This is to certify that the

dissertation entitled

BOUNDEDNESS OF INTEGRAL OPERATORS
IN THE UPPER-HALF SPACE WITH
CARLESON MEASURES

presented by

Naim Saiti

has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Mathematics

William T. Sledd
Major professor

Date 08/18/97

BOUNDEDNESS OF INTEGRAL OPERATORS IN THE UPPER-HALF SPACE WITH CARLESON MEASURES

By

Naim Saiti

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1997

ABSTRACT

BOUNDEDNESS OF INTEGRAL OPERATORS IN THE UPPER-HALF SPACE WITH CARLESON MEASURES

BY

Naim Saiti

In this text we study the boundedness of the integral operator T, as a mapping between the Lebesgue spaces: $T : L^p(X, dm) \to L^p(X^+, d\mu)$ for $p > 1$, where X is a space of homogeneous type with a doubling measure m, and μ is a Carleson measure on X^+, upper-half space over X.

In Chapter 1 we study the case when the kernel of the operator T is admissible. It is known that for such a kernel the operator T is dominated by the Hörmander Maximal Function, H, pointwise. Therefore it is bounded whenever H is.

The case when the kernel of the operator T has a singularity is studied in Chapters 2 and 3. In Chapter 2 we prove that if the operator T satisfies the Calderòn-Zygmund conditions and if the trace of the operator T, the operator T_0, is bounded for some $p_0 > 1$, and if T and T_0 are related by the formula:

$$|Tf(x,t) - T_{0,t}f(x)| \leq CHf(x,t) \qquad \text{for every } (x,t) \in X^+, f \in C_0(X),$$

then $T : L^{p_0}(X, dm) \to L^{p_0}(X^+, d\mu)$ is bounded. In Chapter 3 we give sufficient conditions for the boundedness of the operator T when $p = 2$, and $X = \mathbb{R}^n$.

In Chapter 4 we have given an application of the theory of singular integral operators to the generalized Tent Spaces.

DEDICATION

To my father Isa Saiti, who has not got formal education, but has strongly believed in power of it.

ACKNOWLEDGMENTS

I wish to thank Dr. William T. Sledd for his help and guidance. Without his encouragement, mentoring, and unselfish efforts this thesis would have never been possible.

I would also like to thank the members of my committee for their involvement.

Contents

Introduction

In the course of proving his famous corona theorem, L. Carleson, (see [4]) characterized those positive finite Borel measures μ on the unit ball U in the complex plane $\mathbb{C}$ such that

$$\left(\int_U |f|^p d\mu\right)^{1/p} \leq C\|f\|_{H^p},$$

for every function f in the Hardy Space H^p ($0 < p < \infty$), showing that this holds if and only if the measure μ satisfies the property $\mu S \leq C(1-s)$, for every set S of the form

$$S = S_{s,\theta_0} = \{re^{i\theta} : s \leq r < 1, \theta_0 - \pi(1-s) \leq \theta < \theta_0 + \pi(1-s)\}.$$

Such a measure is called a Carleson measure, and such sets S Carleson sets.

In this text we will study under what conditions an integral operator T is bounded as a linear mapping between the Lebesgue spaces

$$T : L^p(X, dm) \to L^p(X^+, d\mu), \qquad p > 1. \tag{0.1}$$

over a space of homogeneous type X, supplied with a doubling measure m, where X^+ denotes upper-half space over X, and μ a Carleson measure. A space of homogeneous type, first defined by R.R. Coifman and G. Weiss (see [7]), is a generalization of Euclidean space, with the doubling measure being the Lebesgue measure. The technique of working in such a space may be quite different, due to, for example, not having dyadic cubes. This particular difficulty is overcome by using Calderón's covering lemma (see [1]).

It is a well known fact (see [11]) that the nontangential maximal function Nf and a Carleson measure μ are related by the formula

$$\mu(\{f > \lambda\}) \leq C|\{Nf > \lambda\}| \quad \text{is true for every } \lambda > 0 \text{ iff } \mu \text{ is a Carleson measure.}$$

There is also another maximal function, the Hörmander maximal function, H, that is related to a Carleson measure, μ, by

$$H : L^p(dm) \to L^p(d\mu) \text{is bounded for every } p > 1 \text{ iff } \mu \text{ is a Carleson measure.}$$

The proof of both results indicates that the distribution sets for both nontangential maximal function and the Hörmander maximal function have a similar structure. We call such sets the sets with the tent structure. Using the properties of such sets, and the L^p-boundedness of corresponding trace operators, the restrictions of the operators mentioned above to the space X, we prove the above statement about the Hörmander maximal function, and we give an interesting generalization of it.

The significance of the Hörmander maximal function is that it estimates integral operators with so-called admissible kernels pointwise (see [12]). Therefore, such integral operators are L^p-bounded whenever the Hörmander maximal function is. An important example of a convolution operator with admissible kernel is the Poisson transform on $\mathbb{R}^n$.

The last two sections of Chapter 1 are dedicated to the vector-valued inequalities for the Hörmander maximal function, of the type studied in [9]. We use the vector-valued inequality proved by C. Fefferman and E. M. Stein (see [9]), and the approach developed in Chapter 1 to give a more elegant proof of the result proved by F. J. Ruiz and J. L. Torrea (see [18]), that, essentially, is a generalization of the Fefferman-Stein result. At the end of the chapter, we give some applications of the vector-valued theory in spaces of homogeneous type to the weighted vector-valued inequalities in

$\mathbb{R}^n$. In that way we obtain the boundedness of the vector-valued convolution operators with admissible kernels.

In Chapter 2, we study integral operators whose kernels may not be absolutely integrable. The boundedness of such operators as mappings $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, was first studied by A. P. Calderón and A. Zygmund (see [2] and [3]), and for that reason, they are called the *Calderón- Zygmund operators.* In this text we will use the name *singular integral operator*, and study under what conditions such an operator, T, is bounded as a linear mapping between the Lebesgue spaces in the formula (0.1) for every $p > 1$. An important example of a singular integral operator in the upper-half space is the operator that assigns the complex-conjugate $\tilde{u}$ of the harmonic function $u = P_t * f$, to the function f on the real line.

Singular integral operators whose range consists of functions with the domain in the upper-half space were studied by F. J. Ruiz and J.L. Torrea (see [18], [19], [20], [21], [22]). We start Chapter 2 by stating their main result. Using the good-λ approach used by R.R. Coifman and C. Fefferman in [5], we have obtained a stronger conclusion, but used more assumptions, most of them are about the restriction of $Tf(x,t)$ to the hyperplane X.

At the end of the chapter we give several examples of such singular integral operators in Euclidean space, as well as the examples of convolution operators with the Cauchy kernel on the unit ball in $\mathbb{C}^n$, and the Cauchy-Szegö kernel on the Heisenberg Group.

In Chapter 3 we study the boundedness of the operator

$$T : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^{n+1}_+, d\mu)$$

when μ is a Carleson measure on $\mathbb{R}^{n+1}_+$. The whole chapter is based on the ideas and techniques developed in [10]. We use the decomposition of an L^2-function into the sum of smooth atoms (by the virtue of the Calderón formula), and we will prove that if a singular integral operator satisfies certain cancelation properties (that play the role of $T1$ and T^*1 conditions), it will map a smooth atom into an equivalent of a smooth molecule in $L^2(\mathbb{R}^{n+1}_+, d\mu)$, which implies the L^2-boundedness. In the last chapter, Chapter 4, we use the theory of singular integral operators to make a conclusion about the boundedness of the linear mapping

$$T : L^p(\mathbb{R}^n, dx) \to T^p_q(d\mu),$$

for every $1 < p, q < \infty$, when μ is a Carleson measure on $\mathbb{R}^{n+1}_+$. The spaces $T^p_q(d\mu)$ are generalized version of the *tent spaces* defined by Coifman, Meyer and Stein (see [6]). The main technical tool in this chapter is Theorem 4.1 proved by F. J. Ruiz and J. L. Torrea (see [21]), which makes it possible to apply the vector-valued theory of singular integral operators to the new situation.

Throughout the whole text we will use the notion of *accumulative constant*, which means that C will represent a constant, not necessarily the same in each two consecutive appearances.

Chapter 1

Boundedness of Integral Operators with Admissible Kernels

In this chapter we study the L^p-boundedness of a so-called non-singular integral operator, or an integral operator with admissible kernel in the upper-half space supplied with a Carleson measure. Such an operator is dominated by the Hörmander maximal function pointwise, and therefore is bounded whenever the Hörmander maximal function is. We begin by introducing spaces of homogeneous type.

1.1 Definitions and Covering Lemmas

A space of homogeneous type was first defined by R. R. Coifman and G. Weiss (see [7]). First, we define *pseudometric*.

Definition: *Let X be a set. A map $\rho : X \times X \to [0, \infty)$ is called a pseudometric if and only if it has the following properties:*

(i) $\rho(x, y) > 0$ if and only if $x \neq y$.

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

(iii) There exists a constant $k_\rho \geq 1$ such that for all $x, y, z \in X$ we have

$$\rho(x, z) \leq k_\rho(\rho(x, y) + \rho(y, z)).$$

In case $k_\rho = 1$, a pseudometric ρ is a metric and will be denoted by $d(x, y)$.

The *quasi-ball* $B(x, r)$, with the center at the point x, and radius $r = r(B)$, is the set $B(x, r) = \{y \in X : \rho(x, y) < r\}$.

Definition: *A space of homogeneous type is a topological space X endowed with a pseudometric ρ such that:*

(a) The family $\{B(x, r) : x \in X, r > 0\}$ is a basis for the topology on X.

(b) There exists a Borel measure m on X which is a doubling measure, i. e. there exist a constant $C > 0$ so that for every $x \in X$, and $r > 0$

$$m(B(x, 2r)) \leq Cm(B(x, r)).$$

Remark: It has been proved (see [7]), that properties (a) and (b) imply the following:

There exists a number $N \in \mathbb{N}$ such that for any $x \in X$, and for each $r > 0$, the quasi-ball $B(x, r)$ contains at most N points x_i with $\rho(x_i, x_j) > r/2$.

In this text we will use the notation: $|B| = m(B)$.

Examples: Obviously every finite dimensional metric space, supplied with any doubling measure, is a space of homogeneous type. A more interesting example is:

$$X = \mathbb{R}^n \qquad \rho(x, y) = \sum_{i=1}^{n} |x - y|^{p_i} \qquad p_i > 1.$$

In Chapter 3 we will give two interesting examples of spaces of homogeneous type: the unit ball in $\mathbb{C}^n$ with the non-isotopic metric ρ, and the Heisenberg Group on $\mathbb{R} \times \mathbb{C}^{n-1}$.

As we can see, there are many similarities between Euclidean space and the spaces of homogeneous type. One of the differences is that on a space of homogeneous type we may not have features like dyadic cubes. Consequently, tools like the Whitney decomposition theorem, or the Calderón-Zygmund Lemmas cannot be used. The covering lemma, which follows is due to Calderón, (see [1]), is the main tool in bridging this difficulty. We use the following version of Calderón's lemma.

Lemma 1.1 *Let E be a subset of X and $\{B(x, r(x))\}_{x\in E}$ a covering of E, such that the radii $\{r(x)\}_{x\in E}$ are uniformly bounded. Then there exist a (possibly finite) sequence of disjoint quasi-balls $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ such that for every index i $x_i \in E$, and*

$$E \subset \cup_i B(x_i, Kr(x_i)),$$

where constant K depends only on the space X. Moreover, for every quasi-ball B from the covering there exists an index i so that $B \subset B(x_i, Kr(x_i))$.

The constant K from the previous lemma is called the space constant. We will use the notation $B^* = KB$, where cB denotes the quasi-ball concentric with B so that $r(cB) = cr(B)$.

We conclude this section by listing the definitions of some of basic maximal functions and Carleson measure.

The upper-half space X^+, over a space of homogeneous type X, is defined by

$$X^+ = \{(x,t) : x \in X, t \geq 0\}.$$

The *tent* over the point $(x,t) \in X^+$ is the set

$$T(x,t) = \{(y,s) \in X^+ : \rho(y,x) + s \leq t\},$$

and the *tent* over a quasi-ball $B \subset X$, centered at the point $x \in X$, and with the radius r, is $T(B(x,r)) = T(x,r)$. A *Carleson square* over the quasi-ball B in X, is the set

$$S(B) = \{(x,t) \in X^+ : x \in B, 0 \leq t < r\}.$$

A measure μ is called a *Carleson measure* on X^+ if and only if there exist a constant $C_\mu > 0$ so that for every quasi-ball B in X we have

$$\mu(S(B)) \leq C_\mu |B|.$$

Notice that this implies that the measure μ is a Carleson measure if and only if $\mu(T(B)) \leq C_\mu |B|$, for every quasi-ball B in X.

An important example of a Carleson measure on X^+ is the measure defined by

$$\mu(E) = |E \cap X| \qquad \text{for every} \quad E \subset X^+,$$

called the *projection measure* on X, which tells us that the measure m can be viewed as a Carleson measure.

