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CONTINUOUS TIME ARBITRAGE APPROACHED AS A PROBLEM IN CONSTRAINED HEDGING

By

James Andre Demopolos

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ABSTRACT

CONTINUOUS TIME ARBITRAGE APPROACHED AS A PROBLEM IN CONSTRAINED HEDGING

By

James Andre Demopolos

I characterize absence of arbitrage with tame portfolios in a model where a finite vector of stock prices is symbolized by a continuous semi-martingale with respect to the completed filtration generated by a vector-valued standard Brownian Motion. Levental and Skorohod (1995) solved this problem using probabilistic methods in the sub-case of invertible volatility matrix. They constructed an arbitrage trading strategy based upon domination at the end of the time horizon of the value of one stochastic process by that of another. This construction through domination suggests a link between the arbitrage problem and the mathematical theory of financial hedging of contingent claims. This dissertation does not assume invertible volatility. In the case of singular volatility, one faces the constraint that the dominating process constructed by Levental and Skorohod cannot always be effectively converted into a process symbolizing the accumulated capital gains of a trading strategy. Therefore, to apply the theory of hedging, one must consider hedging under constraints. This dissertation contains two primary results. First, I generalize Levental and Skorohod's characterization of arbitrage opportunities in terms of a domination relationship between stochastic processes. Second, I apply work by Cvitanic and Karatzas (1993) pertaining to hedging with constrained portfolios to this generalization to provide a new characterization of absence of arbitrage in the case of singular volatility. The proofs are probabilistic. Some examples are provided.

To my wife, Emily, and my daughter, Sofia

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INTRODUCTION

A basic problem in the construction of asset price models in mathematical finance is the determination of the conditions which are necessary and sufficient for a specified model to exhibit the absence of arbitrage, *i.e.*, the absence of risk-free profit opportunities. In this work, a characterization of absence of arbitrage is provided in the context of a specified model for a finite vector of stock prices. The work has been motivated by results in Levental and Skorohod (1995) and Cvitanic and Karatzas (1993). The former paper characterizes absence of arbitrage in a restricted version of the model considered here. As will be explained in detail in Chapter 1, Levental and Skorohod's Corollary 3 [page 920] suggests a link between the problem of characterizing absence of arbitrage and the theory of hedging contingent claims. In the more general setting of this dissertation, fewer stochastic processes can be taken to meaningfully symbolize accumulated discounted capital gains than in Levental and Skorohod's work. This limitation motivates consideration of the problem of hedging under constraints in the course of attempting to provide a hedging based approach to the arbitrage problem. The problem of hedging contingent claims with constrained portfolios is precisely the topic of Cvitanic and Karatzas (1993). The approach of this dissertation is to generalize Corollary 3 of Levental and Skorohod to provide a characterization of arbitrage "reminiscent of hedging" in the setting of this work, and then to apply the framework of Cvitanic and Karatzas to this generalization in order to provide a new characterization of absence of arbitrage.

The paper is organized in the following way:

In Chapter 1, I specify the model, and motivate the link between the theory of hedging with constrained portfolios and the arbitrage problem. The dissertation's major results are stated. Chapter 1 also contains a history of research into the arbitrage problem.

Chapter 2 contains the statement and proof of necessary and sufficient conditions for the absence of a special kind of arbitrage, namely, immediate arbitrage. Loosely speaking, in an immediate arbitrage, an investor does not ever let his capital gains become negative in the process of obtaining almost sure positive capital gains at the end of the time horizon. Although I did have to make modifications to the argument, the core of the proof, particularly on the necessity side, appears in Levental and Skorohod (1995) [Lemma 2, page 914.]

In Chapter 3, I adapt the work of Cvitanic and Karatzas to this arbitrage problem. The details differ from their work in that in this paper, I need to impose constraints which depend upon (t,ω), whereas their constraints do not vary with (t,ω).

Chapter 4 accomplishes the stated objectives of this work. Theorem 3 is the promised generalization of Levental and Skorohod's Corollary 3. Theorem 4 results from application of the constructions derived from Cvitanic and Karatzas' work in Chapter 3 to the characterization given in Theorem 3. Theorem 5 extends the conclusions of Theorem 4 to address the issue of the equivalence of absence of arbitrage and the existence of an absolutely continuous local martingale measure for the stock price processes.

Chapter 5 contains examples. As will be explained herein, Examples 1 and 4 show that the characterization of arbitrage in the setting of this work is meaningfully different from Levental and Skorohod's (1995) characterization.

Chapter 1

Setting and Main Results

1.1 The Model.

Consider a financial market in which one bond, with price process B, and $d \geq 1$ stocks, with price processes $S_1, ..., S_d$, are traded in the time interval $0 \leq t \leq 1$. Unless otherwise specified, all processes herein will be taken to be defined for $0 \leq t \leq 1$. Correspondingly, in definitions of stopping times interpret the infimum of an empty time set as 1. The source of uncertainty in the market is a d-dimensional standard Brownian Motion $W = (W_1, W_2, \ldots, W_d)^*$ defined on a complete probability space (Ω, F, P)[1]. The term "adapted" will refer throughout to the filtration $\{F_t: 0 \leq t \leq 1\}$, the P augmentation of the natural filtration of W, namely

(1.1) $$F_t = \sigma\{W(s): 0 \leq s \leq t\} \vee U$$

where $U = \{A \in F: P(A) = 0\}$. The price processes of the financial instruments evolve according to the equations

(1.2) $$dB(t) = B(t)r(t)dt, \qquad B(0) = 1.$$

[1] * will denote matrix transpose.

$$dS_i(t) = S_i(t)\left[\sum_{1\le k\le d}\sigma_{i,k}(t)dW_k(t) + b_i(t)dt\right], \tag{1.3}$$

$$S_i(0) = s_i \in (0, \infty), \quad i = 1, \ldots, d.$$

Here $r(\bullet)$ is an adapted R-valued process symbolizing the instantaneous force of interest, volatility $\sigma(\bullet)$ is an adapted $d \times d$ matrix-valued process not necessarily invertible for any (t, ω), and drift $b(\bullet)$ is an adapted R^d-valued process.

In order that (1.2) and (1.3) have well-defined solutions, we require that

$$\int_0^1 \left\{ |r(t)| + \sum_i |b_i(t)| + \sum_{i,j} \sigma_{i,j}^2(t) \right\} dt < \infty \text{ a.s.} \tag{1.4}$$

A continuous-time trader chooses a portfolio, namely, the amount of money to invest in each of the d stocks at each time t. Formally define a portfolio by

Definition 1. *A portfolio is an adapted R^d-valued process π which satisfies the integrability constraint*

$$\int_0^1 \left\{ \|\sigma^*(s)\pi(s)\|^2 + |\pi^*(s)a(s)| \right\} ds < \infty \ a.s., \tag{1.5}$$

where $\| \ \|$ denotes the Euclidean norm in R^d and with

$1_d = (1, 1, \ldots, 1)^* \in R^d$, the process a is defined by

$$a(t) = b(t) - r(t)1_d. \tag{1.6}$$

Since (1.4) implies that the paths $B(\bullet)$ satisfy $\inf\{B(t) : 0 \le t \le 1\} > 0$ a.s., constraint (1.5) implies that the semi-martingale X_π in Definition 2 below is a well-defined process.

Definition 2. *The process X_π given by*

$$(1.7)\qquad X_\pi(t) = \int_0^t B^{-1}(s)\left(\pi^*(s)\sigma(s)\right)dW(s) \;+\; \int_0^t B^{-1}(s)\pi^*(s)a(s)ds$$

is the discounted capital gain process associated with the portfolio π.

To motivate Definition 2, begin from the purpose of investing in stocks, namely, the attempt to obtain capital gains in excess of those available from the less risky bond. In this light, $\pi^*(t)\sigma(t)dW(t) + \pi^*(t)b(t)dt - r(t)\pi^*(t)1_d\,dt$ =

$$= \sum_j \left(\frac{\pi_j(t)}{S_j(t)} dS_j(t) \;-\; \frac{\pi_j(t)}{B(t)} dB(t) \right),$$

which is verbally,

{instantaneous gains from portfolio investment in $\pi_j(t)$ / $S_j(t)$ stock shares, j = 1, …, d} –

{opportunity cost of foregone instantaneous gains possible from the bond}.

Multiplying by $B^{-1}(t)$ discounts these excess (or deficient) gains from stock investment to their present value at time 0. Integration across time sums the discounted instantaneous gains.

Common wisdom is that a reasonable model for the processes S_j and B should not allow for risk-free profits. This is the *no arbitrage* principle.

Definition 3. *An arbitrage is a portfolio π such that the associated discounted capital gain process X_π satisfies*

i) *There exists a $C > 0$ such that $P\{X_\pi(t) \geq -C$ for all $0 \leq t \leq 1\} = 1$.*

ii) *$P\{X_\pi(1) \geq 0\} = 1$.*

iii) $P\{X_\pi(1) > 0\} > 0.$

Any portfolio π for which the associated X_π satisfies i) in Definition 3 is called a *tame* portfolio. *C-tameness* means that i) is satisfied with respect to a particular C. Tameness is a restriction that prevents "doubling schemes" and can be interpreted as putting a limit on borrowing. The mathematics underlying "doubling" in continuous time was set forth by Dudley (1977), who showed that in our model with d = 1, an arbitrary F_1 measurable random variable Λ (including, in particular, Λ satisfying $\Lambda > 0$ a.s.) can be represented as $\Lambda = \int_0^1 g(t)dW(t)$ for an adapted process g satisfying $\int_0^1 g^2(t)dt < \infty$ a.s. In his construction, it is possible that for each C > 0,

$$P\left\{\min_{0\le t\le 1} \int_0^t g(s)dW(s) \quad < \quad -C\right\} > 0.$$

The relationship between the absence of a.s. positive capital gains and the requirement of tameness is treated rigorously in Dybvig and Huang (1988) [see Theorem 2, page 390.]

The purpose of this dissertation is the study of conditions equivalent to the absence of arbitrage. To that end, we need define numerous objects. Let λ denote Lebesgue measure on [0, 1]. As Shreve has shown, unless there is a projection-based arbitrage, then we must have

(1.8) $$\lambda\otimes P\{a(t, \omega) \in \mathrm{Ran}[\sigma(t, \omega)]\} = 1.$$

[See Karatzas and Shreve (1998), Theorem 1.4.2, page 12.] Condition (1.8) will be assumed throughout this work[2]. It holds for each (t,ω) that

$\mathrm{Ran}[\sigma(t,\omega)] = \mathrm{Ran}[\sigma(t,\omega)\sigma^*(t,\omega)]$ and that $\sigma(t,\omega)$ is injective on $\mathrm{Ran}[\sigma^*(t,\omega)]$. Therefore, we may uniquely (up to a $\lambda\otimes P$ null set) and adaptedly define a *relative risk* process θ such that $\theta(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$ for all (t,ω) and $\sigma(\bullet)\theta(\bullet) = a(\bullet)$ $\lambda\otimes P$ a.s.[3] Using θ, define a stopping time α:

$$\text{(1.9)} \qquad \alpha = \inf\left\{ t > 0 : \int_t^{t+h} \|\theta(s)\|^2 ds = \infty \quad for \quad all \quad h \in (0, \ 1-t]. \right\}$$

α is a legitimate stopping time because of right-continuity of the Brownian filtration, and is the key object in a characterization of the absence of a special kind of arbitrage.

Definition 4. *An immediate arbitrage is a portfolio π for which there exists a stopping time $0 \le \tau \le 1$ satisfying $P\{\tau < 1\} > 0$ such that*

$$P\{X_\pi(t) = 0 \text{ for all } t \le \tau \text{ and } X_\pi(t) > 0 \text{ for all } t > \tau\} = 1.$$

Theorem 1 (Immediate Arbitrage Theorem.) *There is no immediate arbitrage if and only if $P\{\alpha = 1\} = 1$.*

The primary contribution of this work pertains to characterization of arbitrage when immediate arbitrage does not exist. Therefore, as is consistent with Theorem 1, for the

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[2] See Chapter 2, Proposition 1, for the details of the necessity of (1.8) for the absence of arbitrage.

[3] To define θ such that it is an adapted process, define $\theta(t) = \sigma_+^{-1}(t)\,a(t)$, $0 \le t \le 1$, where σ_+ is an adapted process such that for each (t,ω), $\sigma_+(t,\omega)$ is an invertible $d\times d$ matrix and for each $x \in \mathrm{Ran}[\sigma(t,\omega)]$, $\sigma_+^{-1}(t,\omega)\,x \in \mathrm{Ran}[\sigma^*(t,\omega)]$ and $\sigma(t,\omega)\,\sigma_+^{-1}(t,\omega)\,x = x$. The existence of such an adapted σ_+ is proven in Lemma 1 of Chapter 2.

remainder of this chapter all results will be given under the assumption that $\alpha = 1$ a.s. In the absence of immediate arbitrage, the fundamental objects in results about existence of arbitrage are *exponential local martingales*. For each adapted R^d-valued process ν satisfying $\alpha^\nu = 1$ a.s., where stopping time α^ν is defined as in (1.9) with process ν substituted for process θ, define for each stopping time $0 \le \tau \le 1$ another stopping time $\zeta^\nu(\tau)$ by

$$(1.10) \qquad \zeta^\nu(\tau) = \inf\left\{ t > 0 : \int_0^t \{\tau < s\} \|\nu(s)\|^2 ds = \infty \right\}.$$

(Adopt the convention of denoting the indicator function of a set by the set itself.) Using $\zeta^\nu(\tau)$, define an adapted process $Z^\nu(\tau; \bullet)$ by

$$(1.11) \quad Z^\nu(\tau;t) = \begin{cases} \exp\left\{ -\int_0^t \{\tau < s\} \nu^*(s) dW(s) - \frac{1}{2} \int_0^t \{\tau < s\} \|\nu(s)\|^2 ds \right\}, & if \quad t < \zeta^\nu(\tau) \\ 0, \quad if \quad t > \zeta^\nu(\tau). \end{cases}$$

$$Z^\nu(\tau; \zeta^\nu(\tau)) = \liminf_{t \uparrow \zeta^\nu(\tau)} Z^\nu(\tau;t).$$

We have that $\lim_{t \uparrow \zeta^\nu(\tau)} Z^\nu(\tau;t)$ exists a.s. The limit exists on

$$\left\{ \int_0^1 \{\tau < s\} \|\nu(s)\|^2 ds < \infty \right\} \subseteq \{ \zeta^\nu(\tau) = 1 \}$$

because $\int_0^1 \{\tau < s\} \nu^*(s) dW(s)$ is well-defined on this set and the stochastic integral with respect to Brownian Motion has continuous paths. The limit exists and equals 0 a.s. on

$$\left\{\int_0^1 \{\tau < s\}\|\nu(s)\|^2 ds \;= \infty\right\} \supseteq \left\{\zeta^\nu(\tau) < 1\right\},$$

because if

$$W_0\left(\int_0^t \{\tau < s\}\|\nu(s)\|^2 ds\right) \;=\; \int_0^t \{\tau < s\}\nu^*(s)dW(s),$$

then

$$\left\{W_0(t), \quad t < \int_0^1 \{\tau < s < \zeta^\nu(\tau)\}\|\nu(s)\|^2 ds\right\}$$

is a standard Brownian Motion in R^1 [see Karatzas and Shreve (1981), page 174], and so

(1.12) $$\lim_{t\uparrow\infty} -W_0(t) - t/2 = -\infty \text{ a.s. on } \left\{\int_0^1 \{\tau < s\}\|\nu(s)\|^2 ds \;= \infty\right\}.$$

Observe that (1.12) implies that the paths $Z^\nu(\tau; \bullet)$ are continuous a.s.

The term *exponential local martingale* is appropriate because if $\zeta^\nu(\tau) = 1$ a.s., then that $Z^\nu(\tau; \bullet)$ is a local martingale follows from that for each t, on $\{t < \zeta^\nu(\tau)\}$

(1.13) $$Z^\nu(\tau;t) \;=\; 1 \;-\; \int_0^t \{\tau < s\}Z^\nu(\tau;s)\nu^*(s)dW(s).$$

It is also relevant that $Z^\nu(\tau; \bullet)$ is a nonnegative supermartingale for any ν and τ for which the process is defined. Observe that for each t, $Z^\nu(\tau; t) = Z^\nu(\tau; t \wedge \zeta^\nu(\tau))$ a.s. The integral representation (1.13) implies that there exist stopping times $\zeta_n^\nu(\tau) \le \zeta^\nu(\tau)$,

$n \geq 1$, such that $\lim_n \uparrow \zeta_n^\nu(\tau) = \zeta^\nu(\tau)$ and $Z^\nu(\tau; \bullet \wedge \zeta_n^\nu(\tau))$ is a martingale for each n. So if $0 \leq s < t$, the Fatou Lemma for conditional expectation implies that a.s.,

$$(1.14) \qquad E[Z^\nu(\tau; t) \mid F_s] = E[\lim_n Z^\nu(\tau; t \wedge \zeta_n^\nu(\tau)) | F_s] \leq \lim_n E[Z^\nu(\tau; t \wedge \zeta_n^\nu(\tau)) | F_s]$$

$$= \lim_n Z^\nu(\tau; s \wedge \zeta_n^\nu(\tau)) = Z^\nu(\tau; s \wedge \zeta^\nu(\tau)) = Z^\nu(\tau; s).$$

Since $Z^\nu(\tau; \bullet)$ is a supermartingale, it holds that $Z^\nu(\tau; \bullet)$ is a martingale if and only if $E(Z^\nu(\tau; 1)) = 1$.

Since the process θ is of central importance, simplify notation by denoting for each τ the process $Z^\theta(\tau; \bullet)$ merely as $Z(\tau; \bullet)$. Abbreviate $Z^\nu(0; \bullet)$ as $Z^\nu(\bullet)$ for any adapted process ν satisfying $\alpha^\nu = 1$ a.s. So $Z(\bullet)$ will denote $Z^\theta(0; \bullet)$.

