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Eugene A. Belchev

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Zhengfang Zhou
Major professor

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FINITE-TIME BLOW-UP OF SOLUTIONS TO NONLINEAR WAVE EQUATIONS

By

Eugene A. Belchev

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ABSTRACT

FINITE-TIME BLOW-UP OF SOLUTIONS TO NONLINEAR WAVE EQUATIONS

By

Eugene A. Belchev

This work studies the finite-time blow-up of solutions to the equation $u_{tt} - \Delta u = F(u, \partial u)$ in Minkowski space. Results available so far involve complicated analysis near the wave front. We develop a new technique which simplifies some of the existing arguments. The approach we use is a modification of the so-called method of conformal compactification. In this we are motivated by the work of Christodoulou, and Baez, Segal, and Zhou on nonlinear wave equations, as well as the recent developments in the rigorous theory of nonlinear quantum fields. In Chapter 3 we study the semilinear case $u_{tt} - \Delta u = p^{-k}|u|^l$, where p is a conformal factor approaching 0 at infinity. We show that in this case the solutions blow up in finite time for small powers l, while having an arbitrarily long life-span for large l. In Chapter 4 we prove finite-time blow-up for a class of quasilinear equations and develop a technique to generate more examples of finite-time blow-up.

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Introduction

The Cauchy problem for a nonlinear evolution equation, when well-posed, determines a local-in-time solution. However, an important feature of nonlinear PDE, and nonlinear evolution equations in particular, is that their solutions may develop singularities as they move away from the initial state. First, a solution may contain singularities in its data, of which one would like to follow the propagation. Second, a solution may form a singularity in a given function space, but may still remain "regular" according to a weaker measure of regularity. Finally, beginning from smooth data, a solution may develop a singularity in finite time; this phenomenon is called *blow-up*, and we say that the solution blows up in finite time.

The interpretation of blow-up in physical terms often poses difficulties; blow-up may indicate a real phenomenon, but it may also be a failure of the physical model. A physical example of a finite time blow-up is the solution of the semilinear Schrödinger equation in space dimension one,

$$iu_t - u_{xx} = |u|^{l-1}u,$$

which blows up in finite time at, it is believed, only a single point. This corresponds to the focusing of a laser beam (see [16]).

In this dissertation we concentrate on the finite-time blow-up of solutions to the nonlinear wave equation

$$u_{tt} - \Delta u = F(u, \partial u) \tag{1}$$

for $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, where Δ is the Laplacian in $\mathbf{x}$ and ∂ is any derivative. This equation arises in different areas of applied mathematics, physics, and engineering, and describes such familiar and important processes as the movement of vibrating strings, drum heads, sound and electromagnetic waves, etc. In 1950s nonlinearities like

$$F(u, \partial u) = mu + u^3, \quad m \geq 0,$$

were proposed as models in relativistic quantum mechanics with local self-interaction (see [26] and [27]). To model effects thought to arise in the case, for instance, of spinor fields u, equation (1) also has been considered in space dimensions $n \geq 3$ (see [27]). The so-called σ-model for the pure gauge solutions of the Yang-Mills equations involves nonlinearities of the type

$$F(u, \partial u) = u\left(|u_t|^2 - |\nabla u|^2\right),$$

where ∇u is the spatial gradient of u (see [31]).

Many blow-up results have been proved for nonlinear wave equations beginning with Keller [14] in 1957. Consider first the Cauchy problem for the semilinear wave equation

$$u_{tt} - \Delta u = F(u), \tag{2}$$

$$u(0, \cdot) = f, \quad u_t(0, \cdot) = g. \tag{3}$$

Early results by Keller [14] in 1957, Jörgens [12] in 1961, Glassey [4] in 1973, and Levine [17] in 1974, showed that this problem does not admit a global solution when the initial data are large in some sense. On the other hand, John [9] proved in 1979 that in three space dimensions there are always global solutions of the problem

$$u_{tt} - \Delta u = |u|^l, \tag{4}$$

$$u(0,\cdot) = f, \quad u_t(0,\cdot) = g, \tag{5}$$

for $l > 1 + \sqrt{2}$ and suitably small initial data, whereas for $l < 1 + \sqrt{2}$ a global solution does not exist for any smooth non-trivial data with compact support. Thus he was the first to show that even small (in L^∞ norm) solutions could blow up.

John's result led Strauss [35] to conjecture in 1981 that in dimensions $n \geq 2$ the critical power, dividing between global existence and finite-time blow-up, for the problem (4)-(5) should be the positive root $l_0(n)$ of the polynomial $(n-1)l^2 - (n+1)l - 2$. Interestingly enough, the critical power $l_0(n)$ plays a prominent role in the scattering theory of nonlinear Schrödinger equations; it appears for the first time explicitly in Strauss [34].

Subsequently, Glassey [5, 6] in 1981 verified the conjecture in two dimensions by showing that $l_0(2) = \frac{1}{2}(3 + \sqrt{17})$; in addition, Schaeffer [24] proved in 1985 finite-time blow-up for the critical power. For higher dimensions $n > 3$, Sideris [33] proved finite-time blow-up for initial data satisfying a certain positivity condition while global existence was shown in 1996 by Lindblad and Sogge [18] but for spherically symmetric initial data only; for general initial data they were able to prove global existence only in dimensions $n \leq 8$.

There are also many blow-up results for the Cauchy problem for the quasilinear wave equation

$$u_{tt} - \Delta u = F(u, \partial u), \tag{6}$$

$$u(0,\cdot) = f, \quad u_t(0,\cdot) = g. \tag{7}$$

For instance, the equation

$$u_{tt} - \Delta u = au_t^2 + b|\Delta u|^2, \quad a + b \neq 0,$$

for dimension three has arbitrarily small solutions which blow up in finite time (see [11] and [7]). If we restrict the nonlinearity in equation (6) to time and radial derivatives of u only, we obtain a class of quasilinear problems governed, as in the semilinear case discussed above, by a critical power. Thus it was conjectured for the equation

$$u_{tt} - \Delta u = |u_t|^l, |u_r|^l$$

that the critical power is $l_1(n) = \frac{n+1}{n-1}$ for $n \geq 2$. Toward verifying this conjecture, for $l_1(2) = 3$ finite-time blow-up has been shown by John [10] for nonlinearity $|u_t|^3$ and by Schaeffer [25] for nonlinearity $|u_r|^3$. The case $l_1(3) = 2$ was verified by Sideris [32], who proved in 1983 finite-time blow-up for nonlinearity $|u_r|^2$ and global existence of small radially symmetric solutions for nonlinearity $|u_t|^l$, $l > 2$. For $l_1(5) = \frac{3}{2}$, Schaeffer [23] proved in 1983 finite-time blow-up for fairly large class of initial data. In high dimensions, Rammaha [22] proved in 1987 finite-time blow-up for spherically symmetric solutions; his result also included the critical power for n odd, whereas Jiao [8] showed in 1996 blow-up for the critical power in even dimensions. In addition, global existence for small initial data was proved by Klainerman and Ponce [15] in 1983 but only(!) for powers $l > \frac{n+\sqrt{2n-1}}{n-1} > l_1(n)$.

Most proofs of blow-up reduce the PDE to an ordinary differential inequality for some functional $H(u(t))$ of a solution u. The inequality is then solved, subject to appropriate initial conditions, so as to obtain a lower bound for $H(u(t))$ that blows up at some finite time. If the definition of H assures that it is finite for globally existing u, then the blow-up of the functional establishes the nonexistence of u beyond a finite time. The typical local-in-time existence theorem (see Segal [28] and Kato [13]) asserts that either a solution u exists for all time or else some norm of u becomes unbounded as t approaches some finite time T^*. Thus we obtain that u blows up in time T^*.

The approach we are going to use is a modification of the so-called method of conformal compactification. In this we are motivated by the work of Christodoulou [3], and Baez, Segal, and Zhou [1] on nonlinear wave equations, as well as the recent developments in the rigorous theory of nonlinear quantum fields (see [19, 20, 29, 30]).

The method of conformal compactification is based on an idea by Penrose [21] dating back to 1963. In order to study the nature of infinity in the various cosmological models, he suggested that a given physical space-time be compactified by conformally embedding it into a compact subset of the *Einstein universe* $\mathbf{E} = \mathbb{R} \times S^n$; the "finite" boundary C of this subset would thus represent the "infinity" of the space-time. To be more specific, let us introduce coordinates on S^n by regarding it as a unit sphere in $\mathbb{R}^{n+1}$: $Y_1^2 + Y_2^2 + \cdots + Y_{n+1}^2 = 1$; thus a point in $\mathbf{E}$ is represented by $(T, Y_1, \ldots, Y_{n+1})$, T being the Einstein time. Define the map $c : \mathbf{M}_0 \to \mathbf{E}$ by

$$c(t, \mathbf{x}) = c(t, x_1, \ldots, x_n) = (T, Y_1, \ldots, Y_{n+1}),$$

where

$$\sin T = pt, \quad \cos T = p\left(1 - \frac{t^2 - \mathbf{x}^2}{4}\right), \quad T \in (-\pi, \pi);$$
$$Y_j = px_j, \quad j = 1, \ldots, n; \quad Y_{n+1} = p\left(1 + \frac{t^2 - \mathbf{x}^2}{4}\right);$$

with

$$p = \left[t^2 + \left(1 - \frac{t^2 - \mathbf{x}^2}{4}\right)^2\right]^{-\frac{1}{2}}.$$

We take the point of observation to be the north pole $T = Y_1 = Y_2 = \cdots = Y_n = 0$, $Y_{n+1} = 1$ and denote by $\rho \in [0, \pi)$ the distance on S^n from that point. It is easy to see that the image of $\mathbf{M}_0$ under c is

$$c(\mathbf{M}_0) = \{\rho - \pi < T < \pi - \rho\}.$$

It can be visualized as a "diamond"; its boundary C consists of two lightcones $C_\pm = \{\pm T + \rho = \pi\}$, which represent the limits of spacelike surfaces in $\mathbf{M}_0$ as the Minkowski time $t \to \pm\infty$. The vertices $I_\pm$ of $C_\pm$ represent past and future infinities, whereas the point $I_0 : \; T = 0, \; \rho = \pi$ represents spatial infinity.