For $f : X \to \mathbb{C}$, a locally integrable function with respect to the measure m, we define the *Hardy-Littlewood maximal function* by

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f| dm \qquad x \in X,$$

where the supremum is taken over all quasi-balls $B \ni x$; and the *Hörmander maximal function* by:

$$Hf(x,t) = \sup \frac{1}{|B|} \int_B |f| dm \qquad t \geq 0,$$

now, the supremum is taken over all quasi-balls $B \ni x$, such that the radius $r(B) \geq t$. For a Borel measure β in the upper-half space X^+ we define the *generalized* Hörmander maximal function by

$$H_\beta F(x,t) = \sup_{B:T(B)\ni(x,t)} \frac{1}{\beta(T(B))} \int_{T(B)} |F| d\beta,$$

where we define $\frac{1}{\beta(T(B))} \int_{T(B)} |F| d\beta = 0$ when $\beta(T(B)) = 0$. In the special case, when the measure β is the projection measure on X, H_β becomes the ordinary Hörmander maximal function.

In Euclidean space $\mathbb{R}^n$, with Euclidean distance, and m being ordinary Lebesgue measure, it is more convenient to define the Hörmander maximal function as

$$Hf(x,t) = \sup \frac{1}{|Q|} \int_Q |f| dx \qquad t \geq 0,$$

where the supreme is taken over all cubes $Q \ni x$, such that $\ell(Q) \geq t$ (where $\ell(Q)$ represents the side-length of the cube Q). The Hörmander maximal function defined as above is equivalent to the Hörmander maximal function defined as a supreme over the balls in $\mathbb{R}^n$, since Lebesgue measure is a doubling measure.

We conclude this section by giving the definition of a nontangential maximal function in the upper-half space X^+. For given $\alpha > 0$, we define the upward-pointing cone with the vertex at the point (x,t) and the aperture α, by

$$\Gamma_\alpha(x,t) = \{(y,s) \in X^+ : \rho(x,y) < \alpha(s-t)\}.$$

Let $F : X^+ \to \mathbb{C}$ be a measurable function, and $\alpha > 0$. Then the function

$$N_\alpha F(x,t) = \sup_{(y,s)\in\Gamma_\alpha(x,t)} |F(y,s)|,$$

represents the nontangential maximal function of the function F. The function $N_\alpha F(x) = N_\alpha F(x,0)$ is the traditional nontangential maximal function, used in most texts in harmonic analysis .

1.2 Sets with the Tent Structure

We will see that the distribution sets for each of the maximal functions we have introduced so far have a similar structure. We will say such sets have the tent structure. Once we show that the distribution set of a certain subadditive operator in the upper-half space X^+ has the tent structure, we can easily obtain the boundedness of such an operator, as a mapping

$$L^p(X, dm) \to L^p(X^+, d\mu)$$

when μ is a Carleson measure, provided that the restriction of this function on X is a bounded operator.

Definition: *A set $E \subset X^+$ has the tent structure if and only if*

$$(x,t) \in E \qquad \textit{implies that} \qquad T(x,t) \subset E.$$

The following theorem provides us with the most important property of the sets with tent structure on spaces of homogeneous type.

Theorem 1.2 *Let Borel set $E \subset X^+$ be a set that has the tent structure, and let μ and ν be Borel measures on X^+, such that*

$$\mu(T(B^*)) \leq \nu(T(B)),$$

for every quasi-ball B in X. Then

$$\mu(E) \leq \nu(E).$$

Proof: For fixed $N > 0$, let $E^N = E \cap \{(x,t) \in X^+ : t \leq N\}$. If $(x,t) \in E^N$, then $T(x,t) \subset E^N$, and consequently $B(x,t) \subset \pi(E^N)$, where $\pi(E^N)$ denotes the projection of the set E^N on X. If we do that for every point $(x,t) \in E^N$, we have obtained a covering of the set $\pi(E^N) \subset X$ by a family of quasi-balls, $\{B(x,t)\}_{(x,t)\in E^N}$, whose radii are uniformly bounded. Applying the Calderón covering lemma, we obtain a sequence of pair-wise disjoint quasi-balls, $\{B_i\}$, such that for every quasi-ball B from the original covering there exists an index i so that $B \subset B_i^*$. Thus, we conclude that $E^N \subset \cup_i T(B_i^*)$. To finish proving the theorem we proceed as follows:

$$\mu(E^N) \leq \mu(\cup_i T(B_i^*)) \leq \sum_i \mu(T(B_i^*)).$$

The assumption of the theorem, the facts that $\{B_i\}$ are disjoint sets, and that $T(B_i) \subset E^N$, imply

$$\leq \sum_i \nu(T(B_i)) \leq \nu(E^N).$$

To complete the proof of the theorem, we let $N \to \infty$.

■

Let us consider the set

$$E_\lambda = \{H_\beta F > \lambda\}.$$

Let $(x,t) \in X^+$ and $(y,s) \in T(x,t)$. If B is such a quasi-ball that $T(B) \ni (x,t)$, then clearly $T(B) \ni (y,s)$. Thus, we have established:

$$(y,s) \in T(x,t) \qquad \text{implies} \qquad H_\beta F(x,t) \le H_\beta F(y,s) \tag{1.1}$$

for every locally integrable function $F : X^+ \to \mathbb{C}$. This fact implies that if $H_\beta F(x,t) > \lambda$, then $H_\beta F(y,s) > \lambda$, for every $(y,s) \in T(x,t)$. Consequently, E_λ is a set with the tent structure.

The set

$$F_\lambda = \{N_1 F > \lambda\}$$

also has the tent structure, which is due to the fact

$$(y,s) \in T(x,t) \qquad \text{if and only if} \qquad (x,t) \in \Gamma_1(y,s).$$

which implies that if $(x,t) \in F_\lambda$ and $(y,s) \in T(x,t)$, then $\Gamma_1(x,t) \subset \Gamma_1(y,s)$ which produces the statement

$$(y,s) \in T(x,t) \qquad \text{implies} \qquad N_1 F(x,t) \le N_1 F(y,s), \tag{1.2}$$

for every locally integrable function F.

The above facts imply the following corollary of Theorem 1.2.

Corollary 1.3 *If μ and ν are two Borel measures on X^+, such that*

$$\mu(T(B^*)) \le \nu(T(B)),$$

for every quasi-ball B in X, then there exists a constant $C > 0$ so that

$$\mu(\{H_\beta F > \lambda\}) \leq \nu(\{H_\beta F > \lambda\}),$$

and

$$\mu(\{N_1 F > \lambda\}) \leq C\nu(\{N_1 F > \lambda\}),$$

for every $\lambda > 0$, and locally integrable function F on X^+.

Corollary 1.3 implies the boundedness of the Hörmander maximal function as a mapping:

$$H : L^p(X, dm) \to L^p(X^+, d\mu)$$

for every $p > 1$, when μ is a Carleson measure. We can show it in the following way. Let the measure β be the projection measure. Then the generalized Hörmander maximal function becomes the ordinary Hörmander maximal function. In the case when μ is a Carleson measure, and ν the projection measure on X, Corollary 1.3 yields

$$\mu(\{Hf > \lambda\}) \leq C|\{Mf > c\lambda\}|,$$

because the restriction of the Hörmander maximal function on X is the Hardy-Littlewood maximal function. Using the fact that the Hardy-Littlewood maximal function is bounded as a mapping $M : L^p(X, dm) \to L^p(X, dm)$ for every $p > 1$, and a standard argument, we conclude that there exists a constant $C_p > 0$ (that depends only on p) such that

$$\|Hf\|_{L^p(X^+, d\mu)} \leq C_p \|f\|_{L^p(X, dx)} \qquad \text{for every} \quad f \in L^p(X, dx),$$

is true for every $p > 1$. For $p = 1$, H is a weak-type 1-1 bounded operator; in other words, there exists a positive constant C so that for every $\lambda > 0$ and $f \in L^1(X, dx)$

$$\mu(\{Hf > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(X, dx)}.$$

A function $F : X^+ \to \mathbb{R}^+$ is said to be of *horizontal bounded ratio* if and only if there exists a positive constant A_F such that

$$F(x,t) \leq A_F F(y,t) \qquad \text{whenever} \quad \rho(x,y) < t.$$

Lemma 1.4 *If β is a doubling measure on X^+, then $H_\beta F$ is of horizontal bounded ratio.*

Proof: Let us fix $t > 0$, and let $\rho(x,y) < t$, and let B be a quasi-ball containing the point x, so that $r(B) \geq t$. Then $B^* \ni y$, and also $r(B^*) \geq t$, which means $(y,t) \in T(B^*)$. Thus

$$\begin{aligned} \frac{1}{\beta(T(B))} \int_{T(B)} |F| d\beta &\leq \frac{\beta(T(B^*))}{\beta(T(B))} \frac{1}{\beta(T(B^*))} \int_{T(B^*)} |F| d\beta \\ &\leq C_\beta H_\beta F(y,t). \end{aligned}$$

If we take the supreme of the left side of the above inequality over all quasi-balls B containing the point x, so that $r(B) \geq t$, we obtain

$$H_\beta F(x,t) \leq C_\beta H_\beta F(y,t),$$

which proves the lemma.

■

We define the *vertical maximal function* for a function $F : X^+ \to \mathbb{R}^+$ as

$$F^*(x,t) = \sup_{s \geq t} F(x,s).$$

Obviously, for every $(x,t) \in X^+$ and $\alpha > 0$ we have $F^*(x,t) \leq N_\alpha F(x,t)$. If the function F is of horizontal bounded ratio, then the converse is also true; in other words we have that $N_1 F(x,t) \leq A_F F^*(x,t)$, for every $(x,t) \in X^+$.

The following lemma shows that there is an interesting relationship between the Hörmander maximal function and the nontangential maximal function. Before we state the lemma we introduce the notation $A_\infty F(x) = N_1 F(x,0)$.

Lemma 1.5 *If $p > 1$, then there exists a positive constant C_p (that depends only on p) such that for every $f \in L^p(X, dm)$*

$$||A_\infty(Hf)||_{L^p(X,dm)} \leq C_p||f||_{L^p(X,dm)}.$$

Proof: If β is the projection measure on X, then β is a doubling measure and the corresponding Hörmander maximal function is of horizontal bounded ratio. Hence

$$A_\infty(Hf)(x) \leq C(Hf)^*(x, 0) = CMf(x).$$

Now, we use the fact that for any $p > 1$, the Hardy-Littlewood maximal function is L^p bounded, to complete the proof of the lemma.

■

1.3 Admissible Kernels

Admissible kernels were first defined by W. T. Sledd and S. Gadbois (see [12]). The main property of integral operator whose kernel is an admissible function is that it is dominated by the Hörmander maximal function. Usually, such kernels are absolutely integrable; so we do not call such operators singular. The most important example of such an operator is the Poisson transform on $\mathbb{R}^n$.

First, we define constants

$$C_k = \sup_{(x,t)\in X^+} \frac{|B(x, 2^{k+1}t)|}{|B(x,t)|}, \qquad k \geq 0.$$

Definition: *A function $\Phi : \mathbb{R}^+ \to [0,1]$ is an admissible function if and only if it satisfies all of the following properties:*

(a) $\Phi(0) = 1$ and $\Phi(1) > 0$,

(b) Φ is monotone decreasing, and

(c)

$$\sum_{k=0}^{\infty} C_k \Phi(2^k) < \infty.$$

We define the kernel K by

$$K(x,y,t) = \frac{\Phi(\frac{\rho(x,y)}{t})}{\int_X \Phi(\frac{\rho(x,z)}{t}) dm(z)},$$

where Φ is an admissible function, and the corresponding integral operator

$$\mathcal{K}f(x,t) = \int_X K(x,y,t) f(y) dm(y).$$

The following lemma, from [12], solves the problem of the L^p-boundedness of any integral operator with admissible kernel. Essentially, the lemma tells us that such an operator is bounded whenever the Hörmander maximal function is bounded.

Lemma 1.6 *There exist a positive constant C so that*

$$|\mathcal{K}f(x,t)| \leq CHf(x,t) \qquad \textit{for every} \quad (x,t) \in X^+,$$

and for every continuous function f with compact support in X.

Example: Poisson kernel. Let $X = \mathbb{R}^n$, $\rho(x,y) = |x-y|$, and let m be Lebesgue measure on $\mathbb{R}^n$. In this case $C_k = 2^{(k+1)n}$. Let

$$\Phi(s) = \frac{c_n}{(1+s^2)^{\frac{n+1}{2}}}.$$

Then Φ is an admissible function because $\sum_k \frac{2^k}{(1+2^{2k})^{\frac{n+1}{2}}} < \infty$. The resulting admissible kernel K is the Poisson kernel on $\mathbb{R}^n$

$$K(x,y,t) = P_t(x,y) = \frac{c_n t}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}}.$$

Lemma 1.6 yields the following pointwise estimate for the Poisson transform of a function f.

$$|(P_t * f)(x)| \leq CHf(x,t) \quad \text{for almost every} \quad (x,t) \in (\mathbb{R}^n)^+ = \mathbb{R}^{n+1}_+. \tag{1.3}$$

As a consequence of Lemma 1.5, and Lemma 1.6 we have that for any admissible kernel K there exists a positive constant C_p, that depends only on $p > 1$, so that for every $f \in L^p(X, dm)$

$$||A_\infty(K * f)||_{L^p(X,dm)} \leq C_p||f||_{L^p(X,dm)}.$$

1.4 Vector-Valued Inequalities

The L^p-boundedness of the Hardy-Littlewood maximal function is a well-known fact, established in the first half of this century. However, its vector-valued version, the inequality of type

$$\left\{\int (\sum_k (Mf_k(x))^q)^{\frac{p}{q}} dx\right\}^{\frac{1}{p}} \leq C \left\{\int (\sum_k |f_k(x)|^q)^{\frac{p}{q}} dx\right\}^{\frac{1}{p}}, \tag{1.4}$$

($p, q > 1,$) was established only in 1972 by C. Fefferman and M. Stein (see [9]).

