamma^i],$$

where $L_\gamma^1, \dots, L_\gamma^m$ are the p-components of G_γ which are not contained in R_γ. If V is a finite-dimensional $GF(p)$-module for a finite group X, let $\mathcal{P}^*(X, V)$ be defined as in A.2.

(2.1.1) (a) If $N \trianglelefteq G_\alpha$ and $Q_\alpha \leq N \leq R_\alpha$, then $L_\alpha^1 N/N, \dots, L_\alpha^m N/N$ are the components of G_α/N which are not contained in R_α/N. Moreover, $L_\alpha^i N/N \neq L_\alpha^j N/N$ when $i \neq j$.

(b) $L_\alpha = L_\alpha^1 \cdot \dots \cdot L_\alpha^m$.

(c) $M_{\alpha+1}$ acts transitively on $\{L_\alpha^1, \dots, L_\alpha^m\}$.

(d) $[L_\alpha^i, A] \leq L_\alpha^i$ for all $i \in \{1, \dots, m\}$ and $A \in \mathcal{P}^*(G_\alpha, Z_\alpha/T_\alpha)$.

(e) Let V be a G_α-submodule of Z_α with $[Z_\alpha, L_\alpha] \not\leq V$. Then for each $i \in \{1, \ldots, m\}$ there exists $A \in \mathcal{P}^*(G_\alpha, Z_\alpha/V)$ such that $A \leq Q_{\alpha+1}$ and $[L_\alpha^i, A] \not\leq R_\alpha$.

(f) $C_{Z_\alpha}(L_\alpha, T_\alpha) = T_\alpha$.

(g) $[Z_\alpha/T_\alpha, L_\alpha] = \bigoplus_{i=1}^m U_\alpha^i T_\alpha/T_\alpha$.

(h) $L_\alpha \cap Q_\alpha = (L_\alpha^1 \cap Q_\alpha) \cdot \ldots \cdot (L_\alpha^m \cap Q_\alpha)$.

(i) $L_\alpha \cap R_\alpha = (L_\alpha^1 \cap R_\alpha) \cdot \ldots \cdot (L_\alpha^m \cap R_\alpha)$.

Proof. (a) Clearly $L_\alpha^1 N/N, \ldots, L_\alpha^m N/N$ are components of G_α/N, since $Q_\alpha \leq N$. Let K be a subgroup of G_α containing N such that K/N is a component of G_α/N which is not contained in R_α/N. Then KR_α/R_α is a component of G_α/R_α, i.e., a minimal normal subgroup of $L_\alpha R_\alpha/R_\alpha$. Choose a subnormal subgroup X of L_α which is minimal with respect to $XR_\alpha/R_\alpha = KR_\alpha/R_\alpha$. Then (1.1.2)(d) implies that X is a p-component of G_α. In particular, XN/N is a component of G_α/N. If $XN \neq K$ then $[X, K] \leq N$, a contradiction to $XR_\alpha/R_\alpha = KR_\alpha/R_\alpha$.
If $i, j \in \{1, \ldots, m\}$ and $i \neq j$, then $[L_\alpha^i, L_\alpha^j] \leq Q_\alpha$ and, hence, $[L_\alpha^i N/N, L_\alpha^j N/N] = 1$, which implies $L_\alpha^i N/N \neq L_\alpha^j N/N$.
(b) Let $i \in \{1, \ldots, m\}$. As above, (1.1.2)(d) implies that L_α contains a p-component X such that $XR_\alpha/R_\alpha = L_\alpha^i R_\alpha/R_\alpha$. Since distinct p-components centralize each other modulo Q_α, we get $X = L_\alpha^i$. Hence $L_\alpha^1 \cdot \ldots \cdot L_\alpha^m \leq L_\alpha$. Now the claim follows from the minimality of L_α.
(c) Let $\{L_\alpha^{i_1}, \ldots, L_\alpha^{i_k}\}$ be an orbit of $M_{\alpha+1}$ on $\{L_\alpha^1, \ldots, L_\alpha^m\}$. Then, by (a), (I), and (II), $N := L_\alpha^{i_1} \cdot \ldots \cdot L_\alpha^{i_k} R_\alpha$ is a normal subgroup of G_α with $R_\alpha \leq N \leq L_\alpha R_\alpha$ and, if $\{L_\alpha^{i_1}, \ldots, L_\alpha^{i_k}\} \neq \{L_\alpha^1, \ldots, L_\alpha^m\}$, then $N \neq L_\alpha R_\alpha$. Hence (c) follows from (1.1.2)(b).
(d) This follows from (a) and (A.2.1)(d).
(e) By (1.2.2), there exists $A \in \mathcal{P}^*(G_\alpha, Z_\alpha/V)$ with $A \leq Z_{\alpha'}$.

Suppose that $[L_\alpha^i, A] \le R_\alpha$ for each $i \in \{1, \dots, m\}$. Then (b) implies that $AR_\alpha/R_\alpha \in C_{G_\alpha/R_\alpha}(L_\alpha R_\alpha/R_\alpha)$. Together with $L_\alpha R_\alpha/R_\alpha = F^*(G_\alpha/R_\alpha)$ it follows that $AR_\alpha/R_\alpha \le Z(L_\alpha R_\alpha/R_\alpha) = 1$. Then, since $A \le Z_{\alpha'} \le Q_{\alpha+1}$, we get $A \le R_\alpha \cap Q_{\alpha+1} = Q_\alpha$ by (1.1.2)(c). But then $[Z_\alpha/V, A] = 1$, a contradiction to $A \in \mathcal{P}^*(G_\alpha, Z_\alpha/V)$. Hence

$$[L_\alpha^i, A] \not\le R_\alpha \quad \text{for some } i \in \{1, \dots, m\}.$$

Now (e) follows from (c).

(f) This follows from $[L_\alpha, L_\alpha] = L_\alpha$, the definition of T_α, and the Three-Subgroup Lemma.

(g) From (a) it follows that $L_\alpha^1 C_\alpha/C_\alpha, \dots, L_\alpha^m C_\alpha/C_\alpha$ are components of G_α/C_α. Note that (e) implies that

$$L_\alpha^i C_\alpha/C_\alpha \le \langle \mathcal{P}^*(G_\alpha/C_\alpha, Z_\alpha/T_\alpha)\rangle, \qquad \text{for each } i \in \{1, \dots, m\}.$$

Now (g) follows from (A.3.3) (with $(G_\alpha/C_\alpha, Z_\alpha/T_\alpha, L_\alpha^1 C_\alpha/C_\alpha, \dots, L_\alpha^m C_\alpha/C_\alpha)$ in place of $(G, V, L_1, \dots, L_n)$), (b), and (f).

(h) Put

$$\overline{L_\alpha} = L_\alpha/((L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha)).$$

From (b) and $[L_\alpha^i, L_\alpha^j] \le Q_\alpha \cap L_\alpha^i \cap L_\alpha^j$, for all $i, j \in \{1, \dots, m\}$ with $i \ne j$, it follows that $\overline{L_\alpha}$ is the central product of $\overline{L_\alpha^1}, \dots, \overline{L_\alpha^m}$. Hence $O_p(\overline{L_\alpha})$ is the central product of $O_p(\overline{L_\alpha^1}), \dots, O_p(\overline{L_\alpha^m})$. For each $i \in \{1, \dots, m\}$, let A_i be the subgroup of L_α such that $(L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha) \le A_i$ and $\overline{A_i} = O_p(\overline{L_\alpha^i})$. Since

$$(L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha) \le A_i \le (L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha) L_\alpha^i,$$

for each $i \in \{1, \dots, m\}$, and

$$L_\alpha \cap Q_\alpha = A_1 \cdot \dots \cdot A_m,$$

we get

$$A_i = (L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha)(L_\alpha^i \cap A_i) = (L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha),$$

for each $i \in \{1, \dots, m\}$, and thus $L_\alpha \cap Q_\alpha = (L_\alpha^1 \cap Q_\alpha) \cdot \dots \cdot (L_\alpha^m \cap Q_\alpha)$.

(i) similar to (h). ■

(2.1.2) Let $i \in \{1, \dots, m\}$. Then there exists $A \le N_{Q_{\alpha+1}}(L_\alpha^i)$ such that $[L_\alpha^i, A] \not\le R_\alpha$ and $A \in \mathcal{P}^*(G_\alpha, Z_\alpha/T_\alpha)$. For any such A and for any irreducible $GF(p)(AL_\alpha^i)$-submodule V of $U_\alpha^i T_\alpha/T_\alpha$ the following hold:

(a) $AC_{A\,L_\alpha^i}(V)/C_{A\,L_\alpha^i}(V) \in \mathcal{P}^*(A\,L_\alpha^i, V)$.

(b) $L_\alpha^i\, C_{A\,L_\alpha^i}(V)/C_{A\,L_\alpha^i}(V) = F^*(A\,L_\alpha^i/C_{A\,L_\alpha^i}(V))$.

Proof. Choose A as in (2.1.1)(e) (with T_α in place of V). Then $A \le N_{Q_{\alpha+1}}(L_\alpha^i)$ by (2.1.1)(d). From (2.1.1)(a) it follows that $L_\alpha^i/L_\alpha^i \cap C_\alpha$ is quasisimple and $Z(L_\alpha^i/L_\alpha^i \cap C_\alpha) = L_\alpha^i \cap R_\alpha/L_\alpha^i \cap C_\alpha$. Since $L_\alpha^i \cap C_\alpha \le C_{L_\alpha^i}(V)$ and, by (2.1.1)(f)(g), $[V, L_\alpha^i] \ne 1$, we get

$$(*) \quad C_{L_\alpha^i}(V) \le R_\alpha$$

and $L_\alpha^i/C_{L_\alpha^i}(V)$ is quasisimple. Hence $L_\alpha^i\, C_{A\,L_\alpha^i}(V)/C_{A\,L_\alpha^i}(V)$ is a component of $A\,L_\alpha^i/C_{A\,L_\alpha^i}(V)$. Assuming that $L_\alpha^i\, C_{A\,L_\alpha^i}(V)/C_{A\,L_\alpha^i}(V) \ne F^*(A\,L_\alpha^i/C_{A\,L_\alpha^i}(V))$, we get $O_p(A\,L_\alpha^i/C_{A\,L_\alpha^i}(V)) \ne 1$, since $A\,L_\alpha^i/L_\alpha^i$ is a p-group. But $O_p(A\,L_\alpha^i/C_{A\,L_\alpha^i}(V)) = 1$, since V is irreducible. Thus (b) holds.

From $(*)$ and $[L_\alpha^i, A] \not\le R_\alpha$ it follows that $[V, [L_\alpha^i, A]] \ne 1$. But then $[V, A] \ne 1$, since L_α^i normalizes $C_{A\,L_\alpha^i}(V)$. Now (a) follows from (A.2.1)(b). ■

2.2

In addition to (I)–(V), we assume

(VI) M is a maximal subgroup of G.

(2.2.1) There exists a subgroup E of $\mathrm{Aut}(L_\alpha^1 R_\alpha/R_\alpha)$ and a monomorphism

$$\phi : G_\alpha/R_\alpha \to E \wr \Sigma_m$$

such that the following hold:

(a) $\mathrm{Inn}(L_\alpha^1 R_\alpha/R_\alpha) \le E$.

(b) The following diagram commutes:

$$\begin{array}{ccc} L_\alpha^1 R_\alpha/R_\alpha & \longrightarrow & E \\ \downarrow & & \downarrow \\ G_\alpha/R_\alpha & \xrightarrow{\phi} & E \wr \Sigma_m \end{array} .$$

(c) $(G_\alpha/R_\alpha)^\phi$ acts transitively on $\{E^x \mid x \in E \wr \Sigma_m\}$.

(d) $((M_{\alpha+1} \cap L_\alpha^1 R_\alpha)/R_\alpha)^\phi = X \cap (L_\alpha^1 R_\alpha/R_\alpha)^\phi$ for some maximal subgroup X of E.

Proof. Using the notation of [12], G_α/R_α viewed as a permutation group on the right cosets of $M_{\alpha+1}/R_\alpha$ is either of type II or of type III(a–c). If G_α/R_α is of type II, i.e., $m = 1$, then the monomorphism $\phi : G/R_\alpha \to \mathrm{Aut}(L_\alpha R_\alpha/R_\alpha)$ given by the action of G_α/R_α on $L_\alpha R_\alpha/R_\alpha$ and $E := (G_\alpha/R_\alpha)^\phi$ satisfy (a)–(d). Hence we may assume that G_α/R_α is of type III.

From $Z_{\alpha'} \le Q_{\alpha+1}$, (1.2.2), and (2.1.1)(a,d) it follows that $M_{\alpha+1}/R_\alpha$ contains a nontrivial normal p-subgroup Y/R_α which normalizes all components of G_α/R_α. In particular, $M_{\alpha+1}/R_\alpha$ does not act faithfully on the set of components of G_α/R_α, whence G_α/R_α is not of type III(c).

Suppose G_α/R_α is of type III(a). Then $(M_{\alpha+1}/R_\alpha) \cap (L_\alpha R_\alpha/R_\alpha)$ is a diagonal in the direct product $L_\alpha R_\alpha/R_\alpha = (L_\alpha^1 R_\alpha/R_\alpha) \times \ldots \times (L_\alpha^m R_\alpha/R_\alpha)$. In particular, $(M_{\alpha+1}/R_\alpha) \cap (L_\alpha R_\alpha/R_\alpha)$ is a nonabelian simple normal subgroup of $M_{\alpha+1}/R_\alpha$. But then

$$[(M_{\alpha+1}/R_\alpha) \cap (L_\alpha R_\alpha/R_\alpha), Y/R_\alpha] \le$$

$$[(M_{\alpha+1}/R_\alpha) \cap (L_\alpha R_\alpha/R_\alpha), Q_{\alpha+1}R_\alpha/R_\alpha] = 1.$$

Now Y/R_α centralizes a diagonal and normalizes each factor of the direct product $L_\alpha R_\alpha/R_\alpha = L^1_\alpha R_\alpha/R_\alpha \times \ldots \times L^m_\alpha R_\alpha/R_\alpha$. Therefore,

$$[L_\alpha R_\alpha/R_\alpha, Y/R_\alpha] = 1.$$

Thus $Y/R_\alpha \trianglelefteq G_\alpha/R_\alpha$, a contradiction to (1.1.2)(b).

Hence G_α/R_α is of type III(b). Thus G_α/R_α is obtained from a primitive permutation group of type II or III(a) by the construction described in [12]. If G_α/R_α is obtained from a group of type (II), then (a)–(d) are satisfied. Otherwise $M_{\alpha+1}/R_\alpha \cap L_\alpha R_\alpha/R_\alpha$ is a direct product of nonabelian simple groups, and we get a similar contradiction as above. ∎

(2.2.2) Let $i \in \{1, \ldots, m\}$.

(a) $(M_{\alpha+1} \cap L^i_\alpha R_\alpha)/R_\alpha$ is maximal (with respect to inclusion) amongst the proper $N_{M_{\alpha+1}}(L^i_\alpha)$-invariant subgroups of $L^i_\alpha R_\alpha/R_\alpha$.

(b) $(M_{\alpha+1} \cap L^i_\alpha R_\alpha)/R_\alpha$ is not a p-group.

(c) $Q_{\alpha+1}$ normalizes L^i_α.

(d) $O_p((M_{\alpha+1} \cap L^i_\alpha R_\alpha)/R_\alpha) = (Q_{\alpha+1} \cap L^i_\alpha)R_\alpha/R_\alpha$.

Proof. Put $X := M_{\alpha+1} \cap L^i_\alpha R_\alpha$.

(a) Let Y be a proper subgroup of $L^i_\alpha R_\alpha$ containing X properly. For each $j \in \{1, \ldots, m\}$, put

$$\mathcal{Y}_j := \{Y^x \mid x \in M_{\alpha+1},\ Y^x \le L^j_\alpha R_\alpha\}.$$

Then

$$G_\alpha/R_\alpha = (\langle \mathcal{Y}_1\rangle/R_\alpha \times \ldots \times \langle \mathcal{Y}_m\rangle/R_\alpha)\,(M_{\alpha+1}/R_\alpha).$$

Since by (1.1.2)(b) $L_\alpha R_\alpha/R_\alpha$ is the only minimal normal subgroup of G_α/R_α, it follows that

$$L_\alpha R_\alpha/R_\alpha = \langle \mathcal{Y}_1\rangle/R_\alpha \times \ldots \times \langle \mathcal{Y}_m\rangle/R_\alpha.$$

Hence $|\mathcal{Y}_j| > 1$ for all $j \in \{1, \dots, m\}$. In particular, $Y \neq Y^x \leq L^i_\alpha R_\alpha$, for some $x \in M_{\alpha+1}$. Since $L^i_\alpha R_\alpha/R_\alpha \cap L^j_\alpha R_\alpha/R_\alpha = 1$ if $i \neq j \in \{1, \dots, m\}$, it follows that x normalizes L^i_α. Hence Y is not $N_{M_{\alpha+1}}(L^i_\alpha)$-invariant.

(b) Suppose that X/R_α is a p-group. Then, by (a),

$$X/R_\alpha \in \mathrm{Syl}_p(L^i_\alpha R_\alpha/R_\alpha).$$

By (V) one of the following holds:

(i) $L^i_\alpha R_\alpha/R_\alpha \cong A_n$, $n \geq 7$ and $p = 2$.

(ii) $L^i_\alpha R_\alpha/R_\alpha$ is a group of Lie type in characteristic p (including A_5 ($\cong \mathrm{PSL}_2(4)$) and A_6 ($\cong \mathrm{Sp}_4(2)'$)).

In case (i) we get a contradiction from (a) and (A.1.4)(b). Hence (ii) holds. By (a), the Cartan subgroup of $L^i_\alpha R_\alpha/R_\alpha$ is trivial and $N_{M_{\alpha+1}}(L^i_\alpha)$ acts transitively on the nodes of the Dynkin diagram of $L^i_\alpha R_\alpha/R_\alpha$. Therefore,

$$L^i_\alpha R_\alpha/R_\alpha \cong \mathrm{PSL}_3(2) \quad \text{or} \quad L^i_\alpha R_\alpha/R_\alpha \cong \mathrm{Sp}_4(2)',$$

and $N_{M_{\alpha+1}}(L^i_\alpha)$ contains an element g which does not normalize the two minimal parabolic subgroups of $L^i_\alpha R_\alpha/R_\alpha$ containing the Sylow 2-subgroup X/R_α.

Put $Y := L^i_\alpha N_{M_{\alpha+1}}(L^i_\alpha)$. Let V be an irreducible Y-submodule of $U^i_\alpha T_\alpha/T_\alpha$. Then $L^i_\alpha \not\leq C_Y(V)$ by (2.1.1)(f)(g). From (2.1.1)(a) it follows that $L^i_\alpha C_\alpha/C_\alpha$ is a component of YC_α/C_α. Choose A as in (2.1.2). Then $[A, L^i_\alpha] \not\leq C_Y(V)$. Depending on whether $L^i_\alpha R_\alpha/R_\alpha \cong \mathrm{Sp}_4(2)'$ or $L^i_\alpha R_\alpha/R_\alpha \cong \mathrm{PSL}_3(2)$, either (A.2.7) or (A.2.8) (with $Y/C_Y(V)$, $R_\alpha C_Y(V)/C_Y(V)$, $L^i_\alpha C_Y(V)/C_Y(V)$, and $AC_Y(V)/C_Y(V), V$) in place of G, R, L, A, and V, respectively) shows that g normalizes the two minimal parabolic subgroups of $L^i_\alpha R_\alpha/R_\alpha$ containing X/R_α, a contradiction.

(c) follows from (b).

(d) Since $R_\alpha \leq M_{\alpha+1}$, the claim is equivalent to

$$O_p((M_{\alpha+1} \cap L^i_\alpha)/(R_\alpha \cap L^i_\alpha)) = (Q_{\alpha+1} \cap L^i_\alpha)(R_\alpha \cap L^i_\alpha)/(R_\alpha \cap L^i_\alpha).$$

Clearly

$$(Q_{\alpha+1} \cap L^i_\alpha)(R_\alpha \cap L^i_\alpha)/(R_\alpha \cap L^i_\alpha) \leq O_p((M_{\alpha+1} \cap L^i_\alpha)/(R_\alpha \cap L^i_\alpha)).$$

Let X and Y be the subgroups of L^i_α satisfying $Q_\alpha \cap L^i_\alpha \leq X$, $R_\alpha \cap L^i_\alpha \leq Y$,

$$O_p((M_{\alpha+1} \cap L^i_\alpha)/(Q_\alpha \cap L^i_\alpha)) = X/(Q_\alpha \cap L^i_\alpha) \quad \text{and}$$

$$O_p((M_{\alpha+1} \cap L^i_\alpha)/(R_\alpha \cap L^i_\alpha)) = Y/(R_\alpha \cap L^i_\alpha).$$

Note that any two distinct conjugates of X under $M_{\alpha+1}$ centralize each other modulo Q_α. Hence $X \leq Q_{\alpha+1}$, and it suffices to show that $Y = X(R_\alpha \cap L^i_\alpha)$. Since L^i_α is a p-component of G_α which is not contained in R_α,

$$(R_\alpha \cap L^i_\alpha)/(Q_\alpha \cap L^i_\alpha) \leq Z(L^i_\alpha/(Q_\alpha \cap L^i_\alpha)).$$

Since $O_p(L^i_\alpha Q_\alpha/Q_\alpha) \leq O_p(G_\alpha/Q_\alpha) = 1$, it follows that $(R_\alpha \cap L^i_\alpha)/(Q_\alpha \cap L^i_\alpha)$ is a p'-group. Hence $Y/(Q_\alpha \cap L^i_\alpha)$ is the product of $(R_\alpha \cap L^i_\alpha)/(Q_\alpha \cap L^i_\alpha)$ and a normal Sylow p-subgroup, which must be $X/(Q_\alpha \cap L^i_\alpha)$. ∎

(2.2.3) (a) $(M_{\alpha+1} \cap L_\alpha R_\alpha)/R_\alpha = (M_{\alpha+1} \cap L^1_\alpha R_\alpha)/R_\alpha \times \ldots \times (M_{\alpha+1} \cap L^m_\alpha R_\alpha)/R_\alpha$.

(b) $O_p((M_{\alpha+1} \cap L_\alpha R_\alpha)/R_\alpha) = (Q_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha = (Q_{\alpha+1} \cap L^1_\alpha)R_\alpha/R_\alpha \times \ldots \times (Q_{\alpha+1} \cap L^m_\alpha)R_\alpha/R_\alpha$.

Proof. (a) For each $i \in \{1, \ldots, m\}$, let π_i be the projection from $L_\alpha R_\alpha/R_\alpha$ onto $L^i_\alpha R_\alpha/R_\alpha$. Then

$$((M_{\alpha+1} \cap L_\alpha R_\alpha)/R_\alpha)^{\pi_i} = (M_{\alpha+1} \cap L^i_\alpha R_\alpha)/R_\alpha, \quad \text{for each } i \in \{1, \ldots, m\},$$

by (2.2.2)(a).

(b) follows from (a) and (2.2.2)(d). ∎

(2.2.4) Let $i \in \{1, \ldots, m\}$. Assume that $L^i_\alpha R_\alpha/R_\alpha$ is a group of Lie type. Then one of the following holds:

(a) $Q_{\alpha+1} \cap L_\alpha^i \not\leq R_\alpha$, and $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ is a parabolic subgroup of $L_\alpha^i R_\alpha / R_\alpha$. Moreover, the following hold:

(a1) $N_{M_{\alpha+1}}(L_\alpha^i)$ is transitive on the set of parabolic subgroups of $L_\alpha^i R_\alpha / R_\alpha$ which contain $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ as a maximal subgroup.

(a2) If $L_\alpha^i R_\alpha / R_\alpha$ is of type A_n and $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ is of type A_{n-2}, then $n = 3$ and $U_\alpha^i T_\alpha / T_\alpha$ is the exterior square of a natural $\mathrm{SL}_4(q)$-module for L_α^i.

(a3) If $L_\alpha^i R_\alpha / R_\alpha$ is a rank 2 group, then $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ is a maximal subgroup of $L_\alpha^i R_\alpha / R_\alpha$.

(a4) If $L_\alpha^i R_\alpha / R_\alpha$ is of type ${}^2\mathsf{A}_n$, then $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ is not of type ${}^2\mathsf{A}_{n-1}$.

(b) $Q_{\alpha+1} \cap L_\alpha^i \leq R_\alpha$. In this case, $L_\alpha^i R_\alpha / R_\alpha \cong \Omega_{2n}^\varepsilon(2) \quad (\varepsilon \in \{+,-\})$, and $U_\alpha^i T_\alpha / T_\alpha$ is a natural $\Omega_{2n}^\varepsilon(2)$-module for L_α^i.

Proof. Assume that $Q_{\alpha+1} \cap L_\alpha^i \not\leq R_\alpha$. Then, by (2.2.2)(d),

$$X := O_p((M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha)$$

is nontrivial, whence $N_{L_\alpha^i R_\alpha / R_\alpha}(X)$ is a proper $N_{M_{\alpha+1}}(L_\alpha^i)$-invariant subgroup of $L_\alpha^i R_\alpha / R_\alpha$. Hence, by (2.2.2)(a),

$$(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha = N_{L_\alpha^i R_\alpha / R_\alpha}(X).$$

In particular,

$$X = O_p(N_{L_\alpha^i R_\alpha / R_\alpha}(X)).$$

Now [1](47.8) shows that $(M_{\alpha+1} \cap L_\alpha^i R_\alpha)/R_\alpha$ is a parabolic subgroup of $L_\alpha^i R_\alpha / R_\alpha$. (a1) follows from (2.2.2)(a). (a2) follows from (A.2.8). (a3) follows from (a1), (a2) and (A.2.7). Note that in $\mathrm{SU}_n(p^k)$ the largest normal p-subgroup of the stabilizer of a 1-dimensional isotropic subspace of the natural module does not contain any offending subgroups for the natural module. This implies (a4).

Now assume that $Q_{\alpha+1} \cap L_\alpha^i \leq R_\alpha$. Choose A as in (2.1.2), and let V be an irreducible AL_α^i-submodule of $U_\alpha^i T_\alpha / T_\alpha$. Then $AL_\alpha^i / C_{AL_\alpha^i}(V)$ is one of the groups listed in (A.2.2), and the intersection of the offending subgroup $AC_{AL_\alpha^i}(V)/C_{AL_\alpha^i}(V)$ with $F^*(AL_\alpha^i / C_{AL_\alpha^i}(V))$ is trivial. This implies (b). ∎

(2.2.5) Let $i \in \{1, \ldots, m\}$. Assume that $L_\alpha^i R_\alpha / R_\alpha \cong A_n$, for some $n \in \mathbb{N}$ with $n = 7$ or $n > 8$. Then one of the following holds:

(a) $U_\alpha^i T_\alpha / T_\alpha$ is a natural Σ_n- or A_n-module and one of the following holds:

(a1) $(M_{\alpha+1} \cap L_\alpha^i) R_\alpha / R_\alpha \cong \Sigma_{n-2}$,

(a2) n is even and $(M_{\alpha+1} \cap L_\alpha^i) R_\alpha / R_\alpha$ is isomorphic to the stabilizer in A_n of a partition of $\{1, \ldots, n\}$ into 2-sets.

(b) $n = 7$, U_α^i is the module listed in (A.2.2)(m), and $(M_{\alpha+1} \cap L_\alpha^i) R_\alpha / R_\alpha$ is isomorphic to the stabilizer of $\{1, 2, 3, 4\}$ in A_7.

Proof. From (A.2.2) it follows that $U_\alpha^i T_\alpha / T_\alpha$ is an irreducible L_α^i-module. More precisely, it is either a natural Σ_n- or A_n-module or the module listed in (A.2.2)(m). Note that Σ_n does not act on the latter.

Assume that not all of the automorphisms of $L_\alpha^i R_\alpha / R_\alpha$ induced by $N_{M_{\alpha+1}}(L_\alpha^i)$ are inner, i.e.,

$$N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i / C_{N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i}(U_\alpha^i T_\alpha / T_\alpha) \cong \Sigma_n.$$

Then $U_\alpha^i T_\alpha / T_\alpha$ is a natural module by the preceding paragraph. Moreover, by (2.2.2)(a) $N_{M_{\alpha+1}}(L_\alpha^i)(M_{\alpha+1} \cap L_\alpha^i) C_{N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i}(U_\alpha^i T_\alpha / T_\alpha) / C_{N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i}(U_\alpha^i T_\alpha / T_\alpha)$ is a maximal subgroup of $N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i / C_{N_{M_{\alpha+1}}(L_\alpha^i) L_\alpha^i}(U_\alpha^i T_\alpha / T_\alpha)$. Hence (a) follows from (C.1.2).

Now assume that all of the automorphisms of $L^i_\alpha R_\alpha/R_\alpha$ induced by $N_{M_{\alpha+1}}(L^i_\alpha)$ are inner. Then (2.2.2)(a) implies that $(M_{\alpha+1} \cap L^i_\alpha)C_{L^i_\alpha}(U^i_\alpha T_\alpha/T_\alpha)/C_{L^i_\alpha}(U^i_\alpha T_\alpha/T_\alpha)$ is a maximal subgroup of $L^i_\alpha/C_{L^i_\alpha}(U^i_\alpha T_\alpha/T_\alpha) \cong A_n$. Hence, if $U^i_\alpha T_\alpha/T_\alpha$ is a natural module, then (a) follows from (C.1.2). If $U^i_\alpha T_\alpha/T_\alpha$ is the module listed in (A.2.2)(m), then (b) follows from (C.2.1) and [14](1.5). ■

For $\lambda \sim \alpha + 1$ define

$$\widehat{M}_\lambda := \bigcap_{\mu \in \Delta(\lambda)} \bigcap_{i=1}^{m} N_{M_\lambda}(L^i_\mu).$$

(2.2.6) (a) Every p-component of $M_{\alpha+1}$ is contained in R_α or L_α.

(b) Assume that $M_{\alpha+1}$ has p-components. Then $M_{\alpha+1} \cap L_\alpha \leq \widehat{M}_{\alpha+1}$.

Proof. (a) Let K be a p-component of $M_{\alpha+1}$ with $K \not\leq L_\alpha$. Then $[K, M_{\alpha+1} \cap L_\alpha] \leq Q_{\alpha+1}$. Hence by (2.2.3) and (2.2.2)(b) K normalizes L^i_α for each $i \in \{1, \ldots, m\}$. Since K is perfect, it follows that K induces inner automorphisms on $L_\alpha R_\alpha/R_\alpha$, i.e., $K \leq L_\alpha R_\alpha$. Therefore

$$K = [K, K] \leq [K, (M_{\alpha+1} \cap L_\alpha)R_\alpha] \leq Q_{\alpha+1}[K, R_\alpha].$$

Since $K \not\leq Q_{\alpha+1}$, we get $[K, R_\alpha] \not\leq Q_{\alpha+1}$ and hence $K \leq R_\alpha$.

(b) Let $\mu \in \Delta(\alpha+1)$ and $j \in \{1, \ldots, m\}$. Since the p-components of $M_{\alpha+1}$ generate a characteristic subgroup of $M_{\alpha+1}$, there exists a p-component K of $M_{\alpha+1}$ with $K \not\leq R_\mu$. Then $K \leq L_\mu$ by (a). Moreover, by (2.1.1)(c), (2.2.3) and (1.1.2)(d) we can choose K such that $K \leq L^j_\mu$.

Suppose that $M_{\alpha+1} \cap L_\alpha$ does not normalize L^j_μ. Then (2.2.3)(a) implies that

$$[M_{\alpha+1} \cap L_\alpha, K] \not\leq Q_{\alpha+1}K.$$

Hence K is a non-normal p-component of $M_{\alpha+1} \cap L_\alpha$. But by (2.2.3)(a), (2.2.4) and (2.2.5) $M_{\alpha+1} \cap L_\alpha$ normalizes all of its p-components. ■

(2.2.7) Assume that $p = 3$ and there exist integers k_1, k_2, k_3, k_4 such that the following hold:

(i) $(M_{\alpha+1} \cap L^1_\alpha)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1})$ is a central product of k_1 copies of $\mathrm{GL}_2(3)$, k_2 copies of Σ_4, and k_3 copies of $\mathrm{SL}_2(3)$.

(ii) $|Z((M_{\alpha+1} \cap L^1_\alpha)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}))| = 2^{k_4}$.

Then $M_{\alpha+1} \cap L_\alpha \leq \widehat{M}_{\alpha+1}$.

Proof. Let $\mu \in \Delta(\alpha+1)$. Put

$$\overline{M_{\alpha+1}} := M_{\alpha+1}/R_\mu Q_{\alpha+1}.$$

Then $\overline{R_\alpha \cap L_\alpha}$ is an abelian normal subgroup of $\overline{M_{\alpha+1}}$. Therefore

$$(*) \quad \overline{R_\alpha \cap L_\alpha} \cap \overline{M_{\alpha+1} \cap L_\mu} \leq Z(\overline{M_{\alpha+1} \cap L_\mu}).$$

Assume that $k_1 > 0$ or $k_2 > 0$. Let X be a normal subgroup of $(M_{\alpha+1} \cap L^1_\alpha)(R_\alpha Q_{\alpha+1})$ containing $R_\alpha Q_{\alpha+1}$ such that $X/R_\alpha Q_{\alpha+1}$ is one of the factors listed in (i). Suppose that X has an orbit of size $s > 1$ on $\{L^1_\mu, \ldots, L^m_\mu\}$. Without loss we may assume that $\{L^1_\mu, \ldots, L^s_\mu\}$ is such an orbit. Then by (A.1.1)(a)

$$\overline{M_{\alpha+1} \cap L^1_\mu}' \times \ldots \times \overline{M_{\alpha+1} \cap L^s_\mu}' \leq \overline{[M_{\alpha+1} \cap L_\mu, X]} \quad \text{and}$$

$$(**) \quad \overline{M_{\alpha+1} \cap L^1_\mu}'' \times \ldots \times \overline{M_{\alpha+1} \cap L^s_\mu}'' \leq \overline{[M_{\alpha+1} \cap L_\mu, X, X]}.$$

Now $(*)$ and $(**)$ imply that

$$2^{(2k_1+2k_2)s} \mid |[M_{\alpha+1} \cap L_\mu, X, X]R_\alpha R_\mu Q_{\alpha+1}/R_\alpha R_\mu Q_{\alpha+1}|.$$

On the other hand, by (2.2.3)(a),

$$[M_{\alpha+1} \cap L_\mu, X, X] \leq [M_{\alpha+1} \cap L_\alpha, X] \leq X' R_\alpha Q_{\alpha+1}.$$

Since $k_1 > 0$ or $k_2 > 0$, it follows that the order of the Sylow 2-subgroups of $X'R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$ is at least 2^{2s}, a contradiction to $s > 1$ and the choice of X. Hence

$$X \leq \bigcap_{i=1}^{m} N_{M_{\alpha+1}}(L_\mu^i).$$

Since this holds for any such X and any $\mu \in \Delta(\alpha+1)$, we get $M_{\alpha+1} \cap L_\alpha^1 \leq \widehat{M}_{\alpha+1}$.

Assume now that $k_1 = k_2 = 0$. Note that this implies that $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{PSp}_4(3)$ and $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha/R_\alpha$ is a parabolic subgroup of cotype 1. Then

$$M_{\alpha+1} \cap L_\alpha^1 \leq Q_{\alpha+1}(R_\alpha \cap L_\alpha)O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1})).$$

Pick $i \in \{1,\ldots,m\}$. By (2.2.2)(c) $Q_{\alpha+1}$ normalizes L_μ^i. Since $\overline{R_\alpha \cap L_\alpha}$ is an abelian normal subgroup of $\overline{M_{\alpha+1}}$, also $R_\alpha \cap L_\alpha$ normalizes L_μ^i. From

$$[Z_{\alpha+1}, O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1}))] =$$

$$[Z_{\alpha+1}, O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1})), O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1}))] \leq$$

$$[U_\alpha \cap Z_{\alpha+1}, O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1}))] = 1$$

and (2.1.1)(g) it follows that also $O^3(C_{M_{\alpha+1}\cap L_\alpha^1}(U_\alpha \cap Z_{\alpha+1}))$ normalizes L_μ^i. Hence $M_{\alpha+1} \cap L_\alpha^1 \leq \widehat{M}_{\alpha+1}$. ■

(2.2.8) Assume that $p = 2$ and $(M_{\alpha+1} \cap L_\alpha)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$ is a direct product of Σ_3's. Then $M_{\alpha+1} \cap L_\alpha \leq \widehat{M}_{\alpha+1}$.

Proof. Let $k \in \mathbb{N}$ such that $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$ is a direct product of k copies of Σ_3. Let $\mu \in \Delta(\alpha+1)$ and put

$$\overline{M_{\alpha+1}} := M_{\alpha+1}/R_\mu Q_{\alpha+1}.$$

Let $D \in \mathrm{Syl}_3(M_{\alpha+1} \cap L_\mu)$. Let A be a 2-subgroup of $M_{\alpha+1} \cap L_\alpha^1$ that is modulo $R_\alpha Q_{\alpha+1}$ contained in an Σ_3. Suppose that A has an orbit of size $s > 1$ on $\{L_\mu^1, \ldots, L_\mu^m\}$.

Without loss we may assume that $\{L_\mu^1, \ldots, L_\mu^s\}$ is such an orbit. Since $\overline{D}$ is the only Sylow 3-subgroup of $\overline{M_{\alpha+1} \cap L_\mu}$, A normalizes $\overline{D}$. From (A.1.1) it follows that $|[\overline{D}, A]| \geq 3^{(s-1)k}$. Moreover, A acts on $[\overline{D}, A]$ by inversion. Hence

$$|[\overline{D}, A, A]| \geq 3^{(s-1)k}.$$

Together with

$$[R_\alpha \cap L_\alpha^1, A] \leq [R_\alpha, L_\alpha] \leq Q_\alpha$$

we get that $[D, A, A]R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$ contains a factor group that is a direct product of at least $(s-1)k$ copies of A_3 and inverted by A. Now

$$[D, A, A] \leq [M_{\alpha+1}, M_{\alpha+1} \cap L_\alpha, M_{\alpha+1} \cap L_\alpha^1] \leq M_{\alpha+1} \cap L_\alpha^1$$

implies that $s = 2$ and

$$[D, A, A]R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1} \in \mathrm{Syl}_3((M_{\alpha+1} \cap L_\alpha^1)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}).$$

Hence A inverts the entire Sylow 3-subgroup of $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$. Therefore

$$k = 1,$$

and then by (2.2.4) and (2.2.5) one of the following holds:

(1) $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{PSL}_3(2)$,

(2) $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{Sp}_4(2)'$,

(3) $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{G}_2(2)'$,

(4) $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{PSL}_4(2)$, $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha/R_\alpha$ is a rank 1 parabolic subgroup of $L_\alpha^1 R_\alpha/R_\alpha$ corresponding to the middle node of the Dynkin diagram, and $U_\alpha^1 T_\alpha/T_\alpha$ is the exterior square of a natural $\mathrm{SL}_4(2)$-module.

Let $X \in \mathrm{Syl}_3([D,A,A])$. From $[Z_{\alpha+1}, X, X] = [Z_{\alpha+1}, X]$ and $X \leq L_\alpha$ it follows that

$$[Z_{\alpha+1}, X] = [Z_{\alpha+1} \cap U_\alpha, X] \quad \text{and}$$

$$[Z_{\alpha+1}, X] \cap T_\alpha = 1.$$

Since $\overline{X}$ is a diagonal between the Sylow 3-subgroups of $\overline{M_{\alpha+1} \cap L_\mu^1}$ and $\overline{M_{\alpha+1} \cap L_\mu^2}$ and $XR_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$ is a Sylow 3-subgroup of $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha Q_{\alpha+1}/R_\alpha Q_{\alpha+1}$, we get

$$|[Z_{\alpha+1}, X]| \geq |[Z_{\alpha+1} \cap U_\mu, X]T_\mu/T_\mu| = |[Z_{\alpha+1} \cap U_\alpha, X]T_\alpha/T_\alpha|^2 = |[Z_{\alpha+1}, X]|^2.$$

Consequently,

$$(*) \quad [Z_{\alpha+1}, X] = 1.$$

In particular, $L_\alpha^1 R_\alpha/R_\alpha$ is not isomorphic to $\mathsf{G}_2(2)'$ or $\mathrm{PSL}_4(2)$.

Suppose that $L_\alpha^1 R_\alpha/R_\alpha \cong \mathrm{Sp}_4(2)'$. By $(*)$, $(M_{\alpha+1} \cap L_\alpha^1)R_\alpha/R_\alpha$ is the centralizer of a 1-dimensional subspace in the natural $\mathrm{Sp}_4(2)$-module. By looking at the 5-dimensional indecomposable $\mathrm{Sp}_4(2)$-module W with a 1-dimensional trivial submodule and a natural module as factor module, we get that this centralizer even centralizes the centralizer of its largest normal 2-subgroup in W. Hence

$$[Z_{\alpha+1} \cap U_\alpha, M_{\alpha+1} \cap L_\alpha^1] = 1.$$

Note that if W^* is the 5-dimensional indecomposable $\mathrm{Sp}_4(2)$-module with a natural module N as submodule and a trivial module as factor module, then the centralizer in W^* of the largest normal 2-subgroup of the centralizer in $\mathrm{Sp}_4(2)'$ of a 1-dimensional subspace in N is contained in N. Therefore

$$Z_\alpha = U_\alpha T_\alpha.$$

But then

$$[Z_{\alpha+1}, M_{\alpha+1} \cap L_\alpha^1] = [(Z_{\alpha+1} \cap U_\alpha)T_\alpha, M_{\alpha+1} \cap L_\alpha^1] = 1.$$

On the other hand, $M_{\alpha+1} \cap L_\alpha^1$ switches $Z_{\alpha+1} \cap U_\mu^1$ and $Z_{\alpha+1} \cap U_\mu^2$, a contradiction. Hence $L_\alpha^1 R_\alpha / R_\alpha \cong \mathrm{PSL}_3(2)$. Then $U_\alpha^1 T_\alpha / T_\alpha$ is a natural $\mathrm{SL}_3(2)$-module, and by $(*)$ $M_{\alpha+1} \cap L_\alpha^1$ centralizes a 1-dimensional subspace in this module. Similarly to the above, we get

$$Z_\alpha = U_\alpha T_\alpha.$$

Then the case $U_\alpha \cap T_\alpha = 1$ leads to a similar contradiction as above. Hence U_α^1 is a 4-dimensional indecomposable module for L_α^1, with a 1-dimensional trivial submodule and a natural $\mathrm{PSL}_3(2)$-module as factor module. Now

$$(**) \quad [Z_{\alpha+1}, M_{\alpha+1} \cap L_\alpha^1] = U_\alpha^1 \cap T_\alpha.$$

In particular,

$$|[Z_{\alpha+1}, M_{\alpha+1} \cap L_\alpha^1]| = 2.$$

But also

$$(***) \quad |[Z_{\alpha+1} \cap U_\mu^1, M_{\alpha+1} \cap L_\alpha^1] T_\mu / T_\mu| = 2.$$

It follows that $M_{\alpha+1} \cap L_\alpha^1$ centralizes $U_\mu^1 \cap T_\mu$. If $M_{\alpha+1} \cap L_\alpha^1$ normalizes L_μ^j, for some $j \in \{1, \ldots, m\}$, then $M_{\alpha+1} \cap L_\alpha^1$ also normalizes U_μ^j and thus centralizes $U_\mu^j \cap T_\mu$. Hence

$$T_\mu \le C_{Z_{\alpha+1}}(M_{\alpha+1} \cap L_\alpha^1) = T_\alpha (Z_{\alpha+1} \cap U_\alpha^2) \cdot \ldots \cdot (Z_{\alpha+1} \cap U_\alpha^m).$$

Since $T_\mu \trianglelefteq M_{\alpha+1}$, this implies $T_\mu \le T_\alpha$, i.e.,

$$T_\mu = T_\alpha,$$

contrary to $(**)$ and $(***)$. ∎

(2.2.9) Assume that $L_\alpha^1 R_\alpha / R_\alpha \cong A_7$ and U_α^1 is the module listed in (A.2.2)(m). Then $M_{\alpha+1} \cap L_\alpha^1 \le \widehat{M}_{\alpha+1}$.

Proof. Let $\mu \in \Delta(\alpha+1)$. Suppose that $M_{\alpha+1} \cap L_\alpha^1$ has an orbit of size $s > 1$ on $\{L_\mu^1, \ldots, L_\mu^m\}$. Without loss we may assume that $\{L_\mu^1, \ldots, L_\mu^s\}$ is such an orbit. Then $M_{\alpha+1} \cap L_\alpha^1$ also permutes $U_\mu^1 \cap Z_{\alpha+1}, \ldots, U_\mu^s \cap Z_{\alpha+1}$ transitively. Then (A.1.1)(b) and $[Z_{\alpha+1}, M_{\alpha+1} \cap L_\alpha^1] \leq U_\alpha^1 \cap Z_{\alpha+1}$ imply that $s = 2$ and

$$(*) \quad [(U_\mu^1 \cap Z_{\alpha+1})(U_\mu^2 \cap Z_{\alpha+1}), M_{\alpha+1} \cap L_\alpha^1] = U_\alpha^1 \cap Z_{\alpha+1}.$$

Pick $x \in M_{\alpha+1} \cap L_\alpha^1$ with $(L_\mu^1)^x = L_\mu^2$. Note that $s = 2$ and (A.1.1)(c) imply that $[(U_\mu^1 \cap Z_{\alpha+1})(U_\mu^2 \cap Z_{\alpha+1}), M_{\alpha+1} \cap L_\alpha^1]$ is a diagonal between $U_\mu^1 \cap Z_{\alpha+1}$ and $U_\mu^2 \cap Z_{\alpha+1}$. Together with $(L_\mu^1)^{x^2} = L_\mu^1$, $(L_\mu^2)^{x^2} = L_\mu^2$, and $(*)$ it follows that

$$[(U_\mu^1 \cap Z_{\alpha+1})(U_\mu^2 \cap Z_{\alpha+1}), x^2] = 1.$$

Hence x acts on $(U_\mu^1 \cap Z_{\alpha+1})(U_\mu^2 \cap Z_{\alpha+1})$ as an involution. Now $(*)$ implies that

$$[U_\alpha^1 \cap Z_{\alpha+1}, x] = 1,$$

a contradiction to (C.2.1). ■

(2.2.10) $M_{\alpha+1} \cap L_\alpha^1 \leq \widehat{M}_{\alpha+1}$.

Proof. This follows from (2.2.4), (2.2.5), (2.2.6), (2.2.7), (2.2.8), and (2.2.9). ■

(2.2.11) Put

$$\widehat{G} := \widehat{M}_{\alpha+1} L_\alpha^1 \quad \text{and}$$

$$\widehat{R} := \bigcap_{g \in \widehat{G}} \widehat{M}_{\alpha+1}^g.$$

Then the following hold:

(a) No nontrivial characteristic subgroup of $\widehat{M}_{\alpha+1}$ is normal in $\widehat{G}$.

(b) $\widehat{R} \cap L_\alpha^1 = R_\alpha \cap L_\alpha^1$.

(c) $[L_\alpha^1, \widehat{R}] \neq 1$.

(d) L_α^1 is the unique minimal member of $\{N \trianglelefteq \widehat{G} \mid N\widehat{R}/\widehat{R} = \mathrm{Soc}(\widehat{G}/\widehat{R})\}$. In particular, $\mathrm{Soc}(\widehat{G}/\widehat{R})$ is a perfect simple group.

Proof. (a) Let C be a characteristic subgroup of $\widehat{M}_{\alpha+1}$ that is normal in $\widehat{G}$. Since $\widehat{M}_{\alpha+1}$ is a characteristic subgroup of $M_{\alpha+1}$, so is C. From

$$G_\alpha = \langle M_{\alpha+1}, L_\alpha^1\rangle = \langle M_{\alpha+1}, \widehat{G}\rangle$$

it follows that C is normal in G_α. Hence $C = 1$.

(b) By (2.2.10), $R_\alpha \cap L_\alpha^1 \leq M_{\alpha+1} \cap L_\alpha^1 \leq \widehat{M}_{\alpha+1}$. Since $R_\alpha \cap L_\alpha^1 \trianglelefteq \widehat{G}$, we get

$$R_\alpha \cap L_\alpha^1 \leq \widehat{R} \cap L_\alpha^1.$$

Suppose that $R_\alpha \cap L_\alpha^1 \neq \widehat{R} \cap L_\alpha^1$. Then $L_\alpha^1 \leq \widehat{R}$, since $L_\alpha^1/R_\alpha \cap L_\alpha^1$ is simple. But then $G_\alpha = \langle M_{\alpha+1}, L_\alpha^1\rangle = M_{\alpha+1}$, a contradiction.

(c) Note that (2.2.2)(c) implies $Z_\alpha \leq \widehat{R}$. Hence $[L_\alpha^1, \widehat{R}] \geq [L_\alpha^1, Z_\alpha] = U_\alpha^1 \neq 1$.

(d) Let $\widehat{L}$ be a normal subgroup of $\widehat{G}$ that is minimal subject to $\widehat{L}\widehat{R}/\widehat{R} = \mathrm{Soc}(\widehat{G}/\widehat{R})$. Then

$$[L_\alpha^1, \widehat{L}]\widehat{R}/\widehat{R} = L_\alpha^1\widehat{R}/\widehat{R}.$$

In particular, $[L_\alpha^1, \widehat{L}] \not\leq O_p(\widehat{G})$. Since L_α^1 is a p-component of $\widehat{G}$, this implies

$$L_\alpha^1 \leq \widehat{L}.$$

Hence, by (c),

$$[\widehat{R}, \widehat{L}] \neq 1.$$

Let $\widehat{H}$ be he group of automorphisms of $\widehat{G}$ that are induced by $N_{M_{\alpha+1}}(L_\alpha^1)$. Note that (b) and (A.1.2)(b)(d) imply that each minimal normal subgroup of $\widehat{G}/\widehat{R}$ is isomorphic to $L_\alpha^1 R_\alpha/R_\alpha$ and normalized by $\widehat{H}$. Now from (2.1.1)(a-c) applied to $(\widehat{G}, \widehat{M}_{\alpha+1}, \widehat{R}, \widehat{L})$ instead of $(G_\alpha, M_{\alpha+1}, R_\alpha, L_\alpha)$ it follows that

- $\widehat{L}$ is the product of the p-components of $\widehat{G}$ that are not contained in $\widehat{R}$,

- $\widehat{M}_{\alpha+1}$ permutes these p-components transitively.

Since L^1_α is a p-component of $\widehat{G}$ with $\widehat{R} \not\geq L^1_\alpha \trianglelefteq \widehat{G}$, we get $\widehat{L} = L^1_\alpha$. ■

From (2.2.2)(a) and (2.2.11) it follows that (I)–(V) are satisfied for $\widehat{M}_{\alpha+1}L^1_\alpha$, $\widehat{M}_{\alpha+1}$, L^1_α, and the group of automorphisms of $\widehat{M}_{\alpha+1}L^1_\alpha$ that are induced by $N_{M_{\alpha+1}}(L^1_\alpha)$ in place of G, M, L, and H, respectively.

2.3

In this section we assume (I)–(V) and

(VI) $m = 1$,

(VII) $L_\alpha R_\alpha / R_\alpha$ is not isomorphic to $\mathrm{PSL}_n(q)$.

(2.3.1) Assume that $U_\alpha \cap T_\alpha \neq 1$. Then $p = 2$ and one of the following holds:

(a) (a1) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{PSp}_{2n}(2^k)'$ for some $n, k \in \mathbb{N}$ with $n \geq 2$,

(a2) $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a parabolic subgroup of type C_{n-1} of $L_\alpha R_\alpha/R_\alpha$,

(a3) $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{Sp}_{2n}(2^k)$-module for L_α,

(a4) $|U_\alpha \cap T_\alpha| \leq 2^k$.

(b) (b1) $L_\alpha R_\alpha/R_\alpha \cong \mathsf{G}_2(2^k)'$, for some $k \in \mathbb{N}$,

(b2) $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a rank 1 parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$ corresponding to the node of the long simple root in the Dynkin diagram,

(b3) $U_\alpha T_\alpha/T_\alpha$ is the irreducible $\mathsf{G}_2(2^k)$-module described in (A.2.2)(i) for L_α,

(b4) $|U_\alpha \cap T_\alpha| \leq 2^k$.

(c) (c1) $L_\alpha R_\alpha/R_\alpha \cong A_{2n}$ for some $n \in \mathbb{N}$ with $n > 2$,

(c2) $U_\alpha T_\alpha/T_\alpha$ is a natural A_{2n}-module over $GF(2)$ for L_α, (c3) $|U_\alpha \cap T_\alpha| = 2$.

Proof. First assume that $L_\alpha R_\alpha/R_\alpha$ is an alternating group. Then $p = 2$ by (VII). By (A.2.6) $U_\alpha T_\alpha/T_\alpha$ is irreducible, and from [14](1.5) it follows that (c) holds.

Now assume that $L_\alpha R_\alpha/R_\alpha$ is not an alternating group. Let s is the number of nontrivial composition factors of U_α, regarded as a module for L_α. By [10], [14](1.5), and (A.2.6) one of the following holds:

(i) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{PSp}_{2n}(2^k)'$ and $|U_\alpha \cap T_\alpha| \leq 2^{ks}$, for some $n, k \in \mathbb{N}$ with $n \geq 2$.

(ii) $L_\alpha R_\alpha/R_\alpha \cong \mathsf{G}_2(2^k)'$ and $|U_\alpha \cap T_\alpha| \leq 2^k$, for some $k \in \mathbb{N}$.

(iii) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{U}_4(2)$ and $|U_\alpha \cap T_\alpha| \leq 4$.

(iv) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{U}_n(3)$, for some $n \in \mathbb{N}$ with $n \geq 4$.

(v) $L_\alpha R_\alpha/R_\alpha \cong \Omega^-_{2n}(3)$, for some $n \in \mathbb{N}$ with $n \geq 3$.

Put $k := 1$ if (iv) or (v) holds, and $k := 2$ if (iii) holds. By (2.2.4) $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a maximal parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$. Let $i \in \{1, \ldots, n\}$ be such that the type of this parabolic subgroup is obtained by omitting the i-th node from the Dynkin diagram. Note that in the cases (i), (iii), (iv), and (v) $Q_{\alpha+1}$ contains an offending subgroup for at least i natural modules only if (i) holds and $i = 1$. Moreover, even in this case $Q_{\alpha+1}$ contains no offending subgroup for more than one natural module. (This can be seen by considering matrices, obtained e.g. from the description of the natural modules given in Appendix B.) In particular, if $i = 1$ then $s = 1$ and (i) or (ii) holds, and therefore (a) or (b) holds. Hence suppose that

$$i > 1.$$

Since $\langle U_\mu \cap T_\mu \mid \mu \in \Delta(\alpha + 1)\rangle$ is a nontrivial characteristic subgroup of $M_{\alpha+1}$, there exists $\mu \in \Delta(\alpha + 1)$ such that

$$U_\alpha \cap T_\alpha \not\leq T_\mu.$$

Put

$$Y := (U_\alpha \cap T_\alpha)T_\mu/T_\mu.$$

Suppose that $i = 2$ and $k = 1$. By the remark above, $s = 1$. Hence $U_\mu T_\mu/T_\mu$ contains a unique irreducible $(M_{\alpha+1} \cap L_\mu)$-submodule, and that one is a natural $\mathrm{SL}_2(p)$-module. Suppose that Y does not contain this module. Then Y is an $(M_{\alpha+1} \cap L_\mu)$-complement to $U_\mu T_\mu/T_\mu$ in the L_μ-submodule $(U_\mu T_\mu/T_\mu)Y$ of Z_μ/T_μ. Since $M_{\alpha+1} \cap L_\mu$ contains a Sylow p-subgroup of L_μ, it follows that $U_\mu T_\mu/T_\mu$ has an L_μ-complement in $(U_\mu T_\mu/T_\mu)Y$, contrary to $C_{Z_\mu/T_\mu}(L_\mu) = 1$.

Hence Y contains a natural $\mathrm{SL}_2(p)$-submodule for $M_{\alpha+1} \cap L_\mu$. Hence (i) and (ii) are impossible and $p = 3$. By the action of $\mathrm{SL}_2(3)$ on its natural module it follows that

$$(*) \quad [U_\alpha \cap T_\alpha, (M_{\alpha+1} \cap L_\mu)''] \neq 1.$$

On the other hand, $(M_{\alpha+1} \cap L_\mu)' \leq R_\alpha$ by the structure of G_α/R_α, and

$$(M_{\alpha+1} \cap L_\mu)'' \leq C_{R_\alpha}(U_\alpha, T_\alpha),$$

by (1.1.2)(d). But

$$C_{R_\alpha}(U_\alpha, T_\alpha) = C_{R_\alpha}(U_\alpha),$$

again by (1.1.2)(d) and the Three-Subgroup Lemma, a contradiction to $(*)$.

