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FINDING ALL ISOLATED ROOTS OF POLYNOMIAL SYSTEMS IN $\mathbb{C}^n$ VIA STABLE MIXED VOLUME

By

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ABSTRACT

FINDING ALL ISOLATED ROOTS OF POLYNOMIAL SYSTEMS IN $\mathbb{C}^n$ VIA STABLE MIXED VOLUME

By

Tangan Gao

To find all the isolated zeroes of a polynomial system $P(\mathbf{x})$ in $\mathbb{C}^n$ (as opposed to in $(\mathbb{C}^*)^n$) via the polyhedral homotopy method of Huber and Sturmfels [7], one first finds all *stable mixed cells* in a *stable mixed subdivision* and establishes a *fine mixed subdivision* for each stable mixed cell. One then solves a collection of polynomial subsystems corresponding to the stable mixed cells, and uses their solutions as starting points for the homotopy paths of a set of nonlinear homotopies which lead to all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$. This method offers a dramatic computational improvement over earlier homotopy algorithms at the cost of many costly recursive liftings at the preprocessing step of finding the stable mixed cells and their fine mixed subdivisions.

The main goal of this dissertation is to present a new strategy which can quickly (and simultaneously) find the stable mixed subdivision, the fine mixed subdivisions of the stable mixed cells, and the necessary subsystems by means of a single lifting.

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Introduction

Polynomial systems arose quite commonly in many fields of science and engineering, such as formula construction, geometric intersection, inverse kinematics, power flow with PQ-specified bases, computation of equilibrium states, etc.. Elimination theory-based methods, most notably the Buchberger algorithm [2] for constructing Gröbner bases, are the classical approach to solving multivariate polynomial systems, but their reliance on symbolic manipulation makes those methods somewhat unsuitable for all but small problems.

In 1977, Garcia and Zangwill [5] and Drexler [3] independently presented theorems suggesting that homotopy continuation could be used to find the full set of isolated zeros of a polynomial system numerically. During the last two decades this method has been developed into a reliable and efficient numerical algorithm for approximating all isolated zeros of polynomial systems. See [10] for a survey.

For a system of polynomials $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ with $\mathbf{x} = (x_1, \ldots, x_n)$, write

$$p_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n,$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $c_{i,\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Here $\mathcal{A}_i$, a finite subset of $\mathbb{N}^n$, is called the *support* of $p_i(\mathbf{x})$, and the convex hull of $\mathcal{A}_i$, denoted by Q_i, is called the *Newton polytope* of $p_i(\mathbf{x})$. We call $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the *support* of $P(\mathbf{x})$.

The *Minkowski sum* of polytopes $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ is defined by

$$\mathcal{Q}_1 + \cdots + \mathcal{Q}_n = \{\mathbf{a}_1 + \cdots + \mathbf{a}_n \mid \mathbf{a}_1 \in \mathcal{Q}_1, \ldots, \mathbf{a}_n \in \mathcal{Q}_n\}.$$

It can be shown that the n-dimensional Euclidean volume of the polytope $\lambda_1\mathcal{Q}_1 + \cdots + \lambda_n\mathcal{Q}_n$ with nonnegative variables $\lambda_1, \ldots, \lambda_n$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ of degree n. The coefficient of $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$ in this polynomial is defined to be the *mixed volume* of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, denoted by $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ or $\mathcal{M}(\mathcal{A})$ when no ambiguity exists.

Theorem 1 *The number of isolated zeros in $(\mathbb{C}^*)^n$, counting multiplicities, of a polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ is bounded above by the mixed volume $\mathcal{M}(\mathcal{A})$. For generically chosen coefficients, the system $P(\mathbf{x}) = \mathbf{0}$ has exactly $\mathcal{M}(\mathcal{A})$ zeros in $(\mathbb{C}^*)^n$.*

The root count in the above theorem was discovered by Bernshtein [1], Khovanskii [8] and Kushnirenko [9] and is sometimes referred to as the BKK bound. While this bound is, in general, significantly sharper than the classical Bézout number and its variants, a limitation is that it only counts zeros of $P(\mathbf{x})$ in the algebraic torus $(\mathbb{C}^*)^n$. Root count in $\mathbb{C}^n$ via mixed volume was first attempted in [14] where an upper bound was derived by introducing the notion of a *shadowed set.* Later, a significantly much tighter bound was given by the following theorem (and was generalized soon after in [16]).

Theorem 2 [12] *The number of isolated zeros in $\mathbb{C}^n$, counting multiplicities, of a polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ with supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is bounded above by the mixed volume $\mathcal{M}(\mathcal{A}_1 \bigcup\{\mathbf{0}\}, \ldots, \mathcal{A}_n \bigcup\{\mathbf{0}\})$.*

We shall call the set $(\mathcal{A}_1 \bigcup\{\mathbf{0}\}, \ldots, \mathcal{A}_n \bigcup\{\mathbf{0}\})$, denoted by $\mathcal{A} \bigcup\{\mathbf{0}\}$, the *extended support* of $P(\mathbf{x})$. In [7], an even tighter bound was obtained: the number of isolated zeros of a polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ in $\mathbb{C}^n$ is bounded above by its *stable mixed volume*. This number is always smaller than the mixed volume of the extended support of $P(\mathbf{x})$. This bound has since been generalized to the root count of polynomial systems over any algebraically closed fields, and various criteria have been established for the equality in this bound [15].

Based on Theorem 1, a *polyhedral homotopy* was proposed in [6] to approximate all the isolated zeros of a polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ in $(\mathbb{C}^*)^n$ by homotopy continuation methods. A random *lifting* ω is applied to the support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of $P(\mathbf{x})$ to obtain a *fine mixed subdivision* S_ω of $\mathcal{A}$ as well as the *supporting systems* induced by the *mixed cells* of type $(1, \ldots, 1)$ in S_ω. These supporting systems are the start systems for a finite set of nonlinear homotopies induced by ω.

To find all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$, rather than in $(\mathbb{C}^*)^n$, a modified algorithm, based on Theorem 2, was formulated in [10, 12]. By the revised algorithm, one can locate all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$ numerically, at the expense of following extraneous homotopy curves frequently. This wasteful computation may be eliminated by following the procedures suggested in [7]: First, identify the *stable mixed cells* of the extended support $\mathcal{A} \cup \{\mathbf{0}\}$ of $P(\mathbf{x})$ by applying an initial simple lifting on $\mathcal{A} \cup \{\mathbf{0}\}$. Followed by applying secondary recursive liftings to the stable mixed cells one obtains fine mixed subdivisions on these cells. Then standard polyhedral homotopies are applied to solve the polynomial subsystems corresponding to the resulting stable mixed cells. Finally, one may trace homotopy paths originated from these solutions of the subsystems to the zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$.

When polyhedral homotopy algorithms are used to find all the isolated zeros of polynomial systems, the most intensive computation rests upon the preprocessing step of identifying of proper mixed cells induced by the liftings. Therefore, the algorithm proposed in [7] may require a heavy preprocessing effort for its demand of recursive liftings. In order to produce a more efficient algorithm, we wish to avoid this scheme of recursive liftings.

The purpose of this dissertation is to present the strategy of a *single* lifting which can accomplish the goals of the multiple liftings of the above procedures simultaneously, so the preprocessing cost of applying polyhedral homotopy algorithms can be reduced considerably. As a by-product, in addition to solving all isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$, the *stable mixed volume* of $\mathcal{A}$ can easily be assembled without recursive liftings. Our single lifting, along with its theoretical justifications, will be given in Chapters 3 and 4 after the necessary terminology is introduced in Chapters 1 and 2. In accordance with our lifting, a new algorithm to find all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$ has been successfully implemented, and numerical results on a substantial variety of examples are presented in Chapter 5.

CHAPTER 1

Polyhedral Homotopy Method

Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ where for each $i = 1, \ldots, n$, $\mathcal{A}_i$ is a nonempty finite subset of $\mathbb{N}^n$. By a *cell* of $\mathcal{A}$ we mean a tuple $C = (C_1, \ldots, C_n)$ of subsets $C_i \subset \mathcal{A}_i$, for $i = 1, \ldots, n$. Define the short hand notations:

$$\text{type}(C) := (\dim(\text{conv}(C_1)), \ldots, \dim(\text{conv}(C_n))),$$

which is called the type of the cell C,

$$\text{conv}(C) := \text{conv}(C_1) + \cdots + \text{conv}(C_n),$$

the Minkowski sum of the convex hulls of $C_1, \ldots, C_n$, and

$$\text{Vol}_n(C) := \text{Vol}_n(\text{conv}(C)),$$

the n-dimensional Euclidean volume of $\text{conv}(C)$. A *face* of C is a subcell $F = (F_1, \ldots, F_n)$ of C where $F_i \subset C_i$ and some linear functional $\alpha \in (\mathbb{R}^n)^\vee$ attains its minimum over C_i at F_i for each $i = 1, \ldots, n$. We call such an α an *inner normal* of F. If F is a face of C then $\text{conv}(F_i)$ is a face of the polytope $\text{conv}(C_i)$ for each $i = 1, \ldots, n$.

Definition 1 [6] *A* subdivision *of $\mathcal{A}$ is a collection $\{C^{(1)}, \ldots, C^{(m)}\}$ of cells of $\mathcal{A}$ such that*

(a) *For all* $j = 1, \dots, m$, $\dim(\mathrm{conv}(C^{(j)})) = n$,

(b) $\mathrm{conv}(C^{(j)}) \cap \mathrm{conv}(C^{(k)})$ *is a proper common face of* $\mathrm{conv}(C^{(j)})$ *and* $\mathrm{conv}(C^{(k)})$ *when it is nonempty for* $j \neq k$,

(c) $\bigcup_{j=1}^{m} \mathrm{conv}(C^{(j)}) = \mathrm{conv}(\mathcal{A})$.

Furthermore, we call the collection a fine mixed subdivision *of* $\mathcal{A}$ *if it also satisfies the following condition:*

(d) *For* $j = 1, \dots, m$, *write* $C^{(j)} = (C_1^{(j)}, \dots, C_n^{(j)})$. *Then, each* $\mathrm{conv}(C_i^{(j)})$ *is a simplex of dimension* $\#C_i^{(j)} - 1$ *and for each* j,

$$\dim(\mathrm{conv}(C_1^{(j)})) + \cdots + \dim(\mathrm{conv}(C_n^{(j)})) = n.$$

A fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ can be found by the following standard process [6, 10]: Choose a real-valued function $\omega_i : \mathcal{A}_i \to \mathbb{R}$ for each $i = 1, \dots, n$. We call the n-tuple $\omega = (\omega_1, \dots, \omega_n)$ a *lifting function* on $\mathcal{A}$, and ω *lifts* $\mathcal{A}_i$ to its graph $\hat{\mathcal{A}}_i(\omega) = \{(\mathbf{q}, \omega_i(\mathbf{q})) : \mathbf{q} \in \mathcal{A}_i\} \subset \mathbb{R}^{n+1}$. This notation is extended in the obvious way: $\hat{\mathbf{q}}(\omega) = (\mathbf{q}, \omega_i(\mathbf{q}))$, $\hat{\mathcal{A}}(\omega) = (\hat{\mathcal{A}}_1(\omega), \dots, \hat{\mathcal{A}}_n(\omega))$, $\hat{\mathcal{Q}}_i(\omega) = \mathrm{conv}(\hat{\mathcal{A}}_i(\omega))$, $\hat{\mathcal{Q}}(\omega) = \hat{\mathcal{Q}}_1(\omega) + \cdots + \hat{\mathcal{Q}}_n(\omega)$, etc.

A *lower face* of a polytope in $\mathbb{R}^{n+1}$ is a face having an inner normal with positive $(n+1)$-th coordinate and a *lower facet* is an n-dimensional lower face. The collection

$$S_\omega = \left\{ C = (C_1, \dots, C_n) \text{ cells of } \mathcal{A} \;\middle|\; \begin{array}{l} \mathrm{conv}(\hat{C}(\omega)) \text{ is a lower facet of} \\ \hat{\mathcal{Q}}_1(\omega) + \cdots + \hat{\mathcal{Q}}_n(\omega) \end{array} \right\}$$

is the *subdivision of* $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ *induced by the lifting function* ω [6]. When $\omega = (\omega_1, \dots, \omega_n)$ is chosen generically, S_ω gives a fine mixed subdivision of $\mathcal{A}$ [6].