1.2 The Link Between Arbitrage and Constrained Hedging.

The importance of the processes $Z(\tau; \bullet)$ in the arbitrage problem has been studied by many. Levental and Skorohod (1995) proved that in the absence of immediate arbitrage, under the additional assumption that $\sigma(t,\omega)$ is invertible for all (t,ω), that absence of arbitrage is equivalent to $E(Z(r; 1)) = 1$ for all constant times $0 \leq r \leq 1$. Their proof uses in a substantial way the invertibility of σ. This work extends that of Levental and Skorohod in that it removes the assumption of invertible volatility, allowing $\lambda \otimes P\{\sigma(t,\omega) \text{ is singular}\} > 0$. Of fundamental importance in Levental and Skorohod's proof that an arbitrage exists if there exists a stopping time τ such that $E(Z(\tau; 1)) < 1$ is the existence in that case of an exponential local martingale $Z^\varphi(\bullet)$ which satisfies

$P\{Z^{\varphi}(1) > Z(\tau;\ 1)\} = 1$. In fact, their Corollary 3 [page 920] states that in the case of invertible σ, with the added assumption that $\int_0^1 \|\theta(t)\|^2 dt < \infty$ a.s., that the existence of arbitrage is equivalent to the existence of an adapted R^d-valued process φ satisfying $\int_0^1 \|\varphi(t)\|^2 dt < \infty$ a.s. such that

$$(1.15) \qquad P\{Z^{\varphi}(1) \geq Z(1)\} = 1 \quad \text{and} \quad P\{Z^{\varphi}(1) > Z(1)\} > 0.$$

Equation (1.15) suggests a link between the arbitrage problem and the theory of hedging. To understand this link, begin with consideration of a "seller's objective in hedging." [See Karatzas (1996), Section 0.4 for more detail than is given here.] Define a *contingent claim* to be a non-negative F_1-measurable random variable[4]. One can view a contingent claim as a financial obligation at time 1 to which a seller commits himself in exchange for money at time 0. Let Λ be a contingent claim. For each $x \geq 0$ such that there exists an adapted R^d-valued process π such that

$$(1.16) \qquad \int_0^1 \left\{ \left\| \sigma^*(t)\pi(t) \right\|^2 + \left| \pi^*(t)b(t) \right| \right\} dt < \infty \text{ a.s.,}$$

$$(1.17) \qquad x + \int_0^1 \pi^*(t)\sigma(t)dW(t) + \int_0^1 \pi^*(t)b(t)dt \geq \Lambda \text{ a.s.,}$$

and there exists a constant $C > 0$ such that we have the tameness constraint

[4] Typically in work focusing on hedging, additional constraints which imply absence of arbitrage are assumed to apply with respect to process θ. A requirement related to these additional constraints is then included in the definition of a contingent claim [see Karatzas (1996), page 10]. Since mention of hedging is intended to be motivational here, and since I have not assumed absence of arbitrage, I have chosen to omit these additional details.

(1.18) $$P\left\{\min_{0\le t\le 1}\left(\int_0^t \pi^*(s)\sigma(s)dW(s) \;+\; \int_0^t \pi^*(s)b(s)ds\right) \;<\; -C\right\} = 0,$$

we have the interpretation that a seller can "hedge" his obligation to pay Λ at time 1 starting with the purchase price x at time 0.[5] Examination of (1.13), the integral representation of process $Z^{\varphi}(\bullet)$, suggests the link between (1.15) and this "seller's objective in hedging." Consider an "auxiliary market" in which asset price processes are characterized by the original invertible volatility σ, but the drift b is replaced by the zero vector process. Then define an adapted R^d-valued process π by

$$\pi(t) = [\sigma^*(t)]^{-1}(-Z^{\varphi}(t)\varphi(t)).$$

π satisfies (1.16) for the "auxiliary market," since the paths $Z^{\varphi}(\bullet)$ are continuous and we have $\int_0^1 \|\varphi(t)\|^2 dt < \infty$ a.s. for the process φ in (1.15). Because

$$\int_0^t \pi^*(s)\sigma(s)dW(s) = Z^{\varphi}(t) - 1; \quad 0 \le t \le 1,$$

(1.15) implies that (1.17) holds with x = 1 and $\Lambda = Z(1)$. (1.18) holds with C = 1. Loosely speaking, we may conclude that the existence of arbitrage is the same as the existence of a portfolio which hedges Z(1) starting from initial wealth 1 in an "auxiliary zero drift market."

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[5] The left hand side of (1.17) symbolizes initial wealth plus non-discounted capital gains through time 1. Since the seller is obligated to pay Λ at time 1, not at time 0, it would be inappropriate to discount the capital gains to their present value at time 0 here.

The following theorem is a generalization of Levental and Skorohod's Corollary 3. It gives a similar characterization of the existence of arbitrage in terms of domination at time 1 without integrability constraints applied to θ beyond $\alpha = 1$ a.s. σ is not assumed to be invertible.

Theorem 3. *Assume absence of immediate arbitrage. Arbitrage exists if and only if there exist both an adapted R^d-valued process φ satisfying*

$$P\{\alpha^{\varphi} = 1\} = 1 \text{ and } \lambda\otimes P\{\varphi(t,\omega) \in Ran[\sigma^{*}(t,\omega)]\} = 1$$

and a stopping time $0 \leq \tau \leq 1$ such that processes $Z^{\varphi}(\tau;\, \bullet)$ and $Z(\tau;\, \bullet)$ are not indistinguishable and $P\{Z^{\varphi}(\tau;\, 1) \geq Z(\tau;\, 1)\} = 1$.

Here, the additional condition of primary importance beyond those stated in the characterization of arbitrage in Levental and Skorohod's Corollary 3 is the range requirement, $\varphi(t,\omega) \in Ran[\sigma^{*}(t,\omega)]$ $\lambda\otimes P$ a.s. Recall that in the preceding discussion linking Corollary 3 to the concept of hedging, the portfolio which hedges $Z(1)$ in the "auxiliary zero drift market" is a linear function of the process φ for which $P\{Z^{\varphi}(1) \geq Z(1)\} = 1$. Therefore, this additional range condition suggests that research into the problem of "hedging with constrained portfolios" may be useful for this text's arbitrage problem. In their 1993 paper, Cvitanic and Karatzas give a control theoretic characterization of the minimal initial wealth level required for a seller to hedge a contingent claim under the constraint that the hedging portfolio must take values for all (t,ω) in a fixed convex subset of R^d. The range condition in Theorem 3 places a similar constraint on process φ, although the convex set varies with (t,ω). In fact, dependence on

(t,ω) of the set $\mathrm{Ran}[\sigma^*(t,\omega)]$ is not an insurmountable obstacle to application of the work of Cvitanic and Karatzas. This dissertation's primary accomplishment lies in bridging their results and Levental and Skorohod's approach to the arbitrage problem.

To incorporate Cvitanic and Karatzas' theory of hedging with constrained portfolios into a solution of the arbitrage problem, we need to introduce various objects. Let D denote the class of all adapted R^d-valued processes $\nu(\bullet)$ satisfying (1.19), (1.20) and (1.21):

(1.19) $$\lambda \otimes P\{\nu(t,\omega) \in \mathrm{Ker}[\sigma(t,\omega)]\} = 1.$$

(1.20) $$P\left\{ \int_0^1 \|\nu(s)\|^2 ds \quad < \infty \right\} \quad = 1.$$

(1.21) $$E(Z^\nu(1)) = 1.$$

For each $\nu \in D$, let P^ν denote the probability measure on (Ω, F) with Radon-Nikodym derivative $dP^\nu/dP = Z^\nu(1)$. Let E^ν *($E^\nu[\bullet \mid F_t]$)* denote expectation (*conditional expectation "given F_t"*) with respect to P^ν.

For each stopping time τ, define for each $t \in [0, 1]$ random variable $V_0(\tau; t)$ to be a version[6]

(1.22) $$V_0(\tau;t) \quad = \quad \operatorname*{ess\,sup}_{\nu \in D} E^\nu\left[Z(\tau;1) \middle| F_t\right]$$

[6] By "ess sup" it is meant the following: If $\{X_i : i \in I\}$ is a collection of random variable measurable with respect to a σ-field G, where the index set I is of arbitrary cardinality, then there exists an a.s. unique G measurable extended random variable Y taking values in $(-\infty, \infty]$ which satisfies the following two conditions:

(i) For each $i \in I$, $Y \geq X_i$ a.s.

(ii) If Y' is a G measurable extended random variable satisfying (i), then $Y' \geq Y$ a.s

Denote $Y = \operatorname{ess\,sup}\{X_i : i \in I\}$.

For each τ, the adapted process $V_0(\tau; \bullet)$ admits a cadlag modification $V(\tau; \bullet)$, which by right-continuity is unique up to indistinguishability. Use the abbreviation $V(\bullet)$ to denote the process $V(0; \bullet)$. The processes $V(\tau; \bullet)$ and $Z(\tau; \bullet)$ are the central objects in this work's characterization of arbitrage:

Theorem 4. *Assume absence of immediate arbitrage. Then there is no arbitrage if and only if processes $Z(r; \bullet)$ and $V(r; \bullet)$ are indistinguishable for all constant times $0 \leq r \leq 1$.*

Theorem 5 below is an equivalent formulation of Theorem 4. In a sense, Theorem 4 is a characterization of absence of arbitrage in terms of a stochastic supremum, while Theorem 5 recasts this result in terms of an attained maximum. Theorem 5 is an important tool in addressing the problem of equivalence of absence of arbitrage and the existence of a probability measure $Q << P$ such that the asset prices $S_i(\bullet)$, $i = 1, \ldots, d$ are local martingales with respect to $(\Omega, F, \{F_t\}_{0 \leq t \leq 1}, Q)$. Worthy of note is that the proof of Theorem 5 contrasts with approaches in the literature to the problem of existence of an absolutely continuous local martingale measure in that it does not rely upon functional analysis.

Theorem 5. *Assume absence of immediate arbitrage. There is no arbitrage if and only if for each constant time $0 \leq r \leq 1$ there exists a $\mu \in D$ such that $E(Z^{\theta+\mu}(r; 1)) = 1$.*

In the final chapter, I give examples illustrating the properties of the processes $V(\tau; \bullet)$ and their relationship to the processes $Z(\tau; \bullet)$. Example 1 demonstrates that the equivalent condition for absence of arbitrage in Theorem 4 is not equivalent to $E(Z(r;1)) = 1$ for all constant times r. If there is no immediate arbitrage, then the latter condition implies absence of arbitrage, but Example 1 serves as a counter-example to the reverse implication by exhibiting both $E(Z(1)) < 1$ and no arbitrage. Example 2 shows that we cannot simplify Theorem 4 by reducing the conditions equivalent to absence of arbitrage to the behavior of the processes $V(r; \bullet)$ at time 0: in Example 2, we have $V(0) = 1$ a.s., but $V(\bullet)$ and $Z(\bullet)$ are not indistinguishable. Although the processes $V(\tau; \bullet)$ have cadlag paths, it is not true in general that they have continuous paths. In Example 3, $P\{V(1/2) \neq \lim_{t\uparrow 1/2} V(t)\} > 0$. Example 4 is due to Delbaen and Schachermayer (1998b). It is similar to Example 1, which I constructed before discovering their paper. In Example 4, there exists a $\nu \in D$ such that $V(\bullet)$ and $E^{\nu}[Z(1) \mid F_{\bullet}]$ are indistinguishable.

1.3 The History of the Arbitrage Problem.

Since the late 1970's researchers have actively investigated the question of which properties of possible asset price models correspond to the absence of arbitrage opportunities. Most of the resulting articles have focused in one way or another on the notion of an *equivalent martingale measure,* namely, a probability measure Q equivalent to the measure P such that the discounted asset price processes (denoted $S_i(\bullet)/B(\bullet)$, $i = 1, \ldots, d$ in this text's notation) are martingales on the original filtered probability space with measure Q replacing P. That the existence of such a martingale measure is a sufficient condition for absence of arbitrage in a wide variety of circumstances was established early in research on this topic. For discrete time asset price models defined on a finite probability space with finitely many time values, Harrison and Kreps (1979) showed that the existence of such an equivalent martingale measure is necessary and sufficient for the absence of arbitrage. An early result for continuous time trading models appeared in Harrison and Pliska (1981); therein, the authors show that with a discounted price model that is a cadlag strictly positive semi-martingale, the existence of an equivalent martingale measure implies absence of arbitrage.

The question of whether absence of arbitrage implies the existence of a martingale measure is complex, particularly in the case of continuous time process models. Almost all proofs of the existence of a martingale measure have employed the Hahn-Banach Theorem or one of its corollaries. In the discrete-time case, Harrison and Kreps (1979), and similarly, Harrison and Pliska (1981) employed the separating hyper-plane theorem to generate a linear functional symbolizing a pricing system with which one can construct an equivalent martingale measure. Both of these papers worked with finite probability spaces for the discrete-time problem. Taqqu and Willinger (1987) also established the equivalence of absence of arbitrage and existence of an equivalent martingale measure for a discrete time, finite probability space framework; their proof differed from preceding works in that it used a geometric reformulation of the no arbitrage assumption. Dalang, Morton and Willinger (1990) established the equivalence for discrete time trading in general (non-finite) probability spaces. The equivalence for general probability spaces was subsequently proved using somewhat simpler arguments than those in

Dalang, *et. al.* by Kabanov and Kramkov (1994) and Rogers (1995). Note that the theorems referred to above all pertained to a finite number of trading times.

In discrete-time asset models, the issue for the infinite time horizon case is more complicated. Back and Pliska (1991) provide an example allowing trading in the infinite horizon which does not permit arbitrage, but for which there is no equivalent martingale measure. The notion of "no free lunch," sometimes called "no approximate arbitrage," is a stronger assertion than no arbitrage and becomes relevant here. There are several formulations of "free lunch" in the literature. Early definitions of the existence of "free lunch" require a sequence of random variables, namely, terminal wealth levels for discounted capital gain processes, to converge topologically to a nonnegative random variable that is not a.s. 0. The topologies used to define the convergence vary by paper. [See, for example, Kreps (1981).] Because a sequence of trading strategies which require a trader to risk increasingly large losses, none of which produce probability one positive capital gains, seems undesirable as an approximation to arbitrage, tameness requirements were added to the definition of "free lunch." Schachermayer (1994) defines the property of "free lunch with bounded risk" as the existence of a sequence of arbitrage approximants which are each C-tame for a single $C > 0$. He proves that in the infinite time horizon discrete trading problem, "no free lunch with bounded risk" (NFLBR) is equivalent to the existence of an equivalent martingale measure. Furthermore, in the infinite horizon discrete case, the need to prevent "doubling scheme" based arbitrage becomes apparent. Harrison and Kreps (1979) provide an example of probability one positive capital gains where the minimum value of the wealth process across time is not bounded below a.s.

In continuous time trading, "doubling" based a.s. positive capital gains are possible in a model admitting an equivalent martingale measure even with a finite time horizon. Harrison and Pliska (1981) correct for this phenomenon by requiring tameness. Dybvig and Huang (1988) provide a rigorous analysis of the impossibility of a.s. positive discounted capital gains in a market admitting an equivalent martingale measure if one adds the requirement of portfolio tameness. Regarding the problem of the existence of a martingale measure, in the context of a continuous bounded semi-martingale model for the discounted asset prices on time set [0, 1], Delbaen (1992) proved the equivalence of NFLBR and the existence of an equivalent martingale measure. The results of Fritelli and Lakner (1995) include that under only the assumption that the discounted asset price processes are adapted and right continuous, existence of an equivalent martingale measure is necessary and sufficient for absence of "free lunch with stopping times." In their work, the stochastic processes are defined on an arbitrary index subset of $[0, \infty)$, and "free lunch with stopping times" is defined as a sequence of arbitrage approximants for which the portfolios processes lie in the linear span of

$$\{\pi(t) = g\{\tau < t \leq \beta\};\ \tau \leq \beta \text{ are stopping times, } g \in L^{\infty}(P),\ g \text{ is } \mathcal{F}_{\tau-} \text{ measurable}\}.$$

Duffie and Huang (1986) and Stricker (1990) study the relationship between "no free lunch" and the existence of an equivalent martingale measure under the assumption that the discounted asset price processes are in L^p, $1 \leq p < \infty$. Duffie and Huang (1986) also prove some interesting results about the relationship between "no free lunch" and the relative sizes of filtrations generated by different agents' information. Delbaen and Schachermayer (1994b) establish that if the discounted asset price process

$\{S(t);\ 0 \le t < \infty\}$ is a bounded real valued semi-martingale, then there is an equivalent martingale measure if and only if S satisfies "no free lunch with vanishing risk" (NFLVR). NFLVR is defined to hold if for any sequence of positive constants δ_n satisfying $\lim_n \delta_n = 0$, each sequence of δ_n-tame portfolios π_n (where portfolios are defined as predictable processes π for which the stochastic integral

$$\left\{ \int_0^s \pi(t)dS(t); \quad 0 \le s < \infty \right\}$$

is well-defined and converges a.s. to a limit as $s \to \infty$) must satisfy $\text{P-lim}_n \int_0^\infty \pi_n(t)dS(t) = 0$. As a corollary, they obtain that if S is a locally bounded semi-martingale, then NFLVR is equivalent to the existence of an equivalent probability measure under which S is a local martingale. This corollary complements Delbaen and Schachermayer (1994a) in which examples are provided showing that for unbounded continuous discounted price processes, NFLBR is not equivalent to the equivalent martingale measure property.

The proofs of Delbaen and Schachermayer rely heavily upon functional analysis. For the model in this thesis with invertible volatility, Levental and Skorohod (1995) prove that an equivalent martingale measure exists if and only if there is "no approximate arbitrage," a condition which means roughly the same thing as NFLVR. Their proof is more probabilistic than that of Delbaen and Schachermayer. Levental and Skorohod (1995) and Delbaen and Schachermayer (1995) both investigate the relationship between the existence of an absolutely continuous measure $Q << P$ under which the discounted asset prices are martingales, and absence of arbitrage (as opposed to absence of "free lunch.") Levental and Skorohod (1995) use the martingale representation theorem to show that in their model, assuming absence of immediate arbitrage, no arbitrage is equivalent to the existence of an absolutely continuous probability measure $Q_r << P$ for each $0 \leq r \leq 1$ under which (with the expression given for the one-dimensional case)

$$\{S(t)/B(t),\ r \leq t < \zeta^{\theta}(r),\ \{F_t\}_{r \leq t \leq 1},\ Q_r\}$$

is a local martingale. Delbaen and Schachermayer (1995) show, referring back to the (1994b) result proven using the Hahn-Banach theorem, that if $\{S(t);\ 0 \leq t < \infty\}$ is a locally bounded semi-martingale , then absence of arbitrage implies the existence of an absolutely continuous probability measure $Q << P$ under which the discounted asset price process is a local martingale. Delbaen and Schachermayer (1998a) consider the case of unbounded asset price processes. Assuming that $\{S(t);\ 0 \leq t < \infty\}$ is a semi-martingale, they prove that NFLVR is equivalent to the existence of a measure Q equivalent to P under which the discounted price process is a martingale transform, *i.e.*,

$\left\{\int_0^s \varphi(t)dM(t);\quad 0 \leq s < \infty\right\}$, where M is an R^d valued martingale, and φ is a predictable M-integrable R_+-valued process.

Chapter 2

Immediate Arbitrage

2.1 Preliminaries.

The following result, due to Shreve, demonstrates why we assume condition (1.8), $\lambda \otimes P\{a(t, \omega) \in \mathrm{Ran}[\sigma(t, \omega)]\} = 1$.