Hence $i > 2$ or $k > 1$. Let K be the p-component of $M_{\alpha+1} \cap L_\mu$ that corresponds to the nodes $1, \ldots, i-1$ of the Dynkin diagram of $L_\mu R_\mu/R_\mu$. Then $K \leq L_\alpha$ or $K \leq R_\alpha$ by (2.2.6)(a). In the latter case it follows from $K = K'$ and arguments similar to the above that K centralizes U_α. Hence in any case

$$(**) \quad [U_\alpha \cap T_\alpha, K] = 1.$$

If $s = 1$, then it follows as above that Y contains a nontrivial K-submodule, contrary to $(**)$. Hence $s > 1$. In particular, (ii) does not hold. Pick an L_μ-submodule X of

$$E := \langle (U_\alpha \cap T_\alpha)^{L_\mu} \rangle T_\mu/T_\mu$$

which is maximal subject to

$$X \cap Y = 1.$$

Let X_1 denote the L_μ-submodule of E with $X \leq X_1$ and $X_1/X = C_{E/X}(L_\mu)$. Then X has an $(M_{\alpha+1} \cap L_\mu)$-complement in the L_μ-module $X(Y \cap X_1)$, namely $Y \cap X_1$. Since $M_{\alpha+1} \cap L_\mu$ contains a Sylow p-subgroup of L_μ, it follows that X has an L_μ-complement in $X(Y \cap X_1)$. Now (2.1.1)(f) and the choice of X imply that $X_1 = X$, i.e.,

$$C_{E/X}(L_\mu) = 1.$$

Pick an L_μ-submodule X_2 of E such that $X \leq X_2$ and X_2/X is irreducible. Then X_2/X is a natural module for L_μ. Moreover, $YX/X \cap X_2/X$ is a nontrivial K-submodule of X_2/X, contrary to (**). ■

(2.3.2) $U_\alpha T_\alpha/T_\alpha$ is irreducible as L_α-module.

Proof. Suppose that $U_\alpha T_\alpha/T_\alpha$ is not irreducible as L_α-module. Then by (2.3.1)

$$U_\alpha \cap T_\alpha = 1.$$

Let X be an irreducible L_α-submodule of U_α and Y an irreducible $L_{\alpha'}$-submodule of $U_{\alpha'}$. From $[X, L_\alpha] \neq 1$ and $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$ it follows that $X \not\leq Q_{\alpha'}$. Hence

$$(*) \quad L_{\alpha'} \leq \langle X^{L_{\alpha'}} \rangle.$$

Note that $[Y, L_{\alpha'}] \neq 1$ implies that $C_X(Y) \leq R_{\alpha'}$, i.e.,

$$(**) \quad C_X(Y) = X \cap Q_{\alpha'} = C_X(U_{\alpha'}).$$

By (A.2.5) X is $U_{\alpha'}$-invariant. From (VII) and (A.2.6) it follows that there exists a nondegenerate $GF(p)$-bilinear form on X that is invariant under $U_{\alpha'}$. Now (A.1.5)(a) and (**) imply that

$$(***) \quad [X, Y] = [X, U_{\alpha'}].$$

Again by (A.2.5) Y is X-invariant. But then by $(*)$ and $(***)$

$$[U_{\alpha'}, L_{\alpha'}] \leq [U_{\alpha'}, \langle X^{L_{\alpha'}} \rangle] \leq Y,$$

i.e., $U_{\alpha'} = Y$. Hence $U_{\alpha'}$ is irreducible. But then also U_α is irreducible, a contradiction.

■

Chapter 3

Determining the action of L on R, Part 1

In this chapter we assume (I)–(IV) and

(V) L_α is perfect.

(VI) One of the following holds:

(1) $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^+(q)$ $(q = p^k)$ for some $n, k \in \mathbb{N}$ with $n \geq 3$, and $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{O}_{2n}^+(q)$-module for G_α.

(2) $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^-(q)$ $(q = p^k)$ for some $n, k \in \mathbb{N}$ with $n \geq 3$, and $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{O}_{2n}^-(q)$-module for G_α.

(3) $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n+1}(q)$ $(q = p^k$ odd$)$ for some $n, k \in \mathbb{N}$, and $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{O}_{2n+1}(q)$-module for G_α.

(4) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{U}_n(q)$ $(q = p^k)$ for some $n, k \in \mathbb{N}$ with $n \geq 4$, and $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{SU}_n(q)$-module for L_α.

(5) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{PSp}_{2n}(q)$ $(q = p^k)$ for some $n, k \in \mathbb{N}$ with $n \geq 2$, and $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{Sp}_{2n}(q)$-module for L_α.

(VII) $(Z_{\alpha+1} \cap U_\alpha)T_\alpha/T_\alpha$ is a singular subspace of $U_\alpha T_\alpha/T_\alpha$ with $|(Z_{\alpha+1} \cap U_\alpha)T_\alpha/T_\alpha| > q$.

Let r be the $GF(q)$-dimension of the singular subspace $(Z_{\alpha+1} \cap U_\alpha)T_\alpha/T_\alpha$ of $U_\alpha T_\alpha/T_\alpha$. Since $M_{\alpha+1} \cap L_\alpha$ is the stabilizer in L_α of this singular subspace, either $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a maximal parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$ or case (1) in (VI) holds and $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a parabolic subgroup of cotype $\{n-1, n\}$. When $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a maximal parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$, let t be its cotype. Note that (VII) implies $r > 1$ and hence $t \neq 1$.

Let s be the number of connected components of the Dynkin diagram of $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$. Define $\tau_1, \ldots, \tau_s$, corresponding to these connected components, as follows: If $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a maximal parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$, put

$$\tau_1 := \begin{cases} \{1, \ldots, t-1\} & \text{if } 1 < t < n-1 \\ \{1, \ldots, n-2\} & \text{if } t = n-1 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is not of type } \mathsf{D}_n \\ \{1, \ldots, n-2, n\} & \text{if } t = n-1 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is of type } \mathsf{D}_n \\ \{1, \ldots, n-1\} & \text{if } t = n \end{cases}$$

$$\tau_2 := \begin{cases} \{t+1, \ldots, n\} & \text{if } t < n-2 \\ \{n-1, n\} & \text{if } t = n-2 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is not of type } \mathsf{D}_n \\ \{n-1\} & \text{if } t = n-2 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is of type } \mathsf{D}_n \\ \{n\} & \text{if } t = n-1 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is not of type } \mathsf{D}_n \end{cases}$$

$$\tau_3 := \{n\} \quad \text{if } t = n-2 \text{ and } L_\alpha R_\alpha/R_\alpha \text{ is of type } \mathsf{D}_n.$$

If $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is not a maximal parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$, put

$$\tau_1 := \{1, \ldots, n-2\}.$$

For each $\mu \sim \alpha$ and $\lambda \in \Delta(\mu)$ let $L_{\mu,\lambda,1}, \ldots, L_{\mu,\lambda,s}$ be subgroups of $M_\lambda \cap L_\mu$ containing $L_\mu \cap R_\mu$ such that, for each $i \in \{1, \ldots, s\}$, $L_{\mu,\lambda,i}R_\mu/R_\mu$ is a Levi complement of a parabolic subgroup of type τ_i in $L_\mu R_\mu/R_\mu$.

For each $\gamma \sim \alpha$ define

$$\Lambda(\gamma) := \{\mu \in \Delta^{(2)}(\gamma) \mid U_\mu Z_\gamma \ntrianglelefteq G_\gamma\},$$

$$X_{\gamma,\lambda} := \langle [U_\mu, Q_\lambda] \mid \mu \in \Delta(\lambda) \cap \Lambda(\gamma) \rangle Z_\gamma \quad \text{for each } \lambda \in \Delta(\gamma),$$

$$X_\gamma := \langle X_{\gamma,\lambda} \mid \lambda \in \Delta(\gamma) \rangle,$$

$$Y_{\gamma,\lambda} := \langle U_\mu \mid \mu \in \Delta(\lambda) \cap \Lambda(\gamma) \rangle Z_\gamma \quad \text{for each } \lambda \in \Delta(\gamma) \text{ and}$$

$$Y_\gamma := \langle U_\mu \mid \mu \in \Lambda(\gamma) \rangle Z_\gamma = \langle Y_{\gamma,\lambda} \mid \lambda \in \Delta(\gamma) \rangle.$$

3.1

(3.1.1) (a) $U_\alpha \cap T_\alpha = 1$.

(b) $[U_\alpha, U_{\alpha'}]$ is an isotropic subspace of U_α.

(c) $$[U_\alpha, U_{\alpha'}] \cap Z_{\alpha+1} \geq \begin{cases} q^3 & \text{if } L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q) \text{ and } r = n \\ q^2 & \text{if } L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q) \text{ and } U_{\alpha'} \leq L_\alpha R_\alpha \\ q^2 & \text{if } L_\alpha R_\alpha / R_\alpha \cong \Omega^-_{2n}(q) \text{ and } U_{\alpha'} \leq L_\alpha R_\alpha \\ q & \text{else} \end{cases}.$$

(d) Let X be a subspace of U_α with $X \cap [U_\alpha, Q_{\alpha+1}] = 1$. If $x \in X$, then

$$\{g \in Q_{\alpha+1} \mid x \in X^g\} = C_{Q_{\alpha+1}}(x).$$

Moreover, if $x \neq 1$ then

$$|Q_{\alpha+1} \cap L_\alpha : C_{Q_{\alpha+1} \cap L_\alpha}(x)| = \begin{cases} q^{2n-r-1} & \text{if (1) holds in (VI)} \\ q^{2n-r-1} & \text{if (2) holds in (VI)} \\ q^{2n-r} & \text{if (3) holds in (VI)} \\ q^{2n-r-1} & \text{if (4) holds in (VI)} \\ q^{2n-r} & \text{if (5) holds in (VI)} \end{cases}$$

(e) If A is a normal p-subgroup of $M_{\alpha+1}$, then one of the following holds:

(e1) $[U_\alpha, Q_{\alpha+1}] = [U_\alpha, A]$,

(e2) $[U_\alpha, A]$ is an isotropic subspace of U_α.

(f) $C_{Q_{\alpha+1}}([U_\alpha, Q_{\alpha+1}]) = C_{Q_{\alpha+1}}(U_\alpha, U_\alpha \cap Z_{\alpha+1})$.

(g) $|U_\alpha \cap Q_{\alpha'}| = |U_{\alpha'} \cap Q_\alpha|$.

(h) If X is a proper $M_{\alpha+1}$-submodule of U_α, then $U_\alpha \cap Z_{\alpha+1} \leq X \leq [U_\alpha, Q_{\alpha+1}]$.

In particular,

(h1) If $\mu \in \Delta(\alpha+1)$, then $U_\alpha \cap T_\mu = 1$ or $U_\alpha \cap Z_{\alpha+1} \leq T_\mu$.

(h2) If $L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q)$ and $r = n$, then $U_\alpha \cap Z_{\alpha+1}$ is the unique proper $M_{\alpha+1}$-submodule of U_α.

(i) Assume that X is a proper $M_{\alpha+1}$-submodule of U_α with $U_\alpha \cap Z_{\alpha+1} \neq X \neq [U_\alpha, Q_{\alpha+1}]$. Then

(i1) $L_\alpha R_\alpha / R_\alpha \cong \Omega_{2n}^{\varepsilon}(q)$ ($\varepsilon \in \{+,-\}$) and $r = n-1$,

(i2) X is a maximal isotropic subspace of U_α.

Proof. (a) This follows from (VI) and (2.3.1).

(b) This follows from (A.1.5)(a).

(c) Suppose that $[U_\alpha, U_{\alpha'}] \cap Z_{\alpha+1} = 1$. Since $[U_\alpha, Q_{\alpha+1}, Q_{\alpha+1}] \leq Z_{\alpha+1}$, it follows that $U_{\alpha'}$ centralizes $[U_\alpha, Q_{\alpha+1}]$. Now (a) and (A.1.5)(a) imply that

$$[U_\alpha, U_{\alpha'}] \leq C_{U_\alpha}(Q_{\alpha+1}) \leq Z_{\alpha+1},$$

contrary to $[U_\alpha, U_{\alpha'}] \neq 1$. Hence

$$|[U_\alpha, U_{\alpha'}] \cap Z_{\alpha+1}| \geq q.$$

The rest of (c) follows from (B.5.1.6)(a)(c) and (B.5.2.5)(a).

(d) If $x, y \in X$, $g \in Q_{\alpha+1}$ and $x = y^g$, then $[y,g] \in X \cap [U_\alpha, Q_{\alpha+1}] = 1$ and hence $[x,g] = 1$.

Since $M_{\alpha+1}$ acts transitively on $\{xC_{U_\alpha}(Q_{\alpha+1} \cap L_\alpha) \mid x \in U_\alpha \setminus [U_\alpha, Q_{\alpha+1} \cap L_\alpha]\}$, it suffices to check the rest of (d) for only one element $x \in U_\alpha \setminus [U_\alpha, Q_{\alpha+1} \cap L_\alpha]$. This can be done by a simple matrix calculation.

(g) From (A.1.5)(a) it follows that

$$|U_\alpha \cap Q_{\alpha'}| = |C_{U_\alpha}(U_{\alpha'})| = |U_\alpha : [U_\alpha, U_{\alpha'}]| =$$

$$|U_{\alpha'} : [U_\alpha, U_{\alpha'}]| = |C_{U_{\alpha'}}(U_\alpha)| = |U_{\alpha'} \cap Q_\alpha|.$$

(e), (f), (h), and (i) follow by matrix calculations, using e.g. the description of the natural modules given in Appendix B. ■

(3.1.2) If $\mu \in \Delta(\alpha+1)$ and $j \in \{1, \ldots, s\}$, then $O^p(L_{\mu,\alpha+1,j}) \leq R_\mu R_\alpha (M_{\alpha+1} \cap L_\alpha)$.

Proof. First assume that $L_{\mu,\alpha+1,j}R_\mu/R_\mu$ is perfect. Then $L_{\mu,\alpha+1,j} \le R_\mu K$ for some p-component K of $M_{\alpha+1}$. Hence (2.2.6)(a) implies that

$$L_{\mu,\alpha+1,j} \le R_\mu R_\alpha L_\alpha \cap M_{\alpha+1} = R_\mu R_\alpha (L_\alpha \cap M_{\alpha+1}).$$

Now assume that $L_{\mu,\alpha+1,j}R_\mu/R_\mu$ is not perfect, and suppose that $O^p(L_{\mu,\alpha+1,j}) \not\leq R_\mu R_\alpha L_\alpha$. Then one of the following holds:

(1) $q = 2$ and $L_{\mu,\alpha+1,j}R_\mu/R_\mu \cong \Sigma_3$,

(2) $q = 3$ and $L_{\mu,\alpha+1,j}R_\mu/R_\mu \cong \mathrm{SL}_2(3)$,

(3) $q = 3$ and $L_{\mu,\alpha+1,j}R_\mu/R_\mu \cong \mathrm{PSL}_2(3)$,

(4) $q = 2$ and $L_{\mu,\alpha+1,j}R_\mu/R_\mu \cong \mathrm{U}_3(2)$ or $\mathrm{SU}_3(2)$.

Suppose that (1) holds. Then $L_{\mu,\alpha+1,j}$ induces a Σ_3 of graph automorphisms on $L_\alpha R_\alpha/R_\alpha$. This is only possible if $L_\alpha R_\alpha/R_\alpha$ is of type D_4, $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$, and $L_{\mu,\alpha+1,j}$ permutes these three A_1's transitively. On the other hand, the centralizer of $U_\alpha \cap Z_{\alpha+1}$ in $M_{\alpha+1} \cap L_\alpha$ is (modulo R_α) of type $\mathsf{A}_1 \times \mathsf{A}_1$, and $L_{\mu,\alpha+1,j}$ normalizes it. This contradiction shows that (1) does not hold.
If (2) or (3) holds, then $L_{\mu,\alpha+1,j}$ induces at least an A_4 of graph automorphisms on $L_\alpha R_\alpha/R_\alpha$, a contradiction similar to the above.
If (4) holds, then $L_{\mu,\alpha+1,j}$ induces at least $\mathrm{U}_3(2)$ or $\mathrm{SU}_3(q)$ of outer automorphisms on $L_\alpha R_\alpha/R_\alpha$. But $\mathrm{Out}(L_\alpha R_\alpha/R_\alpha)$ is isomorphic to Σ_3 or C_2, a contradiction. ∎

(3.1.3) Let $\lambda \in \Delta(\alpha)$. Assume that $U_{\alpha'}R_\alpha \cap L_\alpha R_\alpha = R_\alpha$ and $U_{\alpha'} \not\leq M_\lambda$. Then the following hold:

(a) $G_\alpha = \langle M_\lambda, U_{\alpha'} \rangle$.

(b) Assume that $b > 2$ and $\mu \in \Delta(\lambda) \cap \Lambda(\alpha)$. Then $(\mu, \alpha' - 2)$ is a critical pair.

Proof. (a) Note that $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a maximal subgroup of $L_\alpha R_\alpha/R_\alpha$, unless $L_\alpha R_\alpha/R_\alpha \cong \Omega^+_{2n}(q)$ and $r = n-1$. If $L_\alpha R_\alpha/R_\alpha \cong \Omega^+_{2n}(q)$ and $r = n-1$, then $U_{\alpha'} \not\leq L_\alpha R_\alpha$ implies that $U_{\alpha'}$ switches the two maximal subgroups of $L_\alpha R_\alpha/R_\alpha$ containing $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$. Hence in any case $M_{\alpha+1}$ is a maximal subgroup of G_α. Then also M_λ is maximal in G_α, and (a) follows from $U_{\alpha'} \not\leq M_\lambda$.

(b) Suppose that $(\mu, \alpha'-2)$ is not a critical pair.

Assume that $[U_\alpha, U_{\alpha'}] \not\leq [U_\mu, U_{\alpha'} \cap Q_\alpha]$. Then $[U_\alpha, U_{\alpha'}] \cap [U_\mu, U_{\alpha'} \cap Q_\alpha] = 1$, since $|[U_\alpha, U_{\alpha'}]| = 2$. Now (A.1.5)(b) implies that $U_{\alpha'}$ normalizes $U_\mu U_\alpha$, a contradiction to (a). Hence

$$[U_\alpha, U_{\alpha'}] \leq [U_\mu, U_{\alpha'} \cap Q_\alpha].$$

In particular, $[U_\alpha, U_{\alpha'}] \leq U_\alpha \cap U_\mu$, whence $U_\alpha \cap U_\mu$ is normalized by $U_{\alpha'}$. But then (a) implies that $U_\alpha = U_\mu$, a contradiction to $\mu \in \Lambda(\alpha)$. ■

(3.1.4) Let $\lambda \in \Delta(\alpha)$. Assume that $U_{\alpha'}R_\alpha \cap L_\alpha R_\alpha \neq R_\alpha$ and

$$(*) \quad |[U_\alpha, Q_{\alpha+1}] \cap Z_\lambda| \leq \begin{cases} q & \text{if } L_\alpha R_\alpha/R_\alpha \cong \Omega^+_{2n}(q) \text{ and } r = n \\ 1 & \text{else} \end{cases}.$$

Then the following hold:

(a) $G_\alpha = \langle M_\lambda, U_{\alpha'} \rangle$.

(b) Assume that $b > 2$ and $|[U_\alpha, Q_{\alpha+1}] \cap Z_\lambda| = 1$. If $\mu \in \Delta(\lambda) \cap \Lambda(\alpha)$ and $[U_\mu, Q_\alpha] = [U_\mu, Q_\lambda]$, then $(\mu, \alpha'-2)$ is a critical pair.

(c) Assume that $b > 2$. If $\mu \in \Delta(\lambda) \cap \Lambda(\alpha)$ and $[U_\mu, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] = 1$, then $(\mu, \alpha'-2)$ is a critical pair.

(d) Assume that the following hold:

(i) $b > 2$.

(ii) If (γ, γ') is any critical pair, then $U_\gamma U_{\gamma+2} \not\trianglelefteq G_\gamma$ and $U_{\gamma+2} \cap Z_{\gamma+1} \leq T_\gamma$, where $\gamma+1 \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$ and $\gamma+2 \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$.

Then $b = 4$.

(e) Assume that the following hold:

(i) $b = 4$.

(ii) There exists $\mu \in \Delta(\lambda)$ such that $(\mu, \alpha' - 2)$ is a critical pair.

Then $U_\mu \cap Z_\lambda \leq T_\alpha$.

Proof. (a) Suppose that

$$E := \langle M_\lambda, U_{\alpha'} \rangle \neq G_\alpha.$$

First assume that $|[U_\alpha, Q_{\alpha+1}] \cap Z_\lambda| \neq 1$. Then $L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q)$ and $r = n$. By (3.1.1)(h), this implies that

$$[U_\alpha, Q_{\alpha+1}] = U_\alpha \cap Z_{\alpha+1}$$

Since M_λ is a maximal subgroup of G_α, it follows that $U_{\alpha'}$ is contained in M_λ and hence normalizes Z_λ. Thus

$$[(U_\alpha \cap Z_{\alpha+1})(U_\alpha \cap Z_\lambda), U_{\alpha'}] = [U_\alpha \cap Z_\lambda, U_{\alpha'}] \leq [U_\alpha, U_{\alpha'}] \cap Z_\lambda \leq U_\alpha \cap Z_{\alpha+1} \cap Z_\lambda.$$

This means that $U_{\alpha'} R_\alpha / R_\alpha$ is contained in the largest normal p-subgroup of $N_{L_\alpha}(U_\alpha \cap Z_{\alpha+1} \cap Z_\lambda) R_\alpha / R_\alpha$, which is a parabolic subgroup of cotype 1 of $L_\alpha R_\alpha / R_\alpha$. But in $O^+_{2n}(q)$ no offending subgroup for the natural module is contained in the largest normal p-subgroup of a parabolic subgroup of cotype 1, by (B.5.1.4). This contradiction shows that

$$(*') \quad U_\alpha = [U_\alpha, Q_{\alpha+1}] \times (U_\alpha \cap Z_\lambda).$$

Assume that $U_{\alpha'}$ is contained in M_λ and hence normalizes Z_λ. Since $U_{\alpha'} \leq Q_{\alpha+1}$, it follows from $(*')$ that

$$(**) \quad [U_\alpha, U_{\alpha'}] = [U_\alpha, Q_{\alpha+1}, U_{\alpha'}].$$

Note that $[U_\alpha, Q_{\alpha+1}, Q_{\alpha+1} \cap L_\alpha R_\alpha] \leq Z_{\alpha+1}$. Then (**) and (3.1.1)(f) imply that $[U_\alpha, U_{\alpha'} \cap L_\alpha R_\alpha] = 1$, a contradiction to $U_{\alpha'} R_\alpha \cap L_\alpha R_\alpha \neq R_\alpha$.

Now assume that $U_{\alpha'}$ is not contained in M_λ. Then M_λ is not a maximal subgroup of G_α. It follows that $L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q)$, $r = n-1$, and E is the stabilizer in G_α of a maximal singular subspace X of U_α with $U_\alpha \cap Z_\lambda \leq X$. Moreover, as in the proof of (3.1.3) it follows that $U_{\alpha'} \leq L_\alpha R_\alpha$, since $M_{\alpha+1}$ is not a maximal subgroup of G_α. Note that (B.5.1.5) implies that

$$U_{\alpha'} R_\alpha / R_\alpha \cap Z(Q_{\alpha+1} R_\alpha / R_\alpha) \neq 1,$$

since $U_{\alpha'}$ acts as an offending subgroup on U_α. Pick $u \in U_{\alpha'} \setminus R_\alpha$ such that $[u, Q_{\alpha+1}] \leq R_\alpha$. Then $[U_\alpha, u]$ is a subspace of $U_\alpha \cap Z_{\alpha+1}$ with $|[U_\alpha, u]| \geq q^2$. Since $[U_\alpha, Q_{\alpha+1}, u] = 1$, it follows from $(*')$ that $[U_\alpha, u] \leq X$. But then

$$(U_\alpha \cap Z_{\alpha+1}) \cap (U_\alpha \cap Z_\lambda) \neq 1,$$

since $|X : U_\alpha \cap Z_\lambda| = q$. Now $(*')$ implies that

$$U_\alpha \cap Z_{\alpha+1} \not\leq [U_\alpha, Q_{\alpha+1}],$$

a contradiction.

(b),(c) Suppose this is false. First assume that $[U_\mu, Q_\alpha] = [U_\mu, Q_\lambda]$ and $[U_\alpha, Q_{\alpha+1}] \cap Z_\lambda = 1$. Then (A.1.5)(a) implies that $C_{U_\mu}(Q_\alpha) = U_\mu \cap Z_\lambda$. Hence

$$U_\mu \cap U_\alpha = U_\mu \cap U_\alpha \cap Z_\lambda.$$

Together with $[U_\alpha, Q_{\alpha+1}] \cap Z_\lambda = 1$ we get

$$(**) \quad [C_{U_{\alpha'}}(U_\alpha), U_\mu] \cap [U_{\alpha'}, U_\alpha] \leq [Q_\alpha, U_\mu] \cap [U_{\alpha'}, U_\alpha] \leq U_\mu \cap [U_\alpha, Q_{\alpha+1}] = 1.$$

Since $(\mu, \alpha' - 2)$ is not a critical pair, $U_\mu \leq Q_{\alpha'-2} \leq G_{\alpha'}$. Then it follows from (**) and (A.1.5)(b) that

$$[U_{\alpha'}, U_\mu U_\alpha] \leq U_\mu U_\alpha.$$

Now $\mu \in \Lambda(\alpha)$ implies that $L_\alpha \not\leq \langle M_\lambda, U_{\alpha'} \rangle$, contrary to (a).

Now assume that $[U_\mu, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] = 1$. It follows that

$$[C_{U_{\alpha'}}(U_\alpha), U_\mu] \cap [U_{\alpha'}, U_\alpha] = [U_{\alpha'} \cap Q_\alpha, U_\mu] \cap [U_{\alpha'}, U_\alpha] = 1,$$

and we get the same contradiction as above.

(d) Suppose that $b > 4$. Pick $h_\alpha \in L_\alpha$ such that $\lambda = (\alpha+1)^{h_\alpha}$. Put $\alpha - 1 := \lambda$ and $\alpha - 2 := (\alpha+2)^{h_\alpha}$. If $U_{\alpha+2} \cap U_\alpha \neq 1$, then $U_{\alpha+2} \cap Z_{\alpha+1} \leq U_\alpha$ by (3.1.1)(h), contrary to (ii) and (3.1.1)(a). Thus

$$U_{\alpha+2} \cap U_\alpha = 1.$$

Note that $(*)$ is satisfied for λ^g in place of λ for each $g \in M_{\alpha+1}$. Together with $U_{\alpha-2} \cap U_\alpha = (U_{\alpha+2} \cap U_\alpha)^{h_\alpha} = 1$ and (c) it follows that $((\alpha-2)^g, \alpha' - 2)$ is a critical pair for each $g \in M_{\alpha+1}$.

Since $(\alpha - 2, \alpha - 1)$ is conjugate to $(\alpha, \alpha+1)$, there exists $\alpha - 3 \in \Delta(\alpha - 2)$ such that

$$(*') \quad |[U_{\alpha-2}, Q_{\alpha-1}] \cap Z_{\alpha-3}| \leq \begin{cases} q & \text{if } L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q) \text{ and } r = n \\ 1 & \text{else} \end{cases}.$$

Pick $h_{\alpha-2} \in L_{\alpha-2}$ such that $\alpha - 3 = (\alpha-1)^{h_{\alpha-2}}$. Put $\alpha - 4 := \alpha^{h_{\alpha-2}}$. As above we get $U_{\alpha-4} \cap U_{\alpha-2} = 1$. Note that $(*')$ is satisfied for $((\alpha-3)^g, (\alpha-2)^g, (\alpha-1)^g)$ in place of $(\alpha - 3, \alpha - 2, \alpha - 1)$. Hence, if $g \in M_{\alpha+1}$, then (c) implies that $((\alpha-4)^g, \alpha'-4)$ is a critical pair, provided $U_{\alpha'-2} R^g_{\alpha-2} \cap L^g_{\alpha-2} R^g_{\alpha-2} \neq R^g_{\alpha-2}$. Note that $\alpha - 4 \in \Lambda(\alpha-2)$, since $\alpha \in \Lambda(\alpha+2)$ and $(\alpha-4, \alpha-2) = (\alpha^{h_\alpha h_{\alpha-2}}, (\alpha+2)^{h_\alpha h_{\alpha-2}})$. Hence, if $g \in M_{\alpha+1}$, then (3.1.3) implies that $((\alpha-4)^g, \alpha'-4)$ is a critical pair, provided $U_{\alpha'-2} R^g_{\alpha-2} \cap L^g_{\alpha-2} R^g_{\alpha-2} = R^g_{\alpha-2}$. For each $g \in M_{\alpha+1}$ define

$$Y_g := \begin{cases} U^g_{\alpha-4} & \text{if } [U^g_{\alpha-4}, U_{\alpha'-4}] \leq Z^g_{\alpha-3} \\ [U^g_{\alpha-4}, Q_{\alpha-3}] & \text{if } [U^g_{\alpha-4}, U_{\alpha'-4}] \not\leq Z^g_{\alpha-3} \end{cases}.$$

Let A be a set of representatives for the cosets of $Q_\alpha \cap L_\alpha$ in $Q_{\alpha+1} \cap L_\alpha$. Let E be a complement to $U_\alpha \cap Z_{\alpha+1} \cap Z_{\alpha-1}$ in $U_\alpha \cap Z_{\alpha-1}$. Hence, by $(*)$,

$$|U_\alpha \cap Z_{\alpha-1} : E| \leq q \quad \text{and}$$

$$E = U_\alpha \cap Z_{\alpha-1} \quad \text{unless } L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q) \text{ and } r = n.$$

Choose $e \in \mathbb{N}$ such that

$$q^e = \min\{|[Y_a, U_{\alpha'-4}] \cap E^a| \mid a \in A\}.$$

Define

$$X_a := \{(x,a) \mid 1 \neq x \in [Y_a, U_{\alpha'-4}] \cap E^a\} \quad \text{for each } a \in A \text{ and}$$

$$X := \bigcup_{a \in A} [Y_a, U_{\alpha'-4}] \cap E^a \setminus \{1\}.$$

Then

$$|\bigcup_{a \in A} X_a| = \sum_{a \in A} |X_a| \geq |A|(q^e - 1).$$

Note that, by (3.1.1)(h), $[U_\alpha, Q_{\alpha+1}] = U_\alpha \cap Z_{\alpha+1}$, if $[U_\alpha, Q_{\alpha+1}] \cap Z_{\alpha-1} \neq 1$. Hence

$$E^a \cap [U_\alpha, Q_{\alpha+1}] = 1 \quad \text{for each } a \in A.$$

Therefore by (3.1.1)(d) each $x \in X$ is contained in at most $q^{-2n+r+1}|A|$ of the subgroups $[Y_a, U_{\alpha'-4}] \cap E^a$ with $a \in A$, if (1), (2) or (4) holds in (VI), and in at most $q^{-2n+r}|A|$ of these subgroups, if (3) or (5) holds in (VI). Hence

$$|\bigcup_{a \in A} X_a| = \sum_{x \in X} |\{a \in A \mid (x,a) \in X_a\}| \leq$$

$$\begin{cases} q^{-2n+r+1}|A|\,|X| & \text{, in the cases (1),(2) and (4)} \\ q^{-2n+r}|A|\,|X| & \text{, in the cases (3) and (5)} \end{cases}.$$

Therefore,

$$(**) \quad |X| \geq \begin{cases} (q^e - 1)q^{2n-r-1} & \text{, in the cases (1),(2) and (4)} \\ (q^e - 1)q^{2n-r} & \text{, in the cases (3) and (5)} \end{cases}.$$

Suppose that $b > 6$. Then

$$W := \langle Y_a \mid a \in A \rangle$$

acts quadratically or trivially on $U_{\alpha'-4}$, even if $b = 8$. By (A.1.5)(a), it follows that $[W, U_{\alpha'-4}]$ is an isotropic subspace of $U_{\alpha'-4}$. Hence

$$(***) \quad |X| \leq |[W, U_{\alpha'-4}]| - 1 \leq q^n - 1.$$

From $(**)$ and $(***)$ it follows that $r = n$, $e = 1$ and in (VI) neither (3) nor (5) holds. Since $r = n$ is impossible in case (2), either (1) or (4) holds. Also $r = n$ implies that

$$[U^a_{\alpha-4}, U_{\alpha'-4}] \leq Z^a_{\alpha-3}, \quad \text{for each } a \in A.$$

In particular, $Y_a = U^a_{\alpha-4}$ for each $a \in A$. From (ii) we get

$$U_{\alpha-4} \cap Z_{\alpha-3} = (U_\alpha \cap Z_{\alpha-1})^{h_{\alpha-2}} = U_\alpha \cap Z_{\alpha-1}$$

and hence

$$[Y_a, U_{\alpha'-4}] \leq U_\alpha \cap Z^a_{\alpha-1} \quad \text{for each } a \in A.$$

Therefore (3.1.1)(c) gives a contradiction to $e = 1$ in case (1), even if $E \neq U_\alpha \cap Z_{\alpha-1}$. Hence (4) holds and

$$|U_{\alpha'-4}/U_{\alpha'-4} \cap Q_{\alpha-4}| \leq q < q^2 = |[U_{\alpha-4}, U_{\alpha'-4}]| = |U_{\alpha-4}/C_{U_{\alpha-4}}(U_{\alpha'-4})|,$$

a contradiction.

Hence $b = 6$. In particular, $Y_a^g = Y_{ag}$, for all $a, g \in M_{\alpha+1}$. Thus

$$Y := \langle [Y_h, U_{\alpha+2}] \mid h \in M_{\alpha+1} \rangle$$

is an $M_{\alpha+1}$-submodule of $U_{\alpha+2}$. On the other hand, since $U_{\alpha-4} \cap Z_{\alpha-3} = (U_\alpha \cap Z_{\alpha-1})^{h_{\alpha-2}} \leq T_{\alpha-2}$ and α is conjugate to $\alpha - 4$ under $L_{\alpha-2}$, Y is an $M_{\alpha+1}$-submodule of U_α. Moreover, $(*)$ implies that this submodule is not contained in $[U_\alpha, Q_{\alpha+1}]$, since $U_\alpha \cap Z_{\alpha-1} = U_{\alpha-4} \cap Z_{\alpha-3}$ and

$$|[Y_1, U_{\alpha+2}] \cap Z_{\alpha-3}| \geq \begin{cases} q^3 & \text{if } L_\alpha R_\alpha / R_\alpha \cong \Omega^+_{2n}(q) \text{ and } r = n \\ q & \text{else} \end{cases}.$$

But $[U_\alpha, Q_{\alpha+1}]$ is the unique maximal $M_{\alpha+1}$-submodule of U_α by (3.1.1)(h). Hence

$$U_\alpha = Y \le U_{\alpha+2},$$

a contradiction.

(e) Suppose that $U_\mu \cap Z_\lambda \not\le T_\alpha$. Then

$$(**) \quad U_\mu \cap T_\alpha = 1$$

by (3.1.1)(h). Put

$$E := \langle([U_\mu, U_{\alpha+2}] \cap Z_\lambda)^{M_{\alpha+1}}\rangle T_\alpha.$$

Note that $E \le Z_{\alpha+2}$ and so $U_\alpha \not\le E$. Hence (3.1.1)(h) implies that

$$E \le [U_\alpha, Q_{\alpha+1}]T_\alpha.$$

Together with $(*)$ it follows that

$$|([U_\mu, U_{\alpha+2}] \cap Z_\lambda)T_\alpha/T_\alpha| \le$$

$$|[U_\alpha, Q_{\alpha+1}]T_\alpha/T_\alpha \cap Z_\lambda/T_\alpha| \le \begin{cases} q & \text{if } L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^+(q) \text{ and } r = n \\ 1 & \text{else} \end{cases}.$$

Therefore,

$$(***) \quad |[U_\mu, U_{\alpha+2}] \cap Z_\lambda| \le \begin{cases} q & \text{if } L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^+(q) \text{ and } r = n \\ 1 & \text{else} \end{cases}$$

by $(**)$. Since (3.1.1)(g) implies that (α, α') is an arbitrary critical pair, we can apply (3.1.1)(c) to $(\mu, \alpha+2)$ in place of (α, α') and get a contradiction to $(***)$. ■

(3.1.5) Let $\lambda \in \Delta(\alpha)$ and $\mu \in \Delta(\lambda)$ such that $U_\mu \cap Z_\lambda \le T_\alpha$.

(a) $O^p(L_{\mu,\lambda,1}) \le R_\alpha R_\mu$.

(b) $U_\mu \le O^p(R_\alpha)$.

Proof. Put $A := O^p(L_{\mu,\lambda,1})R_\mu R_\alpha$. Then (3.1.2) implies that

$$A = R_\mu R_\alpha(A \cap L_\alpha).$$

Note that $B := L_{\mu,\lambda,1}Q_\lambda$ is a normal subgroup of M_λ with $C_B(U_\mu \cap Z_\lambda) = Q_\lambda$. Therefore, by $U_\mu \cap Z_\lambda \leq T_\alpha$,

$$A = [A, L_{\mu,\lambda,1}]R_\mu R_\alpha = [A \cap L_\alpha, L_{\mu,\lambda,1}]R_\mu R_\alpha \leq C_B(U_\mu \cap Z_\lambda)R_\mu R_\alpha = Q_\lambda R_\mu R_\alpha.$$

Since $A/R_\mu R_\alpha$ has no nontrivial p-factor group, we get $A = R_\mu R_\alpha$. Hence (a) holds. Note that $r > 1$ implies that $[U_\mu, O^p(L_{\mu,\lambda,1}] \not\leq [U_\mu, Q_\lambda]$. Therefore, by (3.1.1)(h),

$$U_\mu = [U_\mu, \langle O^p(L_{\mu,\lambda,1})^{M_\lambda}\rangle].$$

Hence (b) follows from (a). ∎

3.2

In addition to (I)–(VII) we now assume

(VIII) If $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^\varepsilon(q)$ $(\varepsilon \in \{+,-\})$, then $r < n-1$.

In particular, $M_{\alpha+1}$ is a maximal subgroup of G_α.

(3.2.1) (a) $Q_{\alpha+1} \leq Q_\alpha L_\alpha$.

(b) If $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^\varepsilon(q)$ $(\varepsilon \in \{+,-\})$, then $n \geq 4$.

Proof. (a) Suppose that $Q_{\alpha+1} \not\leq Q_\alpha L_\alpha$. Then $Q_{\alpha+1} \not\leq R_\alpha L_\alpha$ by (1.1.2)(e). Pick $a \in Q_{\alpha+1} \setminus R_\alpha L_\alpha$. Then a induces an outer automorphism on $L_\alpha R_\alpha/R_\alpha$ witch centralizes the parabolic subgroup $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ modulo its largest normal p-subgroup. Hence (A.4.1) applies. Since a is a p-element, case (a) of (A.4.1) does not hold. Thus a induces a graph automorphism that fixes each node in the Dynkin diagram that belongs to $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$. But then (VIII) implies that this graph automorphism fixes each node in the Dynkin diagram and hence is trivial, a contradiction.

(b) Since $r > 1$, this follows from (VIII). ∎

(3.2.2) Let $(\gamma, \gamma+1, \ldots, \gamma+b)$ be a path such that $(\gamma, \gamma+b)$ is a critical pair. Then there exists $g \in L_\gamma$ such that the following hold:

(a) The intersection of the parabolic subgroups $(M_{\gamma+1} \cap L_\gamma)R_\gamma/R_\gamma$ and $(M_{\gamma+1} \cap L_\gamma)^g R_\gamma/R_\gamma$ of $L_\gamma R_\gamma/R_\gamma$ is the product of a Cartan subgroup and a Levi complement in both of them.

(b) If $b > 2$, then $(\nu, \gamma+b-2)$ is a critical pair, for all $x \in Q_{\gamma+1}$ and $\nu \in \Delta((\gamma+1)^g x) \cap \Lambda(\gamma)$.

Proof. Note that (3.1.1)(g) implies that (α, α') is an arbitrary critical pair. Hence it suffices to prove this for $(\gamma, \gamma') = (\alpha, \alpha')$.

Since $(M_{\gamma+1} \cap L_\gamma)R_\gamma/R_\gamma$ is a parabolic subgroup of $L_\gamma R_\gamma/R_\gamma$, there exists a root system Φ for $L_\gamma R_\gamma/R_\gamma$ such that

$$(M_{\gamma+1} \cap L_\gamma)R_\gamma/R_\gamma = QLH,$$

where Q is the product of the root subgroups corresponding to the positive roots which are not contained in the root system Ψ spanned by the simple roots that belong to the nodes $1, \ldots, t-1, t+1, \ldots, n$ of the Dynkin diagram, L is generated by the root subgroups corresponding to the roots in Ψ, and H is a Cartan subgroup in the normalizer of the product of the root subgroups corresponding to the positive roots.

Let Q^- be the product of the root subgroups corresponding to the negative roots which are not contained in Ψ. Then Q^-LH is a parabolic subgroup of $L_\gamma R_\gamma/R_\gamma$ that has the same type as QLH by (VIII). Hence there exists $g \in L_\gamma$ such that

$$(M_{\gamma+1} \cap L_\gamma)^g R_\gamma/R_\gamma = Q^-LH.$$

Then (a) holds.

Assume that $b > 2$. Pick $x \in Q_{\alpha+1}$. Note that by (a)

$$(*) \quad [U_\gamma, Q_{\gamma+1}] \cap Z_{\gamma+1}^{gx} = 1.$$

Let $\nu \in \Delta((\gamma+1)^g) \cap \Lambda(\gamma)$. From (3.1.4)(b) and (*) it follows that $(\nu, \gamma+b-2)$ is a critical pair, provided $[U_\nu, Q_{\gamma+1}^{gx}] = [U_\nu, Q_\gamma]$. Assume that

$$[U_\nu, Q_{\gamma+1}^{gx}] \neq [U_\nu, Q_\gamma].$$

Then (VIII) and (3.1.1)(i) imply that

$$[U_\nu, Q_\gamma] = U_\nu \cap Z_{\gamma+1}^g.$$

Together with (*) it follows that

$$[U_\nu, Q_\gamma] \cap [U_\gamma, U_{\gamma+b}] \leq [U_\nu, Q_\gamma] \cap [U_\gamma, Q_{\gamma+1}] \leq Z_{\gamma+1}^g \cap [U_\gamma, Q_{\gamma+1}] = 1.$$

Now (3.1.4)(c) implies that $(\nu, \gamma+b-2)$ is a critical pair. ■

Choose $g_\alpha \in L_\alpha$ such that (a) and (b) of (3.2.2) are satisfied for $(\alpha, \alpha', g_\alpha)$ in place of (γ, γ', g). Define

$$\alpha - 1 := (\alpha+1)^{g_\alpha} \quad \text{and}$$

$$\alpha - 2 := (\alpha+2)^{g_\alpha}.$$

(3.2.3) $b \leq 4$.

Proof. Suppose that $b > 4$. Note that (3.2.2) and (1.2.1)(e) imply that if (γ, γ') is any critical pair and $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$, then $U_\gamma U_\mu \ntrianglelefteq G_\gamma$. Hence it follows from $b > 4$ and (3.1.4)(d) that there exists a critical pair (γ, γ') such that $U_\mu \cap Z_\lambda \nleq T_\gamma$, where $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$ and $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$. Since (3.1.1)(g) implies that (α, α') is an arbitrary critical pair, we may assume that this is the case for $(\alpha, \alpha') = (\gamma, \gamma')$. Then, by (3.1.1)(h),

$$(*) \quad U_{\alpha-2} \cap T_\alpha = 1.$$

Let A be a set of representatives for the cosets of Q_α in $Q_{\alpha+1}$. Choose $e \in \mathbb{N}$ such that

$$q^e = \min\{|[U_{\alpha-2}^a, U_{\alpha'-2}] \cap Z_{\alpha-1}^a| \mid a \in A\}.$$

Define

$$X_a := \{(x,a) \mid 1 \neq x \in [U^a_{\alpha-2}, U_{\alpha'-2}] \cap Z^a_{\alpha-1}\}, \quad \text{for each } a \in A, \text{ and}$$

$$X := \bigcup_{a\in A} [U^a_{\alpha-2}, U_{\alpha'-2}] \cap Z^a_{\alpha-1} \setminus \{1\}.$$

Then

$$|\bigcup_{a\in A} X_a| = \sum_{a\in A} |X_a| \geq |A|(q^e - 1).$$

On the other hand, (3.2.1)(a), (3.1.1)(d), and $(*)$ imply that each $x \in X$ is contained in at most $q^{-2n+r+1}|A|$ of the subgroups $[U^a_{\alpha-2}, U_{\alpha'-2}]$ with $a \in A$, if (1), (2) or (4) holds in (VI), and in at most $q^{-2n+r}|A|$ of these subgroups, if (3) or (5) holds in (VI). Hence

$$|\bigcup_{a\in A} X_a| =$$

$$\sum_{x\in X} |\{a \in A \mid (x,a) \in X_a\}| \leq \begin{cases} q^{-2n+r+1}|A|\,|X| & \text{,in the cases (1),(2) and (4)} \\ q^{-2n+r}|A|\,|X| & \text{,in the cases (3) and (5)} \end{cases}.$$

Therefore,

$$(**) \quad |X| \geq \begin{cases} (q^e-1)q^{2n-r-1} & \text{, in the cases (1),(2) and (4)} \\ (q^e-1)q^{2n-r} & \text{, in the cases (3) and (5)} \end{cases}.$$

Since $b > 4$, V_α acts quadratically on $U_{\alpha'-2}$. By (A.1.5)(a), it follows that $[V_\alpha, U_{\alpha'-2}]$ is an isotropic subspace of $U_{\alpha'-2}$. Hence

$$(***) \quad |X| \leq |[V_\alpha, U_{\alpha'-2}]| - 1 \leq q^n - 1.$$

From $(**)$ and $(***)$ it follows that $r = n$, $e = 1$ and in (VI) neither (3) nor (5) holds. By (VIII) and $r = n$ we get that (4) holds. Then $e = 1$ implies that

$$|U_{\alpha'-2}/U_{\alpha'-2} \cap Q_{\alpha-2}| \leq q < q^2$$

$$= |[U_{\alpha-2}, U_{\alpha'-2}]| = |U_{\alpha-2}/C_{U_{\alpha-2}}(U_{\alpha'-2})| = |U_{\alpha-2}/U_{\alpha-2} \cap Q_{\alpha'-2}|,$$

a contradiction. ∎

Choose $g_{\alpha-2} \in L_{\alpha-2}$ such that (a) and (b) of (3.2.2) are satisfied for $(\alpha-2, \alpha'-2, g_{\alpha-2})$ in place of (γ, γ', g). Define

$$\alpha - 3 := (\alpha - 1)^{g_{\alpha-2}} \quad \text{and}$$

$$\alpha - 4 := \alpha^{g_{\alpha-2}}.$$

(3.2.4) Assume that $b = 4$. Let $i \in \{1, 2\}$, $j \in \{1, \ldots, s\}$ and $\mu \in \Delta(\alpha - 2i + 1)$.

(a) $O^p(L_{\mu,\alpha-2i+1,j}) \leq R_\mu Q_{\alpha-2i+1}(M_{\alpha-2i+1} \cap M_{\alpha-2i+3})$. Hence all p'-elements of some Levi complement of $(M_{\alpha-2i+1} \cap L_\mu)R_\mu/R_\mu$ in $L_\mu R_\mu/R_\mu$ are contained in $(M_{\alpha-2i+1} \cap M_{\alpha-2i+3})R_\mu/R_\mu$.

(b) $O^p(L_{\alpha-2i+4,\alpha-2i+3,j}) \leq R_{\alpha-2i+4}Q_{\alpha-2i+3}(M_{\alpha-2i+1} \cap M_{\alpha-2i+3})$. Hence all p'-elements of some Levi complement of $(M_{\alpha-2i+3} \cap L_{\alpha-2i+4})R_{\alpha-2i+4}/R_{\alpha-2i+4}$ in $L_{\alpha-2i+4}R_{\alpha-2i+4}/R_{\alpha-2i+4}$ are contained in $(M_{\alpha-2i+1} \cap M_{\alpha-2i+3})R_{\alpha-2i+4}/R_{\alpha-2i+4}$.

Proof. Since the choice of $g_{\alpha-2i+2}$ implies that $M_{\alpha-2i+1} \cap L_{\alpha-2i+2} \leq (M_{\alpha-2i+1} \cap M_{\alpha-2i+3})Q_{\alpha-2i+1}$, it follows from (3.1.2) that

$$O^p(L_{\mu,\alpha-2i+1,j}) \leq R_\mu Q_{\alpha-2i+1}(M_{\alpha-2i+1} \cap M_{\alpha-2i+3}).$$

Hence (a) holds. The proof of (b) is similar. ■

(3.2.5) Assume that $b = 4$. Let $i \in \{1, 2\}$ and $\mu \in \Delta(\alpha - 2i + 1) \cap \Lambda(\alpha - 2i + 2)$.

(a) $U_\mu \cap Z_{\alpha-2i+1} = [U_\mu, U_{\alpha-2i+4}]$.

(b) $U_\mu \cap Z_{\alpha-2i+1} \leq T_{\alpha-2i+2}$.

(c) $O^p(L_{\mu,\alpha-2i+1,1}) \leq R_\mu R_{\alpha-2i+2}$.

(d) $Q_{\alpha-2i+1} = Q_\mu Q_{\alpha-2i+2}$.

Proof. (a) By (3.2.2), $(\mu, \alpha - 2i + 4)$ is a critical pair. Hence $U_{\alpha-2i+4}R_\mu/R_\mu$ is a nontrivial elementary abelian p-subgroup of $(M_{\alpha-2i+1} \cap L_\mu)R_\mu/R_\mu$. Moreover, (3.2.4) implies that $U_{\alpha-2i+4}R_\mu/R_\mu$ is normalized by all p'-elements of some Levi complement of $(M_{\alpha-2i+1} \cap L_\mu)R_\mu/R_\mu$. Since $r > 1$, it follows that (a) holds.

(b) Note that we can apply (3.1.4) with $(\mu, \alpha - 2i + 1, \alpha - 2i + 2, \alpha' - 2i + 2)$ in place of (μ, α, α'). Hence (b) follows from (3.1.4)(e).

(c) This follows from (b) and (3.1.5).

(d) Note that $r > 1$ implies that

$$Q_{\alpha-2i+1}R_\mu = [Q_{\alpha-2i+1}, O^p(L_{\mu,\alpha-2i+1,1})]R_\mu.$$

Also, by (c),

$$[Q_{\alpha-2i+1}, O^p(L_{\mu,\alpha-2i+1,1})] \leq [Q_{\alpha-2i+1}, R_{\alpha-2i+2}R_\mu] \leq [Q_{\alpha-2i+1}, R_{\alpha-2i+2}]R_\mu \leq$$

$$Q_{\alpha-2i+2}R_\mu.$$

Therefore,

$$Q_{\alpha-2i+1} = Q_{\alpha-2i+2}(R_\mu \cap Q_{\alpha-2i+1} = Q_{\alpha-2i+2}Q_\mu.$$

■

(3.2.6) Assume that $b = 4$. Then $[X_\alpha, Q_\alpha] \leq T_\alpha$.

Proof. From (3.2.5)(b) we get $[X_{\alpha,\alpha-1}, Q_\alpha] \leq T_\alpha$. Now the claim follows from the definition of X_α. ■

(3.2.7) Assume that $b = 4$. Let μ and ν be vertices with $\mu \sim \alpha \sim \nu$ and $d(\mu, \nu) = 2$. Then $X_\mu \leq Q_\nu$ or $X_\nu \leq Q_\mu$.

Proof. Suppose that $X_\mu \not\leq Q_\nu$ and $X_\nu \not\leq Q_\mu$. Let λ be the common neighbor of μ and ν. Without loss we may assume that

$$|X_\nu \cap Q_\mu| \leq |X_\mu \cap Q_\nu|.$$

Note that (3.2.6) implies that X_μ/Z_μ is a module for G_μ/Q_μ.

Suppose that $[X_\mu, L_\mu] \leq Z_\mu$. Then $X_\mu = X_{\mu,\lambda}$. Now $b > 2$ implies $X_\mu \leq Q_\nu$, a contradiction. Hence

$$[X_\mu, L_\mu] \not\leq Z_\mu.$$

From this we get that $C_{G_\mu}(X_\mu, Z_\mu) \leq R_\mu$, and so

$$C_{X_\nu}(X_\mu, Z_\mu) = X_\nu \cap Q_\mu.$$

By (3.2.6),

$$X_\mu \cap Q_\nu \leq C_{X_\mu}(X_\nu, Z_\mu).$$

Hence X_μ/Z_μ is an FF-module for G_μ/Q_μ, and X_ν acts as an offending subgroup on X_μ/Z_μ.

Let E be a subgroup of $[X_\mu, L_\mu]Z_\mu$ containing Z_μ such that E/Z_μ is a maximal $L_\mu R_\mu$-submodule of $[X_\mu, L_\mu]Z_\mu/Z_\mu$. Then $[X_\mu, L_\mu]Z_\mu/E$ is an irreducible FF-module for $L_\mu R_\mu$, and X_ν acts as an offending subgroup on $[X_\mu, L_\mu]Z_\mu/E$.

Suppose that $\nu \notin \Lambda(\mu)$. Then (ν, ρ) is not a critical pair, for any $\rho \in \Delta^{(2)}(\mu)$, whence $V_\mu \leq Q_\nu$. In particular, $X_\mu \leq Q_\nu$, a contradiction. Hence $\nu \in \Lambda(\mu)$. Now (3.2.5)(d) implies that

$$O^p(L_{\nu,\lambda,1}) \leq R_\nu R_\mu.$$

Suppose that $[[X_\mu, L_\mu], O^p(L_{\nu,\lambda,1})] \not\leq ER_\nu$. Then it follows from $O^p(L_{\nu,\lambda,1}) \leq R_\nu R_\mu$., (1.1.2)(d), and (A.3.1) that $[X_\mu, L_\mu]Z_\mu/E$, regarded as L_μ-module, is the direct sum of at least r irreducible submodules. But then Q_λ, being the largest normal p-subgroup of the stabilizer of an r-dimensional singular subspace of the natural module, contains no offending subgroup for $[X_\mu, L_\mu]Z_\mu/E$, a contradiction.

Hence $O^p(L_{\nu,\lambda,1})$ centralizes $[X_\mu, L_\mu]Z_\mu/E$ modulo R_ν. Note that

$$X_\mu \leq C_{Q_\lambda}(Q_\lambda, Q_\nu),$$

since $X_\mu \trianglelefteq M_\lambda$ and, by (3.2.6), $X'_\mu \leq Q_\nu$. Now the structure of $C_{Q_\lambda}(Q_\lambda, Q_\nu)/Q_\nu$ as $O^p(L_{\nu,\lambda,1})$-module implies that EQ_ν/Q_ν is a hyperplane in $[X_\mu, L_\mu]Q_\nu/Q_\nu$. Therefore $(EQ_\nu \cap [X_\mu, L_\mu]Z_\mu)/E$ is a hyperplane in $[X_\mu, L_\mu]Z_\mu/E$, and X_ν centralizes it by (3.2.6). But this is a contradiction to $X_\nu \trianglelefteq M_\lambda$ and (VII). ■

(3.2.8) Assume that $b = 4$.

(a) $X_\alpha \leq Q_\mu$ for each $\mu \in \Delta^{(2)}(\alpha)$. In particular, X_α is elementary abelian.

(b) $[Y_\alpha, Q_\alpha] \leq X_\alpha$.

Proof. (a) Suppose that $X_\alpha \not\leq Q_\mu$, for some $\mu \in \Delta^{(2)}(\alpha)$. Then (1.1.2)(a) implies that

$$L_\mu \leq \langle X_\alpha^{L_\mu} \rangle.$$

From (3.2.7) and (3.2.6) we get $[X_\mu, X_\alpha] \leq [Q_\alpha, X_\alpha] \leq T_\alpha \leq Z_\mu$, and therefore

$$[X_\mu, L_\mu] \leq [X_\mu, \langle X_\alpha^{L_\mu} \rangle] \leq Z_\mu.$$

Since μ is conjugate to α, this implies

$$[X_\alpha, L_\alpha] \leq Z_\alpha.$$

Hence, if λ is the common neighbor of α and μ, then $X_\alpha = X_{\alpha,\lambda}$, a contradiction to $X_\alpha \not\leq Q_\mu$.

(b) Since $Q_\alpha \leq Q_\lambda$, for each $\lambda \in \Delta(\alpha)$, (b) follows from the definition of Y_α and X_α. ■

(3.2.9) Assume that $b = 4$.

(a) $Y'_\alpha \leq T_\alpha$.

(b) $(Y_\alpha \cap Q_{\alpha-2})R_{\alpha-4}/R_{\alpha-4} \leq Z(Q_{\alpha-3}R_{\alpha-4}/R_{\alpha-4})$.

Proof. (a) From (3.2.8)(b) it follows that

$$Y'_\alpha \leq X_\alpha.$$

Pick $\mu \in \Delta(\alpha-1) \cap \Lambda(\alpha)$. Then $Y'_\alpha \leq X_\alpha$ and (3.2.8)(a) imply that $Y_\alpha R_\mu / R_\mu$ is an elementary abelian normal p-subgroup of $M_{\alpha-1}/R_\mu$. Hence by $r > 1$

$$Y_\alpha R_\mu / R_\mu \leq Z(Q_{\alpha-1} R_\mu / R_\mu).$$

Now it follows from (3.2.5) that

$$[U_\mu, Y_\alpha] \leq U_\mu \cap Z_{\alpha-1} \leq T_\alpha.$$

Since L_α acts transitively on $\Delta(\alpha)$ and normalizes Y_α, (a) follows.