To find all isolated zeros of $P(\mathbf{x}) = (p_1(\mathbf{x}), \dots, p_n(\mathbf{x}))$ (with support $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$) in $(\mathbb{C}^*)^n$, we will use two homotopies. The first homotopy, called the *polyhedral homotopy*, is used to solve for all the isolated zeros in $(\mathbb{C}^*)^n$ of a new generic system G with the same support as P. The second homotopy, a more standard linear homotopy, uses these zeros of G to find all the isolated zeros of P in $(\mathbb{C}^*)^n$.

To form the new generic polynomial system mentioned above, we assign generic coefficients to all the monomials in $P(\mathbf{x})$. Denote the new system by $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ where

$$g_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} \bar{\mathbf{c}}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n,$$

and $\bar{\mathbf{c}}_{i,\mathbf{a}}$'s are randomly chosen complex numbers. We wish to find all the isolated zeros of this system in $(\mathbb{C}^*)^n$ in the first place. Then, by following all the homotopy paths of the homotopy

$$H(\mathbf{x}, t) = (1-t)G(\mathbf{x}) + tP(\mathbf{x}) = \mathbf{0}$$

emanating from those zeros of $G(\mathbf{x})$, one can obtain all the isolated zeros of $P(\mathbf{x})$ in $(\mathbb{C}^*)^n$ [6, 11].

To solve $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = \mathbf{0}$, we lift its support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ by a generically chosen real lifting function $\omega = (\omega_1, \ldots, \omega_n)$ and consider the polynomial system $\hat{G}(\mathbf{x}, t) = (\hat{g}_1(\mathbf{x}, t), \ldots, \hat{g}_n(\mathbf{x}, t))$ in the $n+1$ variables $x_1, \ldots, x_n, t$, where

$$\hat{g}_i(\mathbf{x}, t) = \sum_{\mathbf{a} \in \mathcal{A}_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{\omega_i(\mathbf{a})}, \quad i = 1, \ldots, n. \tag{1.1}$$

$\hat{G}(\mathbf{x}, t)$ provides a homotopy with t as the parameter and when $t = 1$, $\hat{G}(\mathbf{x}, 1) = G(\mathbf{x})$. It can be shown that for each $t \in (0, 1]$, the isolated zeros of $\hat{G}(\mathbf{x}, t)$ are all nonsingular, and by Theorem 1, the total number of these zeros is equal to the mixed volume $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. We write these zeros as $\mathbf{x}^1(t), \ldots, \mathbf{x}^k(t)$ where $k = \mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, so $\hat{G}(\mathbf{x}^j(t), t) = \mathbf{0}$ for each $t \in (0, 1]$ and $j = 1, \ldots, k$. Let $\mathbf{x}(t)$ represent any one of $\mathbf{x}^1(t), \ldots, \mathbf{x}^k(t)$, and write $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$.

The lifting function $\omega = (\omega_1, \ldots, \omega_n)$ induces a fine mixed subdivision S_ω of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and the mixed volume $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ equals the sum of the volumes of cells of type $(1,\ldots,1)$ in S_ω [6]. Let $C = (\{\mathbf{a}_{10},\mathbf{a}_{11}\},\ldots,\{\mathbf{a}_{n0},\mathbf{a}_{n1}\})$ be a cell of type $(1,\ldots,1)$ in S_ω and $\mathbf{v}_i = \mathbf{a}_{i1} - \mathbf{a}_{i0}$, $i = 1,\ldots,n$. Since S_ω is a fine mixed subdivision, $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is linearly independent, and a simple calculation shows that

$$\mathrm{Vol}_n(C) = \left| \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \right|. \tag{1.2}$$

Let $\hat{\alpha} = (\alpha, 1) = (\alpha_1,\ldots,\alpha_n,1)$ be the inner normal of

$$\mathrm{conv}(\hat{C}(\omega)) = \mathrm{conv}(\{\hat{\mathbf{a}}_{10}(\omega),\hat{\mathbf{a}}_{11}(\omega)\},\ldots,\{\hat{\mathbf{a}}_{n0}(\omega),\hat{\mathbf{a}}_{n1}(\omega)\}).$$

Substituting $\mathbf{x} = \mathbf{y}t^\alpha$, or $x_1 = y_1 t^{\alpha_1},\ldots,x_n = y_n t^{\alpha_n}$, into (1.1) yields,

$$\begin{aligned} \hat{g}_i(\mathbf{y},t) &= \sum_{\mathbf{a}\in\mathcal{A}_i} \bar{c}_{i,\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{\langle \mathbf{a},\alpha\rangle + \omega_i(\mathbf{a})} \\ &= \sum_{\mathbf{a}\in\mathcal{A}_i} \bar{c}_{i,\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{\langle \hat{\mathbf{a}}(\omega),\hat{\alpha}\rangle}, \quad i = 1,\ldots,n, \end{aligned} \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product in $\mathbb{R}^n$. Since $\hat{\alpha}$ is the inner normal of $\mathrm{conv}(\hat{C}(\omega))$, by factoring out the lowest power in t, $\hat{g}_i(\mathbf{y},t)$ becomes

$$r_i(\mathbf{y},t) := \bar{c}_{i,\mathbf{a}_{i0}} \mathbf{y}^{\mathbf{a}_{i0}} + \bar{c}_{i,\mathbf{a}_{i1}} \mathbf{y}^{\mathbf{a}_{i1}} + \text{ higher order terms in } t, \quad i = 1,\ldots,n. \tag{1.4}$$

Write $R(\mathbf{y},t) = (r_1(\mathbf{y},t),\ldots,r_n(\mathbf{y},t))$. Apparently,

$$r_i(\mathbf{y},0) = \bar{c}_{i,\mathbf{a}_{i0}} \mathbf{y}^{\mathbf{a}_{i0}} + \bar{c}_{i,\mathbf{a}_{i1}} \mathbf{y}^{\mathbf{a}_{i1}}, \quad i = 1,\ldots,n, \tag{1.5}$$

and $R(\mathbf{y},1) = \hat{G}(\mathbf{x},1) = G(\mathbf{x})$. The system $R(\mathbf{y},0) = \mathbf{0}$ in (1.5) is a binomial system with generic coefficients. This type of system can easily be solved [10] and it can be shown that the total number of its zeros equals $\mathrm{Vol}_n(C)$ in (1.2). So, by following the solution curves of $R(\mathbf{y},t) = \mathbf{0}$ starting from the solutions of $R(\mathbf{y},0) = \mathbf{0}$ in (1.5) we find $\mathrm{Vol}_n(C)$ isolated zeros of $G(\mathbf{x})$ in $(\mathbb{C}^*)^n$. Repeating the same procedure for each cell of type $(1,\ldots,1)$ in S_ω, all $\mathcal{M}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ isolated zeros of $G(\mathbf{x})$ in $(\mathbb{C}^*)^n$ can be found.

To find all the isolated zeros of $P(\mathbf{x}) = (p_1(\mathbf{x}),\ldots,p_n(\mathbf{x}))$ in $\mathbb{C}^n$, rather than in $(\mathbb{C}^*)^n$, we may modify the above procedure as follows: According to Theorem 2, when $(\mathcal{A}_1\cup\{\mathbf{0}\},\ldots,\mathcal{A}_n\cup\{\mathbf{0}\}) = (\mathcal{A}_1,\ldots,\mathcal{A}_n)$, i.e., all p_i's have nonzero constant terms, then the mixed volume $\mathcal{M}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ also serves as a bound for the number of isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$, and the algorithm we described above finds all isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$ indeed. When $(\mathcal{A}_1\cup\{\mathbf{0}\},\ldots,\mathcal{A}_n\cup\{\mathbf{0}\}) \neq (\mathcal{A}_1,\ldots,\mathcal{A}_n)$, we augment the monomial $\mathbf{x}^{\mathbf{0}}(=1)$ to those p_i's which do not have constant terms and randomly choose the coefficients of all monomials in $P(\mathbf{x})$ as well as augmented monomials $\mathbf{x}^{\mathbf{0}}$, obtaining the system $\bar{G}(\mathbf{x}) = (\bar{g}_1(\mathbf{x}),\ldots,\bar{g}_n(\mathbf{x}))$ where

$$\bar{g}_i(\mathbf{x}) = \sum_{\mathbf{a}\in\mathcal{A}_i\cup\{\mathbf{0}\}} \bar{c}_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}}, \quad i = 1,\ldots,n.$$

By Lemma 2.1 in [12], all isolated zeros of $\bar{G}(\mathbf{x})$ are in $(\mathbb{C}^*)^n$ and, by Theorem 1, the total number of its isolated zeros is equal to $\mathcal{M}(\mathcal{A}_1\cup\{\mathbf{0}\},\ldots,\mathcal{A}_n\cup\{\mathbf{0}\})$. It was shown in [11] that by following exactly the same procedure as we described above with $G(\mathbf{x})$ replaced by $\bar{G}(\mathbf{x})$, all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$ can be found.

In summary, to find all the isolated zeros of a given polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}),\ldots,p_n(\mathbf{x}))$ with support $\mathcal{A} = (\mathcal{A}_1,\ldots,\mathcal{A}_n)$ in $\mathbb{C}^n$ by the above method, which we will refer to as the Li-Wang algorithm in the remainder of this dissertation, one may proceed with the following steps:

- Lift the extended support $\mathcal{A}\cup\{\mathbf{0}\}$ by a randomly chosen real lifting function $\omega = (\omega_1,\ldots,\omega_n)$.
- Find all the cells of type $(1,\ldots,1)$ in the induced fine mixed subdivision S_ω for the extended support $\mathcal{A}\cup\{\mathbf{0}\}$.

- For a polynomial system $\bar{G}(\mathbf{x})$ with support $\mathcal{A} \cup \{\mathbf{0}\}$ and randomly chosen complex coefficients, trace the homotopy curves of $R(\mathbf{x}, t) = \mathbf{0}$ in (1.4) determined by the cells of type $(1, \ldots, 1)$ in S_ω to find all isolated zeros of $\bar{G}(\mathbf{x})$ in $(\mathbb{C}^*)^n$.

- Use the linear homotopy

$$H(\mathbf{x}, t) = (1-t)\bar{G}(\mathbf{x}) + tP(\mathbf{x}) = \mathbf{0} \tag{1.6}$$

  to find all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$. Isolated zeros of $H(\mathbf{x}, 0) = \bar{G}(\mathbf{x})$ in $\mathbb{C}^n$ are available after the last step.

As we can see, the main computation of this method is on

(a) Finding the cells of type $(1, \ldots, 1)$ in S_ω for the extended support $\mathcal{A} \cup \{\mathbf{0}\}$,

(b) Tracing $2\mathcal{M}(\mathcal{A} \cup \{\mathbf{0}\})$ homotopy curves.

The computation in (a) is quite time consuming. In general, cells of type $(1, \ldots, 1)$ in a subdivision induced by a lifting function ω are determined by an exhausting search among all the possible Minkowski sums of edges from $\hat{\mathcal{A}}_1(\omega), \ldots, \hat{\mathcal{A}}_n(\omega)$ by linear programming techniques [18] which requires an intensive computational effort. In (b), some of the homotopy curves in (1.6) may be extraneous. For instance, consider the bivariate system [7],

$$\begin{aligned} p_1(x,y) &= ay + by^2 + cxy^3, \\ p_2(x,y) &= dx + ex^2 + fx^3y. \end{aligned} \tag{1.7}$$

For generic coefficients $\{a, b, c, d, e, f\}$, this system has six isolated zeros in $\mathbb{C}^2$ and three isolated zeros in $(\mathbb{C}^*)^2$. However, its augmented system

$$\begin{aligned} \bar{g}_1(x,y) &= \varepsilon_1 + ay + by^2 + cxy^3, \\ \bar{g}_2(x,y) &= \varepsilon_2 + dx + ex^2 + fx^3y \end{aligned}$$

has eight isolated zeros in $\mathbb{C}^2$. So, one needs to follow eight homotopy paths of the homotopy $H(\mathbf{x}, t) = \mathbf{0}$ in (1.6) to find all six isolated zeros of system (1.7) in $\mathbb{C}^2$, and two of them are obviously extraneous.