The objective of this section is to prove the vector-valued version of the L^p-boundedness theorem for the Hörmander maximal function in the settings of the space of homogeneous type when the measure in the upper-half space X^+ is a Carleson measure. As a consequence of the theorem we conclude that the vector-valued integral operator with admissible kernels

$$\mathcal{K} : L^p_{\ell^q}(X, dm) \to L^p_{\ell^q}(X^+, d\mu),$$

is bounded for every $1 < p, q < \infty$.

We begin by stating the main result, that was proved by F. J. Ruiz and J. L. Torrea (see [18]).

Theorem 1.7 *Let μ be a Carleson measure on X^+, and $1 < p, q < \infty$. Then there exists a constant $C > 0$ so that for every vector-function $f(x) = (f_1(x), f_2(x), ...)$*

such that $\{\int(\sum_k |f_k|^q)^{\frac{p}{q}} dm\}^{\frac{1}{p}} < \infty$,

$$\left\{\int(\sum_k (Hf_k)^q)^{\frac{p}{q}} d\mu\right\}^{\frac{1}{p}} \le C\left\{\int(\sum_k |f_k|^q)^{\frac{p}{q}} dm\right\}^{\frac{1}{p}}. \tag{1.5}$$

In the course of proving this result F. J. Ruiz and J. L. Torrea considered three different cases: $p = q$, $p < q$, and $p > q$. The case $p = q$ was trivial, and in the case $p < q$ F. J. Ruiz and J. L. Torrea adapted the argument used by M. Stein and C. Fefferman (see [9]) to the new situation successfully. The key ingredient in the proof of the theorem in the case $p > q$ is the following maximal operator. For $\varphi \in L^p(X^+, d\mu)$, we set

$$\varphi^*(x) = \sup \frac{1}{|B|}\int_{S(B)} |\varphi(x,t)| d\mu(x,t),$$

where the supreme is taken over all quasi-balls B in X containing x.

We will show that the same result can be proved by using formula (1.4) and the approach developed in the preceding sections of this chapter. Theorem 1.7 follows from the following theorem.

Theorem 1.8 *Let μ and ν be Borel measures on the upper-half space X^+, so that for every quasi-ball B in X we have*

$$\mu(T(B^*)) \le C\nu(T(B)).$$

Then, for every $p, q \ge 1$ there exists a positive constant C_{pq} so that for every vector-valued function $(F_1(x,t), F_2(x,t), \ldots.)$ the following two inequalities hold:

$$\|(\sum_k (H_\beta F_k)^q)^{\frac{1}{q}}\|_{L^p(X^+,d\mu)} \le C_{pq}\|(\sum_k (H_\beta F_k)^q)^{\frac{1}{q}}\|_{L^p(X^+,d\nu)} \tag{1.6}$$

and

$$\|(\sum_k (N_1 F_k)^q)^{\frac{1}{q}}\|_{L^p(X^+,d\mu)} \le C_{pq}\|(\sum_k (N_1 F_k)^q)^{\frac{1}{q}}\|_{L^p(X^+,d\nu)}. \tag{1.7}$$

Proof: We need to prove that each of the sets

$$\{\sum_k (H_\beta F_k)^q > \lambda^q\} \qquad \text{and} \qquad \{\sum_k (N_1 F_k)^q > \lambda^q\}$$

is a set with the tent structure, which is provided by the virtue of the formulas (1.1) and (1.2). Hence, Theorem 1.8 follows from the Theorem 1.2.

■

Now, by considering a special case of the previous theorem and by using formula (1.4), we obtain the result proved by F. J. Ruiz and J. L. Torrea in the following way.

Let the measure β be the projection measure. Then the operator H_β becomes the ordinary Hörmander maximal function. If μ is a Carleson measure and ν the projection measure on X, having in mind that $Hf(x,0) = Mf(x)$, from Theorem 1.8 we deduce

$$\|(\sum_k (Hf_k)^q)^{\frac{1}{q}}\|_{L^p(X^+,d\mu)} \leq C_{pq} \| \sum_k (Mf_k)^q)^{\frac{1}{q}}\|_{L^p(X,dx)}.$$

Applying the Fefferman-Stein's result and formula (1.4) to the last inequality, we obtain the statement of Theorem 1.7.

1.5 Weighted Vector-Valued Inequalities in $\mathbb{R}^n$.

In this section we apply the vector-valued theory in a space of homogeneous type to Euclidean spaces.

A non-negative function w on $\mathbb{R}^n$ is an A_1-*weight* if and only if there exists a constant $C > 0$ so that

$$\frac{1}{|Q|}\int_Q w(x)dx \leq C \inf_{x\in Q} w(x),$$

for every cube Q in $\mathbb{R}^n$. The last condition can also be interpreted as $Mw(x) \leq Cw(x)$ for almost every $x \in \mathbb{R}^n$.

We define the *weighted Hörmander maximal function* on $\mathbb{R}^n$ as

$$H_w f(x,t) = \sup \frac{1}{w(Q)} \int_Q |f(x)| w(x) dx \qquad t \geq 0,$$

where the supreme is taken over all cubes $Q \ni x$, $\ell(Q) \geq t$ and $w(Q) = \int_Q w(x)dx$. (If $w(Q) = 0$, then we set $\frac{1}{w(Q)} \int_Q |f(x)|w(x)dx = 0$.)

The following lemma provides two important properties of A_1 weights.

Lemma 1.9 *If $w \in A_1$, then there exists a constant $C > 0$ so that for every function $f \in L^1(\mathbb{R}^n, dw) \cap L^1(\mathbb{R}^n, dx)$, and every $(x,t) \in \mathbb{R}^{n+1}_+$ we have*

$$Hf(x,t) \leq C H_w f(x,t),$$

and (the doubling condition)

$$w(Q^*) \leq C w(Q),$$

where Q^* represents the cube concentric to Q, but $\ell(Q^*) = 3\ell(Q)$.

For the proof see [24].

A positive measure μ on $\mathbb{R}^{n+1}_+$ belongs to the *class* $C_1(w)$ if and only if

$$\sup_{x \in Q} \frac{\mu(S(Q))}{|Q|} \leq C w(x) \qquad \text{for almost every} \qquad x \in \mathbb{R}^n,$$

for every cube Q in $\mathbb{R}^n$.

Notice that the last condition implies that the measure μ is a Carleson measure over the space of homogeneous type endowed with the measure w, which is a doubling measure when $w \in A_1$, because

$$\frac{\mu(S(Q))}{w(Q)} = \frac{\mu(S(Q))}{|Q|} \frac{|Q|}{w(Q)} \leq C_\mu \{\inf_{x \in Q} w(x)\} |Q| \frac{1}{w(Q)} \leq C,$$

due to the fact that $|Q| \inf_Q w \leq \int_Q w$.

Let $w \in A_1$, and $\mu \in C_1(w)$. On the space of homogeneous type $(\mathbb{R}^n, w)$, the corresponding Hörmander maximal function is the weighted Hörmander maximal function H_w. We apply Theorem 1.7 to this situation to obtain the inequality

$$\left\{\int (\sum_k H_w f_k(x,t)^q)^{\frac{p}{q}} d\mu(x,t)\right\}^{\frac{1}{p}} \leq C\left\{\int (\sum_k |f_k(x)|^q)^{\frac{p}{q}} w(x)dx\right\}^{\frac{1}{p}},$$

which is true for every vector function $f(x) = (f_1(x), f_2(x), ...)$ such that

$$\left\{\int (\sum_k |f_k(x)|^q)^{\frac{p}{q}} w(x)dx\right\}^{\frac{1}{p}} < \infty.$$

Applying Lemma 1.9 to the last formula we obtain the following corollary.

Corollary 1.10 *If $w \in A_1$ and $\mu \in C_1(w)$, then for every vector-valued function $f(x)$ we have*

$$\left\{\int (\sum_k H f_k(x,t)^q)^{\frac{p}{q}} d\mu(x,t)\right\}^{\frac{1}{p}} \leq C\left\{\int (\sum_k |f_k(x)|^q)^{\frac{p}{q}} w(x)dx\right\}^{\frac{1}{p}},$$

when $p, q > 1$.

Chapter 2

Singular Integral Operators

In this chapter we study integral operators whose kernels may not be absolutely integrable. We will call such operators *singular integral operators* or *Calderón-Zygmund operators.* The most important example of a singular integral operator in the upper-half space is the harmonic conjugate operator in the upper-half complex plane. Precisely, we consider a real-valued continuous function f that has compact support in the real line, and define the harmonic function $u(x,t) = P_t * f(x)$, the Poisson transform of f. The harmonic conjugate of u, the function $\tilde{u}$, is given by

$$\tilde{u}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2+t^2} f(y)dy,$$

taken in the principal value sense. (The integral above does not have a singularity when $t > 0$ and at ∞ since the function f is compactly supported. Thus, $\tilde{u}(x,t) = \frac{1}{\pi}\lim_{r\to 0}\int_{|x-y|>r} \frac{x-y}{(x-y)^2+t^2} f(y)dy$.) This gives us the idea to consider the operator T that assigns the function $\tilde{u}$ to function f, i. e. $Tf = \tilde{u}$. Since we are going to study the operator T in a more general context of the space of homogeneous type, the harmonicity of the function Tf will not play any role.

The main result of this chapter concerns a singular integral operator in the upper half space whose trace in X is a singular integral operator.

2.1 The Ruiz-Torrea result

The following result, due to F. J. Ruiz and J. L. Torrea (see [18]), is proved in a much more general context than we need right now. In order to state the vector-valued version of this theorem, we need to introduce the following notation.

If E and F are Banach spaces, then

$$L^p_F(X^+, d\mu) = \{f : X^+ \to F; \quad f \text{ is Borel measurable and } \int_{X^+} \|f\|^p_F d\mu < \infty\},$$

$$L^p_E(X, dx) = \{f : X \to E; \quad f \text{ is Borel measurable and } \int_X \|f\|^p_E dm < \infty\},$$

and let $L(E, F)$ denote the set of all bounded linear operators from E to F. Let the kernel K be a continuous map

$$K : X \times X \times [0, \infty) \setminus \{x = y, t = 0\} \to L(E, F)$$

such that there exist positive constants C and ϵ so that

$$\|K(x, y, t) - K(x, y', t)\|_{L(E,F)} \leq \frac{C\rho(y, y')^\epsilon}{\rho(x, y)^\epsilon |B(x, \rho(x, y) + t)|},$$

whenever $\rho(x, y) + t > 2\rho(y, y')$.

Let f be a continuous function with compact support in X, let B be a quasi-ball that contains the support of f, and $x \in X$ a point such that $x \in X \setminus 2B$. Then, for $t \geq 0$ we set

$$Tf(x, t) = \int K(x, y, t) f(y) dm(y), \tag{2.1}$$

where the integral symbol represents the vector integral. Notice that $f(y) \in E$ is a vector, and $K(x, y, t)$ is a linear mapping $E \to F$. Therefore the expression $K(x, y, t)f(y) \in F$ represents a vector. (Recall, (see [16]), that for a vector function $g : A \subset \mathbb{C} \to F$ over a Banach space F, we say that $\int_A g dm = v$, where $v \in F$ is a vector, if for every linear functional $\Lambda : F \to \mathbb{C}$ we have $\int_A \Lambda(g) dm = \Lambda(v)$.)

Now, we state the theorem

Theorem 2.1 *Let μ be a Carleson measure on X^+, and let T be a vector-valued singular integral operator that satisfies the assumptions above, and that has a continuous extension to $L_E^{p_0}(X, dm)$, i. e.*

$$T : L_E^{p_0}(X, dm) \to L_F^{p_0}(X^+, d\mu)$$

is bounded for some $p_0 > 1$. Then:

T is a weak type 1-1 bounded operator; namely there exist a constant $C > 0$, so that for every $\lambda > 0$

$$\mu\{(x,t) \in X^+ : \|Tf(x,t)\|_F > \lambda\} \leq \frac{C}{\lambda}\|f\|_{L_E^1(X,dm)},$$

for every $f \in L_E^1(X, dm)$, and T is an L^p-bounded operator for every $p \in (1, p_0]$, in other words there exists a constant $C_p > 0$ (that depends only on p) so that

$$\|Tf\|_{L_F^p(X^+,d\mu)} \leq C_p\|f\|_{L_E^p(X,dm)},$$

for every $f \in L_E^p(X, dm)$.

In this chapter we will use the scalar case of Theorem 2.1. The vector case of the theorem will be used in Chapter 4.

2.2 The operator T_0

In the special case when the measure μ is the projection measure on X, we denote the corresponding singular integral operator by T_0. In other words we set $T_0f(x) = Tf(x, 0)$. In the next section we are going to deal with the L^p-boundedness of the operator T, which will depend heavily on the properties of its trace, the operator T_0. In this section we are going to provide some information about the operator T_0.

In this chapter we restrict ourselves to the operators T, so that the restriction of $Tf(x, t)$ to X, the expression $T_0f(x)$, is an operator represented by the formula (2.1) for every $x \in X$, not only for those $x \in X \setminus 2B$, where $B \subset X$ is the quasi-ball that contains the support of the function f. For $x \in 2B$ we have to explain the meaning of the representation formula (2.1).