(b) Note that by (a) $(Y_\alpha \cap Q_{\alpha-2}) R_{\alpha-4}/R_{\alpha-4}$ is an elementary abelian subgroup of $Q_{\alpha-3} R_{\alpha-4}/R_{\alpha-4}$, and (3.2.4) implies that $(Y_\alpha \cap Q_{\alpha-2}) R_{\alpha-4}/R_{\alpha-4}$ is normalized by each p'-element of some Levi complement of $(M_{\alpha-3} \cap L_{\alpha-4}) R_{\alpha-4}/R_{\alpha-4}$. Now the claim follows from $r > 1$. ∎

(3.2.10) $b = 2$.

Proof. Suppose that $b > 2$. From (3.2.9)(b) and (3.2.5)(b) it follows that

$$[Y_\alpha \cap Q_{\alpha-2}, U_{\alpha-4}] \leq T_{\alpha-2} \leq X_\alpha.$$

Note that (3.2.9)(a) implies that $Y_\alpha R_{\alpha-2}/R_{\alpha-2}$ is an elementary abelian normal p-subgroup of $M_{\alpha-1}/R_{\alpha-2}$. By $r > 1$, we get that

$$|Y_\alpha : Y_\alpha \cap Q_{\alpha-2}| = |Y_\alpha R_{\alpha-2}/R_{\alpha-2}| \leq q^{\frac{1}{2}r(r+1)}.$$

Therefore,

$$(*) \quad |Y_\alpha : C_{Y_\alpha}(U_{\alpha-4}, X_\alpha)| \leq q^{\frac{1}{2}r(r+1)}.$$

If $[Y_\alpha, L_\alpha] \leq X_\alpha$, then $U_{\alpha-2} \leq Y_\alpha = Y_{\alpha,\alpha+1} \leq Q_{\alpha+2}$, a contradiction. Hence

$$[Y_\alpha, L_\alpha] \not\leq X_\alpha.$$

Let W be a $GF(p)L_\alpha R_\alpha$-composition factor of Y_α/X_α that is not centralized by L_α. Let D be an irreducible $GF(p)R_\alpha$-submodule of W. From $r > 1$ and (3.2.5)(c) it follows that Y_α/X_α is the direct sum of r-dimensional absolutely irreducible $GF(q)R_\alpha$-modules, each of which is irreducible as $GF(p)R_\alpha$-module. Hence $|D| = q^r$ and

$$\mathrm{End}_{GF(p)R_\alpha}(D) \cong GF(q).$$

From (A.3.1) (with $R_\alpha L_\alpha/Q_\alpha$, R_α/Q_α, $L_\alpha Q_\alpha/Q_\alpha$, $GF(p)$, $GF(q)$, W and D in place of G, A, B, F, K, V and X, respectively) we get that

$$W \cong D \otimes_{GF(q)} E$$

for some irreducible $GF(q)L_\alpha$-module E. Thus, regarded as $GF(p)L_\alpha$-module, W is the direct sum of r copies of E. Together with $(*)$ this implies that

$$(**) \quad |E : C_E(U_{\alpha-4})| \leq q^{\frac{1}{2}(r+1)}.$$

On the other hand,

$$|U_{\alpha-4} : C_{U_{\alpha-4}}(E)| = |U_{\alpha-4}R_\alpha/R_\alpha| \geq q^r,$$

since (3.2.4) implies that $U_{\alpha-4}R_\alpha/R_\alpha$ is normalized by each p'-element of some Levi complement of $M_{\alpha-1}/R_\alpha$. In particular, E is an FF-module and $U_{\alpha-4}$ acts as an offending subgroup. Note that the 'exceptional' irreducible FF-modules for the orthogonal groups (cases (g) and (n) in (A.2.2)) have no quadratically acting offender in common with the natural module. Moreover, if E is as in (A.2.2)(d) then $|E : C_E(U_{\alpha-4})| = q^4$ and $r \leq 3 = \frac{n}{2}$, contrary to $(**)$. Hence E is a natural module. But then

$$(***) \quad |E : C_E(U_{\alpha-4})| = q^r,$$

again since (3.2.4) implies that $U_{\alpha-4}R_\alpha/R_\alpha$ is normalized by each p'-element of some Levi complement of $M_{\alpha-1}/R_\alpha$. From $(**)$ and $(***)$ it follows that $r = 1$, a contradiction. ■

3.3

In addition to (I)–(VII) we now assume

(VIII) $L_\alpha R_\alpha/R_\alpha \cong \Omega^\varepsilon_{2n}(q)$ ($\varepsilon \in \{+,-\}$) and $r = n-1$.

(3.3.1) Assume that $U_{\alpha'}R_\alpha \cap L_\alpha R_\alpha = R_\alpha$.

(a) $|[U_\alpha, U_{\alpha'}]| = 2$.

(b) $b > 2$.

(c) $U_\alpha = (U_\alpha \cap Q_{\alpha'})[U_\alpha, Q_{\alpha+1}]$.

Proof. Note that $U_{\alpha'}R_\alpha \cap L_\alpha R_\alpha = R_\alpha$ and $U_{\alpha'} \le Q_{\alpha+1}$ imply that $U_{\alpha'}$ induces outer automorphisms on $L_\alpha R_\alpha/R_\alpha$ that centralize the parabolic subgroup $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ modulo its largest normal p-subgroup. Hence, by (A.4.1),

$$|U_{\alpha'}R_\alpha/R_\alpha| = 2.$$

Therefore

$$(*) \quad |[U_\alpha, U_{\alpha'}]| = |U_{\alpha'} : C_{U_{\alpha'}}(U_\alpha)| = |U_{\alpha'} : U_{\alpha'} \cap Q_\alpha| = |U_{\alpha'} : U_{\alpha'} \cap R_\alpha| = 2.$$

Thus (a) holds.

Let P_1 and P_2 be the two maximal subgroups of $L_\alpha R_\alpha$ that contain $M_{\alpha+1} \cap L_\alpha R_\alpha$. Since $U_{\alpha'}$ induces on $L_\alpha R_\alpha/R_\alpha$ a graph automorphism that switches P_1/R_α and P_2/R_α, it follows that $U_{\alpha'}$ switches $O_p(P_1/R_\alpha)$ and $O_p(P_2/R_\alpha)$, and hence acts non-trivially on $Q_{\alpha+1}R_\alpha/R_\alpha$ Thus $U_{\alpha'}R_\alpha/R_\alpha$ is not normalized by $M_{\alpha+1}/R_\alpha$, whence $b > 2$. Also the two maximal isotropic subspaces $[U_\alpha, O_p(P_1)]$ and $[U_\alpha, O_p(P_2)]$ of U_α are switched by $U_{\alpha'}$, whence $[U_\alpha, Q_{\alpha+1}] \not\le Q_{\alpha'}$. Since (*) and (3.1.1)(g) implies that

$$|U_\alpha : U_\alpha \cap Q_{\alpha'}| = 2,$$

it follows that (c) holds. ∎

(3.3.2) Let (γ, γ') be a critical pair, $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$, $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$ and $\delta \in \Delta^{(2)}(\gamma') \cap \Delta^{(b-2)}(\gamma)$. Then there exists $g \in L_\gamma$ such that the following hold:

(a) The intersection of the parabolic subgroups $(M_\lambda \cap L_\gamma)R_\gamma/R_\gamma$ and $(M_\lambda \cap L_\gamma)^g R_\gamma/R_\gamma$ of $L_\gamma R_\gamma/R_\gamma$ is the product of a Cartan subgroup and a Levi complement in both of them.

(b) Assume that $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma \neq R_\gamma$. Then one of the following holds:

(b1) (μ^{gx}, δ) is a critical pair for each $x \in M_\lambda$.

(b2) $[U_\gamma, Q_\mu]$ is a maximal isotropic subspace of U_γ, and (μ^{gx}, δ) is a critical pair for each $x \in C_{M_\lambda}([U_\gamma, Q_\mu])$.

Proof. Note that (3.1.1)(g) implies that (α, α') is an arbitrary critical pair. Hence it suffices to prove this for $(\gamma, \gamma') = (\alpha, \alpha')$. Since $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$, there exists a root system Φ for $L_\alpha R_\alpha/R_\alpha$ such that

$$(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha = QLH,$$

where Q is the product of the root subgroups corresponding to the positive roots which are not contained in the root system Ψ spanned by the simple roots that belong to the nodes $1, \ldots, n-2$ of the Dynkin diagram, L is generated by the root subgroups corresponding to the roots in Ψ, and H is a Cartan subgroup in the normalizer of the product of the root subgroups corresponding to the positive roots.

Let Q^- be the product of the root subgroups corresponding to the negative roots which are not contained in Ψ. Then Q^-LH is a parabolic subgroup of $L_\alpha R_\alpha/R_\alpha$ that has the same type as QLH. Hence there exists $h \in L_\alpha$ such that

$$(M_{\alpha+1} \cap L_\alpha)^h R_\alpha/R_\alpha = Q^-LH.$$

Then (a) is satisfied for each $g \in h(M_{\alpha+1} \cap L_\alpha)$.

Assume that $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma \neq R_\gamma$. Clearly (b1) holds if $b = 2$. Hence assume that $b > 2$. Note that

$$(*) \quad [U_\alpha, Q_{\alpha+1}] \cap Z^g_{\alpha+1} = 1, \quad \text{and}$$

$$(*') \quad [U_\alpha, Q^g_{\alpha+1}] \cap Z_{\alpha+1} = 1, \quad \text{for each } g \in hM_{\alpha+1}.$$

From (3.1.4)(b) and $(*)$ it follows that (b1) holds, provided $[U_{\alpha+2}, Q_{\alpha+1}] = [U_{\alpha+2}, Q_\alpha]$. Assume that

$$(**) \quad [U_{\alpha+2}, Q_{\alpha+1}] \neq [U_{\alpha+2}, Q_\alpha].$$

Then (3.1.1)(e) implies that $[U_\alpha, Q_{\alpha+2}]$ is an isotropic subspace of U_α. Note that $[U^{hx}_{\alpha+2}, Q_\alpha] \cap U_\alpha$ is a proper $M^{hx}_{\alpha+1}$-submodule of U_α for each $x \in G_\alpha$. Hence, if $[U_\alpha, U_{\alpha'}] \leq Z_{\alpha+1}$, then by (3.1.1)(h) and $(*')$

$$[U^{hx}_{\alpha+2}, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] \leq [U^{hx}_{\alpha+2}, Q_\alpha] \cap U_\alpha \cap Z_{\alpha+1} \leq [U_\alpha, Q^{hx}_{\alpha+1}] \cap Z_{\alpha+1} = 1,$$

for each $x \in M_{\alpha+1}$. Therefore, (b1) follows from (3.1.4)(c) in this case. Hence we may assume that

$$[U_\alpha, U_{\alpha'}] \not\leq Z_{\alpha+1}.$$

In particular, $[U_\alpha, Q_{\alpha+2}] \neq U_\alpha \cap Z_{\alpha+1}$. Then $[U_\alpha, Q_{\alpha+2}]$ is a maximal isotropic subspace by (3.1.1)(h)(i).

If $[U^g_{\alpha+2}, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] = 1$ for some $g \in h(M_{\alpha+1} \cap L_\alpha)$, then also

$$[U^{gx}_{\alpha+2}, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] = 1 \quad \text{for each } x \in C_{M_{\alpha+1}}([U_\alpha, Q_{\alpha+2}]),$$

since $[U_\alpha, U_{\alpha'}] \leq [U_\alpha, Q_{\alpha+2}]$. Therefore, (b2) follows from (3.1.4)(c) in this case.

Assume that

$$(***) \quad [U^g_{\alpha+2}, Q_\alpha] \cap [U_\alpha, U_{\alpha'}] \neq 1 \quad \text{for each } g \in h(M_{\alpha+1} \cap L_\alpha).$$

Note that $|[U_{\alpha+2}, Q_\alpha]| \leq q^n$ by $(**)$ and (3.1.1)(e). In particular,

$$[U_{\alpha+2}, Q_\alpha] \cap U_\alpha \neq [U_\alpha, Q_{\alpha+1}].$$

Note that $[U_{\alpha+2}, Q_\alpha] \cap U_\alpha \not\leq Z_{\alpha+1}$ by $(*)$ and $(***)$. Then by (3.1.1)(h)(i) $[U_{\alpha+2}, Q_\alpha] \cap U_\alpha$ is a maximal isotropic subspace of U_α that contains $U_\alpha \cap Z_{\alpha+1}$. Together with $(*)$ we get

$$|[U^g_{\alpha+2}, Q_\alpha] \cap [U_\alpha, Q_{\alpha+1}]| \leq q, \quad \text{for each } g \in h(M_{\alpha+1} \cap L_\alpha),$$

and hence, by $(* * *)$,

$$(* * * *) \quad ([U^h_{\alpha+2}, Q_\alpha] \cap [U_\alpha, Q_{\alpha+1}])^x \leq [U_\alpha, U_{\alpha'}], \quad \text{for each } x \in M_{\alpha+1} \cap L_\alpha.$$

Note that

$$U_\alpha \cap Z_{\alpha+1} \leq \langle ([U^h_{\alpha+2}, Q_\alpha] \cap [U_\alpha, Q_{\alpha+1}])^{M_{\alpha+1} \cap L_\alpha} \rangle.$$

Since $[U_\alpha, U_{\alpha'}] \not\leq Z_{\alpha+1}$, it follows from $(* * * *)$ that

$$|[U_\alpha, U_{\alpha'}]| = q^n.$$

Hence $[U_\alpha, U_{\alpha'}] = [U_\alpha, Q_{\alpha+2}]$. In particular, $|U_{\alpha'} \cap Q_\alpha| = q^n$ by (3.1.1)(g). Therefore,

$$(* * * * *) \quad U_{\alpha'} \cap Q_\alpha = [U_\alpha, U_{\alpha'}].$$

Since $\langle U_\nu \mid \nu \in \Delta(\alpha - 1) \rangle$ is a nontrivial characteristic subgroup of $M_{\alpha-1}$,

$$U_\nu U_\alpha \ntrianglelefteq G_\alpha, \quad \text{for some } \nu \in \Delta(\alpha + 1).$$

Suppose that $(\nu^{hx}, \alpha' - 2)$ is not a critical pair for some $x \in M_{\alpha+1}$. Then $U^{hx}_\nu \leq Q_{\alpha'-2} \leq G_{\alpha'}$. Together with $(* * * * *)$ we get

$$[U^{hx}_\nu, U_{\alpha'}] \leq Q_\alpha \cap U_{\alpha'} \leq U_\alpha,$$

contrary to $(*)$, (3.1.4)(a) and the choice of ν. Hence $(\nu^{hx}, \alpha' - 2)$ is a critical pair for each $x \in M_{\alpha+1}$. Now (1.2.1)(e) implies that $U_{\alpha+2} U_\alpha \ntrianglelefteq G_\alpha$. Hence we can choose $\nu = \alpha + 2$. Then (b1) holds. ■

(3.3.3) Assume that in (3.3.2) (b2) is satisfied. Then $|Q : C_Q(x)| = q^{n-1}$ for each $x \in U_\gamma \cap Z^g_\lambda$, where $Q := C_{Q_\lambda \cap L_\gamma}([U_\gamma, Q_\mu])$.

Proof. Similarly to (3.1.1)(d), this follows by a simple matrix calculation. ∎

(3.3.4) Let (γ, γ') be a critical pair, $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$, $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$, and $\delta \in \Delta^{(2)}(\gamma') \cap \Delta^{(b-2)}(\gamma)$. Choose $g \in L_\gamma$ as in (3.3.2), if $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma \neq R_\gamma$. Choose $g \in L_\gamma$ with $U_{\gamma'} \not\leq M_\lambda^g$, if $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma = R_\gamma$.

(a) $G_\gamma = \langle M_\lambda^g, U_{\gamma'} \rangle$.

(b) Assume that $b > 2$, $U_\gamma \cap U_\mu \leq Z_\lambda$, and $Q_\lambda \neq Q_\gamma Q_\mu$. Then $U_\delta \leq L_\mu^g R_\mu^g$.

(c) Assume that $b > 2$ and $U_\gamma \cap U_\mu \leq Z_\lambda$. $U_\delta R_\mu^g \cap L_\mu^g R_\mu^g \neq R_\mu^g$.

Proof. (a) This follows from (3.1.4)(a), (3.1.3)(a), and the choice of g.

(b) Suppose that $U_\delta \not\leq L_\mu^g R_\mu^g$. In particular, $p = 2$.

Suppose that $U_\mu \cap Z_\lambda \not\leq U_\gamma$. Since $U_\mu \cap Z_\lambda$ is irreducible as M_λ-module, it follows that $U_\mu \cap Z_\lambda$ is isomorphic to an M_λ-submodule of Z_γ/U_γ. Hence $L_{\gamma,\lambda,1}$ centralizes $U_\mu \cap Z_\lambda$. But then $O^p(L_{\gamma,\lambda,1}) \leq R_\gamma R_\mu$, contrary to $Q_\gamma Q_\mu \neq Q_\lambda$. Hence

$$(*) \quad U_\mu \cap Z_\lambda = U_\gamma \cap Z_\lambda = U_\gamma \cap U_\mu.$$

Since $[U_\mu^g, Q_\lambda^g, Q_\lambda^g] = (U_\mu^g \cap Z_\lambda^g)[U_\mu^g, U_\delta]$, it follows from $(*)$ and $b > 2$ that $U_{\gamma'}$ normalizes the M_λ^g-module

$$Y := [U_\mu^g, Q_\lambda^g, Q_\lambda^g]U_\gamma.$$

Hence Y is G_γ-invariant by (a). Note that Q_γ centralizes Y, since $Q_\gamma Q_\mu \neq Q_\lambda$. Hence Y is a module for $L_\gamma Q_\gamma/Q_\gamma$. Now

$$[Y, U_{\gamma'}] = [[U_\mu, Q_\lambda, Q_\lambda]U_\gamma, U_{\gamma'}] = [U_\gamma, U_{\gamma'}]$$

and $|[U_\gamma, U_{\gamma'}]| = U_{\gamma'}Q_\gamma/Q_\gamma$ imply that the dual of Y is an FF-module for L_γ, and $U_{\gamma'}$ acts as an offending subgroup. Thus [14](1.5) shows that $Y = U_\gamma$. Since $[U_\mu, Q_\lambda, Q_\lambda] \not\leq Z_\lambda$, this is a contradiction to $(*)$.

(c) Suppose that $U_\delta R_\mu^g \cap L_\mu^g R_\mu^g = R_\mu^g$. Then $Q_\gamma Q_\mu = Q_\lambda$ by (3.3.1)(b) and (b). If $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma \neq R_\gamma$, then $[U_\gamma, Q_\lambda] \cap Z_\lambda^g = 1$ by the choice of g. If $U_{\gamma'}R_\gamma \cap L_\gamma R_\gamma =$

R_γ, then $[U_\gamma, U_{\gamma'}]$ is not singular, but $U_\gamma \cap Z_\lambda^g$ is a singular subspace of U_γ, whence $[U_\gamma, U_{\gamma'}] \cap Z_\lambda^g = 1$. Therefore, in any case $U_\gamma \cap U_\mu \leq Z_\lambda$ implies that

$$[U_\gamma, U_{\gamma'}] \cap U_\mu^g = 1.$$

From (A.1.5)(b) (with U_γ, $(U_\mu^g \cap Q_\delta)U_\gamma$, and $U_{\gamma'}$ in place of A, B, and V) it follows that

$$(*) \quad [(U_\mu^g \cap Q_\delta)U_\gamma, U_{\gamma'}] \leq U_\mu^g U_\gamma.$$

Put

$$Y := \langle U_\mu^{G_\gamma} \rangle \, U_\gamma.$$

From $Q_\lambda = Q_\gamma Q_\mu$, $(*)$, (a), and (3.3.1)(c) we get

$$Y = [Y, Q_\gamma] U_\gamma U_\mu^g.$$

But then $Y/(U_\gamma U_\mu^g) = [Y/(U_\gamma U_\mu^g), Q_\gamma]$ and hence

$$U_\gamma U_\mu^g = Y \trianglelefteq G_\gamma,$$

a contradiction. ∎

(3.3.5) Assume that if $(\gamma, \gamma+1, \ldots, \gamma+b)$ is any path such that $(\gamma, \gamma+b)$ is a critical pair, then $U_\gamma \cap U_{\gamma+2} = 1$ or $U_\gamma \cap U_{\gamma+2} \not\leq Z_{\gamma+1}$. Then $b \leq 4$.

Proof. Suppose that $b > 4$. Note that (3.3.2) and (3.1.3) imply that if (γ, γ') is any critical pair and $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$, then $U_\gamma U_\mu \ntrianglelefteq G_\gamma$. Hence it follows from $b > 4$ and (3.1.4)(d) that there exists a path $(\gamma, \gamma+1, \ldots, \gamma+b)$ such that $(\gamma, \gamma+b)$ is a critical pair and

$$U_\gamma \cap U_{\gamma+2} \neq 1.$$

Hence $U_\gamma \cap U_{\gamma+2} \not\leq Z_{\gamma+1}$. Note that this means that

$$U_\gamma \cap U_{\gamma+2} = [U_\gamma, Q_{\gamma+1}, Q_{\gamma+1}] = [U_{\gamma+2}, Q_{\gamma+1}, Q_{\gamma+1}].$$

Now extend $(\gamma, \ldots, \gamma+b)$ to a path $(\gamma-b-2, \gamma-b+1, \ldots, \gamma+b)$ such that $(\gamma-2i, \gamma+b-2i)$ is a critical pair, for each $i \in \{0, \ldots, \frac{b+4}{2}\}$, as follows. Assume that $(\gamma-2i, \ldots, \gamma+b)$ has already been defined, for some $i \in \{0, \ldots, \frac{b}{2}\}$. If $U_{\gamma+b-2i}R_{\gamma-2i} \cap L_{\gamma-2i}R_{\gamma-2i} \neq R_{\gamma-2i}$, then choose $g_{\gamma-2i} \in L_{\gamma-2i}$ such that (3.3.2) is satisfied for $(\gamma-2i, \gamma+b-2i)$ and $g_{\gamma-2i}$ in place of (γ, γ') and g. If $U_{\gamma+b-2i}R_{\gamma-2i} \cap L_{\gamma-2i}R_{\gamma-2i} = R_{\gamma+b-2i}$, then choose $g_{\gamma-2i} \in L_{\gamma-2i}$ such that $U_{\gamma-2i} = (U_{\gamma-2i} \cap U_{\gamma-2i+2}) \times (U_{\gamma-2i} \cap U_{\gamma-2i+2})^{g_{\gamma-2i}})$. (A simple calculation shows that this is possible.) Then put

$$\gamma - 2i - 1 := (\gamma - 2i + 1)^{g_{\gamma-2i}} \quad \text{and}$$

$$\gamma - 2i - 2 := (\gamma - 2i + 2)^{g_{\gamma-2i}}.$$

Suppose that $(\gamma - 2i - 2, \gamma + b - 2i - 2)$ is not a critical pair. Then (3.3.2) and (3.1.3) imply that $U_{\gamma+b-2i}R_{\gamma-2i} \cap L_{\gamma-2i}R_{\gamma-2i} = R_{\gamma+b-2i}$ and $U_{\gamma+b-2i} \leq M_{\gamma-2i-1}$. Since $U_{\gamma-2i} \cap Z_{\gamma-2i-1}$ is the unique proper $M_{\gamma-2i-1}$-submodule of $U_{\gamma-2i} \cap U_{\gamma-2i-2}$, it follows that

$$[U_{\gamma-2i}, U_{\gamma+b-2i}] = [(U_{\gamma-2i} \cap U_{\gamma-2i+2})(U_{\gamma-2i} \cap U_{\gamma-2i-2}), U_{\gamma+b-2i}] \leq$$

$$U_{\gamma-2i} \cap Z_{\gamma-2i-1}.$$

Since $U_{\gamma-2i} \cap Z_{\gamma-2i-1}$ is a singular subspace of $U_{\gamma-2i}$, but $U_{\gamma+b-2i}R_{\gamma-2i} \cap L_{\gamma-2i}R_{\gamma-2i} = R_{\gamma+b-2i}$ implies that $[U_{\gamma-2i}, U_{\gamma+b-2i}]$ is not singular, this is impossible. Hence $(\gamma - 2i - 2, \gamma + b - 2i - 2)$ is a critical pair.

Define

$$V := \langle U_\mu \mid \mu \in \Delta^{(2)}(\gamma - b + 2), \ (\gamma, \mu) \text{ is a critical pair} \rangle.$$

By a counting argument as in the proof of (3.1.4)(d) (and again shown in detail in the proof of (3.3.7)) we get

$$(*) \quad |[V, U_\gamma]| \geq q^n.$$

Note that $U_{\gamma-2} \cap U_\gamma \not\leq Z_{\gamma+1}$ implies that $[U_\gamma, Q_{\gamma-2}] \leq U_{\gamma-2}$ and hence

$$[V, U_\gamma] \leq [Q_{\gamma-2}, U_\gamma] \leq U_\gamma \cap U_{\gamma-2}.$$

Together with $(*)$ it follows that $[V, U_\gamma] = U_\gamma \cap U_{\gamma-2}$. Since $[V, Q_{\gamma-b+2}] \leq U_{\gamma-b+2}$ and $b > 4$, we get

$$U_\gamma \cap U_{\gamma-2} \leq U_{\gamma-b+2} \leq Q_{\gamma-b-2}.$$

Since $(\gamma - b - 2, \gamma - 2)$ is a critical pair, it now follows the construction of the path $(\gamma - b - 2, \ldots, \gamma + b)$ that

$$U_{\gamma+b-2}R_{\gamma-2} \cap L_{\gamma+b-2}R_{\gamma+b-2} \neq R_{\gamma+b-2}.$$

Hence, if $(\mu - 2, \mu - 1, \mu, \ldots, \mu + b)$ is any path such that $(\mu, \mu + b)$ is a critical pair with $U_\mu \cap U_{\mu+2} \neq 1$ and $(\mu - 2, \mu - 1, \ldots, \mu + b)$ is constructed from $(\mu, \ldots, \mu + b)$ by the method above, then

$$(**) \quad U_{\mu+b-2}R_{\mu-2} \cap L_{\mu+b-2}R_{\mu+b-2} \neq R_{\mu+b-2}.$$

Note that in the construction of $(\gamma - b - 2, \ldots, \gamma + b)$, we can replace $g_{\gamma-b}$ by $g_{\gamma-b}x$, for each $x \in C_{Q_{\gamma-b+1} \cap L_{\gamma-b}}([U_{\gamma-b}, Q_{\gamma-b+2}])$. Define

$$Y := \langle U^x_{\gamma-b-2} \mid x \in C_{Q_{\gamma-b+1} \cap L_{\gamma-b}}([U_{\gamma-b}, Q_{\gamma-b+2}]) \rangle.$$

Then again a counting argument as in the proof of (3.1.4)(d) shows that

$$|[Y, U_\gamma]| > (q^2 - 1)q^{n-1},$$

using $(**)$ and (3.1.1)(c). But since Y acts quadratically on U_γ and hence $[Y, U_\gamma]$ is an isotropic subspace of U_γ, this is impossible. Hence $b \leq 4$. ∎

(3.3.6) There exists a path $(\gamma - 4, \gamma - 3, \ldots, \gamma + b)$ with the following properties:

(a) $\gamma - 2 = (\gamma + 2)^{g_\gamma}$ for some $g_\gamma \in L_\gamma$.

(b) $\gamma - 4 = \gamma^{g_{\gamma-2}}$ for some $g_{\gamma-2} \in L_{\gamma-2}$.

(c) $(\gamma, \gamma+b)$ is a critical pair and $U_{\gamma+b}R_\gamma \cap L_\gamma R_\gamma \neq R_\gamma$.

(d) If $Q_{\gamma+1} = Q_\gamma Q_{\gamma+2}$, then for each $x \in M_{\gamma+1}$ the following hold:

(d1) $((\gamma-2)^x, \gamma+b-2)$ is a critical pair.

(d2) If $U_\gamma \cap U_{\gamma+2} \leq Z_{\gamma+1}$, then $U_{\gamma+b-2}R^x_{\gamma-2} \cap L^x_{\gamma-2}R^x_{\gamma-2} \neq R^x_{\gamma-2}$.

(e) If $Q_{\gamma+1} \neq Q_\gamma Q_{\gamma+2}$, then for each $x \in C_{M_{\gamma+1}}([U_\gamma, Q_{\gamma+1}, Q_{\gamma+1}])$ the following hold:

(e1) $((\gamma-2)^x, \gamma+b-2)$ is a critical pair.

(e2) If $U_\gamma \cap U_{\gamma+2} \leq Z_{\gamma+1}$, then $U_{\gamma+b-2} \leq L^x_{\gamma-2}R^x_{\gamma-2}$.

(f) If there exists a critical pair (μ, μ') such that $1 \neq U_\mu \cap U_\nu \leq Z_\lambda$ where $\nu \in \Delta^{(2)}(\mu) \cap \Delta^{(b-2)}(\mu')$ and $\lambda \in \Delta(\mu) \cap \Delta^{(b-1)}(\mu')$, then $1 \neq U_\gamma \cap U_{\gamma+2} \leq Z_{\gamma+1}$.

Proof. This follows from (3.3.2), (3.1.3), and (3.3.4). ∎

Assume that the critical pair (α, α') is chosen such that the path $(\alpha, \ldots, \alpha')$ can be extended to a path $(\alpha-4, \ldots, \alpha')$ as in (3.3.6). Choose $g_\alpha \in L_\alpha$ and $g_{\alpha-2} \in L_{\alpha-2}$ such that

$$\alpha - 2 = (\alpha+2)^{g_\alpha} \quad \text{and}$$

$$\alpha - 4 = \alpha^{g_{\alpha-2}}.$$

(3.3.7) $b \leq 4$.

Proof. Suppose that $b > 4$. Then it follows from (3.3.5), (3.3.6)(f), and the choice of the path $(\alpha-4, \ldots, \alpha')$ that

$$1 \neq U_{\alpha+2} \cap U_\alpha \leq Z_{\alpha+1}.$$

In particular, by (3.1.1)(a)(h)

$$U_{\alpha+2} \cap T_\alpha = 1.$$

Put

$$Q := \begin{cases} Q_{\alpha+1} \cap L_\alpha & \text{if } Q_{\alpha+1} = Q_\alpha Q_{\alpha+2} \\ C_{Q_{\alpha+1}\cap L_\alpha}([U_\alpha, Q_{\alpha+2}]) & \text{if } Q_{\alpha+1} \neq Q_\alpha Q_{\alpha+2}. \end{cases}$$

Let A be a set of representatives for the cosets of $Q \cap Q_\alpha$ in Q. Define

$$X_a := \{(x, a) \mid 1 \neq x \in [U^a_{\alpha-2}, U_{\alpha'-2}] \cap Z^a_{\alpha-1}\}, \quad \text{for each } a \in A, \text{ and}$$

$$X := \bigcup_{a\in A} [U^a_{\alpha-2}, U_{\alpha'-2}] \cap Z^a_{\alpha-1} \setminus \{1\}.$$

Then by (3.1.1)(c) and $U_{\alpha+2} \cap U_\alpha \leq Z_{\alpha+1}$

$$|\bigcup_{a\in A} X_a| = \sum_{a\in A} |X_a| \geq \begin{cases} |A|(q-1) & \text{if } Q_{\alpha+1} = Q_\alpha Q_{\alpha+2} \\ |A|(q^2-1) & \text{if } Q_{\alpha+1} \neq Q_\alpha Q_{\alpha+2}. \end{cases}$$

Note that $U_{\alpha+2} \cap T_\alpha = 1$ implies that

$$U^a_{\alpha-2} \cap T_\alpha = 1 \quad \text{for each } a \in A.$$

Therefore by (3.1.1)(d) and (3.3.3), each $x \in X$ is contained in at most $q^{-n}|A|$ of the subgroups $[U^a_{\alpha-2}, U_{\alpha'-2}]$ with $a \in A$, if $Q_{\alpha+1} = Q_\alpha Q_{\alpha+2}$, and in at most $q^{1-n}|A|$ of these subgroups, if $Q_{\alpha+1} \neq Q_\alpha Q_{\alpha+2}$. Hence

$$|\bigcup_{a\in A} X_a| = \sum_{x\in X} |\{a \in A \mid (x, a) \in X_a\}| \leq \begin{cases} q^{-n}|A|\,|X| & \text{if } Q_{\alpha+1} = Q_\alpha Q_{\alpha+2} \\ q^{1-n}|A|\,|X| & \text{if } Q_{\alpha+1} \neq Q_\alpha Q_{\alpha+2}. \end{cases}$$

Therefore,

$$(*) \quad |X| \geq \begin{cases} (q-1)q^n & \text{if } Q_{\alpha+1} = Q_\alpha Q_{\alpha+2} \\ (q^2-1)q^{n-1} & \text{if } Q_{\alpha+1} \neq Q_\alpha Q_{\alpha+2}. \end{cases}$$

Since $b > 4$, V_α acts quadratically on $U_{\alpha'-2}$. By (A.1.5)(a), it follows that $[V_\alpha, U_{\alpha'-2}]$ is an isotropic subspace of $U_{\alpha'-2}$. Hence

$$(**) \quad |X| \leq |[V_\alpha, U_{\alpha'-2}]| - 1 \leq q^n - 1.$$

From $(*)$ and $(**)$ it follows that $q^{n+1} - q^n \leq q^n - 1$, a contradiction. ■

(3.3.8) Assume that $b = 4$.

(a) If $i \in \{1,2\}$, then $U_{\alpha-2i} \cap Z_{\alpha-2i+1} \leq T_{\alpha-2i+2}$.

(b) If $i \in \{1,2\}$, then $\langle U_{\alpha-2i}^{L_{\alpha-2i+2}} \rangle \leq \bigcap_{x \in L_{\alpha-2i+2}} L_{\alpha-2i}^{x} R_{\alpha-2i}^{x}$.

(c) If $i \in \{1,2\}$ and $x \in L_{\alpha-2i+2}$ with $U_{\alpha-2i}^{x} \not\leq Q_{\alpha-2i}$, then

$$Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i}) \leq U_{\alpha-2i}^{x} R_{\alpha-2i}/R_{\alpha-2i}.$$

(d) $n \leq 4$.

Proof. (a) By the choice of the path $(\alpha - 4, \ldots, \alpha')$ we can apply (3.1.4)(e) with $(\alpha - 2i, \alpha - 2i + 2, \alpha' - 2i + 2)$ in place of (μ, α, α').

(b) Suppose this is false. Then $U_{\alpha-2i} \not\leq L_{\alpha-2i}^{x} R_{\alpha-2i}^{x}$ for some $x \in L_{\alpha-2i+2}$. In particular,

$$p = 2.$$

Suppose that $M_{\alpha-2i+1} \cap L_{\alpha-2i}$ has a p-component K. Then $U_{\alpha-2i} \leq K$ and, by (a) and (3.1.5)(a), $K \leq R_{\alpha-2i+2}$. Since (1.1.2)(d) implies that $L_{\alpha-2i+2}$ normalizes each p-component of $R_{\alpha-2i+2}$, it follows that

$$U_{\alpha-2i} \leq K^{x} \leq L_{\alpha-2i}^{x},$$

a contradiction.

Hence $n = 3$ and $q = 2$. Then $O^{2}(G_{\alpha-2i}^{x}) \leq L_{\alpha-2i}^{x} R_{\alpha-2i}^{x}$. But (a) and (3.1.5)(b) imply that $U_{\alpha-2i} \leq O^{2}(R_{\alpha-2i+2})$. Since $R_{\alpha-2i+2} \leq G_{\alpha-2i}^{x}$, we get

$$U_{\alpha-2i} \leq L_{\alpha-2i}^{x} R_{\alpha-2i}^{x},$$

a contradiction.

(c),(d) Note that (B.5.1.5) implies that

$$U_{\alpha-2i}^{x} R_{\alpha-2i}/R_{\alpha-2i} \cap Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i}) \neq 1,$$

since $U^x_{\alpha-2i}$ acts as an offending subgroup on $U_{\alpha-2i}$ and is contained in $L_{\alpha-2i}R_{\alpha-2i}$ by (b). Since $Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i})$ is an irreducible $O^p(L_{\alpha-2i,\alpha-2i+1,1})$-module and $U^x_{\alpha-2i}$ is $R_{\alpha-2i+2}$-invariant, (a) and (3.1.5)(a) imply that

$$Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i}) \leq U^x_{\alpha-2i}R_{\alpha-2i}/R_{\alpha-2i}.$$

Hence (c) holds. Moreover,

$$(*)\quad |U^x_{\alpha-2i} : U^x_{\alpha-2i} \cap Q_{\alpha-2i}| = |U^x_{\alpha-2i}R_{\alpha-2i}/R_{\alpha-2i}| \geq$$

$$|Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i})| = q^{\frac{1}{2}(n-1)(n-2)}.$$

Since $[U^x_{\alpha-2i}, U_{\alpha-2i}]$ is an isotropic subspace of $U^x_{\alpha-2i}$, we also have

$$(**)\quad |U^x_{\alpha-2i} : U^x_{\alpha-2i} \cap Q_{\alpha-2i}| = |[U^x_{\alpha-2i}, U_{\alpha-2i}]| \leq q^n.$$

From $(*)$ and $(**)$ it follows that $n \leq 4$. ■

Define

$$Y := \langle U^{L_\alpha}_{\alpha+2}\rangle.$$

(3.3.9) Assume that $b = 4$ and $n = 4$.

(a) $Y R_{\alpha-2}/R_{\alpha-2} = U_{\alpha+2}R_{\alpha-2}/R_{\alpha-2} = Z(Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2})$.

(b) $Y' = [U_{\alpha+2}, U_{\alpha-2}] \leq T_\alpha$.

Proof. Pick $x \in L_\alpha$ with $U^x_{\alpha+2} \not\leq Q_{\alpha-2}$. Suppose that $|[U^x_{\alpha+2}, U_{\alpha-2}]| = q^4$. Then, since $O^p(L_{\alpha-2,\alpha-1,1})$ normalizes $U^x_{\alpha+2}R_{\alpha-2}/R_{\alpha-2}$ by (3.3.8)(a) and (3.1.5)(a), it follows from (3.3.8)(c) that $|U^x_{\alpha+2}R_{\alpha-2}/R_{\alpha-2}| = q^6$. But then

$$q^4 = |[U^x_{\alpha+2}, U_{\alpha-2}]| = |U^x_{\alpha+2} : C_{U^x_{\alpha+2}}(U_{\alpha-2})| =$$

$$|U^x_{\alpha+2} : U^x_{\alpha+2} \cap Q_{\alpha-2}| = |U^x_{\alpha+2}R_{\alpha-2}/R_{\alpha-2}| = q^6,$$

a contradiction. Hence, by (3.3.8)(c)

$$U^x_{\alpha+2}R_{\alpha-2}/R_{\alpha-2} = Z(Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2}).$$

Since this holds for each $x \in L_\alpha$ with $U^x_{\alpha+2} \not\leq Q_{\alpha-2}$, (a) follows. Moreover,

$$[U_{\alpha-2}, Y] = [U_{\alpha-2}, U_{\alpha+2}] \leq T_\alpha$$

by (3.1.4)(e). Hence (b) holds. ∎

Define

$$X := C_{Q_\alpha}(Q_\alpha, T_\alpha).$$

(3.3.10) Assume that $b = 4$ and $n = 3$. Let $i \in \{1, 2\}$.

(a) $[U_{\alpha-2i}, U_{\alpha-2i+4}]$ is a maximal isotropic subspace of $U_{\alpha-2i}$.

(b) $U_{\alpha-2i+4}R_{\alpha-2i}/R_{\alpha-2i} \trianglelefteq O^p(O^{p'}((M_{\alpha-2i+1} \cap L_{\alpha-2i})R_{\alpha-2i}/R_{\alpha-2i}))$.

(c) $X\,R_{\alpha-2}/R_{\alpha-2} = [U_{\alpha+2}, Q_\alpha]R_{\alpha-2}/R_{\alpha-2} = Z(Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2})$.

Proof. (a) By (A.1.5)(a) $[U_{\alpha-2i}, U_{\alpha-2i+2}]$ is an isotropic subspace of $U_{\alpha-2i}$. If it is not maximal, then (3.3.8)(c) implies that $U_{\alpha-2i+4}R_{\alpha-2i}/R_{\alpha-2i} = Z(Q_{\alpha-2i+1}R_{\alpha-2i}/R_{\alpha-2i})$ and hence

$$|U_{\alpha-2i+4}R_{\alpha-2i}/R_{\alpha-2i}| = q < q^2 = |[U_{\alpha-2i}, U_{\alpha-2i+4}]|,$$

contrary to

$$|[U_{\alpha-2i+4}, U_{\alpha-2i}]| = |U_{\alpha-2i+4} : C_{U_{\alpha-2i+4}}(U_{\alpha-2i})| = |U_{\alpha-2i+4} : U_{\alpha-2i+4} \cap Q_{\alpha-2i}| =$$

$$|U_{\alpha-2i+4}R_{\alpha-2i}/R_{\alpha-2i}|.$$

Hence (a) holds.

(b) follows from (3.3.8)(a) and (3.1.5)(a).

(c) Note that $[U_{\alpha-2}, X] \leq T_\alpha \leq Z_{\alpha_1}$, since $b > 2$. Therefore,

$$XR_{\alpha-2}/R_{\alpha-2} \leq C_{Q_{\alpha-1}}(U_{\alpha-2}, U_{\alpha-2} \cap Z_{\alpha-1})R_{\alpha-2}/R_{\alpha-2} = Z(Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2}).$$

Moreover, by (a) and (b),

$$[U_{\alpha+2}, Q_\alpha]R_{\alpha-2}/R_{\alpha-2} = Z(Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2}).$$

Since (3.1.4)(e) implies that $[U_{\alpha+2}, Q_\alpha] \leq X$, (c) holds. ∎

(3.3.11) $b = 2$.

Proof. Suppose that $b > 2$ and $n = 4$. From (3.3.9)(a) it follows that

$$(*) \quad |Y : Y \cap Q_{\alpha-2}| = q^3.$$

Let A be the subgroup of $M_{\alpha-3}$ with $R_{\alpha-4} \leq A$ and $A/R_{\alpha-4} = Z(Q_{\alpha-3}R_{\alpha-4}/R_{\alpha-4})$. By (3.3.9)(b), $Y \cap Q_{\alpha-2}$ acts quadratically on $U_{\alpha-4}$, whence

$$(**) \quad |Y \cap Q_{\alpha-2} : Y \cap Q_{\alpha-2} \cap A| \leq q^3.$$

Note that, by (3.3.8)(a) and the definition of A,

$$[U_{\alpha-4}, Y \cap Q_{\alpha-2} \cap A] \leq T_{\alpha-2}.$$

Thus $(*)$ and $(**)$ imply that

$$(***) \quad |Y : C_Y(U_{\alpha-4}, Z_\alpha[Y, Q_\alpha])| \leq q^6.$$

Suppose that $[Y, L_\alpha] \leq Z_\alpha[Y, Q_\alpha]$. Then $Y \leq [Y, Q_\alpha]Z_\alpha U_{\alpha+2}$. Since $[Y, Q_\alpha] \leq Z(Y)$ by (3.3.9)(a), it follows that Y is abelian, contrary to (3.3.9)(b). Hence

$$[Y, L_\alpha] \not\leq Z_\alpha[Y, Q_\alpha].$$

Let W be a $GF(p)L_\alpha R_\alpha$-composition factor of $Y Z_\alpha/[Y, Q_\alpha]Z_\alpha$ that is not centralized by L_α. Let D be an irreducible $GF(p)R_\alpha$-submodule of W. From (3.3.8)(a) and (3.1.5)(a) it follows that $Y Z_\alpha/[Y, Q_\alpha]Z_\alpha$ is the direct sum of 3-dimensional absolutely irreducible $GF(q)R_\alpha$-modules, each of which is irreducible as $GF(p)R_\alpha$-module. Hence $|D| = q^3$ and

$$\mathrm{End}_{GF(p)R_\alpha}(D) \cong GF(q).$$

From (A.3.1) (with $R_\alpha L_\alpha/Q_\alpha$, R_α/Q_α, $L_\alpha Q_\alpha/Q_\alpha$, $GF(p)$, $GF(q)$, W and D in place of G, A, B, F, K, V and X, respectively) we get that

$$W \cong D \otimes_{GF(q)} E$$

for some irreducible $GF(q)L_\alpha$-module E. Thus, regarded as $GF(p)L_\alpha$-module, W is the direct sum of three copies of E. Together with $(***)$ this implies that

$$(****) \quad |E : C_E(U_{\alpha-4})| \leq q^2.$$

On the other hand, it follows from (3.3.8)(c) that

$$|U_{\alpha-4}R_\alpha/R_\alpha| = |U_{\alpha-4} : U_{\alpha-4} \cap Q_\alpha| = |[U_{\alpha-4}, U_\alpha]| =$$

$$|U_\alpha : U_\alpha \cap Q_{\alpha-4}| = |U_\alpha R_{\alpha-4}/R_{\alpha-4}| \geq q^3.$$

Since $\Omega_8^\varepsilon(q)$ has no nontrivial irreducible module over $GF(q)$ in which the index of the centralizer of a subgroup of order q^3 is at most q^2, this contradicts $(****)$.

Suppose that $b > 2$ and $n = 3$. If $[X, L_\alpha] \leq Z_\alpha$, then $[U_{\alpha+2}, Q_\alpha] \leq [U_{\alpha-2}, Q_\alpha]Z_\alpha \leq R_{\alpha-2}$, contrary to (3.3.10)(c). Hence

$$[X, L_\alpha] \not\leq Z_\alpha.$$

From (3.3.10)(a),(b) it follows that

$$U_\alpha R_{\alpha-4}/R_{\alpha-4} = C_{Q_{\alpha-3}R_{\alpha-4}/R_{\alpha-4}}(U_\alpha R_{\alpha-4}/R_{\alpha-4}).$$

Therefore,

$$[U_{\alpha-4}, X \cap Q_{\alpha-2}] = [U_{\alpha-4}, U_\alpha] \leq U_\alpha.$$

Together with (3.3.10)(c) we get

$$|X : C_X(U_{\alpha-4}, Z_\alpha)| \leq |X : X \cap Q_{\alpha-2}| \leq q.$$

From $|U_{\alpha-4}R_\alpha/R_\alpha| = |U_\alpha R_{\alpha-4}/R_{\alpha-4}$ and (3.3.10)(a),(b) it follows that

$$|U_{\alpha-4}R_\alpha/R_\alpha| = q^3.$$

Hence X/Z_α contains exactly one non-central L_α-chief factor E, and E is a natural $\mathrm{SL}_4(q)$-module for L_α.

Let a be an automorphism of G_α which normalizes $M_{\alpha-1}$. Then X, Z_α and L_α are a-invariant. Hence also E is a-invariant. Thus a normalizes the two maximal parabolic subgroups of $L_\alpha R_\alpha/R_\alpha$ that contain $(M_{\alpha-1} \cap L_\alpha)R_\alpha/R_\alpha$. Since this holds for each $a \in \mathrm{Aut}(G_\alpha)$ with $M_{\alpha-1}^a = M_{\alpha-1}$, we get a contradiction to (II).

Now the claim follows from (3.3.8)(c). ■

3.4

In addition to (I)–(VII) we now assume

(VIII) $L_\alpha R_\alpha/R_\alpha \cong \Omega_{2n}^+(q)$ and $r = n$.

(3.4.1) Let (γ, γ') be a critical pair, $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$, $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$ and $\delta \in \Delta^{(2)}(\gamma') \cap \Delta^{(b-2)}(\gamma)$. Then there exists $g \in L_\gamma$ such that the following hold:

(a) $|U_\gamma \cap Z_\lambda \cap Z_\lambda^g| \leq q$.

(b) (μ^{gx}, δ) is a critical pair for each $x \in Q_\lambda$.

Proof. If n is even, then the proof for this is similar to the proof of (3.3.2). Hence assume that n is odd. Let P be a subgroup of M_λ containing R_γ such that P/R_γ is a parabolic subgroup of $L_\gamma R_\gamma/R_\gamma$ of type $\{1, \ldots, n-2\}$. As in the proof of (3.3.2) we can pick $g \in L_\gamma$ such that the intersection of the parabolic subgroups P/R_γ and P^g/R_γ is a Levi complement in both of them. Then

$$(*) \quad |U_\gamma \cap Z_\lambda \cap Z_\lambda^{gx}| = q, \quad \text{for each } x \in M_\lambda.$$

Suppose that $G_\gamma \neq \langle M_\lambda^{gx}, U_{\gamma'} \rangle$ for some $x \in M_\lambda$. Since M_λ^{gx} is a maximal subgroup of G_γ, it follows that $U_{\gamma'}$ is contained in M_λ^{gx} and hence normalizes Z_λ^{gx}. Thus

$$[(U_\gamma \cap Z_\lambda)(U_\gamma \cap Z_\lambda^{gx}), U_{\gamma'}] = [U_\gamma \cap Z_\lambda^{gx}, U_{\gamma'}] \leq [U_\gamma, U_{\gamma'}] \cap Z_\lambda^{gx} \leq U_\gamma \cap Z_\lambda \cap Z_\lambda^{gx}.$$

This means that $U_{\gamma'}R_\gamma/R_\gamma$ is contained in the largest normal p-subgroup of $N_{L_\gamma}(U_\gamma \cap Z_\lambda \cap Z_\lambda^{gx})R_\gamma/R_\gamma$, which is a parabolic subgroup of cotype 1 of $L_\gamma R_\gamma/R_\gamma$. But in $\mathrm{O}_{2n}^+(q)$ no offending subgroup for the natural module is contained in the largest normal p-subgroup of a parabolic subgroup of cotype 1, by (B.5.1.4). This contradiction shows that

$$(**) \quad G_\gamma = \langle M_\lambda^{gx}, U_{\gamma'} \rangle, \quad \text{for each } x \in M_\lambda.$$

Since $\langle U_\nu \mid \nu \in \Delta(\lambda) \rangle$ is a nontrivial characteristic subgroup of M_λ, we can choose $\nu \in \Delta(\lambda)$ such that

$$U_\nu U_\gamma \ntrianglelefteq G_\gamma.$$

Assume that $|[U_\gamma, U_{\gamma'}]| = q^n$. Then

$$[U_\gamma, U_{\gamma'}] = [Q_\delta, U_{\gamma'}].$$

Hence, if (ν^{gx}, δ) is not a critical pair for some $x \in Q_\lambda$, then

$$[U_\nu^{gx}, U_{\gamma'}] \leq [Q_\delta, U_{\gamma'}] \leq U_\gamma,$$

contrary to $(**)$ and the choice of ν. Now (1.2.1)(e) implies that we can choose $\nu = \mu$. Now assume that $|[U_\gamma, U_{\gamma'}]| \neq q^n$. Then $[U_\gamma, U_{\gamma'}]$ is a proper $GF(q)$-subspace of $U_\gamma \cap Z_\lambda$. Since $M_\lambda \cap L_\gamma$ acts transitively on the set of 1-dimensional $GF(q)$-subspaces of $U_\gamma \cap Z_\lambda$, $(*)$ implies that there exists $y \in M_\lambda \cap L_\gamma$ such that

$$[U_\gamma, U_{\gamma'}] \cap Z_\lambda \cap Z_\lambda^{gy} = 1.$$

Since $[U_\gamma, U_{\gamma'}] \leq Z_\lambda$ and $[Z_\lambda, Q_\lambda] = 1$, it follows that

$$[U_\gamma, U_{\gamma'}] \cap Z_\lambda^{gyx} = 1, \quad \text{for each } x \in Q_\lambda.$$

Therefore,

$$(* * *) \quad [U_\gamma, U_{\gamma'}] \cap [U_\nu^{gyx}, U_{\gamma'} \cap Q_\gamma] \leq [U_\gamma, U_{\gamma'}] \cap [U_\nu^{gyx}, Q_\gamma] \leq$$

$$[U_\gamma, U_{\gamma'}] \cap Z_\lambda^{gyx} = 1, \quad \text{for each } x \in Q_\lambda.$$

From $(**)$, $(* * *)$, (A.1.5)(b), and the choice of ν it follows that (ν^{gyx}, δ) is a critical pair for each $x \in Q_\lambda$. As above, (1.2.1)(e) implies that we can choose $\nu = \mu$. Now the claim holds with gy in place of g. ∎

Choose $g_\alpha \in L_\alpha$ such that (a) and (b) of (3.4.1) are satisfied for $(\alpha, \alpha', g_\alpha)$ in place of (γ, γ', g). Define

$$\alpha - 1 := (\alpha + 1)^{g_\alpha} \quad \text{and}$$

$$\alpha - 2 := (\alpha + 2)^{g_\alpha}.$$

(3.4.2) $b \leq 4$.

Proof. Suppose that $b > 4$. Note that (3.4.1) implies that if (γ, γ') is any critical pair and $\mu \in \Delta^{(2)}(\gamma) \cap \Delta^{(b-2)}(\gamma')$, then $U_\gamma U_\mu \ntrianglelefteq G_\gamma$. As in the proof of (3.2.2) it follows that we may assume that

$$U_{\alpha+2} \cap T_\alpha = 1.$$

Let A be a set of representatives for the cosets of Q_α in $Q_{\alpha+1}$. Let E be a complement to $U_\alpha \cap Z_{\alpha+1} \cap Z_{\alpha-1}$ in $U_\alpha \cap Z_{\alpha-1}$. Define

$$X_a := \{(x, a) \mid 1 \neq x \in [U_{\alpha-2}^a, U_{\alpha'-2}] \cap T_\alpha E\}, \quad \text{for each } a \in A, \text{ and}$$

$$X := \bigcup_{a \in A} [U_{\alpha-2}^a, U_{\alpha'-2}] \cap T_\alpha E^a \setminus \{1\}.$$

Then by (3.1.1)(c)

$$|\bigcup_{a\in A} X_a| = \sum_{a\in A} |X_a| \geq |A|(q^2-1).$$

Note that $U_{\alpha+2}\cap T_\alpha = 1$ implies that

$$U_{\alpha-2}^a \cap T_\alpha = 1, \quad \text{for each } a \in A.$$

Therefore by (3.1.1)(d) each $x \in X$ is contained in at most $q^{1-n}|A|$ of the subgroups $[U_{\alpha-2}^a, U_{\alpha'-2}] \cap T_\alpha E^a$ with $a \in A$. Hence

$$|\bigcup_{a\in A} X_a| = \sum_{x\in X} |\{a \in A \mid (x,a) \in X_a\}| \leq q^{1-n}|A|\,|X|.$$

Therefore,

$$(*) \quad |X| \geq (q^2-1)q^{n-1}.$$

Since $b > 4$, V_α acts quadratically on $U_{\alpha'-2}$. By (A.1.5)(a), it follows that $[V_\alpha, U_{\alpha'-2}]$ is an isotropic subspace of $U_{\alpha'-2}$. Hence

$$(**) \quad |X| \leq |[V_\alpha, U_{\alpha'-2}]| - 1 \leq q^n - 1.$$

From $(*)$ and $(**)$ it follows that $q^{n+1} - q^{n-1} \leq q^n - 1$, a contradiction. ∎

(3.4.3) Assume that $b > 2$.

(a) $U_{\alpha-2} \cap Z_{\alpha-1} \leq T_\alpha$.

(b) $L_{\alpha-2,\alpha-1,1} \leq R_{\alpha-2}R_\alpha$.

(c) $U_{\alpha+2}R_{\alpha-2}/R_{\alpha-2} = Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2}$.

(d) $n = 3$.

Proof. (a) and (b) follow from (3.4.2) and (3.1.4)(e). Since $Q_{\alpha-1}R_{\alpha-2}/R_{\alpha-2}$ is an irreducible $L_{\alpha-2,\alpha-1,1}$-module, (c) follows from (b) and (3.4.2). Note that (c) implies that

$$|U_{\alpha-2} : U_{\alpha-2} \cap Q_{\alpha+2}| = q^n \quad \text{and} \quad |U_{\alpha+2} : U_{\alpha+2} \cap Q_{\alpha-2}| = q^{\frac{1}{2}n(n-1)}.$$

Hence (d) follows from (3.1.1)(g). ∎

Choose $g_{\alpha-2} \in L_{\alpha-2}$ such that (a) and (b) of (3.4.1) are satisfied for $(\alpha-2, \alpha'-2, g_{\alpha-2})$ in place of (γ, γ', g). Define

$$\alpha - 3 := (\alpha - 1)^{g_{\alpha-2}} \quad \text{and}$$

$$\alpha - 4 := \alpha^{g_{\alpha-2}}.$$

(3.4.4) $b = 2$.