By using the algorithm suggested by Huber and Sturmfels in [7] which we will describe in the next chapter, one can skip following those extraneous paths. Furthermore, by their method, isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n \backslash (\mathbb{C}^*)^n$ can be determined without following any paths in many situations or by following homotopy paths of much smaller systems. However, the trade-off is the requirement of the recursive liftings of the method, which drastically increases the computation effort in (a).

CHAPTER 2

Stable Mixed Volumes

For a *generic* polynomial system $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ with support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, where

$$g_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n,$$

define the homotopy $\hat{G}(\mathbf{x}, t) = (\hat{g}_1(\mathbf{x}, t), \ldots, \hat{g}_n(\mathbf{x}, t)) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ by

$$\hat{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}) + t^k \varepsilon_i, \quad i = 1, \ldots, n, \tag{2.1}$$

where k is a positive integer and $\varepsilon_i = 0$ if $g_i(\mathbf{x})$ has a nonzero constant term, otherwise ε_i is a randomly chosen complex number. This homotopy induces a lifting function $\omega^{0k} = (\omega_1^{0k}, \ldots, \omega_n^{0k})$ on the extended support $\mathcal{A} \cup \{\mathbf{0}\} = (\mathcal{A}_1 \cup \{\mathbf{0}\}, \ldots, \mathcal{A}_n \cup \{\mathbf{0}\})$ given by

$$\begin{aligned} \omega_i^{0k}(\mathbf{a}) &= 0 \quad \text{if} \quad \mathbf{a} \in \mathcal{A}_i, \\ \omega_i^{0k}(\mathbf{0}) &= k \quad \text{if} \quad \mathbf{0} \notin \mathcal{A}_i, \end{aligned} \qquad i = 1, \ldots, n. \tag{2.2}$$

Let $\mathcal{A}_i^0 = \mathcal{A}_i \cup \{\mathbf{0}\}$ for $i = 1, \ldots, n$ and $\mathcal{A}^0 = (\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0)$. Recall that for any cell $C = (C_1, \ldots, C_n)$ of $\mathcal{A}^0$, $\hat{C}(\omega^{0k}) = (\hat{C}_1(\omega^{0k}), \ldots, \hat{C}_n(\omega^{0k}))$ is a cell of $\hat{\mathcal{A}}^0(\omega^{0k})$, where

$$\hat{C}_i(\omega^{0k}) = \{(\mathbf{a}, \omega_i^{0k}(\mathbf{a})) : \mathbf{a} \in C_i\}, \quad i = 1, \ldots, n;$$

and

$$S_{\omega^{0k}} = \left\{ C = (C_1, \ldots, C_n) \text{ cells of } \mathcal{A}^0 \;\middle|\; \begin{array}{l} \operatorname{conv}(\hat{C}(\omega^{0k})) \text{ is a lower facet of} \\ \operatorname{conv}(\hat{\mathcal{A}}^0(\omega^{0k})) \end{array} \right\}$$

gives the *stable mixed subdivision* of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ [7].

The coefficients of $G(\mathbf{x})$ are assumed to be sufficiently generic in the sense of Theorem 1 so that system (2.1) has $\mathcal{M}(\mathcal{A}^0)$ isolated zeros in $(\mathbb{C}^*)^n$ for all but finitely many t and has no zeros in $\mathbb{C}^n \backslash (\mathbb{C}^*)^n$ for $t \neq 0$. The zeros of (2.1) as algebraic functions $\mathbf{x}(t)$ can be written by the Puiseux series expansion near $t = 0$ as

$$\mathbf{x}(t) = \mathbf{e}t^{\alpha} + \text{ higher order terms in } t, \tag{2.3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$ and $(\alpha, 1)$ is the unique inner normal of $\operatorname{conv}(\hat{C}(\omega^{0k}))$ whose last coordinate is equal to one for some cell $C = (C_1, \ldots, C_n)$ of $S_{\omega^{0k}}$ and $\mathbf{e} = (e_1, \ldots, e_n) \in (\mathbb{C}^*)^n$ is a root of the system

$$g_{i\alpha}(\mathbf{x}) := \sum_{\mathbf{a} \in C_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} = 0, \quad i = 1, \ldots, n,$$

which is determined by the cell C. A branch $\mathbf{x}(t)$ converges to a solution of $G(\mathbf{x}) = \mathbf{0}$ in $\mathbb{C}^n$ as $t \to 0$ precisely when the exponents $\alpha = (\alpha_1, \ldots, \alpha_n)$ are nonnegative, while the i-th coordinate of such a solution can vanish only when $\alpha_i > 0$. This observation leads to the following definitions [7]:

Let $C = (C_1, \ldots, C_n)$ be a cell of $S_{\omega^{0k}}$ and $(\alpha^C, 1) = (\alpha_1^C, \ldots, \alpha_n^C, 1)$ be the unique inner normal of $\operatorname{conv}(\hat{C}(\omega^{0k}))$ in $\operatorname{conv}(\hat{\mathcal{A}}^0(\omega^{0k}))$ whose last coordinate is equal to 1. In general, when C is a cell of S_ω induced by a lifting ω, we shall call such α^C the *inner normal* of the cell C with respect to ω. A cell C of $S_{\omega^{0k}}$ is said to be *stable* if α^C is nonnegative. A cell C of $S_{\omega^{0k}}$ is called a *stable mixed cell* of $\mathcal{A}$ if it is stable and has nonzero mixed volume. For support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we define its *stable mixed volume*, denoted by $\mathcal{SM}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, to be the sum of the mixed volumes $\mathcal{M}(C_1, \ldots, C_n)$ over all stable mixed cells $C = (C_1, \ldots, C_n)$ of $\mathcal{A}$ in $S_{\omega^{0k}}$.

Since the points of $\mathcal{A}_i$ remain unlifted under ω^{0k}, the cell $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ appears as a cell of the subdivision $S_{\omega^{0k}}$. It is, in fact, the unique cell C in $S_{\omega^{0k}}$ with $\alpha^C = \mathbf{0}$. This stable mixed cell contributes $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ branches in (2.3) which converge to points in $(\mathbb{C}^*)^n$ when $t \to 0$. Each other stable mixed cell C of $\mathcal{A}$ in $S_{\omega^{0k}}$ contributes, by Theorem 1, $\mathcal{M}(C)$ branches converging to points in $\mathbb{C}^n \backslash (\mathbb{C}^*)^n$ as $t \to 0$. By (2.1), those points constitute the full set of isolated zeros of $G(\mathbf{x})$ in $\mathbb{C}^n$.

We summarize the above discussion in the following theorem.

Theorem 3 [7] *Counting multiplicities, the number of isolated zeros of* $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ *in* $\mathbb{C}^n$ *with support* $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *is bounded above by the stable mixed volume* $\mathcal{SM}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. *This bound is exact for* $P(\mathbf{x})$ *with generic coefficients, provided that* $P(\mathbf{x})$ *has only finitely many isolated zeros in* $\mathbb{C}^n$.

Remark 1 The stable mixed volume was originally defined in [7] with $k = 1$. It is easy to see that if C is a cell of $S_{\omega^{01}}$ with inner normal α^C with respect to ω^{01}, then C is also a cell of $S_{\omega^{0k}}$ with inner normal $k\alpha^C$ with respect to ω^{0k} for any real $k > 0$. Consequently, the set of stable mixed cells remains invariant as k varies since $k\alpha^C$ is nonnegative as long as α^C is nonnegative. This variation plays an important role in our construction in the next chapter.

Based on the derivation of Theorem 3, an algorithm for finding all isolated zeros of a polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ in $\mathbb{C}^n$ with support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ where

$$p_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n,$$

which we will refer to as the Huber-Sturmfels algorithm, was suggested in [7] as follows: First of all, if all p_i's have nonzero constant terms, then, as indicated before, the standard polyhedral homotopy described in the beginning of Chapter 1 can find all the isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$. When some of the p_i's have no constant terms, namely, $(\mathcal{A}_1 \cup \{\mathbf{0}\}, \ldots, \mathcal{A}_n \cup \{\mathbf{0}\}) \neq (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, then

- Let

$$\hat{p}_i(\mathbf{x}) = \sum_{\mathbf{a}\in\mathcal{A}_i} c_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} + \varepsilon_i, \quad i = 1, \ldots, n,$$

where ε_i is randomly chosen and is set to be zero if $p_i(\mathbf{x})$ already has a nonzero constant term.

- Use lifting function ω^{0k} on the extended support $\mathcal{A} \cup \{\mathbf{0}\} = (\mathcal{A}_1 \cup \{\mathbf{0}\}, \ldots, \mathcal{A}_n \cup \{\mathbf{0}\})$ and identify all the stable mixed cells of $\mathcal{A}$ in the induced subdivision $S_{\omega^{0k}}$. Let S be the set of those cells.

- For each cell $C = (C_1, \ldots, C_n) \in S$, let $\alpha^C = (\alpha_1^C, \ldots, \alpha_n^C)$ be its inner normal with respect to ω^{0k} with nonnegative components, and find the zeros of $\hat{P}_{\alpha^C}(\mathbf{x}) = (\hat{p}_{1\alpha^C}(\mathbf{x}), \ldots, \hat{p}_{n\alpha^C}(\mathbf{x}))$ where

$$\hat{p}_{i\alpha^C}(\mathbf{x}) = \sum_{\mathbf{a}\in C_i\cap\mathcal{A}_i} c_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} + \beta_i\varepsilon_i, \quad i = 1, \ldots, n, \tag{2.4}$$

$$\beta_i = \begin{cases} 1 & \text{if } \mathbf{0} \notin \mathcal{A}_i \text{ but } \mathbf{0} \in C_i, \\ 0 & \text{otherwise} \end{cases}$$

in $(\mathbb{C}^*)^n$ by the standard polyhedral homotopy described in Chapter 1. For each zero $\mathbf{e} = (e_1, \ldots, e_n)$ of (2.4), let $\bar{\mathbf{e}} = (\bar{e}_1, \ldots, \bar{e}_n)$ where

$$\bar{e}_i = \begin{cases} e_i & \text{if } \ \alpha_i^C = 0, \\ 0 & \text{if } \ \alpha_i^C > 0. \end{cases}$$

Then $\bar{\mathbf{e}}$ is a zero of $P(\mathbf{x})$.

If a stable mixed cell $C = (C_1, \ldots, C_n)$ in S is of type $(1, \ldots, 1)$, then system (2.4) becomes a binomial system which can be solved easily by conventional techniques. For a stable mixed cell of type different from $(1, \ldots, 1)$ whose inner normal has some zero components, such as the cell $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, further lifting is required to find a fine mixed subdivision of this cell before the polyhedral homotopy method described in Chapter 1 can be used to obtain all isolated zeros of the system (2.4) in $(\mathbb{C}^*)^n$.

As we mentioned before, multiple liftings and the identification of cells of type $(1,\ldots,1)$ in their induced subdivisions require an intensive computation effort and occupy a great majority of the computation of this algorithm. Therefore, compared to the Li-Wang algorithm in Chapter 1, this algorithm may cost more in many situations despite it follows no extraneous homotopy paths.

Remark 2 Isolated zeros $\mathbf{e} = (e_1,\ldots,e_n)$ of system (2.4) in $(\mathbb{C}^*)^n$ involve parameter $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$. However, from the proof of Theorem 3 in [7], it can be easily shown that the transition from $\mathbf{e}$ to $\bar{\mathbf{e}}$ in the last step of the algorithm makes the ε-dependent components of $\mathbf{e} = (e_1,\ldots,e_n)$ zero, and the zero $\bar{\mathbf{e}}$ of $P(\mathbf{x})$ we obtain eventually is independent of ε.

In [7], it was suggested to solve

$$\sum_{\mathbf{a}\in C_i\cap\mathcal{A}_i} c_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} = 0, \quad i = 1,\ldots,n \tag{2.5}$$

without ε instead of finding zeros of (2.4). In that case, one must find all isolated solutions of (2.5) in $\mathbb{C}^n$ rather than in $(\mathbb{C}^*)^n$.