Consider on $\mathbf{M}_0$ the Minkowski metric

$$g = dt^2 - d\mathbf{x}^2 = dt^2 - \sum_{i=1}^{n} dx_i^2$$

and on $\mathbf{E}$ the metric

$$\tilde{g} = dT^2 - dS^2,$$

where dS^2 is the canonical metric on S^n. The map c is a conformal map between the Lorentz manifolds $(\mathbf{M}_0, g)$ and $(\mathbf{E}, \tilde{g})$ with a conformal factor p, i.e., $c^*\tilde{g} = p^2 g$.

Let $\Box = \partial_t^2 - \Delta$ and $\tilde{\Box} = \partial_T^2 - \Delta_{S^n}$ be the d'Alembertians relative to g and $\tilde{g}$ respectively, Δ_{S^n} is the Laplace-Beltrami operator on S^n. We will see that the operators $\Box$ and $\Box_c = \tilde{\Box} + s^2$, $s = \frac{n-1}{2}$, are conformally covariant; for this reason $\Box_c$ is called the conformal d'Alembertian. In fact, the solutions of the wave equation on

$\mathbf{M}_0$ and the conformal wave equation on $\mathbf{E}$ are in one-to-one correspondence via the relation $u \mapsto p^s u|_{c(\mathbf{M}_0)}$.

We will modify the conformal transform c by composing it with a one-parameter family of dilations thus obtaining a one-parameter family of conformal transformations. We will then use these mappings to transform into the Einstein universe $\mathbf{E}$ the equation

$$\Box u = p^{-k}|Lu|^l, \quad l > 1, \tag{8}$$

where $k = sl - \frac{n+3}{2}$, and Lu is defined by

$$Lu := a(t,r)u_t + b(t,r)u_r + c(t,r)u, \quad r = |\mathbf{x}|.$$

In order to be able to solve this equation we have to prescribe initial data over some space-like hypersurface. We choose the hypersurface given by the equation $t = 0$ and prescribe

$$u(0,\mathbf{x}) = f(\mathbf{x}), \quad u_t(0,\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{9}$$

We will prove that for a fairly large class of nonlinear interaction terms and compactly supported initial data every solution of the Cauchy problem (8)-(9) blows up in finite time.

The factor p^{-k} in equation (8) shows up there for a rather technical reason; it allows us to avoid having to deal with singularities along the boundary in $\mathbf{E}$ of the compactified Minkowski space $c(\mathbf{M}_0)$. However, as the parameter R increases without bound, this factor tends to 1 uniformly on any compact subset of $\mathbf{M}_0$. In the semilinear case $Lu = u$, this renders equation (8) as a good approximation for the "classical" equation

$$\Box u = |u|^l.$$

In fact, we will see that in this case equation (8) is also governed by a critical power.

The parameter R plays an important role in proving the blow-up results. We use the fact that as R increases, the support of the initial data for the transformed in $\mathbf{E}$ equation decreases; therefore, for any Einstein time $T < \frac{\pi}{2}$, we may choose R large enough so that the support of the solution at time T is contained in $c(\mathbf{M}_0)$. Thus a finite-time blow-up in $\mathbf{E}$ would imply a finite-time blow-up in Minkowski space.

Our work is organized as follows. In Chapter 1 we discuss the compactification of Minkowski space with a conformal transformation c and prove some of the properties of this compactification. In particular, we show that the d'Alembertian $\Box$ transforms "nicely" under c into the conformal d'Alembertian $\Box_c$. In Chapter 2 we modify the transform c and then use it to transform the Minkowski space $\mathbf{M}_0$ into the Einstein universe $\mathbf{E}$. In Chapter 3 we study the semilinear case $Lu = u$ and show that in this case the solutions of (8) blow up in finite time for small powers l, while having an arbitrarily long life-span for large l. Finally, in Chapter 4 we prove finite-time blow-up for a class of quasilinear equations and develop a technique to generate more examples of finite-time blow-up. Our main results are the two finite-time blow-up theorems stated as Theorem 3.1 and Theorem 4.1.

Chapter 1

Compactification of Minkowski Space

Penrose [21] suggested that to study infinity in Minkowski space $\mathbf{M}_0 = \mathbb{R} \times \mathbb{R}^n$, it would be beneficial to conformally embed $\mathbf{M}_0$ into a compact subset of the Einstein universe $\mathbf{E} = \mathbb{R} \times S^n$. The conformal map $c : \mathbf{M}_0 \to \mathbf{E}$ he proposed is the Lorentzian metric analog of the usual stereographic transformation from $\mathbb{R}^n$ to S^n. In this chapter we define the map c and prove some of its basic properties.

1.1 Conformal Transform

We consider the $(n+1)-$dimensional Minkowski space $\mathbf{M}_0$, which is $\mathbb{R} \times \mathbb{R}^n$ endowed with the metric

$$g = dt^2 - d\mathbf{x}^2 = dt^2 - \sum_{i=1}^{n} dx_i^2.$$

We also define the *Einstein universe* $\mathbf{E}$ to be $\mathbb{R} \times S^n$ endowed with the metric

$$\tilde{g} = dT^2 - dS^2,$$

where dS^2 is the canonical metric on S^n.

We introduce coordinates on S^n by regarding it as a unit sphere in $\mathbb{R}^{n+1}$: $Y_1^2 + Y_2^2 + \cdots + Y_{n+1}^2 = 1$. Thus a point in $\mathbf{E}$ is represented by $(T, Y_1, \ldots, Y_{n+1})$; we will call T the *Einstein time*.

Definition 1.1. The map $c : \mathbf{M}_0 \to \mathbf{E}$ is defined by

$$c(t, \mathbf{x}) = c(t, x_1, \ldots, x_n) = (T, Y_1, \ldots, Y_{n+1}),$$

where

$$\sin T = pt, \quad \cos T = p\left(1 - \frac{t^2 - \mathbf{x}^2}{4}\right), \quad T \in (-\pi, \pi);$$
$$Y_j = px_j, \quad j = 1, \ldots, n; \quad Y_{n+1} = p\left(1 + \frac{t^2 - \mathbf{x}^2}{4}\right);$$

with

$$p = \left[t^2 + \left(1 - \frac{t^2 - \mathbf{x}^2}{4}\right)^2\right]^{-\frac{1}{2}}.$$

One checks that

$$p^2 t^2 + p^2 \left(1 - \frac{t^2 - \mathbf{x}^2}{4}\right)^2 = 1,$$

which implies that T is well-defined. Also, $c(t, \mathbf{x}) \in \mathbf{E}$ since $\sum_{j=1}^{n+1} Y_j^2 = 1$.

It is often more convenient to write the map c in spherical coordinates. To do that we represent a point $(t, \mathbf{x}) \in \mathbf{M}_0$ as (t, r, ω), where $r = |\mathbf{x}|$ and $\omega = \frac{\mathbf{x}}{|\mathbf{x}|} \in S^{n-1}$. For a point $(T, Y_1, \ldots, Y_{n+1}) \in \mathbf{E}$ we write (T, ρ, ω), where $\rho \in [0, \pi)$ is the distance on S^n from the north pole; thus ρ and ω are defined by the embedding $[0, \pi) \times S^{n-1} \to S^n$ given by

$$(\omega, \rho) \mapsto (Y_1, Y_2, \dots, Y_{n+1}),$$

where

$$(Y_1, \dots, Y_n) = \sin\rho \cdot \omega, \quad \text{and} \quad Y_{n+1} = \cos\rho. \tag{1.1}$$

Definition 1.2. In spherical coordinates the map $c : \mathbf{M}_0 \to \mathbf{E}$ is defined by

$$c(t, r, \omega) = (T, \rho, \omega),$$

where

$$\begin{aligned} \sin T &= pt, \quad \cos T = p\left(1 - \frac{t^2 - r^2}{4}\right), \quad T \in (-\pi, \pi); \\ \sin\rho &= pr, \quad \cos\rho = p\left(1 + \frac{t^2 - r^2}{4}\right), \quad \rho \in [0, \pi); \end{aligned} \tag{1.2}$$

with the angular variables $\omega \in S^{n-1}$ unchanged and

$$p = \left[t^2 + \left(1 - \frac{t^2 - r^2}{4}\right)^2\right]^{-\frac{1}{2}}.$$

In the following proposition we prove the conformality of c.