Proof. Suppose that $b > 2$. Define

$$Y := \langle U_{\alpha-2}^{L_\alpha} \rangle U_\alpha.$$

Then, by (3.4.3)(c)(d)

$$|Y : Y \cap Q_{\alpha-2}| = q^3.$$

Since $\alpha - 4, \ldots, \alpha$ satisfy the same assumptions as $\alpha - 2, \ldots, \alpha + 2$, (3.4.3)(c) implies that

$$[U_{\alpha-4}, Y \cap Q_{\alpha-2}] \leq [U_{\alpha-4}, Q_{\alpha-3}] = [U_{\alpha-4}, U_\alpha] \leq U_\alpha.$$

Therefore,

$$(*) \quad |Y : C_Y(U_{\alpha-4}, [Y, Q_\alpha]U_\alpha)| \leq q^3.$$

If $[Y, L_\alpha] \leq [Y, Q_\alpha]U_\alpha$, then $U_{\alpha+2} \leq [Y, Q_\alpha]U_\alpha U_{\alpha-2} \leq Q_{\alpha-2}$, a contradiction. Hence

$$\overline{Y} := Y/C_Y(L_\alpha, [Y, Q_\alpha]U_\alpha) \neq 1.$$

Let W be a subgroup of Y containing $C_Y(L_\alpha, [Y, Q_\alpha]U_\alpha)$ such that

$$\overline{W} := W/C_Y(L_\alpha, [Y, Q_\alpha]U_\alpha)$$

is an irreducible $L_\alpha R_\alpha$-submodule of $\overline{Y}$. Note that (3.4.3)(b) implies that $\overline{Y}$, regarded as R_α-module, is a direct sum of natural $\mathrm{SL}_3(q)$-modules. As in the proof of (3.3.11), it follows from (A.3.1) that

$$\overline{W} \cong E \otimes_{GF(q)} D,$$

where E is an irreducible L_α-submodule of $\overline{W}$ and D is a natural $\mathrm{SL}_3(q)$-module for R_α. Now $(*)$ implies that

$$(**) \quad [\overline{Y}, L_\alpha] = \overline{W}$$

and E is a natural $\mathrm{SL}_4(q)$-module for L_α with

$$|C_E(Q_{\alpha-1})| = |C_E(U_{\alpha-4})| = q^3.$$

In particular, $C_{\overline{W}}(Q_{\alpha-1})$ is the unique irreducible $M_{\alpha-1} \cap L_\alpha$-submodule of $\overline{W}$ and has order q^9. Since $\overline{U_{\alpha-2}} := U_{\alpha-2}C_Y(L_\alpha, [Y, Q_\alpha]U_\alpha)/C_Y(L_\alpha, [Y, Q_\alpha]U_\alpha)$ is an $M_{\alpha-1}$-submodule of $\overline{Y}$ with $|\overline{U_{\alpha-2}}| = q^3$, it follows that

$$(***) \quad \overline{W} \cap \overline{U_{\alpha-2}} = 1.$$

From $(**)$ we get that $\overline{W}\,\overline{U_{\alpha-2}}$ is an L_α-submodule of $\overline{Y}$. Then $(***)$ implies that, regarded as L_α-module, $\overline{W}$ has a complement in $\overline{W}\,\overline{U_{\alpha-2}}$, since $M_{\alpha-1} \cap L_\alpha$ contains a Sylow p-subgroup of L_α. By $(**)$, this complement is a trivial L_α-module of order q^3. But the definition of $\overline{Y}$ implies that $\overline{Y}$ has no trivial L_α-submodule, a contradiction.

∎

3.5

In this section we assume that (I)–(VII) hold. Note that by the results of the previous sections we have $b = 2$.

(3.5.1) $[U_{\alpha'}, Q_{\alpha+1}, L_\alpha] = U_\alpha$.

Proof. Suppose that $[U_{\alpha'}, Q_{\alpha+1}, L_\alpha] \neq U_\alpha$. In particular, we are not in the situation of section (3.4). Then by (3.2.2), (3.3.2), and (3.3.4) we can choose $g \in L_\alpha$ such that

$$G_\alpha = \langle M_{\alpha+1}^g, U_{\alpha'} \rangle \quad \text{and}$$

$$Z_{\alpha+1}^g \cap Z_{\alpha+1} \cap U_\alpha = 1.$$

Note that with respect to the nondegenerate $L_{\alpha'}$-invariant bilinear form on $U_{\alpha'}$ we have

$$[U_{\alpha'}, Q_{\alpha+1}]^\perp = U_{\alpha'} \cap Z_{\alpha+1} = [U_{\alpha'}, U_\alpha],$$

by (A.1.5)(a). Again by (A.1.5)(a) and

$$[[U_{\alpha'}^g, Q_{\alpha+1}^g], [U_{\alpha'}, Q_{\alpha+1}]] \leq [U_{\alpha'}^g, Q_{\alpha+1}^g, Q_\alpha] \cap [U_{\alpha'}, Q_{\alpha+1}, Q_\alpha] \leq$$

$$Z_{\alpha+1}^g \cap Z_{\alpha+1} \cap U_\alpha = 1$$

it follows that

$$[U_{\alpha'}, [U_{\alpha'}^g, Q_{\alpha+1}^g]] \leq [U_{\alpha'}, U_\alpha].$$

Now the choice of g implies that $[U_{\alpha'}^g, Q_{\alpha+1}^g]U_\alpha$ and hence also $[U_{\alpha'}, Q_{\alpha+1}]U_\alpha$ is normalized by G_α. But then $[[U_{\alpha'}, Q_{\alpha+1}]U_\alpha, U_{\alpha'}] \leq U_\alpha$ and $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$ imply that $[U_{\alpha'}, Q_{\alpha+1}, L_\alpha] \leq U_\alpha$, a contradiction. ■

(3.5.2) Assume that $L_\alpha R_\alpha / R_\alpha \not\cong \Omega_6^+(q)$. Then one of the following holds:

(a) Case (1),(2), or (5) holds in (VI), $r = 3$, $q = 2$, and $[Q_\alpha, L_\alpha] = U_\alpha$.

(b) Case (5) holds in (VI), $r = 2$, $q \in \{2, 4\}$, and $[Q_\alpha, L_\alpha] = U_\alpha$.

Proof. Since $b = 2$, it follows as in [14](4.1) that case (4) in (VI) is impossible. From (3.5.1) and $L_\alpha' = L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$ it follows that

$$[Q_\alpha, L_\alpha] = [Q_\alpha, L_\alpha, L_\alpha] \leq \langle [Q_\alpha, U_{\alpha'}, L_\alpha]^{L_\alpha} \rangle = U_\alpha.$$

If $Q_{\alpha+1} = Q_\alpha Q_{\alpha'}$, then $[Q_{\alpha+1}, U_\alpha] = [Q_{\alpha'}, U_\alpha] \leq Q_{\alpha'}$. If $Q_{\alpha+1} \neq Q_\alpha Q_{\alpha'}$, then $[Q_{\alpha+1}, Q_\alpha] \leq Q_{\alpha'}$, since $Q_\alpha Q_{\alpha'} \trianglelefteq M_{\alpha+1}$. Hence in any case $U_{\alpha'} R_\alpha / R_\alpha$ is an $M_{\alpha+1}$-submodule of $Z(Q_{\alpha+1} R_\alpha / R_\alpha)$.

Note that $U_{\alpha'} R_\alpha / R_\alpha$, regarded as $GF(p)(M_{\alpha+1} \cap L_{\alpha'})$-module, is isomorphic to the dual of $U_{\alpha'} \cap Z_{\alpha+1}$. By (B.3.1.5), (B.4.1.5), (B.5.1.7), and (B.5.2.6) we get the following:

$r \leq 3$ and

$r = 3$, unless case (5) holds in (IV) and $p = 2$.

Suppose that neither (a) nor (b) holds. In particular, $(r, q) \neq (2, 2)$. Let K be the p-component of $M_{\alpha+1} \cap L_\alpha$ that acts nontrivially on $U_\alpha \cap Z_{\alpha+1}$. Then $K \leq L_{\alpha'}$, for otherwise $K \leq R_{\alpha'}$ and hence $U_\alpha \leq [U_\alpha, K] \leq R_{\alpha'}$, a contradiction. Since $L_\alpha R_\alpha / R_\alpha \not\cong \Omega_6^+(q)$, each automorphism of $K/O_p(K)$ is induced by some element of L_α. Together with (A.4.2) this implies that $M_{\alpha+1}$ induces an inner automorphism on $L_\alpha R_\alpha / R_\alpha$. Hence

$(*)$ $G_\alpha = L_\alpha R_\alpha$.

Since $M_{\alpha+1}$ induces $\mathrm{GL}_3(q)$ on $U_{\alpha'}/[U_{\alpha'}, Q_{\alpha+1}]$, it follows from $(*)$ and $U_{\alpha'} \cap R_\alpha = [U_{\alpha'}, Q_{\alpha+1}]$ that $M_{\alpha+1} \cap L_\alpha$ induces $\mathrm{GL}_3(q)$ on $U_{\alpha'} R_\alpha / R_\alpha$. By the structure of $(M_{\alpha+1} \cap L_\alpha) R_\alpha / R_\alpha$ this implies that

$p = 2$.

Suppose that $U_\alpha \cap Z_{\alpha+1} \neq [U_\alpha, Q_{\alpha+1}]$. Let $g \in A := C_{M_{\alpha+1} \cap L_{\alpha'}}([U_{\alpha'}, Q_{\alpha+1}], U_{\alpha'} \cap Z_{\alpha+1})$. If $[U_{\alpha'}, Q_{\alpha+1}] = [U_\alpha, Q_{\alpha+1}]$, then $[U_\alpha, Q_{\alpha+1}, g] \leq U_\alpha \cap Z_{\alpha+1}$ by the choice of g. If $[U_{\alpha'}, Q_{\alpha+1}] \neq [U_\alpha, Q_{\alpha+1}]$, then $[U_\alpha, Q_{\alpha+1}] \cap U_{\alpha'} \leq Z_{\alpha+1}$, and since $[U_\alpha, Q_{\alpha+1}] \leq Q_{\alpha'}$, we get $[U_\alpha, Q_{\alpha+1}, g] \leq [U_\alpha, Q_{\alpha+1}] \cap [Q_{\alpha'}, L_{\alpha'}] \leq U_\alpha \cap Z_{\alpha+1}$. Hence in any case

$(**)$ $[U_\alpha, Q_{\alpha+1}, g] \leq U_\alpha \cap Z_{\alpha+1}$.

From $(*)$ it follows that there exist $g_1 \in M_{\alpha+1} \cap L_\alpha$ and $g_2 \in R_\alpha$ such that $g = g_1 g_2$. By (1.1.2)(d), g_2 induces an $GF(p)L_\alpha$-endomorphism on U_α. Together with $p = 2$ and $(**)$ it follows that $[U_\alpha, Q_{\alpha+1}, g_1] \leq U_\alpha \cap Z_{\alpha+1}$. Now it follows from $(**)$ that g_2 centralizes $[U_\alpha, Q_{\alpha+1}]/(U_\alpha \cap Z_{\alpha+1})$. Since $U_\alpha \cap Z_{\alpha+1} \neq [U_\alpha, Q_{\alpha+1}]$, g_2 centralizes also U_α. Therefore,

$$A \leq (M_{\alpha+1} \cap L_\alpha) C_{G_\alpha}(U_\alpha).$$

In particular, $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ is dual to $U_\alpha \cap Z_{\alpha+1}$ as a module for A. But now (B.4.1.5), (B.5.1.7), and (B.5.2.6) imply that (a) or (b) holds, a contradiction.

Hence $U_\alpha \cap Z_{\alpha+1} = [U_\alpha, Q_{\alpha+1}]$. This implies that $r = 3$ and (5) holds in (IV) Note that R_α centralizes $U_{\alpha'}/[U_{\alpha'}, Q_{\alpha+1}]$, since $[U_{\alpha'}, R_\alpha] \leq U_{\alpha'} \cap R_\alpha = U_{\alpha'} \cap Q_\alpha$. Then in $G_{\alpha'}/R_{\alpha'}$ we see that this implies that R_α does not centralize $Q_{\alpha+1}/Q_\alpha Q_{\alpha'}$ unless $R_\alpha \leq Q_{\alpha+1} R_{\alpha'}$. Since $[Q_{\alpha+1}, R_\alpha] \leq Q_\alpha$, we get

$$R_\alpha \leq Q_{\alpha+1} R_{\alpha'}.$$

In particular, R_α centralizes $U_\alpha R_{\alpha'}/R_{\alpha'}$, i.e.,

$$[U_\alpha, R_\alpha] \leq U_\alpha \cap R_{\alpha'} = U_\alpha \cap Q_{\alpha'}.$$

Since U_α is an irreducible L_α-module, it follows that $[U_\alpha, R_\alpha] = 1$. Together with $(*)$ this implies that $U_\alpha \cap Z_{\alpha+1}$ is dual to $U_\alpha/[U_\alpha, Q_{\alpha+1}]$, regarded as a module for $M_{\alpha+1}$. Now (B.4.1.5) implies that (a) holds, a contradiction. ∎

Note that (D.2.2) shows that the assumption $L_\alpha R_\alpha/R_\alpha \not\cong \Omega_6^+(q)$ in (3.5.2) is necessary.

Chapter 4

Determining the action of L on R, Part 2

In this chapter we assume (I)–(IV) and

(V) $L_\alpha R_\alpha/R_\alpha \cong \mathrm{PSp}_{2n}(q)'$ $(q = p^k)$ for some $n, k \in \mathbb{N}$ with $n \geq 2$.

(VI) $U_\alpha T_\alpha/T_\alpha$ is a natural $\mathrm{Sp}_{2n}(q)$-module for L_α.

(VII) $(M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha$ is a parabolic subgroup of cotype 1 in $L_\alpha R_\alpha/R_\alpha$.

For each $\mu \sim \alpha$ and $\lambda \in \Delta(\mu)$, let $L_{\mu,\lambda}$ be a subgroup of $M_\lambda \cap L_\mu$ containing $L_\mu \cap R_\mu$ such that $L_{\mu,\lambda}R_\mu/R_\mu$ is a Levi complement of the parabolic subgroup $(M_\lambda \cap L_\mu)R_\mu/R_\mu$ in $L_\mu R_\mu/R_\mu$.

For each $\gamma \sim \alpha$ define

$$\Lambda(\gamma) := \{\mu \in \Delta^{(2)}(\gamma) \mid U_\mu Z_\gamma \ntrianglelefteq G_\gamma\},$$

$$X_{\gamma,\lambda} := \langle [U_\mu, Q_\lambda] \mid \mu \in \Delta(\lambda) \cap \Lambda(\gamma)\rangle Z_\gamma, \quad \text{for each } \lambda \in \Delta(\gamma),$$

$$X_\gamma := \langle X_{\gamma,\lambda} \mid \lambda \in \Delta(\gamma)\rangle,$$

$$Y_{\gamma,\lambda} := \langle U_\mu \mid \mu \in \Delta(\lambda) \cap \Lambda(\gamma)\rangle Z_\gamma, \quad \text{for each } \lambda \in \Delta(\gamma), \text{ and}$$

$$Y_\gamma := \langle U_\mu \mid \mu \in \Lambda(\gamma)\rangle Z_\gamma.$$

4.1

(4.1.1) (a) $|U_\alpha \cap T_\alpha| \in \{1, q\}$.

(b) If A is an offending subgroup for U_α with $A \leq Q_{\alpha+1}$, then

$$C_{Q_{\alpha+1}}(M_{\alpha+1}, Q_\alpha) \leq AQ_\alpha.$$

Moreover, $U_\alpha \cap T_\alpha \not\leq [U_\alpha, A]$ if and only if $q = 2$, $U_\alpha \cap T_\alpha \neq 1$, and $C_{Q_{\alpha+1}}(M_{\alpha+1}, Q_\alpha) = AQ_\alpha$.

(c) $|U_\alpha : U_\alpha \cap Q_{\alpha'}| = |U_{\alpha'} : U_{\alpha'} \cap Q_\alpha|$.

Proof. (a) Assume that $U_\alpha \cap T_\alpha \neq 1$. Then $\langle U_\mu \cap T_\mu \mid \mu \in \Delta(\alpha+1)\rangle$ is a nontrivial characteristic subgroup of $M_{\alpha+1}$. Hence there exists $\mu \in \Delta(\alpha+1)$ such that $U_\mu \cap T_\mu \not\leq T_\alpha$. Then $(U_\mu \cap T_\mu)T_\alpha/T_\alpha$ is an $M_{\alpha+1}$-submodule of Z_α/T_α.
Suppose that $(U_\mu \cap T_\mu) \cap U_\alpha T_\alpha \leq T_\alpha$, i.e.,

$$(*) \quad U_\alpha(U_\mu \cap T_\mu)T_\alpha/T_\alpha = (U_\alpha T_\alpha/T_\alpha) \times (U_\mu \cap T_\mu)T_\alpha/T_\alpha.$$

Note that $U_\alpha(U_\mu \cap T_\mu)T_\alpha/T_\alpha$ is an L_α-submodule of Z_α/T_α. By $(*)$, $U_\alpha T_\alpha/T_\alpha$ has an $(M_{\alpha+1} \cap L_\alpha)$-complement in this module. Since $M_{\alpha+1} \cap L_\alpha$ contains a Sylow 2-subgroup of L_α, it follows that $U_\alpha T_\alpha/T_\alpha$ has also an L_α-complement in this module. Now $[Z_\alpha, L_\alpha] = U_\alpha$ and $C_{Z_\alpha}(L_\alpha, T_\alpha) = T_\alpha$ imply that $U_\mu \cap T_\mu \leq T_\alpha$, contrary to the choice of μ.
Hence $(U_\mu \cap T_\mu) \cap U_\alpha T_\alpha \not\leq T_\alpha$. Since $(U_\alpha \cap Z_{\alpha+1})T_\alpha/T_\alpha$ is the only minimal $M_{\alpha+1}$-submodule of $U_\alpha T_\alpha/T_\alpha$ and has order q, it follows that

$$|U_\mu \cap T_\mu| \geq |((U_\mu \cap T_\mu) \cap U_\alpha T_\alpha)T_\alpha/T_\alpha| = q.$$

By [10] we also have $|U_\mu \cap T_\mu| \leq q$. Hence $|U_\alpha \cap T_\alpha| = |U_\mu \cap T_\mu| = q$.

(b) This follows from (B.3.1.4)(b) and (B.4.1.3)(b).

(c) The proof for this is tha same as the proof for (3.1.1)(g). ∎

(4.1.2) If $\mu \in \Delta(\alpha+1)$, then $O^p(L_{\mu,\alpha+1}) \leq R_\mu R_\alpha(M_{\alpha+1} \cap L_\alpha)$.

Proof. This follows as in the proof of (3.1.2). ∎

(4.1.3) Assume that $b > 2$. Let (γ, γ') be a critical pair and $\gamma + i = \gamma' - b + i \in \Delta^{(i)}(\gamma) \cap \Delta^{(b-i)}(\gamma')$ for each $i \in \{1, \ldots, b\}$. Then there exists $g \in L_\gamma$ such that

(a) The intersection of the parabolic subgroups $(M_{\gamma+1} \cap L_\gamma)R_\gamma/R_\gamma$ and $(M^g_{\gamma+1} \cap L_\gamma)R_\gamma/R_\gamma$ of $L_\gamma R_\gamma/R_\gamma$ is the product of a Levi complement and a Cartan subgroup in both of them.

Moreover, for any such $g \in L_\gamma$ the following hold:

(b) $G_\gamma = \langle M^g_{\gamma+1}, U_{\gamma'} \rangle$.

(c) Assume that $(\mu, \gamma + b - 2)$ is not a critical pair for some $\mu \in \Delta((\gamma + 1)^g) \cap \Lambda(\gamma)$. Then $U_\gamma \cap T_\gamma \neq 1$, $q = 2$, $1 \neq [U_\mu, U_{\gamma'} \cap Q_\gamma] \leq T_\gamma$, and $[U_\mu, Q^g_{\gamma+1}]U_\gamma \trianglelefteq G_\gamma$.

(d) $(\mu, \gamma + b - 2)$ is a critical pair for some $\mu \in \Delta((\gamma + 1)^g)$.

Proof. By (4.1.1)(c) it suffices to prove this for $(\gamma, \gamma') = (\alpha, \alpha')$. As in the proof of (3.2.2) it follows that we can choose $g \in L_\alpha$ such that (a) holds. Then

$$(*) \quad U_\alpha T_\alpha/T_\alpha = [U_\alpha, Q_{\alpha+1}]T_\alpha/T_\alpha \times (U_\alpha \cap Z^g_{\alpha+1})T_\alpha/T_\alpha.$$

Since $[U_\alpha, U_{\alpha'}] \not\leq T_\alpha$, it follows from $(*)$ that $U_{\alpha'}$ does not normalize $Z^g_{\alpha+1}$ and hence $U_{\alpha'} \not\leq M^g_{\alpha+1}$. Thus (b) holds.

From (4.1.1)(c) it follows that we can apply (4.1.1)(b) with (α', α) in place of (α, α') and get

$$C_{Q_{\alpha'-1}}(M_{\alpha'-1}, Q_{\alpha'}) \leq U_\alpha Q_{\alpha'},$$

Therefore,

$$(**) \quad U_{\alpha'} \cap Q_\alpha = C_{U_{\alpha'}}(U_\alpha Q_{\alpha'}) \leq C_{U_{\alpha'}}(C_{Q_{\alpha'-1}}(M_{\alpha'-1}, Q_{\alpha'})) = [U_{\alpha'}, Q_{\alpha'-1}].$$

Assume that $U_\mu \leq Q_{\alpha'-2}$ for some $\mu \in \Delta((\alpha+1)^g) \cap \Lambda(\alpha)$. From $U_\mu \leq Q_{\alpha'-2}$ and $(**)$ we get

$$|[U_{\alpha'} \cap Q_\alpha, U_\mu]| \leq |[U_{\alpha'}, Q_{\alpha'-1}, Q_{\alpha'-1}]| = q.$$

From this it follows that $(U_{\alpha'} \cap Q_\alpha)Q_\mu/Q_\mu$ is a subgroup of order at most 2 in $C_{Q^g_{\alpha+1}}(O^{p'}(M^g_{\alpha+1}), Q_\mu)/Q_\mu$, centralizing $[U_\mu, Q^g_{\alpha+1}]$. Therefore (A.1.5)(b) (applied to $[U_\mu, Q^g_{\alpha+1}]U_\alpha$, $[U_\mu, Q^g_{\alpha+1}]$, and $U_{\alpha'}T_{\alpha'}/T_{\alpha'}$ in place of A, B, and V, respectively) shows that $U_{\alpha'}$ normalizes $[U_\mu, Q^g_{\alpha+1}]U_\alpha$. Then (b) implies that

$$[U_\mu, Q^g_{\alpha+1}]U_\alpha \trianglelefteq G_\alpha.$$

If $[U_\mu, U_{\alpha'} \cap Q_\alpha] = 1$, then again (A.1.5)(b) (with $U_\mu U_\alpha$, U_μ, and $U_{\alpha'}T_{\alpha'}/T_{\alpha'}$ in place of A, B, and V, respectively) shows that $U_{\alpha'}$ normalizes $U_\mu U_\alpha$, contrary to (b) and the choice of μ. Hence

$$[U_\mu, U_{\alpha'} \cap Q_\alpha] \neq 1.$$

Together with $|(U_{\alpha'} \cap Q_\alpha)Q_\mu/Q_\mu| \leq 2$ it follows that

$$|U_{\alpha'} \cap Q_\alpha : U_{\alpha'} \cap Q_\alpha \cap Q_\mu| = 2.$$

Note that $T_{\alpha'} \leq Q_\alpha \cap Q_\mu$, $(U_{\alpha'} \cap Q_\alpha)/(U_{\alpha'} \cap T_{\alpha'}) = C_{U_{\alpha'}/(U_{\alpha'} \cap T_{\alpha'})}(U_\alpha)$, and $(U_{\alpha'} \cap Q_\alpha \cap Q_\mu)/(U_{\alpha'} \cap T_{\alpha'}) = C_{U_{\alpha'}/(U_{\alpha'} \cap T_{\alpha'})}(U_\alpha U_\mu)$. Since $\mathrm{End}_{GF(2)G_{\alpha'}}(U_{\alpha'}/(U_{\alpha'} \cap T_{\alpha'})) \cong GF(q)$, it follows that both $|U_{\alpha'} \cap Q_\alpha|$ and $|U_{\alpha'} \cap Q_\alpha \cap Q_\mu|$ are powers of q. Hence

$$q = 2.$$

Now $(U_{\alpha'} \cap Q_\alpha)Q_\mu = C_{Q^g_{\alpha+1}}(M_{\alpha+1}, Q_\mu) \trianglelefteq M^g_{\alpha+1}$, and hence $[U_\mu, U_{\alpha'} \cap Q_\alpha]$ is a subgroup of $Z^g_{\alpha+1}$ that is normalized by $\langle M^g_{\alpha+1}, U_{\alpha'}\rangle = G_\alpha$. Thus $[U_\mu, U_{\alpha'} \cap Q_\alpha] \leq T_\alpha$. Suppose that $U_\alpha \cap T_\alpha = 1$. Then $[U_\alpha, U_{\alpha'}] = [U_\mu, U_{\alpha'} \cap Q_\alpha]$, and once again (A.1.5)(b) shows that $U_{\alpha'}$ normalizes $U_\alpha U_\mu$, contrary to (b) and the choice of μ. Hence (c) holds.

Suppose that (d) is false. Then (c) implies that $[U_\mu, Q^g_{\alpha+1}]U_\alpha \trianglelefteq G_\alpha$, for each $\mu \in \Delta((\alpha+1)^g) \cap \Lambda(\alpha)$. Hence, by conjugation we get for each $\nu \sim \alpha$ that

$$(***) \quad [U_\eta, Q_\lambda]U_\nu \trianglelefteq G_\nu, \quad \text{for all } \lambda \in \Delta(\nu) \text{ and } \eta \in \Delta(\lambda) \cap \Lambda(\nu).$$

Pick $\mu \in \Delta((\alpha+1)^g) \cap \Lambda(\alpha)$. Choose $x \in L_{\alpha'-2}$ such that $\alpha'-3 = (\alpha'-1)^x$. Since $[U_{\alpha'}, Q_{\alpha'-1}]U_{\alpha'-2} \trianglelefteq G_{\alpha'-2}$ by $(***)$, we get

$$[U_{\alpha'}, Q_{\alpha'-1}] \leq [U^x_{\alpha'}, Q_{\alpha'-3}]U_{\alpha'-2}.$$

Now (c) shows that U_μ does not centralize $[U^x_{\alpha'}, Q_{\alpha'-3}]$. But $(***)$ (with $\alpha'-4$, $\alpha'-3$, and $(\alpha')^x$ in place of ν, λ, and η, respectively) implies that $[U^x_{\alpha'}, Q_{\alpha'-3}] \leq U_{\alpha'-4}U_\delta$, for some $\delta \in \Delta^{(2)}(\alpha'-4) \cap \Delta^{(b-2)}(\mu)$, a contradiction. ∎

(4.1.4) Assume that $b > 2$. There exists a path $(\gamma-4, \gamma-3, \ldots, \gamma+b)$ with the following properties:

(a) $(\gamma, \gamma+b)$, $(\gamma-2, \gamma+b-2)$ and $(\gamma-4, \gamma+b-4)$ are critical pairs.

(b) $\gamma - 4 = \gamma^g$ for some $g \in L_{\gamma-2}$.

(c) For each $i \in \{0, \ldots, \frac{b}{2}\}$, the intersection of the parabolic subgroups

$$(M_{\gamma+2i-1} \cap L_{\gamma+2i-2})R_{\gamma+2i-2}/R_{\gamma+2i-2} \quad \text{and}$$

$$(M_{\gamma+2i-3} \cap L_{\gamma+2i-2})R_{\gamma+2i-2}/R_{\gamma+2i-2}$$

of $L_{\gamma+2i-2}R_{\gamma+2i-2}/R_{\gamma+2i-2}$ is the product of a Levi complement and a Cartan subgroup in both of them.

Proof. By (4.1.3), we can inductively choose $g_{\alpha-2i} \in L_{\alpha-2i}$ and $\alpha-2i-2 \in \Delta(\alpha-2i+1)^{g_{\alpha-2i}}$, for each $i \in \{0, \ldots, b\}$, such that $(\alpha-2i-2, \alpha+b-2i-2)$ is a critical pair, and (a)–(c) of (4.1.3) are satisfied for $g_{\alpha-2i}, \alpha-2i, \ldots, \alpha-2i+b$ in place of $g, \gamma, \ldots, \gamma+b$, and then put

$$\alpha - 2i - 1 := (\alpha-2i+1)^{g_{\alpha-2i}}.$$

If there exists $i \in \{\frac{b}{2}, \ldots, b-1\}$ such that $((\alpha - 2i + 2)^{g_{\alpha-2i}}, \alpha + b - 2i - 2)$ is a critical pair, then $\alpha - 2i - 2$ can be chosen as $(\alpha - 2i + 2)^{g_{\alpha-2i}}$, and then $(\alpha - 2i - 2, \alpha - 2i - 1, \ldots \alpha + b - 2i + 2)$ is a path with the desired properties.

Suppose that $((\alpha - 2i + 2)^{g_{\alpha-2i}}, \alpha + b - 2i - 2)$ is not a critical pair for any $i \in \{\frac{b}{2}, \ldots, b-1\}$. In particular, by (4.1.3)(c),

$$(*) \quad [U_{\alpha-b+2}, Q_{\alpha-b+1}]U_{\alpha-b} \trianglelefteq G_{\alpha-b} \quad \text{and}$$

$$(**) \quad [U_{\alpha-2b+4}^{g_{\alpha-2b+2}}, Q_{\alpha-2b+1}]U_{\alpha-2b+2} = [U_{\alpha-2b+4}, Q_{\alpha-2b+3}]U_{\alpha-2b+2} \trianglelefteq G_{\alpha-2b+2}.$$

Put $\mu := (\alpha - 2b + 4)^{g_{\alpha-2b+2}}$ and $\nu := (\alpha - b + 2)^{g_{\alpha-b}}$. Since U_μ does not centralize $[U_{\alpha-b+2}, Q_{\alpha-b+1}]$ by (4.1.3)(c), it follows from $(*)$ that (μ, ν) is a critical pair with $|U_\nu : U_\nu \cap Q_\mu| > q$. But $(**)$ implies that $|U_\mu : U_\mu \cap Q_\nu| = q$, a contradiction to (4.1.1)(c). ■

In the following, assume that $b > 2$ and (α, α') is chosen such that the path $(\alpha, \ldots, \alpha')$ between α and α' can be extented to a path $(\alpha - 4, \alpha - 3, \alpha - 2, \alpha - 1, \alpha, \ldots, \alpha')$ as in (4.1.4). Then

$$\alpha - 4 = \alpha^{g_{\alpha-2}}$$

for some $g_{\alpha-2} \in L_{\alpha-2}$.

(4.1.5) $b \leq 4$.

Proof. Suppose that $b > 4$. Then the same argument as in the proof of (1.2.5) shows that

$$A_{\alpha,\alpha+2} \leq T_\alpha.$$

On the other hand, (4.1.1)(b) implies that

$$[U_\alpha, C_{Q_{\alpha+1}}(M_{\alpha+1}, Q_\alpha)] \leq A_{\alpha,\alpha+2},$$

a contradiction. ■

4.2

(4.2.1) Assume that $b = 4$.

(a) $[X_\alpha, Q_\alpha] \leq T_\alpha$.

(b) $[X_\alpha, L_\alpha] = U_\alpha$.

Proof. (a) Suppose that $[X_\alpha, Q_\alpha] \not\leq T_\alpha$. Pick $\mu \in \Delta(\alpha+1) \cap \Lambda(\alpha)$ such that $[U_\mu, Q_{\alpha+1}, Q_\alpha] \not\leq T_\alpha$.

If $[U_\mu, U_{\alpha-2}] = U_\mu \cap Z_{\alpha+1}$, then (1.2.3) implies that $U_\mu \cap Z_{\alpha+1} \leq T_\alpha$, contrary to the choice of μ. Hence

$$[U_\mu, U_{\alpha-2}] \neq U_\mu \cap Z_{\alpha+1}.$$

i.e., either $(\alpha-2, \mu)$ is not a critical pair, or $q = 2$, $U_\alpha \cap T_\alpha \neq 1$, and U_μ and $U_{\alpha-2}$ act as transvections on each other. In any case we have

$$(*) \quad [U_\mu, Q_{\alpha+1}] \leq Q_{\alpha-2}.$$

From $[U_\alpha, U_{\alpha-4}, U_{\alpha-4}] = 1$, (4.1.4)(c) and the choice of the path $(\alpha-4, \ldots, \alpha')$ it follows that

$$[U_{\alpha-4}, Q_{\alpha-3}] \leq Q_\alpha.$$

Suppose that $[[U_\mu, Q_{\alpha+1}], [U_{\alpha-4}, Q_{\alpha-3}]] \neq 1$. Then

$$[[U_\mu, Q_{\alpha+1}], [U_{\alpha-4}, Q_{\alpha-3}]]T_\alpha/T_\alpha = [U_\mu, Q_{\alpha+1}, Q_\alpha]T_\alpha/T_\alpha = C(Q_{\alpha+1}).$$

Since $C_{U_\alpha T_\alpha/T_\alpha}(Q_{\alpha+1})$ is not contained in $[U_\alpha, Q_{\alpha-1}]T_\alpha/T_\alpha$, which is the unique maximal $M_{\alpha-1}$-submodule of $U_\alpha T_\alpha/T_\alpha$, it follows that

$$U_\alpha T_\alpha \leq \langle [[U_\mu, Q_{\alpha+1}], [U_{\alpha-4}, Q_{\alpha-3}]]^{M_{\alpha-1}} \rangle T_\alpha \leq [X_\alpha, X_{\alpha-2}]T_\alpha \leq X_{\alpha-2}.$$

Since $Q_{\alpha-2}$ acts quadratically on $X_{\alpha-2}$, but not on U_α, this is impossible. Hence

$$[[U_\mu, Q_{\alpha+1}], [U_{\alpha-4}, Q_{\alpha-3}]] = 1.$$

Together with (*) we get

$$[U_{\alpha-4},[U_\mu,Q_{\alpha+1}]] \leq [U_{\alpha-4},U_\alpha] \leq U_\alpha.$$

Since $G_\alpha = \langle M_{\alpha+1}, U_{\alpha-4}\rangle$, it follows that

$$[U_\mu,Q_{\alpha+1}]U_\alpha \trianglelefteq G_\alpha.$$

But then

$$[U_\mu,Q_{\alpha+1},Q_\alpha] = [[U_\mu,Q_{\alpha+1}]U_\alpha,Q_\alpha] \trianglelefteq G_\alpha.$$

Since $[U_\mu,Q_{\alpha+1},Q_{\alpha+1}] \leq Z_{\alpha+1}$, it follows that $[U_\mu,Q_{\alpha+1},Q_\alpha] \leq T_\alpha$, contrary to the choice of μ.

(b) Pick $g_{\alpha-4} \in L_{\alpha-4}$ such that (a)–(c) of (4.1.3) are satisfied for $g_{\alpha-4}, \alpha-4,\dots,\alpha$ in place of $g,\gamma,\dots,\gamma'$. Put

$$\alpha-5 := (\alpha-3)^{g_{\alpha-4}}.$$

Suppose that $[U_\mu,Q_{\alpha-5}]U_{\alpha-4} \ntrianglelefteq G_{\alpha-4}$ for some $\mu \in \Delta(\alpha-5)\cap\Lambda(\alpha-4)$. Then $(\mu,\alpha-2)$ is a critical pair by (4.1.3)(c). Note that (4.1.3)(a) implies that

$$M_{\alpha-5} \leq M_{\alpha-3}Q_{\alpha-5}.$$

Together with $U_{\alpha-2} \trianglelefteq M_{\alpha-3}$ and $[U_\mu,U_{\alpha-2},U_{\alpha-2}] = 1$ it follows that $U_{\alpha-2}$ is contained in $C_{Q_{\alpha-5}}(O^{p'}(M_{\alpha-5}),Q_\mu)$ and hence

$$[U_\mu,Q_{\alpha-5}] \leq Q_{\alpha-2}.$$

Note that

$$U_\alpha \cap Q_{\alpha-4} \leq [U_\alpha,Q_{\alpha-1}] \leq X_{\alpha-2}.$$

Now (a) and $\alpha-4 = \alpha^{g_{\alpha-2}}$ imply that

$$(*)\quad [[U_\mu,Q_{\alpha-5}],U_\alpha\cap Q_{\alpha-4}] \leq [Q_{\alpha-2},X_{\alpha-2}]\cap[X_{\alpha-4},Q_{\alpha-4}] \leq T_{\alpha-2}\cap T_{\alpha-4} \leq T_\alpha.$$

From $(*)$ and (A.1.5)(b) (with $[U_\mu, Q_{\alpha-5}]U_{\alpha-4}$, $U_{\alpha-4}$, and $U_\alpha/(U_\alpha \cap T_\alpha)$ in place of A, B, and V, respectively) it follows that

$$[[U_\mu, Q_{\alpha-5}]U_{\alpha-4}, U_\alpha] \leq [[U_\mu, Q_{\alpha-5}]U_{\alpha-4}, U_\alpha \cap Q_{\alpha-4}][U_{\alpha-4}, U_\alpha](U_\alpha \cap T_\alpha).$$

Since $[U_\mu, Q_{\alpha-5}]U_{\alpha-4} \ntrianglelefteq G_{\alpha-4} = \langle M_{\alpha-5}, U_\alpha\rangle$, this implies that $U_\alpha \cap T_\alpha \not\leq [U_\mu, Q_{\alpha-5}]U_{\alpha-4}$. Together with (4.1.1)(b) we get $[U_\mu, Q_{\alpha-5}, U_\alpha] \leq [U_{\alpha-4}, U_\alpha]$. Thus U_α normalizes $[U_\mu, Q_{\alpha+5}]U_{\alpha-4}$, a contradiction. Hence

$$[U_\mu, Q_{\alpha-5}]U_{\alpha-4} \trianglelefteq G_{\alpha-4},$$

for each $\mu \in \Delta(\alpha-5) \cap \Lambda(\alpha-4)$. Since $L_{\alpha-4}$ is transitive on $\Delta(\alpha-4)$, we get $X_{\alpha-4} = X_{\alpha-4,\alpha-3}$ and hence

$$[X_{\alpha-4}, U_\alpha] = [X_{\alpha-4,\alpha-3}, U_\alpha] \leq [Z_{\alpha-4}[V_{\alpha-3}, Q_{\alpha-3}], U_\alpha] \leq$$

$$[Z_{\alpha-4}, U_\alpha][V_{\alpha-3}, Q_{\alpha-3}, Q_{\alpha-3}] \leq Z_{\alpha-4}.$$

Now the claim follows from $\alpha - 4 \sim \alpha$ and $L'_{\alpha-4} = L_{\alpha-4} \leq \langle U_\alpha^{L_{\alpha-4}}\rangle$. ∎

(4.2.2) Assume that $b = 4$. Let $(\gamma, \gamma+1, \gamma+2, \gamma+3, \gamma+4)$ be a path such that $(\gamma, \gamma+4)$ is a critical pair.

(a) $U_{\gamma+4}Q_\gamma = C_{Q_{\gamma+1}}(O^{p'}(M_{\gamma+1}), Q_\gamma)$.

(b) $[U_\gamma, U_{\gamma+4}] \leq Z_{\gamma+1}$.

(c) Assume that $g \in L_\gamma$ satisfies (4.1.3)(a). Then $(\mu, \gamma+2)$ is a critical pair, for each $\mu \in \Delta((\gamma+1)^g) \cap \Lambda(\gamma)$

Proof. (a) follows from (4.1.1)(b) and (4.2.1)(b). (b) follows from (a).

(c) Suppose that $(\mu, \gamma+2)$ is not a critical pair. Then U_μ does not centralize $[U_{\gamma+4}, Q_{\gamma+3}]$ by (4.1.3)(c). Pick $x \in L_{\gamma+2}$ such that $(\gamma+3)^x = \gamma+1$. Since (b) (with

$\gamma+2$ in place of α) implies that $[U_{\gamma+4}, Q_{\gamma+3}]U_{\gamma+2} \trianglelefteq G_{\gamma+2}$, it follows that U_μ does not centralize $[U^x_{\gamma+4}, Q_{\gamma+1}]$. Since (b) and $b > 2$ imply that

$$X_\gamma = U_\gamma X^g_{\gamma,\gamma+1} \leq V^g_{\gamma+1} \leq Q_\mu,$$

we get $[U^x_{\gamma+4}, Q_{\gamma+1}] \not\leq X_\gamma$, a contradiction. ∎

(4.2.3) Assume that $b = 4$.

(a) $Y'_\alpha \leq T_\alpha$.

(b) $Y_\alpha \leq C_{Q_{\alpha-1}}(O^{p'}(M_{\alpha-1}), Q_{\alpha-2})$.

(c) $Y_\alpha \cap Q_{\alpha-2} \leq C_{Q_{\alpha-3}}(O^{p'}(M_{\alpha-3}), Q_{\alpha-4})$.

Proof. (a) Pick $\lambda \in \Delta(\alpha)$, $\mu \in \Delta(\lambda)$, and $\gamma \in \Delta(\alpha-1) \cap \Lambda(\alpha)$. By the choice of the path $(\alpha-4, \ldots, \alpha')$, it follows from (4.2.2)(c) that $(\gamma, \alpha+2)$ is a critical pair. Hence

$$[U_\gamma, U_\mu] \leq [U_\gamma, C_{Q_{\alpha-1}}(O^{p'}(M_{\alpha-1}), Q_\gamma)] = \bigcap_{g \in M_{\alpha-1}} [U_\gamma, U^g_{\alpha+2}] \leq T_\alpha,$$

by (4.2.2)(a)(b) and (1.2.4). Therefore, $[Y_{\alpha,\alpha-1}, Y_\alpha] \leq T_\alpha$. Since L_α is transitive on $\Delta(\alpha)$, (a) holds.

(b) From (a) we get $[U_{\alpha-2}, Y_\alpha, Y_\alpha] \leq [Y_\alpha, Y_\alpha, Y_\alpha] = 1$. Since $Y_\alpha \trianglelefteq M_{\alpha-1}$, it follows that (b) holds.

(c) Since $\alpha - 4 \in \Lambda(\alpha-2)$, (4.2.1)(a) implies that

$$[U_{\alpha-4}, Y_\alpha \cap Q_{\alpha-2}, Y_\alpha \cap Q_{\alpha-2}] \leq [Y_{\alpha-2}, Q_{\alpha-2}, Q_{\alpha-2}] \leq [X_{\alpha-2}, Q_{\alpha-2}] \leq T_{\alpha-2}.$$

From (a) we get

$$[U_{\alpha-4}, Y_\alpha \cap Q_{\alpha-2}, Y_\alpha \cap Q_{\alpha-2}] \leq Y'_\alpha \leq T_\alpha$$

and therefore

$$[U_{\alpha-4}, Y_\alpha \cap Q_{\alpha-2}, Y_\alpha \cap Q_{\alpha-2}] \leq T_\alpha \cap T_{\alpha-2} = (T_{\alpha-4} \cap T_{\alpha-2})^{g_{\alpha-2}} = T_{\alpha-4} \cap T_{\alpha-2}.$$

Hence $Y_\alpha \cap Q_{\alpha-2}$ acts quadratically or trivially on $U_{\alpha-4}T_{\alpha-4}/T_{\alpha-4}$. Since $Y_\alpha \cap Q_{\alpha-2} \trianglelefteq M_{\alpha-1}$, (c) now follows from (4.1.2), (4.1.4)(c) and the choice of $(\alpha-4, \ldots, \alpha')$. ∎

(4.2.4) Assume that $b = 4$.

(a) Y_α/X_α is an FF-module for $L_\alpha Q_\alpha/Q_\alpha$, and $[Y_\alpha, L_\alpha]X_\alpha/C_{[Y_\alpha,L_\alpha]X_\alpha}(L_\alpha, X_\alpha)$ is a natural $\mathrm{Sp}_{2n}(q)$-module for L_α.

(b) If $\mu \in \Delta(\alpha-1) \cap \Lambda(\alpha)$, then $[U_\mu, Q_\alpha, L_\alpha] = 1$.

Proof. (a) Note that Y_α/X_α is a module for $L_\alpha Q_\alpha/Q_\alpha$. From (4.1.1)b) and (4.2.3)(c) it follows that $[U_{\alpha-4}, Y_\alpha \cap Q_{\alpha-2}] \leq [U_{\alpha-4}, U_\alpha] \leq U_\alpha$. Thus $U_{\alpha-4}$ centralizes $(Y_\alpha \cap Q_{\alpha-2})X_\alpha/X_\alpha$. Hence

$$|Y_\alpha : C_{Y_\alpha}(U_{\alpha-4}, X_\alpha)| \leq q,$$

by (4.2.3)(b), and then (4.1.1)(b) implies that Y_α/X_α is an FF-module for $L_\alpha Q_\alpha$ and $U_{\alpha-4}$ acts as an offending subgroup. Moreover, (4.2.2)(a) implies that if $U_{\alpha-4}$ acts as an offending subgroup on an FF-module W for $L_\alpha Q_\alpha/Q_\alpha$, then W contains exactly one nontrivial L_α-composition factor and that one is a natural $\mathrm{Sp}_{2n}(q)$-module for L_α.

(b) Pick $\mu \in \Delta(\alpha-1) \cap \Lambda(\alpha)$. By the choice of the path $(\alpha-4, \ldots, \alpha')$, it follows from (4.2.3)(c) that $(\mu, \alpha+2)$ is a critical pair. Hence

$$(*) \quad [U_\mu, C_{Q_{\alpha-1}}(O^{p'}(M_{\alpha-1}), Q_\mu)] \leq T_\alpha,$$

by (4.2.2)(a) and (4.2.3)(a). If $Q_\alpha Q_\mu \neq Q_{\alpha-1}$, then $(*)$ implies that $[U_\mu, Q_\alpha, L_\alpha] = 1$. Hence we may assume that

$$(**) \quad Q_\alpha Q_\mu = Q_{\alpha-1}.$$

By (4.2.1)(a),

$$A := X_\alpha/T_\alpha$$

is a module for G_α/Q_α. From (4.2.1)(b) and [10] it follows that

$$(***) \quad |A : C_A(L_\alpha)[A, L_\alpha]| \leq q.$$

Note that, if $U_\mu \cap T_\mu = 1$ or $q \neq 2$, then $(*)$ implies that $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha$ is an irreducible $M_{\alpha-1}$-module of order q^{2n-2}. If $U_\mu \cap T_\mu \neq 1$ and $q = 2$, then $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha$ is an indecomposable $M_{\alpha-1}$-module of order 2^{2n-1}, $(U_\mu \cap Z_{\alpha-1})T_\alpha/T_\alpha$ is the unique proper $M_{\alpha-1}$-submodule and has order 2. Hence in any case $(***)$ implies that

$$[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha \leq C_A(L_\alpha)[A, L_\alpha].$$

Suppose that $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha \not\leq C_A(L_\alpha)$. Since

$$C_{Z_{\alpha-1}}(L_\alpha, T_\alpha) \leq T_\alpha,$$

the stucture of $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha$ as $M_{\alpha-1}$-module implies that

$$[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha \cap C_A(L_\alpha) = 1.$$

Hence $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha$ is isomorphic to an $M_{\alpha-1}$-submodule of $[A, L_\alpha]/C_{[A,L_\alpha]}(L_\alpha)$. Note that (4.2.1)(b) implies that

$$[A, L_\alpha] = U_\alpha T_\alpha/T_\alpha \cong U_\alpha/(U_\alpha \cap T_\alpha).$$

In particular, $q = 2$ and

$$(****) \quad |[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha| = q^{2n-1},$$

since $U_\alpha/(U_\alpha \cap T_\alpha)$ contains no irreducible $M_{\alpha-1}$-submodule of order q^{2n-2}. Furthermore, the $M_{\alpha-1}$-composition factor of order q^{2n-2} in $[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha$ also appears in U_α, whence $M_{\alpha-1} \cap L_\alpha$ acts nontrivially on this composition factor. Now (4.2.1)(b) implies that this composition factor does not appear in $A/[A, L_\alpha]$. Hence

$$[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha \leq [A, L_\alpha].$$

Since Q_α centralizes $[A, L_\alpha]$, this is a contradiction to $(**)$ and $(****)$. Thus

$$[U_\mu, Q_{\alpha-1}]T_\alpha/T_\alpha \leq C_A(L_\alpha),$$

i.e., $[U_\mu, Q_{\alpha-1}, L_\alpha] \leq T_\alpha$. Now the claim follows from $L_\alpha = L'_\alpha$ and the Three-Subgroup Lemma. ∎

(4.2.5) Assume that $b = 4$. Then $[U_{\alpha+2}, U_{\alpha-4}, U_{\alpha-4}] = 1$.

Proof. Suppose that $[U_{\alpha+2}, U_{\alpha-4}, U_{\alpha-4}] \neq 1$. Define

$$Y := \langle U_{\alpha+2}^{L_\alpha} \rangle.$$

Since $[Y, U_{\alpha-4}] \leq Y_{\alpha-2}$, (4.2.3)(b) applied to $(\alpha-2, \alpha-4)$ in place of $(\alpha, \alpha-2)$ implies that

$$(*) \quad [Y, U_{\alpha-4}, U_{\alpha-4}] \leq T_{\alpha-2}.$$

Suppose that $[Y, U_{\alpha-4}, U_{\alpha-4}] \leq T_\alpha$. Then $(*)$ implies that

$$[Y, U_{\alpha-4}, U_{\alpha-4}] \leq T_\alpha^{g_{\alpha-2}} = T_{\alpha-4}.$$

But then $[Y, U_{\alpha-4}] \leq Q_{\alpha-4}$, contrary to $[Y, U_{\alpha-4}, U_{\alpha-4}] \neq 1$. Hence

$$(**) \quad [Y, U_{\alpha-4}, U_{\alpha-4}] \not\leq T_\alpha.$$

Since $(*)$ implies that

$$[Y, U_{\alpha-4}, U_{\alpha-4}] \leq U_{\alpha-4}^{g_{\alpha-2}} = U_\alpha,$$

it follows from $(**)$ that

$$U_\alpha \leq [Y, L_\alpha].$$

Put

$$A := [Y, L_\alpha]/C_{[Y,L_\alpha]}(L_\alpha) \quad \text{and}$$

$$B := U_\alpha C_{[Y,L_\alpha]}(L_\alpha)/C_{[Y,L_\alpha]}(L_\alpha).$$

Suppose that $A = B$. Pick $g \in L_\alpha$ such that $(\alpha+1)^g = \alpha-1$. Then $Y = U_{\alpha+2}^g [Y, L_\alpha] = U_{\alpha+2}^g U_\alpha C_{[Y,L_\alpha]}(L_\alpha)$. Since $d(\alpha-4, (\alpha+2)^g) \leq 4$ and $U_{\alpha-4} \leq L_\alpha Q_\alpha$, it follows that

$$[Y, U_{\alpha-4}, U_{\alpha-4}] \leq [[U_{\alpha+2}^g, U_{\alpha-4}][U_\alpha, U_{\alpha-4}] C_{[Y,L_\alpha]}(L_\alpha), U_{\alpha-4}] = 1,$$

a contradiction. Therefore

$$A \neq B.$$

Now (4.2.4)(a) and (4.2.1)(b) imply that $A/C_A(L_\alpha, B)$ is a natural $\mathrm{Sp}_{2n}(q)$-module for $L_\alpha Q_\alpha/Q_\alpha$. Put

$$C := C_A(U_{\alpha-4}).$$

Note that C is an $M_{\alpha-1}$-submodule of A by (4.2.2)(a). From (4.2.3)(b)(c) it follows that $|Y_\alpha : Y_\alpha \cap Q_{\alpha-2} \cap Q_{\alpha-4}| \leq q^2$ and hence

$$(* * *) \quad |A : C| \leq q^2.$$

Suppose that $L_\alpha R_\alpha/R_\alpha \cong \mathrm{Sp}_4(2)'$. Pick $x, y \in L_\alpha$ such that KR_α/R_α is a Levi complement of $Q_{\alpha-1}R_\alpha/R_\alpha$ in $(M_{\alpha-1} \cap L_\alpha)R_\alpha/R_\alpha$. where $K := \langle U_{\alpha-4}^x, U_{\alpha-4}^y \rangle$. Note that A, regarded as a module for $K/C_A(K)$ ($\cong \Sigma_3$), consists of two natural Σ_3-modules and trivial modules. Hence $(* * *)$ and the projectivity of the natural Σ_3-module imply that A is completely reducible as K-module. But then $U_{\alpha-4}^x$ and hence also $U_{\alpha-4}$ acts quadratically on A. Since $T_{\alpha-2} \leq Z_\alpha$, this is a contradiction to $(*)$ and $(**)$. Hence

$$(* * *) \quad L_\alpha R_\alpha/R_\alpha \not\cong \mathrm{Sp}_4(2)'.$$

Since $U_{\alpha-4}$ does not centralize the natural $\mathrm{Sp}_{2n}(q)$-modules B and $A/C_A(L_\alpha, B)$, $(* * *)$ implies that A/C picks up the two $M_{\alpha-1}$-composition factors on top of B and $A/C_A(L_\alpha, B)$. Hence

$$[A, O^{p'}(M_{\alpha-1}), O^{p'}(M_{\alpha-1})] \leq C.$$

Since $(* * *)$ implies that $U_{\alpha-4} \leq Q_{\alpha-1} \leq Q_\alpha O^{p'}(M_{\alpha-1})'$, it follows that $U_{\alpha-4}$ centralizes A/C, a contradiction to $(*)$ and $(**)$. ∎

(4.2.6) $b = 2$.

Proof. Suppose that $b > 2$. Then $b = 4$ by (4.1.5). By (4.2.3)(c) there exist $\alpha - 5 \in \Delta(\alpha - 4)$ and $\alpha - 6 \in \Delta(\alpha - 5)$ such that $(\alpha - 6, \alpha - 2)$ is a critical pair. Note that it follows from (4.1.1)(b) and the choice of the path $(\alpha - 4, \dots, \alpha')$ that we can choose $x \in U_{\alpha+2}$ such that (4.1.3)(a) is satisfied for $(\alpha - 2, \alpha - 6)$ in place of (γ, γ') and some $g \in xR_{\alpha-2} \cap L_{\alpha-2}$. Then (4.2.3)(c) implies that $(\alpha - 4, (\alpha - 4)^x)$ is a critical pair, i.e.,

$$U_{\alpha-4}^x \not\leq Q_{\alpha-4}.$$

But from (4.2.5) we get

$$\langle U_{\alpha-4}^{U_{\alpha+2}} \rangle = U_{\alpha-4}[U_{\alpha-4}, U_{\alpha+2}] \leq Q_{\alpha-4},$$

a contradiction. ∎

4.3

(4.3.1) (a) $X_\alpha \leq Q_\alpha$.

(b) $Z_\alpha = U_\alpha T_\alpha$.

(c) $p = 2$.

(d) $G_\alpha = L_\alpha R_\alpha$.

(e) $[Q'_\alpha, L_\alpha] = 1$.

(f) $[X_\alpha, Q_\alpha] \leq T_\alpha$.

(g) $[Q_\alpha, L_\alpha] \leq X_\alpha$.

Proof. (a),(b) Note that

$$(*) \quad [Z_\mu, O^{p'}(M_\lambda)] \leq Q_\alpha \quad \text{for all } \lambda \in \Delta(\alpha) \text{ and } \mu \in \Delta(\lambda),$$

since $Z_\mu \trianglelefteq M_\lambda$ and $[U_\alpha, Z_\mu, Z_\mu] = 1$. In particular, (a) holds. If $Z_\alpha \neq U_\alpha T_\alpha$, then $U_{\alpha'} \leq [Z_{\alpha'}, O^{p'}(M_\lambda)]$, contrary to $(*)$ and $U_{\alpha'} \not\leq Q_\alpha$.

(c) Note that each eigenvalue of an element of $M_{\alpha+1} \cap L_\alpha$ on $C_{Q_{\alpha+1}}(O^{p'}(M_{\alpha+1}), Q_\alpha)/Q_\alpha$ is a square in $GF(q)$, but each element of $GF(q)$ is an eigenvalue of some element of $M_{\alpha+1} \cap L_{\alpha'}$ on $U_{\alpha'}/[U'_\alpha, Q_{\alpha+1}]$.

Suppose that p is odd. Then there exists $\lambda \in GF(q)$ such that λ is not a square in $GF(q)$. Pick $g \in M_{\alpha+1} \cap L_{\alpha'}$ such that g acts on $U_{\alpha'}/[U_{\alpha'}, Q_{\alpha+1}]$ by multiplication by λ.

Suppose that g induces an outer automorphism on $L_\alpha R_\alpha/R_\alpha$. Put

$$A := O^{p'}((M_{\alpha+1} \cap L_\alpha)R_\alpha/R_\alpha).$$

Assume that $n > 2$ or $q > 3$, and hence $A = KR_\alpha/R_\alpha$ for some p-component K of $M_{\alpha+1}$. Then $K \leq R_{\alpha'}$ or $K \leq L_{\alpha'}$ by (2.2.6)(a). In both cases there exists $h \in L_\alpha$ that induces tha same automorphism on $A/O_p(A)$ as g. But then gh^{-1} induces an outer automorphism on $L_\alpha R_\alpha/R_\alpha$ that centralizes $A/O_p(A)$, contrary to (A.4.2).

Assume now that $n = 2$ and $q = 3$. In this case $\lambda = -1$ and we may choose g such that $g \in R_{\alpha'} \cap L_{\alpha'}$. If g induces an outer automorphism on $A/O_p(A)$, then $A = [A, g]O_p(A) \leq (R_{\alpha'}R_\alpha/R_\alpha)\, O_p(A)$ and hence

$$A = [A, g]O_p(A) \leq ([R_{\alpha'}, L_{\alpha'}]R_\alpha/R_\alpha)\, O_p(A) \leq Q_{\alpha'}R_\alpha/R_\alpha,$$

a contradiction. Hence g induces an inner automorphism on $A/O_p(A)$, contrary to (A.4.2).