CHAPTER 3

A Single Lifting

In this chapter, we shall present the strategy of a single lifting on the extended support $\mathcal{A}^0 = (\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0) = (\mathcal{A}_1 \cup \{\mathbf{0}\}, \ldots, \mathcal{A}_n \cup \{\mathbf{0}\})$ of $P(\mathbf{x})$ when $\mathcal{A}^0 \neq \mathcal{A}$. This lifting can identify all the stable mixed cells of $\mathcal{A}$ and, in the mean time, provide a fine mixed subdivision for each stable mixed cell. Consequently, the stable mixed volume of the support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ can be calculated by those fine mixed subdivisions induced by this lifting. Most importantly, recursive liftings are no longer needed as opposed to the Huber-Sturmfels algorithm described in Chapter 2.

Let $\mathcal{B}_i$ be a nonempty finite subset of $\mathbb{N}^n$ for $i = 1, \ldots, n$, and $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_n)$. Let S_ω be the fine mixed subdivision of $\mathcal{B}$ induced by a lifting function $\omega = (\omega_1, \ldots, \omega_n)$ applied to $\mathcal{B}$. For a cell $D = (D_1, \ldots, D_n)$ of S_ω, write

$$D_i = \{\mathbf{a}_{i0}, \ldots, \mathbf{a}_{ik_i}\}, \quad i = 1, \ldots, n,$$

where $k_i \geq 0$ and $k_1 + \cdots + k_n = n$. Let $V(\hat{D}(\omega))$ be the $n \times (n+1)$ matrix whose rows consist of $\hat{\mathbf{a}}_{ij}(\omega) - \hat{\mathbf{a}}_{i0}(\omega)$ for $i = 1, \ldots, n$, $j = 1, \ldots, k_i$ with $k_i \geq 1$, and $V(D)$ be the corresponding $n \times n$ matrix by deleting the last column of $V(\hat{D}(\omega))$. It is easy to see that

$$\mathrm{Vol}_n(\mathrm{conv}(D)) = |\det(V(D))|$$

which is nonzero since $\dim(\mathrm{conv}(D)) = n$.

Consider the linear function

$$f_{\hat{D}(\omega)}(\mathbf{x},t) := \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1} t := \det \begin{pmatrix} x_1 \ \ldots \ x_n \ t \\ V(\hat{D}(\omega)) \end{pmatrix}.$$

We may assume that α_{n+1}, the cofactor of t, is positive, namely $(-1)^n \det(V(D)) > 0$, otherwise, we exchange two rows of $V(\hat{D}(\omega))$. Let $\hat{\mathbf{a}}_0(\omega) = \hat{\mathbf{a}}_{10}(\omega) + \cdots + \hat{\mathbf{a}}_{n0}(\omega)$. The following lemma is the main tool in our analysis.

Lemma 1 *The hyperplane* $L : f_{\hat{D}(\omega)}(\mathbf{x},t) = f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_0(\omega))$ *is the supporting hyperplane of* $\text{conv}(\hat{\mathcal{B}}(\omega))$ *which contains the lower facet* $\text{conv}(\hat{D}(\omega))$ *and* $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ *is an inner normal of* $\text{conv}(\hat{D}(\omega))$.

PROOF: To prove the hyperplane L contains $\text{conv}(\hat{D}(\omega))$, it suffices to show that the points of the form

$$\hat{\mathbf{a}}_{1j_1}(\omega) + \cdots + \hat{\mathbf{a}}_{nj_n}(\omega) \quad \text{where } 0 \le j_i \le k_i,\ i = 1, \ldots, n$$

all belong to L. Since

$$\begin{aligned} l &:= f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{1j_1}(\omega) + \cdots + \hat{\mathbf{a}}_{nj_n}(\omega)) - f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_0(\omega)) \\ &= f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{1j_1}(\omega) + \cdots + \hat{\mathbf{a}}_{nj_n}(\omega)) - f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{10}(\omega) + \cdots + \hat{\mathbf{a}}_{n0}(\omega)) \\ &= f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{1j_1}(\omega) - \hat{\mathbf{a}}_{10}(\omega)) + \cdots + f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{nj_n}(\omega) - \hat{\mathbf{a}}_{n0}(\omega)) \end{aligned}$$

and for $i = 1, \ldots, n$, $\hat{\mathbf{a}}_{ij_i}(\omega) - \hat{\mathbf{a}}_{i0}(\omega)$ is either $\mathbf{0}$ or a row of $V(\hat{D}(\omega))$, so, $f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_{ij_i}(\omega) - \hat{\mathbf{a}}_{i0}(\omega)) = 0$ for all i, and therefore $l = 0$. Hence, $\text{conv}(\hat{D}(\omega)) \subset L$.

Since $\text{conv}(\hat{D}(\omega))$ is a lower facet of $\text{conv}(\hat{\mathcal{B}}(\omega))$ and $\alpha_{n+1} > 0$, $(\alpha_1, \ldots, \alpha_{n+1})$, the normal of L, is an inner normal of $\text{conv}(\hat{D}(\omega))$. □

We now define our single lifting $\omega = (\omega_1, \ldots, \omega_n)$ on $\mathcal{A}^0 = (\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0)$ as follows: For $i = 1, \ldots, n$,

$$\begin{aligned} &\omega_i(\mathbf{0}) = k \quad \text{if } \mathbf{0} \notin \mathcal{A}_i, \quad \text{here } k > 0 \text{ is randomly chosen,} \\ &\omega_i(\mathbf{a}) = \text{a randomly chosen number in } (0,1) \quad \text{if } \mathbf{a} \in \mathcal{A}_i. \end{aligned} \tag{3.1}$$

Since the values of ω are generically chosen, the induced subdivision

$$S_\omega = \left\{ D = (D_1, \ldots, D_n) \text{ cells of } \mathcal{A}^0 \;\middle|\; \begin{array}{l} \text{conv}(\hat{D}(\omega)) \text{ is a lower facet of} \\ \text{conv}(\hat{\mathcal{A}}^0(\omega)) \end{array} \right\}$$

is a fine mixed subdivision of $\mathcal{A}^0$ [6].

Recall that the stable mixed volume of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is derived from the lifting $\omega^{0k} = (\omega_1^{0k}, \ldots, \omega_n^{0k})$ on $\mathcal{A}^0$, as defined in (2.2), along with its induced subdivision

$$S_{\omega^{0k}} = \left\{ C = (C_1, \ldots, C_n) \text{ cells of } \mathcal{A}^0 \;\middle|\; \begin{array}{l} \text{conv}(\hat{C}(\omega^{0k})) \text{ is a lower facet of} \\ \text{conv}(\hat{\mathcal{A}}^0(\omega^{0k})) \end{array} \right\}.$$

For a cell $C = (C_1, \ldots, C_n)$ of $S_{\omega^{0k}}$, let $\omega^C = (\omega_1^C, \ldots, \omega_n^C)$ be the restriction of the function $\omega = (\omega_1, \ldots, \omega_n)$ on C. Its induced subdivision

$$S_{\omega^C} = \left\{ D = (D_1, \ldots, D_n) \text{ cells of } C \;\middle|\; \begin{array}{l} \text{conv}(\hat{D}(\omega)) \text{ is a lower facet of} \\ \text{conv}(\hat{C}(\omega)) \end{array} \right\}$$

gives a fine mixed subdivision of C since ω^C is generic on C. Our main claim is that when the value of k in the lifting ω is sufficiently large, then $S_{\omega^C} \subset S_\omega$. That is, the subdivision S_ω induced by the lifting ω on $\mathcal{A}^0$ does not alter the original configuration of the subdivision $S_{\omega^{0k}}$ induced by the lifting ω^{0k} on $\mathcal{A}^0$. More precisely, S_ω is *finer* than $S_{\omega^{0k}}$ in the sense that any cell $D = (D_1, \ldots, D_n)$ of S_ω is a subcell of a cell C of $S_{\omega^{0k}}$. Consequently, subcollections of cells of S_ω provide fine mixed subdivisions of cells of $S_{\omega^{0k}}$.

In the remainder of this dissertation, we let $d = \max_{1 \le i \le n} \deg p_i(\mathbf{x})$.

Proposition 1 *When $k > n(n+1)d^n$, then for cells $C = (C_1, \ldots, C_n)$ of $S_{\omega^{0k}}$, we have $S_{\omega^C} \subseteq S_\omega$.*

To prove Proposition 1, we first present the preliminaries. Let $D = (D_1, \ldots, D_n)$ be a cell of S_{ω^C} and

$$D_i = \{\mathbf{a}_{i0}, \ldots, \mathbf{a}_{ik_i}\}, \quad i = 1, \ldots, n,$$

where $k_i \geq 0$ and $k_1 + \cdots + k_n = n$. Then

$$V(D) = \begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{10} \\ \vdots \\ \mathbf{a}_{1k_1} - \mathbf{a}_{10} \\ \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} \\ \vdots \\ \mathbf{a}_{nk_n} - \mathbf{a}_{n0} \end{pmatrix}.$$

Note that if $k_i = 0$, then D_i does not contribute any row to the $n \times n$ matrix $V(D)$. Let $\mathbf{a}_i \in \mathcal{A}_i^0$ and E be a matrix obtained by replacing one row of $V(D)$ by $\mathbf{a}_i - \mathbf{a}_{i0}$.

Lemma 2 $|\det(E)| \leq d^n$.

PROOF: We assume the rows of the matrix E are linearly independent, otherwise, $\det(E) = 0$. For notational simplicity, we rewrite the rows of E as,

$$\begin{pmatrix} \mathbf{b}_{11} - \mathbf{b}_{10} \\ \vdots \\ \mathbf{b}_{n1} - \mathbf{b}_{n0} \end{pmatrix}.$$

Since $\{\mathbf{b}_{i0}, \mathbf{b}_{i1}\} \subseteq \mathcal{A}_{j_i}^0$ for $i = 1, \ldots, n$ and $1 \leq j_i \leq n$, $C^E = (\{\mathbf{b}_{10}, \mathbf{b}_{11}\}, \ldots, \{\mathbf{b}_{n0}, \mathbf{b}_{n1}\})$ is a cell of the support $(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0)$ of the new polynomial system $P^E(\mathbf{x}) = (p_{j_1}(\mathbf{x}) + \varepsilon_{j_1}, \ldots, p_{j_n}(\mathbf{x}) + \varepsilon_{j_n})$. Here, $j_1, \ldots, j_n$ may not be all different. A positive random lifting on $(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0)$ can always be arranged for which the cell C^E remains unlifted, that is, the points in C^E receive the lifting value 0 and lifting values for points not in C^E are all positive. For such a lifting, C^E becomes a type $(1, \ldots, 1)$ cell of the resulting induced fine mixed subdivision of $(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0)$. Consequently,

$$|\det(E)| = \mathrm{Vol}_n(\mathrm{conv}(C^E)) \leq \mathcal{M}(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0)$$

$$\begin{aligned} &\leq \Pi_{i=1}^{n} \deg p_{j_i}(\mathbf{x}) \quad \text{(the total degree of } P^E(\mathbf{x})) \\ &\leq d^n. \qquad\qquad\qquad\qquad \square \end{aligned}$$

Remark 3 The most important case is when the cell $D = (D_1, \ldots, D_n)$ is of type $(1, \ldots, 1)$. Let $d_i = \deg p_i(\mathbf{x})$ for $i = 1, \ldots, n$. Without loss of generality, we assume $d_1 \leq d_2 \leq \cdots \leq d_n$. Then a similar argument shows

$$\begin{aligned} |\det(E)| &\leq \mathcal{M}(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0) \\ &\leq d_2 \times \cdots \times d_n \times d_n. \end{aligned}$$

For sparse polynomial systems arose in applications, both $\mathcal{M}(\mathcal{A}_{j_1}^0, \ldots, \mathcal{A}_{j_n}^0)$ and $d_2 \times \cdots \times d_n \times d_n$ are usually much smaller than d^n.

Lemma 3 *For any* $\mathbf{a}_i, \mathbf{b}_i \in \mathcal{A}_i^0$, *let*

$$\beta_i = [\omega_i(\mathbf{a}_i) - \omega_i(\mathbf{b}_i)] - [\omega_i^{0k}(\mathbf{a}_i) - \omega_i^{0k}(\mathbf{b}_i)], \quad i = 1, \ldots, n.$$

Then $-1 \leq \beta_i \leq 1$ *and* β_i *is independent of* k *for all* $i = 1, \ldots, n$.