Proposition 1.1. *The map c is conformal. More precisely, $c^* \tilde{g} = p^2 g$.*

Proof. Denote by ∂_t, ∂_r, ∂_T, and ∂_ρ the respective partial derivative operators. Considered as sections of the tangent bundle $\mathbf{TM}_0$ of $\mathbf{M}_0$, ∂_t and ∂_r induce vector fields on the compactified Minkowski space $c(\mathbf{M}_0)$ defined by $\tilde{\partial}_t = c_* \partial_t$ and $\tilde{\partial}_r = c_* \partial_\rho$. The metric g in $\mathbf{M}_0$ is given in spherical coordinates by

$$g = dt^2 - dr^2 - r^2 d\omega^2, \tag{1.3}$$

where $d\omega^2$ is the canonical metric on S^{n-1}. On the other hand, from formulae (1.1) we have for the metric $\tilde{g}$ on $\mathbf{E}$,

$$\tilde{g} = dT^2 - d\rho^2 - \sin^2 \rho \, d\omega^2 = dT^2 - d\rho^2 - p^2 r^2 d\omega^2. \tag{1.4}$$

We compare (1.3) and (1.4) and take into consideration that $\mathbf{TM}_0$ decomposes orthogonally into $\mathbf{T}S^{n-1}$ and the span of $\{\partial_t, \partial_r\}$, to conclude that it suffices to show,

$$\tilde{g}(\tilde{\partial}_t, \tilde{\partial}_t) = p^2, \quad \tilde{g}(\tilde{\partial}_r, \tilde{\partial}_r) = -p^2, \quad \text{and} \quad \tilde{g}(\tilde{\partial}_t, \tilde{\partial}_r) = 0. \tag{1.5}$$

To this end, we first write

$$\tilde{\partial}_t = \frac{\partial T}{\partial t} \partial_T + \frac{\partial \rho}{\partial t} \partial_\rho \quad \text{and} \quad \tilde{\partial}_r = \frac{\partial T}{\partial r} \partial_T + \frac{\partial \rho}{\partial r} \partial_\rho,$$

and then differentiate equations (1.2) to calculate $\frac{\partial T}{\partial t}$, $\frac{\partial \rho}{\partial t}$, $\frac{\partial T}{\partial r}$, and $\frac{\partial \rho}{\partial r}$. Thus we arrive at the following expressions for $\tilde{\partial}_t$ and $\tilde{\partial}_r$:

$$\begin{aligned} \tilde{\partial}_t &= \frac{1}{2}(1 + \cos T \cos \rho) \, \partial_T - \frac{1}{2} \sin T \sin \rho \, \partial_\rho, \\ \tilde{\partial}_r &= -\frac{1}{2} \sin T \sin \rho \, \partial_T + \frac{1}{2}(1 + \cos T \cos \rho) \, \partial_\rho. \end{aligned} \tag{1.6}$$

Now, using the fact that ∂_T and ∂_ρ are orthonormal in the metric $\tilde{g}$, the verification of (1.5) is straightforward. □

It follows from (1.2) that in spherical coordinates

$$p = \frac{1}{2}(\cos T + \cos \rho),$$

which shows that p can be extended on the whole Einstein universe $\mathbf{E}$. As we mentioned in the Introduction, the compactified Minkowski space $c(\mathbf{M}_0)$ is bounded by the lightcones $C_{\pm} = \{\pm T + \rho = \pi\}$. Therefore p vanishes on the boundary of $c(\mathbf{M}_0)$, in other words, this boundary is given by the equation $p = 0$.

1.2 Transforming the d'Alembertian

We consider in $\mathbf{M}_0$ the d'Alembertian $\Box$ relative to the metric g and in $\mathbf{E}$ the operator $\Box_c = \tilde{\Box} + s^2$, $s = \frac{n-1}{2}$, where $\tilde{\Box}$ is the d'Alembertian relative to the metric $\tilde{g}$. We will prove that $\Box$ and $\Box_c$ are conformally covariant. Let us first agree on the following notation: throughout the remainder of this chapter we will use u or ϕ to denote a function in $\mathbf{M}_0$ and $\tilde{\phi}$ to denote a function in $\mathbf{E}$. We will always assume that these functions have enough smoothness so that the derivatives invoked exist.

Proposition 1.2. *Let u and $\tilde{\phi}$ be related by $u = p^s(\tilde{\phi} \circ c)$. Then*

$$\left(\Box_c \tilde{\phi}\right) \circ c = p^{-\frac{n+3}{2}} \Box u. \tag{1.7}$$

To prove this proposition we will need the following two lemmas.

Lemma 1.1. *If $\phi = \tilde{\phi} \circ c$ then the following formula is true:*

$$\left(\tilde{\Box}\tilde{\phi}\right) \circ c = p^{-2}\Box\phi + (n-1)p^{-3} g(dp, d\phi).$$

Proof. Here we adopt the summation convention whereby a repeated index implies summation over all values of that index. The indices used have their values in the set $\{1, 2, \ldots, n+1\}$. Let $\{\partial_1, \ldots, \partial_{n+1}\}$ be a local basis for $\mathbf{TM}_0$. Denote by $\tilde{\partial}_i = c_*(\partial_i)$ the vector field induced on $c(\mathbf{M}_0)$ by ∂_i. Then $\{\tilde{\partial}_1, \ldots, \tilde{\partial}_{n+1}\}$ forms a local basis for the restriction of $\mathbf{TE}$ to $c(\mathbf{M}_0)$. As customary, we denote $g_{ij} = g(\partial_i, \partial_j)$ and $\tilde{g}_{ij} = \tilde{g}(\tilde{\partial}_i, \tilde{\partial}_j)$; then (g^{ij}) denotes the inverse matrix of (g_{ij}) and similarly, $(\tilde{g}^{ij})$ denotes the inverse matrix of $(\tilde{g}_{ij})$.

Analogously to the Riemannian case, we have in terms of local coordinates

$$\tilde{\Box}\tilde{\phi} = \frac{1}{\sqrt{\tilde{G}}}\,\tilde{\partial}_i\left(\tilde{g}^{ij}\sqrt{\tilde{G}}\,\tilde{\partial}_j\tilde{\phi}\right), \tag{1.8}$$

where $\tilde{G} = |\det(\tilde{g}_{ij})|$. Since $c^*\tilde{g} = p^2 g$, we have $\tilde{g}_{ij}\circ c = p^2 g_{ij}$ and hence, $\tilde{g}^{ij}\circ c = p^{-2}g^{ij}$ and $\tilde{G}\circ c = p^{n+1}G$, G being $|\det(g_{ij})|$. Therefore using (1.8) we obtain

$$\left(\tilde{\Box}\tilde{\phi}\right)\circ c = \frac{1}{p^{n+1}}\cdot\frac{1}{\sqrt{g}}\,\partial_i\left(p^{-2}g^{ij}p^{n+1}\sqrt{g}\,\partial_j\phi\right).$$

After differentiating the expression in the parentheses above, we get

$$\begin{aligned}\left(\tilde{\Box}\tilde{\phi}\right)\circ c &= \frac{1}{p^2}\cdot\frac{1}{\sqrt{g}}\,\partial_i\left(g^{ij}\sqrt{g}\,\partial_j\phi\right) + (n-1)p^{-3}g^{ij}(\partial_i p)(\partial_j\phi)\\ &= p^{-2}\Box\phi + (n-1)p^{-3}g(dp, d\phi). \quad \Box\end{aligned}$$

For the second lemma we use the fact that in terms of the coordinates $(t, \mathbf{x})$ in $\mathbf{M}_0$ the operator $\Box$ is simply $\Box = \partial_t^2 - \Delta$, Δ being the Laplacian in $\mathbf{x}$.

Lemma 1.2. *The following formula is true:*

$$\Box p^s = s^2 p^{\frac{n+3}{2}}.$$

Proof. Observe first that

$$\Box p = sp^3 - \frac{n-3}{2}\cdot\frac{p^3(t^2-\mathbf{x}^2)}{4}.$$

Hence,

$$\Box p^s = \frac{n-1}{2}\cdot\frac{n-3}{2}p^{\frac{n-5}{2}}g(dp, dp) + \frac{n-1}{2}p^{\frac{n-3}{2}}\Box p,$$

which combined with

$$g(dp, dp) = \frac{p^4 \mathbf{x}^2}{4}$$

implies the result. $\square$

Proof of Proposition 1.2. Assume that $u = p^s(\tilde{\phi} \circ c)$. From Lemma 1.1 we have

$$p^{\frac{n+3}{2}} \left(\square_c \tilde{\phi}\right) \circ c = p^s \square \phi + (n-1) p^{\frac{n-3}{2}} g(dp, d\phi) + p^{\frac{n+3}{2}} s^2 \phi. \tag{1.9}$$

On the other hand, applying the identity

$$\square(\phi \cdot \psi) = \psi \square \phi + \phi \square \psi + 2g(d\phi, d\psi),$$

we obtain

$$\square(p^s \phi) = p^s \square \phi + (\square p^s)\phi + 2g(dp^s, d\phi),$$

which, using Lemma 1.2, implies

$$\square(p^s \phi) = p^s \square \phi + s^2 p^{\frac{n+3}{2}} \phi + (n-1) p^{\frac{n-3}{2}} g(dp, d\phi). \tag{1.10}$$

Combining (1.9) and (1.10), we obtain

$$p^{\frac{n+3}{2}} \left(\square_c \tilde{\phi}\right) \circ c = \square(p^s \phi). \quad \square$$

Remark. A well-known result in Riemannian geometry, generalized for the case of Lorentzian metrics, states that for a conformal transform $c : (M, g) \to (N, \tilde{g})$ between two Lorentz manifolds of dimension $n+1$ we have the following formula for the conformal d'Alembertians on M and N:

$$\left[\left(\tilde{\Box} - \tilde{R}\frac{n-1}{4n}\right)\tilde{\phi}\right] \circ c = p^{-\frac{n+3}{2}}\left(\Box - R\frac{n-1}{4n}\right)\left(p^{\frac{n-1}{2}}u\right).$$

Here R, $\tilde{R}$ and $\Box$, $\tilde{\Box}$ are respectively the scalar curvatures and the d'Alembertians for M and N. Taking into account that $(\mathbf{M}_0, g)$ is flat, i.e., its curvature is $R = 0$, and that the curvature of $(\mathbf{E}, \tilde{g})$ is $\tilde{R} = -n(n-1)$, the above formula simplifies to formula (1.7).