Thus g induces an inner automorphism on $L_\alpha R_\alpha/R_\alpha$. Since $L_\alpha R_\alpha/R_\alpha = F^*(G_\alpha/R_\alpha)$, it follows that

$$g = g_1 g_2 \quad \text{for some } g_1 \in (M_{\alpha+1} \cap L_\alpha) \text{ and } g_2 \in R_\alpha.$$

Since $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ is dual to $U_\alpha \cap U_{\alpha'}$ as $(M_{\alpha+1} \cap L_\alpha)$-module, there exists $\lambda_1 \in GF(q)$ such that g_1 acts on $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ by multiplication by λ_1^{-1} and on $U_\alpha \cap U_{\alpha'}$ by multiplication by λ_1. Note that this implies that g_1 acts on $C_{Q_{\alpha+1}}(O^{p'}(M_{\alpha+1}), Q_\alpha)/Q_\alpha$ by multiplication by λ_1^2. Since $[Q_{\alpha+1}, R_\alpha] \leq Q_\alpha$, also g acts on $C_{Q_{\alpha+1}}(O^{p'}(M_{\alpha+1}), Q_\alpha)/Q_\alpha$ by multiplication by λ_1^2. Since (a) implies that

$$U_{\alpha'} Q_\alpha = C_{Q_{\alpha+1}}(O^{p'}(M_{\alpha+1}), Q_\alpha),$$

it follows that λ is a square in $GF(q)$, a contradiction.

(d) By (c), $Q'_\alpha \leq Q_{\alpha'}$ and hence $[Q'_\alpha, U_{\alpha'}] = 1$. Now (d) follows from $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$.

(e) Note that $[X_{\alpha,\lambda}, Q_\alpha] \leq Z_\lambda$ for each $\lambda \in \Delta(\alpha)$. Hence (e) follows from (d).

(f) From the definition of X_α and Y_α it follows that $[Y_\alpha, Q_\alpha] \leq X_\alpha$. Since (4.2.6) implies that $L_\alpha \leq Y_\alpha$, (f) holds. ∎

(4.3.2) Assume that $[Q_\alpha, L_\alpha] \neq U_\alpha$.

(a) $Q_{\alpha+1} = X_\alpha Q_{\alpha'} = X_{\alpha'} Q_\alpha$.

(b) $U_\alpha \cap T_\alpha \neq 1$.

(c) $n = 3$.

(d) $[Q_\alpha, L_\alpha]/C_{[Q_\alpha, L_\alpha]}(L_\alpha, U_\alpha)$ is an $O_7(q)$-spin-module for L_α.

Proof. (a) Suppose that $Q_{\alpha+1} \neq X_\alpha Q_{\alpha'}$. Then $X_\alpha \leq U_\alpha Q_{\alpha'}$, since $X_\alpha \trianglelefteq M_{\alpha+1}$. Hence $[X_\alpha, U_{\alpha'}] \leq U_\alpha$. Since $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$, we get

$$[X_\alpha, L_\alpha] \leq U_\alpha,$$

contrary to (4.3.1)(f) and $[Q_\alpha, L_\alpha] \neq U_\alpha$. The proof of $Q_{\alpha+1} = X_{\alpha'} Q_\alpha$ is similar.

(b) Suppose that $U_\alpha \cap T_\alpha = 1$. Then

$$U_{\alpha'} \cap Z_{\alpha+1} = [U_{\alpha'}, U_\alpha] \leq U_\alpha.$$

But also, by (a),

$$U_{\alpha'} \cap Z_{\alpha+1} = [U_{\alpha'}, Q_{\alpha+1}, X_\alpha] \leq X'_\alpha.$$

Now (4.3.1)(e) implies that $U_{\alpha'} \cap Z_{\alpha+1} \leq U_\alpha \cap T_\alpha$, a contradiction.

(c),(d) By (4.3.1)(e),

$$A := [X_\alpha, L_\alpha]/C_{[X_\alpha, L_\alpha]}(L_\alpha, U_\alpha)$$

is a module for G_α/Q_α. Since α is conjugate to α', (4.3.1)(f) implies that $[X_{\alpha'}, Q_{\alpha'}] \leq T_{\alpha'}$ and hence

$$|A : C_A(X_{\alpha'})| \leq |[X_\alpha, L_\alpha] : [X_\alpha, L_\alpha] \cap Q_{\alpha'}| \leq |X_\alpha : X_\alpha \cap Q_{\alpha'}|.$$

Note that

$$|X_\alpha : X_\alpha \cap Q_{\alpha'}| = q^{2n-1} = |X_{\alpha'} : X_{\alpha'} \cap Q_\alpha|.$$

Hence A is an FF-module for L_α on which $X_{\alpha'}$ acts as an offending subgroup. Moreover, by (4.3.1)(a)(e),

$$[A, X_{\alpha'}, X_{\alpha'}] = 1.$$

Since (a) implies that $X_{\alpha'}$ does not act quadratically on the natural $\mathrm{Sp}_{2n}(q)$-module, it follows that $n = 3$ and A is an $\mathrm{O}_7(q)$-spin-module for L_α. ■

(4.3.3) Assume that $[Q_\alpha, L_\alpha] = U_\alpha$. Then $q \in \{2, 4\}$.

Proof. Let $g \in A := C_{M_{\alpha+1} \cap L_{\alpha'}}([U_{\alpha'}, Q_{\alpha+1}], U_{\alpha'} \cap Z_{\alpha+1})$. If $[U_{\alpha'}, Q_{\alpha+1}] = [U_\alpha, Q_{\alpha+1}]$, then $[U_\alpha, Q_{\alpha+1}, g] \leq U_\alpha \cap Z_{\alpha+1}$ by the choice of g. If $[U_{\alpha'}, Q_{\alpha+1}] \neq [U_\alpha, Q_{\alpha+1}]$, then $[U_\alpha, Q_{\alpha+1}] \cap U_{\alpha'} \leq Z_{\alpha+1}$, and since $[U_\alpha, Q_{\alpha+1}] \leq Q_{\alpha'}$ by (4.3.1)(a), we get

$$[U_\alpha, Q_{\alpha+1}, g] \leq [U_\alpha, Q_{\alpha+1}] \cap [Q_{\alpha'}, L_{\alpha'}] \leq U_\alpha \cap Z_{\alpha+1}.$$

Hence in any case

$$(*) \quad [U_\alpha, Q_{\alpha+1}, g] \leq U_\alpha \cap Z_{\alpha+1}.$$

In particular, g centralizes $O^{p'}(M_{\alpha+1} \cap L_\alpha)Q_{\alpha+1}/Q_{\alpha+1}$ and hence induces an inner automorphism on $L_\alpha R_\alpha/R_\alpha$, by (A.4.2). Thus there exist $g_1 \in M_{\alpha+1} \cap L_\alpha$ and $g_2 \in R_\alpha$ such that $g = g_1 g_2$. By (1.1.2)(d), g_2 induces $GF(p)L_\alpha$-endomorphisms on $U_\alpha/(U_\alpha \cap$

T_α). Together with (*) it follows that g_1 centralizes $O^{p'}(M_{\alpha+1} \cap L_\alpha)Q_{\alpha+1}/Q_{\alpha+1}$. Then $[U_\alpha, Q_{\alpha+1}, g_1] \leq U_\alpha \cap Z_{\alpha+1}$ by (4.3.1)(c). Now it follows from (*) that g_2 centralizes $[U_\alpha, Q_{\alpha+1}]/(U_\alpha \cap Z_{\alpha+1})$ and hence also $U_\alpha/(U_\alpha \cap T_\alpha)$. Therefore,

$$(**) \quad A \leq (M_{\alpha+1} \cap L_\alpha)C_{G_\alpha}(U_\alpha, U_\alpha \cap T_\alpha).$$

In particular, $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ is dual to $(U_\alpha \cap Z_{\alpha+1})/(U_\alpha \cap T_\alpha)$ as a module for A.
Assume that $U_\alpha \cap Z_{\alpha+1} \leq T_\alpha T_{\alpha'}$. Then A centralizes $(U_\alpha \cap Z_{\alpha+1})/(U_\alpha \cap T_\alpha)$ and hence also $U_\alpha/[U_\alpha, Q_{\alpha+1}]$. Since $C_{Q_{\alpha+1}}(O^{p'}(M_{\alpha+1}, Q_{\alpha'})$ is isomorphic to $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ as A-module, it follows that $q = 2$.
Assume that $U_\alpha \cap Z_{\alpha+1} \not\leq T_\alpha T_{\alpha'}$. Then $(U_\alpha \cap Z_{\alpha+1})/(U_\alpha \cap T_\alpha)$ is isomorphic to $(U_{\alpha'} \cap Z_{\alpha+1})/(U_{\alpha'} \cap T_\alpha)$ as A-module. Hence $U_\alpha/[U_\alpha, Q_{\alpha+1}]$ is dual to $(U_{\alpha'} \cap Z_{\alpha+1})/(U_{\alpha'} \cap T_\alpha)$ and isomorphic to $U_{\alpha'}/[U_{\alpha'}, Q_{\alpha+1}]$ as A-module. Now (B.4.1.4)(b) implies that $q \in \{2, 4\}$. ■

Chapter 5

Determining the action of L on R, Part 3

5.1

In this section we assume (I)–(IV) and

(V) $L_\alpha R_\alpha/R_\alpha \cong \mathsf{G}_2(q)'$ ($q = 2^k$ for some $k \in \mathbb{N}$), and $p = 2$.

Note that (2.3.2) implies that $U_\alpha T_\alpha/T_\alpha$ is the $\mathsf{G}_2(q)$-module listed in (A.2.2)(i).

(5.1.1) Let (γ, γ') be a critical pair.

(a) $[U_\gamma, U_{\gamma'}] = U_{\gamma'} \cap Q_\gamma = U_\gamma \cap Q_{\gamma'}$.

(b) $|U_\gamma : U_\gamma \cap Q_{\gamma'}| = |U_{\gamma'} : U_{\gamma'} \cap Q_\gamma| = q^3$.

(c) $U_{\gamma'}R_\gamma/R_\gamma = C_{G_\gamma/R_\gamma}(U_{\gamma'}R_\gamma/R_\gamma)$.

Proof. [14](1.2). ∎

(5.1.2) $b \leq 4$.

Proof. Suppose that $b > 4$. By (5.1.1)(c) and (1.2.6) assumption (iii) of (1.2.5) is satisfied. Since $M_{\alpha+1}$ is a maximal subgroup of G_α, also assumption (ii) of (1.2.5) is satisfied. Hence (1.2.5) implies that there exists $\nu \in \Xi_{\alpha,\alpha+2}$ such that

$$[\bigcap_{g \in M_{\alpha+1}} Z_\nu^g, L_\alpha] = 1.$$

Define

$$D := \bigcap_{g \in M_{\alpha+1}} [U_\alpha, U_\nu^g].$$

Then $D \leq T_\alpha$. But either $U_\nu R_\alpha / R_\alpha$ is normalized by $M_{\alpha+1}/R_\alpha$ or $|U_\nu R_\alpha/R_\alpha \cap Z(M_{\alpha+1}/R_\alpha)| = q$, and therefore in any case $D \not\leq T_\alpha$, a contradiction. ■

Choose $g_\alpha \in L_\alpha$ such that $U_{\alpha'} \not\leq M_{\alpha+1}^{g_\alpha}$, i.e.,

$$G_\alpha = \langle M_{\alpha+1}^{g_\alpha}, U_{\alpha'} \rangle.$$

Define

$$\alpha - 1 := (\alpha+1)^{g_\alpha} \quad \text{and}$$

$$\alpha - 2 := (\alpha+2)^{g_\alpha}.$$

(5.1.3) Assume that $b = 4$.

(a) $(\alpha - 2, \alpha + 2)$ is a critical pair.

(b) $Q_{\alpha-1} = Q_{\alpha-2} Q_\alpha$.

Proof. (a) follows from (1.2.6), (1.2.1)(e) and (5.1.2).

(b) Assume first that $U_{\alpha+2} R_{\alpha-2} \ntrianglelefteq M_{\alpha-1}$. Then $Z(M_{\alpha-1}/R_{\alpha-2}) \leq U_{\alpha+2} R_{\alpha-2}/R_{\alpha-2}$. Since $Q_{\alpha-1} R_{\alpha-2}/C_{Q_{\alpha-1}R_{\alpha-2}}(M_{\alpha-1}, R_{\alpha-2})$ consists of two irreducible $M_{\alpha-1}$-modules of order q^2, it follows that $Q_{\alpha-1} \leq \langle U_{\alpha+2}^{M_{\alpha-1}} \rangle R_{\alpha-2}$. Now $Q_\alpha R_{\alpha-2} \trianglelefteq M_{\alpha-1}$ implies that $Q_{\alpha-1} \leq Q_\alpha R_{\alpha-2}$ and hence $Q_{\alpha-1} = Q_\alpha (Q_{\alpha-1} \cap R_{\alpha-2}) = Q_\alpha Q_{\alpha-2}$.

Hence we may assume that

$$(*) \quad U_{\alpha+2} R_{\alpha-2} \trianglelefteq M_{\alpha-1}.$$

Let K be a Sylow 3-subgroup of $M_{\alpha-1} \cap L_\alpha$ or the 2-component of $M_{\alpha-1} \cap L_\alpha$, depending on whether $q = 2$ or $q > 2$. From $G_\alpha = \langle M_{\alpha-1}, U_{\alpha'} \rangle$, $b = 4$, and $(*)$ it follows that

$$[U_{\alpha-2}, U_{\alpha+2}] \leq T_\alpha.$$

Hence K centralizes $[U_{\alpha-2}, U_{\alpha+2}]$, which implies that

$$K \leq R_{\alpha-2}.$$

Since the two $M_{\alpha-1}/R_\alpha$-chief factors of order q^2 in $Q_{\alpha-1}R_\alpha/R_\alpha = [Q_{\alpha-1}, M_{\alpha-1}]R_\alpha/R_\alpha$ are contained in $[Q_{\alpha-1}, K]R_\alpha/R_\alpha$, we get $Q_{\alpha-1} \leq [Q_{\alpha-1}, R_{\alpha-2}]R_\alpha \leq Q_{\alpha-2}R_\alpha$ and hence $Q_{\alpha-1} = Q_{\alpha-2}(Q_{\alpha-1} \cap R_\alpha) = Q_{\alpha-2}Q_\alpha$. ∎

(5.1.4) $b = 2$.

Proof. Suppose that $b > 2$. Define

$$Y := [\langle U_{\alpha+2}^{L_\alpha}\rangle, Q_\alpha]U_\alpha.$$

Note that $[Y, Q_\alpha] \leq R_{\alpha-2}$ by (5.1.1)(b). Hence

$$(*) \quad |Y : Y \cap Q_{\alpha-2}| \leq q^2$$

by (5.1.3)(b). Pick $g_{\alpha-2} \in L_{\alpha-2}$ such that $U_{\alpha+2} \not\leq M_{\alpha-1}^{g_{\alpha-2}}$ and put

$$\alpha - 4 := \alpha^{g_{\alpha-2}}.$$

As in (5.1.3)(a) it follows that $(\alpha-4, \alpha)$ is a critical pair. Since Y centralizes U_α, (5.1.1)(c) implies that

$$(**) \quad Y \cap Q_{\alpha-2} \leq C_Y(U_{\alpha-4}, [Y, Q_\alpha]U_\alpha).$$

Since $\mathsf{G}_2(q)$ does not have an FF-module in which the index of the centralizer of an offending subgroup is smaller than q^3, it follows from $(*)$, $(**)$, and (5.1.1)(b) that

$$[Y, L_\alpha] \leq [Y, Q_\alpha]U_\alpha.$$

But then

$$[U_{\alpha+2}, Q_\alpha] \leq [U_{\alpha-2}, Q_\alpha][Y, Q_\alpha]U_\alpha \leq Q_{\alpha-2},$$

contrary to (5.1.1)(b) and (5.1.3)(b). ∎

(5.1.5) (a) $[U_\alpha, Q'_{\alpha+1}] = U_\alpha \cap Q_{\alpha'} = U_\alpha \cap U_{\alpha'} = U_{\alpha'} \cap Q_\alpha = [U_{\alpha'}, Q'_{\alpha+1}]$.

(b) $[Q_\alpha, L_\alpha] = U_\alpha$.

Proof. (a) This follows from (5.1.1)(a)(b) and (5.1.4).

(b) From (a) and $L_\alpha = \langle U_{\alpha'}^{L_\alpha} \rangle$ we get

$$[Q_\alpha, L_\alpha] \leq \langle [Q_\alpha, U_{\alpha'}]^{L_\alpha} \rangle = \langle (U_\alpha \cap U_{\alpha'})^{L_\alpha} \rangle = U_\alpha.$$

Hence (b) holds. ■

(5.1.6) Assume that $q \neq 2$. Let K be the 2-component of $M_{\alpha+1} \cap L_\alpha$.

(a) $K \leq L_{\alpha'}$.

(b) $R_{\alpha'} = Q_{\alpha'}(R_{\alpha'} \cap R_\alpha)$.

(c) $[U_{\alpha'}, R_{\alpha'}] = 1$.

Proof. (a) Note that (5.1.5)(a) implies that $[U_\alpha, K] \not\leq U_\alpha \cap Q_{\alpha'}$. Hence $K \not\leq R_{\alpha'}$. Now the claim follows from (2.2.6)(a).

(b) Suppose that $R_{\alpha'} \not\leq Q_{\alpha'}(R_{\alpha'} \cap R_\alpha)$. Since $R_{\alpha'}R_\alpha/R_\alpha \trianglelefteq M_{\alpha+1}/R_\alpha$, it follows that $Q_{\alpha+1} \leq R_\alpha R_{\alpha'}$. Hence $[K, R_{\alpha'}] \not\leq Q_{\alpha'}$, contrary to (a) and (1.1.2)(d).

(c) From (b) and (5.1.4) it follows that $[U_{\alpha'}, R_{\alpha'}] \neq U_{\alpha'}$, i.e.,

$$[U_{\alpha'}, R_{\alpha'}] \leq T_{\alpha'}.$$

Now the claim follows from (1.1.2)(d) and the Three-Subgroup Lemma. ■

(5.1.7) $q = 2$.

Proof. Suppose that $q > 2$. Let S be a Sylow 2-subgroup of $M_{\alpha+1}$. Let H be a subgroup of $M_{\alpha+1} \cap L_\alpha$ such that HR_α/R_α is the intersection of $C_{G_\alpha/R_\alpha}(Z(SR_\alpha/R_\alpha))$ and a Cartan subgroup in $N_{M_{\alpha+1}/R_\alpha}(SR_\alpha/R_\alpha)$. Then $q > 2$ implies that $[U_\alpha, Q'_{\alpha+1}]T_\alpha/T_\alpha$ contains no trivial H-composition factor.

Suppose that $H \not\leq L_{\alpha'}R_{\alpha'}$. Define

$$\overline{M_{\alpha+1}} := M_{\alpha+1}/Q_{\alpha+1}.$$

Let K be the 2-component of $M_{\alpha+1} \cap L_{\alpha}$. Then

$$\overline{K} \cong \mathrm{SL}_2(q) \quad \text{and} \quad \overline{HK} \cong \mathrm{GL}_2(q).$$

Pick $h \in H$ and $k \in K$ such that $h \notin L_{\alpha'}R_{\alpha'}$ and $\overline{hk} \in Z(\overline{HK})$. From (5.1.6)(a) it follows that $hk \notin L_{\alpha'}R_{\alpha'}$, i.e., hk induces an outer automorphism on $L_{\alpha'}R_{\alpha'}/R_{\alpha'}$. But also by (5.1.6)(a), K is the 2-component of $M_{\alpha+1} \cap L_{\alpha'}$ and hence $[K, hk] \leq Q_{\alpha+1}$ implies that hk induces an inner automorphism on $L_{\alpha'}R_{\alpha'}/R_{\alpha'}$. This contradiction shows that

$$(*) \quad H \leq (M_{\alpha+1} \cap L_{\alpha'})R_{\alpha'}.$$

Since, regarded as $(M_{\alpha+1} \cap L_{\alpha'})$-module, $[U_{\alpha'}, Q'_{\alpha+1}]Z_{\alpha+1}/Z_{\alpha+1}$ is dual to the centralizer of $Q_{\alpha+1}$ in $U_{\alpha'}/[U_{\alpha'}, Q'_{\alpha+1}]$, it follows from $(*)$, (5.1.6)(c)(a), and the first paragraph that the centralizer of $Q_{\alpha+1}$ in $U_{\alpha'}/[U_{\alpha'}, Q'_{\alpha+1}]$ contains no trivial H-composition factor. But

$$Z(SR_{\alpha}/R_{\alpha}) \leq C_{U_{\alpha'}}(Q_{\alpha+1}, [U_{\alpha'}, Q'_{\alpha+1}])R_{\alpha}/R_{\alpha},$$

and H centralizes $Z(SR_{\alpha}/R_{\alpha})$, a contradiction. ∎

5.2

In this section we assume (I)–(IV) and

(V) $L_{\alpha}R_{\alpha}/R_{\alpha} \cong \Omega^{+}_{10}(q)$ $(q = p^k)$ for some $k \in \mathbb{N}$.

(VI) U_{α} is a half-spin module for L_{α}.

(5.2.1) Let (γ, γ') be a critical pair.

(a) $[U_{\gamma}, U_{\gamma'}] = U_{\gamma'} \cap Q_{\gamma} = U_{\gamma} \cap Q_{\gamma'}$.

(b) $|U_\gamma : U_\gamma \cap Q_{\gamma'}| = |U_{\gamma'} : U_{\gamma'} \cap Q_\gamma| = q^8$.

(c) $U_{\gamma'}Q_\gamma = Q_\lambda$ where $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$.

(d) $(M_\lambda \cap L_\gamma)R_\gamma/R_\gamma$ is a parabolic subgroup of cotype 1 in $L_\gamma R_\gamma/R_\gamma$ where $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$.

Proof. [14](1.2) and [16]. ∎

(5.2.2) $b \leq 4$.

Proof. Suppose that $b > 4$. By (5.2.1)(c) and (1.2.6) assumption (iii) of (1.2.5) is satisfied. Since $M_{\alpha+1}$ is a maximal subgroup of G_α, also assumption (ii) of (1.2.5) is satisfied. Hence (1.2.5) and (5.2.1)(c) imply that

$$[U_\alpha, U_{\alpha'}] \leq T_\alpha,$$

contrary to $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$. ∎

Choose $g_\alpha \in L_\alpha$ such that $U_{\alpha'} \not\leq M_{\alpha+1}^{g_\alpha}$, i.e.,

$$G_\alpha = \langle M_{\alpha+1}^{g_\alpha}, U_{\alpha'} \rangle.$$

Define

$$\alpha - 1 := (\alpha+1)^{g_\alpha} \quad \text{and}$$

$$\alpha - 2 := (\alpha+2)^{g_\alpha}.$$

(5.2.3) Assume that $b > 2$.

(a) $(\alpha - 2, \alpha + 2)$ is a critical pair.

(b) $[U_{\alpha-2}, U_{\alpha+2}] \leq T_\alpha$.

(c) $O^{p'}(M_{\alpha-1}) \leq R_{\alpha-2}R_\alpha$.

Proof. (a) follows from (1.2.6), (1.2.1)(e) and (5.2.2).

(b) From (1.2.4) and (5.2.1)(c) it follows that L_α centralizes $[U_{\alpha-2}, U_{\alpha+2}]$. Also (5.2.1)(c) implies that $Q_{\alpha-1}$ centralizes $[U_{\alpha-2}, U_{\alpha+2}]$.

(c) Let K be the (by (5.2.1)(d) unique) p-component of $M_{\alpha-1} \cap L_{\alpha-2}$. Then

$$[U_{\alpha-2}, U_{\alpha+2}, K] \neq 1.$$

Thus (b) implies that $K \not\leq L_\alpha$. Hence $K \leq R_\alpha$ by (2.2.6)(a). ■

(5.2.4) $b = 2$.

Proof. Suppose that $b > 2$. Define

$$Y := \langle U_{\alpha+2}^{L_\alpha} \rangle \, Z_\alpha.$$

From (5.2.1)(c) we get

$$(*) \quad |Y : Y \cap Q_{\alpha-2}| \leq q^8.$$

Pick $g_{\alpha-2} \in L_{\alpha-2}$ such that $U_{\alpha+2} \not\leq M_{\alpha-1}^{g_{\alpha-2}}$ and put

$$\alpha - 4 := \alpha^{g_{\alpha-2}}.$$

As in (5.2.3)(a) it follows that $(\alpha - 4, \alpha)$ is a critical pair. From (5.2.1)(c) it follows that

$$(**) \quad Y \cap Q_{\alpha-2} \leq C_Y(U_{\alpha-4}, U_\alpha).$$

Note that Y/Z_α is a module for G_α/Q_α by (5.2.3)(b). If $[Y, L_\alpha] \leq Z_\alpha$, then $U_{\alpha+2} \leq U_{\alpha-2}Z_\alpha \leq Q_{\alpha-2}$, contrary to (5.2.3)(a). Hence

$$[Y, L_\alpha] \not\leq Z_\alpha.$$

Let W be a $GF(p)(L_\alpha R_\alpha)$-composition factor of Y/Z_α that is not centralized by L_α. By the same argument as in the proof of (3.2.10) it follows that, regarded as $GF(p)L_\alpha$-module, W is the direct sum of 8 copies of some irreducible $GF(p)L_\alpha$-submodule E of W. Then $(*)$ and $(**)$ imply that

$$(***) \quad |E : C_E(U_{\alpha-4})| \leq q.$$

In particular, E is an FF-module for $L_\alpha U_{\alpha-4}$ and $U_{\alpha-4}$ acts as an offending subgroup. By (5.2.1)(c), $U_{\alpha-4}$ does not act as an offending subgroup on the natural module. Hence (A.2.2) implies that E is a half-spin module. But then $(***)$ contradicts (5.2.1)(b). ■

(5.2.5) $[Q_\alpha, L_\alpha] = U_\alpha$.

Proof. From (5.2.1)(c) (with (α', α) in place of (γ, γ')), (5.2.4), and $L_\alpha \leq \langle U_{\alpha'}^{L_\alpha} \rangle$ it follows that

$$[Q_\alpha, L_\alpha] \leq \langle [Q_\alpha, U_{\alpha'}]^{L_\alpha} \rangle \leq \langle [Q_{\alpha+1}, U_{\alpha'}]^{L_\alpha} \rangle = \langle [U_\alpha Q_{\alpha'}, U_{\alpha'}]^{L_\alpha} \rangle \leq U_\alpha.$$

■

5.3

In this section we assume (I)–(IV) and

(V) $Q_{\alpha+1} \cap L_\alpha \leq R_\alpha$.

(5.3.1) (a) One of the following holds:

(a1) $G_\alpha / R_\alpha \cong \mathrm{O}_{2n}^\varepsilon(2)$ and U_α is a natural $\Omega_{2n}^\varepsilon(2)$-module for L_α for some $\varepsilon \in \{+, -\}$ and $n \in \mathbb{N}$ with $n \geq 4$.

(a2) $G_\alpha / R_\alpha \cong \Sigma_n$ and $U_\alpha T_\alpha / T_\alpha$ is a natural A_n-module for L_α ($n \in \mathbb{N}$ with $n \geq 5$).

(b) $Q_{\alpha+1} = Q_\alpha U_{\alpha'}$.

(c) $Q_{\alpha+1}$ acts as a transvection on U_α. In particular, $|U_\alpha : U_\alpha \cap Z_{\alpha+1}| = 2$.

(d) $M_{\alpha+1} = C_{G_\alpha}([U_\alpha, Q_{\alpha+1}] T_\alpha / T_\alpha)$.

(e) $M_{\alpha+1}$ is a maximal subgroup of G_α.

Proof. This follows from (2.2.4) and (2.3.1). ■

In chapter 1 the critical pair (α, α') was chosen such that $Z_{\alpha'}$ acts as an offending subgroup on Z_α. By (5.3.1)(c) we now have symmetry in α and α'.

(5.3.2) Let (γ, γ') be a critical pair and $\gamma + i = \gamma' - b + i \in \Delta^{(i)}(\gamma) \cap \Delta^{(b-i)}(\gamma')$ for each $i \in \{1, \ldots, b\}$. Let $\lambda \in \Delta(\gamma)$

(a) $G_\gamma = \langle M_\lambda, Q_{\gamma+1} \rangle$ if and only if $[Q_\lambda, Q_{\gamma+1}] \not\leq Q_\gamma$.

(b) Assume that $G_\gamma = \langle M_\lambda, Q_{\gamma+1} \rangle$. Then $(\mu, \gamma' - 2)$ is a critical pair for each $\mu \in \Delta(\lambda)$ with $Z_\mu Z_\gamma \ntrianglelefteq G_\gamma$.

Proof. Since (5.3.1)(c) implies that (α, α') is an arbitrary critical pair, it suffices to prove this for $(\gamma, \gamma') = (\alpha, \alpha')$.

(a) Note that $[Q_\lambda, Q_{\alpha+1}] \leq Q_\alpha$ is equivalent to $Q_{\alpha+1} \leq N_{G_\alpha}(Q_\lambda)$, since Q_α has index 2 in Q_λ and is normalized by $Q_{\alpha+1}$. Hence (a) follows from (5.3.1)(e).

(b) Since $Z_\mu Z_\alpha$ is normalized by M_λ, (5.3.1)(b) implies

$$[Z_\mu, U_{\alpha'}] \not\leq Z_\mu Z_\alpha.$$

Again by (5.3.1)(b) and the remark following (5.3.1),

$$[Q_{\alpha'-1}, U_{\alpha'}] = [U_\alpha Q_{\alpha'}, U_{\alpha'}] = [U_\alpha, U_{\alpha'}] \leq Z_\alpha.$$

Hence $Z_\mu \not\leq Q_{\alpha'-1}$. In particular, $Z_\mu \not\leq Q_{\alpha'-2}$. ■

(5.3.3) Assume that $b > 2$.

(a) $b = 4$.

(b) $[U_\alpha, Q_{\alpha+1}] \leq T_{\alpha+2}$.

Proof. Note that (5.3.1)(a) implies $A_{\alpha,\alpha+2} = [U_\alpha, Q_{\alpha+1}] \not\leq T_\alpha$. Moreover, by (5.3.2) assumptions (ii) and (iii) of (1.2.5) are satisfied. Hence the claim follows from from (1.2.4) and (1.2.5). ■

For $\gamma \sim \alpha$ define

$$X_\gamma := Z_\gamma \langle U_\delta \mid d(\gamma, \delta) = 2 \rangle.$$

(5.3.4) Assume that $b = 4$.

(a) $[X_\alpha, Q_\alpha] \leq T_\alpha$.

(b) $Q'_{\alpha+1}$ is elementary abelian.

Proof. (a) follows from (5.3.3)(b).

(b) From $b = 4$ and (5.3.1)(b) it follows that

$$Q_{\alpha+1} = Q_\alpha X_{\alpha+2}$$

and, hence,

$$Q'_{\alpha+1} = Q'_\alpha X'_{\alpha+2}[Q_\alpha, X_{\alpha+2}] = Q'_\alpha[Q_{\alpha+1}, X_{\alpha+2}].$$

Note that

$$[Q_{\alpha+1}, X_{\alpha+2}] = [X_\alpha Q_{\alpha+2}, X_{\alpha+2}] \leq [X_\alpha, X_\alpha + 2]T_{\alpha+2} \leq X_\alpha.$$

Hence

$$[Q_{\alpha+1}, X_{\alpha+2}, Q'_\alpha] \leq [X_\alpha, Q_\alpha] \leq T_\alpha \quad \text{and}$$

$$(*) \quad [Q_{\alpha+1}, X_{\alpha+2}]' \leq X'_\alpha \leq T_\alpha,$$

by (a). Clearly

$$A := [Q_{\alpha+1}, X_{\alpha+2}, Q'_\alpha][Q_{\alpha+1}, X_{\alpha+2}]'$$

is a normal subgroup of $M_{\alpha+1}$. Since $[A, L_\alpha] \leq [T_\alpha, L_\alpha] = 1$, we get $A \trianglelefteq G_\alpha$. Now $Q''_{\alpha+1} = Q''_\alpha A$ implies that

$$Q''_{\alpha+1} = 1.$$

In particular,

$$(**) \quad \Phi(Q'_{\alpha+1}) = \Phi(Q'_\alpha)\Phi([Q_{\alpha+1}, X_{\alpha+2}]).$$

Since X_α is generated by involutions, $(*)$ implies

$$\Phi([Q_{\alpha+1}, X_{\alpha+2}]) \leq T_\alpha.$$

As above we get that $\Phi([Q_{\alpha+1}, X_{\alpha+2}]) \trianglelefteq G_\alpha$, and then (b) follows from $(**)$. ■

(5.3.5) There exists $\lambda \in \Delta(\alpha)$ such that $\langle Q_{\alpha+1}, Q_\lambda\rangle R_\alpha/R_\alpha \cong \Sigma_3$.

Proof. It suffices to show that $Q_{\alpha+1}R_\alpha/R_\alpha$ is contained in a subgroup of G_α/R_α that is isomorphic to Σ_3. This is evident if (a2) holds in (5.3.1)(a).

Assume first that G_α/R_α is isomorphic to $\mathrm{O}^+_{2n}(2)$ for some $n \geq 4$. Then $Q_{\alpha+1}$ induces a nontrivial graph automorphism on a Levi complement X of a parabolic subgroup of type $\{n-2, n-1, n\}$ of $L_\alpha R_\alpha/R_\alpha$. Since $X \cong \mathrm{PSL}_4(2) \cong A_8$, it follows that $(Q_{\alpha+1}R_\alpha/R_\alpha)X \cong \Sigma_8$ and then the claim is obvious.

Assume now that G_α/R_α is isomorphic to $\mathrm{O}^-_{2n}(2)$ for some $n \geq 4$. Then $Q_{\alpha+1}$ induces a nontrivial graph automorphism on a Levi complement X of a parabolic subgroup of type $\{n-1, n\}$ of $L_\alpha R_\alpha/R_\alpha$. Since $X \cong \mathrm{PSL}_2(4) \cong A_5$, it follows that $(Q_{\alpha+1}R_\alpha/R_\alpha)X \cong \Sigma_5$ and again the claim is obvious. ■

Choose $g_\alpha \in L_\alpha$ such that $\langle Q_{\alpha+1}, Q_{\alpha+1}^{g_\alpha}\rangle R_\alpha/R_\alpha \cong \Sigma_3$ and put

$$\alpha - 1 := (\alpha+1)^{g_\alpha} \quad \text{and}$$

$$\alpha - 2 := (\alpha+2)^{g_\alpha}.$$

(5.3.6) $b = 2$.

Proof. Suppose $b \neq 2$. Then $b = 4$ by (5.3.3)(a). Pick $u \in U_{\alpha'} \setminus Q_\alpha$. Note that the choice of g_α implies that

$$[Q_{\alpha-1}, Q_{\alpha-1}^u] \not\leq Q_\alpha.$$

Therefore

$$G_\alpha = \langle U_{\alpha'}^{g_\alpha}, M_{\alpha-1}^u \rangle$$

by (5.3.2)(a). From (5.3.2) and (1.2.1)(e) it follows that $U_{\alpha+2}U_\alpha \ntrianglelefteq G_\alpha$, and hence

$$U_{(\alpha-2)^u} U_\alpha \ntrianglelefteq G_\alpha.$$

Now from (5.3.2) (with $(\alpha, (\alpha')^{g_\alpha}, (\alpha-1)^u, (\alpha-2)^u)$ in place of $(\gamma, \gamma', \lambda, \mu)$) we get that $(\alpha - 2, (\alpha-2)^u)$ is a critical pair.

Pick $v \in U_{\alpha-2} \setminus Q_{\alpha-2}^u$. Then $v^u \notin Q_{\alpha-2}$, i.e., $[v, v^u] \neq 1$. But then $[v, u]$ is not an involution. Since $v, u \in Q_{\alpha+1}$, this is a contradiction to (5.3.4)(b). ∎

(5.3.7) $[Q_\alpha, L_\alpha] = U_\alpha$.

Proof. This follows from (5.3.1)(b) and (5.3.6) the same way (5.2.5) follows from (5.2.1)(c) and (5.2.4).

5.4

In this section we assume (I)–(IV) and

(V) $L_\alpha R_\alpha / R_\alpha \cong \Omega_7(q)$ $(q = p^k)$ for some $k \in \mathbb{N}$.

(VI) U_α is a spin module for L_α.

(5.4.1) Let (γ, γ') be a critical pair.

(a) $[U_\gamma, U_{\gamma'}] = [U_\gamma, Q_\lambda] = U_{\gamma'} \cap Q_\gamma = U_\gamma \cap Q_{\gamma'}$ where $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$

(b) $|U_\gamma : U_\gamma \cap Q_{\gamma'}| = |U_{\gamma'} : U_{\gamma'} \cap Q_\gamma| = q^4$.

(c) $(M_\lambda \cap L_\gamma) R_\gamma / R_\gamma$ is a parabolic subgroup of type B_2 in $L_\gamma R_\gamma / R_\gamma$ where $\lambda \in \Delta(\gamma) \cap \Delta^{(b-1)}(\gamma')$

Proof. [14](1.2), [16], and (B.3.2.1). ∎

(5.4.2) $U_{\alpha'}R_\alpha/R_\alpha \ntrianglelefteq M_{\alpha+1}/R_\alpha$. In particular, $b \neq 2$.

Proof. By (5.4.1)(c) $Q_{\alpha+1}R_\alpha/R_\alpha$ contains no $M_{\alpha+1}$-submodule of order q^4. Hence the claim follows from (5.4.1)(b). ■

(5.4.3) A contradiction.

Proof. Using (5.4.1) in place of (5.2.1), it follows as in the proof of (5.2.2) that $b \leq 4$. Hence, by (5.4.2),

$$b = 4.$$

Pick $g \in L_\alpha$ such that $U_{\alpha'} \not\leq M^g_{\alpha+1}$. Then (5.4.1) and (1.2.6) imply that $((\alpha+2)^g, \alpha+2)$ is a critical pair.

Suppose that $[U^g_{\alpha+2}, U_{\alpha+2}] \leq U_\alpha$. Then (5.4.1)(a) and $[U_{\alpha+2}, U_{\alpha'}] = 1$ imply that $[U_\alpha, Q^g_{\alpha+1}] = [U_\alpha, Q_{\alpha+2}]$. Since $N_{L_\alpha}([U_\alpha, Q_{\alpha+1}]) = M_{\alpha+1} \cap L_\alpha$, this is a contradiction to the choice of g. Hence

$$[U^g_{\alpha+2}, U_{\alpha+2}] \leq T_\alpha.$$

But this implies that the p-component of $M^g_{\alpha+1} \cap L^g_{\alpha+2}$ is contained in R_α and hence normalizes $U_{\alpha+2}$, contrary to (5.4.2) (applied to $((\alpha+2)^g, \alpha+2)$ in place of (α, α')).

■

Chapter 6

Proof of Theorem 1

Let G, M, R, L, $L_1, \ldots, L_m$, and p satisfy the assumptions of Theorem 1. By (2.2.1)(b)(c) L is the product of $L_1, \ldots, L_m$, and M permutes $L_1, \ldots, L_m$ transitively. Put

$$\widehat{M} := \bigcap_{\phi \in \mathrm{Aut}(M)} \bigcap_{i=1}^{m} N_M(L_i)^{\phi},$$

$$\hat{G} := \widehat{M} L_1,$$

$$\hat{R} := \bigcap_{g \in \hat{G}} \widehat{M}^g,$$

$$\hat{Z} := \langle \Omega_1(Z(O_p(M)))^G \rangle,$$

$$\hat{U} := [\hat{Z}, L_1], \quad \text{and}$$

$$\hat{T} := C_{\hat{Z}}(L_1).$$

By (2.2.2)(a)(c) and (2.2.11) assumptions (I)–(IV) of chapter 1 are satisfied for $\hat{G}$, $\widehat{M}$, $\hat{R}$, L_1, and the group of automorphisms of $\hat{G}$ that are induced by $N_M(L_1)$ in place of G, M, R, L, and H. Moreover, $\hat{Z}$, $\hat{U}$, and $\hat{T}$ play the role of Z_α, U_α, and T_α, respectively. Note that $O_p(M) = O_p(\widehat{M})$ by (2.2.2)(c), and hence $O_p(G) \leq R \cap O_p(\widehat{M}) = O_p(\hat{G})$. Now $L_1 = L_1'$ and (1.1.2)(d) imply that

$$(*) \quad [R, L_1] = [R, L_1, L_1] \leq [O_p(G), L_1] \leq [O_p(\hat{G}), L_1].$$

Assume $L_1\widehat{R}/\widehat{R}$ is not an orthogonal, symplectic, or unitary group. Then (A.2.2) implies that $L_1\widehat{R}/\widehat{R} \cong \mathsf{G}_2(2^k)'$ for some $k \in \mathbb{N}$, since U_1 is an FF-module for L_1 and we assume that $L_1\widehat{R}/\widehat{R}$ is a group of Lie type. Now (5.1.5), (5.1.7), and $(*)$ show that case 10 of Theorem 1 holds for $i = 1$.

Assume $L_1\widehat{R}/\widehat{R}$ is an orthogonal, symplectic, or unitary group. If $\widehat{U} \cap \widehat{T} \neq 1$, then by (2.2.11)(d) and the assumption that L_1R/R belongs to $\tilde{\mathcal{L}}_p$ we can apply (2.3.1), and thus (4.3.2), (4.3.3), and $(*)$ show that one of the cases 1 or 4 of Theorem 1 holds for $i = 1$.

Assume now that $\widehat{U} \cap \widehat{T} = 1$. If $(\widehat{M} \cap L_1)\widehat{R}/\widehat{R}$ is not a parabolic subgroup of $L_1\widehat{R}/\widehat{R}$, then (2.2.4), (5.3.6), (5.3.7), and $(*)$ imply that case 8 of Theorem 1 holds for $i = 1$. Hence assume that $(\widehat{M} \cap L_1)\widehat{R}/\widehat{R}$ is a parabolic subgroup of $L_1\widehat{R}/\widehat{R}$. If U_1 is not a natural module for L_1, then (A.2.2), (5.2.5), (5.4.3), and $(*)$ show that one of the case 9 of Theorem 1 holds for $i = 1$.

Hence assume that U_1 is a natural module for L_1. Suppose that $\widehat{M}$ is not the stabilizer of a singular subspace of U_1. Then by (2.2.4) $L_1\tilde{R}/\tilde{R}$ is of type D_4, and the parabolic subgroup $(\widehat{M} \cap L_1)\widehat{R}/\widehat{R}$ belongs to the interior node of the Dynkin diagram. Moreover, $N_M(L_1)$ permutes the maximal parabolic subgroups of $L_1\widehat{R}/\widehat{R}$ containing $(\widehat{M} \cap L_1)\widehat{R}/\widehat{R}$ transitively. But this is a contradiction to (A.2.9). Hence $\widehat{M}$ is the stabilizer of a singular subspace of U_1. Now (3.5.2), (4.3.2), (4.3.3), and $(*)$ show that one of the cases 1,2,3,5,6, or 7 of Theorem 1 holds for $i = 1$.

Since M permutes $L_1, \ldots, L_m$ transitively, all of this holds for any $i \in \{1, \ldots, m\}$ in place of 1.

Suppose that $[L_i, L_j] \neq 1$, for some $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Since L_i and L_j are p-components,

$$[L_i, L_j, L_j] = [L_j, L_i, L_i] = [L_i, L_j] \leq O_p(G) \leq R.$$

Note that in the cases 1–3 and 5–10 of Theorem 1 we have $[[R, L_i] \cap [R, L_j], L] = 1$ by (2.1.1)(g), and hence

$$[L_i, L_j, L] = [[L_i, L_j, L_j] \cap [L_j, L_i, L_i], L] \leq [[R, L_j] \cap [R, L_i], L] = 1,$$

contrary to $1 \neq [L_i, L_j] = [L_i, L_j, L_j]$. Hence we are in case 4 of Theorem 1. Pick a subgroup X of $[L_i, L_j]$ such that $[L_i, L_j]/X$ is an irreducible L_iL_j-module. Then (A.3.1) implies that $[L_i, L_j]/X$ contains 8 nontrivial L_i-composition factors. But $O_p(G)$ contains only 2 nontrivial L_i-chief factors. This contradiction shows that L is the central product of $L_1, \ldots, L_m$.

Appendix A

General Lemmas

A.1 Various Results

(A.1.1) Let X and Y be finite groups such that Y acts on X. Assume that X is the direct product

$$X = X_1 \times \ldots \times X_m$$

of at least two subgroups $X_1, \ldots, X_m$, that are transitively permuted by Y.

(a) $X' \leq [X, Y]$.

(b) $[X, Y]/X'$ contains a subgroup that is isomorphic to a direct product of $m-1$ copies of X_1/X_1'.

(c) $X = [X, Y]X_i$, for each $i \in \{1, \ldots, m\}$.

Proof. (a) Let $i \in \{1, \ldots, m\}$. Pick $y \in Y$ such that $X_i^y \neq X_i$. Since $[X_i, X_i^y] = 1$, we get

$$[a, y][b, y][ab, y]^{-1} = a^{-1}a^y b^{-1} b^y ((ab)^y)^{-1} ab = a^{-1}b^{-1}aba^y b^y (a^y b^y)^{-1} = [a, b],$$

for all $a, b \in X_i$.

(b) For each $i \in \{1, \ldots, m-1\}$, pick $y_i \in Y$ with $X_i^{y_i} = X_m$. Then

$$A_i := [X_i, y_i]X'$$

is a subgroup of $[X,Y]$ with $A_i/X' \cong X_1/X_1'$ and

$$A_i \cap A_1 \cdot \ldots \cdot A_{i-1}A_{i+1} \cdot \ldots \cdot A_{m-1} \leq A_i \cap X_1 \cdot \ldots \cdot X_{i-1}X_i'X_{i+1} \cdot \ldots \cdot X_m \leq X'$$

for each $i \in \{1,\ldots,m-1\}$. Hence $A_1 \cdot \ldots \cdot A_{m-1}/X'$ has the desired properties.

(c) If $i,j \in \{1,\ldots,m\}$, then there exists $y \in Y$ such that $X_j^y = X_i$ and hence $X_j \subseteq [X_j,y]X_i \subseteq [X,Y]X_i$. ■

(A.1.2) Let G be a finite group, M a subgroup of G, and H a subgroup of $\mathrm{Aut}(G)$. Assume that the following hold:

(i) M is H-invariant.

(ii) MH is a maximal subgroup of GH.

Put $R := \bigcap_{g\in G} M^g$ and $\tilde{R} := \bigcap_{g\in GH}(MH)^g$. Let $\mathcal{N}$ be the set of all minimal normal subgroups of G/R. Put

$$\mathcal{N}_0 := \{N_0 \in \mathcal{N} \mid N \cap \langle N_0^H \rangle = 1 \text{ for each } N \in \mathcal{N} \setminus N_0^H\}.$$

For each $N \in \mathcal{N}$ define

$$\phi(N) := \langle X^H \rangle \tilde{R}/\tilde{R},$$

where $R \leq X \leq G$ and $X/R = N$.

(a) If $N \in \mathcal{N}$ is perfect, then $N \in \mathcal{N}_0$.

(b) Let $N \in \mathcal{N}_0$. Then $\phi(N)$ is a minimal normal subgroup of $GH/\tilde{R}$. Moreover, if the H-orbit of N in $\mathcal{N}$ has size k, then $\phi(N)$ is isomorphic to a direct product of at least $\min\{2,k\}$ and at most k copies of N. In particular, $\phi(N)$ is perfect if and only if N is perfect.

(c) If $N_1 \in \mathcal{N}_0$ and $N_2 \in \mathcal{N}$, then $\phi(N_1) = \phi(N_2)$ if and only if N_1 and N_2 are conjugate under H.

(d) The elements of $\mathcal{N}$ are pairwise isomorphic.

(e) If $GH/\tilde{R}$ has a nonabelian simple normal subgroup A, then each elements of $\mathcal{N}$ is isomorphic to A and normalized by H.

Proof. (a) This is clear, since the set of components of a direct product is the union of the sets of components of the direct factors.

(b) Let $X \leq G$ such that $R \leq X$ and $X/R = N$. Let $Y \leq \langle X^H \rangle \tilde{R}$ such that $\tilde{R} \leq Y$ and $Y/\tilde{R}$ is a minimal normal subgroup of $GH/\tilde{R}$. Then

$$Y = (Y \cap \langle X^H \rangle)\tilde{R} \quad \text{and} \quad Y \not\subseteq \tilde{R}.$$

Hence $Y \cap \langle X^H \rangle \not\subseteq R$. Choose $X_1 \leq Y \cap \langle X^H \rangle$ such that $R \leq X_1$ and $X_1/R \in \mathcal{N}$. Then N and X_1/R are in the same H-orbit of $\mathcal{N}$, since $N \in \mathcal{N}_0$. Hence $\phi(N) = \phi(X_1/R) = Y/\tilde{R}$ is a minimal normal subgroup of $GH/\tilde{R}$. The second assertion follows from

$$\phi(N) = \langle X^H \rangle \tilde{R}/\tilde{R} \cong \langle X^H \rangle / \langle X^H \rangle \cap \tilde{R} = \langle X^H \rangle / R.$$

(c) Clearly $\phi(N_1) = \phi(N_2)$ if N_1 and N_2 are conjugate under H. Now assume that $\phi(N_1) = \phi(N_2)$. Let $X_1, X_2 \leq G$ such that $R \leq X_1, X_2$, $X_1/R = N_1$ and $X_2/R = N_2$. Then $\langle X_1^H \rangle \leq \langle X_2^H \rangle \tilde{R}$. Since $\tilde{R} \cap G = R$, we get

$$\langle X_1^H \rangle \leq \langle X_2^H \rangle R = \langle X_2^H \rangle,$$

and likewise $\langle X_2^H \rangle \leq \langle X_1^H \rangle$. Hence

$$\langle N_1^H \rangle = \langle N_2^H \rangle.$$

Since $N_1 \in \mathcal{N}_0$, it follows that N_2 is conjugate to N_1 under H.

(d) Note that if $N \in \mathcal{N}$ is abelian, then $\phi(N)$ is an abelian normal subgroup of $GH/\tilde{R}$ and hence $GH/\tilde{R}$ has an abelian minimal normal subgroup. Since, by [12], the minimal normal subgroups of $GH/\tilde{R}$ (at most two) are pairwise isomorphic, (d) now follows from (a) and (b). ∎

(A.1.3) Let G be a symmetric group of degree 2^k for some $k \in \mathbb{N} \setminus \{0\}$. Put

$$\mathcal{X}_i := \left\{ \{r \mid 2^i(j-1) < r \leq 2^i j\} \mid 1 \leq j \leq 2^{k-i} \right\}$$

for each $i \in \{0, \ldots, k\}$. Then

(a) $N_G(\bigcup_{i=0}^k \mathcal{X}_i) \in \mathrm{Syl}_2(G)$.

(b) $Z(N_G(\bigcup_{i=0}^k \mathcal{X}_i)) = \langle (1\ \ 2)\ \ (3\ \ 4) \ldots (2^k - 1\ \ 2^k) \rangle$

Proof. This is obvious when $k = 1$, so assume that $k > 1$. Put

$$A := C_G(\{2^{k-1}+1, \ldots, 2^k\}),$$

$$B := C_G(\{1, \ldots, 2^{k-1}\}) \quad \text{and}$$

$$x := (1\ \ 2^{k-1}+1)\ \ (2\ \ 2^{k-1}+2)\ \ \ldots\ \ (2^{k-1}\ \ 2^k).$$

By induction on k we may assume that

$$N_A(\bigcup_{i=0}^k \mathcal{X}_i) \in \mathrm{Syl}_2(A),$$

$$N_B(\bigcup_{i=0}^k \mathcal{X}_i) \in \mathrm{Syl}_2(B),$$

$$Z(N_A(\bigcup_{i=0}^k \mathcal{X}_i)) = \langle (1\ \ 2)\ \ (3\ \ 4) \ldots (2^{k-1} - 1\ \ 2^{k-1}) \rangle \quad \text{and}$$

$$Z(N_B(\bigcup_{i=0}^k \mathcal{X}_i)) = \langle (2^{k-1}+1\ \ 2^{k-1}+2)\ \ (2^{k-1}+3\ \ 2^{k-1}+4) \ldots (2^k - 1\ \ 2^k) \rangle.$$

Now

$$N_G(\bigcup_{i=0}^k \mathcal{X}_i) = (N_A(\bigcup_{i=0}^k \mathcal{X}_i) \times N_B(\bigcup_{i=0}^k \mathcal{X}_i))\langle x \rangle \cong N_A(\bigcup_{i=0}^k \mathcal{X}_i) \wr C_2$$

is a Sylow 2-subgroup, since it has the appropriate order. Moreover,

$$Z(N_G(\bigcup_{i=0}^k \mathcal{X}_i)) = C_{Z(N_A(\bigcup_{i=0}^k \mathcal{X}_i)) \times Z(N_B(\bigcup_{i=0}^k \mathcal{X}_i))}(x) =$$

$$\langle (1\ \ 2)\ \ (3\ \ 4) \ldots (2^k - 1\ \ 2^k) \rangle.$$

■

(A.1.4) Let G be a finite symmetric group of degree $n \geq 7$. Then for each $S \in \mathrm{Syl}_2(G)$ the following hold:

(a) $Z(S \cap G') \leq Z(S)$.

(b) $C_{G'}(Z(S \cap G')) \neq S \cap G'$.

Proof. (a) Assume that $n = 2^k$ for some $k \in \mathbb{N}$. Put

$$H := C_G(\{1, \ldots, 2^{k-1}\}) \quad \text{and}$$

$$x := (1 \;\; 2^{k-1}+1) \;\; (2 \;\; 2^{k-1}+2) \;\; \ldots \;\; (2^{k-1} \;\; n).$$

Let T be a Sylow 2-subgroup of H. Then

$$S := \langle T, x \rangle = (T^x \times T)\langle x \rangle$$

is a Sylow 2-subgroup of G, and

$$(*) \quad Z(S) = C_{Z(T^x) \times Z(T)}(x).$$

If $Z(S \cap G') \not\leq T^x \times T$ then $T \cap G' \trianglelefteq S$, a contradiction to $T^x \cap T = 1$ and $|T \cap G'| \geq 2^{k-2} \geq 2$. Hence $Z(S \cap G') \leq T^x \times T$. Let $\alpha : T^x \times T \to T^x$ and $\beta : T^x \times T \to T$ be defined by

$$y = y^\alpha y^\beta \quad \text{for each } y \in T^x \times T.$$

Then $(T^x \times T) \cap G'$ centralizes y^α and y^β for each $y \in Z(S \cap G')$. Since $Z(S \cap G') \cap T$ is centralized by $((T^x \times T) \cap G')^\beta = T$, we get

$$Z(S \cap G') \cap T \leq Z(T)$$

and likewise

$$Z(S \cap G') \cap T^x \leq Z(T^x).$$

Hence

$$Z(S \cap G') = Z(S \cap G') \cap (T^x \times T) \leq$$

$$(Z(S \cap G') \cap (T^x \times T))^\alpha \times (Z(S \cap G') \cap (T^x \times T))^\beta \leq$$

$$(Z(S \cap G') \cap T^x) \times (Z(S \cap G') \cap T) \leq Z(T^x) \times Z(T).$$

Since 2^{k-1} is even, $S \cap G'$ contains x. Now (a) follows from $(*)$.

Assume that $n = \sum_{i=1}^{m} 2^{k_i}$ for some integers $k_1, \ldots, k_m$ with $m > 1$ and $1 \leq k_1 < \ldots < k_m$. For each $i \in \{1, \ldots, m\}$ let T_i be a Sylow 2-subgroup of

$$H_i := C_G(\{r \mid \textstyle\sum_{j=1}^{i-1} 2^{k_j} < r \leq \sum_{j=1}^{i} 2^{k_j}\}).$$

Then

$$S := \langle T_1, \ldots, T_m \rangle = T_1 \times \ldots \times T_m$$

is a Sylow 2-subgroup of G. Define $\alpha_i : S \to T_i \quad (i \in \{1, \ldots, m\})$ by

$$y = y^{\alpha_1} y^{\alpha_2} \cdot \ldots \cdot y^{\alpha_m} \qquad \text{for each } y \in S.$$

Note that $m > 1$ implies $(S \cap G')^{\alpha_i} = T_i$ for each $i \in \{1, \ldots, m\}$, whence

$$Z(S \cap G') \leq Z(S \cap G')^{\alpha_1} \times \ldots \times Z(S \cap G')^{\alpha_m} \leq$$

$$C_{T_1}((S \cap G')^{\alpha_1}) \times \ldots \times C_{T_m}((S \cap G')^{\alpha_m}) = Z(T_1) \times \ldots \times Z(T_m) = Z(S).$$

(b) If $n = 7$ and $S \in \mathrm{Syl}_2(G)$ then $Z(S \cap G')$ has 3 fixed points on $\{1, \ldots, 7\}$, so $C_{G'}(S \cap G')$ contains an element of order 3 permuting the fixed points of $Z(S \cap G')$. Now assume $n \geq 8$. Let $k_1, \ldots, k_m, T_1, \ldots, T_m$ and S be defined as in the proof of (a) (except that now possibly $m = 1$). From (A.1.3) and $n \geq 8$ it follows that $Z(T_m)$ has order 2 and at least 4 orbits of size 2 on $\{1, \ldots, n\}$. Hence $C_{G'}(Z(S))$ contains a subgroup isomorphic to A_4 permuting the $Z(T_m)$-orbits of size 2. Now the claim follows from (a). ∎

The following lemma from [14] will be particularly useful.