PROOF: For fixed i, if $\mathbf{0} \in \mathcal{A}_i$, or $\mathbf{0} \notin \mathcal{A}_i$ but none of $\mathbf{a}_i, \mathbf{b}_i$ equals $\mathbf{0}$, then $\omega_i^{0k}(\mathbf{a}_i) = \omega_i^{0k}(\mathbf{b}_i) = 0$ and both $\omega_i(\mathbf{a}_i)$ and $\omega_i(\mathbf{b}_i)$ are between 0 and 1. So, $\beta_i \in (-1, 1)$. If one of $\mathbf{a}_i, \mathbf{b}_i$ is $\mathbf{0} \notin \mathcal{A}_i$, say $\mathbf{a}_i$, then $\omega_i(\mathbf{a}_i) = \omega_i^{0k}(\mathbf{a}_i) = k$ and $\omega_i^{0k}(\mathbf{b}_i) = 0$, and so, $\beta_i = -\omega_i(\mathbf{b}_i) \in (-1, 1)$. $\square$

Proof of Proposition 1: Since D is a cell of C which by itself is a cell of the subdivision $S_{\omega^{0k}}$ of $\mathcal{A}^0$, so, by Lemma 1 the supporting hyperplane which contains

$\text{conv}(\hat{D}(\omega^{0k}))$ is $f_{\hat{D}(\omega^{0k})}(\mathbf{x},t) = f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}_0(\omega^{0k}))$ where

$$f_{\hat{D}(\omega^{0k})}(\mathbf{x},t) = \det\begin{pmatrix} (\mathbf{x},t) \\ \hat{\mathbf{a}}_{11}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{1k_1}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{nk_n}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \end{pmatrix}$$

and $\hat{\mathbf{a}}_0(\omega^{0k}) = \hat{\mathbf{a}}_{10}(\omega^{0k}) + \cdots + \hat{\mathbf{a}}_{n0}(\omega^{0k})$. Again, only those D_i's with $k_i \geq 1$ contribute to the rows of the above matrix. Assume the normal of this supporting hyperplane is an inner normal of $\text{conv}(\hat{D}(\omega^{0k}))$, i.e., the coefficient of t in $f_{\hat{D}(\omega^{0k})}(\mathbf{x},t)$ is positive.

On the other hand, since D is a cell of S_{ω^C}, the supporting hyperplane of $\hat{C}(\omega)$ containing $\text{conv}(\hat{D}(\omega))$ is $f_{\hat{D}(\omega)}(\mathbf{x},t) = f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_0(\omega))$ where

$$f_{\hat{D}(\omega)}(\mathbf{x},t) = \det\begin{pmatrix} (\mathbf{x},t) \\ \hat{\mathbf{a}}_{11}(\omega) - \hat{\mathbf{a}}_{10}(\omega) \\ \vdots \\ \hat{\mathbf{a}}_{1k_1}(\omega) - \hat{\mathbf{a}}_{10}(\omega) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega) - \hat{\mathbf{a}}_{n0}(\omega) \\ \vdots \\ \hat{\mathbf{a}}_{nk_n}(\omega) - \hat{\mathbf{a}}_{n0}(\omega) \end{pmatrix},$$

and $\hat{\mathbf{a}}_0(\omega) = \hat{\mathbf{a}}_{10}(\omega) + \cdots + \hat{\mathbf{a}}_{n0}(\omega)$.

To prove D is a cell of S_ω, we need to show that $\text{conv}(\hat{D}(\omega))$ is a lower facet of $\text{conv}(\hat{\mathcal{A}}^0(\omega))$. That is, for any $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_n \in \text{conv}(\mathcal{A}^0)\backslash\text{conv}(D)$ where $\mathbf{a}_i \in \mathcal{A}_i^0$

for $i = 1, \ldots, n$, we want to prove $f_{\hat{D}(\omega)}(\hat{\mathbf{a}}(\omega)) > f_{\hat{D}(\omega)}(\hat{\mathbf{a}}_0(\omega))$, or,

$$f_{\hat{D}(\omega)}(\hat{\mathbf{a}}(\omega) - \hat{\mathbf{a}}_0(\omega)) > 0. \tag{3.2}$$

If $\mathbf{a} \in \operatorname{conv}(C)$, then the inequality in (3.2) is true since D is a cell of S_{ω^C} and $\omega^C \equiv \omega$ on C. If $\mathbf{a} \notin \operatorname{conv}(C)$, we actually have

$$f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}(\omega^{0k})) > f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}_0(\omega^{0k})).$$

Namely,

$$f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}(\omega^{0k}) - \hat{\mathbf{a}}_0(\omega^{0k})) = \det \begin{pmatrix} \hat{\mathbf{a}}(\omega^{0k}) - \hat{\mathbf{a}}_0(\omega^{0k}) \\ \hat{\mathbf{a}}_{11}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{1k_1}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{nk_n}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \end{pmatrix} > 0.$$

The entries of the last column of the matrix above consist of either 0 or $\pm k$ except the first entry which equals $\pm m_1 k$ for certain integer $0 \leq m_1 \leq n$. So, expending the determinant by its last column gives

$$f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}(\omega^{0k}) - \hat{\mathbf{a}}_0(\omega^{0k})) = mk,$$

where m is a positive integer. Now,

$$f_{\hat{D}(\omega)}(\hat{\mathbf{a}}(\omega) - \hat{\mathbf{a}}_0(\omega)) = \det \begin{pmatrix} \mathbf{a} - \mathbf{a}_0 & \sum_{i=1}^{n}[\omega_i(\mathbf{a}_i) - \omega_i(\mathbf{a}_{i0})] \\ \mathbf{a}_{11} - \mathbf{a}_{10} & \omega_1(\mathbf{a}_{11}) - \omega_1(\mathbf{a}_{10}) \\ \vdots & \\ \mathbf{a}_{1k_1} - \mathbf{a}_{10} & \omega_1(\mathbf{a}_{1k_1}) - \omega_1(\mathbf{a}_{10}) \\ \vdots & \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} & \omega_n(\mathbf{a}_{n1}) - \omega_n(\mathbf{a}_{n0}) \\ \vdots & \\ \mathbf{a}_{nk_n} - \mathbf{a}_{n0} & \omega_n(\mathbf{a}_{nk_n}) - \omega_n(\mathbf{a}_{n0}) \end{pmatrix}$$

$$= \det \begin{pmatrix} \mathbf{a} - \mathbf{a}_0 & \sum_{i=1}^{n}[\omega_i^{0k}(\mathbf{a}_i) - \omega_i^{0k}(\mathbf{a}_{i0})] + \beta_0 \\ \mathbf{a}_{11} - \mathbf{a}_{10} & \omega_1^{0k}(\mathbf{a}_{11}) - \omega_1^{0k}(\mathbf{a}_{10}) + \beta_{11} \\ \vdots & \\ \mathbf{a}_{1k_1} - \mathbf{a}_{10} & \omega_1^{0k}(\mathbf{a}_{1k_1}) - \omega_1^{0k}(\mathbf{a}_{10}) + \beta_{1k_1} \\ \vdots & \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} & \omega_n^{0k}(\mathbf{a}_{n1}) - \omega_n^{0k}(\mathbf{a}_{n0}) + \beta_{n1} \\ \vdots & \\ \mathbf{a}_{nk_n} - \mathbf{a}_{n0} & \omega_n^{0k}(\mathbf{a}_{nk_n}) - \omega_n^{0k}(\mathbf{a}_{n0}) + \beta_{nk_n} \end{pmatrix}$$

where for $i = 1, \ldots, n$ and $1 \leq j \leq k_i$,

$$\beta_0 = \sum_{i=1}^{n}[\omega_i(\mathbf{a}_i) - \omega_i(\mathbf{a}_{i0})] - \sum_{i=1}^{n}[\omega_i^{0k}(\mathbf{a}_i) - \omega_i^{0k}(\mathbf{a}_{i0})]$$

and

$$\beta_{ij} = [\omega_i(\mathbf{a}_{ij}) - \omega_i(\mathbf{a}_{i0})] - [\omega_i^{0k}(\mathbf{a}_{ij}) - \omega_i^{0k}(\mathbf{a}_{i0})].$$

It follows that

$$
\begin{aligned}
f_{\hat{D}(\omega)}(\hat{\mathbf{a}}(\omega)-\hat{\mathbf{a}}_0(\omega)) &= \det\begin{pmatrix} \hat{\mathbf{a}}(\omega^{0k})-\hat{\mathbf{a}}_0(\omega^{0k}) \\ \hat{\mathbf{a}}_{11}(\omega^{0k})-\hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{1k_1}(\omega^{0k})-\hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega^{0k})-\hat{\mathbf{a}}_{n0}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{nk_n}(\omega^{0k})-\hat{\mathbf{a}}_{n0}(\omega^{0k}) \end{pmatrix} + \det\begin{pmatrix} \mathbf{a}-\mathbf{a}_0 & \beta_0 \\ \mathbf{a}_{11}-\mathbf{a}_{10} & \beta_{11} \\ \vdots & \\ \mathbf{a}_{1k_1}-\mathbf{a}_{10} & \beta_{1k_1} \\ \vdots & \\ \mathbf{a}_{n1}-\mathbf{a}_{n0} & \beta_{n1} \\ \vdots & \\ \mathbf{a}_{nk_n}-\mathbf{a}_{n0} & \beta_{nk_n} \end{pmatrix} \\
&= f_{\hat{D}(\omega^{0k})}(\hat{\mathbf{a}}(\omega^{0k})-\hat{\mathbf{a}}_0(\omega^{0k})) + \det\begin{pmatrix} \mathbf{a}-\mathbf{a}_0 & \beta_0 \\ \mathbf{a}_{11}-\mathbf{a}_{10} & \beta_{11} \\ \vdots & \\ \mathbf{a}_{1k_1}-\mathbf{a}_{10} & \beta_{1k_1} \\ \vdots & \\ \mathbf{a}_{n1}-\mathbf{a}_{n0} & \beta_{n1} \\ \vdots & \\ \mathbf{a}_{nk_n}-\mathbf{a}_{n0} & \beta_{nk_n} \end{pmatrix} \\
&= mk + \sum_{i=1}^{n} \det\begin{pmatrix} \mathbf{a}_i-\mathbf{a}_{i0} & \beta_{i0} \\ \mathbf{a}_{11}-\mathbf{a}_{10} & \beta_{11} \\ \vdots & \\ \mathbf{a}_{1k_1}-\mathbf{a}_{10} & \beta_{1k_1} \\ \vdots & \\ \mathbf{a}_{n1}-\mathbf{a}_{n0} & \beta_{n1} \\ \vdots & \\ \mathbf{a}_{nk_n}-\mathbf{a}_{n0} & \beta_{nk_n} \end{pmatrix}
\end{aligned}
$$

where, by Lemma 3, for all $1 \leq i \leq n$ and $1 \leq j \leq k_i$,

$$\beta_{ij} \in (-1, 1) \quad \text{and} \quad \beta_{i0} = [\omega_i(\mathbf{a}_i) - \omega_i(\mathbf{a}_{i0})] - [\omega_i^{0k}(\mathbf{a}_i) - \omega_i^{0k}(\mathbf{a}_{i0})] \in (-1, 1).$$

Expanding the determinants above according to their last columns whose elements all have absolute values less than one, we have, by Lemma 2,

$$\left| \det \begin{pmatrix} \mathbf{a}_i - \mathbf{a}_{i0} & \beta_{i0} \\ \mathbf{a}_{11} - \mathbf{a}_{10} & \beta_{11} \\ \vdots & \\ \mathbf{a}_{1k_1} - \mathbf{a}_{10} & \beta_{1k_1} \\ \vdots & \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} & \beta_{n1} \\ \vdots & \\ \mathbf{a}_{nk_n} - \mathbf{a}_{n0} & \beta_{nk_n} \end{pmatrix} \right| < (n+1)d^n, \quad i = 1, \ldots, n.$$

Thus, $f_{\hat{D}(\omega)}(\hat{\mathbf{a}}(\omega) - \hat{\mathbf{a}}_0(\omega)) > mk - n(n+1)d^n > 0$ since $k > n(n+1)d^n$ and m is a positive integer, and thus (3.2) is proved. $\square$