Chapter 2

Transforming the Equation into Einstein Universe

For a function u in $\mathbf{M}_0$ we denote

$$Lu := a(t,r)u_t + b(t,r)u_r + c(t,r)u$$

and consider the equation

$$\Box u = p^{-k}|Lu|^l, \quad l > 1, \tag{2.1}$$

where $k = sl - \frac{n+3}{2}$.

In this chapter we modify the conformal transform c and then use it to transform equation (2.1) into the Einstein universe $\mathbf{E}$.

2.1 Modifying the Transform

In this section we will compose the conformal transform c with an one-parameter family of dilations thus obtaining an one-parameter family of conformal transforms c_R.

We will then reformulate Propositions 1.1 and 1.2 in terms of c_R.

We consider the one-parameter family of dilations on $\mathbf{M}_0$

$$d_R : (t, \mathbf{x}) \mapsto (\frac{t}{R}, \frac{\mathbf{x}}{R}), \quad R > 0.$$

We first note that d_R is a conformal map, i.e., $d_R^* g = R^{-2} g$. Indeed, for a vector field $X \in \mathbf{TM}_0$ and a function ϕ in $\mathbf{M}_0$ we have, using the Chain Rule,

$$(d_{R*}X)\phi = X(\phi \circ d_R) = \frac{1}{R} X\phi,$$

which implies for $X, Y \in \mathbf{TM}_0$,

$$d_R^* g(X, Y) = g(d_{R*}X, d_{R*}Y) = \frac{1}{R^2} g(X, Y).$$

With this in mind, we are ready now to restate Proposition 1.1.

Proposition 2.1. *The map c_R, $R > 0$, is conformal. More precisely, $c_R^* \tilde{g} = R^{-2} p^2 g$.*

Proof. It follows immediately from the definition of the pull-back and the Chain Rule that a composition of two conformal maps with conformal factors λ and μ is also a conformal map with a conformal factor $\lambda\mu$. $\square$

Remark. Note that in the above proposition p should be understood as

$$p_R = p \circ d_R = \left[\frac{t^2}{R^2} + \left(1 - \frac{t^2 - \mathbf{x}^2}{4R^2} \right)^2 \right]^{-\frac{1}{2}}.$$

From now on we will consistently suppress R in p_R to avoid unnecessary pile-up of notation.

We now restate Proposition 1.2.

Proposition 2.2. *Let u and $\tilde{\phi}$ be related by $u = R^2p^s(\tilde{\phi}\circ c_R)$. Then*

$$\left(\Box_c\tilde{\phi}\right)\circ c_R = p^{-\frac{n+3}{2}}\Box u.$$

Proof. Assume that $u = R^2p^s(\tilde{\phi}\circ c_R)$. Denote for convenience $\phi = \tilde{\phi}\circ c_R$. From the proof of Lemma 1.1 it is clear that the lemma holds true for every conformal map. Therefore, using $\frac{p}{R}$ for p, we obtain from Lemma 1.1,

$$\left(\tilde{\Box}\tilde{\phi}\right)\circ c_R = R^2p^{-2}\Box\phi + (n-1)R^3p^{-3}g(d\frac{p}{R}, d\phi),$$

which implies

$$p^{\frac{n+3}{2}}\left(\tilde{\Box}\tilde{\phi}\right)\circ c_R = R^2p^s\Box\phi + (n-1)R^2p^{\frac{n-3}{2}}g(dp, d\phi),$$

and hence,

$$p^{\frac{n+3}{2}}\left(\Box_c\tilde{\phi}\right)\circ c_R = R^2p^s\Box\phi + (n-1)R^2p^{\frac{n-3}{2}}g(dp, d\phi) + p^{\frac{n+3}{2}}s^2\phi. \tag{2.2}$$

On the other hand, as in the proof of Proposition 1.2, we have

$$\Box(R^2p^s\phi) = R^2p^s\Box\phi + R^2(\Box p^s)\phi + 2R^2g(dp^s, d\phi). \tag{2.3}$$

For $\Box p^s$ we obtain from Lemma 1.2 and the Chain Rule

$$\Box p^s = \frac{s^2}{R^2}p^{\frac{n+3}{2}},$$

which we substitute in (2.3) to get

$$\Box(R^2p^s\phi) = R^2p^s\Box\phi + s^2p^{\frac{n+3}{2}}\phi + (n-1)R^2p^{\frac{n-3}{2}}g(dp, d\phi). \tag{2.4}$$

Combining (2.2) and (2.4), we obtain

$$p^{\frac{n+3}{2}}\left(\Box_c\tilde{\phi}\right)\circ c_R=\Box(R^2p^s\phi).$$

which completes the proof of the proposition. $\Box$

2.2 Transformed Equation

For a function v in $\mathbf{E}$, let us denote

$$\Lambda v := A(T,\rho)v_T + B(T,\rho)v_\rho + C(T,\rho)v. \tag{2.5}$$

Identifying $\mathbf{M}_0$ and $c_R(\mathbf{M}_0)$, we have the following transformation relation between Lu and Λv.

Proposition 2.3. *Let u and v be related by $u=R^{-\frac{2}{l-1}}p^s v$. Then,*

$$\Lambda v = R^{\frac{2}{l-1}}p^{-s}Lu, \tag{2.6}$$

where the relation between the coefficients a, b, and c of L and A, B, and C of Λ is given by

$$\begin{aligned}
a &= A\left(R+\frac{t^2+r^2}{4R}\right)+B\frac{tr}{2R},\\
b &= A\frac{tr}{2R}+B\left(R+\frac{t^2+r^2}{4R}\right),\\
c &= A\frac{st}{2R}+B\frac{sr}{2R}+C.
\end{aligned}$$

Proof. Since, as we already noted in Section 2.1, d_R maps a vector field $X\in\mathbf{TM}_0$

to $\frac{1}{R}X$, we have from formulae (1.6) on page 12,

$$\begin{aligned}\tilde{\partial}_t &= \frac{1}{2R}(1+\cos T\cos\rho)\,\partial_T - \frac{1}{2R}\sin T\sin\rho\,\partial_\rho,\\ \tilde{\partial}_r &= -\frac{1}{2R}\sin T\sin\rho\,\partial_T + \frac{1}{2R}(1+\cos T\cos\rho)\,\partial_\rho,\end{aligned}$$

where we identify on $c_R(\mathbf{M}_0)$ the vector fields $\tilde{\partial}_t = c_*\partial_t \equiv \partial_t$ and $\tilde{\partial}_r = c_*\partial_r \equiv \partial_r$.

We solve for ∂_T and $\partial\rho$ to get

$$\begin{aligned}\partial_T &= \frac{R}{2}p^{-2}\left[(1+\cos T\cos\rho)\partial_t + \sin T\sin\rho\,\partial_r\right],\\ \partial_\rho &= \frac{R}{2}p^{-2}\left[\sin T\sin\rho\,\partial_t + (1+\cos T\cos\rho)\partial_r\right].\end{aligned} \tag{2.7}$$

We next employ the fact that c_R is given by the formulae

$$\begin{aligned}\sin T &= \frac{pt}{R}, \quad \cos T = p\left(1-\frac{t^2-r^2}{4R^2}\right), \quad T\in(-\pi,\pi),\\ \sin\rho &= \frac{pr}{R}, \quad \cos\rho = p\left(1+\frac{t^2-r^2}{4R^2}\right), \quad \rho\in[0,\pi),\end{aligned}$$

to obtain

$$\begin{aligned}1+\cos T\cos\rho &= 1+p^2\left(1-\frac{t^2-r^2}{4R^2}\right)\left(1+\frac{t^2-r^2}{4R^2}\right)\\ &= p^2\left[\frac{t^2}{R^2}+\left(1-\frac{t^2-r^2}{4R^2}\right)^2+\left(1-\frac{t^2-r^2}{4R^2}\right)\left(1+\frac{t^2-r^2}{4R^2}\right)\right]\\ &= 2p^2\left(1+\frac{t^2-r^2}{4R^2}\right),\end{aligned}$$

and

$$\sin T\sin\rho = \frac{p^2tr}{R^2}$$

which, substituted in (2.7), gives us

$$\partial_T = \left(R + \frac{t^2+r^2}{4R}\right)\partial_t + \frac{tr}{2R}\partial_r,$$
$$\partial_\rho = \frac{tr}{2R}\partial_t + \left(R + \frac{t^2+r^2}{4R}\right)\partial_r.$$

Therefore,

$$\begin{aligned} Lv = {} & R^{\frac{2}{l-1}} p^{-s}\left[A\left(1+\frac{t^2+r^2}{4R}\right) + B\frac{tr}{2R}\right] u_t \\ & + R^{\frac{2}{l-1}} p^{-s}\left[A\frac{tr}{2R} + B\left(1+\frac{t^2+r^2}{4R}\right)\right] u_r \\ & + R^{\frac{2}{l-1}} p^{-s}\left\{-sp^{-1}A\left[\left(R+\frac{t^2+r^2}{4R}\right)\partial_t p + \frac{tr}{2R}\partial_r p\right]\right. \\ & \left. -sp^{-1}B\left[\left(R+\frac{t^2+r^2}{4R}\right)\partial_r p + \frac{tr}{2R}\partial_t p\right] + C\right\} u. \end{aligned}$$

Finally, we use that

$$\partial_t p = -\frac{p^3 t}{2R^2}\left(1 + \frac{t^2-r^2}{4R^2}\right),$$

and

$$\partial_r p = -\frac{p^3 r}{2R^2}\left(1 - \frac{t^2-r^2}{4R^2}\right),$$

to complete the proof. □

Proposition 2.4. *The equation*

$$\Box u = p^{-k}|Lu|^l$$

transforms under c_R into the equation

$$\Box_c v = |\Lambda v|^l,$$

where u and v are related by $u = R^{-\frac{2}{l-1}} p^s v$, and $k = sl - \frac{n+3}{2}$.