(A.1.5) Let G be a group, p a prime, and V a faithful $GF(p)G$-module. Assume that there exists a non-degenerate G-invariant symmetric $GF(p)$-bilinear form on V.

(a) $[V, A]^{\perp} = C_V(A)$, for each $A \leq G$.

(b) Assume that $B \leq A \leq G$, $[V, B, A] = 1$, and $[C_V(B), A] \cap [V, B] = 1$. Then $[V, A] = [C_V(B), A][V, B]$.

Proof. If $p = 2$, then this is [14](1.7). The proof for an arbitrary prime is the same. ∎

A.2 FF-Modules

Let p be prime and G a finite group. Define

$$\mathcal{E}_p(G) := \{A \leq G \mid A \text{ is a nontrivial elementary abelian } p\text{-group}\}.$$

If V is a faithful finite-dimensional $GF(p)G$-module, put

$$\mathcal{P}(G, V) := \{A \in \mathcal{E}_p(G) \mid B \leq A \ \Rightarrow \ |A|\,|C_V(A)| \geq |B|\,|C_V(B)|\} \quad \text{and}$$

$$\mathcal{P}^*(G, V) := \{A \in \mathcal{P}(G, V) \mid A \geq B \in \mathcal{P}(G, V) \ \Rightarrow \ A = B\}.$$

If V is any finite-dimensional $GF(p)G$-module, put

$$\mathcal{P}(G, V) := \{A \leq G \mid A\,C_G(V)/C_G(V) \in \mathcal{P}(G/C_G(V), V)\} \quad \text{and}$$

$$\mathcal{P}^*(G, V) := \{A \leq G \mid A\,C_G(V)/C_G(V) \in \mathcal{P}^*(G/C_G(V), V)\}.$$

(A.2.1) Let G be a finite group, p a prime, V a faithful $GF(p)G$-module, and $A \in \mathcal{P}^*(G, V)$.

(a) $[V, A, A] = 0$.

(b) If $V \geq W \geq [W, A] \neq 0$, then $AC_G(W)/C_G(W) \in \mathcal{P}^*(N_G(W)/C_G(W), W)$.

(c) If W is an A-invariant subspace of V such that $C_A(V/W) = 1$, then V/W is an FF-module for $N_G(W)/C_{N_G(W)}(V/W)$ and $A\,C_{N_G(W)}(V/W)/C_{N_G(W)}(V/W)$ is an offending subgroup acting quadratically on V/W.

(d) A normalizes each component of G.

Proof. (a) This follows from [5](4.2).

(b) If $B \leq A$, then

$$|B\,C_A(W)|\,|C_V(B\,C_A(W))| \leq |A|\,|C_V(A)|,$$

since $A \in \mathcal{P}(G,V)$, and therefore

$$|B\,C_G(W)/C_G(W)|\,|C_W(B)| = |B\,C_A(W)C_G(W)/C_G(W)|\,|C_W(B\,C_A(W))| =$$

$$\frac{|B\,C_A(W)|}{|C_{B\,C_A(W)}(W)|}\,|C_W(B\,C_A(W))| = \frac{|B\,C_A(W)|}{|C_A(W)|}\,|C_W(B\,C_A(W))| =$$

$$\frac{|B\,C_A(W)|}{|C_A(W)|}\,|C_V(B\,C_A(W)) \cap W| = \frac{|B\,C_A(W)|}{|C_A(W)|}\,\frac{|C_V(B\,C_A(W))|\,|W|}{|C_V(B\,C_A(W)) + W|} \leq$$

$$\frac{|B\,C_A(W)|}{|C_A(W)|}\,\frac{|C_V(B\,C_A(W))|\,|W|}{|C_V(A) + W|} \leq \frac{|A|}{|C_A(W)|}\,\frac{|C_V(A)|\,|W|}{|C_V(A) + W|} =$$

$$|A\,C_G(W)/C_G(W)|\,|C_W(A)|.$$

(c) This is obvious.

(d) [4] ■

(A.2.2) Let G be a finite group, p a prime, and V a faithful and irreducible FF-module for $GF(p)G$. Assume that the following hold:

(i) $F^*(G)$ is quasisimple, and $F^*(G)/Z(F^*(G))$ belongs to the class $\mathcal{L}_p$, as defined in the introduction.

(ii) $G = AF^*(G)$ for some $A \in \mathcal{P}^*(G,V)$.

Then one of the following holds:

(a) $G \cong \mathrm{SL}_n(q)$ $(q = p^k)$, and V is a natural $\mathrm{SL}_n(q)$-module.

(b) $G \cong \mathrm{SL}_n(q)$ $(q = p^k,\ n \geq 4)$, and V is the second exterior power of a natural $\mathrm{SL}_n(q)$-module.

(c) $G \cong \mathrm{Sp}_{2n}(q)$ $(q = p^k)$, and V is a natural $\mathrm{Sp}_{2n}(q)$-module.

(d) $G \cong \mathrm{Sp}_6(q)$ $(p = 2,\ q = 2^k)$, $|V| = q^8$, $|C_V(A)| = |[V,A]| = q^4$, and $[V,A] = [V, C_G([V,A])]$.

(e) $G \cong \Omega_n^{\varepsilon}(q)$ $(q = p^k,\ \varepsilon \in \{0,+,-\})$, and V is a natural $\Omega_n^{\varepsilon}(q)$-module.

(f) $G \cong \mathrm{O}_{2n}^{\varepsilon}(q)$ $(p = 2,\ q = 2^k,\ \varepsilon \in \{+,-\})$, and V is a natural $\mathrm{O}_{2n}^{\varepsilon}(q)$-module.

(g) $G \cong \Omega_{10}^{+}(q)$ $(q = p^k)$, and V is a half-spin module.

(h) $G \cong \mathrm{SU}_n(q)$ $(q = p^k,\ n \geq 4)$, and V is a natural $\mathrm{SU}_n(q)$-module.

(i) $G \cong \mathsf{G}_2(q)$ $(p = 2,\ q = 2^k)$, $|V| = q^6$, $|C_V(A)| = |[V,A]| = |A| = q^3$, and $C_G(A) = A$.

(j) $G \cong \Sigma_n$, $p = 2$, and V is a natural Σ_n-module.

(k) $G \cong A_{2n}$, $p = 2$, and V is a natural A_{2n}-module.

(l) $G \cong \hat{A}_6$, $p = 2$, $|V| = 2^6$, $|V/C_V(A)| = |A| = 4$, $|[V,A]| = 16$, and $C_G(A) = Z(G)A$.

(m) $G \cong A_7$, $p = 2$, and $|V| = 2^4$, $|V/C_V(A)| = |A| = |[V,A]| = 4$.

(n) $G \cong \Omega_7^0(q)$ $(p \neq 2,\ q = p^k)$, and V is a spin module.

Proof. This follows from [14](1.2), [6], and [16]. ■

(A.2.3) Let G be a finite group and V a faithful FF-module for $GF(p)G$. Let L be a component of G. Assume that $[A,L] \not\leq Z(L)$ for some $A \in \mathcal{P}^*(G,V)$. Then for each $GF(p)(AL)$-composition factor W of V with $[W,L] \neq 0$ the following hold:

(a) $AC_{AL}(W)/C_{AL}(W) \in \mathcal{P}^*(AL,W)$,

(b) $LC_{AL}(W)/C_{AL}(W) = F^*(AL/C_{AL}(W))$,

(c) $[L, C_{AL}(W)] \leq Z(L)$.

Proof. From $[W, L] \neq 0$ it follows that $LC_{AL}(W)/C_{AL}(W)$ is a component of $AL/C_{AL}(W)$. If $LC_{AL}(W)/C_{AL}(W) \neq F^*(AL/C_{AL}(W))$, then $O_p(AL/C_{AL}(W)) \neq 1$, since AL/L is a p-group. But $O_p(AL/C_{AL}(W)) = 1$, since W is irreducible. Thus (b) holds.

(c) follows from $[L, C_{AL}(W)] \leq C_L(W)$ and $[W, L] \neq 0$.

(a) follows from (c), $[L, A] \not\leq Z(L)$ and (A.2.1)(b). ■

(A.2.4) Let G be a finite group, F a finite field of characteristic p, and V an irreducible FG-module. Assume that there exist subgroups A and L of G such that $G = AL$, $[V, A, A] = 0$ and $L \trianglelefteq G$. Then one of the following holds:

(a) V is irreducible as FL-module.

(b) $|A/C_A(V)| = |G : C_G(V)L| = 2$ and

$$V \cong W \otimes_{F(C_G(V)L)} FG$$

for some irreducible FL-submodule W of V. In particular, $|A/C_A(V)| < |V/C_V(A)|$.

Proof. Without loss we may assume that G acts faithfully on V. Let W be an irreducible FL-submodule of V and assume that $W \neq V$. Then $N_A(W)$ is a proper subgroup of A.

Choose $a \in A \setminus N_A(W)$. Then $W + W^a$ is the direct sum of W and W^a. Since A, acting faithfully and quadratically on V, is abelian, $N_A(W)$ normalizes W^a. Now $[W, a, N_A(W)] \leq [V, A, A] = 0$ implies $N_A(W) = C_A(W)$. Since $C_G(V) = 1$, $N_A(W) \trianglelefteq A$ and $\langle W^A \rangle = V$, we get

(∗) $N_A(W) = 1$.

In particular,

$$A \cap L = 1.$$

As FL-module,

$$V = W_1 \oplus \ldots \oplus W_n,$$

where $W_1, \ldots, W_n$ are irreducible FL-submodules conjugate to W. Without loss, $W_i \cong W$ precisely when $1 \leq i \leq m$. Put $X := W_1 + \ldots + W_m$. Since

$$\mathrm{Hom}_{FL}(W, X) \cong \bigoplus_{i=1}^{m} \mathrm{Hom}_{FL}(W, W_i) \cong \bigoplus_{i=1}^{m} \mathrm{End}_{FL}(W)$$

is an m-dimensional vector space over $\mathrm{End}_{FL}(W)$, the number of irreducible FL-submodules of X is $\sum_{i=0}^{m-1} |\mathrm{End}_{FL}(W)|^i$, which is congruent 1 modulo p. Consequently, the p-group $N_A(X)$ fixes some irreducible FL-submodule of X. But then, since W is an arbitrary irreducible FL-submodule of V, $(*)$ implies

$$N_A(X) = 1.$$

From this it follows that

$$V \cong \bigoplus_{a \in A} W^a \quad \text{as } FL\text{-module.}$$

Hence $\dim V = |A| \dim W = |G : L| \dim W = \dim(W \otimes_{FL} FG)$. Since V is generated as FG-module by the FL-submodule W, we get

$$V \cong W \otimes_{FL} FG.$$

From $A \cap L = 1$ it follows that V is a free FA-module. Since $[V, A, A] = 0$, this implies $|A| = 2$.

Suppose $|A| \geq |V/C_V(A)|$. Since V is free as FA-module and $|A| = 2$, we get $|F| = 2$ and $\dim V = 2$. Then W is a trivial FL-module and $[V, L] = 0$, whence $L = 1$ and $G = A$. But this is a contradiction to the irreducibility of V as FG-module. ∎

(A.2.5) Let G be a finite group, K a finite field of characteristic p, and V a faithful KG-module. Assume that there exist subgroups A and L of G such that

(i) $L \trianglelefteq G$ and $G = AL$,

(ii) V is completely reducible as KL-module,

(iii) $[V, L] = V$,

(iv) $[V, A, A] = 0$,

(v) $|V/C_V(A)| \leq |A/C_A(V)|$,

(vi) L is quasisimple,

(vii) $O_p(G) = 1$.

Then V is completely reducible as KG-module. Moreover, each irreducible KG-submodule of V is also irreducible as KL-module.

Proof. Suppose this is false. Let (G, V) be a counterexample with $|G| + |V|$ minimal. Then

$$C_G(V) = 1.$$

Suppose that there exists an irreducible KG-submodule X of V which is also irreducible as KL-module. Let Y be any KL-submodule of V with $X \neq Y$. Then

$$V = X \oplus Y$$

and Y is not A-invariant, for otherwise there would be a counterexample for the same group G and a KG-module W strictly smaller than V. Pick $a \in A$ such that $Y \neq Y^a$. Then Y^a is a diagonal between X and Y. Hence there exist bijective K-linear maps $\alpha : Y \to X$ and $\beta : Y \to Y$ such that

$$y^a = y\alpha + y\beta \quad \text{for each } y \in Y$$

and $\beta^{-1}\alpha$ is a KL-isomorphism from Y onto X. Then

$$C_V(a) \subseteq \{x + y \mid x \in X,\ y = -[x,a]\alpha^{-1}\} \quad \text{and}$$

$$[V,a] = \{([x,a] + y\alpha) + (y\beta - y) \mid x \in X,\ y \in Y\}.$$

Since α is bijective, we get

$$|C_V(a)| \leq |X| = |Y| \leq |[V,a]|.$$

Now $[V,a] \leq C_V(A) \leq C_V(a)$ implies

$$[V,a] = C_V(a) = C_V(A).$$

Then

$$[Y, N_A(Y)] \leq C_Y(A) = C_Y(a) = 0.$$

Since A is abelian, $N_A(Y) = N_A(Y^a)$. Hence we also have $[Y^a, N_A(Y)] = 0$. Now $V = Y + Y^a$ implies

$$N_A(Y) \leq C_G(V) = 1.$$

Thus A acts fixed point freely on the set $\mathcal{Y}$ of irreducible KL-submodules of V other than X. Put $E := \mathrm{End}_{KL}(X)$. Then $\mathcal{Y} = |E|$ and therefore

$$|A| \leq |E|.$$

On the other hand,

$$|A| \geq |V/C_V(A)| = \frac{|V|}{|C_V(a)|} = \frac{|V|}{|X|} = |X| = |E|^{\dim_E X}.$$

Hence X is a 1-dimensional EL-module, contrary to (iii).

Thus no irreducible KG-submodule of V is irreducible as KL-module. Let X be any irreducible KG-submodule of V. Then X is like V in (A.2.4)(b). In particular,

$$|A : C_A(X)| = 2.$$

Put $H := C_A(X)L$. Then X is not irreducible as KH-module. Hence H is a proper subgroup of G. Note that the assumptions of the lemma are satisfied for $H, C_A(X), L, V$ in place of G, A, L, V, respectively. Therefore the minimality of $|G| + |V|$ implies that

$$V = W_1 \oplus \ldots \oplus W_m$$

for some irreducible KH-submodules $W_1, \ldots, W_m$ which are also irreducible as KL-modules. Note that V is indecomposable as KG-module by the minimality of $|G|+|V|$. Since H has index 2 in G, it follows that $m = 2$, i.e.,

$$V = X,$$

a contradiction to (A.2.4)(b) and (v). ■

(A.2.6) Let G be a finite group, $L := F^*(G)$, p a prime, and V a faithful FF-module for $GF(p)G$. Assume that the following hold:

(i) L is quasisimple,

(ii) $G = AL$ for some $A \in \mathcal{P}^*(G, V)$,

(iii) $Z(G) = Z(L)$,

(iv) $O_p(G) = 1$,

(v) V contains more than one nontrivial G-composition factor.

Then one of the following holds:

(a) $G \cong \mathrm{SL}_n(q)$ $(q = p^k)$.

(b) $G \cong \mathrm{Sp}_{2n}(q)$ $(n \geq 3,\ q = p^k)$, and each nontrivial G-composition factor of V is a natural $\mathrm{Sp}_{2n}(q)$-module.

(c) $G \cong \Omega_n^{\varepsilon}(q)$ $(q = p^k,\ \varepsilon \in \{0, +, -\})$, and each nontrivial G-composition factor of V is a natural $\Omega_n^{\varepsilon}(q)$-module.

(d) $G \cong \mathrm{O}_{2n}^{\varepsilon}(q)$ $(p = 2,\ q = 2^k,\ \varepsilon \in \{+,-\})$, and each nontrivial G-composition factor of V is a natural $\mathrm{O}_{2n}^{\varepsilon}(q)$-module.

(e) $G \cong \mathrm{SU}_n(q)$ $(q = p^k,\ n \geq 4)$, and each nontrivial G-composition factor of V is a natural $\mathrm{SU}_n(q)$-module.

Proof. Let W be a a nontrivial G-composition factor of V. By (A.2.3) and (iii), $[L, C_G(W)] \leq Z(G)$. Hence $C_A(W)Z(G) \trianglelefteq G$, and then (iv) implies that

(*) $C_G(W) \leq Z(L)$.

Note that $(G/C_G(W), W)$ appears in (A.2.2). Since (*) implies that A acts faithfully on each nontrivial G-composition factor, (v) excludes the cases (d), (g) and (i)-(n) in (A.2.2). Thus one of the following holds:

(1) $G/C_G(W) \cong \mathrm{SL}_n(q)$ $(q = p^k)$,

(2) $G/C_G(W) \cong \mathrm{Sp}_{2n}(q)$ $(q = p^k)$, and W is a natural $\mathrm{Sp}_{2n}(q)$-module,

(3) $G/C_G(W) \cong \Omega_n^{\varepsilon}(q)$ $(q = p^k,\ \varepsilon \in \{0,+,-\})$, and W is a natural $\Omega_n^{\varepsilon}(q)$-module,

(4) $G/C_G(W) \cong \mathrm{O}_{2n}^{\varepsilon}(q)$ $(p = 2,\ q = 2^k,\ \varepsilon \in \{+,-\})$, and W is a natural $\mathrm{O}_{2n}^{\varepsilon}(q)$-module,

(5) $G/C_G(W) \cong \mathrm{SU}_n(q)$ $(q = p^k,\ n \geq 4)$, and W is a natural $\mathrm{SU}_n(q)$-module.

Note that the isomorphism type of $L/C_L(W)$ does not depend on the choice of W. Since $Z(L)$ is cyclic, it follows that $C_G(W)$ $(= C_L(W))$ does not depend on the choice of W. Hence $C_G(W) = C_G(V) = 1$, since $Z(L)$ is a p'-group. The restriction $n \geq 3$ in (b) follows from (B.4.1.6). ∎

(A.2.7) Let G be a finite group, R a normal subgroup of G, L a component of G, and V a faithful irreducible FF-module for $GF(2)G$. Let S be a subgroup of LR containing R such that S/R is a Sylow 2-subgroup of LR/R. Assume that the following hold:

(i) $LR/R = F^*(G/R) \cong \mathrm{Sp}_4(2^k)'$ for some $k \in \mathbb{N}$.

(ii) There exists $A \in \mathcal{P}^*(N_G(S), V)$ such that $[A, L] \not\leq Z(L)$.

(iii) $O_2(G) = 1$.

Let P_1, P_2 be subgroups of LR containing R such that P_1/R and P_2/R are the minimal parabolic subgroups of LR/R containing S/R. Then $N_G(S) \leq N_G(P_i)$ for each $i \in \{1, 2\}$.

Proof. If K is any component of G with $K \not\leq R$ then, by (i), $KR/R = LR/R$ and, hence, $[K, L] \neq 1$. It follows that L is the unique component of G which is not contained in R. In particular,

$$L \trianglelefteq G.$$

Put $H := AL$. Let W be an irreducible H-submodule of V. Note that

$$(*) \quad \bigcap_{g \in G} C_G(W)^g = 1,$$

since $C_G(V) = 1$ and $V = \langle W^G \rangle \leq C_V(\bigcap_{g \in G} C_G(W)^g)$. In particular, $[W, L] \neq 0$ and, hence, $C_L(W) \leq Z(L)$. Since G acts faithfully and irreducibly on V, $O_2(L) \leq O_2(G) = 1$. Then by tables 6.1.2 and 6.1.3 in [8], $Z(L)$ is a cyclic group of order 1 or 3. Thus $C_L(W)$, being a characteristic subgroup of $Z(L)$, is normal in G. Now $(*)$ implies

$$(**) \quad C_L(W) = 1.$$

But then $[C_G(W), L] \leq C_L(W) = 1$. Since (i) implies $C_G(L)R/R \leq C_{G/R}(LR/R) = 1$, we get

$$(***) \quad C_G(W) \leq C_G(L) \leq R.$$

Since $[W, L] \neq 0$, (A.2.3) implies that $(H/C_H(W), W)$ appears in (A.2.2). Note that the natural Σ_6-module is isomorphic to a natural $\mathrm{Sp}_4(2)$-module. Hence we are left with the following cases:

(1) $H/C_H(W) \cong \mathrm{Sp}_4(2^k)$, and W is a natural $\mathrm{Sp}_4(2^k)$-module.

(2) $H/C_H(W) \cong \mathrm{Sp}_4(2)'$, and W is a natural $\mathrm{Sp}_4(2)$-module.

(3) $H/C_H(W) \cong \hat{A}_6$, $|W| = 2^6$ and $|W/C_W(A)| = |A| = 4$.

Together with $(***)$ it follows that $H \cap R/C_H(W)$ is a $2'$-group. Since W is an arbitrary irreducible H-submodule of V and (A.2.5) implies that V is the direct sum of its irreducible H-submodules, it follows that $H \cap R$ is a $2'$-group. Hence $H \cap R \le L$ and therefore by $(**)$

$$C_H(W) = 1.$$

Note that by (A.2.6) none of $\mathrm{Sp}_4(2^k)$, $\mathrm{Sp}_4(2)'$ or $\hat{A}_6$ has an FF-module with more than one nontrivial composition factor. Thus in each of the cases (1)–(3) $C_H(W) = 1$ implies that

$$V = W.$$

But if $g \in N_G(S) \setminus N_G(P_1)$, then W^g is not isomorphic to W as L-module. Hence $N_G(S) = N_G(P_1)$. ∎

(A.2.8) Let G be a finite group, R a normal subgroup of G, L a component of G, and p a prime such that

(i) $LR/R = F^*(G/R) \cong \mathrm{PSL}_n(p^k)$, for some $n, k \in \mathbb{N}$ with $n \ge 3$,

(ii) $O_p(G) = 1$.

Let M, Q, P_1 and P_2 be subgroups of LR containing R such that

(iii) M/R is a parabolic subgroup of LR/R corresponding to the $n-2$ interior nodes of the Dynkin diagram,

(iv) $Q/R = O_p(M/R)$,

(v) P_1/R and P_2/R are the maximal parabolic subgroups of LR/R containing M/R.

Let V be a faithful irreducible FF-module for $GF(p)G$. Assume that there exists $A \in \mathcal{P}^*(N_G(M), V)$ such that

(vi) $[A, L] \not\leq Z(L)$ and

(vii) $[A, M] \leq QA$.

Then one of the following holds:

(a) $N_G(M) \leq N_G(P_i)$ for each $i \in \{1, 2\}$.

(b) $L \cong \mathrm{SL}_4(p^k)$, and V is the exterior square of a natural module for L.

Proof. Put $H := AL$. Let W be an irreducible H-submodule of V. The same argument as in the proof of (A.2.7) shows that

(*) $\quad C_L(W) = 1 \quad$ and

(**) $\quad C_G(W) \leq R$.

Since $[W, L] \neq 0$, (A.2.3) implies that $(H/C_H(W), W)$ appears in (A.2.2). Hence one of the following holds:

(1) $H/C_H(W) \cong \mathrm{SL}_n(p^k)$, and W is a natural $\mathrm{SL}_n(p^k)$-module,

(2) $H/C_H(W) \cong \mathrm{SL}_n(p^k)$, and W is the exterior square of a natural $\mathrm{SL}_n(p^k)$-module,

(3) $p^k = 2$, $n = 4$, $H/C_H(W) \cong \Sigma_8$, and W is a natural Σ_8-module.

Together with (**) it follows that $H \cap R/C_H(W)$ is a p'-group. Since W is an arbitrary irreducible H-submodule of V and (A.2.5) implies that V is the direct sum of its irreducible H-submodules, it follows that $H \cap R$ is a p'-group. Hence $H \cap R \leq L$ and therefore by $(*)$

$C_H(W) = 1.$

Assume that (3) holds. Since V is the direct sum of L-submodules conjugate to W and also the direct sum of irreducible H-submodules, we get that V is a direct sum of natural Σ_8-modules for H. Note that no subgroup of Σ_8 acts as an offender on a direct sum of two natural modules. Hence $V = W$ and (b) holds.

Assume that (1) or (2) holds. Similarly to the above we get that as H-module V is a direct sum of irreducible H-submodules that are isomorphic to W and irreducible H-submodules that are dual to W. If no irreducible H-submodule of V is dual to W, then (a) holds. Hence assume that both types occur. Note that $A \leq C_L(M \cap L, Q_\cap L)$ implies that A does not act as an offender on a direct sum of a natural module and its dual. Thus W is isomorphic to its dual and $V = W$. Hence (b) holds. ∎

(A.2.9) Let G be a finite group, R a normal subgroup of G, L a component of G, and p a prime such that

(i) $LR/R = F^*(G/R) \cong \Omega_8^+(p^k)$, for some $k \in \mathbb{N}$,

(ii) $O_p(G) = 1$.

Let M, Q, P_1, P_2, and P_3 be subgroups of LR containing R such that

(iii) M/R is a rank 1 parabolic subgroup of LR/R corresponding to the interior node of the Dynkin diagram,

(iv) $Q/R = O_p(M/R)$,

(v) P_1/R, P_2/R and P_3/R are the maximal parabolic subgroups of LR/R containing M/R.

Let V be a faithful irreducible FF-module for $GF(p)G$. Assume that there exists $A \in \mathcal{P}^*(N_G(M), V)$ such that

(vi) $[A, L] \not\leq Z(L)$ and

(vii) $[A, M] \leq QA$.

Then $N_G(M) \leq N_G(P_i)$ for some $i \in \{1, 2, 3\}$.

Proof. Put $H := AL$. Let W be an irreducible H-submodule of V. As in the proof of (A.2.7) we get

$$C_L(W) = 1 \quad \text{and}$$

$$C_G(W) \leq R.$$

Since $[W, L] \neq 0$, (A.2.3) implies that $(H/C_H(W), W)$ appears in (A.2.2). Hence $H/C_W(H)$ is isomorphic to $\Omega_8^+(q)$ or $\mathrm{O}_8^+(q)$, and W is a natural $\Omega_8^+(q)$-module. As in the proof of (A.2.8) it follows that

$$C_H(W) = 1.$$

Suppose that $N_G(M)$ does not normalize any one of P_1, P_2, or P_3. Then V contains at least three pairwise non-isomorphic natural $\Omega_8^+(q)$-modules for H. Without loss, we may assume that W is chosen such that $|W : C_W(A)|$ is minimal. Note that $|A : A \cap L| \leq p$. Then

$$|A \cap L| \geq p^{-1}|A| \geq p^{-1}|V/C_V(A)| \geq p^{-1}|W/C_W(A)|^3,$$

and hence

$$|A \cap L| \geq |W/C_W(A)|^2 = |[W, A]|^2,$$

contrary to (B.5.1.8). ■

A.3 Modules for central products

Let p be a prime.

(A.3.1) Let G be a finite group which is the central product of two subgroups A and B. Let K be a field, F a subfield of K, X an irreducible KA-module, and V an irreducible FG-module. Assume that the following hold:

(i) There exists an FA-monomorphism $\phi : X \to V$.

(ii) X is irreducible as FA-module.

(iii) $\mathrm{End}_{FA}(X) \cong K$.

Then there exists a KB-module Y such that the following hold:

(a) $X \otimes_K Y$ is a KG-module, where G acts on $X \otimes_K Y$ as follows:

$$(x \otimes_K y)ab = (xa) \otimes_K (yb), \quad \text{for all } x \in X,\ y \in Y,\ a \in A, \text{ and } b \in B.$$

(b) $X \otimes_K Y$ and V are isomorphic as FG-modules.

(c) Y is irreducible as FB-module.

Proof. Let H be the direct product of A and B. Clearly $X \otimes_K KB$ is a KH-module, where the action of H is given by

$$(x \otimes_K y)ab = (xa) \otimes_K (yb), \quad \text{for all } x \in X,\ y \in KB,\ a \in A, \text{ and } b \in B.$$

Note that $x \otimes_K b \mapsto x \otimes_{KA} b$ $(x \in X,\ b \in B)$ defines a KH-isomorphism α from $X \otimes_K KB$ onto $X \otimes_{KA} KH$.

Since $X \otimes_{KA} KH$ is generated as FH-module by the FA-submodule $X \otimes_{KA} 1$, there exists an FH-epimorphism β from $X \otimes_{FA} FH$ onto $X \otimes_{KA} KH$ which maps $x \otimes_{FA} 1$ to $x \otimes_{KA} 1$ for each $x \in X$. As

$$\dim_F(X \otimes_{FA} FH) = |B| \dim_F X = |B| \dim_F K \ \dim_K X =$$

$$\dim_F K \dim_K(X \otimes_{KA} KH) = \dim_F(X \otimes_{KA} KH),$$

β is an FH-isomorphism.

Hence $\alpha\beta^{-1}$ is an FH-isomorphism from $X \otimes_K KB$ to $X \otimes_{FA} FH$. Note that there is an epimorphism γ from H onto G. Regarding V via γ as an FH-module, it follows that the map

$$\eta : X \otimes_K KB \to V,\ x \otimes_K b \mapsto (x\phi)\, b \qquad (x \in X, b \in B)$$

is an FH-epimorphism. Let U be the kernel of η. Then

$$W := \{b \in KB \mid X \otimes_K b \leq U\}$$

is a KB-module. As FA-module, $X \otimes_K KB$ is the direct sum of the submodules $X \otimes_K b$ with $b \in B$. Now it follows from (ii) and (iii) that any irreducible FA-submodule S of $X \otimes_K KB$ is of the form

$$S = X \otimes_K \sum_{b \in B} k_b b$$

for some $k_b \in K$. Since U is the sum of its irreducible FA-submodules, we get

$$U = X \otimes_K W.$$

Put $Y := KB/W$. Then

$$V \cong (X \otimes_K KB)/(X \otimes_K W) \cong X \otimes_K (KB/W) = X \otimes_K Y \ \text{ (as } FH\text{-modules).}$$

By definition of the action of H on $X \otimes_K KB$, (a) and (b) are satisfied with H instead of G. But since H acts on V via γ, it follows that (a) and (b) hold.

The irreducibility of V implies (c). ∎

(A.3.2) Let G be a finite group which is the central product of two subgroups A and B. Let F be a field, V an irreducible FG-module, and $K := \mathrm{End}_{FG}(V)$. Let X be a KA-submodule and Y a KB-submodule of V.

(a) $X \otimes_K Y$ is a KG-module, where G acts on $X \otimes_K Y$ as follows:

$$(x \otimes_K y)ab = (xa) \otimes_K (yb), \quad \text{for all } x \in X,\ y \in Y,\ a \in A, \text{ and } b \in B.$$

(b) If X and Y are irreducible, then V is isomorphic to $X \otimes_K Y$ as KG-module.

Proof. (a) It suffices to show that $(xc) \otimes_K y = x \otimes_K (yc)$, for all $x \in X$, $y \in Y$, and $c \in A \cap B$. Since $A \cap B \leq Z(G)$, for each $c \in A \cap B$ the map

$$\phi_c : V \to V,\ v \mapsto vc$$

is an FG-endomorphism. Hence

$$(xc) \otimes_K y = (x\phi_c) \otimes_K y = x \otimes_K (y\phi_c) = x \otimes_K (yc),$$

for all $x \in X$, $y \in Y$, and $c \in A \cap B$.

(b) Let X_0 be an irreducible FA-submodule of X. From (A.3.1) (with $G, A, B, \mathrm{End}_{FA}X_0, F, X_0$ in place of G, A, B, K, F, X, respectively) it follows that V has the structure of an $\mathrm{End}_{FA}X_0$-module. Hence $E := N_K(X_0)$ is a subfield of K with $E \cong \mathrm{End}_{FA}X_0$. Since X_0 is absolutely irreducible as EA-module, X ($\cong K \otimes_E X_0$) is absolutely irreducible as KA-module. From (A.3.1) (with G, A, B, K, K, X in place of G, A, B, K, F, X, respectively) it follows that $V \cong X \otimes_K W$ for some irreducible KB-module W, and G acts on $X \otimes_K W$ as described in (A.3.1)(a). Hence, by (a), V is isomorphic to $X \otimes_K U$ for any KB-submodule U of V with $U \cong W$. But any irreducible KB-submodule of $X \otimes_K W$ is isomorphic to W. In particular, $Y \cong W$ and (b) holds. ∎

(A.3.3) Let G be a finite group, F a finite field of characteristic p, and V an FF-module for FG. Let $\{L_1, \ldots, L_n\}$ be a G-invariant set of components of G satisfying

$$L \leq \langle \mathcal{P}^*(G, V/C_V(L)) \rangle$$

where $L := \langle L_1, \ldots, L_n \rangle$. Then

$$[V/C_V(L), L] = \bigoplus_{i=1}^{n} [V/C_V(L), L_i].$$

Proof. Let $(G, V, L_1, \ldots, L_n)$ be a counterexample with $|G| + |V|$ minimal. Then $C_V(L) = 0$, $C_G(V) = 1$, and $G = \langle \mathcal{P}^*(G,V) \rangle$. If $i \neq j$ implies $[V, L_i, L_j] = 0$, for all $i, j \in \{1, \ldots, n\}$, then $[V, L_i] \cap \sum_{j \neq i} [V, L_j] \leq C_V(L) = 0$, for all $i \in \{1, \ldots, n\}$, and $(G, V, L_1, \ldots, L_n)$ is not a counterexample. Hence, without loss we may assume that

$$[V, L_1, L_2] \neq 0.$$

From $G = \langle \mathcal{P}^*(G,V) \rangle$ and (A.2.1)(d) it follows that $L_k \trianglelefteq G$ for each $k \in \{1, \ldots, n\}$. Suppose that V is not irreducible. Let $W \neq 0$ be a proper submodule of V. Choose $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. If $L_k \leq \langle \mathcal{P}^*(G,W) \rangle$ for each $k \in \{i, j\}$, then $[W, L_i, L_j] = 0$ by the minimality of $|G| + |V|$. If $L_k \not\leq \langle \mathcal{P}^*(G,W) \rangle$ for some $k \in \{i, j\}$, then, by (A.2.1)(b), $[W, A] = 0$, for each $A \in \mathcal{P}^*(G,V)$ with $[L_k, A] \not\leq Z(L_k)$, and therefore $[W, L_k] = 0$. Hence in any case we get

$$[W, L_i, L_j] = 0.$$

A similar argument (with V/W instead of W, using (A.2.1)(c) instead of (A.2.1)(b)) shows that

$$[V, L_i, L_j] \leq W.$$

Note that

$$[V, L_1, L_2, L_2] = [V, L_1, L_2] = [V, L_2, L_1] = [V, L_2, L_1, L_1]$$

by the Three-Subgroup Lemma. Hence

$$[V, L_1, L_2, L] = [V, L_1, L_2, L_1] + [V, L_2, L_1, L_2] + [V, L_1, L_2, L_3 \cdot \ldots \cdot L_n] =$$

$$[V, L_1, L_2, L_2, L_1] + [V, L_2, L_1, L_1, L_2] + [V, L_1, L_2, L_2, L_3 \cdot \ldots \cdot L_n] \leq$$

$$[W, L_2, L_1] + [W, L_1, L_2] + [W, L_2, L_3 \cdot \ldots \cdot L_n] = 0,$$

contrary to $[V, L_1, L_2] \neq 0 = C_V(L)$. Hence V is irreducible.

For the rest of the proof let

$$\{i, j\} = \{1, 2\}.$$

Put

$$\widehat{L}_i := L_j L_3 \cdots L_n,$$

and choose $A_i \in \mathcal{P}^*(G, V)$ such that

$$[L_i, A_i] \not\leq Z(L_i).$$

Let W be an irreducible $F(A_iL)$-submodule of V. Put $K := \mathrm{End}_{F(A_iL)}W$, and let X be an irreducible $K(A_iL_i)$-submodule of W. Note that (A.2.1)(b) and (A.2.4) imply that W is irreducible as FL-module and X is irreducible as KL_i-module. Thus, by (A.3.2),

$$W \cong X \otimes_K Y, \quad \text{for some irreducible } K\widehat{L}_i\text{-submodule } Y \text{ of } W.$$

Since $L \trianglelefteq G$ and V is an irreducible FG-module, we get that, regarded as FL-module, V is a direct sum of conjugates of W. In particular, X is not a trivial FL_i-module, since $[V, L_i] \neq 0$. Let

$$0 = W_0 < W_1 < \ldots < W_m = W$$

be a $K(A_iL_i)$-composition series of W.

We assert that $C_{A_i}(W_k/W_{k-1}) \leq C_{A_i}(W)$ for each $k \in \{1, \ldots, m\}$. As KL_i-modules both W_k/W_{k-1} and W are direct sums of submodules isomorphic to X, whence $C_{L_i}(W_k/W_{k-1})$ centralizes W. Then $C_{L_i}(W_k/W_{k-1})$ centralizes V, since as FL-module V is a direct sum of conjugates of W. Since G acts faithfully on V, we get $C_{L_i}(W_k/W_{k-1}) = 1$. In particular, L_i centralizes $C_{A_i}(W_k/W_{k-1})$. Since A_i, acting faithfully and quadratically on V, is elementary abelian, we get $C_{A_i}(W_k/W_{k-1}) \leq Z(A_iL_i)$. Hence $C_W(C_{A_i}(W_k/W_{k-1}))$ is a $K(A_iL_i)$-submodule of W. From the quadratic action of A_i on W it follows that A_i centralizes the $K(A_iL_i)$-module

$W/C_W(C_{A_i}(W_k/W_{k-1}))$. But $L\langle A_i^{L_i}\rangle$ contains L_i, since $[A_i, L_i] \not\leq Z(L_i)$. Therefore, $W/C_W(C_{A_i}(W_k/W_{k-1}))$ is centralized by L_i. Since all KL_i-composition factors of W are isomorphic to X, which is a nontrivial KL_i-module, this implies $W = C_W(C_{A_i}(W_k/W_{k-1}))$.

Now it follows from (A.2.1)(c) and (A.2.4) that $W_k/W_{k-1} \cong X$ as KL_i-modules for each $k \in \{1, \ldots, m\}$. In particular,

$$x := \dim_K X = \dim_K W_k/W_{k-1}, \qquad \text{for each } k \in \{1, \ldots, m\}.$$

Since, for each $k \in \{1, \ldots, m\}$, $A_i/C_{A_i}(W)$ acts faithfully and quadratically on W_k/W_{k-1}, we get

$$(*) \quad |A_i/C_{A_i}(W)| \leq |K|^{(x-c_k)c_k},$$

where

$$c_k := \dim_K C_{W_k/W_{k-1}}(A_i), \quad \text{for each } k \in \{1, \ldots, m\}.$$

From (A.2.1)(b) it follows that

$$A_i/C_{A_i}(W) \cong A_iC_{A_iL_i}(W)/C_{A_iL_i}(W) \in \mathcal{P}^*(A_iL_i/C_{A_iL_i}(W), W).$$

Since

$$|C_W(A_i)| \leq \prod_{k=1}^{m} |C_{W_k/W_{k-1}}(A_i)| = |K|^{c_1+\ldots+c_m},$$

this implies

$$(**) \quad |A_i/C_{A_i}(W)| \geq |W/C_W(A_i)| \geq |K|^{mx-c_1-c_2-\ldots-c_m} \geq |K|^{m(x-c)},$$

where

$$c := \max\{c_1, \ldots, c_m\}.$$

Thus $m \leq c$ by $(*)$ and $(**)$. Since $W_k/W_{k-1} \cong X$ for each $k \in \{1, \ldots, m\}$ and $W \cong X \otimes_K Y$, we get

$$\dim_K Y < \dim_K X.$$

Since V, regarded as KL-module, is a direct sum of conjugates of W, this implies that the dimension of any $K\widehat{L}_i$-composition factor of V is smaller than the dimension of any KL_i-composition factor of V. In particular, the dimension of any KL_j-composition factor of V is smaller than the dimension of any KL_i-composition factor of V. Since all assumptions are symmetric in L_1 and L_2, this is a contradiction. ∎

A.4 Automorphisms of finite simple groups of Lie Type

In this section we adopt the notation of chapter 2 in [8]. In particular, let r be a prime and K a finite simple group of Lie type in characteristic r. Let q be the power of r such that $GF(q)$ is the field of definition for K. Let a be an automorphism of K. As in Theorem 2.5.1 of [8], write $a = idfg$ where i is an inner automorphism, d is a diagonal automorphism, f is a field automorphism and g is a graph automorphism. Moreover, if K is a twisted group then g is the identity.

(A.4.1) Let P be a parabolic subgroup of K containing the Borel subgroup B. Assume that $[P, a] \leq O_r(P)$. Then one of the following holds:

(a) $a = id$,

(b) $q = 2$, $a = ig$, and g fixes each node of the Dynkin diagram that belongs to P.

Proof. Note that d, f and g all normalize B. Since $[B, a] \leq [P, a] \leq O_r(P) \leq B$, it follows that also i normalizes B. Hence i is induced by an element of B. In particular,

$$(*) \quad [h_{\widehat{\alpha}}(t), i] \leq U, \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma} \text{ and } 0 \neq t \in GF(q).$$

Note that we also have

$$(**) \quad [h_{\widehat{\alpha}}(t), d] = 1, \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma} \text{ and } 0 \neq t \in GF(q), \text{ and}$$

$$f \in N_K(\langle h_{\widehat{\alpha}}(t) \mid 0 \neq t \in GF(q)\rangle), \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma}.$$

Since

$$[\langle h_{\widehat{\alpha}}(t) \mid t \in GF(q)\rangle U, a] \leq [P, a] \leq O_r(P) \leq U \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma},$$

it follows that

$$(***) \quad [\langle h_{\widehat{\alpha}}(t) \mid t \in GF(q)\rangle U, g] \leq U, \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma}.$$

Assume that g is not the identity. Then $(***)$ implies that $q = 2$. Hence both d and f are the identity. If $\alpha \in \Sigma$ is a fundamental root corresponding to a node in the Dynkin diagram of P, then

$$[x_\alpha(t), i] \leq [U, B] = [U, U] \quad \text{and}$$

$$[x_\alpha(t), a] \leq O_r(P), \quad \text{for each } t \in GF(q).$$

Hence

$$[x_\alpha(t), g] \leq O_r(P)[U, U], \quad \text{for each } t \in GF(q).$$

This implies that g fixes the node corresponding to α in the Dynkin diagram. Hence (b) holds.

Now assume that g is the identity. As in Theorem 2.5.1 of [8], let ϕ be the automorphism of $GF(q)$ that induces f on K. Then

$$[h_{\widehat{\alpha}}(t), f] = h_{\widehat{\alpha}}(t)^{-1} h_{\widehat{\alpha}}(t^\phi) \in H, \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma} \text{ and } 0 \neq t \in GF(q).$$

On the other hand, by $(*)$ and $(**)$,

$$[h_{\widehat{\alpha}}(t), f] \leq U, \quad \text{for all } \widehat{\alpha} \in \widehat{\Sigma} \text{ and } t \in GF(q).$$

Since $U \cap H = 1$, it follows that ϕ is the identity. Hence (a) holds. ∎

(A.4.2) Assume that K is of type C_n with $n \geq 2$. Let P be the parabolic subgroup of cotype 1 in K containing the Borel subgroup B. Assume that $[P, a] \leq P$. Then $a \notin \mathrm{Inn}(K)$ if and only if a induces an outer automorphism on $O^{r'}(P)/O_r(P)$.

Proof. Let A be the normalizer of P in $\mathrm{Aut}(K)$. Let A_1 be the intersection of A and $\mathrm{Inn}(K)$. Let A_2 be the subgroup of A that induces inner automorphisms on $O^{r'}(P)/O_r(P)$. Since $P = N_K(P)$, each element of A_1 induces an inner automorphism on $P/O_r(P)$. Together with

$$P/O_r(P) = O^{r'}(P)/O_r(P)C_{P/O_r(P)}(O^{r'}(P)/O_r(P))$$

we get that

$$(*) \quad A_2 \leq A_1.$$

If $\phi \in \mathrm{Aut}(GF(q))$, then the field automorphism of K defined by

$$x_\alpha(t) \mapsto x_\alpha(t^\phi) \quad \text{for each } \alpha \in \Sigma \text{ and } t \in GF(q)$$

induces on $O^{r'}(P)/O_r(P)$ a field automorphism of the same type. If q is odd, then it follows from section 7.1 in [3] that K has an outer diagonal automorphism which acts as follows:

$$x_\alpha(t) \mapsto \begin{cases} x_\alpha(t) & \text{if the long simple root is not involved in } \alpha \\ x_\alpha(-t) & \text{if the long simple root is involved in } \alpha \end{cases}$$

for each $\alpha \in \Sigma$ and $t \in GF(q)$. Note that this automorphism induces on $O^{r'}(P)/O_r(P)$, which is of type C_{n-1}, a diagonal automorphism of the same type. Thus

$$(**) \quad |A : A_2| \geq \begin{cases} 2|\mathrm{Aut}(GF(q))| & \text{if } q \text{ is odd} \\ |\mathrm{Aut}(GF(q))| & \text{if } q \text{ is even} \end{cases}$$

Since A fixes P, no element of A involves a graph automorphism of K. Hence by Table 2.1.C in [11],

$$|A : A_1| = \begin{cases} 2|\mathrm{Aut}(GF(q))| & \text{if } q \text{ is odd} \\ |\mathrm{Aut}(GF(q))| & \text{if } q \text{ is even} \end{cases}.$$

Together with $(*)$ and $(**)$ it follows that $A_1 = A_2$. ∎

Appendix B

FF-modules for groups of Lie type

Let $\mathcal{L}$ be a complex simple Lie algebra of type T, where T is one of $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n$ or D_n. Let $\eta : \mathcal{L} \to \mathcal{U}$ be the embedding of $\mathcal{L}$ in its universal enveloping algebra $\mathcal{U}$. Let $\mathcal{H}$ be a Cartan subalgebra. Let $\Phi = \Phi(\mathsf{T})$ be the corresponding root system and $\Pi = (\alpha_i)_{1 \leq i \leq n}$ a system of fundamental roots, where α_i corresponds to the node i in the Dynkin diagram, and the numbering of the nodes is as in [9]. Also, the fundamental weights are labeled as in Table 1 of [9]. We denote the height of a root β by $\mathrm{ht}(\beta)$. Let $\prec$ be the total ordering on the euclidean space spanned by Φ defined by

$$\sum_{i=1}^{n} a_i \alpha_i \prec \sum_{i=1}^{n} b_i \alpha_i \quad \Leftrightarrow$$

there exists $j \in \{1, \ldots, n\}$ with $a_i = b_i$ for all $i \in \{1, \ldots, j-1\}$ and $a_j < b_j$.

Let h_α $(\alpha \in \Pi)$, e_β $(\beta \in \Phi)$ be a Chevalley basis of $\mathcal{L}$. Since we will write modules as right modules, we choose this Chevalley basis such that, for each $\alpha \in \Pi$, we have $h_\alpha = [e_{-\alpha}, e_\alpha]$ rather than $h_\alpha = [e_\alpha, e_{-\alpha}]$ as in [9]. (Pick a Chevalley basis as in [9] and replace h_α by $-h_\alpha$).

Let $\mathcal{U}_{\mathbf{Z}}$ be the $\mathbf{Z}$-form of $\mathcal{U}$ corresponding to this Chevalley basis, i.e., $\mathcal{U}_{\mathbf{Z}}$ is the $\mathbf{Z}$-subalgebra of $\mathcal{U}$ generated by the elements $\frac{1}{k!}(e_\alpha \eta)^k$ $(k \in \mathbb{N}, \alpha \in \Phi)$.

Let K be a finite field of characteristic p. Put $q := |K|$.

B.1 Construction of the groups and modules

Let λ be a dominant integral weight, $W = W(\lambda)$ the irreducible $\mathcal{L}$-module of highest weight λ, and Λ the set of weights of W with respect to $\mathcal{H}$. Let ϕ be the representation of $\mathcal{U}$ on W such that $v(a\eta\phi) = va$, for all $v \in W$ and $a \in \mathcal{L}$.

W has a $\mathbf{C}$-basis B which is also a $\mathbf{Z}$-basis for a $\mathcal{U}_{\mathbf{Z}}$-submodule $W_{\mathbf{Z}}$ of W.

Put $V := V(\mathsf{T}, K, \lambda) := K \otimes_{\mathbf{Z}} W_{\mathbf{Z}}$. For all $t \in K$ and $\alpha \in \Phi$ put

$$x_\alpha(t) := x_\alpha(t; \mathsf{T}, K, \lambda) := \sum_{k=1}^{\infty} \left(t^k \mathrm{id}_K \otimes_{\mathbf{Z}} \left(\frac{1}{k!}(e_\alpha \eta)^k\right)\phi \right) \quad (\in \mathrm{GL}(V)).$$

Define

$$X_\alpha := X_\alpha(\mathsf{T}, K, \lambda) := \langle x_\alpha(t) \mid t \in K \rangle, \quad \text{for each } \alpha \in \Phi, \text{ and}$$

$$G := G(\mathsf{T}, K, \lambda) := \langle X_\alpha \mid \alpha \in \Phi \rangle.$$

Let $\Phi^+ = \Phi(\mathsf{T})^+$ be the set of positive roots. For each $J \subseteq \{1, \ldots, n\}$, let $\Phi_J = \Phi_J(\mathsf{T})$ be the root system spanned by $\{\alpha_j \mid j \in J\}$, and put

$$Q_J := Q_J(\mathsf{T}) := \langle X_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_J \rangle,$$

$$L_J := L_J(\mathsf{T}) := \langle X_\alpha \mid \alpha \in \Phi_J \rangle \quad \text{and}$$

$$P_J := P_J(\mathsf{T}) := N_G(Q_J).$$

B.2 A_n

Assume that $\mathcal{L}$ is of type A_n. Then $\Phi^+ = \{\beta_{i,j} \mid i, j \in \{1, \ldots, n\},\ i \leq j\}$, where

$$\beta_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ with } i \leq j.$$

Note that with respect to $\prec$ the pairs $(\alpha_i, \beta_{i+1,j})$ are extraspecial in the sense of [3] for all $i, j \in \{1, \ldots, n\}$ with $i < j$. Hence without loss we may assume that

$$e_{\beta_{i,j}} = -[e_{\alpha_i}, e_{\beta_{i+1,j}}], \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ with } i < j.$$

B.2.1

In the construction of section B.1, let

$$\lambda = \lambda_1(\mathsf{A}_n) = \frac{1}{n+1}\sum_{i=1}^{n}(n-i+1)\alpha_i.$$

Then $\Lambda = \{\mu_1, \dots, \mu_{n+1}\}$, where

$$\mu_i := \frac{1}{n+1}\left(-\sum_{k=1}^{i-1} k\alpha_k + \sum_{k=i}^{n}(n-k+1)\alpha_k\right), \qquad \text{for all } i \in \{1, \dots, n+1\}.$$

Let v_{μ_1} be a nonzero weight vector of weight μ_1. Then the basis B can be chosen as follows: Put

$$v_{\mu_{i+1}} := v_{\mu_i} e_{-\alpha_i}, \qquad \text{for each } i \in \{1, \dots, n\}.$$

Then $\{v_{\mu_1}, \dots, v_{\mu_n}\}$ is a basis with the desired properties. We obtain a group G with generators $x_\beta(t)$ ($\beta \in \Phi$, $t \in K$) acting on a module V.

(B.2.1.1) Let $i, j \in \{1, \dots, n\}$ with $i \le j$.

(a) $\beta_{i,j} = \mu_i - \mu_{j+1}$.

(b) Let $\mu \in \Lambda$. Then

$$v_\mu e_{\beta_{i,j}} = \begin{cases} v_{\mu_i} & \text{if } \mu = \mu_{j+1} \\ 0 & \text{else.} \end{cases}$$

and

$$v_\mu e_{-\beta_{i,j}} = \begin{cases} v_{\mu_{j+1}} & \text{if } \mu = \mu_i \\ 0 & \text{else.} \end{cases} .$$

Proof. (a) This is clear.

(b) From (a) it follows that

$$v_\mu e_{\beta_{i,j}} = 0 \quad \text{unless } \mu = \mu_{j+1}, \text{ and}$$

$$v_\mu e_{-\beta_{i,j}} = 0 \quad \text{unless } \mu = \mu_i.$$

First assume that $i = j$. Then $\beta_{i,j} = \alpha_i$, and from the definition of $v_{\mu_{i+1}}$ we get

$$v_{\mu_i} e_{-\beta_{i,j}} = v_{\mu_{i+1}} \quad \text{and}$$

$$v_{\mu_{i+1}} e_{\beta_{i,j}} = v_{\mu_i} e_{-\alpha_i} e_{\alpha_i} = v_{\mu_i} h_{\alpha_i} + v_{\mu_i} e_{\alpha_i} e_{-\alpha_i} = \langle \mu_i, \alpha_i \rangle v_{\mu_i} + 0 = v_{\mu_i}.$$

Hence (b) holds in this case.

If $i < j$, then by induction on $\mathrm{ht}(\beta_{i,j})$ we get

$$v_{\mu_{j+1}} e_{\beta_{i,j}} = -v_{\mu_{j+1}} [e_{\alpha_i}, e_{\beta_{i+1,j}}] = -v_{\mu_{j+1}} e_{\alpha_i} e_{\beta_{i+1,j}} + v_{\mu_{j+1}} e_{\beta_{i+1,j}} e_{\alpha_i} = v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i}$$

and

$$v_{\mu_i} e_{-\beta_{i,j}} = v_{\mu_i} [e_{-\alpha_i}, e_{-\beta_{i+1,j}}] = v_{\mu_i} e_{-\alpha_i} e_{-\beta_{i+1,j}} - v_{\mu_i} e_{-\beta_{i+1,j}} e_{-\alpha_i} = v_{\mu_{i+1}} e_{-\beta_{i+1,j}} =$$

$$v_{\mu_{j+1}}.$$

■

Put

$$v_i := 1 \otimes_{\mathbf{Z}} v_{\mu_i}, \quad \text{for each } i \in \{1, \ldots, n+1\}.$$

(B.2.1.2) Let $t \in K$, $i, j \in \{1, \ldots, n\}$ with $i \leq j$, and $k \in \{1, \ldots, n+1\}$. Then

$$v_k x_{\beta_{i,j}}(t) = \begin{cases} v_k + t v_i & \text{if } k = j+1 \\ v_k & \text{else.} \end{cases}$$

and

$$v_k x_{-\beta_{i,j}}(t) = \begin{cases} v_k + t v_{j+1} & \text{if } k = i \\ v_k & \text{else.} \end{cases} .$$

Proof. This follows from (B.2.1.1). ■

B.3 B_n

Assume that $\mathcal{L}$ is of type B_n. Then

$$\Phi^+ = \{\beta_{i,j} \mid i,j \in \{1,\ldots,n\},\ i \le j\} \cup \{\beta'_{i,j} \mid i,j \in \{1,\ldots,n-1\},\ i \le j\} \text{ where}$$

$$\beta_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad \text{for all } i,j \in \{1,\ldots,n\} \text{ with } i \le j, \text{ and}$$

$$\beta'_{i,j} := \sum_{k=i}^{j} \alpha_k + 2\sum_{k=j+1}^{n} \alpha_k, \quad \text{for all } i,j \in \{1,\ldots,n-1\} \text{ with } i \le j.$$

Note that with respect to $\prec$ the following pairs are extraspecial in the sense of [3]:

$$(\alpha_i, \beta_{i+1,j}), \quad \text{for all } i,j \in \{1,\ldots,n\} \text{ with } i < j,$$

$$(\alpha_i, \beta'_{i+1,j}), \quad \text{for all } i,j \in \{1,\ldots,n-1\} \text{ with } i < j,$$

$$(\alpha_{i+1}, \beta'_{i,i+1}), \quad \text{for all } i \in \{1,\ldots,n-2\}, \text{ and}$$

$$(\alpha_n, \beta_{n-1,n}).$$

Hence without loss we may assume that

(i) $e_{\beta_{i,j}} = -[e_{\alpha_i}, e_{\beta_{i+1,j}}]$, for all $i,j \in \{1,\ldots,n\}$ with $i < j$,

(ii) $e_{\beta'_{i,j}} = -[e_{\alpha_i}, e_{\beta'_{i+1,j}}]$, for all $i,j \in \{1,\ldots,n-1\}$ with $i < j$,

(iii) $e_{\beta'_{i,i}} = -[e_{\alpha_{i+1}}, e_{\beta'_{i,i+1}}]$, for all $i \in \{1,\ldots,n-2\}$,

(iv) $2e_{\beta'_{n-1,n-1}} = -[e_{\alpha_n}, e_{\beta_{n-1,n}}]$.