We are now in a position to identify the stable mixed cells of $\mathcal{A}^0$ and their fine mixed subdivisions induced by the lifting ω. Let

$$S_{\omega\mathcal{A}} = \{D = (D_1, \ldots, D_n) \in S_\omega \mid D_i \subseteq \mathcal{A}_i, i = 1, \ldots, n\}.$$

Apparently, $S_{\omega\mathcal{A}}$ gives a fine mixed subdivision of the stable mixed cell $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. For $D = (D_1, \ldots, D_n) \in S_\omega \backslash S_{\omega\mathcal{A}}$, where $D_i = \{\mathbf{a}_{i0}, \ldots, \mathbf{a}_{ik_i}\}$ for

$i = 1, \ldots, n$, the $n \times (n+1)$ matrix

$$V(\hat{D}(\omega^{0k})) = \begin{pmatrix} \hat{\mathbf{a}}_{11}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{1k_1}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{nk_n}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \end{pmatrix}$$

is of rank n. Therefore, one can find a vector $\hat{\alpha}^D = (\alpha^D, 1) = (\alpha_1^D, \ldots, \alpha_n^D, 1)$ such that

$$V(\hat{D}(\omega^{0k}))\hat{\alpha}^D = \mathbf{0}. \tag{3.3}$$

Clearly, $\alpha^D = (\alpha_1^D, \ldots, \alpha_n^D)$ is the inner normal of D with respect to the lifting ω^{0k}. For the collection of cells $\{D^{(1)}, \ldots, D^{(s)}\} \subseteq S_\omega \backslash S_{\omega^{\mathcal{A}}}$ with the same inner normal $\alpha = (\alpha_1, \ldots, \alpha_n)$ with respect to ω^{0k}, let

$$C_i = \bigcup_{j=1}^{s} D_i^{(j)}, \quad i = 1, \ldots, n$$

and $C = (C_1, \ldots, C_n)$. It is clear that $\text{conv}(\hat{C}(\omega^{0k}))$ is a lower facet of $\text{conv}(\hat{\mathcal{A}}^0(\omega^{0k}))$ with inner normal $(\alpha, 1)$. So, C becomes a cell of $S_{\omega^{0k}}$ with inner normal $\alpha = (\alpha_1, \ldots, \alpha_n)$ with respect to ω^{0k} and has a fine mixed subdivision

$$S_{\omega^C} := \{D^{(1)}, \ldots, D^{(s)}\}.$$

When α is nonnegative, C is a stable mixed cell of $\mathcal{A}$ in $S_{\omega^{0k}}$ and the mixed volume of C, $\mathcal{M}(C)$, is equal to the sum of the volumes of cells of type $(1, \ldots, 1)$ in S_{ω^C}.

In this way, we have discovered all the stable mixed cells of $\mathcal{A}$ with their fine mixed subdivisions. And, the stable mixed volume, defined to be the sum of the mixed volumes of all those stable mixed cells, can easily be calculated by adding all the volumes of cells of type $(1, \ldots, 1)$ in those subdivisions.

CHAPTER 4

The Main Algorithm

From what we have derived in Chapter 3, we now propose a new algorithm for finding all isolated zeros of the polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ in $\mathbb{C}^n$. As indicated in Chapter 2, the Huber-Sturmfels algorithm requires recursive liftings, which is computationally costly. Our algorithm, a refinement of the Huber-Sturmfels algorithm, employs only one lifting in the whole course.

The main algorithm consists of three major parts: First of all, by assigning lifting $\omega = (\omega_1, \ldots, \omega_n)$ defined in (3.1) on the extended support $\mathcal{A}^0 = (\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0) \neq \mathcal{A}$, we find all stable mixed cells C of $\mathcal{A}$ with $\mathcal{M}(C) > 0$ along with all cells of type $(1, \ldots, 1)$ in their fine mixed subdivisions induced by ω^C, the restriction of ω on C. Secondly, we choose a generic polynomial system $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ with the support $\mathcal{A}^0$, where

$$g_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i^0} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n.$$

And for a given stable mixed cell $C = (C_1, \ldots, C_n)$ with $\mathcal{M}(C) > 0$ and inner normal $\alpha^C = (\alpha_1, \ldots, \alpha_n)$ with respect to ω^{0k} (α^C can be found simply by solving the linear system in (3.3) with a cell of S_{ω^C} as D), we solve all isolated zeros of $G_{\alpha^C}(\mathbf{x}) = (g_{1\alpha^C}(\mathbf{x}), \ldots, g_{n\alpha^C}(\mathbf{x}))$ where

$$g_{i\alpha^C}(\mathbf{x}) = \sum_{\mathbf{a} \in C_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n.$$

in $(\mathbb{C}^*)^n$. Third, we find all isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^n$ by linear homotopies between $G_{\alpha^C}(\mathbf{x})$ and $P_{\alpha^C}(x) = (p_{1\alpha^C}(\mathbf{x}), \ldots, p_{n\alpha^C}(\mathbf{x}))$ where

$$p_{i\alpha^C}(\mathbf{x}) = \sum_{\mathbf{a} \in C_i \cap \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} + \beta_i \bar{c}_{i,\mathbf{0}}, \quad i = 1, \ldots, n,$$

with

$$\beta_i = \begin{cases} 1 & \text{if } \mathbf{0} \notin \mathcal{A}_i \text{ but } \mathbf{0} \in C_i, \\ 0 & \text{otherwise.} \end{cases}$$

Algorithm :

0. Let $d = \max_{1 \le i \le n} \deg p_i(\mathbf{x})$. Choose a real number $k > n(n+1)d^n$ at random.

1. Lift the extended support $\mathcal{A}^0 = (\mathcal{A}_1^0, \ldots, \mathcal{A}_n^0)$ by a random lifting $\omega = (\omega_1, \ldots, \omega_n)$ as defined in (3.1), that is, for $i = 1, \ldots, n$,

$$\omega_i(\mathbf{0}) = k \quad \text{if } \mathbf{0} \notin \mathcal{A}_i,$$
$$\omega_i(\mathbf{a}) = \text{a randomly chosen number in } (0,1) \quad \text{if } \mathbf{a} \in \mathcal{A}_i.$$

   Find the cells of type $(1, \ldots, 1)$ in the induced fine mixed subdivision S_ω of $\mathcal{A}^0$.

2. Choose a generic polynomial system $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ where

$$g_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i^0} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n.$$

   The collection of cells

$$S_{\omega^{\mathcal{A}}} = \{(D_1, \ldots, D_n) \in S_\omega \mid D_i \subseteq \mathcal{A}_i \text{ for all } 1 \le i \le n\}$$

   gives a fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Let $S(\mathbf{0})$ be the set of cells of type $(1, \ldots, 1)$ in $S_{\omega^{\mathcal{A}}}$ found at step 1. Use these cells in $S(\mathbf{0})$ to find all the isolated zeros of $P(\mathbf{x})$ in $(\mathbb{C}^*)^n$ by the polyhedral homotopy described in Chapter 1 with lifting $\omega^{\mathcal{A}}$, the restriction of ω on $\mathcal{A}$.

3. For a cell $D = (D_1, \ldots, D_n)$ of type $(1, \ldots, 1)$ in $S_\omega \backslash S_{\omega\mathcal{A}}$, write

$$D_i = \{\mathbf{a}_{i0}, \mathbf{a}_{i1}\}, \quad i = 1, \ldots, n. \tag{4.1}$$

For $i = 1, \ldots, n$ and $j = 0, 1$, let $\hat{\mathbf{a}}_{ij}(\omega^{0k}) = (\mathbf{a}_{ij}, \omega^{0k}(\mathbf{a}_{ij}))$ as before, where ω^{0k} is defined in (2.2), and form the $n \times (n+1)$ matrix

$$V = \begin{pmatrix} \hat{\mathbf{a}}_{11}(\omega^{0k}) - \hat{\mathbf{a}}_{10}(\omega^{0k}) \\ \vdots \\ \hat{\mathbf{a}}_{n1}(\omega^{0k}) - \hat{\mathbf{a}}_{n0}(\omega^{0k}) \end{pmatrix}.$$

Find the unique vector $(\alpha^D, 1) = (\alpha_1^D, \ldots, \alpha_n^D, 1)$ in the kernel of V. Apparently, α^D is the inner normal of D with respect to ω^{0k}. Let $S(\alpha)$ be the collection of all cells of type $(1, \ldots, 1)$ in $S_\omega \backslash S_{\omega\mathcal{A}}$ with the same nonnegative inner normal $\alpha = (\alpha_1, \ldots, \alpha_n)$ with respect to ω^{0k}.

4. (a) Choose a cell D from $S(\alpha)$, using the same notations as in (4.1), let

$$C_i = \{\mathbf{a} \in \mathcal{A}_i^0 \mid \langle \hat{\mathbf{a}}(\omega^{0k}), \hat{\alpha} \rangle = \langle \hat{\mathbf{a}}_{i0}(\omega^{0k}), \hat{\alpha} \rangle\}, \quad i = 1, \ldots, n.$$

Then $C = (C_1, \ldots, C_n)$ is a stable mixed cell of $\mathcal{A}$ in $S_{\omega^{0k}}$ with mixed volume $\mathcal{M}(C) > 0$. Let

$$S_{\omega C} = \{(D_1, \ldots, D_n) \in S_\omega \mid D_i \subseteq C_i \text{ for all } 1 \le i \le n\}.$$

Then $S_{\omega C}$ is a fine mixed subdivision of C and $S(\alpha)$ consists of all the cells of type $(1, \ldots, 1)$ of C in $S_{\omega C}$.

We now solve the system

$$G_{\alpha C}(\mathbf{x}) = (g_{1\alpha C}(\mathbf{x}), \ldots, g_{n\alpha C}(\mathbf{x})) \tag{4.2}$$

in $(C^*)^n$, where

$$g_{i\alpha C}(\mathbf{x}) = \sum_{\mathbf{a} \in C_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n.$$

When system (4.2) is not a binomial system, then the polyhedral homotopy method described in Chapter 1 can be applied to find all its isolated zeros in $(\mathbb{C}^*)^n$ with the cells of type $(1,\ldots,1)$ in $S(\alpha)$ grouped at step 3, and lifting ω^C, the restriction of the lifting ω on C.

(b) Let

$$P_{\alpha^C}(\mathbf{x}) = (p_{1\alpha^C}(\mathbf{x}),\ldots,p_{n\alpha^C}(\mathbf{x})), \tag{4.3}$$

where

$$p_{i\alpha^C}(\mathbf{x}) = \sum_{\mathbf{a}\in C_i\cap\mathcal{A}_i} c_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} + \beta_i\bar{c}_{i,\mathbf{0}}, \quad i=1,\ldots,n,$$

with

$$\beta_i = \begin{cases} 1 & \text{if } \mathbf{0}\notin\mathcal{A}_i \text{ but } \mathbf{0}\in C_i, \\ 0 & \text{otherwise.} \end{cases}$$

All the isolated zeros of $P_{\alpha^C}(\mathbf{x})$ in $(\mathbb{C}^*)^n$ can be found by following the homotopy curves of the linear homotopy

$$H(\mathbf{x},t) = (1-t)G_{\alpha^C}(\mathbf{x}) + tP_{\alpha^C}(\mathbf{x})$$

starting from the zeros of $G_{\alpha^C}(\mathbf{x})$ at $t=0$.

(c) For zeros $\mathbf{e}=(e_1,\ldots,e_n)$ of system (4.3), let $\bar{\mathbf{e}}=(\bar{e}_1,\ldots,\bar{e}_n)$, where

$$\bar{e}_i = \begin{cases} e_i & \text{if } \alpha_i = 0, \\ 0 & \text{if } \alpha_i \neq 0. \end{cases}$$

Then $\bar{\mathbf{e}}$ is a zero of $P(\mathbf{x})$ in $\mathbb{C}^n\backslash(\mathbb{C}^*)^n$.

Remark 4 We can see that in our algorithm, only cells of type $(1,\ldots,1)$ in S_ω are used. Therefore, as suggested in Remark 3, choosing $k > n(n+1)d_2\times\cdots d_n\times d_n$ is sufficient for our need, where d_i's are defined in Remark 3. Currently, we are not aware of better lower bounds for k which are also easy to obtain computationally.

Remark 5 It is commonly known that when the polyhedral homotopy method is used to solve polynomial systems, large differences between powers of t in the polyhedral homotopies may cause computational instability when homotopy curves are followed. In our algorithm, the point $\mathbf{0}$ often receives very large lifting value k, compared to the rest of the lifting values in $(0,1)$. We will show in the following that the stability of our algorithm is independent of the large lifting value k.