Proposition 1. *If $\lambda \otimes P\{a(t, \omega) \in \mathrm{Ran}[\sigma(t, \omega)]\} < 1$, then an immediate arbitrage exists.*

Proof. For each (t,ω), $R^d = \mathrm{Ker}[\sigma^*(t,\omega)] \oplus \mathrm{Ran}[\sigma(t,\omega)]$, where $\oplus$ denotes orthogonal sum. Define an adapted R^d-valued process a_1 by defining $a_1(t,\omega)$ to be the projection of $a(t,\omega)$ on $\mathrm{Ker}[\sigma^*(t,\omega)]$. Then define another adapted R^d-valued process π by

(2.1) $$\pi(t) = \frac{1}{1+\|a_1(t)\|^2} a_1(t).$$

π is a portfolio. That π meets the integrability constraint in the definition of a portfolio follows from that for each (t,ω)

(2.2) $$\sigma^* \pi \equiv 0 \in R^d \quad \text{and}$$

$$a_1^* a = \|a_1\|^2 \text{ so that } 0 \le \pi^* a < 1.$$

Now define a stopping time τ by

(2.3) $$\tau = \inf\{t > 0:\ \lambda(\{s: a_1(s) \neq 0\} \cap [t, t+\varepsilon]) > 0 \text{ for all } \varepsilon > 0\}.$$

That τ is a stopping time follows from right-continuity of $\{F_t\}$. That (1.8) does not hold implies that $P\{\tau < 1\} > 0$. Furthermore, (2.2) yields that X_π satisfies

$$P\{X_\pi(t) = 0 \text{ for all } t \leq \tau \text{ and } X_\pi(t) > 0 \text{ for all } t > \tau\} = 1.$$

So π is an immediate arbitrage. ■

Let us now attend to some technical details used in the proof of the immediate arbitrage theorem.

Lemma 1. *Let σ be an adapted $d\times d$ matrix valued process. Then there exists a (non-unique) adapted process σ_+ such that for each (t,ω), $\sigma_+(t,\omega)$ is an invertible $d\times d$ matrix and for each $x \in Ran[\sigma(t,\omega)]$ we have both $\sigma_+^{-1}(t,\omega)x \in Ran[\sigma^*(t,\omega)]$ and $\sigma(t,\omega)\sigma_+^{-1}(t,\omega)x = x$.*

Proof. Let $k(t,\omega) = \dim(Ran[\sigma^*(t,\omega)]) = \dim(Ran[\sigma(t,\omega)])$, and let $\{e_j,\ f_j,\ g_j, ; j = 1, \ldots, d\}$be a set of adapted R^d-valued processes such that for each (t,ω), $\{e_1(t,\omega), \ldots, e_k(t,\omega)\}$, $\{f_1(t,\omega), \ldots, f_{d-k}(t,\omega)\}$, and $\{g_1(t,\omega), \ldots, g_{d-k}(t,\omega)\}$ are bases for $Ran[\sigma^*(t,\omega)]$, $(Ran[\sigma^*(t,\omega)])^\perp$, and $(Ran[\sigma(t,\omega)])^\perp$, respectively. Take $\sigma_+(t,\omega)$ to be the matrix representation of the full-rank linear map on R^d defined by

$$\sigma_+(t,\omega)e_j(t,\omega) = \sigma(t,\omega)e_j(t,\omega), \quad j = 1, \ldots, k(t,\omega),$$

$$\sigma_+(t,\omega)f_j(t,\omega) = g_j(t,\omega),\ j = 1, \ldots, d - k(t,\omega).$$

That $Ran[\sigma(t,\omega)] = Ran[\sigma(t,\omega)\sigma^*(t,\omega)]$ justifies that $\sigma_+(t,\omega)$ is invertible. The remaining assertions made about σ_+ are evident from its construction. ■

In this chapter and the next, the Girsanov Theorem will be a useful tool. [See Karatzas and Shreve (1991), Theorem 3.5.1, page 191.] For each adapted R^d-valued process ν satisfying $\int_0^1 \|\nu(t)\|^2 dt < \infty$ a.s, define another adapted R^d-valued-process

$W^\nu = (W_1^\nu, \ldots W_d^\nu)^*$ by

$$(2.4) \qquad W_i^\nu(t) = W_i(t) + \int_0^t \nu_i(s)ds; \quad 1 \le i \le d.$$

If $E(Z^\nu(1)) = 1$, and probability measure P^ν is defined by $dP^\nu/dP = Z^\nu(1)$, then W^ν is a d-dimensional standard Brownian motion on $(\Omega, F, \{F_t\}, P^\nu)$.

1.2 Proof of the Immediate Arbitrage Theorem.

Theorem 1 (Immediate Arbitrage Theorem.) *There is no immediate arbitrage if and only if $P\{\alpha = 1\} = 1$.*

Proof. (Necessity.) Suppose that $P(\alpha < 1) > 0$. Start the construction of an immediate arbitrage by selecting a sequence of constants $r_k \downarrow 0$ such that if stopping times α_k, $k = 0, 1, \ldots$, are defined by $\alpha_k = (\alpha + r_k) \wedge 1$, then

$$(2.5) \quad \sum_{k=1}^{\infty} P\left(\left\{\int_0^1 \{\alpha_k < t \le \alpha_{k-1}\}\left[\|\theta(t)\|^2 \wedge \frac{1}{r_k}\right]dt \le 1\right\} \cap \{\alpha < 1\}\right) < \infty.$$

Get such constants satisfying (2.5) as follows: Let $r_0 = 1$. After selecting *$\{r_i, i = 0, \ldots, k-1\}$ and defining $\{\alpha_i, i = 0, \ldots, k-1\}$ as stipulated above, the divergence*

$$\lim_{r\downarrow 0} \int_0^1 \{\alpha + r < t \le \alpha_{k-1}\}\left[\|\theta(t)\|^2 \wedge \frac{1}{r}\right]dt = \infty \text{ a.s. on } \{\alpha < 1\}$$

allows choice of $0 < r_k < \frac{1}{2} r_{k-1}$ such that

$$(2.6) \qquad P\left(\left\{\int_0^1 \{\alpha + r_k < t < \alpha_{k-1}\}\left[\|\theta(t)\|^2 \wedge \frac{1}{r_k}\right]dt \le 1\right\} \cap \{\alpha < 1\}\right) \le \frac{1}{k^2}.$$

With the sequences $\{\alpha_k\}$ and $\{r_k\}$ define a sequence of stopping times $\{\tau_k; k \ge 1\}$ by

$$(2.7) \qquad \tau_k = \inf\left\{t > 0 : \int_0^t \{\alpha_k < s \le \alpha_{k-1}\}\left[\|\theta(s)\|^2 \wedge \frac{1}{r_k}\right]ds = 1\right\} \wedge \alpha_{k-1}.$$

The Borel-Cantelli Lemma implies that a.s. on $\{\alpha < 1\}$, $\tau_k < \alpha_{k-1}$ for all k sufficiently large.

Now fix $c \in (1, 2)$ and define two adapted scalar processes

$$\beta(t) = \sum_{k=1}^{\infty} \{\alpha_k < t \le \tau_k\} k^{-c} \quad and \quad \gamma(t) = \sum_{k=1}^{\infty} \{\alpha_k < t \le \tau_k\}\left(\|\theta(t)\| \wedge r_k^{-1/2}\right)$$

Then it holds a.s. on $\{\alpha < 1\}$ that

$$(2.8) \quad \lim_{t \downarrow \alpha} \frac{\int_0^t \beta(s)\gamma^2(s)ds}{\left(\int_0^t \beta^2(s)\gamma^2(s)ds\right)^{1/3}} = \lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} k^{-c}}{\left(\sum_{k=n}^{\infty} k^{-2c}\right)^{1/3}} = \lim_{n \to \infty} \frac{(2c-1)^{1/3}}{c-1} n^{(2-c)/3} = \infty.$$

Now define an adapted R^d-valued process ψ by

$$(2.9) \qquad \psi(t) = \begin{cases} \dfrac{\beta(t)\gamma^2(t)}{\|\theta(t)\|^2}\theta(t), & if \quad \theta(t) \ne 0 \\ 0, \quad if \quad \theta(t) = 0. \end{cases}$$

$0 \le \gamma \le \|\theta\|$ implies that a.s.

(2.10) $$\int_0^1 \|\psi(t)\|^2 dt \;\le\; \int_0^1 \beta^2(t)\gamma^2(t)dt \;\le\; \sum_{k=1}^{\infty} k^{-2c} \;<\; \infty.$$

So the stochastic integral in (2.11) and (2.12) below is well defined. If we put

(2.11) $$W_0\left(\int_0^t \|\psi(s)\|^2 ds\right) \;=\; \int_0^t \psi^*(s)dW(s),$$

then W_0 is a standard Brownian Motion in R^1, so that

$$P\{t^{-1/3} W_0(t) \to 0 \text{ as } t \downarrow 0\} = 1.$$

Then, observing that $\psi(t) = 0$ on $\{t < \alpha\}$ and that

$$\lambda(\{\psi(s) \neq 0\} \cap (\alpha, \alpha + \varepsilon]) > 0 \text{ for all } \varepsilon > 0$$

holds a.s. on $\{\alpha < 1\}$, we can conclude that

(2.12) $$\lim_{t \downarrow \alpha} \frac{\int_0^t \psi^*(s)dW(s)}{\left(\int_0^t \|\psi(s)\|^2 ds\right)^{1/3}} = 0 \quad a.s. \quad on \quad \{\alpha < 1\}.$$

Using (2.10) and (2.12) for the inequality, and then (2.8), proves that a.s. on $\{\alpha < 1\}$

(2.13) $$\lim_{t \downarrow \alpha} \frac{\int_0^t \psi^*(s)dW(s) + \int_0^t \psi^*(s)\theta(s)ds}{\left(\int_0^t \|\psi(s)\|^2 ds\right)^{1/3}} \;\ge\; \lim_{t \downarrow \alpha} \frac{\int_0^t \beta(s)\gamma^2(s)ds}{\left(\int_0^t \beta^2(s)\gamma^2(s)ds\right)^{1/3}} \;=\; \infty.$$

Let τ be the stopping time

(2.14) $$\tau \;=\; \inf\left\{t > \alpha : \; \int_0^t \psi^*(s)dW(s) \;+\; \int_0^t \psi^*(s)\theta(s)ds \;=\; \left(\int_0^t \|\psi(s)\|^2 ds\right)^{1/3}\right\}.$$

Expression (2.13) implies that a.s. on $\{\alpha < 1\}$, both $\tau > \alpha$ and if $\alpha < t \le \tau$, then

$$(2.15) \qquad \int_0^t \psi^*(s)dW(s) + \int_0^t \psi^*(s)\theta(s)ds \ge \left(\int_0^t \|\psi(s)\|^2 ds\right)^{1/3} > 0.$$

Now observe that for an adapted scalar process g, $\psi = g\theta$, so that for all (t,ω), $\psi(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$. Therefore Lemma 1 implies that there exists an adapted process $(\sigma^*)_+$ taking values in the invertible d×d matrices such that $\sigma^*(\bullet)[(\sigma^*)_+(\bullet)]^{-1}\psi(\bullet) = \psi(\bullet)$. Define an adapted vector process π_0 by $\pi_0(t) = [(\sigma^*)_+(t)]^{-1}\psi(t)$. Then define the immediate arbitrage π by

$$\pi(t) = B(t)\{t \le \tau\}\pi_0(t).$$

Regarding integrability constraints on a portfolio, the paths $B(\bullet)$ are bounded in t a.s., and

$$(2.16) \qquad \int_0^1 \left(\|B^{-1}(t)\sigma^*(t)\pi(t)\|^2 + |B^{-1}(t)\pi^*(t)a(t)|\right)dt = \int_0^1 \{t \le \tau\}\left(\|\psi(t)\|^2 + |\psi^*(t)\theta(t)|\right)dt$$

$$\le \int_0^1 \left(\beta^2(t) + \beta(t)\right)\gamma^2(t)dt \le \sum_{k=1}^{\infty}\left(k^{-2c} + k^{-c}\right) \text{ a.s.}$$

Since $B^{-1}(\bullet)\pi^*(\bullet)\sigma(\bullet) = \{\bullet \le \tau\}\psi^*(\bullet)$, and X_π satisfies the SDE

$$(2.17) \qquad dX_\pi(t) = B^{-1}(t)\sigma(t)dW(t) + B^{-1}(t)\sigma(t)\theta(t)dt, \quad X_\pi(0) = 0,$$

it follows from (2.15) that a.s.

$$(2.18) \qquad X_\pi(t) = 0, \text{ if } t \le \alpha \quad \text{and} \quad X_\pi(t) > 0, \text{ if } t > \alpha.$$

So π is an immediate arbitrage with α serving as the stopping time required in the definition.

(Sufficiency.) Suppose $\alpha = 1$ a.s. Let $\pi(\bullet)$ be a portfolio such that there exists a stopping time τ for which

$$P\{X_\pi(t) = 0 \text{ for all } 0 \le t \le \tau \text{ and } X_\pi(t) > 0 \text{ for all } t > \tau\} = 1.$$

Define another stopping time β by

$$\beta = \inf\left\{ t > 0 : \int_0^t \{\tau < s\} \|\theta(s)\|^2 ds \ge 1 \right\}.$$

Note that $\alpha = 1$ a.s. implies that $\tau < \beta$ a.s. on $\{\tau < 1\}$. We also have $Z(\tau; \beta) > 0$ a.s. Further observe that the Novikov Condition implies $E(Z(\tau; \beta)) = 1$ [See Karatzas and Shreve (1991), Corollary 3.5.13, page 199.] Define an adapted vector process $\hat{\theta}$ by

$$\hat{\theta}(t) = \theta(t)\{\tau < t \le \beta\}.$$

Then the process $Z^{\hat{\theta}}(\bullet)$ is $Z(\tau; \bullet \wedge \beta)$, so that $E(Z^{\hat{\theta}}(1)) = 1$. Because $P\{X_\pi(t) = 0 \text{ for } t \le \tau\} = 1$ and X_π satisfies the SDE (2.17), the Girsanov Theorem implies that on $(\Omega, F_1, \{F_t\}, P^{\hat{\theta}})$, the process $X_\pi(\bullet \wedge \beta)$ is a stochastic integral with respect to $W^{\hat{\theta}}$, a standard Brownian Motion in R^d [see (2.4)]. In particular, $X_\pi(\bullet \wedge \beta)$ is a $(P^{\hat{\theta}})$ local martingale. Then, since $P^{\hat{\theta}}$ is equivalent to P, we have

(2.19) $$P^{\hat{\theta}}\{X_\pi(t) = 0 \text{ for all } 0 \le t \le \tau \text{ and } X_\pi(t) > 0 \text{ for all } t > \tau\} = 1.$$

Therefore, $X_\pi(\bullet \wedge \beta)$ is a $(P^{\hat{\theta}})$ supermartingale.[1] Then $E^{\hat{\theta}}(X_\pi(\beta)) \le E^{\hat{\theta}}(X_\pi(\tau)) = 0$, so that $P^{\hat{\theta}}(X_\pi(\beta) = 0) = 1$ by (2.19). By probability equivalence, $P(X_\pi(\beta) = 0) = 1$.

[1] It follows from the Fatou Lemma for conditional expectation that any local martingale which is bounded below is also a supermartingale.

So $\tau < \beta$ a.s. on $\{\tau < 1\}$ and $P\{X_\pi(t) > 0 \text{ for all } t > \tau\} = 1$ imply that $P\{\tau = 1\} = 1$. Since π and τ were chosen arbitrarily, no immediate arbitrage exists.

■

Chapter 3

A Construction from the Theory of Constrained Hedging

3.1 The Construction and Closely Related Properties.

Throughout Chapters 3 and 4, assume absence of immediate arbitrage, *i.e.*, assume $\alpha = 1$ a.s. The results in this chapter specialize work of Cvitanic and Karatzas (1992) and (1993) to the problem of this dissertation. This chapter contains little that Cvitanic and Karatzas did not prove. The usefulness of Corollary 1 is specific to this arbitrage problem; so it did not appear in their work. All proofs, barring that of the existence of the cadlag modification of V_0 in Theorem 2 draw at least their key probabalistic content from Cvitanic and Karatzas. They give a different proof, which also appears in El Karoui and Quenez (1995), justifying the existence of the cadlag modification, but to the best of my knowledge the one given here has not appeared elsewhere. Their papers constructed the V process for the problem of pricing contingent claims in incomplete markets, and did not recognize its usefuleness for the arbitrage problem. Comparison with their papers will show that my approach is to view $Z(\tau; 1)$ as a contingent claim to be hedged in a market with d risky assets characterized by our original volatility process σ and drift process identically zero.

The results and proofs of this chapter will use the notation $V(\bullet)$ and $Z(\bullet)$ otherwise reserved for $\tau = 0$. All of the proofs go through without alteration for $V(\tau;\bullet)$ given any stopping time $0 \le \tau \le 1$. I have chosen the simpler notation since nothing herein depends upon τ.

Recall that we define D to be the class of adapted R^d-valued processes ν for which

(1.19) $$\lambda \otimes P\,\{\nu(t,\omega) \in \mathrm{Ker}[\,\sigma(t,\omega)\,]\} = 1,$$

(1.20) $$P\left\{\int_0^1 \|\nu(s)\|^2 ds \quad < \infty\right\} \quad = 1,$$

(1.21) $$E(Z^{\nu}(1)) = 1.$$

For each $\nu \in D$, E^{ν} denotes expectation and conditional expectation with respect to the probability measure P^{ν} with Radon-Nikodym derivative $dP^{\nu}/dP = Z^{\nu}(1)$.

Let V_0 be an adapted R^1-valued process such that for each $0 \le t \le 1$, $V_0(t)$ is a version of $\text{ess sup}_{\nu\in D}(E^{\nu}[Z(1)|\, F_t])$. In working with essential suprema, the following lemma will be of use.

Lemma 2. *For each stopping time $0 \le \tau \le 1$, there exists a sequence $\{\nu_n;\, n \ge 1\}$ in D for which we have the a.s. monotone convergence*

(3.1) $$\text{ess sup}_{\nu\in D}(E^{\nu}[Z(1)|\, F_\tau]) = \lim\uparrow_n E^{\nu_n}[Z(1)|\, F_\tau].$$

Proof. If a family of random variables is *directed upward*, then the essential supremum of the family is the a.s. increasing limit of a sequence in the family [see

Neveu (1975), Proposition VI-1-1, page 121.][1] So fix τ and show that

$\{E^{\nu}[Z(1) \mid F_\tau]: \nu \in D\}$ is directed upward. Let ν_1 and ν_2 be in D and define

$$A = \left\{E^{\nu_1}\left[Z(1)\middle|F_\tau\right] \geq E^{\nu_2}\left[Z(1)\middle|F_\tau\right]\right\} \in F_\tau .$$

Then define an adapted process μ by

$$\mu(t) = \nu_1(t)[A \cap \{t > \tau\}] + \nu_2(t)[A^c \cap \{t > \tau\}].$$

$\mu \in D$: μ obviously satisfies (1.19) and (1.20). For (1.21), because $A \in F_\tau$,

$$(3.2) \qquad E\left(Z^{\mu}(1)\right) = E\left(AE\left[Z^{\nu_1}(\tau;1)\middle|F_\tau\right]\right) + E\left(A^c E\left[Z^{\nu_2}(\tau;1)\middle|F_\tau\right]\right)$$

Then, for j = 1, 2, using that (1.20) implies $P\{\min_{0 \leq t \leq 1} Z^{\nu j}(t) > 0\} = 1$, and optionally

stopping martingale $Z^{\nu j}$,

$$(3.3) \quad E\left[Z^{\nu_j}(\tau;1)\middle|F_\tau\right] = E\left[\frac{Z^{\nu_j}(1)}{Z^{\nu_j}(\tau)}\middle|F_\tau\right] = \frac{1}{Z^{\nu_j}(\tau)}E\left[Z^{\nu_j}(1)\middle|F_\tau\right] = 1.$$

So $E(Z^{\mu}(1)) = P(A) + P(A^c) = 1$. Since for each $\nu \in D$,

$$(3.4) \qquad E^{\nu}[Z(1) \mid F_\tau] = E[Z(1)\, Z^{\nu}(\tau;1) \mid F_\tau],$$

$A \in F_\tau$ implies

$$(3.5) \qquad E^{\mu}[Z(1) \mid F_\tau] = AE^{\nu_1}[Z(1) \mid F_\tau] + A^c E^{\nu_2}[Z(1) \mid F_\tau]$$

$$\geq E^{\nu_1}[Z(1) \mid F_\tau] \vee E^{\nu_2}[Z(1) \mid F_\tau] \text{ a.s.}$$

■

[1] A family H of random variables is said to be directed upward if for each $h_1, h_2 \in H$, there exists an $h_3 \in H$ with $h_3 \geq h_1 \vee h_2$ a.s.