B.3.1

This subsection is about the natural $O_{2n+1}(q)$-module. In the construction of section B.1, let

$$\lambda = \lambda_1(\mathsf{B}_n) = \sum_{k=1}^{n} \alpha_k.$$

Then $\Lambda = \{\mu_0, \mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n\}$, where

$$\mu_0 := 0, \quad \text{and}$$

$$\mu_i := \sum_{k=i}^{n} \alpha_k, \quad \text{for all } i \in \{1, \ldots, n\}.$$

Let v_{μ_1} be a nonzero weight vector of weight μ_1. Then the basis B can be chosen as follows: Put

$$v_{\mu_{i+1}} := v_{\mu_i} e_{-\alpha_i}, \quad \text{for each } i \in \{1, \ldots, n-1\},$$

$$v_{-\mu_n} := v_{\mu_n} e_{-\alpha_n},$$

$$v_{\mu_0} := v_{\mu_n} e_{-\alpha_n},$$

$$v_{-\mu_n} := -\frac{1}{2} v_{\mu_0} e_{-\alpha_n}, \quad \text{and}$$

$$v_{-\mu_{n-i}} := -v_{-\mu_{n-i+1}} e_{-\alpha_{n-i}}, \quad \text{for each } i \in \{1, \ldots, n-1\}.$$

Then $\{v_{\mu_0}, v_{\mu_1}, \ldots, v_{\mu_n}, v_{-\mu_1}, \ldots, v_{-\mu_n}\}$ is a basis with the desired properties.

(B.3.1.1) (a) $\beta_{i,j} = \mu_i - \mu_{j+1}$, for all $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$.

(b) $\beta_{i,n} = \mu_i - \mu_0$, for all $i \in \{1, \ldots, n\}$.

(c) $\beta'_{i,j} = \mu_i + \mu_{j+1}$, for all $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$.

(d) Let $\mu \in \Lambda$, $\varepsilon \in \{1, -1\}$, and $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$. Then

$$v_\mu e_{\varepsilon\beta_{i,j}} = \begin{cases} \varepsilon v_{\varepsilon\mu_i} & \text{if } \mu = \varepsilon\mu_{j+1} \\ -\varepsilon v_{-\varepsilon\mu_{j+1}} & \text{if } \mu = -\varepsilon\mu_i \\ 0 & \text{else.} \end{cases}$$

(e) Let $\mu \in \Lambda$, $\varepsilon \in \{1,-1\}$, and $i \in \{1,\dots,n\}$. Then

$$v_\mu e_{\varepsilon\beta_{i,n}} = \begin{cases} 2\varepsilon v_{\varepsilon\mu_i} & \text{if } \mu = \varepsilon\mu_0 \\ -\varepsilon v_{\mu_0} & \text{if } \mu = -\varepsilon\mu_i \\ 0 & \text{else.} \end{cases}$$

(f) Let $\mu \in \Lambda$, $\varepsilon \in \{1,-1\}$ and $i,j \in \{1,\dots,n-1\}$ with $i \le j$. Then

$$v_\mu e_{\varepsilon\beta'_{i,j}} = \begin{cases} \varepsilon v_{\varepsilon\mu_i} & \text{if } \mu = -\varepsilon\mu_{j+1} \\ -\varepsilon v_{\varepsilon\mu_{j+1}} & \text{if } \mu = -\varepsilon\mu_i \\ 0 & \text{else.} \end{cases}$$

Proof. (a),(b),(c) This is clear.

(d) From (a) it follows that

$$v_\mu e_{\varepsilon\beta_{i,j}} = 0 \quad \text{unless } \mu \in \{-\varepsilon\mu_i, \varepsilon\mu_{j+1}\}.$$

First assume that $i = j$. Then $\beta_{i,j} = \alpha_i$. From the definition of $v_{\mu_{i+1}}$ and $v_{-\mu_i}$ we get

$$v_{\mu_i} e_{-\alpha_i} = v_{\mu_{i+1}},$$

$$v_{-\mu_{i+1}} e_{-\alpha_i} = -v_{-\mu_i},$$

$$v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i} e_{-\alpha_i} e_{\alpha_i} = v_{\mu_i} h_{\alpha_i} + v_{\mu_i} e_{\alpha_i} e_{-\alpha_i} = \langle \mu_i, \alpha_i \rangle v_{\mu_i} + 0 = v_{\mu_i}, \quad \text{and}$$

$$v_{-\mu_i} e_{\alpha_i} = -v_{-\mu_{i+1}} e_{-\alpha_i} e_{\alpha_i} = -v_{-\mu_{i+1}} h_{\alpha_i} - v_{-\mu_{i+1}} e_{\alpha_i} e_{-\alpha_i} =$$

$$-\langle -\mu_{i+1}, \alpha_i \rangle v_{-\mu_{i+1}} - 0 = -v_{-\mu_{i+1}}.$$

Hence (d) holds if $i = j$.

Now assume that $i < j$. Then by (i) and induction on $\mathrm{ht}(\beta_{i,j})$ we get

$$v_{-\mu_i} e_{\beta_{i,j}} = -v_{-\mu_i} e_{\alpha_i} e_{\beta_{i+1,j}} + v_{-\mu_i} e_{\beta_{i+1,j}} e_{\alpha_i} = v_{-\mu_{i+1}} e_{\beta_{i+1,j}} + 0 = -v_{-\mu_{j+1}},$$

$$v_{\mu_{j+1}} e_{\beta_{i,j}} = -v_{\mu_{j+1}} e_{\alpha_i} e_{\beta_{i+1,j}} + v_{\mu_{j+1}} e_{\beta_{i+1,j}} e_{\alpha_i} = 0 + v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i},$$

$$v_{\mu_i} e_{-\beta_{i,j}} = v_{\mu_i} e_{-\alpha_i} e_{-\beta_{i+1,j}} - v_{\mu_i} e_{-\beta_{i+1,j}} e_{-\alpha_i} = v_{\mu_{i+1}} e_{-\beta_{i+1,j}} - 0 = v_{\mu_{j+1}},$$

and

$$v_{-\mu_{j+1}} e_{-\beta_{i,j}} = v_{-\mu_{j+1}} e_{-\alpha_i} e_{-\beta_{i+1,j}} - v_{-\mu_{j+1}} e_{-\beta_{i+1,j}} e_{-\alpha_i} = 0 + v_{-\mu_{i+1}} e_{-\alpha_i} = -v_{-\mu_i}.$$

(e) From (b) it follows that

$$v_\mu e_{\varepsilon\beta_{i,n}} = 0 \quad \text{unless } \mu \in \{-\varepsilon\mu_i, \mu_0\}.$$

First assume that $i = n$. Then $\beta_{i,n} = \alpha_n$. From the definition of v_{μ_0} and $v_{-\mu_n}$ we get

$$v_{\mu_n} e_{-\alpha_n} = v_{\mu_0},$$

$$v_{\mu_0} e_{-\alpha_n} = -2v_{-\mu_n},$$

$$v_{\mu_0} e_{\alpha_n} = v_{\mu_n} e_{-\alpha_n} e_{\alpha_n} = v_{\mu_n} h_{\alpha_n} + v_{\mu_n} e_{\alpha_n} e_{-\alpha_n} = \langle \mu_n, \alpha_n \rangle v_{\mu_n} + 0 = 2v_{\mu_n}, \quad \text{and}$$

$$v_{-\mu_n} e_{\alpha_i} = -\frac{1}{2} v_{\mu_0} e_{-\alpha_n} e_{\alpha_n} = -\frac{1}{2} v_{\mu_0} h_{\alpha_n} - \frac{1}{2} v_{\mu_0} e_{\alpha_n} e_{-\alpha_n} =$$

$$-\frac{1}{2} \langle \mu_0, \alpha_i \rangle v_{-\mu_0} - v_{\mu_n} e_{-\alpha_n} = 0 - v_{\mu_0} = -v_{\mu_0}.$$

Hence (e) holds if $i = n$.

Now assume that $i < n$. Then by (i), (d), and induction on $\mathrm{ht}(\beta_{i,n})$ we get

$$v_{-\mu_i} e_{\beta_{i,n}} = -v_{-\mu_i} e_{\alpha_i} e_{\beta_{i+1,n}} + v_{-\mu_i} e_{\beta_{i+1,n}} e_{\alpha_i} = v_{-\mu_{i+1}} e_{\beta_{i+1,n}} + 0 = -v_{-\mu_0},$$

$$v_{\mu_0} e_{\beta_{i,n}} = -v_{\mu_0} e_{\alpha_i} e_{\beta_{i+1,n}} + v_{\mu_0} e_{\beta_{i+1,n}} e_{\alpha_i} = 0 + 2v_{\mu_{i+1}} e_{\alpha_i} = 2v_{\mu_i},$$

$$v_{\mu_i} e_{-\beta_{i,n}} = v_{\mu_i} e_{-\alpha_i} e_{-\beta_{i+1,n}} - v_{\mu_i} e_{-\beta_{i+1,n}} e_{-\alpha_i} = v_{\mu_{i+1}} e_{-\beta_{i+1,n}} - 0 = v_{\mu_0},$$

and

$$v_{\mu_0} e_{-\beta_{i,n}} = v_{\mu_0} e_{-\alpha_i} e_{-\beta_{i+1,n}} - v_{\mu_0} e_{-\beta_{i+1,n}} e_{-\alpha_i} = 0 + 2v_{-\mu_{i+1}} e_{-\alpha_i} = -2v_{-\mu_i}.$$

(f) From (c) it follows that

$$v_\mu e_{\varepsilon\beta'_{i,j}} = 0 \quad \text{unless } \mu \in \{-\varepsilon\mu_i, -\varepsilon\mu_{j+1}\}.$$

From (iv), (d), and (e) it follows that

$$v_{-\mu_n} e_{\beta'_{n-1,n-1}} = -\frac{1}{2} v_{-\mu_n} e_{\alpha_n} e_{\beta_{n-1,n}} + \frac{1}{2} v_{-\mu_n} e_{\beta_{n-1,n}} e_{\alpha_n} = \frac{1}{2} v_{\mu_0} e_{\beta_{n-1,n}} + 0 = v_{\mu_{n-1}},$$

$$v_{-\mu_{n-1}}e_{\beta'_{n-1,n-1}} = -\frac{1}{2}v_{-\mu_{n-1}}e_{\alpha_n}e_{\beta_{n-1,n}} + \frac{1}{2}v_{-\mu_{n-1}}e_{\beta_{n-1,n}}e_{\alpha_n} = 0 - \frac{1}{2}v_{\mu_0}e_{\alpha_n} =$$

$$-v_{\mu_n},$$

$$v_{\mu_n}e_{-\beta'_{n-1,n-1}} = \frac{1}{2}v_{\mu_n}e_{-\alpha_n}e_{-\beta_{n-1,n}} - \frac{1}{2}v_{\mu_n}e_{-\beta_{n-1,n}}e_{-\alpha_n} = \frac{1}{2}v_{\mu_0}e_{-\beta_{n-1,n}} - 0 =$$

$$-v_{-\mu_{n-1}},$$

and

$$v_{\mu_{n-1}}e_{-\beta'_{n-1,n-1}} = \frac{1}{2}v_{\mu_{n-1}}e_{-\alpha_n}e_{-\beta_{n-1,n}} - \frac{1}{2}v_{\mu_{n-1}}e_{-\beta_{n-1,n}}e_{-\alpha_n} = 0 - \frac{1}{2}v_{\mu_0}e_{-\alpha_n} =$$

$$v_{-\mu_n}.$$

Hence (f) holds if $i = j = n-1$.

If $i < j$, then by (ii), (d), and induction on $\mathrm{ht}(\beta'_{i,j})$ we get

$$v_{-\mu_i}e_{\beta'_{i,j}} = -v_{-\mu_i}e_{\alpha_i}e_{\beta'_{i+1,j}} + v_{-\mu_i}e_{\beta'_{i+1,j}}e_{\alpha_i} = v_{-\mu_{i+1}}e_{\beta'_{i+1,j}} + 0 = -v_{\mu_{j+1}},$$

$$v_{-\mu_{j+1}}e_{\beta'_{i,j}} = -v_{-\mu_{j+1}}e_{\alpha_i}e_{\beta'_{i+1,j}} + v_{-\mu_{j+1}}e_{\beta'_{i+1,j}}e_{\alpha_i} = 0 + v_{\mu_{i+1}}e_{\alpha_i} = v_{\mu_i},$$

$$v_{\mu_i}e_{-\beta'_{i,j}} = v_{\mu_i}e_{-\alpha_i}e_{-\beta'_{i+1,j}} - v_{\mu_i}e_{-\beta'_{i+1,j}}e_{-\alpha_i} = v_{\mu_{i+1}}e_{-\beta'_{i+1,j}} - 0 = v_{-\mu_{j+1}},$$

and

$$v_{\mu_{j+1}}e_{-\beta'_{i,j}} = v_{\mu_{j+1}}e_{-\alpha_i}e_{-\beta'_{i+1,j}} - v_{\mu_{j+1}}e_{-\beta'_{i+1,j}}e_{-\alpha_i} = 0 + v_{-\mu_{i+1}}e_{-\alpha_i} = -v_{-\mu_i}.$$

If $i < n-1$, then by (iii), (d), and induction on $\mathrm{ht}(\beta'_{i,j})$ we get

$$v_{-\mu_i}e_{\beta'_{i,i}} = -v_{-\mu_i}e_{\alpha_{i+1}}e_{\beta'_{i,i+1}} + v_{-\mu_i}e_{\beta'_{i,i+1}}e_{\alpha_{i+1}} = 0 - v_{\mu_{i+2}}e_{\alpha_{i+1}} = -v_{\mu_{i+1}},$$

$$v_{-\mu_{i+1}}e_{\beta'_{i,i}} = -v_{-\mu_{i+1}}e_{\alpha_{i+1}}e_{\beta'_{i,i+1}} + v_{-\mu_{i+1}}e_{\beta'_{i,i+1}}e_{\alpha_{i+1}} = v_{-\mu_{i+2}}e_{\beta'_{i,i+1}} + 0 = v_{\mu_i},$$

$$v_{\mu_i}e_{-\beta'_{i,i}} = v_{\mu_i}e_{-\alpha_{i+1}}e_{-\beta'_{i,i+1}} - v_{\mu_i}e_{-\beta'_{i,i+1}}e_{-\alpha_{i+1}} = 0 - v_{-\mu_{i+2}}e_{-\alpha_{i+1}} = v_{-\mu_{i+1}},$$

and

$$v_{\mu_{i+1}} e_{\beta'_{i,i}} = v_{\mu_{i+1}} e_{-\alpha_{i+1}} e_{-\beta'_{i,i+1}} - v_{\mu_{i+1}} e_{-\beta'_{i,i+1}} e_{-\alpha_{i+1}} = v_{\mu_{i+2}} e_{-\beta'_{i,i+1}} - 0 = -v_{-\mu_i}.$$

■

Put

$$v_i := 1 \otimes_{\mathbf{Z}} v_{\mu_i} \quad \text{and}$$

$$v_{-i} := 1 \otimes_{\mathbf{Z}} v_{-\mu_i}$$

for each $i \in \{0, \ldots, n\}$.

(B.3.1.2) (a) Let $k \in \{-n, \ldots, n\}$, $\varepsilon \in \{1, -1\}$, $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$, and $t \in K$. Then

$$v_k x_{\varepsilon\beta_{i,j}}(t) = \begin{cases} v_k + \varepsilon t v_{\varepsilon i} & \text{if } k = \varepsilon(j+1) \\ v_k - \varepsilon t v_{-\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

(b) Let $k \in \{-n, \ldots, n\}$, $\varepsilon \in \{1, -1\}$, $i \in \{1, \ldots, n\}$, and $t \in K$. Then

$$v_k x_{\varepsilon\beta_{i,n}}(t) = \begin{cases} v_k + 2t\varepsilon v_{\varepsilon i} & \text{if } k = 0 \\ v_k - t\varepsilon v_0 - t^2 v_{\varepsilon i} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

(c) Let $k \in \{-n, \ldots, n\}$, $\varepsilon \in \{1, -1\}$, $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$, and $t \in K$. Then

$$v_k x_{\varepsilon\beta'_{i,j}}(t) = \begin{cases} v_k + t\varepsilon v_{\varepsilon i} & \text{if } k = -\varepsilon(j+1) \\ v_k - t\varepsilon v_{\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

Proof. This follows from (B.3.1.1). ■

(B.3.1.3) (a) Let $r \in \{1, \ldots, n\}$ and put $J := \{1, \ldots, n\} \setminus \{r\}$.

(a1) $[V, Q_J] = \bigoplus_{i=0}^{n} K v_i \oplus \bigoplus_{i=r+1}^{n} K v_{-i}$,

(a2) $[V, Q_J, Q_J] = C_V(Q_J) = \bigoplus_{i=1}^{r} K v_i$.

(a3) $|[V, a] \cap C_V(Q_J)| \neq 1$, for each $a \in Q_J \setminus \{1\}$.

(b) If q is odd, then the map

$$\xi : V \times V \to K,\ (\sum_{i=-n}^{n} a_i v_i, \sum_{i=-n}^{n} b_i v_i) \mapsto \sum_{i=0}^{n} a_i b_{-i} + a_{-i} b_i$$

is a nondegenarete G-invariant symmetric bilinear form.

Proof. This follows from (B.3.1.2). ∎

(B.3.1.4) Let $J := \{2, \ldots, n\}$. Assume that A is a subgroup of Q_J with $1 \neq |A| \geq |V : C_V(A)|$.

(a) q is even.

(b) $Kv_0 \not\subseteq [V, A]$ if and only if $q = 2$ and $A = X_{\beta_{1,n}}$.

Proof. Note that (B.3.1.3) implies that

$$(*) \quad \overline{[v_{-1}, ab]} = \overline{[v_{-1}, a]}\,\overline{[v_{-1}, b]} \quad \text{for all } a, b \in Q_J \text{ where}$$

$\overline{V} := V/Kv_1$.

From (B.3.1.2) we get that

$$tv_0 - t^2 v_1 = [v_{-1}, x_{\beta_{1,n}}(-t)] \in [v_{-1}, Q_J],$$

$$tv_{-i} = [v_{-1}, x_{\beta_{1,i-1}}(-t)] \in [v_{-1}, Q_J], \quad \text{and}$$

$$tv_i = [v_{-1}, x_{\beta'_{1,i-1}}(-t)] \in [v_{-1}, Q_J] \quad \text{for all } t \in K \text{ and } i \in \{2, \ldots, n\}.$$

Then $(*)$ implies that the map

$$\kappa : Q_J \to \overline{[V, Q_J]},\ a \mapsto \overline{[v_{-1}, a]}$$

is surjective. Since $|Q_J| = |\overline{[V, Q_J]}|$ by (B.3.1.3)(a1), it follows that κ is bijective. In particular,

$$|[V, A]| \geq |\kappa(A)| = |A|.$$

Assume that $|[V,A]| \leq |A|$. Then $|[V,A]| = |\kappa(A)|$ and hence $[V,A] \cap Kv_1 = 0$. Now (B.3.1.2) implies that q is even, $A \leq X_{\beta 1,n}$, and $|A| = 2$. Since $|V : C_V(A)|$ is a power of q, it follows that $q = 2$.

Assume that $|[V,A]| > |A|$. Then (B.3.1.3)(b) and the assumption $|V : C_V(A)| \leq |A|$ imply that q is even. Since $|[V/Kv_0, A]| = |(V/Kv_0) : C_{V/Kv_0}(A)| \leq |A| < |[V,A]|$, it follows that $Kv_0 \leq [V,A]$.

From (B.3.1.2) it follows that $X_{\beta_{1,n}}$ is a subgroup of Q_J with $|X_{\beta_{1,n}}| \geq |V : C_V(X_{\beta_{1,n}})|$ and $Kv_0 \not\leq [V, X_{\beta_{1,n}}]$, provided $q = 2$. ∎

(B.3.1.5) Assume that q is odd. Let $r \in \{2, \ldots, n\}$. Put $J := \{1 \ldots, n\} \setminus \{r\}$.

(a) $Z(Q_J)$ is a vector space over K with basis $\{x_{\beta'_{i,j}}(1) \mid 1 \leq i \leq j \leq r-1\}$ where

$$t\,x_{\beta'_{i,j}}(1) = x_{\beta'_{i,j}}(t), \quad \text{for all } t \in K \text{ and } 1 \leq i \leq j \leq r-1.$$

(b) $Z(Q_J)$ and $C_V(Q_J) \wedge C_V(Q_J)$ are isomorphic as KP_J-modules.

(c) $V/[V,Q_J]$ and $Z(Q_J)$ are not isomorphic as $GF(p)P_J$-modules.

Proof. (a) follows from (B.3.1.2). Again by (B.3.1.2), the K-linear map which sends $x_{\beta'_{i,j}}(1)$ to $v_i \wedge v_j$, for all $1 \leq i \leq j \leq r$, is a KP_J-isomorphism from $Z(Q_J)$ to $C_V(Q_J) \wedge C_V(Q_J)$. Hence (b) holds.

Suppose that $Z(Q_J)$ and $V/[V,Q_J]$ are isomorphic as $GF(p)P_J$-modules. Then $r = 3$, since $\dim_K Z(Q_J) = \frac{r(r-1)}{2}$ and $\dim_K V/[V,Q_J] = r$. For each $t \in K \setminus \{0\}$, put

$$h(t) := x_{\beta'_{2,2}}(t)x_{-\beta'_{2,2}}(-t^{-1})x_{\beta'_{2,2}}(t)x_{\beta'_{2,2}}(1)x_{-\beta'_{2,2}}(-1)x_{\beta'_{2,2}}(1).$$

Then $h(t)$ has on $Z(Q_J)$ the eigenvalues t^{-1} and t^{-2}, and on $V/[V,Q_J]$ the eigenvalues 1 and t, for each $t \in K \setminus \{0\}$, contrary to the assumption that q is odd. ∎

B.3.2

This subsection is about the spin module for $O_7(q)$. Assume that $n = 3$. In the construction of section B.1, let

$$\lambda = \lambda_3(\mathsf{B}_3) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3).$$

Then $\Lambda = \{\mu_1, \dots, \mu_4, -\mu_1, \dots, -\mu_4\}$, where

$$\mu_1 := \lambda,$$

$$\mu_2 := \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3),$$

$$\mu_3 := \frac{1}{2}(\alpha_1 + \alpha_3), \quad \text{and}$$

$$\mu_4 := \frac{1}{2}(\alpha_1 - \alpha_3).$$

Let v_{μ_1} be a nonzero weight vector of weight μ_1. Then the basis B can be chosen as follows: Put

$$v_{\mu_2} := v_{\mu_1} e_{-\alpha_3},$$

$$v_{\mu_3} := v_{\mu_2} e_{-\alpha_2},$$

$$v_{\mu_4} := v_{\mu_3} e_{-\alpha_3},$$

$$v_{-\mu_4} := v_{\mu_3} e_{-\alpha_1},$$

$$v_{-\mu_3} := v_{-\mu_4} e_{-\alpha_3},$$

$$v_{-\mu_2} := v_{-\mu_3} e_{-\alpha_2}, \quad \text{and}$$

$$v_{-\mu_1} := v_{-\mu_2} e_{-\alpha_3}.$$

Then $\{v_{\mu_1}, \dots, v_{\mu_4}, v_{-\mu_1}, \dots, v_{-\mu_4}\}$ is a basis with the desired properties. We obtain a group G with generators $x_\beta(t)$ $(\beta \in \Phi,\ t \in K)$ acting on a module V.

Using (i)–(iv) and the definition of $v_{\mu_1}, \ldots, v_{\mu_4}, v_{-\mu_1}, \ldots, v_{-\mu_4}$ we get

$$v_{-\mu_4} e_{\alpha_1} = v_{\mu_3} e_{-\alpha_1} e_{\alpha_1} = v_{\mu_3} h_{\alpha_1} = \langle \mu_3, \alpha_1 \rangle v_{\mu_3} = v_{\mu_3},$$

$$v_{-\mu_3} e_{\alpha_1} = v_{-\mu_4} e_{-\alpha_3} e_{\alpha_1} = v_{-\mu_4} e_{\alpha_1} e_{-\alpha_3} = v_{\mu_3} e_{-\alpha_3} = v_{\mu_4},$$

$$v_{\mu_4} e_{-\alpha_1} = v_{\mu_3} e_{-\alpha_3} e_{-\alpha_1} = v_{\mu_3} e_{-\alpha_1} e_{-\alpha_3} = v_{-\mu_4} e_{-\alpha_3} = v_{-\mu_3},$$

$$v_{\mu_3} e_{\alpha_2} = v_{\mu_2} e_{-\alpha_2} e_{\alpha_2} = v_{\mu_2} h_{\alpha_2} = \langle \mu_2, \alpha_2 \rangle v_{\mu_2} = v_{\mu_2},$$

$$v_{-\mu_2} e_{\alpha_2} = v_{-\mu_3} e_{-\alpha_2} e_{\alpha_2} = v_{-\mu_3} h_{\alpha_2} = \langle -\mu_3, \alpha_2 \rangle v_{-\mu_3} = v_{-\mu_3},$$

$$v_{\mu_2} e_{\alpha_3} = v_{\mu_1} e_{-\alpha_3} e_{\alpha_3} = v_{\mu_1} h_{\alpha_3} = \langle \mu_1, \alpha_3 \rangle v_{\mu_1} = v_{\mu_1},$$

$$v_{-\mu_1} e_{\alpha_3} = v_{-\mu_2} e_{-\alpha_3} e_{\alpha_3} = v_{-\mu_2} h_{\alpha_3} = \langle -\mu_2, \alpha_3 \rangle v_{-\mu_2} = v_{-\mu_2},$$

$$v_{\mu_4} e_{\alpha_3} = v_{\mu_3} e_{-\alpha_3} e_{\alpha_3} = v_{\mu_3} h_{\alpha_3} = \langle \mu_3, \alpha_3 \rangle v_{\mu_3} = v_{\mu_3},$$

$$v_{-\mu_3} e_{\alpha_3} = v_{-\mu_4} e_{-\alpha_3} e_{\alpha_3} = v_{-\mu_4} h_{\alpha_3} = \langle -\mu_4, \alpha_3 \rangle v_{-\mu_4} = v_{-\mu_4},$$

$$v_{-\mu_4} e_{\beta_{1,2}} = v_{-\mu_4} [e_{\alpha_2}, e_{\alpha_1}] = v_{-\mu_4} e_{\alpha_2} e_{\alpha_1} - v_{-\mu_4} e_{\alpha_1} e_{\alpha_2} = 0 - v_{\mu_3} e_{\alpha_2} = -v_{\mu_2},$$

$$v_{-\mu_2} e_{\beta_{1,2}} = v_{-\mu_2} [e_{\alpha_2}, e_{\alpha_1}] = v_{-\mu_2} e_{\alpha_2} e_{\alpha_1} = v_{-\mu_3} e_{\alpha_1} = v_{\mu_4},$$

$$v_{\mu_3} e_{\beta_{2,3}} = v_{\mu_3} [e_{\alpha_3}, e_{\alpha_2}] = -v_{\mu_3} e_{\alpha_2} e_{\alpha_3} = -v_{\mu_2} e_{\alpha_3} = -v_{\mu_1},$$

$$v_{-\mu_1} e_{\beta_{2,3}} = v_{-\mu_1} [e_{\alpha_3}, e_{\alpha_2}] = v_{-\mu_1} e_{\alpha_3} e_{\alpha_2} = v_{-\mu_2} e_{\alpha_2} = v_{-\mu_3},$$

$$v_{\mu_4} e_{\beta_{2,3}} = v_{\mu_4} [e_{\alpha_3}, e_{\alpha_2}] = v_{\mu_4} e_{\alpha_3} e_{\alpha_2} = v_{\mu_3} e_{\alpha_2} = v_{\mu_2},$$

$$v_{-\mu_2} e_{\beta_{2,3}} = v_{-\mu_2} [e_{\alpha_3}, e_{\alpha_2}] = -v_{-\mu_2} e_{\alpha_2} e_{\alpha_3} = -v_{-\mu_3} e_{\alpha_3} = -v_{-\mu_4},$$

$$v_{-\mu_4} e_{\beta_{1,3}} = v_{-\mu_4} [e_{\beta_{2,3}}, e_{\alpha_1}] = -v_{-\mu_4} e_{\alpha_1} e_{\beta_{2,3}} = -v_{\mu_3} e_{\beta_{2,3}} = v_{\mu_1},$$

$$v_{-\mu_1} e_{\beta_{1,3}} = v_{-\mu_1} [e_{\beta_{2,3}}, e_{\alpha_1}] = v_{-\mu_1} e_{\beta_{2,3}} e_{\alpha_1} = v_{-\mu_3} e_{\alpha_1} = v_{\mu_4},$$

$$v_{-\mu_3} e_{\beta_{1,3}} = v_{-\mu_3}[e_{\beta_{2,3}}, e_{\alpha_1}] = -v_{-\mu_3} e_{\alpha_1} e_{\beta_{2,3}} = -v_{\mu_4} e_{\beta_{2,3}} = -v_{\mu_2},$$

$$v_{-\mu_2} e_{\beta_{1,3}} = v_{-\mu_2}[e_{\beta_{2,3}}, e_{\alpha_1}] = v_{-\mu_2} e_{\beta_{2,3}} e_{\alpha_1} = -v_{-\mu_4} e_{\alpha_1} = -v_{\mu_3},$$

$$v_{\mu_4} e_{\beta'_{2,2}} = \frac{1}{2} v_{\mu_4}[e_{\beta_{2,3}}, e_{\alpha_3}] = \frac{1}{2} v_{\mu_4} e_{\beta_{2,3}} e_{\alpha_3} - \frac{1}{2} v_{\mu_4} e_{\alpha_3} e_{\beta_{2,3}} = \frac{1}{2} v_{\mu_2} e_{\alpha_3} - \frac{1}{2} v_{\mu_3} e_{\beta_{2,3}} =$$

$$v_{\mu_1},$$

$$v_{-\mu_1} e_{\beta'_{2,2}} = \frac{1}{2} v_{-\mu_1}[e_{\beta_{2,3}}, e_{\alpha_3}] = \frac{1}{2} v_{-\mu_1} e_{\beta_{2,3}} e_{\alpha_3} - \frac{1}{2} v_{-\mu_1} e_{\alpha_3} e_{\beta_{2,3}} =$$

$$\frac{1}{2} v_{-\mu_3} e_{\alpha_3} - \frac{1}{2} v_{-\mu_2} e_{\beta_{2,3}} = v_{-\mu_4},$$

$$v_{-\mu_1} e_{\beta'_{1,2}} = -v_{-\mu_1}[e_{\alpha_1}, e_{\beta'_{2,2}}] = v_{-\mu_1} e_{\beta'_{2,2}} e_{\alpha_1} = v_{-\mu_4} e_{\alpha_1} = v_{\mu_3},$$

$$v_{-\mu_3} e_{\beta'_{1,2}} = -v_{-\mu_3}[e_{\alpha_1}, e_{\beta'_{2,2}}] = -v_{-\mu_3} e_{\alpha_1} e_{\beta'_{2,2}} = -v_{\mu_4} e_{\beta'_{2,2}} = -v_{\mu_1},$$

$$v_{-\mu_1} e_{\beta'_{1,1}} = -v_{-\mu_1}[e_{\alpha_2}, e_{\beta'_{1,2}}] = v_{-\mu_1} e_{\beta'_{1,2}} e_{\alpha_2} = v_{\mu_3} e_{\alpha_2} = v_{\mu_2}, \quad \textit{and}$$

$$v_{-\mu_2} e_{\beta'_{1,1}} = -v_{-\mu_2}[e_{\alpha_2}, e_{\beta'_{1,2}}] = -v_{-\mu_2} e_{\alpha_2} e_{\beta'_{1,2}} = -v_{-\mu_3} e_{\beta'_{1,2}} = v_{\mu_1}.$$

Put

$$v_i := 1 \otimes_{\mathbf{Z}} v_{\mu_i} \quad \text{and}$$

$$v_{-i} := 1 \otimes_{\mathbf{Z}} v_{-\mu_i}$$

for each $i \in \{1, \ldots, 4\}$.

(B.3.2.1) Let $J := \{2, 3\}$. Assume that A is a subgroup of Q_J with $|V : C_V(A)| \leq |A| \neq 1$. Then $[V, A] = [V, Q_J]$.

Proof. From the calculations above it follows that with respect to the basis $v_1, \dots, v_4$, $v_{-4}, \dots, v_{-1}$ of V the action of the generators $x_{\alpha_1}(t)$, $x_{\beta_{1,1}}(t)$, $x_{\beta_{1,3}}(t)$, $x_{\beta'_{1,1}}(t)$, and $x_{\beta'_{1,2}}(t)$ $(t \in K)$ of Q_J is given by the following matrices:

$$x_{\alpha_1}(t) : \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & t & & 1 & & & \\ & & & t & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \quad x_{\beta_{1,2}}(t) : \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & -t & & & 1 & & & \\ & & & & & 1 & & \\ & & & t & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

$$x_{\beta_{1,3}}(t) : \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ t & & & & 1 & & & \\ & -t & & & & 1 & & \\ & & -t & & & & 1 & \\ & & & t & & & & 1 \end{pmatrix} \quad x_{\beta'_{1,1}}(t) : \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ t & & & & & & 1 & \\ & t & & & & & & 1 \end{pmatrix}$$

and

$$x_{\beta'_{1,2}}(t) : \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ -t & & & & & 1 & & \\ & & & & & & 1 & \\ & & t & & & & & 1 \end{pmatrix}.$$

Suppose that $[V, A] \neq [V, Q_J]$. Since L_J induces $\mathrm{Sp}_4(q)$ on $[V, Q_J]$ and therefore acts tranisively on the 3-dimensional subspaces of $[V, Q_J]$, we may assume that $[V, A] \leq Kv_1 + Kv_2 + Kv_3$. By the matrices above it follows that

$$A \leq \{x_{\beta'_{1,1}}(s)\, x_{\beta'_{1,2}}(t) \mid s, t \in K\}.$$

In particular, $|A| \leq |K|^2$ and hence

$$(*) \quad C_V(A) \neq [V, Q_J].$$

Pick $a \in A \setminus \{1\}$. Then $a = x_{\beta'_{1,1}}(s)\, x_{\beta'_{1,2}}(t)$, for some $s, t \in K$. By the matrices above we get

$$(**) \quad C_V(a) = [V, Q_J] + Kv_{-4} + K(tv_{-2} - sv_{-3}).$$

If $A \leq \{x_{\beta'_{1,1}}(su)\, x_{\beta'_{1,2}}(tu) \mid u \in K\}$, then $|A| \leq |K|$ and, by (**), $|V : C_V(A)| = |K|^2$, a contradiction. If $A \not\leq \{x_{\beta'_{1,1}}(su)\, x_{\beta'_{1,2}}(tu) \mid u \in K\}$, then $|A| \leq |K|^2$ and, by (**), $|V : C_V(A)| = |K|^3$, again a contradiction. ■

B.4 C_n

Assume that $\mathcal{L}$ is of type C_n, and

$$\lambda = \sum_{i=1}^{n-1} \alpha_i + \frac{1}{2}\alpha_n.$$

Then

$$\Phi^+ = \{\beta_{i,j} \mid i, j \in \{1, \ldots, n-1\},\ i \leq j\} \cup \{\beta'_{i,j} \mid i, j \in \{1, \ldots, n\},\ i \leq j\} \quad \text{and}$$

$$\Lambda = \{\mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n\},$$

where

$$\beta_{i,j} := \sum_{k=i}^{j} \alpha_k, \qquad \text{for all } i, j \in \{1, \ldots, n-1\} \text{ with } i \leq j,$$

$$\beta'_{i,j} := \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{n-1} \alpha_k + \alpha_n, \qquad \text{for all } i, j \in \{1, \ldots, n\} \text{ with } i \leq j, \text{ and}$$

$$\mu_i := \sum_{k=i}^{n-1} \alpha_k + \frac{1}{2}\alpha_n, \qquad \text{for all } i \in \{1, \ldots, n\}.$$

Let v_{μ_1} be a nonzero weight vector of weight μ_1. Then the basis B can be chosen as follows: Put

$$v_{\mu_{i+1}} := v_{\mu_i} e_{-\alpha_i}, \qquad \text{for each } i \in \{1, \ldots, n-1\},$$

$$v_{-\mu_n} := v_{\mu_n} e_{-\alpha_n}, \qquad \text{and}$$

$$v_{-\mu_{n-i}} := -v_{-\mu_{n-i+1}} e_{-\alpha_{n-i}}, \quad \text{for each } i \in \{1, \dots, n-1\}.$$

Then $\{v_{\mu_1}, \dots, v_{\mu_n}, v_{-\mu_1}, \dots, v_{-\mu_n}\}$ is a basis with the desired properties. Note that with respect to $\prec$ the following pairs are extraspecial in the sense of [3]:

$$(\alpha_i, \beta_{i+1,j}), \quad \text{for all } i, j \in \{1, \dots, n-1\} \text{ with } i < j,$$

$$(\alpha_i, \beta'_{i+1,j}), \quad \text{for all } i, j \in \{1, \dots, n\} \text{ with } i < j, \text{ and}$$

$$(\alpha_i, \beta'_{i,i+1}), \quad \text{for all } i \in \{1, \dots, n-1\}.$$

Hence without loss we may assume that

(i) $e_{\beta_{i,j}} = -[e_{\alpha_i}, e_{\beta_{i+1,j}}]$, for all $i, j \in \{1, \dots, n-1\}$ with $i < j$,

(ii) $e_{\beta'_{i,j}} = -[e_{\alpha_i}, e_{\beta'_{i+1,j}}]$, for all $i, j \in \{1, \dots, n\}$ with $i < j$,

(iii) $2e_{\beta'_{i,i}} = -[e_{\alpha_i}, e_{\beta'_{i,i+1}}]$, for all $i \in \{1, \dots, n-1\}$.

B.4.1

This subsection is about the natural $\mathrm{Sp}_{2n}(q)$-module.

(B.4.1.1) (a) $\beta_{i,j} = \mu_i - \mu_{j+1}$, for all $i, j \in \{1, \dots, n-1\}$ with $i \leq j$.

(b) $\beta'_{i,j} = \mu_i + \mu_j$, for all $i, j \in \{1, \dots, n\}$ with $i \leq j$.

(c) Let $\mu \in \Lambda$, $\varepsilon \in \{1, -1\}$, and $i, j \in \{1, \dots, n-1\}$ with $i \leq j$. Then

$$v_\mu e_{\varepsilon\beta_{i,j}} = \begin{cases} \varepsilon v_{\varepsilon\mu_i} & \text{if } \mu = \varepsilon\mu_{j+1} \\ -\varepsilon v_{-\varepsilon\mu_{j+1}} & \text{if } \mu = -\varepsilon\mu_i \\ 0 & \text{else.} \end{cases}$$

(d) Let $\mu \in \Lambda$, $\varepsilon \in \{1, -1\}$ and $i, j \in \{1, \dots, n\}$ with $i \leq j$. Then

$$v_\mu e_{\varepsilon\beta'_{i,j}} = \begin{cases} v_{\varepsilon\mu_i} & \text{if } \mu = -\varepsilon\mu_j \\ v_{\varepsilon\mu_j} & \text{if } \mu = -\varepsilon\mu_i \\ 0 & \text{else.} \end{cases}$$

Proof. (a),(b) This is clear.

(c) From (a) it follows that

$$v_\mu e_{\varepsilon\beta_{i,j}} = 0 \quad \text{unless } \mu \in \{-\varepsilon\mu_i, \varepsilon\mu_{j+1}\}.$$

First assume that $i = j$. Then $\beta_{i,j} = \alpha_i$. From the definition of $v_{\mu_{i+1}}$ and $v_{-\mu_i}$ we get

$$v_{\mu_i} e_{-\alpha_i} = v_{\mu_{i+1}},$$

$$v_{-\mu_{i+1}} e_{-\alpha_i} = -v_{-\mu_i},$$

$$v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i} e_{-\alpha_i} e_{\alpha_i} = v_{\mu_i} h_{\alpha_i} + v_{\mu_i} e_{\alpha_i} e_{-\alpha_i} = \langle \mu_i, \alpha_i \rangle v_{\mu_i} + 0 = v_{\mu_i}, \quad \text{and}$$

$$v_{-\mu_i} e_{\alpha_i} = -v_{-\mu_{i+1}} e_{-\alpha_i} e_{\alpha_i} = -v_{-\mu_{i+1}} h_{\alpha_i} - v_{-\mu_{i+1}} e_{\alpha_i} e_{-\alpha_i} =$$

$$-\langle -\mu_{i+1}, \alpha_i \rangle v_{-\mu_{i+1}} - 0 = -v_{-\mu_{i+1}}.$$

Hence (c) holds if $i = j$.

Now assume that $i < j$. Then by (i) and induction on $\mathrm{ht}(\beta_{i,j})$ we get

$$v_{-\mu_i} e_{\beta_{i,j}} = -v_{-\mu_i} e_{\alpha_i} e_{\beta_{i+1,j}} + v_{-\mu_i} e_{\beta_{i+1,j}} e_{\alpha_i} = v_{-\mu_{i+1}} e_{\beta_{i+1,j}} + 0 = -v_{-\mu_{j+1}},$$

$$v_{\mu_{j+1}} e_{\beta_{i,j}} = -v_{\mu_{j+1}} e_{\alpha_i} e_{\beta_{i+1,j}} + v_{\mu_{j+1}} e_{\beta_{i+1,j}} e_{\alpha_i} = 0 + v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i},$$

$$v_{\mu_i} e_{-\beta_{i,j}} = v_{\mu_i} e_{-\alpha_i} e_{-\beta_{i+1,j}} - v_{\mu_i} e_{-\beta_{i+1,j}} e_{-\alpha_i} = v_{\mu_{i+1}} e_{-\beta_{i+1,j}} - 0 = v_{\mu_{j+1}},$$

and

$$v_{-\mu_{j+1}} e_{-\beta_{i,j}} = v_{-\mu_{j+1}} e_{-\alpha_i} e_{-\beta_{i+1,j}} - v_{-\mu_{j+1}} e_{-\beta_{i+1,j}} e_{-\alpha_i} = 0 + v_{-\mu_{i+1}} e_{-\alpha_i} = -v_{-\mu_i}.$$

(d) From (b) it follows that

$$v_\mu e_{\varepsilon\beta'_{i,j}} = 0 \quad \text{unless } \mu \in \{-\varepsilon\mu_i, -\varepsilon\mu_j\}.$$

First assume that $i = j = n$. Then $\beta'_{i,j} = \alpha_n$, and from the definition of $v_{-\mu_n}$ we get

$$v_{\mu_n} e_{-\beta'_{n,n}} = v_{-\mu_n} \quad \text{and}$$

$$v_{-\mu_n} e_{\beta'_{n,n}} = v_{\mu_n} e_{-\alpha_n} e_{\alpha_n} = v_{\mu_n} h_{\alpha_n} + v_{\mu_n} e_{\alpha_n} e_{-\alpha_n} = \langle \mu_n, \alpha_n \rangle v_{\mu_n} - 0 = v_{\mu_n}.$$

Hence (d) holds in this case.

Now assume that $i < j$. Then by (ii), (c), and induction on $\mathrm{ht}(\beta'_{i,j})$ we get

$$v_{-\mu_i} e_{\beta'_{i,j}} = -v_{-\mu_i} e_{\alpha_i} e_{\beta'_{i+1,j}} + v_{-\mu_i} e_{\beta'_{i+1,j}} e_{\alpha_i} = v_{-\mu_{i+1}} e_{\beta'_{i+1,j}} + 0 = v_{\mu_j},$$

$$v_{-\mu_j} e_{\beta'_{i,j}} = -v_{-\mu_j} e_{\alpha_i} e_{\beta'_{i+1,j}} + v_{-\mu_j} e_{\beta'_{i+1,j}} e_{\alpha_i} = 0 + v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i},$$

$$v_{\mu_i} e_{-\beta'_{i,j}} = v_{\mu_i} e_{-\alpha_i} e_{-\beta'_{i+1,j}} - v_{\mu_i} e_{-\beta'_{i+1,j}} e_{-\alpha_i} = v_{\mu_{i+1}} e_{-\beta'_{i+1,j}} - 0 = v_{-\mu_j},$$

and

$$v_{\mu_j} e_{-\beta'_{i,j}} = v_{\mu_j} e_{-\alpha_i} e_{-\beta'_{i+1,j}} - v_{\mu_j} e_{-\beta'_{i+1,j}} e_{-\alpha_i} = 0 - v_{-\mu_{i+1}} e_{-\alpha_i} = v_{-\mu_i}.$$

Now assume that $i = j < n$. Then by (iii), (c) and induction on $\mathrm{ht}(\beta'_{i,j})$ we get

$$v_{-\mu_i} e_{\beta'_{i,j}} = \frac{1}{2}(-v_{-\mu_i} e_{\alpha_i} e_{\beta'_{i,i+1}} + v_{-\mu_i} e_{\beta'_{i,i+1}} e_{\alpha_i}) = \frac{1}{2}(v_{-\mu_{i+1}} e_{\beta'_{i,i+1}} + v_{\mu_{i+1}} e_{\alpha_i}) =$$

$$\frac{1}{2}(v_{\mu_i} + v_{\mu_i}) = v_{\mu_i}$$

and

$$v_{\mu_i} e_{-\beta'_{i,j}} = \frac{1}{2}(v_{\mu_i} e_{-\alpha_i} e_{-\beta'_{i,i+1}} - v_{\mu_i} e_{-\beta'_{i,i+1}} e_{-\alpha_i}) = \frac{1}{2}(v_{\mu_{i+1}} e_{-\beta'_{i,i+1}} - v_{-\mu_{i+1}} e_{-\alpha_i}) =$$

$$\frac{1}{2}(v_{-\mu_i} + v_{\mu_i}) = +v_{-\mu_i}.$$

■

Put

$$v_i := 1 \otimes_{\mathbf{Z}} v_{\mu_i} \quad \text{and}$$

$$v_{-i} := 1 \otimes_{\mathbf{Z}} v_{-\mu_i}$$

for each $i \in \{1, \ldots, n\}$.

(B.4.1.2) Let $k \in \{1,\ldots,n\} \cup \{-1,\ldots,-n\}$, $i,j \in \{1,\ldots,n-1\}$ with $i \leq j$, $\varepsilon \in \{1,-1\}$, and $t \in K$. Then

$$v_k x_{\varepsilon\beta_{i,j}}(t) = \begin{cases} v_k + \varepsilon t v_{\varepsilon i} & \text{if } k = \varepsilon(j+1) \\ v_k - \varepsilon t v_{-\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

and

$$v_k x_{\varepsilon\beta'_{i,j}}(t) = \begin{cases} v_k + \varepsilon t v_{\varepsilon i} & \text{if } k = -\varepsilon j \\ v_k + \varepsilon t v_{\varepsilon j} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

Proof. This follows from (B.4.1.1). ■

(B.4.1.3) Let $J := \{2,\ldots,n\}$.

(a) $C_V(Q_J) \leq [V,a]$, for each $a \in Q_J \setminus \{1\}$.

(b) Assume that $1 \neq A \leq Q_J$ and $|A| \geq |[V,A]|$. Then $C_{Q_J}([V,Q_J]) \leq A$.

Proof. Note that (B.4.1.2) this implies that

$$[V,Q_J] = \bigoplus_{i=1}^{n} Kv_i \oplus \bigoplus_{i=2}^{n} Kv_{-i}, \quad \text{and}$$

$$[V,Q_J,Q_J] = C_V(Q_J) = Kv_1.$$

In particular,

$$(*) \quad \overline{[v_{-1},ab]} = \overline{[v_{-1},a]}\,\overline{[v_{-1},b]} \quad \text{for all } a,b \in Q_J \text{ where}$$

$$\overline{V} := V/Kv_1.$$

(a) Pick $a \in Q_J \setminus \{1\}$. Then there exists $t_1,\ldots,t_{n-1},t'_1,\ldots,t'_n \in K$ such that

$$a = x_{\beta_{i,1}}(t_1) \cdot \ldots \cdot x_{\beta_{i,n-1}}(t_{n-1}) \cdot x_{\beta'_{i,1}}(t'_1) \cdot \ldots \cdot x_{\beta'_{i,n-1}}(t'_n).$$

If $t_i \neq 0$ for some $i \in \{1,\ldots,n-1\}$, then

$$tv_1 = [t_i^{-1}tv_{i+1},a] \in [V,a] \quad \text{for each } t \in K.$$

If $t'_i \neq 0$ for some $i \in \{2, \ldots, n\}$, then

$$tv_1 = [(t'_i)^{-1} tv_{-i}, a] \in [V, a] \quad \text{for each } t \in K.$$

If $t_i = t'_{i+1} = 0$ for each $i \in \{1, \ldots, n-1\}$, then $t_1 \neq 0$ and

$$tv_1 = [(t'_1)^{-1} tv_{-1}, a] \in [V, a] \quad \text{for each } t \in K.$$

(b) From (B.4.1.2) we get that

$$tv_{-i} = [v_{-1}, x_{\beta_{1,i-1}}(-t)] \in [v_{-1}, Q_J] \quad \text{and}$$

$$tv_i = [v_{-1}, x_{\beta'_{1,i}}(-t)] \in [v_{-1}, Q_J] \quad \text{for all } t \in K \text{ and } i \in \{2, \ldots, n\}.$$

Then $(*)$ implies that the map

$$\kappa : \; Q_J \to \overline{[V, Q_J]}, \; a \mapsto \overline{[v_{-1}, a]}$$

is a surjective homomorphism. Note that $C_{Q_J}([V, Q_J])$ is the kernel of κ and has order q. Together with (a) and $|[V, A]| \leq |A|$ we get

$$|A| = |A \cap \mathrm{Ker}(\kappa)|\, |\kappa(A)| \leq q |\kappa(A)| \leq |[V, A]| \leq |A|.$$

Hence $|A \cap \mathrm{Ker}(\kappa)| = q$, i.e., $C_{Q_J}([V, Q_J]) \leq A$. ■

(B.4.1.4) Put $J := \{2 \ldots, n\}$.

(a) $C_{Q_J}([V, Q_J])$ is a 1-dimensional vector space over K spanned by $x_{\beta'_{1,1}}(1)$ where

$$t\, x_{\beta'_{1,1}}(1) = x_{\beta'_{i,j}}(t), \quad \text{for all } t \in K.$$

(b) Let $A := C_{P_J}([V, Q_J], C_V(Q_J))$. If $C_{Q_J}([V, Q_J])$ and $V/[V, Q_J]$ are isomorphic as $GF(p)A$-modules, then $q \in \{2, 4\}$.

Proof. (a) This follows from (B.4.1.2).

(b) Assume that $C_{Q_J}([V,Q_J])$ and $V/[V,Q_J]$ are isomorphic as $GF(p)P_J$-modules. For each $t \in K \setminus \{0\}$, put

$$h(t) := x_{\beta'_{1,1}}(t)x_{-\beta'_{1,1}}(-t^{-1})x_{\beta'_{1,1}}(t)x_{\beta'_{1,1}}(-1)x_{-\beta'_{1,1}}(1)x_{\beta'_{1,1}}(-1).$$

Then $h(t) \in A$ has on $C_{Q_J}([V,Q_J])$ the eigenvalue t^{-2}, and on $V/[V,Q_J]$ the eigenvalue t, for each $t \in K \setminus \{0\}$. Put $H := \langle h(t) \mid t \in K\rangle$. Then $K \otimes_{GF(p)} (V/[V,Q_J])$ is the direct sum of 1-dimensional KH-modules $W_0, \ldots, W_{k-1}$ where $h(t)$ has on W_i the eigenvalue t^{p^i}, for all $t \in K$ and $i \in \{0, \ldots, k-1\}$. Since $C_{Q_J}([V,Q_J])$ and $V/[V,Q_J]$ are isomorphic as $GF(p)H$-modules, it follows that there exists $m \in \{0, \ldots, k-1\}$ such that

$$t^{p^m} = t^{-2}, \quad \text{for each } t \in K.$$

Hence $p^k - 1$ divides $p^m + 2$. Since $m < k$, this implies $p = 2$ and $k \leq 2$. ∎

(B.4.1.5) Let $r \in \{2, \ldots, n\}$. Put $J := \{1 \ldots, n\} \setminus \{r\}$.

(a) $Z(Q_J)$ is a vector space over K with basis $\{x_{\beta'_{i,j}}(1) \mid 1 \leq i \leq j \leq r\}$ where

$$t\, x_{\beta'_{i,j}}(1) = x_{\beta'_{i,j}}(t), \quad \text{for all } t \in K \text{ and } 1 \leq i \leq j \leq r.$$

(b) Assume that q is odd. Then $Z(Q_J)$ and the symmetric square of $C_V(Q_J)$ are isomorphic as KP_J-modules.

(c) Assume that q is even. Put $W := \langle X_{\beta'_{i,j}} \mid 1 \leq i < j \leq r\rangle$.

(c1) W is a KP_J-submodule of $Z(Q_J)$ that is isomorphic to the exterior square of $C_V(Q_J)$.

(c2) If $Z(Q_J)/W$ and $V/[V,Q_J]$ are isomorphic as KP_J-modules, then $q \in \{2,4\}$.

Proof. (a) This follows from (B.4.1.2).

(b) Let γ be the natural homomorphism from $C_V(Q_J)\otimes_K C_V(Q_J)$ onto the symmetric square of $C_V(Q_J)$. Then (B.4.1.2) implies that the K-linear map defined by

$$x_{\beta'_{i,j}}(1) \mapsto \begin{cases} \gamma(v_i \otimes v_j) & \text{if } 1 \leq i < j \leq r \\ \frac{1}{2}\gamma(v_i \otimes v_i) & \text{if } 1 \leq i = j \leq r \end{cases}$$

is a KP_J-isomorphism.

(c) By (B.4.1.2) the K-linear map that sends $x_{\beta'_{i,j}}(1)$ to $v_i \wedge v_j$, for all $1 \leq i < j \leq r$, is a KP_J-isomorphism from W to $C_V(Q_J) \wedge C_V(Q_J)$. Hence (c1) holds. The proof of (c2) is similar to the proof of (B.4.1.4)(b). ■

(B.4.1.6) Assume that A is a nontrivial subgroup of G with $|A| \geq |[V, A]|^2$ and $[V, A, A] = 0$. Then $n \geq 3$.

Proof. Note that (A.1.5)(a) implies that $[V, A]$ is an isotropic subspace of V. Let r be the K-dimension of $[V, A]$. Then $r \leq n$, since $[V, A]$ is isotropic. Moreover, $[V, A]$ is conjugate to the span of $v_1, \ldots, v_r$, and hence $|A| \leq q^{\frac{r(r+1)}{2}}$. Thus $\frac{r(r+1)}{2} \geq 2r$, i.e., $r \geq 3$. ■

B.5 D_n

Assume that $n \geq 3$, $\mathcal{L}$ is of type D_n, and

$$\lambda = \sum_{i=1}^{n-2} \alpha_i + \frac{1}{2}(\alpha_{n-1} + \alpha_n).$$

Then $\Phi^+ = \{\beta_{i,j} \mid i, j \in \{1, \ldots, n-1\},\ i \leq j\} \cup \{\beta'_{i,j} \mid i, j \in \{1, \ldots, n-1\},\ i \leq j\}$ and $\Lambda = \{\mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n\}$, where

$$\beta_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad \text{for all } i, j \in \{1, \ldots, n-1\} \text{ with } i \leq j,$$

$$\beta'_{i,j} := \sum_{k=i}^{j} \alpha_k + 2 \sum_{k=j+1}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n, \quad \text{for all } i, j \in \{1, \ldots, n-2\} \text{ with } i \leq j,$$

$$\beta'_{i,n-1} := \sum_{k=i}^{n-2} \alpha_k + \alpha_n, \quad \text{for all } i \in \{1, \ldots, n-1\},$$

$$\mu_i := \sum_{k=i}^{n-2} \alpha_k + \frac{1}{2}(\alpha_{n-1} + \alpha_n), \quad \text{for all } i \in \{1, \dots, n-1\}, \text{ and}$$

$$\mu_n := \frac{1}{2}(\alpha_n - \alpha_{n-1}).$$

Let v_{μ_1} be a nonzero weight vector of weight μ_1. Then the basis B can be chosen as follows: Put

$$v_{\mu_{i+1}} := v_{\mu_i} e_{-\alpha_i}, \quad \text{for each } i \in \{1, \dots, n-1\},$$

$$v_{-\mu_n} := v_{\mu_{n-1}} e_{-\alpha_n}, \quad \text{and}$$

$$v_{-\mu_{n-i}} := -v_{\mu_{n-i+1}} e_{-\alpha_{n-i}}, \quad \text{for each } i \in \{1, \dots, n-1\}.$$

Then $\{v_{\mu_1}, \dots, v_{\mu_n}, v_{-\mu_1}, \dots, v_{-\mu_n}\}$ is a basis with the desired properties. Note that with respect to $\prec$ the following pairs are extraspecial in the sense of [3]:

$$(\alpha_i, \beta_{i+1,j}), \quad \text{for all } i, j \in \{1, \dots, n-1\} \text{ with } i < j,$$

$$(\alpha_i, \beta'_{i+1,j}), \quad \text{for all } i, j \in \{1, \dots, n-2\} \text{ with } i < j, \text{ and}$$

$$(\beta_{i,i+1}, \beta'_{i+1,i+1}), \quad \text{for all } i \in \{1, \dots, n-2\}.$$

Hence without loss we may assume that

(i) $e_{\beta_{i,j}} = -[e_{\alpha_i}, e_{\beta_{i+1,j}}]$, for all $i, j \in \{1, \dots, n-1\}$ with $i < j$,

(ii) $e_{\beta'_{i,j}} = -[e_{\alpha_i}, e_{\beta'_{i+1,j}}]$, for all $i, j \in \{1, \dots, n-2\}$ with $i < j$,

(iii) $e_{\beta'_{i,i}} = -[e_{\beta_{i,i+1}}, e_{\beta'_{i+1,i+1}}]$, for all $i \in \{1, \dots, n-2\}$.