Let $D = (\{\mathbf{a}_{11}, \mathbf{a}_{10}\}, \ldots, \{\mathbf{a}_{n1}, \mathbf{a}_{n0}\})$ be a cell of type $(1, \ldots, 1) \in S_\omega \backslash S_{\omega\mathcal{A}}$ and a subcell of a stable mixed cell $C = (C_1, \ldots, C_n)$ of $\mathcal{A}$. Using the notations introduced in the previous chapters, the inner normal of D, denoted by α^D, with respect to the lifting ω satisfies

$$\begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{10} & \omega_1(\mathbf{a}_{11}) - \omega_1(\mathbf{a}_{10}) \\ \vdots & \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} & \omega_n(\mathbf{a}_{n1}) - \omega_n(\mathbf{a}_{n0}) \end{pmatrix} \begin{pmatrix} \alpha^D \\ \\ 1 \end{pmatrix} = 0,$$

or,

$$V(D)\alpha^D + u(\omega) = 0, \tag{4.4}$$

where

$$V(D) = \begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{10} \\ \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} \end{pmatrix} \quad \text{and} \quad u(\omega) = \begin{pmatrix} \omega_1(\mathbf{a}_{11}) - \omega_1(\mathbf{a}_{10}) \\ \vdots \\ \omega_n(\mathbf{a}_{n1}) - \omega_n(\mathbf{a}_{n0}) \end{pmatrix}.$$

Let the inner normal of stable mixed cell C with respect to the lifting ω^{0k} be α^C, then

$$\begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{10} & \omega_1^{0k}(\mathbf{a}_{11}) - \omega_1^{0k}(\mathbf{a}_{10}) \\ \vdots & \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n0} & \omega_n^{0k}(\mathbf{a}_{n1}) - \omega_n^{0k}(\mathbf{a}_{n0}) \end{pmatrix} \begin{pmatrix} \alpha^C \\ \\ 1 \end{pmatrix} = 0,$$

or,

$$V(D)\alpha^C + u(\omega^{0k}) = 0, \tag{4.5}$$

where

$$u(\omega^{0k}) = \begin{pmatrix} \omega_1^{0k}(\mathbf{a}_{11}) - \omega_1^{0k}(\mathbf{a}_{10}) \\ \vdots \\ \omega_n^{0k}(\mathbf{a}_{n1}) - \omega_n^{0k}(\mathbf{a}_{n0}) \end{pmatrix}.$$

Let $\beta = (\beta_1, \ldots, \beta_n)^T = u(\omega) - u(\omega^{0k})$. Then

$$\beta_i = [\omega_i(\mathbf{a}_{i1}) - \omega_i(\mathbf{a}_{i0})] - [\omega_i^{0k}(\mathbf{a}_{i1}) - \omega_i^{0k}(\mathbf{a}_{i0})], \quad i = 1, \ldots, n.$$

By Lemma 3, $\beta_i \in (-1, 1)$ for $i = 1, \ldots, n$, and they are independent of the value k. Subtracting (4.5) from (4.4) yields

$$V(D)(\alpha^D - \alpha^C) + \beta = 0.$$

It follows that $\|\alpha^D - \alpha^C\|$ is independent of k.

When the polyhedral homotopy

$$\hat{g}_i(\mathbf{y}, t) = \sum_{\mathbf{a} \in C_i} \bar{c}_{i,\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{\langle \mathbf{a}, \alpha \rangle + \omega_i(\mathbf{a})}, \quad i = 1, \ldots, n \tag{4.6}$$

as in (1.3), is used to solve the system

$$g_{i\alpha^C}(\mathbf{x}) = \sum_{\mathbf{a} \in C_i} \bar{c}_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n$$

in $(\mathbb{C}^*)^n$, large differences between exponents of t will result in large exponents of t for certain terms in the final polyhedral homotopy in (1.4) when we factor out the lowest power of t. Large exponents of t in the resulting homotopies sometimes cause numerical instability of the curve-tracing of the homotopy paths.

Now, for $\mathbf{a}$, $\mathbf{b} \in C_i$, since

$$\langle (\alpha^C, 1), (\mathbf{a}, \omega_i^{0k}(\mathbf{a})) \rangle = \langle (\alpha^C, 1), (\mathbf{b}, \omega_i^{0k}(\mathbf{b})) \rangle,$$

we have

$$\langle \alpha^C, \mathbf{a} - \mathbf{b} \rangle = \omega_i^{0k}(\mathbf{b}) - \omega_i^{0k}(\mathbf{a}).$$

Thus,

$$\begin{aligned}
& |[\langle \alpha^D, \mathbf{a} \rangle + \omega_i(\mathbf{a})] - [\langle \alpha^D, \mathbf{b} \rangle + \omega_i(\mathbf{b})]| \\
= \; & |\langle \alpha^D, \mathbf{a} - \mathbf{b} \rangle + [\omega_i(\mathbf{a}) - \omega_i(\mathbf{b})]| \\
= \; & |\langle \alpha^D - \alpha^C, \mathbf{a} - \mathbf{b} \rangle + \beta_i| \\
\leq \; & \|\alpha^D - \alpha^C\| \; \|\mathbf{a} - \mathbf{b}\| + |\beta_i|,
\end{aligned}$$

where $\beta_i = [\omega_i(\mathbf{a}) - \omega_i(\mathbf{b})] - [\omega_i^{0k}(\mathbf{a}) - \omega_i^{0k}(\mathbf{b})] \in (-1, 1)$ and is independent of k by Lemma 3. The right hand side of the above inequality is also independent of k since $\|\alpha^D - \alpha^C\|$ is independent of k, and therefore, the difference between the exponents of t in (4.6) is independent of k.

CHAPTER 5

Numerical Implementation

Our algorithm has been successfully implemented. In this chapter, we will present the numerical results of applying our algorithm to several well-known polynomial systems. The stable mixed cells other than $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ are the ones which contribute zeros of polynomial systems in $\mathbb{C}^n \backslash (\mathbb{C}^*)^n$. We will call them *nontrivial* stable mixed cells. In the tables below, only nontrivial stable mixed cells are addressed and $\mathcal{M}(\mathcal{A})$, $\mathcal{SM}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A}^0)$ denote the mixed volume of $\mathcal{A}$, stable mixed volume of $\mathcal{A}$ and mixed volume of the extended support $\mathcal{A}^0 = (\mathcal{A}_1 \cup \{\mathbf{0}\}, \ldots, \mathcal{A}_n \cup \{\mathbf{0}\})$ respectively. The root counts in the examples are obtained from the numerical results of our algorithm. It is well-known that homotopy curves may converge to solutions in an algebraic variety with nonzero dimension, i.e., they may lead to non-isolated zeros of the target polynomial systems. In our root count, we exclude those numerical solutions at which the Jacobian matrices of the corresponding polynomial systems are almost singular but no other numerical solutions are close to them.

EXAMPLE 1 For the bivariate system [7]

$$p_1(x, y) = ay + by^2 + cxy^3,$$

$$p_2(x, y) = dx + ex^2 + fx^3y$$

with generic coefficients $\{a, b, c, d, e, f\}$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^2$	# of Isolated Zeros in $\mathbb{C}^2\backslash(\mathbb{C}^*)^2$
96	3	6	8	3	3

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
$C^{(1)}$	(k,k)	1
$C^{(2)}$	$(k,0)$	1
$C^{(3)}$	$(0,k)$	1

The supporting polynomial systems corresponding to these stable mixed cells are:

$$(p_{1_{\alpha}C^{(1)}}(x,y), p_{2_{\alpha}C^{(1)}}(x,y)) = (ay + \bar{c}_{1,\mathbf{0}}, dx + \bar{c}_{2,\mathbf{0}}),$$

$$(p_{1_{\alpha}C^{(2)}}(x,y), p_{2_{\alpha}C^{(2)}}(x,y)) = (ay + by^2, dx + \bar{c}_{2,\mathbf{0}}),$$

and

$$(p_{1_{\alpha}C^{(3)}}(x,y), p_{2_{\alpha}C^{(3)}}(x,y)) = (ay + \bar{c}_{1,\mathbf{0}}, dx + ex^2).$$

Each of these three supporting polynomial systems determines an isolated zero of the original system $(p_1(x,y), p_2(x,y))$ in $\mathbb{C}^2\backslash(\mathbb{C}^*)^2$.

Example 2 For the system of E. R. Speer [18]:

$$p_1(\mathbf{x}) = 4\beta(n + 2a_1 - 8x_1)(a_2 - a_3) - x_2x_3x_4 + x_2 + x_4,$$

$$p_2(\mathbf{x}) = 4\beta(n + 2a_1 - 8x_2)(a_2 - a_3) - x_1x_3x_4 + x_1 + x_3,$$

$$p_3(\mathbf{x}) = 4\beta(n + 2a_1 - 8x_3)(a_2 - a_3) - x_1x_2x_4 + x_2 + x_4,$$

$$p_4(\mathbf{x}) = 4\beta(n + 2a_1 - 8x_4)(a_2 - a_3) - x_1x_2x_3 + x_1 + x_3,$$

where β, n are random parameters, $a_1 = x_1 + x_2 + x_3 + x_4$, $a_2 = x_1x_2x_3x_4$, and $a_3 = x_1x_2 + x_2x_3 + x_3x_4 + x_1x_4$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^4$	# of Isolated Zeros in $\mathbb{C}^4\backslash(\mathbb{C}^*)^4$
12500	96	97	97	43	0

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
C	(k,k,k,k)	1

The supporting polynomial system $P_{\alpha^C}(\mathbf{x})$ of $P(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), p_3(\mathbf{x}), p_4(\mathbf{x}))$ corresponding to the stable mixed cell C is

$$P_{\alpha^C}(\mathbf{x}) = \begin{cases} x_2 + x_4 + \bar{c}_{1,\mathbf{0}}, \\ x_1 + x_3 + \bar{c}_{2,\mathbf{0}}, \\ x_2 + x_4 + \bar{c}_{3,\mathbf{0}}, \\ x_1 + x_3 + \bar{c}_{4,\mathbf{0}}. \end{cases}$$

This system has no isolated zeros in $(\mathbb{C}^*)^4$. In fact, the original system $P(\mathbf{x})$ has two 1-dimensional zero sets $\{(0,a,0,-a) \mid a \in \mathbb{C}\}$ and $\{(a,0,-a,0) \mid a \in \mathbb{C}\}$ which contain $(0,0,0,0)$. It was reported in [18] that the system has 50 isolated zeros. Actually, seven of them are not isolated, they belong to the two 1-dimensional zero sets listed above.

EXAMPLE 3 For the planar four-bar mechanism system [13]:

$$\begin{aligned} p_l(\mathbf{x}) \;=\; & a_{l1}x_1^2x_3^2 + a_{l2}x_1^2x_3x_4 + a_{l3}x_1^2x_3 + a_{l4}x_1^2x_4^2 + a_{l5}x_1^2x_4 \\ & + a_{l6}x_1^2 + a_{l7}x_1x_2x_3^2 + a_{l8}x_1x_2x_3x_4 + a_{l9}x_1x_2x_3 + a_{l10}x_1x_2x_4^2 \\ & + a_{l11}x_1x_2x_4 + a_{l12}x_1x_3^2 + a_{l13}x_1x_3x_4 + a_{l14}x_1x_3 + a_{l15}x_1x_4^2 \\ & + a_{l16}x_1x_4 + a_{l17}x_2^2x_3^2 + a_{l18}x_2^2x_3x_4 + a_{l19}x_2^2x_3 + a_{l20}x_2^2x_4^2 \\ & + a_{l21}x_2^2x_4 + a_{l22}x_2^2 + a_{l23}x_2x_3^2 + a_{l24}x_2x_3x_4 + a_{l25}x_2x_3 \\ & + a_{l26}x_2x_4^2 + a_{l27}x_2x_4 + a_{l28}x_3^2 + a_{l29}x_4^2, \qquad l = 1, \ldots, 4 \end{aligned}$$

with generic choice of the parameters of the system, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^4$	# of Isolated Zeros in $\mathbb{C}^4 \backslash (\mathbb{C}^*)^4$
5120	80	96	96	36	0

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
C	$(k/2, k/2, k/2, k/2)$	16

The supporting polynomial system $P_{\alpha^C}(\mathbf{x})$ corresponding to the nontrivial stable mixed cell C is

$$\begin{aligned} p_{l\alpha^C}(\mathbf{x}) &= a_{l6}x_1^2 + a_{l14}x_1x_3 + a_{l16}x_1x_4 + a_{l22}x_2^2 + a_{l25}x_2x_3 \\ &\quad + a_{l27}x_2x_4 + a_{l28}x_3^2 + a_{l29}x_4^2 + \bar{c}_{i,0}, \qquad l = 1, \ldots, 4. \end{aligned}$$

This system has no isolated zeros in $(\mathbb{C}^*)^4$, and the original system $P(\mathbf{x})$ has a 2-dimensional zero set which contains $(0,0,0,0)$. Our root count agrees with the result in [13].