Theorem 2. *(i) For each* $\nu \in D$, V_0 *is a* (P^ν) *supermartingale.*

(ii) V_0 *admits a (P) modification with cadlag paths.*

(iii) Denote the cadlag modification, unique up to indistinguishability, guaranteed by (ii) as V. *If* $\hat{V}$ *is another process with cadlag paths such that* $\hat{V}$ *is a* (P^ν) *supermartingale for each* $\nu \in D$ *and* $\hat{V}(1) = Z(1)$ *a.s., then*

$$P\{V(t) \leq \hat{V}(t) \text{ for all } 0 \leq t \leq 1\} = 1.$$

Proof. The (P^ν) supermartingale property for V_0 for each $\nu \in D$ will follow immediately from the following identity, which holds for all $0 \leq r \leq s \leq 1$.

$$V_0(r) = \text{ess sup}_{\nu\in D}(E^\nu[V_0(s) \mid F_r]). \tag{3.6}$$

In (3.6), "$\leq$" stems merely from monotonicity of conditional expectation:

$$\begin{aligned} V_0(r) &= \text{ess sup}_{\nu\in D} E^\nu[Z(1) \mid F_r] = \text{ess sup}_{\nu\in D} E^\nu[E^\nu[Z(1) \mid F_s] \mid F_r] \\ &\leq \text{ess sup}_{\nu\in D} E^\nu[V_0(s) \mid F_r]. \end{aligned} \tag{3.7}$$

For "$\geq$", fix $\mu \in D$, and prove $V_0(r) \geq E^\mu[V_0(s) \mid F_r]$. We know by Lemma 2 that there exists a sequence $\{\nu_n\} \subseteq D$ such that $V_0(s) = \lim\uparrow_n E^{\nu_n}[Z(1) \mid F_s]$. For each n define an adapted process μ_n by

$$\mu_n(t) = \mu(t)\{t \leq s\} + \nu_n(t)\{t > s\}.$$

Each $\mu_n \in D$ since, with a calculation like (3.3) justifying the third equality in (3.8),

$$E(Z^{\mu_n}(1)) = E(E[Z^{\mu_n}(1) \mid F_s]) = E(Z^\mu(s)E[Z^{\nu_n}(s;1) \mid F_s]) = E(Z^\mu(s)) = 1. \tag{3.8}$$

Observe that for each n

(3.9) $E^{\nu_n}[Z(1) \mid F_s] = E[Z(1)Z^{\nu_n}(s;1) \mid F_s] = E[Z(1)Z^{\mu_n}(s;1) \mid F_s] = E^{\mu_n}[Z(1) \mid F_s]$,

so that $V_0(s) = \lim\uparrow_n E^{\mu_n}[Z(1) \mid F_s]$. Then that $V_0(r) \geq E^{\mu}[V_0(s) \mid F_r]$ follows from $V_0(r) \geq \limsup_n E^{\mu_n}[Z(1) \mid F_r]$ and (3.10): [2]

(3.10) $E^{\mu_n}[Z(1) \mid F_r] = E^{\mu_n}[E^{\mu_n}[Z(1) \mid F_s] \mid F_r] = E[E^{\mu_n}[Z(1) \mid F_s]\, Z^{\mu}(r;s) \mid F_r]$ and

$$\lim\uparrow_n E[E^{\mu_n}[Z(1) \mid F_s]\, Z^{\mu}(r;s) \mid F_r] = E[V_0(s)Z^{\mu}(r;s) \mid F_r] = E^{\mu}[V_0(s) \mid F_r] \text{ a.s.}$$

So (3.6) holds, and V_0 is a (P^{ν}) supermartingale for each $\nu \in D$.

The filtration $\{F_t\}$ is right-continuous and complete. Therefore, to see that V_0 admits a modification with cadlag paths, it suffices to prove the equivalent condition that $t \to E(V_0(t))$ is a right continuous function [See Lipster and Shiryayev (1977), Theorem 3.1, page 55]. Fix t, and suppose $t_n > t$, $n = 1, 2, \ldots$, satisfy $t = \lim\downarrow_n t_n$. For each ν in D, it holds for all n that $V_0(t_n) \geq E^{\nu}[Z(1) \mid F_{t_n}]$ a.s. Also, for each $\nu \in D$, since (1.20) implies that $\min_{0 \leq t \leq 1}(Z^{\nu}(1)) > 0$ a.s., we have

$$E^{\nu}[Z(1) \mid F_\bullet] = \frac{E[Z(1)Z^{\nu}(1) \mid F_\bullet]}{Z^{\nu}(\bullet)}. \tag{3.11}$$

Then the paths $E^{\nu}[Z(1) \mid F_\bullet]$ are continuous a.s., since $E[Z(1)Z^{\nu}(1) \mid F_\bullet]$ is a (P) martingale with respect to the Brownian filtration $\{F_t\}$. Therefore, for each $\nu \in D$,

[2] In the second and final equalities in (3.10) we use the following consequence of that for each $\nu \in D$, Z^{ν} is a (P) martingale. If $\tau \leq \beta$ are stopping times and Y is an F_β-measurable random variable, then for any $\nu \in D$ such that Y is (P^{ν}) integrable,

$$E^{\nu}[Y \mid F_\tau] = E[Y\, Z^{\nu}(\tau;\beta) \mid F_\tau].$$

$\liminf_n V_0(t_n) \geq E^{\nu}[Z(1) \mid F_t]$, implying $\liminf_n V_0(t_n) \geq V_0(t)$ a.s. Then by the Fatou Lemma,

$$(3.12) \qquad \liminf_n E(V_0(t_n)) \geq E(\liminf_n V_0(t_n)) \geq E(V_0(t)).$$

Since V_0 is a (P) supermartingale by (i), $E(V_0(t_n)) \leq E(V_0(t))$ for each n. Therefore (3.12) implies that $\lim_n E(V_0(t_n)) = E(V_0(t))$. So we have the equivalent condition, and the process denoted by V in (iii) exists.

For (iii), suppose $\hat{V}$ is an adapted process with (a.s.) cadlag paths such that $\hat{V}(1) = Z(1)$ a.s. Suppose that there exists a $q \in Q = \{\text{rational } q: 0 \leq q \leq 1\}$ such that $P\{\hat{V}(q) < V(q)\} > 0$. (By right continuity, $\{\hat{V}(t) - V(t) \geq 0 \text{ for all } 0 \leq t \leq 1\}^c$ equals $\bigcup_{q \in Q}\{\hat{V}(q) - V(q) < 0\}$ modulo null sets.) Apply Lemma 2 to obtain the monotone convergence $V(q) = \lim\uparrow_n E^{\nu_n}[Z(1) \mid F_q]$. Then, for n sufficiently large, we see that $\hat{V}$ cannot be a (P^{ν_n}) supermartingale from equivalence of P and P^{ν_n} and both $\hat{V}(1) = Z(1)$ (P) a.s. and $P\{\hat{V}(q) < E^{\nu_n}[Z(1) \mid F_q]\} > 0$. ■

Corollary 1. *$P\{V(t) \leq Z(t)$ for all $0 \leq t \leq 1\} = 1$.*

Proof. The result will follow from (iii) of Theorem 2 once we show that $Z(\bullet)$ is a (P^{ν}) supermartingale for each $\nu \in D$. For each $\nu \in D$, any adapted process $Y(\bullet)$ is a (P^{ν}) supermartingale if and only if $Y(\bullet)Z^{\nu}(\bullet)$ is a (P) supermartingale. If $\nu \in D$, we have

$$\lambda \otimes P\{\theta(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)], \nu(t,\omega) \in (\mathrm{Ran}[\sigma^*(t,\omega)])^{\perp}\} = 1,$$

so that $Z(\bullet)Z^{\nu}(\bullet)$ is (P) supermartingale $Z^{\theta+\nu}(\bullet)$. ■

3.2 An Essential Lemma in the Link to the Arbitrage Problem.

Lemma 3 below will be essential in constructing an arbitrage in the proof of the final theorem in Chapter 4. In the proof of Lemma 3, it is necessary to use the Doob-Meyer Decomposition of a cadlag supermartingale. Below is the formulation of the Doob-Meyer Decomposition Theorem given in Kopp (1984) [Theorem 3.8.10, page 122].

Theorem (Doob-Meyer Decomposition.) *Let X be a right-continuous supermartingale. Then X has a unique decomposition $X = M - A$, where M is a local martingale and A is a predictable increasing process.*

The conclusion of Theorem 2 that for each $\nu \in D$, V is a (P^ν) supermartingale with cadlag paths implies that for each $\nu \in D$ we have the unique (P^ν) Doob-Meyer Decomposition $V = L^\nu - A^\nu$. (Uniqueness is up to (P^ν) indistinguishability, which is the same as (P) indistinguishability by equivalence of P and P^ν.) Each L^ν is a (P^ν) local martingale, and each A^ν is an adapted process with cadlag and non-decreasing paths satisfying $A^\nu(0) = 0$.

Lemma 3. *Let L denote the (P) local martingale in the (P) Doob-Meyer decomposition of V. Then $L(\bullet) = V(0)Z^\varphi(\bullet)$ for an adapted R^d-valued process φ satisfying*

$$\lambda \otimes P\{\varphi(t,\omega) \in Ran[\sigma^*(t,\omega)]\} = 1. \tag{3.13}$$

Proof. Because L is a nonnegative Brownian local martingale satisfying L(0) = V(0), $L(\bullet) = V(0)Z^\varphi(\bullet)$ for an adapted R^d-valued φ satisfying $\alpha^\varphi = 1$ a.s. There exists such a φ satisfying that if $\zeta(L) = \inf\{t > 0:\ L(t) = 0\}$, then for each (t, ω), $t \geq \zeta(L)_\omega$ implies

$\varphi(t,\omega) = 0$. [See Lipster and Shiryayev (1977), Lemma 6.2, page 208.] (3.13) will follow once we prove

(3.14) For each $\nu \in D$, $\lambda \otimes P\{\varphi^*(t,\omega)\nu(t,\omega) \le 0\} = 1$.

To show that (3.14) implies (3.13), suppose that (3.13) does not hold. Then define the process μ_0 by putting $\mu_0(t,\omega)$ the projection of $\varphi(t,\omega)$ on $\mathrm{Ker}[\sigma(t,\omega)] = (\mathrm{Ran}[\sigma^*(t,\omega)])^{\perp}$. Then we have that $\lambda \otimes P\{\mu_0(t,\omega) \neq 0\} > 0$. Let process μ be given by

$$\mu(t) = \frac{1}{\|\mu_0(t)\|}\mu_0(t), \text{ if } \mu_0(t) \neq 0; \qquad \mu(t) = 0 \text{ if } \mu_0(t) = 0.$$

The Novikov Criterion implies that $\mu \in D$ because μ is bounded uniformly in (t,ω). (3.14) does not hold for this μ, since $\varphi^*\mu = \|\mu_0\|$.

To prove (3.14), consider that for each $\nu \in D$, where (P^ν) Brownian Motion W^ν is as defined in (2.4), it is possible to write the (P^ν) Doob-Meyer Decomposition of V as

$$(3.15) \qquad V(t) = V(0) + \int_0^t f^{\nu^*}(s)dW^\nu(s) - A^\nu(t), \quad 0 \le t \le 1,$$

where f^ν is an adapted R^d-valued process such that $\int_0^1 \|f^\nu(t)\|^2 dt < \infty$ a.s. With respect to (3.15) note that for each $\nu \in D$, L^ν is adapted to $\{F_t\}$, and not necessarily adapted the P-augmentation of the natural filtration of W^ν. If this latter filtration is denoted $\{F_t^\nu; 0 \le t \le 1\}$, then it is possible that there exists an adapted ν such that there exist t such that F_t^ν are strict subsets of the corresponding F_t. Therefore, it is not possible to directly apply the Martingale Representation Theorem to obtain the stochastic integral representation of $L^{\nu}(\bullet)$ in (3.15). If $V(0) = 0$, then because $V(\bullet)$ is a nonnegative (P) supermartingale, $V(\bullet)$ is indistinguishable from the zero process, and clearly we may take f^{ν} to be the zero vector process for each $\nu \in D$.[3] If $V(0) > 0$, then for each fixed $\nu \in D$, to construct the process f^{ν} start from that $L^{\nu}(\bullet)$ is a (P^{ν}) local martingale is equivalent to that $L^{\nu}(\bullet)Z^{\nu}(\bullet)$ is a (P) local martingale. Then since $L^{\nu}(\bullet)Z^{\nu}(\bullet)$ is a nonnegative process, there exists an adapted R^d-valued process $g = g^{\nu}$ satisfying $\alpha^{g} = 1$ a.s. such that $L^{\nu}(\bullet)Z^{\nu}(\bullet) = V(0)Z^{g}(\bullet)$. Then the Itô formula gives that

$$(3.16)\qquad dL^{\nu}(t) = d\left(\frac{V(0)Z^{g}(t)}{Z^{\nu}(t)}\right) = \frac{V(0)}{Z^{\nu}(t)}dZ^{g}(t) - \frac{V(0)Z^{g}(t)}{\left(Z^{\nu}(t)\right)^{2}}dZ^{\nu}(t)$$

$$- \frac{V(0)}{\left(Z^{\nu}(t)\right)^{2}}d\left[Z^{g}(\bullet),Z^{\nu}(\bullet)\right](t) + \frac{V(0)Z^{g}(t)}{\left(Z^{\nu}(t)\right)^{3}}d\left[Z^{\nu}(\bullet),Z^{\nu}(\bullet)\right](t)$$

$$= \frac{V(0)Z^{g}(t)}{Z^{\nu}(t)}\left\{ -g^{*}(t)dW(t) + \nu^{*}(t)dW(t) - g^{*}(t)\nu(t)dt + \|\nu\|^{2}(t)dt \right\}$$

$$= L^{\nu}(t)\left(\nu^{*}(t) - g^{*}(t)\right)\left(dW(t) + \nu(t)dt\right).$$

So we have equation (3.15) with $f^{\nu}(\bullet) = L^{\nu}(\bullet)[\nu(\bullet) - g(\bullet)]$.

Then because $dL(t) = -L(t)\varphi(t)dW(t)$, $0 \le t \le 1$, it follows from uniqueness of the (P) Doob-Meyer Decomposition and equivalence of the (P^{ν}) that for each $\nu \in D$

$$(3.17)\qquad \lambda\otimes P\{f^{\nu}(t,\omega) = -L(t,\omega)\varphi(t,\omega)\} = 1.$$

and consequently, up to indistinguishability,

---

[3] If $Y(t)$, $0 \le t \le 1$, a process with cadlag and non-negative paths, is an $(\Omega, F, \{F_t\}, P)$ supermartingale, and $S = \inf\{t > 0: Y(t) = 0\}$, then $P\{Y(t) = 0 \text{ for all } S < t \le 1\} = 1$. [See Elliott (1982), Theorem 4.16, page 38.]

$$A^{\nu}(t) = A(t) - \int_0^t L(s)\varphi^*(s)\nu(s)ds; \quad 0 \le t \le 1. \tag{3.18}$$

Equation (3.18) motivates the completion of the argument. Fix $\nu \in D$. Then define for each n a process $\mu_n \in D$, with $E(Z^{\mu_n}(1)) = 1$ following from uniform boundedness of μ_n in (t,ω):

$$\mu_n(t) = n\frac{\{\varphi^*(t)\nu(t) > 0\}}{1+\|\nu(t)\|}\nu(t) + \frac{\{\varphi^*(t)\nu(t) \le 0\}}{1+\|\nu(t)\|}\nu(t). \tag{3.19}$$

Then, by (3.18),

$$A^{\mu_n}(1) = A(1) - n\int_0^1 \frac{\{\varphi^*(t)\nu(t) > 0\}}{1+\|\nu(t)\|}L(t)\varphi^*(t)\nu(t)dt \tag{3.20}$$

$$- \int_0^1 \frac{\{\varphi^*(t)\nu(t) \le 0\}}{1+\|\nu(t)\|}L(t)\varphi^*(t)\nu(t)dt$$

We must have $\lambda\otimes P\{\varphi^*(t,\omega)\nu(t,\omega) \le 0\} = 1$ because for $\omega \in \{\lambda\{t: \varphi^*(t)\nu(t) > 0\} > 0\}$ the right-hand side of (3.20) tends to $-\infty$ as $n \to \infty$. (Recall that for each (t,ω), $L(t,\omega) = 0$ implies $\varphi(t,\omega) = 0 \in R^d$.) Such divergence would contradict that for each n the paths $A^{\mu_n}(\bullet)$ are non-decreasing.

■

3.3 Two Useful Characterizations of the Process V(•).

It will be useful that the following result holds when we stop process V. Karatzas and Shreve (1998) give a different presentation of this proof which is based upon a similar underlying approach [see Remark 5.6.7, page 215.]

Lemma 4. *For each stopping time* $0 \le \tau \le 1$,

$$V(\tau) = ess\ sup_{\nu \in D}(E^{\nu}[Z(1) \mid F_{\tau}]). \tag{3.21}$$

Proof. For each τ, denote the right-hand side of (3.21) by $Y(\tau)$. First assume that τ is a simple stopping time, $\tau = \sum_{1 \le k \le n} \{\tau = t_k\} t_k$. Then for each $\nu \in D$,

$$E^{\nu}\left[Z(1)\middle|F_{\tau}\right] = \sum_{k=1}^{n}\{\tau = t_k\}E^{\nu}\left[Z(1)\middle|F_{t_k}\right]. \tag{3.22}$$

(The right-hand side of (3.22) is F_{τ}-measurable because $\{\tau = t_k\} \cap A \in F_{\tau}$ for any $A \in F_{t_k}$. Equality of integrals on F_{τ} sets follows from that $\{\tau = t_k\} \cap A \in F_{t_k}$ for any $A \in F_{\tau}$.) Evident from (3.22) is that $Y(\tau) \le \sum_{1 \le k \le n}\{\tau = t_k\}V(t_k) = V(\tau)$.