Proof. Recall that from Proposition 2.2 we have for $\phi = \tilde{\phi} \circ c_R$,

$$\Box_c \tilde{\phi} = p^{-\frac{n+3}{2}} \Box (R^2 p^s \phi).$$

Consequently,

$$p^{\frac{n+3}{2}} \Box_c v = p^{\frac{n+3}{2}} \Box_c \left(R^{\frac{2}{l-1}} p^{-s} u \right) = R^{\frac{2l}{l-1}} \Box u = R^{\frac{2l}{l-1}} p^{-k} |L\, u|^l = p^{\frac{n+3}{2}} |\Lambda\, v|^l$$

where in the last equality we used equation (2.6). $\Box$

Chapter 3

Semilinear Case

In this chapter we will prove finite-time blow-up for the solutions of the following Cauchy problem in $\mathbf{M}_0$:

$$\Box u = p^{-k}|u|^l, \tag{3.1}$$

$$u(0,\cdot) = f, \quad u_t(0,\cdot) = g, \tag{3.2}$$

where $l > 1$ and $k = sl - \frac{n+3}{2}$. We assume that $f, g \in X$ where the space X is defined by

$$X := \{\, f \mid f \in C_0^\infty(\mathbb{R}^n),\ f \geq 0,\ \}\,.$$

Let us first fix some notation and recommit ourselves to some already introduced one. In what follows we will identify, for the sake of brevity, $\mathbf{M}_0$ and $c(\mathbf{M}_0)$; thus, for example, the expression $\tilde{\Box}\tilde{\phi}$ will be understood where appropriate as $(\tilde{\Box}\tilde{\phi}) \circ c$. We will continue to suppress R in $p_R = p \circ d_R$; in addition, we will suppress R in

$c_R = c \circ d_R$. Accordingly, the map $c : \mathbf{M}_0 \to \mathbf{E}$ is defined in spherical coordinates by

$$\begin{aligned} \sin T &= \frac{pt}{R}, \quad \cos T = p\left(1 - \frac{t^2 - r^2}{4R^2}\right), \quad T \in (-\pi, \pi); \\ \sin\rho &= \frac{pr}{R}, \quad \cos\rho = p\left(1 + \frac{t^2 - r^2}{4R^2}\right), \quad \rho \in [0, \pi); \end{aligned} \tag{3.3}$$

with the angular variables unchanged and

$$p = \left[\frac{t^2}{R^2} + \left(1 - \frac{t^2 - r^2}{4R^2}\right)^2\right]^{-\frac{1}{2}}.$$

3.1 Finite-Time Blow-Up

To prove finite-time blow-up we will transform the problem (3.1)-(3.2) to the Einstein universe $\mathbf{E}$.

Let us define $\tilde{f} = f \circ c^{-1}$, $\tilde{g} = g \circ c^{-1}$. We will show that under the transform c problem (3.1)-(3.2) transforms into the following Cauchy problem in $\mathbf{E}$:

$$\Box_c v = |v|^l, \tag{3.4}$$

$$v(0, \cdot) = R^{\frac{2}{l-1}} p_0^{-s} \tilde{f}, \quad v_T(0, \cdot) = R^{\frac{l+1}{l-1}} p_0^{-(s+1)} \tilde{g}, \tag{3.5}$$

where u and v are related by $u = R^{-\frac{2}{l-1}} p^s v$ and $p_0 = \cos^2 \frac{\rho}{2}$. Indeed, equation (3.1) transforms into equation (3.4) by virtue of Proposition 1.2 applied for L and Λ being the identity maps.

To verify the first initial condition it is enough to note only that

$$p_0 := p(0, \cdot) = \frac{1}{2}(\cos T + \cos\rho)\big|_{T=0} = \cos^2 \frac{\rho}{2}.$$

For the second initial condition we use that

$$\tilde{\partial}_t\big|_{t=0} = \frac{1}{2R}(1+\cos\rho)\,\partial_T\big|_{T=0} = \frac{p_0}{R}\,\partial_T\big|_{T=0},$$

and, consequently, $\tilde{\partial}_t\big|_{t=0} p = 0$. Therefore,

$$u_t(0,\cdot) = \tilde{\partial}_t\big|_{t=0}(R^{-\frac{2}{l-1}}p^s v) = R^{-\frac{2}{l-1}}p_0^s(\frac{p_0}{R}\,\partial_T\big|_{T=0} v) = R^{-\frac{2}{l-1}}\frac{p_0^{s+1}}{R}\,v_T(0,\cdot) = g.$$

We now state the first main blow-up result.

Theorem 3.1. *Let $1 < l < \frac{2}{n}+1$, and u be a solution of (3.1)-(3.2) with initial data $f, g \in X$. Then u blows up in finite time.*

To prove this theorem we will need the following proposition which, although stronger than what we need for the proof of Theorem 3.1, we believe is interesting by itself to warrant stating in its full power.

Lemma 3.1. *Consider the initial-value problem*

$$\begin{cases} y' = \sqrt{y^{l+1} - M^{l+1} + M^2}, \\ y(0) = M, \end{cases} \tag{3.6}$$

where the constant $M \geq 2$. Let L_y be the life-span of the solution y of (3.6). Then

$$L_y \sim M^{-\frac{l-1}{2}}.$$

Proof. We set $z(t) = M^{-1}y(M^{-\frac{l-1}{2}}t)$. Hence,

$$z' = M^{-\frac{l+1}{2}}y' = M^{-\frac{l+1}{2}}\sqrt{y^{l+1} - M^{l+1} + M^2} = \sqrt{(M^{-1}y)^{l+1} - 1 + M^{1-l}}.$$

Therefore z is a solution of

$$\begin{cases} z' = \sqrt{z^{l+1} - 1 + M^{1-l}}, \\ z(0) = 1. \end{cases} \tag{3.7}$$

Denote by L_z the life-span of z. It suffices to show that $L_z \sim 1$, i.e., $c_1 \leq L_z \leq c_2$.

Step 1. $L_z \leq c_2$.

By the Mean Value Theorem, we have

$$z^{l+1} - 1 + M^{1-l} \geq (l+1)(z-1)\theta^l + M^{1-l} \geq (z-1) + M^{1-l}.$$

Setting $w = z - 1$ we obtain

$$w' = z' \geq \sqrt{w + M^{1-l}},$$

or equivalently,

$$\frac{w'}{\sqrt{w + M^{1-l}}} \geq 1,$$

which, after integration, gives

$$2\sqrt{w + M^{1-l}} - 2M^{\frac{1-l}{2}} \geq t.$$

Hence

$$w + M^{1-l} \geq \frac{t^2}{4} + M^{1-l},$$

from where it follows that

$$z \geq 1 + \frac{t^2}{4} \geq 2, \quad t \geq 2.$$

Therefore for $t \geq 2$, we have

$$z^{l+1} - 1 + M^{1-l} \geq z^{l+1} - 1 \geq Az^{l+1},$$

where we set $A = 1 - 2^{-l-1} > 0$. Hence we have

$$z' \geq \sqrt{Az^{l+1}},$$

which we integrate between 2 and t to obtain

$$\frac{2}{1-l}\left(z^{\frac{1-l}{2}} - 2^{\frac{1-l}{2}}\right) \geq \sqrt{A}(t-2).$$

This is equivalent to

$$z^{\frac{l-1}{2}} \geq \frac{1}{2^{\frac{1-l}{2}} - \frac{2}{l-1}\sqrt{A}(t-2)},$$

which implies $L_z \leq 2 + cst. = c_2$.

Step 2. $L_z \geq c_1 > 0$.

Assume $z \leq 2$ on $[0, \delta]$. Hence,

$$z^{l+1} - 1 + M^{1-l} \leq (l+1)(z-1)\theta^l + M^{1-l} \leq cw + 1,$$

since $\theta^l \leq 2^l$ and therefore $w' \leq \sqrt{cw+1}$. We integrate the above inequality between 0 and t and arrive at

$$\sqrt{cw+1} \leq \frac{c}{2}t + 1.$$

Thus for any $t \leq \frac{2}{c}$, we have

$$\sqrt{cw+1} \leq 2$$

or, consequently,

$$w \leq \frac{3}{c} = \frac{3}{(l+1)2^l} \leq \frac{3}{4} \leq 1.$$

Hence for $t \leq \frac{2}{c}$, we have $z \leq 2$, which implies that z can not blow up before $c_1 = \frac{2}{c}$, i.e., $L_z \geq c_1$. $\square$

Let us now prove Theorem 3.1. Throughout the proof we will use for convenience C as a generic name for a (strictly) positive constant—its values may be different in the various places it appears; what matters is that it is always independent of the Einstein coordinates, as well as the parameter R.