EXAMPLE 4 For the Caprasse system from PoSSo test suite [17]:

$$\begin{aligned} p_1(\mathbf{x}) &= y^2z + 2xyt - 2x - z, \\ p_2(\mathbf{x}) &= -x^3z + 4xy^2z + 4x^2yt + 2y^3t + 4x^2 - 10y^2 + 4xz - 10yt + 2, \\ p_3(\mathbf{x}) &= 2yzt + xt^2 - x - 2z, \\ p_4(\mathbf{x}) &= -xz^3 + 4yz^2t + 4xzt^2 + 2yt^3 + 4xz + 4z^2 - 10yt - 10t^2 + 2 \end{aligned}$$

with variables $\mathbf{x} = (x, y, z, t)$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^4$	# of Isolated Zeros in $\mathbb{C}^4\backslash(\mathbb{C}^*)^4$
3840	48	56	56	48	8

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
C	$(k, 0, k, 0)$	8

The supporting polynomial system corresponding to the nontrivial stable mixed cell C is

$$P_{\alpha^C}(\mathbf{x}) = \begin{cases} y^2z + 2xyt - 2x - z + \bar{c}_{1,0}, \\ 2y^3t - 10y^2 - 10yt + 2, \\ 2yzt + xt^2 - x - 2z + \bar{c}_{3,0}, \\ 2yt^3 - 10yt - 10t^2 + 2. \end{cases}$$

This system has eight isolated zeros in $(\mathbb{C}^*)^4$ which determine the following eight isolated zeros of the original system $P(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), p_3(\mathbf{x}), p_4(\mathbf{x}))$ in $\mathbb{C}^4 \backslash (\mathbb{C}^*)^4$:

```
(0, -0.318, 0, -0.318),      (0, -i,  0,  i),
(0,  0.318, 0,  0.318),      (0,  i,  0, -i),
(0, -3.146, 0, -3.146),      (0,  1,  0  -1),
(0,  3.146, 0,  3.146),      (0, -1,  0   1),
```

where each component of the isolated zeros is rounded to three decimal places.

EXAMPLE 5 For the Cohn-2 system from PoSSo test suite [17]:

$$\begin{aligned}
p_1(\mathbf{x}) &= x^3y^2 + 4x^2y^2z - x^2yz^2 + 288x^2y^2 + 207x^2yz + 1152xy^2z \\
&\quad +156xyz^2 + xz^3 - 3456x^2y + 20736xy^2 + 19008xyz + 82944y^2z \\
&\quad +432xz^2 - 497664xy + 62208xz + 2985984x, \\
p_2(\mathbf{x}) &= y^3t^3 + 4y^3t^2 - y^2zt^2 + 4y^2t^3 - 48y^2t^2 - 5yzt^2 \\
&\quad +108yzt + z^2t + 144zt - 1728z, \\
p_3(\mathbf{x}) &= -x^2z^2t + 4xz^2t^2 + z^3t^2 + x^3z + 156x^2zt + 207xz^2t + 1152xzt^2 \\
&\quad +288z^2t^2 + 432x^2z + 19008xzt - 3456z^2t + 82944xt^2 \\
&\quad +20736zt^2 + 62208xz - 497664zt + 2985984z, \\
p_4(\mathbf{x}) &= y^3t^3 - xy^2t^2 + 4y^3t^2 + 4y^2t^3 - 5xy^2t - 48y^2t^2 \\
&\quad +x^2y + 108xyt + 144xy - 1728x
\end{aligned}$$

with variables $\mathbf{x} = (x, y, z, t)$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^4$	# of Isolated Zeros in $\mathbb{C}^4\backslash(\mathbb{C}^*)^4$
21600	124	150	150	18	0

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
$C^{(1)}$	$(0,0,k,k/2)$	4
$C^{(2)}$	$(k,k/2,0,0)$	4
$C^{(3)}$	$(k,0,k,k/2)$	8
$C^{(4)}$	$(k,0,k,0)$	2
$C^{(5)}$	$(k,k/2,k,0)$	8

The supporting polynomial systems corresponding to these nontrivial stable mixed cells are

$$P_{\alpha^{C^{(1)}}}(\mathbf{x}) = \begin{cases} x^3y^2 + 288x^2y^2 - 3456x^2y + 20736xy^2 - 497664xy + 2985984x, \\ 4y^3t^2 - 48y^2t^2 - 1728z + \bar{c}_{2,\mathbf{0}}, \\ x^3z + 432x^2z + 82944xt^2 + 62208xz + 2985984z + \bar{c}_{3,\mathbf{0}}, \\ x^2y + 144xy - 1728x, \end{cases}$$

$$P_{\alpha^{C^{(2)}}}(\mathbf{x}) = \begin{cases} xz^3 + 82944y^2z + 432xz^2 + 62208xz + 2985984x + \bar{c}_{1,\mathbf{0}}, \\ z^2t + 144zt - 1728z, \\ z^3t^2 + 288z^2t^2 - 3456z^2t + 20736zt^2 - 497664zt + 2985984z, \\ 4y^2t^3 - 48y^2t^2 - 1728x + \bar{c}_{4,\mathbf{0}}, \end{cases}$$

$$P_{\alpha^{C^{(3)}}}(\mathbf{x}) = \begin{cases} 20736xy^2 + 82944y^2z - 497664xy + 2985984x + \bar{c}_{1,\mathbf{0}}, \\ 4y^3t^2 - 48y^2t^2 - 1728z + \bar{c}_{2,\mathbf{0}}, \\ 2985984z + \bar{c}_{3,\mathbf{0}}, \\ 4y^3t^2 - 48y^2t^2 + 144xy - 1728x + \bar{c}_{4,\mathbf{0}}, \end{cases}$$

$$P_{\alpha^{C^{(4)}}}(\mathbf{x}) = \begin{cases} 20736xy^2 + 82944y^2z - 497664xy + 2985984x + \bar{c}_{1,\mathbf{0}}, \\ y^3t^3 + 4y^3t^2 + 4y^2t^3 - 48y^2t^2, \\ 82944xt^2 + 20736zt^2 - 497664zt + 2985984z + \bar{c}_{3,\mathbf{0}}, \\ y^3t^3 + 4y^3t^2 + 4y^2t^3 - 48y^2t^2, \end{cases}$$

and

$$P_{\alpha^{C^{(5)}}}(\mathbf{x}) = \begin{cases} 2985984x + \bar{c}_{1,\mathbf{0}}, \\ 4y^2t^3 - 48y^2t^2 + 144zt - 1728z + \bar{c}_{2,\mathbf{0}}, \\ 82944xt^2 + 20736zt^2 - 497664zt + 2985984z + \bar{c}_{3,\mathbf{0}}, \\ 4y^2t^3 - 48y^2t^2 - 1728x + \bar{c}_{4,\mathbf{0}}. \end{cases}$$

These five supporting polynomial systems have no isolated zeros in $(\mathbb{C}^*)^4$. From our numerical experiment, $P(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), p_3(\mathbf{x}), p_4(\mathbf{x}))$ has no isolated zeros in $\mathbb{C}^4 \backslash (\mathbb{C}^*)^4$.

EXAMPLE 6 For the Katsura4 system from PoSSo test suite [17]:

$$\begin{aligned} p_1(\mathbf{x}) &= 2x^2 + 2y^2 + 2z^2 + 2t^2 + u^2 - u, \\ p_2(\mathbf{x}) &= xy + 2yz + 2zt + 2tu - t, \\ p_3(\mathbf{x}) &= 2xz + 2yt + t^2 + 2zu - z, \\ p_4(\mathbf{x}) &= 2xt + 2zt + 2yu - y, \\ p_5(\mathbf{x}) &= 2x + 2y + 2z + 2t + u - 1 \end{aligned}$$

with variables $\mathbf{x} = (x, y, z, t, u)$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^5$	# of Isolated Zeros in $\mathbb{C}^5 \backslash (\mathbb{C}^*)^5$
960	12	16	16	12	4

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
$C^{(1)}$	$(0, k, 0, k, 0)$	2
$C^{(2)}$	$(0, k, k, k, 0)$	2

The supporting polynomial systems of $P(\mathbf{x})$ are

$$P_{\alpha^{C^{(1)}}}(\mathbf{x}) = \begin{cases} 2x^2 + 2z^2 + u^2 - u, \\ xy + 2yz + 2zt + 2tu - t + \bar{c}_{2,\mathbf{0}}, \\ 2xz + 2zu - z, \\ 2xt + 2zt + 2yu - y + \bar{c}_{4,\mathbf{0}}, \\ 2x + 2z + u - 1, \end{cases}$$

and

$$P_{\alpha^{C^{(2)}}}(\mathbf{x}) = \begin{cases} 2x^2 + u^2 - u, \\ xy + 2tu - t + \bar{c}_{2,\mathbf{0}}, \\ 2xz + 2zu - z + \bar{c}_{3,\mathbf{0}}, \\ 2xt + 2yu - y + \bar{c}_{4,\mathbf{0}}, \\ 2x + u - 1. \end{cases}$$

Each of these two supporting polynomial systems has two isolated zeros in $(\mathbb{C}^*)^5$ which determine two isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^5$. The four isolated zeros of $P(\mathbf{x})$ in $\mathbb{C}^5 \backslash (\mathbb{C}^*)^5$ are

```
(0.273, 0, 0.113, 0, 0.227),    (-0.131, 0, 0.315, 0, 0.631),
(0, 0, 0, 0, 1),                (0.333, 0, 0, 0, 0.333),
```

where each component of the isolated zeros is rounded to three decimal places.

EXAMPLE 7 For the Moeller4 system from PoSSo test suite [17]:

$$\begin{aligned} p_1(\mathbf{x}) &= y + u + v - 1, \\ p_2(\mathbf{x}) &= z + t + 2u - 3, \\ p_3(\mathbf{x}) &= y + t + 2v - 1, \\ p_4(\mathbf{x}) &= x - y - z - t - u - v, \\ p_5(\mathbf{x}) &= -1569/31250yz^3 + x^2tu, \\ p_6(\mathbf{x}) &= -587/15625yt + zv \end{aligned}$$

with variables $\mathbf{x} = (x, y, z, t, u, v)$, we have

Chosen k	$\mathcal{M}(\mathcal{A})$	$\mathcal{SM}(\mathcal{A})$	$\mathcal{M}(\mathcal{A}^0)$	# of Isolated Zeros in $(\mathbb{C}^*)^6$	# of Isolated Zeros in $\mathbb{C}^6\backslash(\mathbb{C}^*)^6$
1344	7	8	8	7	1

and

Nontrivial Stable Cell	Inner Normal	Mixed Volume
C	$(0,0,k,k,0,0)$	1

The supporting polynomial system of $P(\mathbf{x})$ corresponding to C is

$$P_{\alpha^C}(\mathbf{x}) = \begin{cases} y+u+v-1, \\ 2u-3, \\ y+2v-1, \\ x-y-u-v, \\ x^2tu+\bar{c}_{5,\mathbf{0}}, \\ -587/15625yt+zv+\bar{c}_{6,\mathbf{0}}. \end{cases}$$

$P_{\alpha^C}(\mathbf{x})$ has an isolated zero in $(\mathbb{C}^*)^6$ which determines the isolated zero $(1,-2,0,0,1.5,1.5)$ of $P(\mathbf{x})$ in $\mathbb{C}^6\backslash(\mathbb{C}^*)^6$.

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