In this chapter we will assume that the kernel K is a continuous mapping

$$K : X \times X \times [0, \infty) \setminus \{(x = y, t = 0\} \to \mathbb{C},$$

that satisfies the following two conditions. There exist positive constants C and ϵ, so that for every $x, x', y \in X (x \neq y)$, and $t \geq 0$ we have

$$|K(x, y, t)| \leq \frac{C}{|B(x, \rho(x, y) + t)|}, \tag{2.2}$$

and

$$|K(x, y, t) - K(x', y, t)| + |K(y, x, t) - K(y, x', t)| \leq \frac{C\rho(x, x')^\epsilon}{\rho(x, y)^\epsilon |B(x, \rho(x, y) + t)|}, \tag{2.3}$$

whenever $\rho(x, y) > 2\rho(x, x') + t$.

The condition (2.2), imposed on the kernel K, is needed for the L^2-boundedness of corresponding singular integral operator on Euclidean space $\mathbb{R}^n$ (see [23]). However F. J. Ruiz and J. L. Torrea did not use this condition when proving their theorem. An integral operator whose kernel K satisfies conditions (2.2) and (2.3) is called a Calderón-Zygmund, or singular integral operator. Condition (2.2) also insures that the integral in the following definition makes sense

Let K be the function described as above. For any $s > 0$ we set

$$T_{0,s} f(x) = \int_{\rho(x,y)>s} K(x, y, 0) f(y) dm(y),$$

when f is a continuous function with compact support in X.

We define the *maximal singular integral operator* in the hyperplane, the operator $T_0^{\#}$, by the formula

$$T_0^{\#} f(x) = \sup_{s>0} |T_{0,s} f(x)| \qquad x \in X.$$

If the operator

$$T_0^{\#} : L^{p_0}(X, dm) \to L^{p_0}(X, dm)$$

is a bounded operator for some $p_0 > 1$, the operator T_0 can be defined by

$$T_0 f = \lim_{s \to 0} T_{0,s} f,$$

where the limit is taken in the $L^{p_0}(X, dm)$-sense (see [23]).

The same assumption insures the existence of the adjoint operator T_0^*, defined by

$$T_0^* f = \lim_{s \to 0} T_{0,s}^* f,$$

(where the limit is also taken in the $L^{q_0}(X, dm)$-sense) is bounded as a mapping

$$T_0^* : L^{q_0}(X, dm) \to L^{q_0}(X, dm),$$

where q_0 is the conjugated exponent to p_0, i. e. $\frac{1}{p_0} + \frac{1}{q_0} = 1$. Moreover, T_0^* is the same type of operator as T_0, whose kernel is $\overline{K(y, x, 0)}$, where $\bar{K}$ represents the complex conjugate to the function K.

The following theorem is a simple consequence of Theorem 2.1.

Theorem 2.2 *Let T_0 be the operator associated with the operator T that satisfies the scalar version of the representation formula (2.1), whose kernel K satisfies the conditions (2.2) and (2.3) with $t = 0$. If the operator*

$$T_0^{\#} : L^{p_0}(X, dm) \to L^{p_0}(X, dm),$$

is bounded for one $p_0 > 1$, then the operator

$$T_0 : L^p(X, dm) \to L^p(X, dm)$$

is bounded for every $p > 1$. When $p = 1$, the operator T_0 is a weak-type 1-1 bounded operator; that is, there exist a positive constant C so that for every $\lambda > 0$, and every continuous function f, with compact support in X,

$$|\{x \in X : |T_0 f(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1(X, dm)}.$$

Proof: To prove Theorem 2.2, we consider two different cases $p \leq p_0$ and $p > p_0$.

When $p \leq p_0$, Theorem 2.1 applied to T_0 gives that the operator $T_0 : L^p(X, dm) \rightarrow L^p(X, dm)$ is bounded for every $p \in (1, p_0]$, and weak-type 1-1 bounded when $p = 1$.

In order to prove the case $p > p_0$ we need to consider the adjoint operator to T_0. The operator T_0^* is L^q-bounded, for every $q \geq q_0$, where q_0 is the conjugate exponent to p_0, as an adjoint to a bounded operator. On the other hand, since the operator T_0^* is of the same type as T_0, Theorem 2.1 implies that it is L^q-bounded for every $q \in (1, q_0]$. Thus, T^* is L^q-bounded for every $q > 1$, which implies the L^p-boundedness of the operator T for every $p > 1$.

■

The following result (see [23]) provides a connection between the singular integral operator T, and the corresponding maximal singular integral operator on $\mathbb{R}^n$.

Lemma 2.3 *Let $K(x, y, 0)$ satisfy the conditions (2.2) and (2.3) with $t = 0$, and for every $\lambda > 0$ and $x, y \in \mathbb{R}^n$ we have $K(\lambda x, \lambda y, 0) = \lambda^{-n} K(x, y, 0)$. Then, there exists a positive constant C so that for every $f \in C_0^\infty(\mathbb{R}^n)$, and almost every $x \in \mathbb{R}^n$*

$$T_0^{\#} f(x) \leq CM(T_0 f)(x) + CMf(x),$$

where M represents the Hardy-Littlewood maximal function.

For the proof of the lemma see ([23], page 67).

Remark: Lemma 2.3, combined with Theorem 2.2 implies that if the operator T_0 is $L^p(\mathbb{R}^n)$-bounded for one $p_0 > 1$, then $T_0^{\#}$ is $L^p(\mathbb{R}^n)$-bounded for every $p > 1$, due to the $L^p(\mathbb{R}^n)$-boundedness property of the Hardy-Littlewood maximal function for every $p > 1$.

2.3 Singular Integral Operators on X^+

In this section we are going to focus on the problem of boundedness of the singular integral operator as a mapping:

$$T : L^p(X, dm) \to L^p(X^+, d\mu) \qquad \text{for } p > 1,$$

when μ is a Carleson measure. Unlike the operator T_0, the operator T does not have a nice adjoint. Therefore, we cannot tell whether T is a L^p-bounded operator for $p > p_0$, by using Theorem 2.1. This difficulty can be overcome by using the good-λ principle argument (see [5] and [24]).

In this section we assume that the operator T, whose trace, T_0, is represented by the formula (2.1), and the conditions (2.2) and (2.3) with $t = 0$, imposed on the kernel K, are satisfied. So, T can be any operator whose range consists of functions with the domain in X^+, so that $Tf(x, 0) = T_0 f(x)$, and T_0 satisfies the properties mentioned above. Additionally, we want the operators T and T_0 to be related in the following way. There exists a positive constant B so that

$$|Tf(x,t) - T_{0,t}f(x)| \leq BHf(x,t) \qquad \text{for every } x \in X, \text{ and } t > 0 \tag{2.4}$$

for every continuous function f with compact support in X.

Now, we state the main result of this chapter.

Theorem 2.4 *Let μ be a Carleson measure on X^+,and let T and T_0 be the operators defined as above which satisfy (2.4). If*

$$T_0^{\#} : L^{p_0}(X, dm) \to L^{p_0}(X, dm)$$

is a bounded operator for some $1 < p_0 < \infty$, then the operator

$$T : L^{p_0}(X, dm) \to L^{p_0}(X^+, d\mu)$$

is bounded; in other words, there exist a positive constant $C_{p_0} > 0$ *(that depends only on* p_0*) so that*

$$||Tf||_{L^{p_0}(X^+,d\mu)} \leq C_{p_0}||f||_{L^{p_0}(X,dm)} \qquad \text{for every } f \in L^{p_0}(X,dm).$$

Proof: Without loss of generality we can assume that f is a continuous function with compact support in X, since the set $C_0(X)$ is dense in $L^{p_0}(X)$. Let us fix a $\lambda > 0$. We want to estimate the μ-measure of the set

$$\Delta_\lambda = \{(x,t) \in X^+ : |Tf(x,t)| > 3\lambda, Hf(x,t) \leq \gamma\lambda\},$$

where $0 < \gamma < 1$ is a fixed number to be specified later. We set

$$\Omega_\lambda = \{x \in X : T_0^\# f(x) > \lambda\}.$$

If γ is small, then property (2.4) yields

$$(x,t) \in \Delta_\lambda \qquad \text{implies} \qquad x \in \Omega_\lambda, \tag{2.5}$$

because

$$3\lambda < |Tf(x,t)| \leq |T_{0,t}f(x)| + |T_{0,t}f(x) - Tf(x,t)|,$$

$$\leq T_0^\# f(x) + BHf(x,t) \leq T_0^\# f(x) + B\gamma\lambda.$$

So if we choose γ so small that $3 - B\gamma > 1$, claim (2.5) holds. (More explicitly, we need $\gamma < 2/B$, but we may need γ to be even smaller; so we will not specify its upper bound for now.)

The set Ω_λ is open and bounded. Let the family of quasi-balls $\{B(x,r(x))\}_{x\in\Omega_\lambda}$, where $r(x)$ is chosen so that $B(x,r(x)) \subset \Omega_\lambda$ but $B^*(x,r(x)) \cap \Omega_\lambda^c \neq \emptyset$, be a covering of Ω_λ. (Recall that B^* represents the quasi-ball with the same center as B, but $r(B^*) = Kr(B)$ where K is the space constant from the Calderón covering Lemma.) Using the covering lemma we obtain a sequence of pairwise disjoint quasi-balls $\{B_j\}$

such that $\Omega_\lambda \subset \cup_j B_j^*$. Moreover, for every index j there is a point $w_j \in B_j^*$, so that $T_0^\# f(w_j) \leq \lambda$.

The set $R_\lambda = \{Mf > \gamma\lambda\}$ is also open and bounded. Then, in the same way as in case of Ω_λ, we obtain a sequence of pairwise disjoint quasi-balls $\{D_k\}$, $D_k \subset R_\lambda$, such that $R_\lambda \subset \cup_k D_k^*$, and accordingly, for every index k, there is a point $x_k \in D_k^*$ so that $Mf(x_k) \leq \gamma\lambda$.

We set

$$P_j = (B_j^* \times [0,\infty)) \cap \Delta_\lambda \qquad \text{and} \qquad P_k' = (D_k^* \times [0,\infty)) \cap \Delta_\lambda.$$

Formula (2.5) implies

$$\Delta_\lambda \subset \cup_j P_j.$$

Therefore, in order to estimate $\mu(\Delta_\lambda)$ we need to estimate $\mu(P_j)$ for each j.

Let $(x_0, t_0) \in P_j$ (which means $x_0 \in B_j^*$) for some index j. If $B_j \cap R_\lambda = \emptyset$, then we have found a point $x_0 \in B_j^*$ such that $Mf(x_0) \leq \gamma\lambda$. If $B_j \cap R_\lambda \neq \emptyset$, then $x_0 \in B_j \cap D_k^*$ for some index k. But, for every index k there is a point $x_k \in D_k$ such that $Mf(x_k) \leq \gamma\lambda$.

We distinguish two cases: $r(B_j) \leq r(D_k)$ and $r(B_j) \geq r(D_k)$. In the case $r(B_j) \geq r(D_k)$, which implies $D_k^* \subset B_j^{**}$ (because $x_0 \in B_j \cap D_k^* \neq \emptyset$), we set

$$f(x) = f(x)\chi_{B_j^\#} + f(x)\chi_{X \setminus B_j^\#} = f_1(x) + f_2(x),$$

where $B_j^\#$ denotes the quasi-ball in X concentric to B_j, and $r(B_j^\#) = 20Kk_\rho r(B_j)$. ($K$ is the space constant from the covering lemma, and k_ρ the constant from the condition (iii) in the definition of pseudo-metric.) Notice that the set B_j^* contains points w_j ($T_0^\#(w_j) < \lambda$), and x_k. The formula

$$P_j = P_j \cap (\{|Tf_1| > \lambda, Hf \leq \gamma\lambda\} \cup \{|Tf_2| > 2\lambda, Hf \leq \gamma\lambda\}) = P_j^1 \cup P_j^2,$$

shows us that in order to estimate $\mu(P_j)$, it suffices to estimate each of $\mu(P_j^1)$, and $\mu(P_j^2)$ separately. Let

$$P_j^1 = P_j^1 \cap \left(B_j^* \times (t < Ar(B_j)) \cup B_j^* \times (t \geq Ar(B_j))\right) = P_j^{11} \cup P_j^{12},$$

where $A = 40Kk_\rho^2$. Since the measure μ is a Carleson measure we have

$$\mu(P_j^{11}) \leq \mu(S(AB_j)) \leq C|B_j|.$$

We will show that when $\gamma > 0$ is small, then $P_j^{12} = \emptyset$. Let $(x,t) \in P_j^{12}$. Using conditions (2.4), (2.2), and the fact that $Hf(x,t) \leq \gamma\lambda$, we conclude

$$\lambda < |Tf_1(x,t)| \leq |T_{0,t}f_1(x,t)| + B\gamma\lambda \leq \int_{\rho(x,y)>t} |K(x,y,0)||f_1(y)|dm(y) + B\gamma\lambda,$$

which implies

$$(1 - B\gamma)\lambda = b\lambda < \int_{B_j^{\#}\cap\{\rho(x,y)>t\}} \frac{1}{|B(x,\rho(x,y))|}|f(y)|dm(y).$$

But the integral on the left is 0, which yields a contradiction, if $1 - B\gamma > 0$, which happens when γ is small. We will show that $B_j^{\#} \cap \{\rho(x,y) > t\} = \emptyset$, using the following argument. Let c be the center of the quasi-ball B_j. Then

$$\rho(x,y) \leq k_\rho(\rho(x,c) + \rho(y,c)) \leq k_\rho(\rho(x,c) + 20Kk_\rho r(B_j)),$$

since $y \in B_j^{\#}$. The fact that $\rho(x,y) > t \geq Ar(B_j)$ implies

$$\rho(x,y) \leq k_\rho\left(\rho(x,c) + \frac{20Kk_\rho}{A}\rho(x,y)\right) = k_\rho\rho(x,c) + \frac{1}{2}\rho(x,y).$$

Hence

$$2k_\rho\rho(x,c) \geq \rho(x,y) > 40Kk_\rho^2 r(B_j),$$

which yields

$$\rho(x,c) \geq 20Kk_\rho r(B_j^*),$$

which contradicts the fact that $x \in B_j^*$, so the set over which we integrate must be empty.