Proof. Let v be a solution of the Cauchy problem (3.4)-(3.5). We define the function $H(T)$ by

$$H(T) := \int_{S^n} v(T,\cdot)\, dS.$$

Observe that the definition of the functional H assures that it is finite for globally existing v; therefore, the blow-up of H would establish the nonexistence of v beyond a finite time. Observe also that as the parameter R increases, the support of the initial data (3.5) decreases at a rate of $\frac{1}{R}$. Therefore, for any Einstein time $T < \frac{\pi}{2}$, we may choose R large enough so that the support of $v(T,\cdot)$ is contained in $c(\mathbf{M}_0)$.

Thus we have that a blow-up for $H(T)$ at a finite time $T < \frac{\pi}{2}$ implies a finite-time blow-up for the solution u of (3.1)–(3.2) in Minkowski space.

Remark. Note that had we not included the factor p^{-k} in the interaction term of equation (3.1), we would have had a factor involving some power of p in the transformed equation in $\mathbf{E}$. This would have made the functional H singular along the boundary in $\mathbf{E}$ of the compactified Minkowski space $c(\mathbf{M}_0)$.

We integrate equation (3.4) on S^n and, noticing that by the Divergence Theorem

$$\int_{S^n} \Delta_{S^n} v \, dS = 0,$$

arrive at

$$H''(T) + s^2 H(T) = \int_{S^n} |v(T,\cdot)|^l dS. \tag{3.8}$$

We use Hölder inequality to estimate

$$\int_{S^n} |v(T,\cdot)|^l dS \geq C \left| \int_{S^n} v(T,\cdot)\, dS \right|^l ,$$

which, combined with equation (3.8), gives us

$$H'' + s^2 H \geq C|H|^l.$$

Writing the latter inequality as

$$H'' \geq C|H|^l + \left(C|H|^l - s^2 H\right),$$

we obtain

$$H'' \geq C|H|^l + \left(C|H|^{l-1} - s^2\right)|H|. \tag{3.9}$$

As we showed, the transform c is a conformal transform with a conformal factor $\frac{p}{R}$, i.e., we have in local coordinates

$$\tilde{g}_{\alpha\beta} = \frac{p^2}{R^2} g_{\alpha\beta}, \quad \alpha, \beta = 0, 1, \ldots, n.$$

If we restrict ourselves only to Minkowski time $t = 0$ (or equivalently, to Einstein time $T = 0$), we have the following relation between the determinants of the metrics $\tilde{g}$ and g on $\{0\} \times S^n$ and $\{0\} \times \mathbb{R}^n$ respectively,

$$\det \tilde{g}_{\alpha\beta} = \frac{p^{2n}}{R^{2n}} \det g_{\alpha\beta}.$$

From this we can deduce that the corresponding volume forms are related by

$$dS = \frac{p^n}{R^n} dx. \tag{3.10}$$

Consequently, for $R > 1$ we have from the assumption for the initial values $H_0 := H(0)$ and $H_0' := H'(0)$ of $H(T)$,

$$\begin{aligned} H_0 &= R^{\frac{2}{l-1}} \int_{S^n} (p_0^{-s} \tilde{f})\, dS = R^{\frac{2}{l-1}-n} \int_{\mathbf{R}^n} \left(1 + \frac{r^2}{4R^2}\right)^{-\frac{n+1}{2}} f\, dx \\ &\geq R^{\frac{2}{l-1}-n} \int_{\mathbf{R}^n} \left(1 + \frac{r^2}{4}\right)^{-\frac{n+1}{2}} f\, dx = CR^{\frac{2}{l-1}-n}, \end{aligned}$$

and

$$H_0' = R^{\frac{l+1}{l-1}} \int_{S^n} (R p_0^{-(s+1)} \tilde{g})\, dS = R^{\frac{l+1}{l-1}-n} \int_{\mathbf{R}^n} \left(1 + \frac{r^2}{4R^2}\right)^{-\frac{n-1}{2}} g\, dx$$
$$\geq R^{\frac{l+1}{l-1}-n} \int_{\mathbf{R}^n} \left(1 + \frac{r^2}{4}\right)^{-\frac{n-1}{2}} g\, dx = C R^{\frac{l+1}{l-1}-n}.$$

Since $\frac{2}{l-1} > n$, we have, for $R \gg 1$, in the right-hand side of equation (3.9)

$$C|H(T)|^{l-1} - s^2 \geq 0, \quad \text{for all } T \geq 0.$$

Thus inequality (3.9) leads to

$$H'' \geq C H^l.$$

We integrate this inequality to get

$$H'^2 \geq C\left(H^{l+1} - H_0^{l+1}\right) + H_0'^2,$$

which, setting

$$y(T) = H_0^{-1} H\left(C^{-\frac{1}{2}} H_0^{-\frac{l-1}{2}} T\right),$$

implies that y is a solution of the initial-value problem

$$\begin{cases} y' \geq \sqrt{y^{l+1} - 1 + C M^{1-l}}, \\ y(0) = 1, \end{cases}$$

where $M = R^{\frac{2}{l-1}-n}$.

Choosing again R large enough so that $M \geq 2$, we employ Lemma 3.1 to obtain

$L_y \leq C$ for the life-span L_y of y. Consequently, we have for the life-span T_0 of $H(T)$

$$T_0 \leq CM^{-\frac{l-1}{2}} = CR^{\frac{n(l-1)}{2}-1}.$$

It is important to notice that T_0 decreases at a slower rate compared with the support of the initial data. This allows us to choose $R \gg 1$ so that v blows up at a point inside of $c(\mathbf{M}_0)$.

Finally, it follows from the definition of the map c that for the corresponding Minkowski time we have

$$t_0 \sim RT_0 \leq CR^{\frac{n(l-1)}{2}} \leq CR. \quad \square$$

Remark. As we know from the Introduction, $l < \frac{2}{n} + 1$ is not a sharp critical power dividing between global existence and finite-time blow-up. Having said that, we must note that Theorem 3.1 has the following advantage over the existing blow-up results: the nonlinearity in equation (3.1) is tempered at infinity by the factor p^{-k}, which for $l < \frac{2}{n}+1$ approaches 0, as $|\mathbf{x}| \longrightarrow \infty$; yet, this nonlinearity still produces a finite-time blow-up.

As a generalization of Theorem 3.1, we state the following theorem.

Theorem 3.2. *Let $1 < l < \frac{2}{n} + 1$, and u be a solution of the Cauchy problem*

$$\begin{cases} \square u = F(t, \mathbf{x}, u), \\ u(0, \cdot) = f, \quad u_t(0, \cdot) = g, \end{cases} \tag{3.11}$$

where $F(t, \mathbf{x}, u) \geq c|u|^l$ and the initial data $f, g \in X$. Then u blows up in finite time.

Proof. We write the right-hand side of the above equation as

$$F(t, \mathbf{x}, u) = p^{-k} p^k F(t, \mathbf{x}, u) \geq c p^{-k} |u|^l,$$

then observe that the proof of Theorem 3.1 can be *ad lib.* modified to accomodate this case.

3.2 Comparing the equations $\Box u = |u|^l$ and $\Box u = p^{-k}|u|^l$

Consider the equations

$$\Box u = p^{-k} |u|^l, \quad R > 1, \tag{3.12}$$

and

$$\Box u = |u|^l, \tag{3.13}$$

and the initial conditions

$$u(0, \mathbf{x}) = f(\mathbf{x}), \quad u_t(0, \mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.14}$$

where the Cauchy data $f, g \in X$.

Let u_R be the solution of the Cauchy problem (3.12)-(3.14) and u be the solution of the Cauchy problem (3.13)-(3.14). We will obtain an estimate for u_R and u for which we will need the following lemma. Throughout this section, we will denote by $W^{m,q} = W^{m,q}(\mathbb{R}^n)$ and $H^m = W^{m,2}(\mathbb{R}^n)$ the usual Sobolev spaces.

Lemma 3.2. *Let $u \in W^{m,q}(\mathbb{R}^n)$ have a compact support, $q \geq 1$, and $m > \frac{n}{q}$. Then for every function $h \in C^m(\mathbb{R})$, $h(0) = 0$, there exists a function C such that*

$$\|h(u)\|_{W^{m,q}} \leq C(\|u\|_{W^{m,q}}).$$

When $q = 2$, this result is well-known; for other values of q, see [2].

The following estimate for u_R and u is true.

Proposition 3.1. *Let $l > m > \frac{n}{2}$. Then*

$$\left\|p^{-k}|u_R|^l - |u|^l\right\|_{H^{m-1}} \leq C\left(\|u\|_{H^m}, \|u_R\|_{H^m}\right)\|u - u_R\|_{H^m} + \varepsilon(R), \tag{3.15}$$

where $\varepsilon(R) \to 0$, as $R \to \infty$.