For the reverse inequality, choose by Lemma 2 sequences $\{\nu_{k,m};\ m \ge 1\} \subseteq D$ for $k = 1, \ldots, n$ such that for each k, $\lim_m \uparrow E^{\nu k,m}[Z(1) \mid F_{t_k}] = V(t_k)$ a.s. Then, in light of (3.22), it holds a.s. on $\{\tau = t_k\}$ that

$$Y(\tau) \ge \lim_m \uparrow E^{\nu k,m}[Z(1) \mid F_{\tau}] = V(\tau).$$

Now let τ be an arbitrary stopping time. Stopping times τ_k, $k = 1, 2, \ldots,$ given by

$$\tau_k = \frac{j}{k} \text{ on } \left\{\frac{j-1}{k} < \tau \le \frac{j}{k}\right\}, \text{ for } j = 1, \ldots, k$$

satisfy $\tau_k \geq \tau$ and $\lim_k \downarrow \tau_k = \tau$ a.s. The paths $E^{\nu}[Z(1) \mid F_{\bullet}]$ are continuous for each $\nu \in D$ and the paths $V(\bullet)$ are right-continuous, so that for each $\nu \in D$ it holds a.s. that

(3.23) $$E^{\nu}[Z(1) \mid F_{\tau}] = \lim_k E^{\nu}[Z(1) \mid F_{\tau_k}] \leq \lim_k V(\tau_k) = V(\tau).$$

Therefore, $Y(\tau) \leq V(\tau)$ a.s.

Now take for each $k \geq 1$ a sequence $\{\nu_{k,m}\,;\, m \geq 1\} \subseteq D$ such that

(3.24) $$V(\tau_k) = \lim_m \uparrow E^{\nu_{k,m}}[Z(1) \mid F_{\tau_k}] \text{ a.s.}$$

Define adapted processes $\mu_{k,m}$, $k \geq 1$, $m \geq 1$, by $\mu_{k,m}(t) = \nu_{k,m}(t)\{t > \tau_k\}$. Then each $\mu_{k,m} \in D$. (Calculation (3.3) shows that for any ν in D and stopping time β, $\nu(\bullet)\{\bullet > \beta\}$ is also in D.) For each k, we have (3.24) with the processes $\mu_{k,m}$ replacing the processes $\nu_{k,m}$. Furthermore, for each k,m, with the last equality in (3.25) holding because $Z^{\mu_{k,m}}(\tau; \tau_k) = 1$ a.s,

(3.25) $$Y(\tau) \geq E^{\mu_{k,m}}[Z(1) \mid F_{\tau}] = E^{\mu_{k,m}}[E^{\mu_{k,m}}[Z(1) \mid F_{\tau_k}] \mid F_{\tau}]$$

$$= E[E^{\mu_{k,m}}[Z(1) \mid F_{\tau_k}] \mid F_{\tau}] \text{ a.s.}$$

(Refer to footnote 2 on page 33 for more detail on this calculation.) Letting $m \to \infty$ for fixed k in (3.25) shows that for each k, $Y(\tau) \geq E[\,V(\tau_k) \mid F_{\tau}]$ a.s. Therefore, the Fatou Lemma implies

(3.26) $$E(Y(\tau)) \geq \limsup_k E(E[V(\tau_k) \mid F_{\tau}]) = \lim_k E(V(\tau_k)) \geq E(V(\tau)).$$

Then $Y(\tau) \leq V(\tau)$ a.s. and (3.26) imply $Y(\tau) = V(\tau)$ a.s.

■

Although properties of the processes $Z^{\nu}(\bullet)$ dependent upon $\nu \in D$ were used in proving properties of the process $V(\bullet)$ above, for each t, V(t) is in fact a version of an essential supremum taken over a much wider class than D. Define the class N by

$$N = \{\text{adapted } R^d\text{-valued processes } \nu:\ \alpha^{\nu} = 1 \text{ a.s. and } \lambda\otimes P\{\nu(t,\omega)\in \mathrm{Ker}[\sigma(t,\omega)]\} = 1\}.$$

Lemma 5. *For each stopping time $0 \le \tau \le 1$,*

$$V(\tau) = ess\ sup_{\nu\in N}(E[Z(1)Z^{\nu}(\tau;1)\ /F_{\tau}]). \tag{3.27}$$

Proof. $D \subseteq N$ implies "$\le$". For "$\ge$", fix a stopping time τ and $\nu \in N$. Then define two sequences, one of stopping times β_n and the other of adapted processes ν_n, by

$$\beta_n = \inf\left\{t > 0:\ \int_0^t \{\tau < s\}\|\nu(s)\|^2\, ds \ge n\right\}, \tag{3.28}$$

$$\nu_n(t) = \nu(t)\{\tau < t \le \beta_n\}. \tag{3.29}$$

Each $\nu_n \in D$. Clearly $\lambda\otimes P\{\nu_n(t,\omega) \in \mathrm{Ker}[\sigma(t,\omega)]\} = 1$, and we have for each n that

$\int_0^1 \|\nu_n(t)\|^2 dt \le n$ a.s. The Novikov Criterion implies that $E(Z^{\nu_n}(1)) = 1$.

The paths $Z^{\nu}(\tau;\bullet)$ are continuous a.s., and $Z^{\nu_n}(\tau;1) = Z^{\nu}(\tau;\beta_n)$. Therefore, $\lim_n Z^{\nu_n}(\tau;1) = Z^{\nu}(\tau;\zeta^{\nu}(\tau)) = Z^{\nu}(\tau;1)$ a.s. Conclude through the Fatou Lemma:

$$\begin{aligned} V(\tau) &\ge \limsup_n E^{\nu_n}[Z(1) \mid F_{\tau}] = \limsup_n E[Z(1)Z^{\nu_n}(\tau;1) \mid F_{\tau}] \\ &\ge E[Z(1)Z^{\nu}(\tau;1) \mid F_{\tau}] \text{ a.s.} \end{aligned} \tag{3.30}$$

■

Chapter 4

Characterizations of Arbitrage and Corollaries

4.1 Characterization of Arbitrage in Terms of Domination of $Z(\tau; 1)$.

The following simple fact is presented first to avoid repetitive justification.

Lemma 6. *Let φ be an adapted R^d-valued process satisfying $\alpha^\varphi = 1$ a.s., and let $0 \le \tau \le 1$ be a stopping time satisfying $P\{\tau < 1\} > 0$ such that processes $Z^\varphi(\bullet)$ and $Z(\tau; \bullet)$ are not indistinguishable. Then there exists a constant $0 < \delta < 1$ such that the stopping time β defined by*

$$\beta = \inf\{t > \tau : Z^\varphi(t) \le \delta Z(\tau; t) \text{ and } Z(\tau; t) > 0\}$$

satisfies $P\{\beta < 1\} > 0$.

Proof. Suppose that no such δ exists. Then since the paths $Z^\varphi(\bullet)$ and $Z(\tau; \bullet)$ are nonnegative and continuous,

$$P\{Z^\varphi(t) \ge Z(\tau; t) \text{ for all } \tau \le t \le 1\} = 1. \tag{4.1}$$

In particular, $Z^\varphi(\tau) \ge Z(\tau; \tau) = 1$ a.s. Then that $Z^\varphi(\bullet)$ is a continuous supermartingale with $Z^\varphi(0) = 1$ a.s. implies

$$P\{Z^\varphi(t) = Z(\tau; t) \text{ for all } 0 \le t \le \tau\} = 1.$$

Furthermore, (4.1) implies that that $\zeta^{\varphi}(0) \geq \zeta^{\theta}(\tau)$ a.s. Then there exist localization stopping times η_k with $\lim_k \uparrow \eta_k = \zeta^{\theta}(\tau)$ a.s. such that for each k the process $Z^{\varphi}(\bullet \wedge \eta_k) - Z(\tau; \bullet \wedge \eta_k)$ is a nonnegative martingale. Then $Z^{\varphi}(0) - Z(\tau; 0) = 0$ a.s. implies that each of these localized martingales is indistinguishable from the zero process. Then path continuity implies both that

$$(4.2) \qquad P\{Z^{\varphi}(t) = Z(\tau; t) \text{ for all } 0 \leq t \leq \zeta^{\theta}(\tau)\} = 1$$

and that $Z^{\varphi}(\zeta^{\theta}(\tau)) = 0$ a.s. on $\{\zeta^{\theta}(\tau) < 1\}$. Because $Z^{\varphi}(\bullet)$ is a nonnegative supermartingale, we can replace $\zeta^{\theta}(\tau)$ by 1 in (4.2). So $Z^{\varphi}(\bullet)$ and $Z(\tau; \bullet)$ are indistinguishable. ■

The following theorem is an extension of Corollary 3 of Levental and Skorohod (1995) [page 920]. It serves as the link between the V process detailed in the last section and our arbitrage problem.

Theorem 3. *Assume absence of immediate arbitrage. Arbitrage exists if and only if there exists both an adapted R^d-valued process φ satisfying*

$$\lambda \otimes P\{\varphi(t,\omega) \in Ran[\sigma^*(t,\omega)]\} = 1 \text{ and } P\{\alpha^{\varphi} = 1\} = 1$$

and a stopping time $0 \leq \tau \leq 1$ such that the processes $Z^{\varphi}(\tau; \bullet)$ and $Z(\tau; \bullet)$ are not indistinguishable and $P\{Z^{\varphi}(\tau; 1) \geq Z(\tau; 1)\} = 1$.

Proof. (Sufficiency.) Lemma 6, applied to $\varphi(\bullet)\{\bullet > \tau\}$, implies that there exists a constant $0 < \delta < 1$ such that the stopping time β defined by

$$\beta = \inf\{t > \tau: Z^{\varphi}(\tau; t) \leq \delta Z(\tau; t) \text{ and } Z(\tau; t) > 0\}$$

satisfies $P\{\beta < 1\} > 0$. Assume first that $Z(\tau;\,1) > 0$ a.s. on $\{\beta < 1\}$. Then it also holds a.s. on $\{\beta < 1\}$ that

$$(4.3) \qquad 1 \;\le\; \frac{Z^{\varphi}(\tau;1)}{Z(\tau;1)} \;=\; \left(\frac{Z^{\varphi}(\tau;\beta)}{Z(\tau;\beta)}\right)\left(\frac{Z^{\varphi}(\beta;1)}{Z(\beta;1)}\right) \;=\; \delta\,\frac{Z^{\varphi}(\beta;1)}{Z(\beta;1)}.$$

So $Z^{\varphi}(\beta;1)\,/\,Z(\beta;1) \ge \delta^{-1} > 1$ a.s. on $\{\beta < 1\}$.

It follows from the Itô Formula that on $\{\beta < t < \zeta^{\theta}(\beta)\}$

$$(4.4) \qquad d\left(\frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)}\right) \;=\; \frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)}\Big(\theta^{*}(t) - \varphi^{*}(t)\Big)\Big(dW(t) + \theta(t)dt\Big).$$

(The computational details underlying (4.4) are precisely the same as those given explicitily in calculation (3.16).) Let $(\sigma^{*})_{+}$ denote an adapted process such that for each (t,ω), $(\sigma^{*})_{+}(t,\omega)$ is an invertible $d{\times}d$ matrix and

$$(4.5) \qquad \sigma^{*}(t,\omega)\;(\sigma^{*})_{+}^{-1}(t,\grave{u})\;x \;=x \text{ for all } x \in \mathrm{Ran}[\sigma^{*}(t,\omega)].$$

Lemma 1 guarantees that such an $(\sigma^{*})_{+}$ exists. Then because of the range hypothesis on φ and the construction of θ,

$$(4.6) \qquad \lambda\otimes P\{\,\sigma^{*}(\sigma^{*})_{+}^{-1}(\theta - \varphi) = \theta - \varphi\} = 1.$$

Choose the portfolio π as follows:

$$(4.7) \qquad \pi(t) \;=\; B(t)\frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)}\{\beta < t\}(\sigma^{*})_{+}^{-1}(t)\big(\theta(t) - \varphi(t)\big).$$

We have assumed $Z(\tau;1) > 0$ a.s. on $\{\beta < 1\}$. So $Z^{\varphi}(\tau;\,1) \ge Z(\tau;1) > 0$ a.s. on $\{\beta < 1\}$.

Then since $\beta \ge \tau$,

(4.8) $$\int_0^1 \{\beta < t\}\left(\|\theta(t)\|^2 + \|\varphi(t)\|^2\right)dt \; < \infty \text{ a.s.}$$

The paths $B(\bullet)\{Z^{\varphi}(\beta;\bullet) / Z(\beta;\bullet)\}$ are bounded in t a.s. because it holds a.s. that nonnegative supermartingale $Z(\beta;\bullet)$ cannot hit 0 on $\{Z(\tau;1) > 0\}$. So, in light of (4.8), π satisfies the integrability constraint required of a portfolio. Where random variable M is given by

$$M = \sup_{0\le t\le 1} B(t)\frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)},$$

(4.9) $$\int_0^1 \left(\left\|\sigma^*(t)\pi(t)\right\|^2 + \left|\pi^*(t)a(t)\right|\right)dt \;\le$$

$$\left(M^2+1\right)\int_0^1 \{\beta < t\}\left(\|\theta(t)-\varphi(t)\|^2 \;+\; \left|\left(\theta^*(t)-\varphi^*(t)\right)\theta(t)\right|\right)dt.$$

By construction,

(4.10) $$X_{\pi}(t) \;=\; \int_0^t d\left(\frac{Z^{\varphi}(\beta;s)}{Z(\beta;s)}\right) \;=\; \frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)} \;-\; 1.$$

Observe that π is 1-tame, and that $X_{\pi}(1) = 0$ a.s. on $\{\beta = 1\}$. Then that $Z^{\varphi}(\beta;1) / Z(\beta;1) \ge \delta^{-1} > 1$ a.s. on $\{\beta < 1\}$, where $P\{\beta < 1\} > 0$, completes justification of that π is an arbitrage.

Now drop the assumption that $Z(\tau; 1) > 0$ a.s. on $\{\beta < 1\}$, but treat the situation where $P(\{Z(\tau; 1) = 0\}\cap\{\beta < 1\}) > 0$ under the added assumption that $P\{\zeta^{\varphi}(\beta) \ge \zeta^{\theta}(\beta)\} = 1$. The Itô Formula gives that on $\{\beta < t < \zeta^{\theta}(\beta)\}$,

$$(4.11)\qquad d\left(\frac{1}{Z(\beta;t)}\right) = \frac{1}{Z(\beta;t)}\theta^*(t)\big(dW(t)+\theta(t)dt\big).$$

With constant δ and matrix-valued process $(\sigma^*)_+$ as they were earlier in the proof, define the portfolio π' by

$$(4.12)\qquad \pi'(t) = B(t)\frac{Z^{\varphi}(\beta;t)}{Z(\beta;t)}\{\beta < t \le \gamma\}(\sigma^*)_+^{-1}(t)\big(\theta(t)-\varphi(t)\big)$$

$$+ \; B(t)\frac{K}{Z(\beta;t)}\{\beta < t \le \gamma\}(\sigma^*)_+^{-1}(t)\theta(t),$$

where $K = (\delta^{-1} - 1)/2 > 0$, and stopping time γ is defined

$$\gamma = \inf\{t > 0 : Z(\beta; t) = K/(2+2K)\}.$$

Path-continuity of $Z(\beta; \bullet)$ and $K/(2+2K) < 1$ imply that a.s., $Z(\beta; 1) > 0$ on $\{\gamma = 1\}$ and

$$(4.13)\qquad \gamma < \zeta^{\theta}(\beta) \text{ on } \{Z(\beta; \zeta^{\theta}(\beta)) = 0\} = \left\{\int_0^1 \{\beta < t\}\|\theta(t)\|^2 dt = \infty\right\}.$$

Paths $Z^{\varphi}(\beta; \bullet\wedge\gamma)/Z(\beta; \bullet\wedge\gamma)$ and $K/Z(\beta; \bullet\wedge\gamma)$ are bounded in t a.s. So inequality (4.14) below shows that (4.13) and $P\{\zeta^{\varphi}(\beta) \ge \zeta^{\theta}(\beta)\} = 1$ together imply that π' satisfies the portfolio integrability constraint. With random variable M' defined by

$$M' = \sup_{0\le t\le 1} B(t)\frac{Z^{\varphi}(\beta; t\wedge\gamma) + K}{Z(\beta;t)},$$

(4.14) $$\int_0^1 \left\{ \left\| \sigma^*(t)\pi'(t) \right\|^2 + \left| \pi'^*(t)a(t) \right| \right\} dt \quad \le$$

$$\left(M'^2+1\right)\int_0^1 \{\beta < t \le \gamma\}\left(\left\|\theta(t)-\varphi(t)\right\|^2 + \left|\left(\theta^*(t)-\varphi^*(t)\right)\theta(t)\right|\right)dt \quad +$$

$$2\left(M'^2+1\right)\int_0^1 \{\beta < t \le \gamma\}\left\|\theta(t)\right\|^2 dt\,.$$

(Since $Z(\tau;\beta) > 0$ on $\{\beta < 1\}$ and $Z(\beta;1) > 0$ a.s. on $\{\gamma = 1\}$, $P\{Z^{\varphi}(\tau;1) \ge Z(\tau;1)\} = 1$

implies that $\int_0^1 \{\beta < t\}\left\|\varphi(t)\right\|^2 dt < \infty$ a.s. on $\{\gamma = 1\}$.)

The associated discounted capital gain process satisfies

(4.15) $$X_{\pi'}(t) \;=\; \int_0^t \{\beta < s \le \gamma\} d\left(\frac{Z^{\varphi}(\beta;s)}{Z(\beta;s)} + \frac{K}{Z(\beta;s)} \right)$$

$$= \frac{Z^{\varphi}(\beta;t\wedge\gamma)}{Z(\beta;t\wedge\gamma)} - 1 + \frac{K}{Z(\beta;t\wedge\gamma)} - K.$$

So π' is $(1 + K)$-tame. On $\{\beta = 1\}$, $X_{\pi'}(1) = 0$ a.s. On $\{\beta < 1\}\cap\{\gamma < 1\}$, $X_{\pi'}(1) \ge 1 + K$ a.s.

With regard to $X_{\pi'}(1)$ on $\{\beta < 1\}\cap\{\gamma = 1\}$, recall that we have $Z(\tau;\beta) > 0$ on $\{\beta < 1\}$ and $Z(\beta;1) > 0$ a.s. on $\{\gamma = 1\}$. Then calculation (4.3) shows that

$$Z^{\varphi}(\beta;1)\,/\,Z(\beta;1) \;\ge\; \delta^{-1} \text{ a.s. on } \{\beta < 1\}\cap\{\gamma = 1\},$$

so that $X_{\pi'}(1) \ge \delta^{-1} - 1 - K > 0$ a.s. on this set. So π' is an arbitrage.

All that remains in the sufficiency argument is to produce an arbitrage given $P\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\} > 0$. In this case, define the portfolio π'' by

$$(4.16) \qquad \pi''(t) = \frac{B(t)}{Z(\zeta^\varphi(\beta);t)}\left\{\zeta^\varphi(\beta) < t < \rho, \quad \zeta^\varphi(\beta) < \zeta^\theta(\beta)\right\}(\sigma^*)_+^{-1}(t)\theta(t),$$

where stopping time ρ is defined

$$\rho = \inf\{t > 0 : \ Z(\zeta^\varphi(\beta);\, t) = 1/2\}.$$

If $\zeta^\varphi(\beta) < \zeta^\theta(\beta)$, then $\int_0^1 \{\beta < t\}\|\varphi(t)\|^2 dt = \infty$, so that $Z^\varphi(\beta;\, 1) = 0$ a.s. on $\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\}$.