Now, we claim that $P_j^2 = \emptyset$, for γ sufficiently small. Condition (2.4) implies that for any $(x,t) \in P_j^2$ we have

$$|Tf_2(x,t)| \leq |T_{0,t}f_2(x)| + B\gamma\lambda.$$

So, we need to estimate $|T_{0,t}f_2(x)|$, when $(x,t) \in P_j^2$. Recall that $w_j \in B_j^*$ is such a point that $T_0^\# f(w_j) < \lambda$, which implies

$$|T_{0,t}f_2(x)| = \left| \int_{(X \setminus B_j^\#) \cap \{y: \rho(x,y) > t\}} K(x,y,0) f(y) dm(y) \right|$$

$$\leq \left| \int_{(X \setminus B_j^\#) \cap \{y: \rho(x,y) > t\}} K(w_j,y,0) f(y) dm(y) \right|$$

$$+ \int_{X \setminus B_j^\#} |K(w_j,y,0) - K(x,y,0)||f(y)| dm(y) = I + II,$$

(Notice that in the second integral we integrate over a larger set.)

Let us estimate the term I, first. Let $\eta = 21Kk_\rho^2 r(B_j)$. If $t > \eta$ and $\rho(x,y) > t$, then using the property (iii) of pseudometrics we conclude $\rho(c_j, y) \geq 20Kk_\rho r(B_j) = r(B_j^\#)$ (where c_j is the center of the quasi-ball B_j), which implies $B_j^\# \subset B(x,\eta)$. Hence, when $t > \eta$ we have

$$I = \left| \int_{\{y: \rho(x,y) > t\}} K(w_j,y,0) f(y) dm(y) \right|$$

$$I \leq \left| \int_{\{y: \rho(w_j,y) > t\}} K(w_j,y,0) f(y) dm(y) \right|$$

$$+ \int_{\{y: \rho(w_j,y) \leq t\} \cap \{y: \rho(x,y) > t\}} |K(w_j,y,0)||f(y)| dm(y)$$

$$+ \int_{\{y: \rho(w_j,y) > t\} \cap \{y: \rho(x,y) \leq t\}} |K(w_j,y,0)||f(y)| dm(y)$$

$$= |T_{0,t}f(w_j)| + A + B.$$

If $y \in \{y : \rho(w_j, y) \leq t\} \cap \{y : \rho(x, y) > t\}$, then the property (iii) of pseudo-metrics implies

$$ct \leq \frac{t - 2Kk_\rho r(B_j)}{k_\rho} \leq \rho(w_j, y) < t,$$

for some $c < 1$. Employing the property $|K(w_j, y, 0)| \leq C|B(w_j, \rho(w_j, y))|^{-1}$ we obtain that

$$A \leq C\frac{1}{|B(w_j, ct)|}\int_{B(w_j,t)} |f(y)|dm(y).$$

The fact that m is a doubling measure implies $\frac{B(w_j,t)}{B(w_j,ct)} \leq const$, which yields

$$A \leq C \inf_{x \in B_j^{\#}} Mf(x) \leq CMf(x_k).$$

When $y \in \{y : \rho(w_j, y) > t\} \cap \{y : \rho(x, y) \leq t\}$, then

$$t < \rho(w_j, y) < 2k_\rho t,$$

and in the same way as in the case of the term A we obtain

$$B \leq C \inf_{x \in B_j^{\#}} Mf(x) \leq CMf(x_k).$$

Thus, when $t > \eta$ we have

$$I \leq \lambda + CMf(x_k).$$

If $t \leq \eta$, then

$$I \leq |T_{0,\eta}f(w_j)| + C\int_{\{y:\rho(w_j,y)\leq\eta\}\setminus B_j^{\#}} |B(w_j, \rho(w_j, y))|^{-1}|f(y)|dm(y)$$

$$+C\int_{\{y:\rho(x,y)\leq t\}\setminus\{y:\rho(w_j,y)>\eta\}} |B(w_j, \rho(w_j, y))|^{-1}|f(y)|dm(y) \leq \lambda + E + F.$$

If $y \in \{y : \rho(w_j, y) \leq \eta\} \setminus B_j^{\#}$ then

$$c\eta = 19Kr(B_j)\rho(w_j, y) < \eta.$$

(The last claim is true, because whenever $a \in B_j^*$, and $b \in X \setminus B_j^{\#}$, the property (iii) of pseudo-metric implies $\rho(a,b) \geq \frac{\rho(b,c)-k_\rho \rho(a,c)}{k_\rho} \geq 19Kr(B_j)$, where c is the center of the quasi-ball B_j. In our case $w_j \in B_j^*$ and $y \in X \setminus B_j^{\#}$.) Notice that $B_j^{\#} \subset B(w_j, \eta)$, therefore, using the same argument as when estimating the term A we conclude

$$E \leq C \inf_{x \in B_j^{\#}} Mf(x) \leq CMf(x_k).$$

If $y \in \{y : \rho(x,y) \leq t\} \setminus \{y : \rho(w_j, y) > \eta\}$, then

$$\eta \leq \rho(w_j, y) < 2k_\rho \eta,$$

which yields

$$F \leq C \inf_{x \in B_j^{\#}} Mf(x) \leq CMf(x_k).$$

When we summarize the last four results we have that for $t > \eta$ as well as for $t \leq \eta$ we have

$$I \leq \lambda + CMf(x_k).$$

In order to estimate the term II we use the assumption (2.3) (Notice that $2\rho(x, w_j) \leq \rho(x,y)$, because $x, w_j \in B_j^*$, and $y \in X \setminus B_j^{\#}$.) to get

$$II \leq \int_{X \setminus B_j^{\#}} \frac{\rho(x,w_j)^\epsilon}{\rho(x,y)^\epsilon |B(x, \rho(x,y))|} |f(y)| dm(y)$$

$$\leq (Kr(B_j))^\epsilon \sum_{m=0}^{\infty} \int_{2^{m+1}B_j^{\#} \setminus 2^m B_j^{\#}} \frac{|f(y)| dm(y)}{\rho(x,y)^\epsilon |B(x, \rho(x,y))|}.$$

Since $\rho(x,y) \geq 2^m 20 Kr(B_j)$ on $2^{m+1}B_j^{\#} \setminus 2^m B_j^{\#}$, we have

$$\leq C(Kr(B_j))^\epsilon \sum_{m=0}^{\infty} \frac{1}{|B(x, 2^m r(B_j^{\#}))| 2^{m\epsilon} r(B_j)^\epsilon} \int_{2^{m+1}(B_j^{\#})} |f(y)| dy.$$

Using the fact that $\frac{|B(x,2^{m+1}r(B_j^{\#}))|}{|B(x,2^m r(B_j^{\#}))|} \leq C$, where C is the doubling constant for the measure m, we get

$$II \leq CMf(x_k) \sum_{m=0}^{\infty} 2^{-\epsilon m} \leq C\gamma\lambda,$$

because $x_k \in B_j^* \subset 2^m B_j^{\#}$ for every index m.

We have proved

$$|Tf_2(x,t)| \leq \lambda + C\gamma\lambda + B\gamma\lambda \qquad \text{when } (x,t) \in P_j^2.$$

Now, we choose γ to be so small that $(C+B)\gamma < 1$, (Now we can fix $\gamma > 0$.) to get that the set P_j^2 is empty.

If $r(B_j) \leq r(D_k)$, then $B_j^* \subseteq D_k^{**}$ (Recall $x_0 \in B_j \cap D_k^* \neq \emptyset$.) and we write:

$$f(x) = f(x)\chi_{D_k^{\#}} + f(x)\chi_{X \setminus D_k^{\#}} = f_1(x) + f_2(x).$$

The same argument as in the other case, (now $x_k, w_j \in D_k^*$) leads to

$$\mu(P_k') \leq C\gamma|D_k|.$$

Let $J_k = \{j \in \mathbb{N} : B_j^* \subset D_k^{\#}\}$. Then

$$\mu(\cup_{j \in J_k} P_j) \leq C\mu(D_k^{\#} \times [0,\infty) \cap \Delta_\lambda),$$

which implies

$$\mu(\Delta_\lambda) \leq \sum_{j \in (\cup_k J_k)^c} \mu(P_j) + \sum_k \mu(\cup_{j \in J_k} P_j)$$

$$\leq \sum_{j \in (\cup_k J_k)^c} \mu(P_j) + \sum_k \mu(D_k^{\#} \times [0,\infty) \cap \Delta_\lambda) \leq \sum_{j \in (\cup_k J_k)^c} C\gamma|B_j| + \sum_k C\gamma|D_k|.$$

Thus, we have proved

$$\mu(\Delta_\lambda) \leq C\gamma\left(|\{T_0^{\#}f > c\lambda\}| + |\{Mf > \gamma\lambda\}|\right). \tag{2.6}$$

To finish proving the theorem we proceed as follows.

$$\int |Tf|^{p_0} d\mu = C\int_0^\infty \lambda^{p_0-1}\mu(\{Tf > 3\lambda\})d\lambda$$

$$\leq C\int_0^\infty \lambda^{p_0-1}\mu(\{Hf > \gamma\lambda\})d\lambda + C\int_0^\infty \lambda^{p_0-1}\mu(\{Tf > 3\lambda, Hf \leq \gamma\lambda\})d\lambda,$$

here, we have used the formula $A \subset (A \cap B) \cup B^c$. Then (2.6) yields

$$\leq C(\gamma) \left(\int (Hf)^{p_0} d\mu + \int (T_0^{\#} f)^{p_0} dm + \int (Mf)^{p_0} dm \right).$$

Finally, using the fact that the operator $H : L^{p_0}(X, dm) \to L^{p_0}(X^+, d\mu)$ is bounded (since $p_0 > 1$), the assumption of the L^{p_0}-boundedness of $T_0^{\#}$, the L^{p_0} boundedness of the Hardy-Littlewood maximal function for any $p_0 > 1$, and the fact that γ is fixed, we obtain that the last line is dominated by

$$C||f||^{p_0}_{L^{p_0}(X,dm)},$$

which completes the proof of the theorem.

■

Remark 1: In case when $p_0 = 1$ we consider the case when the operator $T_0^{\#}$ is weak type 1-1 bounded. If all the assumptions of Theorem 2.4 are satisfied then the operator T is also a weak-type 1-1 bounded operator.

Remark 2: We can also prove the vector-valued version of Theorem 2.4. We generalize the operator T in the same way as we did in Theorem 2.1, and also we can define the corresponding vector-valued operator T_0. Essentially, in the proof of Theorem 2.4 we need to replace the absolute value brackets by one of the norms: $||\cdot||_E$, $||\cdot||_F$, or $||\cdot||_{L(E,F)}$ (including the absolute value brackets in the definition of the Hörmander maximal function), and instead of L^p spaces we consider their vector version, either L^p_E or L^p_F. The integrals in the proof of Theorem 2.4 now become vector-integrals.

The following lemma is a useful tool when checking if condition (2.4) is satisfied.

Lemma 2.5 *Let X be such a space of homogeneous type. Suppose there exists a positive integer d such that*

$$|B(x,r)| = Cr^d \qquad \text{for every } x \in X \text{ and } r > 0.$$

Then, condition (2.4), for the operators T and T_0, associated with the kernel K, is satisfied if there exists a positive constant C such that

$$|K(x,y,t) - K(x,y,0)| \le C\left(\frac{1}{\rho(x,y)^d} - \frac{1}{(\rho(x,y)+t)^d}\right), \tag{2.7}$$

for every $x, y \in X$ and $t \ge 0$, such that $\rho(x,y) > t$.

Proof: To prove the lemma we proceed as follows.

$$|Tf(x,t) - T_{0,t}f(x)| \le \int_{\rho(x,y)>t} |K(x,y,t) - K(x,y,0)||f(y)|dm(y)$$

$$+ \int_{\rho(x,y)\le t} |K(x,y,t)||f(y)|dm(y) = I_1 + I_2.$$

Property (2.2) yields the following estimate for the term I_2.

$$I_2 \le C\int_{B(x,t)} \frac{1}{|B(x,\rho(x,y)+t)|}|f(y)|dm(y)$$

$$\le C\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y)|dm(y) \le CHf(x,t).$$

In order to estimate the term I_1, we use (2.7) and some elementary computation. First, the binomial formula implies

$$\frac{(\rho(x,y)+t)^d - \rho(x,y)^d}{\rho(x,y)^d} \le C\frac{t}{\rho(x,y)},$$

whenever $\rho(x,y) > t$. Thus, we have

$$I_1 \le C\int_{\rho(x,y)>t} \frac{t}{\rho(x,y)}\frac{|f(y)|}{|B(x,\rho(x,y)+t)|}dm(y)$$

$$= C\sum_{k=1}^{\infty}\int_{2^{k-1}t<\rho(x,y)\le 2^k t} \frac{t}{\rho(x,y)}\frac{|f(y)|}{|B(x,\rho(x,y))|}dm(y)$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k+1} \frac{1}{|B(x, 2^{k-1}t)|} \int_{\rho(x,y) \leq 2^k t} |f(y)| dm(y).$$

Finally, the facts that $\sum_{k=1}^{\infty} 2^{-k+1} = 2$, $\frac{|B(x,2^k t))|}{|B(x,2^{k-1}t))|} \leq C$, where C is the doubling constant for the measure m, and that for every $k = 1, 2, ...$

$$\frac{1}{|B(x, 2^k t)|} \int_{\rho(x,y) \leq 2^k t} |f(y)| dm(y) \leq BHf(x, t),$$

yield the desired estimate, which completes the proof of the lemma.