Proof. We first observe that

$$\left\|p^{-k}|u_R|^l - |u|^l\right\|_{H^{m-1}} \leq \left\|p^{-k}\left(|u_R|^l - |u|^l\right)\right\|_{H^{m-1}} + \left\||u|^l(p^{-k} - 1)\right\|_{H^{m-1}}. \tag{3.16}$$

Denoting

$$\begin{aligned}\tilde{v} := \frac{|u_R|^l - |u|^l}{u_R - u} &= \frac{1}{u_R - u}\int_0^1 \frac{d}{ds}\sigma\left(su_R + (1-s)u\right)ds \\ &= \int_0^1 \sigma'(su_R + (1-s)u)\,ds,\end{aligned}$$

where $\sigma(x) = |x|^l$, we have for the first term in (3.16),

$$\begin{aligned}\left\|p^{-k}\left(|u_R|^l - |u|^l\right)\right\|_{H^{m-1}} &= \left\|p^{-k}\tilde{v}\left(u_R - u\right)\right\|_{H^{m-1}} \\ &= \sum_{\substack{0 \leq m_i \leq m-1 \\ i=1,2,3}} \left\|\nabla^{m_1}p^{-k}\,\nabla^{m_2}\tilde{v}\,\nabla^{m_3}(u_R - u)\right\|_{L^2}.\end{aligned}$$

Using Hölder's inequality, we estimate every term in the above sum by

$$\begin{aligned}
&\left\|\nabla^{m_1}p^{-k}\,\nabla^{m_2}\tilde{v}\,\nabla^{m_3}(u_R-u)\right\|_{L^2}\\
\le&\left\|\nabla^{m_1}p^{-k}\right\|_{L^\infty}\left\|\nabla^{m_2}\tilde{v}\,\nabla^{m_3}(u_R-u)\right\|_{L^2}\\
\le&C\left\|\nabla^{m_2}\tilde{v}\,\nabla^{m_3}(u_R-u)\right\|_{L^2}.
\end{aligned}$$

We choose numbers $r,s\in(2,\infty)$ so that $\frac{1}{r}>\max\{0,\frac{1}{2}-\frac{m-m_2}{n}\}$, $\frac{1}{s}>\max\{0,\frac{1}{2}-\frac{m-m_3}{n}\}$, and $\frac{1}{r}+\frac{1}{s}=\frac{1}{2}$. This is possible since by the assumption we have

$$\frac{1}{2}-\frac{m-m_2}{n}+\frac{1}{2}-\frac{m-m_3}{n}=1-\frac{2m}{n}+\frac{m_2+m_3}{n}\le 1-\frac{m}{n}<\frac{1}{2},$$

and also

$$\frac{1}{2}-\frac{m-m_i}{n}<\frac{1}{2},\quad i=2,3.$$

Using again Hölder's inequality we obtain

$$\left\|\nabla^{m_2}\tilde{v}\,\nabla^{m_3}(u_R-u)\right\|_{L^2}\le\left\|\nabla^{m_2}\tilde{v}\right\|_{L^r}\left\|\nabla^{m_3}(u_R-u)\right\|_{L^s}.$$

Observe that since $\frac{1}{r}>\frac{1}{2}-\frac{m-m_2}{n}$, we have by Lemma 3.2 and the Sobolev embedding theorem

$$\begin{aligned}
\left\|\nabla^{m_2}\tilde{v}\right\|_{L^r}&\le\left[\int_0^1\left\|\sigma'(su_R+(1-s)u)\right\|_{W^{m_2,r}}^r ds\right]^{\frac{1}{r}}\\
&\le\left(\int_0^1\left\|\sigma'(su_R+(1-s)u)\right\|_{H^m}^r ds\right)^{\frac{1}{r}}\\
&\le C\left(\|u_R\|_{H^m},\|u\|_{H^m}\right).
\end{aligned}$$

On the other hand, since $\frac{1}{s} > \frac{1}{2} - \frac{m-m_3}{n}$, we obtain by the Sobolev embedding theorem

$$\|\nabla^{m_3}(u_R - u)\|_{L^s} \leq C\|u_R - u\|_{H^m}.$$

We now estimate the second term in (3.16).

$$\left\| |u|^l (p^{-k} - 1) \right\|_{H^{m-1}} = \sum_{\substack{0 \leq m_i \leq m-1 \\ i=1,2}} \left\| \nabla^{m_1} |u|^l \, \nabla^{m_2} \left(p^{-k} - 1\right) \right\|_{L^2}.$$

For every term in the above sum we choose numbers $r, s \in (2, \infty)$ as before and again employ the Sobolev embedding theorem and Lemma 3.2 to obtain

$$\begin{aligned}
&\left\| \nabla^{m_1} |u|^l \, \nabla^{m_2} \left(p^{-k} - 1\right) \right\|_{L^2} \\
\leq &\left\| \nabla^{m_1} |u|^l \right\|_{L^r} \left\| \nabla^{m_2} \left(p^{-k} - 1\right) \right\|_{L^s} \\
\leq &C \left\| |u|^l \right\|_{W^{m_1,r}} \|p^{-k} - 1\|_{H^m} \\
\leq &C \left(\|u\|_{H^m} \right) \|p^{-k} - 1\|_{H^m} = \varepsilon(R).
\end{aligned}$$

This completes the proof of Proposition 3.1. $\square$

Assume now that, for $l > m > \frac{n}{2}$, the solution u of problem (3.13)-(3.14) in H^m is defined globally in time, whereas the solution u_R of problem (3.12)-(3.14) in the same space blows up in finite time T_0 for arbitrarily large R. The latter means that $\|u_R(t)\|_{H^m}$ increases without bound as $t \to T_0$. Therefore we may choose $\delta > 0$ so that

$$\|u_R(T_0 - \delta) - u(T_0 - \delta)\|_{H^m} > 1. \tag{3.17}$$

Denoting

$$U = \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \quad \text{and} \quad U_R = \begin{pmatrix} u_R \\ \dot{u}_R \end{pmatrix},$$

we have by the Duhamel's principle

$$U(t) = S(t)U(0) + \int_0^t S(t-s)\,\alpha(U(s))\,ds, \tag{3.18}$$

$$U_R(t) = S(t)U_R(0) + \int_0^t S(t-s)\,\beta(U_R(s))\,ds. \tag{3.19}$$

Here $S(t)$ are the linear bounded operators generated by the linear wave equation, and

$$\alpha(U) = \begin{pmatrix} 0 \\ |u|^l \end{pmatrix}, \quad \beta(U_R) = \begin{pmatrix} 0 \\ p^{-s}|u_R|^l \end{pmatrix},$$

respectively. We let

$$C_1 = \sup_{0 \le t \le T_0} \|S(t)\|,$$

and

$$C_2 = \sup_{0 \le t \le T_0 - \delta} C_1 C(\|u_R(t)\|_{H^m}, \|u(t)\|_{H^m}),$$

and use $0 < T_1 < T_2 < \cdots < T_{k-1} < T_0 - \delta$ to subdivide the interval $[0, T_0 - \delta]$ into k subintervals, each of length at most $\frac{1}{2C_1}$. We next choose $R \gg 1$ so that

$$\varepsilon < \left(\sum_{i=1}^{k} 2^i C_1^i \right)^{-1}.$$

Subtracting equations (3.18) and (3.19) we obtain

$$\|u_R - u\|_{H^m} \leq \|U_R - U\|_{H^m \oplus H^{m-1}} \leq C_1 \int_0^t \|p^{-k}\, |u_R|^l - |u|^l\|_{H^{m-1}}\, ds,$$

which, by virtue of Proposition 3.1, leads us to

$$\sup_{0\leq s\leq T_1} \|u_R - u\|_{H^m} \leq C_2 T_1 \sup_{0\leq s\leq T_1} \|u_R - u\|_{H^m} + C_1 \varepsilon(R).$$

Since $C_2 T_1 < \frac{1}{2}$, we obtain from the last inequality

$$\sup_{0\leq s\leq T_1} \|u_R - u\|_{H^m} \leq 2C_1 \varepsilon(R),$$

and hence,

$$\|U_R(T_1) - U(T_1)\|_{H^m\oplus H^{m-1}} \leq C_2 T_1 \sup_{0\leq s\leq T_1} \|u_R - u\|_{H^m} + C_1\varepsilon(R) \leq 2C_1 \varepsilon(R). \quad (3.20)$$

Using T_1 as a starting point, we rewrite equations (3.18) and (3.19) as

$$\begin{aligned} U(t) &= S(t)U(T_1) + \int_{T_1}^t S(t-s)\, \alpha(U(s))\, ds, \\ U_R(t) &= S(t)U_R(T_1) + \int_{T_1}^t S(t-s)\, \beta(U_R(s))\, ds. \end{aligned}$$

Thus, for $T_1 \leq t \leq T_2$, we have

$$\begin{aligned} &\|u_R - u\|_{H^m} \leq \|U_R - U\|_{H^m\oplus H^{m-1}} \\ &\leq C_1 \|U_R(T_1) - U(T_1)\|_{H^m\oplus H^{m-1}} + C_2(t - T_1) \sup_{T_1\leq t\leq T_2} \|u_R - u\|_{H^m} + C_1 \varepsilon(R). \end{aligned}$$

Consequently, using inequality (3.20), we obtain

$$\sup_{T_1 \le t \le T_2} \|u_R - u\|_{H^m} \le 2C_1^2 \varepsilon(R) + C_2(T_2 - T_1) \sup_{T_1 \le t \le T_2} \|u_R - u\|_{H^m} + C_1 \varepsilon(R),$$

which implies

$$\sup_{T_1 \le t \le T_2} \|u_R - u\|_{H^m} \le \left(2C_1 + 2^2 C_1^2\right) \varepsilon(R),$$

and hence,

$$\|U_R(T_2) - U(T_2)\|_{H^m \oplus H^{m-1}} \le \left(2C_1 + 2^2 C_1^2\right) \varepsilon(R).$$

We continue in this fashion and after $k-2$ steps arrive at

$$\|U_R(T_0 - \delta) - U(T_0 - \delta)\|_{H^m \oplus H^{m-1}} \le \left(\sum_{i=1}^{k} 2^i C_1^i \right)^{-1} \varepsilon(R) \le 1.$$

Finally, we obtain

$$\|u_R(T_0 - \delta) - u(T_0 - \delta)\| \le \|U_R(T_0 - \delta) - U(T_0 - \delta)\|_{H^m \oplus H^{m-1}} \le 1,$$

which contradicts (3.17).