Since $\tau \le \beta$, we also have $Z^\varphi(\tau;\, 1) = 0$ a.s. on this set. Therefore, $Z^\varphi(\tau;\, 1) \ge Z(\tau;\, 1)$ a.s. implies

$$(4.17) \qquad Z(\tau;\, 1) = 0 \text{ a.s. on } \{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\}.$$

On $\{\zeta^\varphi(\beta) < t \le 1\}$,

$$(4.18) \qquad Z(\tau;\, t) = Z(\tau;\, \beta)\, Z(\beta;\, \zeta^\varphi(\beta))\, Z(\zeta^\varphi(\beta);\, t).$$

We have both $Z(\tau;\, \beta) > 0$ on $\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\} \subseteq \{\beta < 1\}$, and $Z(\beta;\, \zeta^\varphi(\beta)) > 0$ a.s. on $\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\}$. Therefore, path continuity, (4.17), and (4.18) imply that $\rho < \zeta^\theta(\beta)$ a.s. on $\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\}$. Then a calculation like (4.14) shows that π'' is a suitably integrable portfolio. π'' is an arbitrage since

$$(4.19) \qquad X_{\pi''}(t) = \left(\frac{1}{Z(\zeta^\varphi(\beta);t\wedge\rho)} - 1\right)\left\{\zeta^\varphi(\beta) < \zeta^\theta(\beta)\right\}$$

Observe that π'' is 1-tame, $X_{\pi''}(1) = 0$ a.s. on $\{\zeta^{\varphi}(\beta) \geq \zeta^{\theta}(\beta)\}$, and $X_{\pi''}(1) = 1$ a.s on $\{\zeta^{\varphi}(\beta) < \zeta^{\theta}(\beta)\}$.

(Necessity.) Suppose that π_0 is a C-tame arbitrage portfolio. Then defining $\pi = (2C)^{-1}\pi_0$ produces an arbitrage for which $P\{X_\pi(t) + 1 > 0 \text{ for all } 0 \leq t \leq 1\} = 1$. Define the adapted vector process φ as follows:

$$\varphi(t) \;=\; \theta(t) \;-\; \frac{1}{1+X_\pi(t)}\sigma^*(t)\pi(t). \tag{4.20}$$

Then $\varphi(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$ for each (t,ω). Because π is 1/2-tame and satisfies the portfolio integrability constraint,

$$\int_0^1 \frac{1}{(1+X_\pi(t))^2}\left\|\sigma^*(t)\pi(t)\right\|^2 dt < \infty \text{ a.s.} \tag{4.21}$$

so that $\alpha^{\varphi} = \alpha = 1$ a.s. (4.21) also implies that $\zeta^{\varphi}(\tau) = \zeta^{\theta}(\tau)$ a.s.

Now define the stopping time τ by

$$\tau = \inf\{\, t > 0 \colon X_\pi(t) \neq 0\}.$$

It follows from the definition of τ and that $dX_\pi(t) = B^{-1}(t)\pi^*(t)\sigma(t)[dW(t) + \theta(t)dt]$ that up to a P-null set,

$$\{\tau < 1\} = \left\{\lambda(0 \leq t \leq 1 \colon \sigma^*(t)\pi(t)\{\tau \leq t < (\tau+\varepsilon)\wedge 1\} \neq 0) > 0 \ \text{ for all } \varepsilon > 0\right\}. \tag{4.22}$$

$P\{\alpha = 1\} = 1$ implies $\zeta^{\theta}(\tau) > \tau$ a.s. on $\{\tau < 1\}$. That π is an arbitrage implies $P\{\tau < 1\} > 0$. So (4.22) implies that

$$\lambda\otimes P\left(\varphi(t,\omega)\{\tau_\omega < t < \zeta^{\varphi}(\tau)_\omega\} \neq \theta(t,\omega)\{\tau_\omega < t < \zeta^{\theta}(\tau)_\omega\}\right) > 0.$$

Then to see that $Z^{\varphi}(\tau; \bullet)$ and $Z(\tau; \bullet)$ are not indistinguishable, examine the explicit formula for process $Z^{\nu}(\tau; \bullet)$, (1.11), and conclude that the map $[\nu] \to Z^{\nu}(\tau; \bullet)$ defined on the equivalence classes of adapted R^d-valued processes ν satisfying $\alpha^{\nu} = 1$ a.s. partitioned by the relation

$$\nu_1 \cong \nu_2 \text{ if } \lambda \otimes P(\nu_1(t,\omega)\{\tau_\omega < t < \zeta^{\nu_1}(\tau)_\omega\} = \nu_2(t,\omega)\{\tau_\omega < t < \zeta^{\nu_2}(\tau)_\omega\}) = 1$$

is an injective map.

Now calculate that on $\{0 \le t < \zeta^{\theta}(\tau)\}$,

$$(4.23) \quad \frac{Z^{\varphi}(\tau;t)}{Z(\tau;t)} = \exp\left\{\int_0^t \frac{\{\tau < s\}}{1 + X_\pi(s)} \pi^*(s)\sigma(s)dW(s)\right\} \times$$

$$\times \exp\left\{-\frac{1}{2}\int_0^t \frac{\{\tau < s\}}{(1 + X_\pi(s))^2} \|\sigma^*(s)\pi(s)\|^2 ds + \int_0^t \frac{\{\tau < s\}}{1 + X_\pi(s)} \pi^*(s)\sigma(s)\theta(s)ds\right\}.$$

So $Z^{\varphi}(\tau; \bullet) / Z(\tau; \bullet)$ is the unique strong solution on the stochastic interval $\{(t,\omega): 0 \le t < \zeta^{\theta}(\tau)_\omega\}$ to the SDE

$$(4.24) \qquad d\Gamma(t) = \Gamma(t)\frac{\{\tau < t\}}{1 + X_\pi(t)} \pi^*(t)\sigma(t)[dW(t) + \theta(t)dt]; \quad \Gamma(0) = 1.$$

Since $1 + X_\pi(\bullet)$ solves (4.24), it follows that $(1 + X_\pi(\bullet))\{\bullet < \zeta^{\theta}(\tau)\}$ and $(Z^{\varphi}(\tau; \bullet) / Z(\tau; \bullet))\{\bullet < \zeta^{\theta}(\tau)\}$ are indistinguishable processes. Then reasoning on $\{Z(\tau; 1) > 0\} \subseteq \{\zeta^{\theta}(\tau) = 1\}$ that X_π is an arbitrage capital gain, and on $\{Z(\tau;1) = 0\}$ that $Z^{\varphi}(\tau; \bullet)$ is a nonnegative process, we obtain that $P\{Z^{\varphi}(\tau; 1) \ge Z(\tau; 1)\} = 1$. ■

Remark. Under the assumption that $P\{\alpha = 1\} = 1$, the proof of necessity in the previous theorem characterizes the capital gain process $X_\pi(\bullet)$ associated with an arbitrary arbitrage $\pi(\bullet)$ as $C(\{Z^\varphi(\tau;\bullet) / Z(\tau;\bullet)\} - 1)$, for a constant $C > 0$, on the stochastic interval $\{(t,\omega): 0 \le t < \zeta^\theta(\tau)_\omega\}$. Here $\tau = \inf\{t > 0: X_\pi(t) \neq 0\}$, and φ is an adapted process satisfying a range requirement. Therefore, it is interesting to question if it is ever necessary in producing arbitrage to hold asset shares at times $t \ge \zeta^\theta(\tau)$ when $\zeta^\theta(\tau) < 1$, *i.e.*, at times t when the ratio $Z^\varphi(\tau; t) / Z(\tau; t)$ is undefined because $Z(\tau; t) = 0$. The answer is no. Assume absence of immediate arbitrage. If arbitrage exists, then there is an arbitrage π such that, where $\tau = \inf\{t > 0: X_\pi(t) \neq 0\}$, there exists a stopping time γ such that $\gamma < \zeta^\theta(\tau)$ a.s. on $\{Z(\tau; \zeta^\theta(\tau)) = 0\}$ and $X_\pi(1) = X_\pi(\gamma)$ a.s.

To justify the assertion in the preceding paragraph, suppose $\hat{\pi}$ is an arbitrage. Put stopping time $\hat{\tau} = \inf\{t > 0: X_{\hat{\pi}}(t) \neq 0\}$. By the proof of necessity in Theorem 3, there exists an adapted R^d-valued φ satisfying $\alpha^\varphi = 1$ a.s., $\lambda \otimes P\{\varphi(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]\} = 1$, $Z^\varphi(\hat{\tau};\bullet)$ is not indistinguishable from $Z(\hat{\tau};\bullet)$, and $P\{Z^\varphi(\hat{\tau}; 1) \ge Z(\hat{\tau}; 1)\} = 1$. The proof demonstrated that this φ satisfies $P\{\zeta^\varphi(\hat{\tau}) = \zeta^\theta(\hat{\tau})\} = 1$. Therefore, we can define an arbitrage π using the same construction as for the arbitrage π' in the proof of sufficiency in Theorem 3. Define

$$(4.12)' \qquad \pi(t) = B(t)\frac{Z^\varphi(\beta;t)}{Z(\beta;t)}\{\beta < t \le \gamma\}(\sigma^*)_+^{-1}(t)\big(\theta(t) - \varphi(t)\big)$$

$$+ \; B(t)\frac{K}{Z(\beta;t)}\{\beta < t \le \gamma\}(\sigma^*)_+^{-1}(t)\theta(t).$$

Here, there exists a $0 < \delta < 1$ such that

$$\beta = \inf\{t > \hat{\tau} : Z^{\varphi}(\hat{\tau}\,; t) \le \delta Z(\hat{\tau}\,; t) \text{ and } Z(\hat{\tau}\,; t) > 0\}$$

satisfies $P\{\beta < 1\} > 0$, $K = (\delta^{-1} - 1)/2 > 0$, and

$$\gamma = \inf\{t > \beta\colon Z(\beta; t) = K/(2 + 2K)\}.$$

π is an arbitrage, and $X_\pi(1) = X_\pi(\gamma)$ a.s. Let $\tau = \inf\{t > 0\colon X_\pi(t) \ne 0\}$. Note that $\hat{\tau} < \beta \le \tau \le \gamma$ holds a.s. on $\{\hat{\tau} < 1\}$. We have $\gamma < \zeta^\theta(\beta)$ a.s. on $\{Z(\beta; \zeta^\theta(\beta)) = 0\}$ [*c.f.* (4.13)]. Therefore, $\beta \le \tau < \zeta^\theta(\beta)$ a.s. on $\{Z(\beta; \zeta^\theta(\beta)) = 0\}$, and consequently, $\zeta^\theta(\tau) = \zeta^\theta(\beta)$ a.s. on $\{Z(\beta; \zeta^\theta(\beta)) = 0\}$ and $\{Z(\tau; \zeta^\theta(\tau)) = 0\} = \{Z(\beta; \zeta^\theta(\beta)) = 0\}$ a.s. So $\gamma < \zeta^\theta(\tau)$ a.s. on $\{Z(\tau; \zeta^\theta(\tau)) = 0\}$, and we have proven the assertions in the remark.

4.2 Characterization of Arbitrage in Terms of the Processes $V(\tau; \bullet)$.

Lemma 7. *Assume $P\{\alpha = 1\} = 1$. Then $Z(r; \bullet)$ is indistinguishable from $V(r; \bullet)$ for all constants $0 \le r \le 1$ if and only if $Z(\tau; \bullet)$ is indistinguishable from $V(\tau; \bullet)$ for all stopping times $0 \le \tau \le 1$.*

Proof. Sufficiency is obvious.

(Necessity.) First prove that for any stopping time τ, in order to prove $Z(\tau; \bullet)$ and $V(\tau; \bullet)$ are indistinguishable it suffices to show

(4.25) For each $0 \le t \le 1$, $V(\tau; t \vee \tau) \ge Z(\tau; t \vee \tau)$ a.s.

Corollary 1 states that $P\{V(\tau; t) \le Z(\tau; t) \text{ for all } 0 \le t \le 1\} = 1$, so that we may replace "$\ge$" in (4.25) by "=". The paths $V(\tau; \bullet)$ are right-continuous, and the paths $Z(\tau; \bullet)$ are continuous. So (4.25) implies

$$P\{V(\tau; t) = Z(\tau; t) \text{ for all } \tau \le t \le 1\} = 1.$$

In particular, $V(\tau;\tau) = Z(\tau;\tau) = 1$ a.s. Since $V(\tau; \bullet)$ is a (P) supermartingale by Theorem 2 and $V(\tau; 0) \le Z(\tau; 0) = 1$ a.s., we then have

$$\text{For each } 0 \le t \le 1,\ V(\tau; t \wedge \tau) = 1 = Z(\tau; t \wedge \tau) \text{ a.s.}$$

Then by right-continuity, $Z(\tau; \bullet)$ and $V(\tau; \bullet)$ are indistinguishable given (4.25).

Now let τ be a simple stopping time, $\tau = \Sigma_j\{\tau = t_j\}t_j$. Fix $0 \le t \le 1$. Because of indistinguishability of $Z(t_j; \bullet)$ and $V(t_j; \bullet)$ for each j, Lemmas 2 and 4 imply that for each j there exists a sequence $\{\nu_{j,k}; k \ge 1\} \subseteq D$ such that a.s.,

$$\text{(4.26)} \qquad \lim_k\uparrow E^{\nu_{j,k}}\left[Z(t_j;1)|F_{t\vee\tau}\right] = V(t_j; t \vee \tau) = Z(t_j; t \vee \tau).$$

Since for each $\nu \in D$,

$$\text{(4.27)} \qquad E^{\nu}\left[Z(\tau;1)|F_{t\vee\tau}\right] = \Sigma_j\{\tau = t_j\}\, E^{\nu}\left[Z(t_j;1)|F_{t\vee\tau}\right],$$

it follows that

$$\text{(4.28)} \qquad \lim_k\uparrow E^{\nu_{j,k}}\left[Z(\tau;1)|F_{t\vee\tau}\right] = Z(\tau; t \vee \tau) \quad \text{a.s. on } \{\tau = t_j\}.$$

Since $\sup_{j,k} E^{\nu_{j,k}}\left[Z(\tau;1)|F_{t\vee\tau}\right] \le V(\tau; t \vee \tau)$ a.s., (4.25) holds for this simple τ.

Now let $0 \le \tau \le 1$ be any stopping time. There exist simple stopping times $\tau_n \ge \tau$ with $\lim_n\downarrow\tau_n = \tau$ a.s. Fix $0 \le t \le 1$. By the preceding paragraph, for each n $V(t \vee \tau_n; \bullet)$ and $Z(t \vee \tau_n; \bullet)$ are indistinguishable processes. So for each n, there exists a sequence $\{\nu_{n,k}; k \ge 1\} \subseteq D$ such that a.s., (4.29) $\lim_k \uparrow E^{\nu_{n,k}}\left[Z(t \vee \tau_n;1)\middle|F_{t\vee\tau_n}\right] = V(t \vee \tau_n; t \vee \tau_n) = Z(t \vee \tau_n; t \vee \tau_n) = 1.$

For each n, k, the adapted process $\mu_{n,k}$, defined by $\mu_{n,k}(s) = \nu_{n,k}(s)\{s > t \vee \tau_n\}$ lies in D [see (3.3)], and we have

$$E^{\mu_{n,k}}\left[Z(t \vee \tau_n;1)\middle|F_{t\vee\tau_n}\right] = E^{\nu_{n,k}}\left[Z(t \vee \tau_n;1)\middle|F_{t\vee\tau_n}\right].$$

For each n, with the last equality below holding because $Z^{\mu_{n,k}}(t \vee \tau; t \vee \tau_n) = 1$ a.s. for each n, k [see footnote 2, page 33],

(4.30) $V(\tau; t \vee \tau) \geq \lim_k \uparrow E^{\mu_{n,k}}\left[Z(\tau;1)\middle|F_{t\vee\tau}\right] = Z(\tau; t \vee \tau) \lim_k \uparrow E^{\mu_{n,k}}\left[Z(t \vee \tau;1)\middle|F_{t\vee\tau}\right]$

$$= Z(\tau; t \vee \tau) \lim_k \uparrow E^{\mu_{n,k}}\left[Z(t \vee \tau; t \vee \tau_n) E^{\mu_{n,k}}\left[Z(t \vee \tau_n;1)\middle|F_{t\vee\tau_n}\right]\middle|F_{t\vee\tau}\right]$$

$$= Z(\tau; t \vee \tau) \lim_k \uparrow E\left[Z(t \vee \tau; t \vee \tau_n) E^{\mu_{n,k}}\left[Z(t \vee \tau_n;1)\middle|F_{t\vee\tau_n}\right]\middle|F_{t\vee\tau}\right].$$

Then monotone convergence (4.29) yields that for each n,

(4.31) $V(\tau; t \vee \tau) \geq Z(\tau; t \vee \tau) E\left[Z(t \vee \tau; t \vee \tau_n)\middle|F_{t\vee\tau}\right].$

We have $\lim_n Z(t \vee \tau; t \vee \tau_n) = 1$ a.s., so that (4.31) and the Fatou Lemma imply $V(\tau; t \vee \tau) \geq Z(\tau; t \vee \tau)$ a.s. ■

Theorem 4. *Assume absence of immediate arbitrage. There is no arbitrage if and only if processes $Z(r; \bullet)$ and $V(r; \bullet)$ are indistinguishable for all constant times $0 \leq r \leq 1$.*

Proof. (Necessity.) Suppose that $Z(r; \bullet)$ and $V(r; \bullet)$ are not indistinguishable. Consider the (P) Doob-Meyer Decomposition, cadlag $V(r; \bullet) = L(\bullet) - A(\bullet)$. Lemma 3 implies that $L(\bullet) = V(r; 0)Z^{\varphi}(\bullet)$ for an adapted R^d-valued process φ satisfying $\alpha^{\varphi} = 1$ a.s.

and $\lambda\otimes P\{\varphi(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]\} = 1$. $A(\bullet)$ is an adapted process with non-decreasing and cadlag paths starting at $A(0) = 0$. From

$$V(r;\,0) \le Z(r;\,0) = 1 \text{ a.s. and } V(r;\,1) = V(r;\,0)Z^{\varphi}(1) - A(1) = Z(r;\,1) \text{ a.s.,}$$

we obtain that $Z^{\varphi}(1) \ge Z(r;\,1)$ a.s. It cannot be that $Z^{\varphi}(\bullet)$ and $Z(r;\,\bullet)$ are indistinguishable: otherwise, in order for $V(r;\,1) = Z(r;\,1)$ a.s. to hold, we would need both $V(r;\,0) = 1$ and $A(1) = 0$ a.s. These values would contradict that $Z(r;\,\bullet)$ and $V(r;\,\bullet)$ are not indistinguishable because the paths $A(\bullet)$ are non-decreasing. By Lemma 6 then, there exists a stopping time $\beta \ge \tau$ for which

$$P\{\beta < 1\} > 0 \quad\text{and}\quad 0 < Z^{\varphi}(\beta) < Z(r;\,\beta) \text{ on } \{\beta < 1\}. \tag{4.32}$$

That $Z^{\varphi}(1) \ge Z(r;\,1)$ a.s and (4.32) imply $Z^{\varphi}(\beta;\,1) \ge Z(\beta;\,1)$ a.s. If $P\{Z^{\varphi}(\beta;\,1) > Z(\beta;\,1)\} > 0$, then $Z^{\varphi}(\beta;\,\bullet)$ and $Z(\beta;\,\bullet)$ are not indistinguishable, so that Theorem 3 implies that an arbitrage exists. If $Z^{\varphi}(\beta;\,1) = Z(\beta;\,1)$ a.s., then $Z^{\varphi}(1) \ge Z(r;\,1)$ a.s. and (4.32) imply that $Z(\beta;\,1) = 0$ a.s. on $\{\beta < 1\}$. In this case, apply Theorem 3 with o, the zero vector process, in the role of φ. We have $o(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$ for each (t,ω) and

$$Z^{o}(\beta;\,1) = 1 \ \ge \{\beta = 1\} = Z(\beta;\,1). \tag{4.33}$$

$Z^{o}(\beta;\,\bullet)$ and $Z(\beta;\,\bullet)$ are not indistinguishable, and so an arbitrage exists.