■

2.4 Singular Integral Operator on Atomic Spaces for $0 < p \leq 1$

Let T be a singular integral operator that satisfies all the assumptions of the scalar version of Theorem 1.2 (Ruiz-Torrea theorem), and is bounded as a mapping

$$T : L^{p_0}(X, dm) \to L^{p_0}(X^+, d\mu)$$

for some $p_0 > 1$. For $p \in (0, 1]$ fixed, we define a p-atom as a Borel measurable function on X whose support is in a quasi-ball B, $\int_X a = 0$, and $|a| \leq C|B|^{-1/p}$.

Let C_k be constants defined as in Section 1.3. If $A > 0$ is the doubling constant for the measure m, i. e. for every quasi-ball $B \subset X$ $|B(z, 2r)| \leq A|B(z, r)|$, then $C_k \leq A^k$.

Theorem 2.6 *Let T be the operator defined as above, with the constant ϵ as in the condition (2.3), and let $p \in (0, 1]$ be such a number that the series $\sum_k C_k^{1-p} 2^{-p\epsilon k}$ converges. Then there exists a positive constant C, that depends on the space X, p, ϵ, and the measure μ, such that for every p-atom a we have*

$$\int |Ta(x, t)|^p d\mu(x, t) \leq C.$$

Proof: Let $B = B(z, r)$ be the quasi-ball associated with the p-atom a. Hölder's inequality applied to the integral

$$\int_{T(B^*)} |Ta(x,t)|^p d\mu(x,t),$$

using the conjugate exponents p_0/p and $1/(1 - p/p_0)$, yields

$$\leq \left(\int_{T(B^*)} |Ta(x,t)|^{p_0} d\mu(x,t) \right)^{p/p_0} (\mu(T(B^*)))^{1-p/p_0}.$$

The L^{p_0}-boundedness of T implies

$$\leq C \left(\int |a(y)|^{p_0} dm(y) \right)^{p/p_0} (\mu(T(B^*)))^{1-p/p_0},$$

$$\leq C|B|^{p/p_0 - 1} (\mu(T(B^*)))^{1-p/p_0} = C \left(\frac{\mu(T(B^*))}{|B|} \right)^{1-p/p_0},$$

which is dominated by a constant due to the fact that μ is a Carleson measure.

When $x \in X \setminus B^*$, the property $\int a = 0$ implies

$$|Ta(x,t)| = \left| \int_B (K(x,y,t) - K(x,z,t)) a(y) dm(y) \right|.$$

(Recall that z is the center of the quasi-ball B.) Notice that $(x,t) \in X^+ \setminus T(B^*)$, and $y \in B$, imply $2\rho(y,z) \leq \rho(x,z) + t$; so we can use the estimate on the integrand in the integral above, which yields

$$|Ta(x,t)| \leq C \int_B \frac{\rho(y,z)^\epsilon}{|B(z, \rho(x,z) + t)| (\rho(x,z) + t)^\epsilon} |a(y)| dm(y).$$

The property $|a| \leq C|B|^{-1/p}$, implies

$$|Ta(x,t)| \leq C \frac{r^\epsilon |B|^{1-1/p}}{|B(z, \rho(x,z) + t)| (\rho(x,z) + t)^\epsilon}. \tag{2.8}$$

Let us define sets

$$A_k = \{(x,t) \in X^+ : 2^{k-1} r(B^*) \leq \rho(x,z) + t < 2^k r(B^*)\},$$

for $k = 1, 2, \ldots$. Then, $X^+ \setminus T(B^*)$ is a disjoint union of the sets $\{A_k\}_{k=1}^{\infty}$. Thus

$$\int_{X^+\setminus T(B^*)} |Ta(x,t)|^p d\mu(x,t) = \sum_{k=1}^{\infty} \int_{A_k} |Ta(x,t)|^p d\mu(x,t).$$

The formula (2.8) implies

$$\int_{X^+\setminus T(B^*)} |Ta(x,t)|^p d\mu(x,t) \leq Cr^{p\epsilon}|B|^{p-1} \sum_{k=1}^{\infty} \int_{A_k} \frac{d\mu(x,t)}{|B(z,\rho(x,z)+t)|^p(\rho(x,z)+t)^{\epsilon p}}.$$

Using the facts that $A_k \subset T(B(z, 2^k r(B^*)))$, and $\rho(x,z) + t > 2^{k-1}r(B^*)$ whenever $(x,t) \in A_k$, we get

$$\int_{X^+\setminus T(B^*)} |Ta(x,t)|^p d\mu(x,t) \leq Cr^{p\epsilon}|B|^{p-1} \sum_{k=1}^{\infty} \frac{\mu(T(B(z,2^k r(B^*))))|2^k B^*|^{1-p}}{|2^k B^*|^p |2^k B^*|^{1-p} r^{p\epsilon}} 2^{-\epsilon pk}.$$

We have also used the fact that $\frac{|2^k B^*|}{|2^{k-1}B^*|} \leq A$. The facts that μ is a Carleson measure, and m is a doubling measure, imply

$$\leq CC_\mu \sum_{k=1}^{\infty} C_k^{(1-p)} 2^{-p\epsilon k},$$

since the series above is convergent by the assumption of the theorem, the expression on the left side of the formula above is dominated by a constant, which completes the proof of the theorem.

■.

Remark: In case when $X = \mathbb{R}^n$, we have that $C_k = 2^k$. Then the assumption about the convergence of $\sum_k C_k^{1-p} 2^{-p\epsilon k}$ becomes $\epsilon > \frac{1}{p} - 1$.

2.5 Examples

Let X be Euclidean space $\mathbb{R}^n$, with Euclidean distance being the metric, provided with the following measure. Let $w : \mathbb{R}^n \to [0, \infty)$ be a bounded homogeneous function with the degree of homogeneity 0, such that $\inf w \geq c > 0$. Then, the function w is bounded, and also an A_1 weight, since

$$\frac{1}{|Q|}\int_Q w(x)dx \le C \le C\frac{w(x)}{c} = Cw(x).$$

Therefore, the measure

$$w(E) = \int_E w(x)dx \qquad E \subset \mathbb{R}^n$$

is a doubling measure, and moreover it is comparable to Lebesgue measure on $\mathbb{R}^n$; i. e. there exist positive constants c and C such that for every cube Q in $\mathbb{R}^n$ we have

$$c|Q| \le w(Q) \le C|Q|,$$

where $|Q|$ represents the Lebesgue measure of the cube Q. Also, it easy to see that such a weight w belongs to the Muckenhoupt class A_p for any $p > 1$.

We define Tf as the principal value of

$$Tf(x,t) = \int_{\mathbb{R}^n} K(x,y,t)f(y)w(y)dy,$$

for every function $f \in C_0^\infty(\mathbb{R}^n)$. The integral above makes sense, because the function w does not create a new, nor eliminates any of the existing singularities.

Moreover, if the kernel K satisfies $K(\lambda x, \lambda y, 0) = \lambda^{-n}K(x,y,0)$ for every $\lambda > 0$ and $x, y \in \mathbb{R}^n$, and if we assume the boundedness of the operator $T_0 : L^p(\mathbb{R}^n, dw) \to L^p(\mathbb{R}^n, dw)$ for some $p > 1$, then the operator $T_0^\#$ is bounded for every $p > 1$, because $w \in A_p$ (Muckenhoupt class, see [15]) for every $p > 1$.

Harmonic Kernels

Harmonic Conjugate in the Complex Plane

As we know the complex conjugate to the harmonic function $u(x,t) = P_t * f(x)$ is given by the formula

$$\tilde{u}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2+t^2} f(y)dy \qquad \text{for} \qquad f \in C_0^\infty(\mathbb{R}^n),$$

where the integral is taken in the principal value sense. Motivated by that fact, we set $k(x) = \frac{1}{\pi}\frac{x}{1+x^2}$, that produces the kernel

$$K(x,y,t) = \frac{1}{t}k\left(\frac{x-y}{t}\right) = \frac{1}{\pi}\frac{x-y}{(x-y)^2+t^2},$$

which for $t = 0$ induces $k_0(x) = \frac{1}{\pi x}$, the kernel of the Hilbert transform, that is a homogeneous function with the degree of homogeneity-1. It is a well known fact that the corresponding maximal singular integral operator

$$T_0^\# : L^p(\mathbb{R}, dx) \to L^p(\mathbb{R}, dx)$$

is bounded for every $p > 1$.

A singular integral operator in the upper half-plane is defined by

$$Tf(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2+t^2} f(y)dy \qquad \text{for} \qquad f \in C_0^\infty(\mathbb{R}^n),$$

taken in the principal value sense. The operator T satisfies all the assumptions in the definition of the singular integral operators on $\mathbb{R}_+^2$. The assumptions (2.2), and (2.3) (with $\epsilon = 1$) are fulfilled because

$$|k(x)| \le \frac{2}{(1+|x|)} \quad \text{and} \quad |(\nabla k)(x)| = |k'(x)| \le \frac{2}{(1+|x|)^2}.$$

Thus, we only need to check if the condition (2.4) is satisfied. Elementary computations yield

$$\Pi|Tf(x,t) - T_{0,t}f(x)| = \left|\int \left(\frac{1}{y}\chi_{|y|>t} - \frac{y}{y^2+t^2}\right) f(x-y)dy\right|$$

$$\leq \int_{|y|\leq t} \frac{|y|}{y^2+t^2}|f(x-y)|dy + \int_{|y|>t} \frac{t^2}{|y|(y^2+t^2)}|f(x-y)|dy$$

$$\leq \int_{|y|\leq t} P_t(y)|f(x-y)|dy + \int_{|y|>t} P_t(y)|f(x-y)|dy \leq Hf(x,t).$$

Therefore, the condition (2.4) is satisfied. Thus, the operator

$$T : L^p(\mathbb{R}, dx) \to L^p(\mathbb{R}^2_+, d\mu),$$

is bounded for every $p > 1$, and weak type 1-1 bounded, when μ is a Carleson measure.

Harmonic Conjugates of Functions in Several Variables.

In his attempt to define the conjugate functions to the function

$$u(x,t) = P_t * f(x) \qquad x \in R^n, t \geq 0, f \in C_0^\infty(\mathbb{R}^n),$$

in several variables, J. Horvath (see [13]) considered the integrals

$$u_j(x,t) = \int_{\mathbb{R}^n} \frac{c_n(x_j - y_j)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy \qquad j = 1, 2, ..., n;$$

taken in the principal value sense, where $x_j - y_j$ denotes j-th coordinate of $x - y = (x_1 - y_1, ..., x_n - y_n)$. Therefore, we will consider the kernels

$$k_j(x) = \frac{c_n x_j}{(|x|^2+1)^{\frac{n+1}{2}}}, \qquad x = (x_1, x_2, ..., x_n), \quad j = 1, 2, ..., n,$$

that each produces the kernel

$$K_j(x,y,t) = \frac{c_n(x_j - y_j)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}},$$

so that $k_{j,0}(x) = c_n x_j |x|^{-n-1}$, is the kernel of the Riesz transform R_j ($j = 1, 2, ..., n$). Notice that each $k_{j,0}$ is homogeneous with the degree $-n$. We will consider the following singular integral operator

$$Tf(x,t) = \int \frac{c_n(x_j - y_j)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)w(y)dy.$$

Since the measures $w(E)$ and $|E|$ are comparable for every set $E \subset \mathbb{R}^n$, μ is a Carleson measure with respect to w.

Let us check if the operator defined by $Tf = u_j$, for any fixed $j \in \{1, 2, ..., n\}$, is a singular integral operator on $\mathbb{R}^{n+1}_+$. The elementary formula $\frac{1}{2}(a+b)^2 \leq a^2 + b^2 \leq (a+b)^2$ for $a, b \geq 0$, yields

$$|k_j(x)| \leq \frac{2^{\frac{n+1}{2}}}{(|x|+1)^n} \qquad \text{and} \qquad |\nabla k_j(x)| \leq \frac{(n+2)2^{\frac{n+1}{2}}}{(|x|+1)^{n+1}},$$

which imply (2.2) and (2.3). Now, we check if the property (2.4) holds.

$$|T_{0,t}f(x) - Tf(x,t)| \leq \int_{|y|<t} \frac{c_n|y_j|}{(|y|^2+t^2)^{\frac{n+1}{2}}} |f(x-y)|w(y)dy +$$

$$+ \int_{|y|\geq t} \frac{c_n|y_j|[(|y|^2+t^2)^{\frac{n+1}{2}} - |y|^{n+1}]}{|y|^{n+1}(|y|^2+t^2)^{\frac{n+1}{2}}} |f(x-y)|w(y)dy.$$

If $|y| > t$, then elementary calculations yield

$$\frac{c_n|y_j|[(|y|^2+t^2)^{\frac{n+1}{2}} - |y|^{n+1}]}{|y|^{n+1}} \leq \frac{c_n[(|y|+t)^{n+1} - |y|^{n+1}]}{|y|^n} \leq c_n(2^n-1)t,$$

this, with the boundedness of w, implies

$$\begin{aligned} |T_{0,t}f(x) - Tf(x,t)| &\leq C\int_{|y|\leq t} P_t(y)|f(x-y)|dy + C\int_{|y|>t} P_t(y)|f(x-y)|dy \\ &= C\int P_t(x)|f(x-y)|dy, \end{aligned}$$

which proves that the property (2.4) holds.