B.5.1

This subsection is about the natural $O_{2n}^{+}(q)$-module. The information given in the lemmas below will also help to construct the natural $O_{2n}^{-}(q)$-module later.

(B.5.1.1) (a) $\beta_{i,j} = \mu_i - \mu_{j+1}$, for all $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$.

(b) $\beta'_{i,j} = \mu_i + \mu_{j+1}$, for all $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$.

(c) Let $\mu \in \Lambda$ and $\beta \in \Phi$. Then

$$v_\mu e_\beta = \begin{cases} v_{\mu+\beta} & \text{if } 2\mu + \beta \text{ is a sum of pos. roots and } \mu + \beta \in \Lambda \\ -v_{\mu+\beta} & \text{if } 2\mu + \beta \text{ is a sum of neg. roots and } \mu + \beta \in \Lambda \\ 0 & \text{else.} \end{cases}$$

Proof. (a),(b) This is clear.

(c) If $\beta = -\alpha_i$ for some $i \in \{1, \ldots, n-1\}$, then (c) holds by definition of $v_{\mu_2}, \ldots, v_{\mu_n}$ and $v_{-\mu_{n-1}}, \ldots, v_{-\mu_1}$. Note that $2\mu_{n-1} - \alpha_n = \alpha_{n-1}$ is a sum of positive roots and $2\mu_n - \alpha_n = -\alpha_{n-1}$ is a sum of negative roots. Hence the definition of $v_{-\mu_n}$ and

$$v_{\mu_n} e_{-\alpha_n} = v_{\mu_{n-1}} e_{-\alpha_{n-1}} e_{-\alpha_n} = v_{\mu_{n-1}} [e_{-\alpha_{n-1}}, e_{-\alpha_n}] + v_{\mu_{n-1}} e_{-\alpha_n} e_{-\alpha_{n-1}} =$$

$$0 + v_{-\mu_n} e_{-\alpha_{n-1}} = -v_{-\mu_{n-1}},$$

imply that (c) holds if $\beta = -\alpha_n$.

Now assume that $\beta = \alpha_i$, for some $i \in \{1, \ldots, n-1\}$. Then

$$v_{\mu_{i+1}} e_\beta = v_{\mu_i} e_{-\alpha_i} e_{\alpha_i} = v_{\mu_i} h_{\alpha_i} + v_{\mu_i} e_{\alpha_i} e_{-\alpha_i} = \langle \mu_i, \alpha_i \rangle v_{\mu_i} + 0 = v_{\mu_i}$$

and

$$v_{-\mu_i} e_\beta = -v_{-\mu_{i+1}} e_{-\alpha_i} e_{\alpha_i} = -v_{-\mu_{i+1}} h_{\alpha_i} - v_{\mu_{i+1}} e_{\alpha_i} e_{-\alpha_i} = -\langle -\mu_{i+1}, \alpha_i \rangle v_{\mu_{i+1}} - 0 =$$

$$-v_{\mu_{i+1}}.$$

Hence (c) also holds in this case.

Let $i, j \in \{1, \ldots, n-1\}$ with $i < j$. Then, by (i) and induction on $\mathrm{ht}\beta_{i,j}$

$$v_{\mu_{j+1}} e_{\beta_{i,j}} = -v_{\mu_{j+1}} [e_{\alpha_i}, e_{\beta_{i+1,j}}] = v_{\mu_{j+1}} e_{\beta_{i+1,j}} e_{\alpha_i} = v_{\mu_{i+1}} e_{\alpha_i} = v_{\mu_i},$$

$$v_{-\mu_i}e_{\beta_{i,j}} = -v_{-\mu_i}[e_{\alpha_i}, e_{\beta_{i+1,j}}] = -v_{-\mu_i}e_{\alpha_i}e_{\beta_{i+1,j}} = v_{-\mu_{i+1}}e_{\beta_{i+1,j}} = -v_{-\mu_{j+1}},$$

$$v_{-\mu_{j+1}}e_{-\beta_{i,j}} = v_{-\mu_{j+1}}[e_{-\alpha_i}, e_{-\beta_{i+1,j}}] = -v_{-\mu_{j+1}}e_{-\beta_{i+1,j}}e_{-\alpha_i} = v_{-\mu_{i+1}}e_{-\alpha_i} = -v_{-\mu_i},$$

and

$$v_{\mu_i}e_{-\beta_{i,j}} = v_{\mu_i}[e_{-\alpha_i}, e_{-\beta_{i+1,j}}] = v_{\mu_i}e_{-\alpha_i}e_{-\beta_{i+1,j}} = v_{\mu_{i+1}}e_{-\beta_{i+1,j}} = v_{\mu_{j+1}}.$$

Hence (c) also holds if $\beta = \beta_{i,j}$ or $\beta = -\beta_{i,j}$.

Let $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$. By (ii), (iii), induction on $\mathrm{ht}\beta'_{i,j}$, and the parts of (c) proven so far, if $i < j$ then

$$v_{-\mu_{j+1}}e_{\beta'_{i,j}} = -v_{-\mu_{j+1}}[e_{\alpha_i}, e_{\beta'_{i+1,j}}] = v_{-\mu_{j+1}}e_{\beta'_{i+1,j}}e_{\alpha_i} = v_{\mu_{i+1}}e_{\alpha_i} = v_{\mu_i},$$

$$v_{-\mu_i}e_{\beta'_{i,j}} = -v_{-\mu_i}[e_{\alpha_i}, e_{\beta'_{i+1,j}}] = -v_{-\mu_i}e_{\alpha_i}e_{\beta'_{i+1,j}} = v_{-\mu_{i+1}}e_{\beta'_{i+1,j}} = -v_{\mu_{j+1}},$$

$$v_{\mu_{j+1}}e_{-\beta'_{i,j}} = v_{\mu_{j+1}}[e_{-\alpha_i}, e_{-\beta'_{i+1,j}}] = -v_{\mu_{j+1}}e_{-\beta'_{i+1,j}}e_{-\alpha_i} = v_{-\mu_{i+1}}e_{-\alpha_i} = -v_{-\mu_i},$$

and

$$v_{\mu_i}e_{-\beta'_{i,j}} = v_{\mu_i}[e_{-\alpha_i}, e_{-\beta'_{i+1,j}}] = v_{\mu_i}e_{-\alpha_i}e_{-\beta'_{i+1,j}} = v_{\mu_{i+1}}e_{-\beta'_{i+1,j}} = v_{-\mu_{j+1}};$$

and if $i = j < n-1$ then

$$v_{-\mu_i}e_{\beta'_{i,j}} = -v_{-\mu_i}[e_{\beta_{i,i+1}}, e_{\beta'_{i+1,i+1}}] = -v_{-\mu_i}e_{\beta_{i,i+1}}e_{\beta'_{i+1,i+1}} = v_{-\mu_{i+2}}e_{\beta'_{i+1,i+1}} = v_{\mu_{i+1}},$$

$$v_{-\mu_{i+1}}e_{\beta'_{i,j}} = -v_{-\mu_{i+1}}[e_{\beta_{i,i+1}}, e_{\beta'_{i+1,i+1}}] = v_{-\mu_{i+1}}e_{\beta'_{i+1,i+1}}e_{\beta_{i,i+1}} = -v_{\mu_{i+2}}e_{\beta_{i,i+1}} = -v_{\mu_i},$$

$$v_{\mu_i}e_{-\beta'_{i,j}} = v_{\mu_i}[e_{-\beta_{i,i+1}}, e_{-\beta'_{i+1,i+1}}] = v_{\mu_i}e_{-\beta_{i,i+1}}e_{-\beta'_{i+1,i+1}} = v_{\mu_{i+2}}e_{-\beta'_{i+1,i+1}} = -v_{-\mu_{i+1}},$$

and

$$v_{\mu_{i+1}}e_{-\beta'_{i,j}} = v_{\mu_{i+1}}[e_{-\beta_{i,i+1}}, e_{-\beta'_{i+1,i+1}}] = -v_{\mu_{i+1}}e_{-\beta'_{i+1,i+1}}e_{-\beta_{i,i+1}} = -v_{-\mu_{i+2}}e_{-\beta_{i,i+1}} =$$

$$v_{-\mu_i}.$$

Hence (c) also holds if $\beta = \beta'_{i,j}$ or $\beta = -\beta'_{i,j}$. ∎

Put

$$v_i := 1 \otimes_{\mathbf{Z}} v_{\mu_i} \quad \text{and}$$

$$v_{-i} := 1 \otimes_{\mathbf{Z}} v_{-\mu_i}$$

for each $i \in \{1, \ldots, n\}$.

(B.5.1.2) Let $k \in \{1, \ldots, n\} \cup \{-1, \ldots, -n\}$, $i, j \in \{1, \ldots, n-1\}$ with $i \leq j$, $\varepsilon \in \{1, -1\}$, and $t \in K$. Then

$$v_k x_{\varepsilon\beta_{i,j}}(t) = \begin{cases} v_k + \varepsilon t v_{\varepsilon i} & \text{if } k = \varepsilon(j+1) \\ v_k - \varepsilon t v_{-\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

and

$$v_k x_{\varepsilon\beta'_{i,j}}(t) = \begin{cases} v_k + \varepsilon t v_{\varepsilon i} & \text{if } k = -\varepsilon(j+1) \\ v_k - \varepsilon t v_{\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ v_k & \text{else.} \end{cases}$$

Proof. This follows from (B.5.1.1)(c). ■

(B.5.1.3) (a) Let $r \in \{1, \ldots, n-2\}$ and put $J := \{1, \ldots, n\} \setminus \{r\}$. Then

(a1) $[V, Q_J] = \bigoplus_{i=1}^{n} K v_i \oplus \bigoplus_{i=r+1}^{n} K v_{-i}$,

(a2) $[V, Q_J, Q_J] = C_V(Q_J) = \bigoplus_{i=1}^{r} K v_i$.

(b) If $J = \{1, \ldots, n-2\}$, then

(b1) $[V, Q_J] = \bigoplus_{i=1}^{n} K v_i \oplus K v_{-n}$,

(b2) $[V, Q_J, Q_J] = C_V(Q_J) = \bigoplus_{i=1}^{n-1} K v_i$.

(c) If $J = \{1, \ldots, n-1\}$, then

$$[V, Q_J] = C_V(Q_J) = \bigoplus_{i=1}^{n} K v_i.$$

(d) The map

$$\xi : V \times V \to K,$$

$$(\sum_{i=1}^{n} a_i v_i + a_{-i}v_{-i}, \sum_{i=1}^{n} b_i v_i + b_{-i}v_{-i}) \mapsto \sum_{i=1}^{n} a_i b_{-i} + a_{-i} b_i$$

is a nondegenarete G-invariant symmetric bilinear form.

Proof. This follows from (B.5.1.2) ∎

(B.5.1.4) Let $J := \{2, \dots, n\}$. If $1 \neq A \leq Q_J$, then $|A| < |[V, A]|$.

Proof. Note that (B.5.1.3) implies that

$$(*) \quad \overline{[v_{-1}, ab]} = \overline{[v_{-1}, a]}\,\overline{[v_{-1}, b]} \quad \text{for all } a, b \in Q_J \text{ where}$$

$$\overline{V} := V/Kv_1.$$

From (B.5.1.2) we get that

$$tv_{-i} = [v_{-1}, x_{\beta_{1,i-1}}(-t)] \in [v_{-1}, Q_J] \quad \text{and}$$

$$tv_i = [v_{-1}, x_{\beta'_{1,i-1}}(-t)] \in [v_{-1}, Q_J] \quad \text{for all } t \in K \text{ and } i \in \{2, \dots, n\}.$$

Then $(*)$ implies that the map

$$\kappa : \ Q_J \to \overline{[V, Q_J]}, \ a \mapsto \overline{[v_{-1}, a]}$$

is surjective. Since $|Q_J| = |\overline{[V, Q_J]}|$ by (B.5.1.3)(a1), it follows that κ is bijective. In particular,

$$|[V, A]| \geq |\kappa(A)| = |A|.$$

Suppose that $|[V, A]| \leq |A|$. Then $|[V, A]| = |\kappa(A)|$ and hence $[V, A] \cap Kv_1 = 0$. Now (B.5.1.2) implies that $A = 1$, a contradiction. ∎

(B.5.1.5) Let $J := \{1, \ldots, n-2\}$. Assume that A is a nontrivial subgroup of Q_J with $[V, A, A] = 0$ and $|A| \geq |[V, A]|$. Then $A \cap C_{Q_J}([V, Q_J]) \neq 1$.

Proof. Put $Z := C_{Q_J}([V, Q_J])$. Suppose that $A \cap Z = 1$. Note that, by (B.5.1.2),

$$[Kv_n + Kv_{-n}, L_J] = 0.$$

Suppose that there exists $a \in A$ such that $[Kv_n + Kv_{-n}, a]$ is 2-dimensional. Since L_J is 2-transitive on $C_V(Q_J) \setminus \{0\}$, we may assume that $[v_n, a] = v_1$ and $[v_{-n}, a] = v_2$, i.e.,

$$a \in x_{\beta_{1,n-1}}(1) x_{\beta'_{2,n-1}}(1) Z.$$

But then, by (B.5.1.2),

$$[-v_{-1}, a, a] = [v_{-n}, a] = v_2,$$

contrary to $[V, A, A] = 0$.

Now let a be any nontrivial element of A. From $A \cap Z = 1$ and the previous paragraph it follows that $[Kv_n + Kv_{-n}, a]$ is 1-dimensional. Since L_J is transitive on $C_V(Q_J) \setminus \{0\}$, we may assume that $[Kv_n + Kv_{-n}, a] = Kv_1$, i.e.,

$$a \in x_{\beta_{1,n-1}}(s) x_{\beta'_{1,n-1}}(t) Z, \quad \text{for some } s, t \in K \text{ with } s \neq 0 \text{ or } t \neq 0.$$

Then, by (B.5.1.2),

$$[v_{-1}, a] \in -sv_{-n} - tv_n + C_V(Q_J).$$

In particular, $[V, A] \not\subseteq C_V(Q_J)$. Together with $|[V, A]| \leq |A|$ it follows that the following homomorphisms are not injective:

$$\kappa_n : A \to C_V(Q_J), \; g \mapsto [v_n, g] \quad \text{and}$$

$$\kappa_{-n} : A \to C_V(Q_J), \; g \mapsto [v_{-n}, g].$$

Pick $b \in \mathrm{Ker}(\kappa_n) \setminus \{1\}$. Then

$$b \in x_{\beta'_{1,n-1}}(t_1) x_{\beta'_{2,n-1}}(t_2) \cdot \ldots \cdot x_{\beta'_{n-1,n-1}}(t_{n-1}) Z,$$

for some $t_1, \ldots, t_{n-1} \in K$. Since $A \cap Z = 1$, there exists $j \in \{1, \ldots, n-1\}$ such that $t_j \neq 0$. Then

$$[v_{-j}, b, a] = [-t_j v_n, a] = -t_j s v_1.$$

Since $[V, A, A] = 0$, this implies that $s = 0$. A similar argument, using $\mathrm{Ker}(\kappa_{-n}) \neq 1$, shows that $t = 0$. But then $a \in Z$, a contradiction. ■

(B.5.1.6) Let $J := \{1 \ldots, n\} \setminus \{r\}$, for some $r \in \{2, \ldots, n-2\} \cup \{n\}$. Assume that A is a nontrivial subgroup of Q_J with $[V, A, A] = 0$ and $|A| \geq |[V, A]|$.

(a) $|[V, A] \cap C_V(Q_J)| \geq q^2$.

(b) $|[V, A]| \geq q^3$.

(c) If $r = n$, then $|[V, A] \cap C_V(Q_J)| \geq q^3$.

Proof. (a) Suppose that $|[V, A] \cap C_V(Q_J)| \leq q$. Since L_J acts transitively on $C_V(Q_J) \setminus \{0\}$, we may assume that

$$(*) \quad [V, A] \cap C_V(Q_J) \leq K v_1.$$

Since $[V, Q_J, A] \leq C_V(Q_J)$ by (B.5.1.3)(a)(c), we get

$$[V, Q_J, A] \leq K v_1.$$

Note that (B.5.1.3)(d) implies that there exists a G-isomorphism ρ from $[V, Q_J]$ to $\mathrm{Hom}_K(V/C_V(Q_J), K)$ defined by

$$(v + C_V(Q_J))(w\rho) := \xi(v, w) \quad , \text{ for all } v \in V \text{ and } w \in [V, Q_J].$$

Together with $[V, Q_J, A] \leq K v_1$ it follows that A centralizes $v_{-2} + C_V(Q_J), \ldots, v_{-n} + C_V(Q_J)$ in $V/C_V(Q_J)$. Now $(*)$ implies that

$$[\bigoplus_{i=2}^{n} K v_i \oplus K v_{-i}, A] \leq K v_1.$$

By (B.5.1.2), this means that $A \leq Q_{\{2,\ldots,n\}}$, contrary to (B.5.1.4).

(b) Suppose that $|[V, A]| \leq q^2$. Then $[V, A] \leq C_V(Q_J)$ by (a). Since L_J acts 2-transitively on $C_V(Q_J) \setminus \{0\}$, we may assume that

$$[V, A] \leq Kv_1 + Kv_2.$$

Now that (B.5.1.3)(d) implies that

$$[\bigoplus_{i=3}^{n} Kv_i \oplus Kv_{-i}, A] = 0.$$

Together with $[V, A] \leq Kv_1 + Kv_2$ and (B.5.1.2) it follows that $A \leq X_{\beta'_{1,1}}$ and hence

$$|[V, A]| \leq |A| \leq |X_{\beta'_{1,1}}| = q,$$

contrary to (a).

(c) follows from (b) and (B.5.1.3)(c). ∎

(B.5.1.7) Let $r \in \{2, \ldots, n\}$. Put $J := \{1 \ldots, n\} \setminus \{r\}$, if $r \neq n-1$, and $J := \{1, \ldots, n-2\}$, if $r = n - 1$.

(a) $Z(Q_J)$ is a vector space over K with basis $\{x_{\beta'_{i,j}}(1) \mid 1 \leq i \leq j \leq r - 1\}$ where

$$t\, x_{\beta'_{i,j}}(1) = x_{\beta'_{i,j}}(t), \quad \text{for all } t \in K \text{ and } 1 \leq i \leq j \leq r - 1.$$

(b) $Z(Q_J)$ and the exterior square of $C_V(Q_J)$ are isomorphic as KP_J-modules.

(c) If $V/[V, Q_J]$ and $Z(Q_J)$ are isomorphic as $GF(p)P_J$-modules, then $r = 3$ and $q = 2$.

Proof. (a) follows from (B.5.1.2). Again by (B.5.1.2), the K-linear map which sends $x_{\beta'_{i,j}}(1)$ to $v_i \wedge v_{j+1}$, for all $1 \leq i \leq j \leq r - 1$, is a KP_J-isomorphism from $Z(Q_J)$ to $C_V(Q_J) \wedge C_V(Q_J)$. Hence (b) holds.

Assume that $Z(Q_J)$ and $V/[V, Q_J]$ are isomorphic as $GF(p)P_J$-modules. Then $r = 3$, since $\dim_K Z(Q_J) = \frac{r(r-1)}{2}$ and $\dim_K V/[V, Q_J] = r$. For each $t \in K \setminus \{0\}$, put

$$h(t) := x_{\beta'_{2,2}}(t)x_{-\beta'_{2,2}}(-t^{-1})x_{\beta'_{2,2}}(t)x_{\beta'_{2,2}}(-1)x_{-\beta'_{2,2}}(1)x_{\beta'_{2,2}}(-1).$$

Then $h(t)$ has on $Z(Q_J)$ the eigenvalues t^{-1} and t^{-2}, and on $V/[V, Q_J]$ the eigenvalues 1 and t, for each $t \in K \setminus \{0\}$. Hence $q = 2$. ∎

(B.5.1.8) Assume that A is a nontrivial subgroup of G with $|A| \geq |[V, A]|^2$ and $[V, A, A] = 0$. Then $n \geq 5$.

Proof. Note that (A.1.5)(a) implies that $[V, A]$ is an isotropic subspace of V. Let r be the K-dimension of $[V, A]$. Then $r \leq n$, since $[V, A]$ is isotropic.

Assume first that $[V, A]$ is singular. Then $[V, A]$ is conjugate to the span of $v_1, \dots, v_r$, and hence $|A| \leq q^{\frac{r(r-1)}{2}}$. Thus $\frac{r(r-1)}{2} \geq 2r$, i.e., $r \geq 5$.

Assume now that $[V, A]$ is not singular. Then $[V, A]$ is conjugate to the span of $v_1, \dots, v_{r-1}, v_n + v_{-n}$, and hence $|A| \leq q^{\frac{(r-1)(r-2)}{2}+r}$. Thus $\frac{(r-1)(r-2)}{2} + r \geq 2r$. It follows that $r(r-2) > (r-1)(r-2) \geq 2r$, i.e., $r > 4$. ∎

B.5.2

This subsection is about the natural $O^-_{2n}(q)$-module. Assume that $K = GF(q^2)$ for some prime power q. Let $\tau : V \to V$ be the K-linear map defined by

$$v_i\tau = \begin{cases} v_i & \text{if } i \in \{1, \dots, n-1\} \cup \{-1, \dots, -n\} \\ v_{-n} & \text{if } i = n \\ v_n & \text{if } i = -n \end{cases} .$$

Put $\widehat{K} := \{t \in K \mid t = t^q\}$. Let $\sigma : V \to V$ be the $\widehat{K}$-linear map defined by

$$tv_i\sigma = t^q v_i, \quad \text{for all } i \in \{1, \dots, n\} \cup \{-1, \dots, -n\} \text{ and } t \in K.$$

Put $\widehat{V} := C_V(\sigma\tau)$. Choose an element s in $K \setminus \widehat{K}$. For each $i \in \{1, \dots, n\} \cup \{-1, \dots, -n\}$ define

$$\widehat{v}_i := \begin{cases} v_i & \text{if } i \in \{1, \dots, n-1\} \cup \{-1, \dots, -n\} \\ v_n + v_{-n} & \text{if } i = n \\ sv_n + s^q v_{-n} & \text{if } i = -n \end{cases} .$$

For all $i, j \in \{1, \dots, n-1\}$ with $i \leq j$ define

$$\widehat{\beta}_{i,j} := \begin{cases} \beta_{i,j} & \text{if } j \leq n-2 \\ \frac{1}{2}(\beta_{i,n-1} + \beta'_{i,n-1}) & \text{if } j = n-1 \end{cases}$$

and

$$\widehat{\beta}'_{i,j} := \begin{cases} \beta'_{i,j} & \text{if } j \leq n-2 \\ \frac{1}{2}(\beta_{i,n-1} + \beta'_{i,n-1}) & \text{if } j = n-1 \end{cases}$$

Put

$$\widehat{\Phi}^+ := \{\widehat{\beta}_{i,j}, \widehat{\beta'_{i,j}} \mid i,j \in \{1,\ldots,n-1\},\ i \leq j\},$$

$$\widehat{\Phi}^- := \{-\widehat{\beta}_{i,j}, \widehat{\beta'_{i,j}} \mid i,j \in \{1,\ldots,n-1\},\ i \leq j\},$$

$$\widehat{\Phi} := \widehat{\Phi}^+ \cup \widehat{\Phi}^-, \quad \text{and}$$

$$\widehat{\Pi} := \{\widehat{\beta}_{1,1},\ldots,\widehat{\beta}_{n-1,n-1}\}.$$

Note that $\widehat{\Phi}$ is a root system of type B_{n-1} and $\widehat{\Pi}$ is a system of fundamental roots in such a way that $\widehat{\beta}_{i,i}$ corresponds to the node i in the Dynkin diagram. For all $i \in \{1,\ldots,n-1\}$ and $t \in K$ define

$$x_{\widehat{\beta}_{i,n-1}}(t) := x_{\beta_{i,n-1}}(t)\, x_{\beta'_{i,n-1}}(t^q) \quad \text{and}$$

$$x_{-\widehat{\beta}_{i,n-1}}(t) := x_{-\beta_{i,n-1}}(t)\, x_{-\beta'_{i,n-1}}(t^q).$$

For all $\widehat{\beta} \in \widehat{\Phi}$ let $\widehat{X}_{\widehat{\beta}}$ be the subgroup of G consisting of all $x_{\widehat{\beta}}(t)$ where t ranges over K or $\widehat{K}$, depending on whether $\widehat{\beta} \in \{\widehat{\beta}_{i,n-1}, -\widehat{\beta}_{i,n-1}\}$ for some $i \in \{1,\ldots,n-1\}$ or not. Define

$$\widehat{G} := \langle \widehat{X}_{\widehat{\beta}} \mid \widehat{\beta} \in \widehat{\Phi} \rangle.$$

For each $J \subseteq \{1,\ldots,n-1\}$, let $\widehat{\Phi}_J$ be the root system spanned by $\{\widehat{\beta}_{j,j} \mid j \in J\}$, and put

$$\widehat{Q}_J := \langle \widehat{X}_{\widehat{\beta}} \mid \widehat{\beta} \in \widehat{\Phi}^+ \setminus \widehat{\Phi}_J \rangle,$$

$$\widehat{L}_J := \langle \widehat{X}_{\widehat{\beta}} \mid \widehat{\beta} \in \widehat{\Phi}_J \rangle.$$

(B.5.2.1) (a) $\widehat{v}_1,\ldots,\widehat{v}_n,\widehat{v}_{-1},\ldots,\widehat{v}_{-n}$ is a $\widehat{K}$-basis of $\widehat{V}$.

(b) $\widehat{V}$ is a $\widehat{K}\widehat{G}$-module.

(c) Let $k \in \{1,\ldots,n\} \cup \{-1,\ldots,-n\}$, $i,j \in \{1,\ldots,n-2\}$ with $i \leq j$, $\varepsilon \in \{1,-1\}$, and $t \in \widehat{K}$. Then

$$\widehat{v}_k x_{\varepsilon\widehat{\beta}_{i,j}}(t) = \begin{cases} \widehat{v}_k + \varepsilon t \widehat{v}_{\varepsilon i} & \text{if } k = \varepsilon(j+1) \\ \widehat{v}_k - \varepsilon t \widehat{v}_{-\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ \widehat{v}_k & \text{else.} \end{cases}$$

and

$$\widehat{v}_k x_{\varepsilon\widehat{\beta}'_{i,j}}(t) = \begin{cases} \widehat{v}_k + \varepsilon t \widehat{v}_{\varepsilon i} & \text{if } k = -\varepsilon(j+1) \\ \widehat{v}_k - \varepsilon t \widehat{v}_{\varepsilon(j+1)} & \text{if } k = -\varepsilon i \\ \widehat{v}_k & \text{else.} \end{cases}$$

(d) Let $k \in \{1,\ldots,n\} \cup \{-1,\ldots,-n\}$, $i \in \{1,\ldots,n-1\}$, $\varepsilon \in \{1,-1\}$, and $t \in K$. Write $t = t_1 + t_2 s$ with $t_1, t_2 \in K$. Then

$$\widehat{v}_k x_{\varepsilon\widehat{\beta}_{i,n-1}}(t) =$$

$$\begin{cases} \widehat{v}_n + \varepsilon(t + t^q)\widehat{v}_{\varepsilon i} & \text{if } k = n \\ \widehat{v}_{-n} + (st + (st)^q)\widehat{v}_i & \text{if } k = -n \text{ and } \varepsilon = 1 \\ \widehat{v}_{-n} + (s^q t + st^q)\widehat{v}_{-i} & \text{if } k = -n \text{ and } \varepsilon = -1 \\ \widehat{v}_{-i} - t^q t \widehat{v}_i - (t_1 + t_2(s^q + s))\widehat{v}_n + t_2 \widehat{v}_{-n} & \text{if } k = -i \text{ and } \varepsilon = 1 \\ \widehat{v}_i - t^q t \widehat{v}_{-i} - t_1 \widehat{v}_n - t_2 \widehat{v}_{-n} & \text{if } k = i \text{ and } \varepsilon = -1 \\ \widehat{v}_k & \text{else.} \end{cases}$$

Proof. This follows from the definitions and (B.5.1.2). ■

(B.5.2.2) (a) Let $r \in \{1,\ldots,n-1\}$ and put $J := \{1,\ldots,n-1\} \setminus \{r\}$. Then

(a1) $[\widehat{V}, \widehat{Q}_J] = \bigoplus_{i=1}^{n} \widehat{K}\widehat{v}_i \oplus \bigoplus_{i=r+1}^{n} \widehat{K}\widehat{v}_{-i}$,

(a2) $[\widehat{V}, \widehat{Q}_J, \widehat{Q}_J] = C_{\widehat{V}}(\widehat{Q}_J) = \bigoplus_{i=1}^{r} \widehat{K}\widehat{v}_i$.

(b) The map

$$\widehat{\xi} : \widehat{V} \times \widehat{V} \to \widehat{K},$$

$$\left(\sum_{i=1}^{n} a_i \widehat{v}_i + a_{-i}\widehat{v}_{-i}, \sum_{i=1}^{n} b_i \widehat{v}_i + b_{-i}\widehat{v}_{-i}\right) \mapsto \sum_{i=1}^{n} a_i b_{-i} + a_{-i} b_i$$

is a nondegenarete G-invariant symmetric bilinear form.

Proof. This follows from (B.5.2.1) ■

(B.5.2.3) Let $J := \{2, \dots, n-1\}$. If $1 \neq A \leq \hat{Q}_J$, then $|A| < |[\hat{V}, A]|$.

Proof. This follows from (B.5.2.1) and (B.5.1.2) the same way as (B.5.1.4) follows from (B.5.1.2) and (B.5.1.3). ■

(B.5.2.4) Let $J := \{1, \dots, n-2\}$. Assume that A is a nontrivial subgroup of $\hat{Q}_J$ with $[\hat{V}, A, A] = 0$ and $|A| \geq |[\hat{V}, A]|$. Then $A \cap C_{\hat{Q}_J}([\hat{V}, \hat{Q}_J]) \neq 1$.

Proof. Put $Z := C_{\hat{Q}_J}([\hat{V}, \hat{Q}_J])$. Suppose that $A \cap Z = 1$. Note that, by (B.5.2.1),

$$[\widehat{K}\hat{v}_n + \widehat{K}\hat{v}_{-n}, L_J] = 0.$$

Suppose that there exists $a \in A$ such that $[\widehat{K}\hat{v}_n + \widehat{K}\hat{v}_{-n}, a]$ is 2-dimensional over $\widehat{K}$. Since $\hat{L}_J$ is 2-transitive on $C_{\hat{V}}(\hat{Q}_J) \setminus \{0\}$, we may assume that $[\hat{v}_n, a] = (s + s^q)\hat{v}_1$ and $[\hat{v}_{-n}, a] = (s + s^q)\hat{v}_2$, i.e.,

$$a \in x_{\hat{\beta}_{1,n-1}}(s) x_{\beta'_{2,n-1}}(1) Z.$$

But then, by (B.5.2.1),

$$[\hat{v}_{-1}, a, a] = [-s^q s\hat{v}_1 - (s + s^q)\hat{v}_n + \hat{v}_{-n}, a] = -(s + s^q)\hat{v}_2,$$

contrary to $[\hat{V}, A, A] = 0$.

Now let a be any nontrivial element of A. From $A \cap Z = 1$ and the previous paragraph it follows that $[\widehat{K}\hat{v}_n + \widehat{K}\hat{v}_{-n}, a]$ is 1-dimensional. Since $\hat{L}_J$ is transitive on $C_{\hat{V}}(\hat{Q}_J) \setminus \{0\}$, we may assume that $[\widehat{K}\hat{v}_n + \widehat{K}\hat{v}_{-n}, a] = \widehat{K}\hat{v}_1$, i.e.,

$$a \in x_{\beta_{1,n-1}}(t) Z, \quad \text{for some } t \in K \text{ with } t \neq 0.$$

Write $t = t_1 + t_2 s$ for some $t_1, t_2 \in \widehat{K}$. Then, by (B.5.2.1),

$$[\hat{v}_{-1}, a] = -t^q t\hat{v}_1 - (t_1 + t_2(s^q + s))\hat{v}_n + t_2\hat{v}_{-n} \notin C_{\hat{V}}(\hat{Q}_J).$$

Together with $|[\hat{V}, A]| \leq |A|$ it follows that the following homomorphisms are not injective:

$$\kappa_n : A \to C_{\hat{V}}(\hat{Q}_J), \ g \mapsto [\hat{v}_n, g] \quad \text{and}$$

$$\kappa_{-n}: A \to C_{\widehat{V}}(\widehat{Q}_J),\ g \mapsto [\widehat{v}_{-n}, g].$$

Pick $b \in \mathrm{Ker}(\kappa_n) \setminus \{1\}$. Then

$$b \in x_{\widehat{\beta}_{1,n-1}}(u_1) x_{\widehat{\beta}_{2,n-1}}(u_2) \cdot \ldots \cdot x_{\widehat{\beta}_{n-1,n-1}}(u_{n-1}) Z,$$

for some $u_1, \ldots, u_{n-1} \in K$ with

$$u_i^q + u_i = 0, \qquad \text{for each } i \in \{1, \ldots, n-1\}.$$

Since $A \cap Z = 1$, there exists $j \in \{1, \ldots, n-1\}$ such that $u_j \neq 0$. Write $u_j = u_{j,1} + u_{j,2}s$ with $u_{j,1}, u_{j,2} \in \widehat{K}$. Then

$$[\widehat{v}_{-j}, b, a] = [-u_j^q u_j \widehat{v}_j - (u_{j,1} + u_{j,2}(s^q + s))\widehat{v}_n + u_{j,2}\widehat{v}_{-n}, a] =$$

$$(-(u_{j,1} + u_{j,2}(s^q + s))(t + t^q) + u_{j,2}(st + (st)^q))v_1 =$$

$$-(u_{j,1}(t + t^q) + u_{j,2}(s^q t + t^q s))v_1 =$$

$$-(u_j^q t + t^q u_j)v_1 = u_j(t - t^q)v_1.$$

Since $[\widehat{V}, A, A] = 0$ and $u_j \neq 0$, this implies that $t = t^q$. A similar argument, using $\mathrm{Ker}(\kappa_{-n}) \neq 1$, shows that $st = (st)^q$. But then

$$s = (st)^q t^{-1} = s^q t^q t^{-1} = s^q t t^{-1} = s^q,$$

a contradiction. ■

(B.5.2.5) Let $J := \{1 \ldots, n-1\} \setminus \{r\}$, for some $r \in \{2, \ldots, n-1\}$. Assume that A is a nontrivial subgroup of $\widehat{Q}_J$ with $[\widehat{V}, A, A] = 0$ and $|A| \geq |[\widehat{V}, A]|$.

(a) $|[\widehat{V}, A] \cap C_{\widehat{V}}(\widehat{Q}_J)| \geq q^2$.

(b) $|[\widehat{V}, A]| \geq q^3$.

Proof. (a) Suppose that $|[\widehat{V},A]\cap C_{\widehat{V}}(\widehat{Q}_J)|\le q$. Since $\widehat{L}_J$ acts transitively on $C_{\widehat{V}}(\widehat{Q}_J)\setminus\{0\}$, we may assume that

$$(*)\quad [\widehat{V},A]\cap C_{\widehat{V}}(\widehat{Q}_J)\le \widehat{K}\widehat{v}_1.$$

Since $[\widehat{V},\widehat{Q}_J,A]\le C_{\widehat{V}}(\widehat{Q}_J)$ by (B.5.2.2)(a), we get

$$[\widehat{V},\widehat{Q}_J,A]\le \widehat{K}\widehat{v}_1.$$

Note that (B.5.2.2)(b) implies that there is a G-isomorphism $\widehat{\rho}$ from $[\widehat{V},\widehat{Q}_J]$ to $\mathrm{Hom}_{\widehat{K}}(\widehat{V}/C_{\widehat{V}}(\widehat{Q}_J),\widehat{K})$ defined by

$$(v+C_{\widehat{V}}(\widehat{Q}_J))(w\rho):=\widehat{\xi}(v,w)\quad \text{, for all } v\in\widehat{V} \text{ and } w\in[\widehat{V},\widehat{Q}_J].$$

Together with $[\widehat{V},\widehat{Q}_J,A]\le\widehat{K}\widehat{v}_1$ it follows that A centralizes $\widehat{v}_{-2}+C_{\widehat{V}}(\widehat{Q}_J),\dots,\widehat{v}_{-n}+C_{\widehat{V}}(\widehat{Q}_J)$ in $\widehat{V}/C_{\widehat{V}}(\widehat{Q}_J)$. Now $(*)$ implies that

$$[\bigoplus_{i=2}^{n}\widehat{K}\widehat{v}_i\oplus\widehat{K}\widehat{v}_{-i},A]\le\widehat{K}\widehat{v}_1.$$

By (B.5.2.1), this means that $A\le\widehat{Q}_{\{2,\dots,n-1\}}$, contrary to (B.5.2.3).

(b) Suppose that $|[\widehat{V},A]|\le q^2$. Then $[\widehat{V},A]\le C_{\widehat{V}}(\widehat{Q}_J)$ by (a). Since $\widehat{L}_J$ acts 2-transitively on $C_{\widehat{V}}(\widehat{Q}_J)\setminus\{0\}$, we may assume that

$$[\widehat{V},A]\le\widehat{K}\widehat{v}_1+\widehat{K}\widehat{v}_2.$$

Now that (B.5.2.2)(b) implies that

$$[\bigoplus_{i=3}^{n}\widehat{K}\widehat{v}_i\oplus\widehat{K}\widehat{v}_{-i},A]=0.$$

Together with $[\widehat{V},A]\le\widehat{K}\widehat{v}_1+\widehat{K}\widehat{v}_2$ and (B.5.2.1) it follows that $A\le\widehat{X}_{\widehat{\beta}'_{1,1}}$ and hence

$$|[\widehat{V},A]|\le|A|\le|\widehat{X}_{\widehat{\beta}'_{1,1}}|=q,$$

contrary to (a). ∎

(B.5.2.6) Let $J:=\{1\dots,n-1\}\setminus\{r\}$, for some $r\in\{2,\dots,n-1\}$.

(a) $Z(\hat{Q}_J)$ is a vector space over K with basis $\{x_{\widehat{\beta}'_{i,j}}(1) \mid 1 \le i \le j \le r-1\}$ where

$$t\, x_{\widehat{\beta}'_{i,j}}(1) = x_{\widehat{\beta}'_{i,j}}(t) \quad \text{for all } t \in K \text{ and } 1 \le i \le j \le r-1.$$

(b) $Z(\hat{Q}_J)$ and the exterior square of $C_{\widehat{V}}(\hat{Q}_J)$ are isomorphic as KP_J-modules.

(c) If $\widehat{V}/[\widehat{V}, \hat{Q}_J]$ and $C_{\widehat{V}}(\hat{Q}_J)$ are isomorphic as P_J-modules, then $r = 3$ and $|\widehat{K}| = 2$.

Proof. This follows from (B.5.2.1) the same way (B.5.1.7) follows from (B.5.1.2). ∎

Appendix C

FF-modules for alternating groups

C.1

Let $G := \Sigma_n$ for some $n \in \mathbb{N}$ with $n \geq 5$ and $n \neq 8$. Let V be the permutation module for $GF(2)G$ constructed from the natural action of G on $\{1, \ldots, n\}$. Put $W := [V, G]C_V(G)/C_V(G)$. Define

$$m := \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

(C.1.1) Let $A \in \mathcal{P}^*(G, W)$. Then there exist pairwise commuting transpositions $t_1, \ldots, t_m$ in G such that one of the following holds:

(a) $A = \langle t_1, \ldots, t_k \rangle$, for some $k \in \{1, \ldots, m\}$.

(b) n is even, and A is a subgroup of index 2 in $\langle t_1, \ldots, t_m \rangle$.

Proof. [14](1.2) ∎

(C.1.2) Let $H \in \{G, G'\}$. Let M be a maximal subgroup of H such that $A \leq O_2(M)$ for some $A \in \mathcal{P}^*(H, W)$. Then one of the following holds:

(a) $H = G$ and $M = C_G(t)$, for some transposition $t \in G$.

(b) n is even, and M is the stabilizer in H of a partition of $\{1, \ldots, n\}$ into 2-sets.

Proof. From [13] and $O_2(M) \neq 1$ it follows that one of the following holds:

(1) M is conjugate to the stabilizer of $\{1,2\}$ in H,

(2) M is conjugate to the stabilizer of $\{1,2,3,4\}$ in H,

(3) n is even and M is conjugate to the stabilizer of $\{\{2i-1,2i\} \mid i \in \{1,\ldots,m\}\}$ in H,

(4) n is divisible by 4, and M is conjugate to the stabilizer in H of the partition

$$\{\{4i-3, 4i-2, 4i-1, 4i\} \mid i \in \{1,\ldots,\tfrac{n}{4}\}\}$$

of $\{1,\ldots,n\}$,

(5) n is a power of 2, and M is the normalizer in H of an elementary abelian 2-subgroup of G acting regularly on $\{1,\ldots,n\}$.

If A is as in (C.1.1)(a), then $H = G$ and (1) or (3) holds, since these are the only cases in which $O_2(M)$ contains transpositions. Now assume that A is as in (C.1.1)(b). Note that A has m orbits of size 2 on $\{1,\ldots,n\}$. In case (2), $O_2(M)$ does not have a subgroup with this property unless $n = 8$, and then such a subgroup has order 4. In case (4), the only subgroups of $O_2(M)$ with this property have oder $2^{\frac{n}{4}}$. In case (5), the only subgroups of $O_2(M)$ acting quadratically on W are of order 2. Since $|A| = 2^{m-1}$, this excludes cases (2), (4) and (5). ∎

Let (,) be the bilinear form on V which has $v_1,\ldots,v_n$ as an orthogonal basis, where v_i denotes the vector of V that corresponds to the element i in $\{1,\ldots,n\}$.

(C.1.3) (,) is a non-degenerate symmetric G-invariant bilinear form.

Proof. This is obvious. ∎

C.2

Let $G := A_7$. Since $A_7 \leq A_8 \cong \mathrm{SL}_4(2)$, the natural $\mathrm{SL}_4(2)$-module V is also a G-module.

(C.2.1) Let M be a maximal subgroup of G. Assume that $|A| < |V/C_V(A)|$, for some nontrivial elementary abelian subgroup A of $O_2(M)$.

(a) M is conjugate to the stabilizer of $\{1, 2, 3, 4\}$ in G.

(b) If $x \in M \setminus O^2(M)$, then x does not centralize $[V, A]$.

Proof. (a) Since $O_2(M) \neq 1$, this follows from [13].

(b) By (A.2.2)(m) $[V, A]$ is a 2-dimensional subspace of V. Hence the centralizer of $[V, A]$ in $\mathrm{SL}_4(2)$ has order $2^5 \cdot 3$. Since (a) implies that $|M| = 2^3 \cdot 3^2$, it follows that M does not centralize $[V, A]$. If $x \in M \setminus O^2(M)$, then also by (*a*) $M = \langle x^M \rangle$ and hence x does not centralize $[V, A]$. ■

Appendix D

Examples

D.1

Let $n \in \mathbb{N}$ with $n \geq 3$. For each $j \in \{0, \ldots, n-1\}$ let U_j be a 2-dimensional vector space over $GF(3)$ with basis $u_j^{(0)}, u_j^{(1)}$ and define $x_j, y_j \in \mathrm{GL}(U_j)$ whose action on U_j is given by the following matrices with respect to the basis $u_j^{(0)}, u_j^{(1)}$:

$$x_j : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad y_j : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Put

$$V := U_0 \otimes \ldots \otimes U_{n-1} \quad \text{and}$$

$$H := \mathrm{GL}(V).$$

For each $j \in \{0, \ldots, n-1\}$ let ϕ_j be the homomorphism from $\mathrm{GL}(U_j)$ to H satisfying

$$(a_0 \otimes \ldots \otimes a_{n-1})(\phi_j(x)) = (a_0 \otimes \ldots \otimes a_{j-1} \otimes a_j x \otimes a_{j+1} \otimes \ldots \otimes a_{n-1})$$

for all $a_0 \in U_0 \ldots, a_{n-1} \in U_{n-1}$ and put

$$e_j := \phi_j(y_j x_j) \quad \text{and} \quad f_j := \phi_j(y_j).$$

Then

$$Q := \langle e_j, f_j \mid j \in \{0, \ldots, n-1\} \rangle$$

is extraspecial of type 2_+^{1+2n}.

For each $m = \sum_{j=0}^{n-1} m_j 2^j$ with $m_0, \dots, m_{n-1} \in \{0,1\}$ put

$$v_m := \bigotimes_{j=0}^{n-1} u_j^{(m_j)}.$$

Then $v_0, \dots, v_{2^n-1}$ is a basis of V. For all $r \in \{0, \dots, n\}$ and $k \in \{0, \dots, 2^r - 1\}$ define

$$W_{r,k} := \langle v_m \mid 2^{n-r}k \le m < 2^{n-r}(k+1) \rangle.$$

(D.1.1) (a) If $j \in \{0, \dots, n-1\}$, then f_j centralizes $\bigoplus_{k=0}^{2^{n-j-1}-1} W_{n-j,2k}$ and inverts $\bigoplus_{k=0}^{2^{n-j-1}-1} W_{n-j,2k+1}$.

(b) If $r \in \{0, \dots, n\}$, then

$$C_H(\langle f_j \mid n-r \le j \le n-1 \rangle) \le \bigcap_{k=0}^{2^r-1} N_H(W_{r,k}).$$

Proof. (a) is an immediate consequence of the definitions. (b) follows from (a). ∎

Let ψ be the homomorphism from Σ_8 to H defined as follows. If $\pi \in \Sigma_8$ and $m = \sum_{j=0}^{n-1} m_j 2^j$ with $m_0, \dots, m_{n-1} \in \{0,1\}$, then

$$v_m \psi(\pi) := v_{m'} \quad \text{where}$$

$$m' = \sum_{j=0}^{n-4} m_j 2^j + 2^{n-3}((m_{n-3} + 2m_{n-2} + 4m_{n-1} + 1)\pi - 1).$$

Define

$$G := N_H(Q),$$

$$F := \langle f_{n-3}, f_{n-2}, f_{n-1} \rangle \, Z(H),$$

$$M := N_G(F),$$

$$X := \bigcap_{k=0}^{7} N_H(W_{3,k}), \quad \text{and}$$

$$Y := \langle \psi((2\,4)(6\,8)), \psi((3\,7)(4\,8)), \psi((3\,4)(7\,8)), \psi((5\,7)(6\,8)) \rangle.$$

Let z be the involution in $Z(N_H(W_{3,0}) \cap \bigcap_{k=1}^{7} C_H(W_{3,k}))$. Let c be the involution in $Z(H)$ $(= Z(Q))$.

(D.1.2) (a) $N_H(F) \leq XYQ$.

(b) $[z, XYQ] \leq M$.

(c) $z \notin G$.

Proof. (a) From (D.1.1)(b) it follows that $C_H(F) \leq X$. Since $|C_H(F, Z(H)) : C_H(F)| \leq 8 = |Q : C_Q(F)|$, we get

$$C_H(F, Z(H)) \leq XQ.$$

Regarding $F/Z(H)$ as a vector space over $GF(2)$ with basis $f_{n-3}Z(H)$, $f_{n-2}Z(H)$, $f_{n-1}Z(H)$, the action of $\psi((2\ 4)(6\ 8))$, $\psi((3\ 7)(4\ 8))$, $\psi((3\ 4)(7\ 8))$, and $\psi((5\ 7)(6\ 8))$ is described by the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $Y/C_Y(F, Z(H)) \cong \mathrm{PSL}_3(2) \cong \mathrm{Aut}(F/Z(H))$ and therefore

$$N_H(F) \leq C_H(F, Z(H))Y \leq XQY.$$

(b),(c) Note that z is centralized by X, Y, $f_0, \ldots, f_{n-1}$, and $e_0, \ldots, e_{n-4}$. Moreover, $[e_{n-3}, z]$, $[e_{n-2}, z]$, and $[e_{n-1}, z]$ act on V as follows:

$$v_m[e_{n-3}, z] = \begin{cases} -v_m & \text{if } m < 2^{n-2} \\ v_m & \text{if } m \geq 2^{n-2} \end{cases},$$

$$v_m[e_{n-2}, z] = \begin{cases} -v_m & \text{if } m < 2^{n-3} \text{ or } 2^{n-2} \leq m < 2^{n-3} + 2^{n-2} \\ v_m & \text{if } 2^{n-3} \leq m < 2^{n-2} \text{ or } m \geq 2^{n-3} + 2^{n-2} \end{cases},$$

$$v_m[e_{n-1}, z] = \begin{cases} -v_m & \text{if } m < 2^{n-3} \text{ or } 2^{n-1} \leq m < 2^{n-3} + 2^{n-1} \\ v_m & \text{if } 2^{n-3} \leq m < 2^{n-1} \text{ or } m \geq 2^{n-3} + 2^{n-1} \end{cases},$$

for each $m \in \{0, \ldots, 2^n - 1\}$. Therefore,

$$[e_{n-3}, z, e_j] = \begin{cases} 1 & \text{if } j \leq n-3 \\ e_{n-2}f_{n-1}c & \text{if } j = n-2 \\ e_{n-1}f_{n-2}c & \text{if } j = n-1 \end{cases},$$

$$[e_{n-2}, z, e_j] = \begin{cases} 1 & \text{if } j \le n-4 \text{ or } j = n-2 \\ e_{n-3} f_{n-1} c & \text{if } j = n-3 \\ e_{n-1} f_{n-3} c & \text{if } j = n-1 \end{cases},$$

$$[e_{n-1}, z, e_j] = \begin{cases} 1 & \text{if } j \le n-4 \text{ or } j = n-1 \\ e_{n-3} f_{n-2} c & \text{if } j = n-3 \\ e_{n-2} f_{n-3} c & \text{if } j = n-2 \end{cases},$$

for each $j \in \{0, \ldots, n-1\}$. In particular, $[Q, z, Q] \not\subseteq Z(Q)$ and hence $z \notin G$. Since $f_0, \ldots, f_{n-1}$ are centralized by $[e_{n-3}, z]$, $[e_{n-2}, z]$, and $[e_{n-1}, z]$, we also get $[Q, z] \le M$.

■

Put

$$\overline{G} := G/Z(H).$$

(D.1.3) (a) $F^*(\overline{G}/\overline{Q}) \cong \Omega_{2n}^+(2)$.

(b) $\overline{M}$ is a maximal subgroup of $\overline{G}$.

(c) No nontrivial characteristic subgroup of $\overline{M}$ is normal in $\overline{G}$.

Proof. (a) From (D.1.1)(b) (with $r = n$) it follows that $C_H(Q)$ normalizes $\langle v_m \rangle$ for each $m \in \{0, \ldots, 2^n - 1\}$. Since $\langle e_0, \ldots, e_{n-1} \rangle$ acts transitively on $\langle v_0 \rangle, \ldots, \langle v_{2^n-1} \rangle$, we get

$$C_H(Q) = Z(H).$$

Hence $\overline{G}/\overline{Q}$ is isomorphic to a subgroup of $\mathrm{Out}(Q)$, which is isomorphic to $\mathrm{O}_{2n}^+(2)$ by [11], Table 4.6.A. Moreover, $\overline{G}$ induces at least $\Omega_{2n}^+(2)$ on Q by Proposition 4.6.8(II) of [11].

(b) With respect to the non-degenerate $\overline{G}$-invariant quadratic form κ on $\overline{Q}$ defined by

$$\kappa(\overline{a}) := \begin{cases} 0 & \text{if } a^2 = 1 \\ 1 & \text{if } a^2 = c \end{cases}$$

for each $a \in Q$, $\overline{F}$ is a singular subspace. Since the stabilizer of any singular subspace of $\overline{Q}$ in $\overline{G}$ is maximal in $\overline{G}$, (b) holds.

(c) By (a), $\overline{Q}$ is the only nontrivial normal subgroup of $\overline{G}$ that is contained in $\overline{M}$. From (D.1.2)(a)(b) it follows that z induces an automorphism on $\overline{M}$, and (D.1.2)(c) implies that $\overline{Q}$ is not invariant under this automorphism. ∎

D.2

(D.2.1) Let X be a finite simple group, G a subgroup of H, M a proper subgroup of G, and $a \in \mathrm{Aut}(H)$. Assume that $X = \langle G^a, G\rangle$ and $M^a = M$. Then no nontrivial characteristic subgroup of M is normal in G.

Proof. Let C be a characteristic subgroup of M that is normal in G. Then $C = C^a \trianglelefteq \langle G, G^a\rangle = X$. Since H is simple and $C \leq M \neq H$, it follows that $C = 1$. ∎

(D.2.2) Let $X := \Omega_8^+(p^k)$, for some $k \in \mathbb{N}$ and some prime p. Let B be a Borel subgroup of X, G the parabolic subgroup of type $\{1,2,3\}$ and M the parabolic subgroup of type $\{1,2\}$ of X containing B.

(a) No nontrivial characteristic subgroup of M is normal in G.

(b) $O_p(G)$ is a natural $\Omega_6^+(q)$-module for $O^{p'}(G)$.

Proof. (a) This follows from (D.2.1), since X has a graph automorphism that normalizes B and acts on the Dynkin diagram by switching the nodes 3 and 4.

(b) This follows from [2]. ∎

(D.2.3) Let X be a simple group of type $\mathsf{E}_6(p^k)$, for some $k \in \mathbb{N}$ and some prime p. Let B be a Borel subgroup of X, G the parabolic subgroup of type $\{1,2,3,4,5\}$ and M the parabolic subgroup of type $\{2,3,4,5\}$ of X containing B.

(a) No nontrivial characteristic subgroup of M is normal in G.

(b) $O_p(G)$ is an $\Omega^+_{10}(q)$-half spin module for $O^{p'}(G)$.

Proof. (a) This follows from (D.2.1), since X has a graph automorphism that normalizes B and induces on the Dynkin diagram the permutation $(1\ 6)(3\ 5)$.

(b) This follows from [2]. ■

(D.2.4) Let X be a simple group of type $\mathsf{F}_4(2^k)$, for some $k \in \mathbb{N}$. Let B be a Borel subgroup of X, G the parabolic subgroup of type $\{1,2,3\}$ and M the parabolic subgroup of type $\{2,3\}$ of X containing B.

(a) No nontrivial characteristic subgroup of M is normal in G.

(b) $O_p(G)$ contains exactly two noncentral $O^{p'}(G)$-chief factors, one natural $\mathrm{Sp}_6(2^k)$-module and one $\mathrm{O}_7(2^k)$-spin module.

Proof. (a) This follows from (D.2.1), since X has a graph automorphism that normalizes B and induces on the Dynkin diagram the permutation $(1\ 4)(2\ 3)$.

(b) If $k > 2$, then this follows from (a) and Theorem 1. From the way $\mathsf{F}_4(2)$ and $\mathsf{F}_4(4)$ are embedded in $\mathsf{F}_4(8)$, this implies that (b) also holds if $k \in \{1,2\}$. ■

Bibliography

[1] M. ASCHBACHER, *Finite Group Theory*, Cambridge University Press 1986

[2] H. AZARD, M. BARRY, G. SEITZ, On the structure of parabolic subgroups, *Comm. Alg.* **18** (1990), 551–562

[3] R.W. CARTER, *Simple Groups of Lie Type*, Wiley 1972

[4] A. CHERMAK, An elementary proof of the $\mathcal{P}(G, V)$-Theorem in arbitrary characteristic, preprint

[5] A. CHERMAK, A.L. DELGADO, A measuring argument for finite groups, *Proc. Am. Math. Soc.* **107**, 907–914

[6] A. CHERMAK, A.L. DELGADO, J-Modules for local BN-Pairs, *Proc. London Math. Soc.* (3) **63** (1991), 69–112

[7] A.L. DELGADO, D. GOLDSCHMIDT, B. STELLMACHER, *Groups and Graphs: New Results and Methods*, Birkhäuser-Verlag, 1985

[8] GORENSTEIN, LYONS, SOLOMON, *The Classification of the Finite Simple Groups*, AMS, Mathematical Surveys and Monographs, Vol.40, Number 3

[9] J.E. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972

[10] W. JONES AND B. PARSHALL, On the 1-cohomology of finite groups of Lie type. *Proceedings of the Conference of Finite Groups (Univ. Utah, Park City, Utah, 1975)*, 313-328, Academic Press, New-York, 1976

[11] P. KLEIDMAN AND M. LIEBECK, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, 1990

[12] LIEBECK, PRAEGER, SAXL, On the O'Nan-Scott Theorem for finite primitive permutation groups, *J. Austral. Math. Soc. (Series A)* **44** (1988), 389–396

[13] LIEBECK, PRAEGER, SAXL, A classification of the maximal subgroups of the finite alternating and symmetric groups, *J. Alg.* **111** (1987), 365–383

[14] U. MEIERFRANKENFELD, Eine Lösung des Pushing-up Problems für eine Klasse endlicher Gruppen, Dissertation Bielefeld 1986

[15] U. MEIERFRANKENFELD, Remarks on vertex transitive groups, preprint

[16] T. MEIXNER, Failure of factorization modules for Lie-type groups in odd characteristic, *Comm. Alg.* **19** (1991), 3193–3222

[17] V.I. TROFIMOV, R.M. WEISS, Graphs with a locally linear group of automorphisms, *Math. Proc. Camb. Phil. Soc.* **118** (1995), 191–206