(Sufficiency.) Suppose that there exists an arbitrage given $\alpha = 1$ a.s. By Theorem 3, there exist both an adapted R^d-valued process φ satisfying $\alpha^{\varphi} = 1$ a.s. and $\lambda\otimes P\left\{\varphi(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]\right\} = 1$ and a stopping time $0 \le \tau \le 1$ such that the processes

$Z^{\varphi}(\tau; \bullet)$ and $Z(\tau; \bullet)$ are not indistinguishable and $Z^{\varphi}(\tau; 1) \geq Z(\tau; 1)$ a.s. Using these two processes, we will construct an adapted process $\hat{V}(\bullet)$ with continuous paths which is a (P^{ν}) supermartingale for each $\nu \in D$, which satisfies $\hat{V}(1) = Z(\tau; 1)$ a.s., and such that there exists a stopping time γ for which $P\{\hat{V}(\gamma) < Z(\tau; \gamma)\} > 0$. The existence of such a $\hat{V}(\bullet)$ implies through (iii) of Theorem 2 that $Z(\tau; \bullet)$ and $V(\tau; \bullet)$ are not indistinguishable. Then Lemma 7 implies that there exists a constant time r such that $Z(r; \bullet)$ and $V(r; \bullet)$ are not indistinguishable.

Lemma 6 implies that there exists a stopping time $\beta > \tau$ such that

$$P\{\beta < 1\} > 0 \text{ and } 0 < Z^{\varphi}(\tau; \beta) < Z(\tau; \beta) \text{ on } \{\beta < 1\}. \tag{4.32$'$}$$

Let $g:[0, \infty) \to [0, 1]$ be a continuous and strictly decreasing deterministic function satisfying $g(0) = 1$. Define stopping times γ and η as follows:

$$\gamma = \inf\{t > \beta: Z^{\varphi}(\tau; t) = g(t - \beta)Z(\tau; t)\} \tag{4.34}$$

$$\eta = \inf\{t > \gamma: Z^{\varphi}(\tau; t) = Z(\tau; t)\}.$$

Path-continuity of $Z^{\varphi}(\tau; \bullet)$ and $Z(\tau; \bullet)$, (4.32)$'$, and that $Z^{\varphi}(\tau; 1) \geq Z(\tau; 1)$ a.s. together imply that a.s. on $\{\beta < 1\}$

$$Z^{\varphi}(\tau; \gamma) = g(\gamma - \beta)Z(\tau; \gamma) < Z(\tau; \gamma) \text{ and } Z^{\varphi}(\tau; \eta) = Z(\tau; \eta). \tag{4.35}$$

Now define $\hat{V}(\bullet)$ by

$$\hat{V}(t) = g\big(([t \wedge \gamma] - \beta)^{+}\big)Z^{\mu}(t), \tag{4.36}$$

where the adapted process $\mu(\bullet)$ is given by

(4.37) $$\mu(t) = \theta(t)\{\tau < t \leq \gamma\} + \varphi(t)\{\gamma < t \leq \eta\} + \theta(t)\{\eta < t\}.$$

Then for $0 \leq t \leq 1$,

(4.38) $$\hat{V}(t) = g\big(([t \wedge \gamma] - \beta)^{+}\big)Z(\tau; t \wedge \gamma)\, Z^{\varphi}(\gamma; t \wedge \eta)\, Z(\eta; t),$$

so that in light of (4.35),

(4.39) $$\hat{V}(\gamma) = g(\gamma - \beta)Z(\tau; \gamma) = Z^{\varphi}(\tau; \gamma) < Z(\tau; \gamma) \text{ a.s. on } \{\beta < 1\}$$

$$\text{and } \hat{V}(1) = Z(\tau; 1) \text{ a.s.}$$

All that remains to accomplish is to show that $\hat{V}(\bullet)$ is a (P^{ν}) supermartingale for each $\nu \in D$. Since $\mu(t, \omega) \in \{\theta(t, \omega), \varphi(t, \omega)\}$ for each (t, ω), we have

$$\lambda \otimes P\left\{\mu(t,\omega) \in \mathrm{Ran}[\sigma^{*}(t,\omega)]\right\} = 1.$$

As in the proof of Corollary 1, this fact and that $\mathrm{Ran}[\sigma^{*}(t,\omega)] = (\mathrm{Ker}[\sigma(t,\omega)])^{\perp}$ for each (t,ω) imply that $Z^{\mu}(\bullet)$ is a (P^{ν}) supermartingale for each $\nu \in D$. We are done, because $\hat{V}(\bullet)$ is the product of $Z^{\mu}(\bullet)$ and an adapted process with non-increasing paths. ■

Theorem 4 characterizes absence of arbitrage in terms of the processes $V(r; \bullet)$, which are, loosely speaking, stochastic suprema. The following theorem improves the characterization by expressing the result in terms of maxima at which these suprema are attained.

Theorem 5. *Assume absence of immediate arbitrage. There is no arbitrage if and only if for each constant time $0 \leq r \leq 1$ there exists a $\mu \in D$ such that $E(Z^{\theta+\mu}(r; 1)) = 1$.*

Proof. (Sufficiency.) As justified in the proof of Lemma 7, for each r, to show that V(r; •) and Z(r; •) are indistinguishable it suffices to show that

(4.40) For each t such that $r \leq t \leq 1$, $V(r; t) = Z(r; t)$ a.s.

Fix $0 \leq r \leq 1$. There exists a $\mu \in D$ such that the process $Z^{\theta+\mu}(r; \bullet)$ is a martingale. Then if $r \leq t \leq 1$,

(4.41) $$E^{\mu}[Z(r; 1) \mid F_t] = E^{\nu}[Z^{\theta+\mu}(r; 1) \mid F_t] / Z^{\mu}(r; t) = Z(r; t).$$

Therefore $V(r; t) = Z(r; t)$ a.s. It follows that V(r; •) and Z(r; •) are indistinguishable for each $0 \leq r \leq 1$, and so Theorem 4 implies that there is no arbitrage.

(Necessity.) Fix $0 \leq r \leq 1$. Theorem 4 implies that V(r; •) and Z(r; •) are indistinguishable. Let $\{\varepsilon_n ; n \geq 1\}$ be a strictly decreasing sequence in (0, 1) such that $\lim_n\downarrow \varepsilon_n = 0$. Since

$$V(r; r) = Z(r; r) = 1 \text{ a.s.}$$

and

$$V(r ;r) = \text{ess sup}_{\nu\in D} E[Z^{\theta+\nu}(r; 1) \mid F_r],$$

by taking $\mu_1 = \nu_n^{(1)}$ for n sufficiently large, where $\{ \nu_n^{(1)} ; n \geq 1\}$ is a sequence in D such that $\lim_n\uparrow E[Z^{\theta+\nu_n^{(1)}}(r;1) \mid F_r] = V(r; r) = 1$ a.s., we obtain that there exists a $\mu_1 \in D$ such that

$$E(Z^{\theta+\mu_1}(r;1)) > 1 - \varepsilon_1.$$

Define a sequence of stopping times $\{\beta_{1,k} ; k \geq 1\}$ by

$$\beta_{1,k} = \inf\{t > r : Z^{\theta+\mu_1}(r;t) \vee Z^{\mu_1}(r;t) = k\}$$

Then $\beta_{1,k} = 1$ for k sufficiently large holds a.s., so that the monotone convergence theorem implies that there exists a k_1 such that

(4.42) $$E(Z^{\theta+\mu_1}(r;1)\{\beta_{1,k_1} = 1\}) \wedge E(Z^{\mu_1}(r;1)\{\beta_{1,k_1} = 1\}) > 1 - \varepsilon_1.$$

($\mu_1 \in D$ implies that $Z^{\mu_1}(\bullet)$ is a martingale process.) Denote $\tau_1 = \beta_{1,k_1}$.

Now consider that the absence of arbitrage, Theorem 4 and Lemma 7 together imply that $V(\tau_1; \bullet)$ and $Z(\tau_1; \bullet)$ are indistinguishable. So there exists a sequence $\{ \nu_n^{(2)} ; n \geq 1\}$ in D such that

(4.43) $$\lim_n \uparrow E[Z^{\theta+\nu_n^{(2)}}(\tau_1;1) | F_{\tau_1}] = 1 \text{ a.s.}$$

Then

(4.44) $$\lim_n E\left(Z^{\theta+\mu_1}(r;\tau_1) Z^{\theta+\nu_n^{(2)}}(\tau_1;1)\right) =$$

$$\lim_n \uparrow E\left(Z^{\theta+\mu_1}(r;\tau_1) E[Z^{\theta+\nu_n^{(2)}}(\tau_1;1) | F_{\tau_1}]\right) = E(Z^{\theta+\mu_1}(r;\tau_1)).$$

Because $Z^{\theta+\mu_1}(r;\bullet \wedge \tau_1)$ is a uniformly bounded local martingale, and therefore a martingale, $E(Z^{\theta+\mu_1}(r;\tau_1)) = 1$. Therefore, there exists a $\mu_2 \in D$ such that

$$E(Z^{\theta+\mu_1}(r;\tau_1) Z^{\theta+\mu_2}(\tau_1;1)) > 1 - \varepsilon_2.$$

By considering the stopping times $\{\beta_{2,k} ; k \geq k_1 + 1\}$ defined by

$$\beta_{1,k} = \inf\{t > \tau_1 : Z^{\theta+\mu_1}(r;\tau_1) Z^{\theta+\mu_2}(\tau_1;t) \vee Z^{\mu_1}(r;\tau_1) Z^{\mu_2}(\tau_1;t) = k\}$$

and monotone convergence, we obtain a stopping time $\tau_2 \geq \tau_1$ such that the processes $Z^{\theta+\mu_1}(r;\bullet\wedge\tau_1)Z^{\theta+\mu_2}(\tau_1;\bullet\wedge\tau_2)$ and $Z^{\mu_1}(r;\bullet\wedge\tau_1)Z^{\mu_2}(\tau_1;\bullet\wedge\tau_2)$ are uniformly integrable martingales, and

$$(4.45)\qquad E(Z^{\theta+\mu_1}(r;\tau_1)Z^{\theta+\mu_2}(\tau_1;1)\{\tau_2=1\}) \wedge E(Z^{\mu_1}(r;\tau_1)Z^{\mu_2}(\tau_1;1)\{\tau_2=1\})$$

$$> 1 - \varepsilon_2.$$

Then by induction, there exist a sequence of processes $\{\mu_n\,;\, n \geq 1\}$ in D and a sequence of stopping times $\tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots$ (put $\tau_0 = r$) such that the processes $G_n(\bullet)$ and $H_n(\bullet)$ defined by

$$(4.46)\qquad G_n(\bullet) = \prod_{j=1}^{n} Z^{\theta+\mu_j}(\tau_{j-1};\bullet\wedge\tau_j), \quad H_n(\bullet) = \prod_{j=1}^{n} Z^{\mu_j}(\tau_{j-1};\bullet\wedge\tau_j)$$

are uniformly bounded martingales satisfying

$$(4.47)\qquad E(G_n(1)\{\tau_n = 1\}) \wedge E(H_n(1)\{\tau_n = 1\}) > 1 - \varepsilon_n.$$

Define the adapted R^d-valued process μ by

$$(4.48)\qquad \mu(t) = \sum_{n=1}^{\infty} \mu_n(t)\{\tau_{n-1} < t \leq \tau_n\}.$$

Then $\lambda\otimes P\{\mu(t,\omega) \in \mathrm{Ker}[\sigma(t,\omega)]\} = 1$. $\alpha^{\mu} = 1$ a.s. so that the processes $Z^{\mu}(\bullet)$ and $Z^{\theta+\mu}(\bullet)$ are well defined and continuous. Our method of selecting the stopping times τ_n imply that there exist integers $N_1 < N_2 < \ldots$ such that for each n,

$$\tau_n = \inf\{t > 0 : Z^{\theta+\mu}(r;t) \vee Z^{\mu}(r;t) = N_n\}.$$

Therefore $P(\bigcup_n\{\tau_n = 1\}) = 1$. So the summation in (4.48) has finitely many non-zero summands a.s, and consequently, $\int_0^1 \|\mu(t)\|^2 dt < \infty$ a.s. By monotone convergence,

$$E(Z^{\mu}(r;1)) = \lim_n \uparrow E(H_n(1)\{\tau_n = 1\}) = 1.$$

So $\mu \in D$. Similarly, to complete the proof,

$$E(Z^{\theta+\mu}(r;1)) = \lim_n \uparrow E(G_n(1)\{\tau_n = 1\}) = 1.$$

■

The remark following the proof of Theorem 3 implies that in the absence of immediate arbitrage, existence of arbitrage implies that there exists an arbitrage π satisfying that for each ω, $\pi(t,\omega) = 0$ for all $t \geq \zeta^{\theta}(\tau)_{\omega}$, where stopping time τ is defined by $\tau = \inf\{t > 0: X_{\pi}(t) \neq 0\}$. The following result is interesting in comparison with that conclusion. Note that the assumption of Corollary 2 below implies that for all stopping times $0 \leq \tau \leq 1$, $\zeta^{\theta}(\tau) = 1$ a.s. Loosely speaking, we need check for indistinguishability of $V(r;\bullet)$ and $Z(r;\bullet)$ for $r > 0$ only if an explosion of θ occurs before time 1 and "kills the market initiated at time 0."

Corollary 2. *Assume $P\{\zeta^{\theta}(0) = 1\} = 1$. Then there is no arbitrage if and only if processes $V(\bullet)$ and $Z(\bullet)$ are indistinguishable. Equivalently, there is no arbitrage if and only if there exists a $\mu(\bullet) \in D$ such that $E(Z^{\theta+\mu}(1)) = 1$.*

Proof. The proof of Theorem 5 establishes the equivalence of the two characterizations. For necessity, apply Theorem 4 with $r = 0$.

(Sufficiency.) Assume V(•) and Z(•) are indistinguishable. Since $P\{\zeta^\theta(0) = 1\} = 1$ implies $P\{\alpha = 1\} = 1$, Theorem 4 applies. Since Z(1; •) is indistinguishable from the constant 1 process, V(1; •) is indistinguishable from Z(1; •). Now fix $0 \le r < 1$. As in the proof of Lemma 7, to prove that V(r;•) and Z(r;•) are indistinguishable it suffices to show

(4.25)′ For each t with $r \le t \le 1$, $V(r; t) \ge Z(r; t)$ a.s.

If $r \le t \le 1$, then by Lemma 2 there exists a sequence ν_n in D such that

(4.49) $$\lim_n \uparrow E^{\nu_n}[Z(1)|F_t] = V(t) = Z(t) \text{ a.s.}$$

Equivalently, since $r \le t$,

(4.50) $$Z(t) = Z(r)(\lim_n \uparrow E^{\nu_n}[Z(r; 1)|F_t]) \text{ a.s.}$$

$P\{\zeta^\theta(0) = 1\} = 1$ and $r < 1$ imply that $Z(r) > 0$ a.s. so that a.s.

(4.51) $$V(r; t) \ge \lim_n \uparrow E^{\nu_n}[Z(r; 1)|F_t] = Z(t) / Z(r) = Z(r; t).$$

■

In the statements of the remaining corollaries, recall that we assume absence of immediate arbitrage, *i.e.*, we assume $\alpha = 1$ a.s. The following result is generally understood; Levental and Skorohod (1995) prove it without reference to invertibility of σ [see page 917]. Here it is an easy corollary.

Corollary 3. *If $E(Z(\tau; 1)) = 1$ for all stopping times $0 \le \tau \le 1$, then no arbitrage exists.*

Proof. The zero vector process lies in D. Apply Theorem 5. ■

The following Corollary explicitly formulates the link between Theorem 5 and the notion of an equivalent local martingale measure for the stock price processes.

Corollary 4. Assume $\int_0^1 \|\theta(t)\|^2 dt < \infty$ a.s. Then there is no arbitrage if and only if for each i = 1, ... d,

$$dS_i(t) = S_i(t) \sum_{1 \le k \le d} \sigma_{i,k}(t) d\widetilde{W}_k(t),$$

where there exists a probability measure Q ~ P such that $\widetilde{W}(\bullet)$ is a d-dimensional standard Brownian Motion with respect to $(\Omega, F, \{F_t\}_{0 \le t \le 1}, Q)$.

Proof. No arbitrage is equivalent to that there exists a $\nu(\bullet) \in D$ such that $E(Z^{\theta+\nu}(1)) = 1$. Let $\widetilde{W}(\bullet)$ be given by

$$\widetilde{W}_i(t) = W_i(t) + \int_0^t (\theta_i(s) + \nu_i(s)) ds; \quad 1 \le i \le d.$$

Apply Girsanov's Theorem with probability measure Q equivalent to P defined by

$dQ / dP = Z^{\theta+\nu}(1)$. ■

Under the assumptions $\int_0^1 \|\theta(t)\|^2 dt < \infty$ a.s., Z(1) is a.s. bounded, and E(Z(1)) < 1, Levental and Skorohod construct an arbitrage which does not require invertible volatility [(1995), Example 5, page 924]. The following corollary accomplishes the same result under a slightly weaker integrability assumption.

Corollary 5. *If $\zeta^{\theta}(0) = 1$ a.s. and there exists a constant C for which $Z(1) \leq C$ a.s., then there is no arbitrage if and only if $E(Z(1)) = 1$.*

Proof. (Sufficiency.) If $E(Z(1)) = 1$, then the proof of Corollary 3 implies that $Z(\bullet)$ and $V(\bullet)$ are indistinguishable. Then Corollary 2 implies absence of arbitrage.