Now, we need to consider the boundedness of the operator

$$T_0^{\#} : L^p(\mathbb{R}^n, dw) \to L^p(\mathbb{R}^n, dw).$$

First, notice that the kernel $k(x,y,0) = k_j(x-y)w(y)$ satisfies $k(\lambda x, \lambda y, 0) = \lambda^{-n}k(x,y,0)$, which implies that the corresponding operator T_0 satisfies the weak boundedness property (see [10]), and due to the oddness of k_0 in both x and y, both conditions $T_0 1 = 0$

and $T_0^*1 = 0$ are satisfied, which makes the operator $T_0 : L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, dx)$, bounded for every $p > 1$. Lemma 2.3 implies that the operator $T_0^{\#} : L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, dx)$ is bounded, and using the Muckenhoupt theory we can conclude the boundedness of the operator $T_0^{\#} : L^p(\mathbb{R}^n, dw) \to L^p(\mathbb{R}^n, dw)$, because $w \in A_p$ for every $p > 1$. Therefore, the operator

$$T : L^p(\mathbb{R}^n, dw) \to L^p(\mathbb{R}_+^{n+1}, d\mu),$$

is bounded for every $p > 1$ and weak-type 1-1 bounded.

In the case of a singular integral operator whose kernel is harmonic, and when $w(x) = 1$ we may obtain the L^p-boundedness of the operator T, in the following way. It is a well known fact that for the complex conjugate $\tilde{u}$ of a harmonic function $u = P_t * f$,

$$\tilde{u}(x,t) = (P_t * \mathcal{H} * f)(x),$$

where $\mathcal{H}$ represents the Hilbert transform. When $n > 1$, J. Horvath proved that

$$u_j(x,t) = (P_t * R_j * f)(x) \qquad j = 1, 2, ..., n,$$

where $R_j f$ is the Riesz transform of the function f in $\mathbb{R}^n$. The formula (1.3) implies

$$||Tf||_{L^p(\mathbb{R}_+^{n+1}, d\mu)} = ||P_t * R_j * f||_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq C||R_j * f||_{L^p(\mathbb{R}^n, dm)},$$

and the L^p-boundedness of the Riesz transform ($p > 1$) yields

$$||Tf||_{L^p(\mathbb{R}_+^{n+1}, d\mu)} \leq C||f||_{L^p(\mathbb{R}^n, dm)}.$$

(Notice that in the case $n = 1$ the Riesz transform becomes the Hilbert transform).

Examples — Non-harmonic Kernels

In case $n = 1$, the kernel

$$K(x,t) = \frac{sign(x)|x|^\alpha}{|x|^{\alpha+1} + t^{\alpha+1}} \qquad x \in \mathbb{R}^n, t \geq 0,$$

is harmonic only for $\alpha = 1$; for $\alpha \neq 1$ the kernel is not harmonic.

We claim that the corresponding convolution operator T satisfies all the assumptions in the definition of the singular integral operator for a convenient α. It is easy to check that conditions (2.2) and (2.3) are satisfied for any $\alpha > 0$. To show that property (2.4) holds, according to Lemma 2.5 we need to determine if

$$\left| \frac{1}{x} - \frac{sign(x)|x|^\alpha}{|x|^{\alpha+1} + t^{\alpha+1}} \right| \leq \frac{t}{|x|(|x|+t)} \qquad \text{when } |x| > t,$$

which follows from $\frac{t^\alpha}{|x|^{\alpha+1}+t^{\alpha+1}} \leq \frac{1}{|x|+t}$, which is true whenever $|x| > t > 0$.

The corresponding operator T_0 is the Hilbert transform, whose maximal singular integral operator is L^p-bounded for every $p > 1$, thus the singular integral operator

$$T : L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^{n+1}_+, d\mu),$$

corresponding to the kernel K is bounded for every $p > 1$ and weak-type 1-1 bounded, when μ is a Carleson measure.

When $n > 1$, let Ω be an odd (that is $\Omega(-x) = -\Omega(x)$), bounded, and homogeneous function on $\mathbb{R}^n$, with degree of homogeneity 0 (i. e. for every $\lambda > 0$ we have $\Omega(\lambda x) = \Omega(x)$), such that

$$|\nabla\Omega(x)| \leq \frac{C}{|x|} \qquad \text{for every } x \in \mathbb{R}^n,\ x \neq 0.$$

Let T be the convolution operator associated with the kernel

$$K(x,t) = \frac{\Omega(x)}{(|x|+t)^n}.$$

It is a well-known fact that the convolution operator T_0 with the kernel $K(x,0)$ has L^p-bounded maximal singular integral operator for every $p > 1$ (see [23]). The function K clearly satisfies condition (2.2) because Ω is a bounded function. Using the formula

$$\frac{\partial K}{\partial x_i} = \frac{\frac{\partial \Omega}{\partial x_i}(|x|+t)^n - \Omega n(|x|+t)^{n-1}\frac{x_i}{|x|}}{(|x|+t)^{2n}},$$

boundedness of Ω, and $|\nabla\Omega(x)| \le \frac{C}{|x|}$ we obtain

$$|\nabla K(x,t)| \le \frac{C}{(|x|+t)^{n+1}},$$

that implies condition (2.3), with $\epsilon = 1$. The formula

$$|K(x,t) - K(x,0)| \le |\Omega(x)| \left(\frac{1}{|x|^n} - \frac{1}{(|x|+t)^n} \right),$$

the fact that Ω is a bounded function, and Lemma 2.5 imply that property (2.4) holds. Thus, the operator

$$T : L^p(\mathbb{R}^n, dw) \to L^p(\mathbb{R}_+^{n+1}, d\mu),$$

defined as above, is bounded for every $p > 1$ and weak-type 1-1 bounded when μ is a Carleson measure.

2.6 Cauchy Kernel on $\mathbb{C}^n$ $(n > 1)$

Let $X = S$ be the unit sphere in $\mathbb{C}^n$, $\rho(x,y) = |1 - \langle x,y\rangle|^{1/2}$ (where $\langle x,y\rangle = \sum_{k=1}^n x_k \bar{y}_k$) be the non-isotopic metric on S, and σ be the rotation invariant measure on S. When $n > 1$, we have

$$\sigma(B(x,r)) \asymp r^{2n} \qquad \text{where} \quad B(x,r) = \{y \in S : \rho(x,y) < r\}, \tag{2.9}$$

(The symbol $\asymp$ means there exist positive constants c and C so that for every $r > 0$ we have $cr^{2n} \le \sigma(B(x,r)) \le Cr^{2n}$.)

The role of X^+ will be played by the closed unit ball $\bar{U}$ in $\mathbb{C}^n$. The fact that $0 \le t \le 1$ makes almost no difference. We take $t^2 + r^2 = 1$, where $r = |z|$, so that S is obtained when $t = 0$.

Before we proceed, let us list some of the properties of the non-isotopic metric (For the proof of these properties and the formula (2.9) see [17]) :

(a) *For every $x, y, z \in \bar{U}$, where $\bar{U}$ is the closed unit ball in $\mathbb{C}^n$ we have*

$$\rho(x, y) \le \rho(x, z) + \rho(z, y).$$

(b) *ρ is a metric on S.*

(c) *For every $0 \le r \le 1$, and $x, z \in S$ we have*

$$\rho(rx, z) = |1 - r\langle x, z\rangle|^{1/2} \ge \sqrt{1-r}.$$

(d) *For every $0 \le r \le 1$, and $x \in S$ we have*

$$\rho(rx, x) = \sqrt{1-r}.$$

(e) *If U is a unitary map, i. e. $\langle Ux, Uy\rangle = \langle x, y\rangle$, then*

$$\rho(Ux, Uy) = \rho(x, y),$$

and for every $x \in S$ there is a unitary map U such that $Ux = e_1 = (1, 0, 0, ...0)$.

(f) *For every $0 \le r \le 1$, and $x, z \in S$ we have*

$$\rho(rx, z) \ge \frac{\rho(x, z) + \sqrt{1-r}}{3}.$$

The last property follows from (a), (b), and (c) as follows.

$$\rho(x, z) + \sqrt{1-r} \le \rho(rx, z) + \rho(rx, x) + \sqrt{1-r} \le 3\rho(rx, z).$$

For every $0 \le r \le 1$, and $x, y \in S$ we define the *Cauchy kernel* as

$$C_t(x,y) = c_n(1 - r\langle x, y\rangle)^{-n} \qquad \text{where} \quad r^2 + t^2 = 1.$$

The Cauchy kernel is a complex valued function and $|C_t(x, \cdot)|$ is not absolutely integrable. Therefore the kernel is not admissible. We will prove that the operator represented by the formula

$$Tf(x,t) = \int C_t(x,y) f(y) d\sigma(y),$$

is a singular integral operator.

First, we check if for every $0 \le r \le 1$, and $x, y \in S$

$$|C_t(x,y)| \le C\sigma(B(x, \rho(x,y) + t))^{-1}.$$

The definition of C_t, property (f), the fact $\sqrt{1-r} \asymp t$, and property (2.9), imply

$$|C_t(x,y)| = \rho(rx,y)^{-2n} \le C(\rho(x,y)+t)^{-2n} = C\sigma(B(x,\rho(x,y)+t))^{-1}.$$

Next, we check if for every $0 \le r \le 1$, and $x, y \in S$ such that $2\rho(x,y) \le \rho(x,z) + t$ $(r^2 + t^2 = 1)$ it follows that

$$|C_t(x,z) - C_t(y,z)| \le C \frac{\rho(x,y)^2}{\sigma(B(x,\rho(x,z)+t))(\rho(x,z)+t)^2}.$$

To show that the above inequality holds, we employ the formula $a^n - b^n = (a-b)\sum_{k=1}^{n} a^{n-k} b^{k-1}$, to get

$$|C_t(x,z) - C_t(y,z)| = r|\langle x - y, z\rangle| \sum_{k=1}^{n} \rho(rx,z)^{-2(n-k)} \rho(ry,z)^{-2(k+1)}.$$

Property (f) implies

$$\le Cr|\langle x-y, z\rangle| \sum_{k=1}^{n} (\rho(x,z)+t)^{-2(n-k)} (\rho(y,z)+t)^{-2(k+1)}.$$

Recalling that $\rho(y,z)+t \geq \rho(x,z)+t-\rho(x,y) \geq \frac{1}{2}(\rho(x,z)+t)$, we obtain

$$= Cr|\langle x-y,z\rangle|(\rho(x,z)+t)^{-2(n+1)},$$

and now we use property (2.9) , again, to conclude

$$= Cr|\langle x-y,z\rangle|\,\sigma(B(x,\rho(x,z)+t))^{-1}(\rho(x,z)+t)^{-2},$$

so we need to prove that $|\langle x-y,z\rangle| \leq C\rho(x,y)^2$. To show it, we use property (e) to restrict ourselves to the case $z=e_1$. Then $|\langle x-y,z\rangle| = |x_1-y_1|$, where $x_1,y_1 \in \mathbb{C}$ are the first components of $x,y \in \mathbb{C}^n$. Now, the desired inequality follows from

$$\rho(x,y)^4 - |\langle x-y,z\rangle|^2 = |1-x_1\bar{y}_1|^2 - |x_1-y_1|^2$$

$$= (1-|x_1|^2)(1-|y_1|^2) \geq 0.$$

Then, we check if the singular integral operator T, associated with the Cauchy kernel, satisfies the property

$$|Tf(x,t) - T_{0,t}f(x)| \leq BHf(x,t).$$

Using $C_t(x,y) = C_0(rx,y)$ we obtain

$$|C_t(x,y)-C_0(x,y)| \leq C|1-r||\langle x,y\rangle| \sum_{k=1}^{n} \rho(x,y)^{-2(n-k)}(\rho(x,y)+t)^{-2(k+1)}.$$

Knowing that $\sqrt{1-r} \asymp t$ we conclude

$$|C_t(x,y)-C_0(x,y)| \leq C\frac{t^2}{\rho(x,y)^2}\frac{1}{\sigma(B(x,\rho(x,y))},$$

when $\rho(x,y) > t$, and then the proof goes the same as in Lemma 2.5.

Now, let us check if the restriction of the operator T to the space X, the operator T_0, has its maximal singular integral operator bounded. Property (2.4) implies

$$|T_{0,t}f(x)| \leq |Tf(x,t)| + BHf(x,t) \qquad \text{for every} \quad x \in X, t \geq 0.$$

When we take the supremum of both sides of the last inequality, with respect to $t \geq 0$ we obtain

$$T_0^{\#} f(x) \leq \sup_{t \geq 0} |Tf(x,t)| + BMf(x),$$

where M, represents the Hardy-Littlewood maximal function on S. Thus

$$T_0^{\#} f(x) \leq N_\alpha(Tf)(x) + BMf(x),$$

where N_α represents the non-tangential maximal function on the unit ball. Theorem 6.3.1 (page 99, [17]), tells us that the operator $N_\alpha(T)$ is an L^p-bounded operator for every $p > 1$, and the L^p-boundedness of the Hardy-Littlewood maximal function is a well known fact. Therefore the operator

$$T_0^{\#} : L^p(S, d\sigma) \to L^p(S, d\sigma),$$

is bounded for every $p > 1$. Hence, the operator

$$T : L^p(X, d\sigma) \to L^p(X^+, d\mu),$$

where μ is a Carleson measure on the closed unit ball, $\bar{U}$ in $\mathbb{C}^n$ $(n > 1)$, is bounded for every $p > 1$ and weak-type 1-1 bounded.