Thus we proved the following result for the Cauchy problem (3.12)-(3.14).

Theorem 3.3. *If the Cauchy problem (3.13)-(3.14) admits a global solution in some Sobolev space H^m, $l > m > \frac{n}{2}$, then the solution u_R of the Cauchy problem (3.12)-(3.14) exists at least on the interval $[0, T_R]$, with $T_R \longrightarrow \infty$, as $R \longrightarrow \infty$.*

Chapter 4

Quasilinear Case

In this chapter we will prove finite-time blow-up for the solutions of a certain type of quasilinear equations. We use the same notation and conventions as in the previous Chapter 3.

4.1 Finite Time Blow-Up

We consider the following Cauchy problem in Minkowski space $\mathbf{M}_0$:

$$\Box u = p^{-k}|Lu|^l, \tag{4.1}$$

$$u(0,\cdot) = f, \quad u_t(0,\cdot) = g, \tag{4.2}$$

where

$$Lu = A\left(R + \frac{t^2+r^2}{4R}\right)u_t + A\frac{tr}{2R}u_r + \left(A\frac{st}{2R} + C\right)u. \tag{4.3}$$

Here in Einstein coordinates,

$$A = \sin sT, \quad \text{and} \quad C = -s\cos sT.$$

We assume that $f \in Y$, where the space Y is defined by

$$Y := \{ f \mid f \in C_0^\infty(\mathbb{R}^n), \ f \leq 0 \} .$$

We are ready now to state our main quasilinear finite-time blow-up result.

Theorem 4.1. *Let $1 < l < \frac{2}{n} + 1$, and u be a solution of (4.1)-(4.2) with Lu given by (4.3), and the initial function $f \in Y$. Then u blows up in finite time.*

Remark. In the spirit of the remark we made on page 33, we observe that for $l < \frac{n+3}{n+1} < \frac{2}{n} + 1$, the coefficients of the nonlinearity of equation (4.1) approach 0, as $|\mathbf{x}| \longrightarrow \infty$. Thus for such powers this nonlinearity, although tempered at infinity, still produces finite-time blow-up. In contrast, Rammaha in [22] proves a sharp result for l, but uses nonlinearities that are not tempered at infinity.

Proof. Let $v = R^{\frac{2}{l-1}} p^{-s} u$. Since u satisfies (4.1), by Proposition 2.4 it follows that

$$\Box_c v = |\Lambda v|^l, \tag{4.4}$$

where, using Proposition 2.3,

$$\Lambda v = (\sin sT) v_T - (s \cos sT) v.$$

Define a function $H(T)$ by

$$H(T) = \int_{S^n} \Lambda v \, dS = \int_{S^n} \left[(\sin sT) v_T - (s \cos sT) v \right] dS$$

From the Divergence Theorem we have,

$$\int_{S^n} \Delta_{S^n} v \, dS = 0,$$

hence,

$$H'(T) = \int_{S^n} \sin sT \cdot \Box_c v \, dS.$$

Therefore, multiplying both sides of equation (4.4) by $\sin sT$ and integrating on S^n, we obtain for $T \in [0, \frac{2\pi}{n-1})$,

$$H'(T) \geq \sin sT |H(T)|^l.$$

Another integration yields

$$\frac{1}{l-1}\left(\frac{1}{H_0^{l-1}} - \frac{1}{H^{l-1}}\right) \geq \frac{1}{s}\sin^2 \frac{s}{2}T, \quad \text{where} \quad H_0 = H(0),$$

or equivalently,

$$\frac{1}{H^{l-1}} \leq \frac{1}{H_0^{l-1}} - C \sin^2 \frac{s}{2}T,$$

where C is a constant depending only upon n and l. The latter inequality implies that v blows up at most in time T_0, where T_0 is the smallest solution of

$$H_0^{-(l-1)} = C \sin^2 \frac{s}{2}T. \tag{4.5}$$

Using relation (3.10) for the volume forms, we observe that

$$H_0 = -s \int_{S^n} v(0, \cdot) \, dS = -sR^{\frac{2}{l-1}-n} \int_{\mathbf{R}^n} \left(1 + \frac{r^2}{4R^2}\right)^{-\frac{n+1}{2}} f \, dx = C' R^{\frac{2}{l-1}-n},$$

where, according to the assumption, C' is a positive constant. This and equation (4.5) imply that $T_0(R) = O(R^{-1+\frac{n(l-1)}{2}})$.

As in the proof of Theorem 3.1, we observe that the support of the initial data (4.2) when transformed by c, decreases at a rate of $\frac{1}{R}$, as the parameter R increases. This rate is slower compared with the rate of T_0; therefore we may choose $R \gg 1$ so that v blows up inside the compactified Minkowski space $c(\mathbf{M}_0)$.

Finally, it follows that the solution u of the Cauchy problem (4.1)-(4.2) blows up in finite time bounded by $t_0 \sim RT_0 = O(R^{\frac{n(l-1)}{2}})$. □

4.2 Generating More Examples of Finite-Time Blow-Up

Let us consider the equation in $\mathbf{E}$

$$\Box_c v = |\Lambda v|^l, \tag{4.6}$$

where $\Lambda v = Av_T + Bv_\rho + Cv$ is defined as in Section 2.2. We multiply both sides by $\mu^2(T, \rho)$ and integrate on S^n to obtain

$$\int_{S^n} \mu^2 \Box_c v \, dS \geq \left| \int_{S^n} \mu^{\frac{2}{l}} \Lambda v \, dS \right|^l .$$

Therefore a choice of A, B, C, and μ^2 so that

$$\frac{\partial}{\partial t} \left(\int_{S^n} \mu^{\frac{2}{l}} \Lambda v \, dS \right) \geq \int_{S^n} \mu^2 \Box_c v \, dS \tag{4.7}$$

would result in

$$H'(T) \geq |H(T)|^l ,$$

where we set

$$H(T) = \int_{S^n} \mu^{\frac{2}{l}} \Lambda v \, dS.$$

Thus a choice of A, B, C, and μ^2 satisfying condition (4.7) would imply a finite time blow-up for a solution v of equation (4.6). This, employing the transformation formulae from Proposition 2.3 for the nonlinear part Λv, will translate into a finite time blow-up for a solution u of the corresponding equation in Minkowski space.

After an integration by parts, we see that inequality (4.7) on A, B, C, and μ^2 is equivalent to

$$\begin{aligned}\int_{S^n} \Big\{ \left(\mu^{\frac{2}{l}} A\right) v_{TT} + \left(\mu^{\frac{2}{l}} B\right)_T v_\rho + \left(\mu^{\frac{2}{l}} C\right)_T v \\ + \left[\mu^{\frac{2}{l}} A_T + \mu^{\frac{2}{l}} C - (n-1)\cot\rho\,\mu^{\frac{2}{l}} B - \left(\mu^{\frac{2}{l}} B\right)_\rho\right] v_T \Big\} \, dS \\ \geq \int_{S^n} \left[\mu^2 v_{TT} + 2(\mu^2)_\rho v_\rho + \left(\Delta_{S^n}\mu^2 + s^2\mu^2\right) v\right] \, dS. \quad (4.8)\end{aligned}$$

Restricting (4.8) to an equality we obtain

$$\mu^2 = \mu^{\frac{2}{l}} A, \quad (4.9)$$

$$2(\mu^2)_\rho = \left(\mu^{\frac{2}{l}} B\right)_T, \quad (4.10)$$

$$\Delta_{S^n}\mu^2 + s^2\mu^2 = \left(\mu^{\frac{2}{l}} C\right)_T, \quad (4.11)$$

$$\mu^{\frac{2}{l}} A_T + \mu^{\frac{2}{l}} C = (n-1)(\cot\rho)\mu^{\frac{2}{l}} B + \left(\mu^{\frac{2}{l}} B\right)_\rho. \quad (4.12)$$

After a differentiation with respect to T, equation (4.9) yields

$$\mu^{\frac{2}{l}} A_T = \frac{l-1}{l}(\mu^2)_T. \quad (4.13)$$

Differentiating equation (4.12) with respect to T, and using (4.13) to substitute for

$\left(\mu^{\frac{2}{l}} A_T\right)_T$, equation (4.11) to substitute for $\left(\mu^{\frac{2}{l}} C\right)_T$, equation (4.10) to substitute for $\left(\mu^{\frac{2}{l}} B\right)_T$, and again equation (4.10), differentiated this time with respect to ρ, to substitute for the mixed derivative $\left(\mu^{\frac{2}{l}} B\right)_{T\rho}$, we arrive at

$$\frac{l-1}{l}(\mu^2)_{TT} + \Delta_{S^n}(\mu^2) + s^2\mu^2 = 2(n-1)(\cot\rho)(\mu^2)_\rho + 2(\mu^2)_{\rho\rho}.$$

This, considering the fact that Δ_{S^n} restricted to a function depending only upon ρ is given by $(n-1)\cot\rho\,\partial_\rho + \partial^2_{\rho\rho}$, leads us to the conformal wave equation

$$\frac{l-1}{l}(\mu^2)_{TT} - \Delta_{S^n}(\mu^2) + s^2\mu^2 = 0.$$

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