(Necessity.) For each $\nu \in D$ and $0 \leq t \leq 1$, $E^{\nu}[\, Z(1) | \, F_t] \leq C$. Then it follows from right-continuity of paths $V(\bullet)$ that $P\{V(t) \leq C \text{ for all } 0 \leq t \leq 1\} = 1$. Theorem 4 states that absence of arbitrage implies that $Z(\bullet)$ and $V(\bullet)$ are indistinguishable. Therefore, $Z(\bullet)$ is a uniformly bounded local martingale. ($\zeta^{\theta}(0) = 1$ a.s. implies that we can take 1 as the limit of the localization stopping times.) A uniformly bounded local martingale is a martingale. [See Revuz and Yor (1991), Proposition IV.1.7, page 118.] Therefore, $E(Z(1)) = 1$. ■

Chapter 5

Examples

In all examples, assume that r is the zero process.

Example 1. This is an example where $E(Z(1)) < 1$ and $Z(1) > 0$ a.s., but $V(\bullet)$ and $Z(\bullet)$ are indistinguishable. This example shows through Corollary 2 that when $\sigma(\bullet)$ can be singular, the result that $E(Z(1)) < 1$ implies existence of arbitrage is not true. The example can be written in one dimension, but is easier to read in a construction with $d = 2$. Let

$$\sigma(t) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

To define θ, let $0 = t_0 < t_1 < t_2 < \ldots$ with $\lim_n \uparrow t_n = 1$. Where $W = (W_1, W_2)^*$, define for $k \geq 1$ sigma algebra

$$G_k = \sigma\big(W_2(t) - W_2(t_{k-1});\, t_{k-1} \leq t < t_k\big).$$

Take for $k \geq 1$, A_k from G_k satisfying both $P(A_k) < 1$ and $P(\bigcap_k A_k) = 1/2$. Define a deterministic process $f(\bullet)$ satisfying $\alpha^f = 1$ a.s. by $f(t) = ((1-t)^{-1}, 0)^*$, $0 \leq t < 1$, and a stopping time τ by $\tau = \inf\{t > t_1 \colon Z^f(t_1; t) = 1/2\}$. Then define $\theta(\bullet)$ by

(5.1) $$\theta(t) = \begin{cases} (0,0)^* & for \quad t \le t_1 \\ \left(\bigcap_{j=1}^{k} A_j \cap \{t \le \tau\}\right) f(t) & for \quad t_k < t \le t_{k+1}, \quad k \ge 1. \end{cases}$$

θ is adapted since each $A_k \in F_{t_k}$. Note that for each (t,ω), $\theta(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$ as is consistent with the general construction of θ. Since $\int_{t_1}^{1} \|f(s)\|^2 ds = \infty$ a.s., $P\{\tau < 1\} = 1$.

Then $\sigma(\bullet)$ and $b(\bullet) = \sigma(\bullet)\theta(\bullet) = \theta(\bullet)$ satisfy the market parameters integrability constraint:

(5.2) $$\int_0^1 \left(\sum_{j,k} \sigma_{j,k}^2(s) + \sum_j |b_j(s)| \right) ds \le 1 + \int_0^1 \{t_1 < s \le \tau\} \frac{1}{1-s} ds < \infty \text{ a.s.}$$

That $\tau < 1$ a.s. and that $\zeta^f(t_1) = 1$ for each ω imply that $Z(1) > 0$ a.s.

To see that $E(Z(1)) < 1$, consider separately integrals over the cells of the partition

$$\Omega = \left(\bigcap_{k\ge1} A_k\right) \cup \left(\bigcup_{k\ge1} B_k\right), \quad \text{where} \quad B_k = A_k^c \setminus \left(\bigcup_{j<k} A_j^c\right) \quad \text{for} \quad k \ge 1.$$

We have

(5.3) $$Z(1) = Z^f(t_1; \tau) = 1/2 \text{ a.s. on } \cap_k A_k.$$

Furthermore, it follows by induction on k that[1]

(5.4) $$E(Z(1); B_k) = P(B_k) \text{ for } k \ge 1.$$

(5.4) holds at k = 1 since $B_1 = A_1^c$ so that on B_1, $Z(1) = Z^f(t_1; t_1) = 1$. Now suppose that (5.4) holds for j = 1, ... k−1. Note that, with F_t^j denoting the P-augmentation of

[1] For this example denote for Y integrable and $A \in F$, $E(Y; A) = \int_A Y dP$.

$\sigma\{W_j(s), 0 \le s \le t\}$ for $j = 1, 2$, $Z^f(t_1; \bullet \wedge \tau)$ is $\{F_t^1\}$-adapted. Calculate, using in the third equality that the $F_1^1 \vee F_{t_{k-1}}^2$ - measurable random variable $Z^f(t_1; t_k \wedge \tau)\{\bigcap_{j<k} A_j\}$ is independent of G_k:

$$(5.5.a) \qquad E(Z(1); B_k) = E(Z^f(t_1; t_k \wedge \tau) ; A_k^c \cap (\bigcap_{j<k} A_j))$$

$$= E(E[Z^f(t_1; t_k \wedge \tau)\{\bigcap_{j<k} A_j\} \mid G_k]; A_k^c)$$

$$= P(A_k^c)\, E(Z^f(t_1; t_k \wedge \tau); \bigcap_{j<k} A_j).$$

Through uniform boundedness in (t,ω) of $f(\bullet)\{\bullet \le t_k\}$, $Z^f(t_1; \bullet \wedge t_k \wedge \tau)$ is a martingale process. Continue (5.5), reasoning in the second equality below that $\theta(t) = 0$ for all $t > t_k$ on $\bigcup_{j<k} B_j$.

$$(5.5.b) \qquad E(Z(1); B_k) = P(A_k^c)\{1 - E(Z^f(t_1; t_k \wedge \tau); (\bigcap_{j<k} A_j)^c)\}$$

$$= P(A_k^c)\{1 - \Sigma_{j<k} E(Z(1); B_j)\} = P(A_k^c)\{1 - \Sigma_{j<k} P(B_j)\}$$

$$= P(A_k^c) P(\bigcap_{j<k} A_j) = P(B_k).$$

Combining (5.3) and (5.4) yields that $E(Z(1)) = 3/4 < 1$.

It remains to show that $V(\bullet)$and $Z(\bullet)$ are indistinguishable. Since

$$P\{V(t) \le Z(t), \text{ for all } 0 \le t \le 1\} = 1,$$

continuity of $Z(\bullet)$ and Lemma 5 imply that it suffices to produce for each fixed $0 \le t \le 1$ a $\varphi \in N$ for which $Z(t) = E[Z(1)Z^{\varphi}(t; 1) \mid F_t]$. To this end, choose k such that $t_{k-1} > t$, and consider the G_k- measurable random variable $Y = A_k^c / P(A_k^c)$, which satisfies $E(Y) = 1$.

There exists an adapted R^1-valued process φ_0 satisfying $\alpha^{\varphi_0} = 1$ a.s. such that if the R^2-valued process φ is defined by $\varphi(t) = (0, \varphi_0(t))^*$, then the processes $E[Y \mid F_\bullet^2]$ and $Z^\varphi(\bullet)$ are indistinguishable. $\varphi \in N$, since the definition of σ results in that

$$N = \{\text{adapted } \nu = (\nu_1, \nu_2)^* : \alpha^\nu = 1 \text{ a.s., } \lambda \otimes P\{\nu_1(t,\omega) = 0\} = 1\}.$$

To prove $Z(t) = E[Z(1)Z^\varphi(t;1) \mid F_t]$, show that $Z(\bullet)Z^\varphi(t;\bullet)$ is a (P) martingale. Since Y is G_k-measurable with $t_{k-1} > t$, and therefore independent of F_t^2,

$$\lambda \otimes P\{s \leq t \text{ implies } \varphi(s,\omega) = 0\} = 1.$$

So $Z^\varphi(t;\bullet)$ and $Z^\varphi(\bullet)$ are indistinguishable. In particular $Z^\varphi(t;1) = Y$ a.s. For any $\nu \in N$, $Z(\bullet)Z^\nu(\bullet)$ is the (P) supermartingale $Z^{\theta+\nu}(\bullet)$. Then that $Z(\bullet)Z^\varphi(t;\bullet)$ is a (P) martingale follows from that

$$E(Z(1)Z^\varphi(t;1)) = P(A_k^c)^{-1}E(Z(1);\ A_k^c) = 1. \tag{5.6}$$

To justify the final equality in (5.6), consider the disjoint union

$$A_k^c = \bigcup_{j=1}^{k}\{A_k^c \cap B_j\}$$

If $j < k$, then

$$E(Z(1);A_k^c \cap B_j) = E\left(E\left[Z(t_j)A_k^c \cap B_j \middle| F_{t_j}\right]\right) \tag{5.7}$$

$$= P(A_k^c)E(Z(1);B_j) = P(A_k^c)P(B_j) = P(A_k^c \cap B_j).$$

In the first and second equalities in (5.7) we use that on B_j, $Z(1) = Z(t_j)$ a.s. For the second equality also reason that if $j < k$, $A_k^c \in G_k$ is independent of F_{t_j}. If $j = k$, observe that $B_k \cap A_k^c = B_k$, so that (5.4) implies

$$E(Z(1);\ A_k^c \cap B_k) \ = \ P(A_k^c \cap B_k).$$

Example 2. By demonstrating a case where $V(0) = 1$ a.s. and $V(\bullet)$ is not indistinguishable from $Z(\bullet)$, this example shows that we cannot reduce the conditions equivalent to the absence of arbitrage to the behavior of the processes $V(r; \bullet)$ at time 0. Note that the example exists under the restriction that $Z(1) > 0$ a.s.

Let $d = 1$, and $\sigma(t) = \{t > 1/2\}$, $0 \le t \le 1$. Let deterministic $f(t) = (1-t)^{-1}$, for $0 \le t < 1$, and define a stopping time τ by

$$\tau = \inf\{t > 1/2:\ Z^f(1/2; t) = 1/2\}.$$

Choose A from $F_{1/2}$ with $P(A) = 1/2$; then define θ as follows:

$$\theta(t) \ = \ f(t)\big(A \cap \{1/2 < t \le \tau\}\big)$$

Take $b = \sigma\theta = \theta$. This model is valid. θ is adapted, because $A \in F_{1/2}$. $\theta \in \mathrm{Ran}[\sigma^*]$ for all (t,ω). Moreover, $P\{\tau < 1\} = 1$, so that σ and b satisfy (1.4), the market parameter integrability constraint. $P\{\tau < 1\} = 1$ also implies that $Z(1) = A^c + 1/2\ A$ a.s.

That $Z(\bullet)$ and $V(\bullet)$ are not indistinguishable follows from $F_{1/2}$ measurability of $Z(1)$. We have $V(1/2) = Z(1)$ a.s., and $Z(1) < Z(1/2)$ a.s. on A. To see that $V(0) = 1$, consider the nonnegative $\{F_t\}$-martingale $Y(\bullet)$, with $Y(t) = E[2A^c \mid F_t]$. Since $Y(0) = E(2A^c) = 1$,

$Y(\bullet) = Z^{\varphi}(\bullet)$ for an adapted process φ satisfying $\alpha^{\varphi} = 1$ a.s. (Take $\varphi(t) = 0$ for $t > \zeta^{\varphi}(0)$ for each ω.) Because $A^c \in F_{1/2}$,

$$\lambda\otimes P\{\varphi(t,\omega) = 0 \text{ for all } t > 1/2\} = 1, \ \textit{i.e.}, \lambda\otimes P\{\varphi(t,\omega) \in \operatorname{Ker}[\sigma(t,\omega)]\} = 1.$$

So $\varphi \in N$. Then by Lemma 5,

$$V(0) \geq E(Z(1)Z^{\varphi}(1)) = E(2A^c) = 1.$$

Example 3. This example shows that we do not have in general that $V(\bullet)$ is a continuous process. Here $P\{V(1/2) \neq \lim_{t\uparrow 1/2} V(t)\} > 0$. Let $d = 1$, and let $0 = t_0 < t_1 < \ldots$, with $\lim_n t_n = 1/2$. Define $G_k = \sigma(W(t) - W(t_{k-1}), t_{k-1} < t \leq t_k)$ for each $k \geq 1$. Choose a sequence of sets A_k, such that $A_k \in G_k$ and $P(A_k) < 1$ for each k, and $P(\bigcap_k A_k) = 1/2$. Define $A = \bigcap_k A_k \in F_{1/2}$, and let $\sigma(\bullet)$, $\theta(\bullet)$ and $b(\bullet)$ be as in Example 2, using this particular A when defining $\theta(\bullet)$.

Fix $0 \leq t < 1/2$. Take k so that $t_{k-1} > t$ and define a martingale process $Y(\bullet)$ by $Y(t) = P(A_k^c)^{-1}E[A_k^c \mid F_t]$. As argued for similar processes in earlier examples, $Y(\bullet) = Z^{\varphi}(\bullet)$ for an adapted process φ satisfying $\alpha^{\varphi} = 1$ a.s. and such that $\varphi(t) = 0$ for $t > \zeta^{\varphi}(0)$ holds for each ω. $A_k \in G_k \subset F_{1/2}$. So $\lambda\otimes P\{s > 1/2 \text{ implies } \varphi(s,\omega) = 0\} = 1$; equivalently, $\varphi \in N$. $A_k \in G_k$ also implies through independence of G_k and $F_{t_{k-1}}$ that $\lambda\otimes P\{s < t_{k-1} \text{ implies } \varphi(s,\omega) = 0\} = 1$. Then $t < t_{k-1}$ implies $Z^{\varphi}(1) = Z^{\varphi}(t; 1)$ a.s. Now, using in the second equality below that $A_k^c \subseteq A^c$, where $Z(1) = 1$ a.s. on A^c, and using independence in the final equality,

$$(5.8) \qquad V(t) \geq E[Z(1)Z^{\varphi}(t;1)|\ F_t] = E[Z(1)Y(1)|\ F_t] = P(A_k^c)^{-1}E[A_k^c \mid F_t]$$

$$= P(A_k^c)^{-1}P(A_k^c).$$

Since $t < 1/2$ implies $Z(t) = 1$ a.s, and $V(t) \leq Z(t)$ a.s., we have $V(t) = 1$ a.s. for each $t < 1/2$. So $\lim_{t\uparrow 1/2} V(t) = 1$ a.s. From Example 2, we know that

$$V(1/2) = Z(1) = A^c + 1/2\ A \ \text{ a.s.}$$

Then $P\{V(1/2) \neq \lim_{t\uparrow 1/2} V(t)\} = 1/2$.

Example 4. This example is the central object of Delbaen and Schachermayer (1998b). The Z process of this example satisfies $EZ(1) < 1$ and $Z(1) > 0$ a.s. Furthermore, there exists a $\nu \in D$ such that $Z(\bullet)Z^{\nu}(\bullet)$ is a martingale. Given such Z and ν, for each $0 \leq t \leq 1$

$$E^{\nu}\left[Z(1)\middle|F_t\right] = \frac{1}{Z^{\nu}(t)}E\left[Z(1)Z^{\nu}(1)\middle|F_t\right] = Z(t),$$

so that the processes $V(\bullet)$, $Z(\bullet)$, and $E^{\nu}[Z(1)\,|\,F_{\bullet}]$ are mutually indistinguishable. Because $Z(1) > 0$ a.s., Corollary 2 implies absence of arbitrage.

One can construct a similar example in a market with $d = 1$, but the expression below using $d = 2$ is nice in terms of readability. Define

$$\sigma(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so that for all (t,ω), $\mathrm{Ker}[\sigma] = \{x \in R^2\colon\ x_1 = 0\}$. Define the deterministic processes f and g for times $0 \leq t < 1$ by

$$f(t) = [1/(1-t),\ 0]^*, \qquad g(t) = [0,\ 1/(1-t)]^*.$$

Define two stopping times τ and β by

$$\tau = \inf\{t > 0 : Z^f(t) = ½\}, \qquad \beta = \inf\{t > 0 : Z^g(t) = 2\}.$$

Since $Z^f(1) = 0$ a.s., we have $\tau < 1$ a.s. Define the process Z by stopping Z^f: put

$$Z(t) = Z^f(\tau \wedge \beta \wedge t).$$

So defining Z is equivalent to defining process θ by

$$\theta(t) = [(1/(1-t))\{t \le \tau \wedge \beta\},\ 0]^*.$$

Observe that θ satisfies $\theta(t,\omega) \in \mathrm{Ran}[\sigma^*(t,\omega)]$ for all (t,ω). If $b = \sigma\theta$, then $b = \theta$, and the market parameters integrability constraint (1.4) holds because $\tau < 1$ a.s:

$$\int_0^1 \left\{ \sum_{i,j} \sigma_{i,j}{}^2(t) \quad + \quad \sum_i |b_i(t)| \right\} dt \quad = \quad 1 \quad + \quad \int_0^1 \{t \le \tau \wedge \beta\} \frac{1}{1-t} dt.$$

Check first that that $E(Z(1)) < 1$:

$$E(Z^f(\tau \wedge \beta)) \quad = \quad \int_{\{\beta=1\}} Z^f(\tau) dP \quad + \quad \int_{\{\beta<1\}} Z^f(\tau \wedge \beta) dP. \tag{5.9}$$

$P\{\beta = 1\} = 1/2$ because bounded local martingale $Z^g(\beta \wedge t)$ is a martingale with $Z^g(\beta) = 2$ on $\{\beta < 1\}$ and $Z^g(\beta) = 0$ a.s. on $\{\beta = 1\}$. Therefore, the first summand in (5.9) is $1/2\ P\{\beta = 1\} = 1/4$. To calculate the integral over $\{\beta < 1\}$, use independence of β and the process $Z^f(\bullet \wedge \tau)$ in the first equality below, and in the second, that for any $t \in [0, 1)$, $\{Z^f(s \wedge \tau), F_s;\ 0 \le s \le t\}$ is a martingale by boundedness of $f(\bullet)\{\bullet \le t\}$.

$$(5.10) \qquad \int_{\{\beta<1\}} Z^f(\beta\wedge\tau)\,dP \;=\; \int_{[0,1)} E\big(Z^f(t\wedge\tau)\big)P\{\beta\in dt\} \;=\; P\{\beta<1\} = 1/2.$$

So E(Z(1)) = 1/4 + 1/2 = 3/4 .

Now consider the adapted process ν given by ν(t) = g(t){t ≤ τ ∧ β} From τ < 1 a.s., it holds that $\int_0^1 \|\nu(t)\|^2 dt < \infty$ a.s. We have ν(t,ω) ∈ Ker[σ(t,ω)] for all (t,ω). Furthermore, $Z^\nu(\bullet)$ is the martingale $Z^g(\tau\wedge\beta\wedge\bullet)$. So ν ∈ D.

Finish the example by showing $E(Z(1)Z^\nu(1)) = 1$. We use the martingale property of the process $Z^g(\beta\wedge\bullet)$ in the second equality below, and refer to (5.10) in the fourth.

$$(5.11) \qquad E(Z(1)Z^\nu(1)) = E(Z^f(\tau\wedge\beta)\,Z^g(\tau\wedge\beta)) \;=\; E(Z^f(\tau\wedge\beta)\,Z^g(\beta))$$

$$= 2\int_{\{\beta<1\}} Z^f(\tau\wedge\beta)\,dP \;=\; 1.$$

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