2.7 Cauchy-Szegö Kernel on the Heisenberg Group

Let $X = \mathbb{R} \times \mathbb{C}^{n-1}$, and let σ be Lebesgue measure on $\mathbb{R}^{2n-1}$. If we write an element $x \in X$ as $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}$, and $x_2 \in \mathbb{C}^{n-1}$, then for $x, y \in X$ $(y = (y_1, y_2))$ we define operation $\circ$ on X as

$$x \circ y = (x_1 + y_1 + 2\Im\langle x_2, y_2 \rangle, x_2 + y_2),$$

where $\langle x_2, y_2 \rangle$ denotes the scalar product in $\mathbb{C}^{n-1}$. $(X, \circ)$ is a group, with $(0,0)$ being the neutral element, and $(-x_1, -x_2)$ the inverse to (x_1, x_2). We define the pseudometric, ρ on this group by

$$\rho(x,y) = \gamma(x \circ y^{-1}) \qquad \text{where} \qquad \gamma(x) = \left(|x_1|^2 + |x_2|^4\right)^{1/2}.$$

The pseudometric ρ is invariant under the group action which means, that for every $x, y, z \in X$, we have $\rho(x,y) = \rho(x \circ z, y \circ z)$.

It has been proved, see [14], that

$$\sigma(B(x,r)) \asymp r^n \qquad \text{where} \qquad B(x,r) = \{y \in X : \rho(x,y) < r\} \tag{2.10}$$

We define the *Cauchy-Szegö Kernel* on X by

$$C_t(x,y) = c_n \left(t + |x_2 - y_2|^2 - i(x_1 - y_1 - 2\Im\langle x_2, y_2\rangle)\right)^{-n},$$

for every $x, y \in X$ and $t \geq 0$. The Cauchy-Szegö Kernel is invariant under the group action, which means that for every $x, y, z \in X$ and $t \geq 0$, we have

$$C_t(x,y) = C_t(x \circ z, y \circ z).$$

Let us prove that the integral operator defined by the Cauchy-Szegö kernel is a singular integral operator on X^+. First, we check if there exists a positive constant C such that

$$C_t(x,y) \leq \frac{C}{\sigma(B(x, \rho(x,y) + t))},$$

for every $x, y \in X$ and $t \geq 0$.

Knowing that the Cauchy-Szegö Kernel is invariant under the group action, it suffices to prove the claim for $y = 0$.

$$\begin{aligned}
|C_t(x,0)| &= c_n \left|t + |x_2|^2 + ix_1\right|^{-n} = c_n \left|(t + |x_2|^2)^2 + x_1^2\right|^{-n/2} \\
&\leq c_n \left|t^2 + |x_2|^4 + x_1^2\right|^{-n/2} = c_n \left|t^2 + \gamma(x)^2\right|^{-n/2} \\
&\leq 2^n c_n |t + \gamma(x)|^{-n} = C\sigma(B(x, \rho(x,0) + t))^{-1},
\end{aligned}$$

the last equation holds due to formula (2.10).

Next, we verify that

$$|C_t(x,z) - C_t(y,z)| \leq C \frac{\rho(x,y)^{1/2}}{\sigma(B(x,\rho(x,z)+t))(\rho(x,z)+t)^{1/2}}, \tag{2.11}$$

for every $x, y, z \in X$ and $t \geq 0$ such that $2\rho(x,y) \leq \rho(x,z) + t$. Again, using the invariance of the Cauchy-Szegö kernel under the group action, it suffices to prove the claim for $z = 0$. Then

$$|C_t(x,0) - C_t(y,0)| = c_n \left|(t + |x_2|^2 + ix_1)^{-n} - (t + |y_2|^2 + iy_1)^{-n}\right|.$$

As in the case of the Cauchy Kernel we obtain

$$|C_t(x,0) - C_t(y,0)| = c_n \left||y_2|^2 - |x_2|^2 - i(y_1 - x_1)\right| \times$$

$$\times \left|\sum_{k=0}^{n-1} (t + |x_2|^2 + ix_1)^{-(n-k)} (t + |y_2|^2)^{-(k+1)}\right|,$$

which leads to the estimate

$$|C_t(x,0) - C_t(y,0)| \leq C \left||y_2|^2 - |x_2|^2 - i(y_1 - x_1)\right| (\rho(x,0) + t)^{-(n+1)}.$$

Thus, in order to prove the formula (2.11) we need to prove

$$\left||y_2|^2 - |x_2|^2 - i(y_1 - x_1)\right| \leq C\rho(x,y)^{1/2}(\rho(x,0) + t)^{1/2}.$$

Simple computations yield

$$\left||y_2|^2 - |x_2|^2 - i(y_1 - x_1)\right| \leq \left||x_2|^2 - |y_2|^2\right| + |x_1 - y_1|.$$

$$\leq |x_2 - y_2|(|x_2| + |y_2|) + |x_1 - y_1 - 2\Im\langle x_2, y_2\rangle| + |2\Im\langle x_2, x_2 - y_2\rangle|.$$

(In the last line we have used the fact that $\Im\langle x_2, x_2\rangle = 0$.) Using that $|x_2 - y_2| \leq \rho(x,y)^{1/2} = (|x_2 - y_2|^4 - |x_1 - y_1 - 2\Im\langle x_2, y_2\rangle|^2)^{1/4}$,

$|x_1 - y_1 - 2\Im\langle x_2, y_2\rangle| \le \rho(x,y)$, $|x_2| \le (\rho(x,0)+t)^{1/2}$, $|y_2| \le (\rho(y,0)+t)^{1/2}$, and the Cauchy-Schwarz inequality we get

$$\Big||y_2|^2 - |x_2|^2 - i(y_1 - x_1)\Big| \le C\rho(x,y)^{1/2}\left[(\rho(x,0)+t)^{1/2} + (\rho(y,0)+t)^{1/2}\right] +$$

$$+\rho(x,y) + 2|x_2||x_2 - y_2|,$$

and using the same estimates on the last term on the right we have that

$$\le C\rho(x,y)^{1/2}\left((\rho(x,0)+t)^{1/2} + (\rho(y,0)+t)^{1/2}\right) + \rho(x,y) + 2\rho(x,y)^{1/2}(\rho(x,0)+t)^{1/2}.$$

Notice that $2\rho(x,y) \le \rho(x,0)+t$ implies $\rho(x,y)^{1/2} \le (\frac{1}{2}(\rho(x,0)+t))^{1/2}$, and $\rho(y,0) + t \le \frac{3}{2}(\rho(x,0)+t)$, which altogether gives

$$\le C\rho(x,y)^{1/2}(\rho(x,0)+t)^{1/2},$$

what we wanted to prove.

To check if T satisfies the property

$$|Tf(x,t) - T_{0,t}f(x)| \le BHf(x,t),$$

for every $x \in X$ and $t \ge 0$, we notice

$$|C_t(x,0) - C_0(x,0)| \le c_n \frac{|(t+|x_2|^2 + ix_1)^n - (|x_2|^2 + ix_1)^n|}{(t+\rho(x,0))^n \rho(x,0)^n}.$$

The binomial formula and elementary computations imply that

$$\frac{(t+|x_2|^2 + ix_1)^n - (|x_2|^2 + ix_1)^n}{\rho(x,0)^n} \le \frac{Ct}{\rho(x,0)},$$

whenever $\rho(x,y) > t$, and then the proof goes the same as in Lemma 2.5.

Thus, we have proved that the convolution operator associated with the Cauchy-Szegö kernel is a singular integral operator.

Chapter 3

Singular Integral Operators on Euclidean Space

In this chapter we study under what conditions imposed on the Carleson measure μ and the kernel K, the singular integral operator

$$T : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^{n+1}_+, d\mu),$$

is bounded. Euclidean space, $\mathbb{R}^n$ is a space of homogeneous type with the quasi-metric being Euclidean distance, and Lebesgue measure being the doubling measure.

The kernel K is still assumed to be a continuous map

$$K : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \setminus \{x = y, t = 0\} \to \mathbb{C},$$

satisfying conditions (2.2) and (2.3). Additionally, we assume that the kernel K satisfies the following two cancelation properties.

$$\int_{B(x,R)\setminus B(x,r)} K(x, y, t)dy = 0, \tag{3.1}$$

for every $R > r > 0$, $t \geq 0$, and $x \in \mathbb{R}^n$; and

$$\int_{S(B(y,R))\setminus S(B(y,\rho))} K(x, y, t)\overline{K(x', z, t)}d\mu(x, t) = 0, \tag{3.2}$$

for every $y \in \mathbb{R}^n$, $|x' - z| > \rho$, $t \geq 0$, and $R > \rho > 0$. The first cancelation property is supposed to play the role of the $T1$-condition together with the weak boundedness property, while the second plays the role of T^*1-condition, that requires $T^*(const) = 0$, but now we ask for more. $T^*\varphi = 0$, for the function $\varphi = Ta(z,t)$, for a certain type of functions $a \in C_0^\infty(\mathbb{R}^n)$. Notice that the function φ is constant in x, but not in t; (see [10] or [8]) and when μ is the projection measure, the condition (3.2) becomes the condition (3.1) in variable y.

The operator T is defined by the formula

$$Tf(x,t) = \int_{\mathbb{R}^n} K(x,y,t)f(y)dy, \qquad \text{for every} \quad x \in \mathbb{R}^n, t \geq 0 \tag{3.3}$$

and $f \in C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ represents the set of all infinitely differentiable functions with compact support in $\mathbb{R}^n$. The integral in (3.3) makes sense for any $f \in L^1(\mathbb{R}^n)$ when $t > 0$ or $|x-y| > 0$, due to (2.2). When it is not the case, we define $Tf(x,0)$ as in the previous chapter.

Using the Calderón formula, whose proof is based on the properties of the Fourier transform on $\mathbb{R}^n$, it is established that any function $f \in L^2(\mathbb{R}^n, dx)$ can be decomposed into the sum of the smooth atoms (see [10]). We define a smooth atom as follows.

Definition: *A function $a_Q \in C_0^\infty(\mathbb{R}^n)$ is a smooth atom in $L^2(\mathbb{R}^n, dx)$ associated with the dyadic cube Q if it satisfies:*

(a) a_Q is supported inside the cube Q^*, (Recall that Q^* denotes the cube concentric with the cube Q such that $\ell(Q^*) = 3\ell(Q)$.),

(b) $\int a_Q = 0$,

(c) $|D^\gamma a_Q(x)| \leq c_\gamma \ell(Q)^{-|\gamma|-n/2}$ for every multi-index γ, and $x \in \mathbb{R}^n$.

The following theorem (see [10]) tells us that any function $f \in L^2(\mathbb{R}^n, dx)$ can be written as an infinite linear combination of smooth atoms.

Theorem 3.1 *For any $f \in L^2(\mathbb{R}^n, dx)$, there exists a sequence $\{s_Q\} \in \ell^2$, and a sequence of smooth atoms $\{a_Q\}$, such that*

$$f = \sum_Q s_Q a_Q \qquad \text{and} \qquad \|f\|^2_{L^2(\mathbb{R}^n,dx)} = C \sum_Q |s_Q|^2,$$

where the convergence of the series is taken in the L^2-sense, and the summation is over the family of all dyadic cubes in $\mathbb{R}^n$.

For the proof see [10].

Now, we are ready to prove the main result of this chapter.

Theorem 3.2 *Let μ be a Carleson measure, and let T be a linear operator represented by the formula (3.3), that satisfies conditions (2.2), (2.3), and the cancelation properties (3.1), and (3.2). Then the operator*

$$T : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^{n+1}_+, d\mu),$$

is bounded.

Proof: The proof of this theorem is an adaptation of the technique developed in [10], applied to this new situation. The proof of the theorem is rather technical, and it will use several lemmas. Before we state and start proving the lemmas, let us fix dyadic cubes P and Q, let x_P and x_Q denote the center of the cube P and Q, respectively, and let a_P and a_Q be smooth atoms associated with cubes P and Q.

Lemma 3.3 *Let μ be a Carleson measure, and $\epsilon > 0$. Then there exists a constant $C > 0$ depending only on μ and ϵ, so that for every cube Q, and every point $z \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^{n+1}_+} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} \leq C|Q|.$$

Proof of Lemma 3.3: Let

$$A_k = \{(x,t) \in \mathbb{R}^{n+1}_+ : 2^{k-1}\ell(Q) < |x-z| + t \le 2^k \ell(Q)\},$$

for $k = 1, 2, \ldots$. Then, $\mathbb{R}^{n+1}_+$ is a disjoint union of the sets $\{A_k\}_{k=0}^{\infty}$, where $A_0 = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x-z| + t \le \ell(Q)\}$. Thus

$$\int_{\mathbb{R}^{n+1}_+} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} = \sum_{k=0}^{\infty} \int_{A_k} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}}.$$

On A_0, we use the estimate $\frac{1}{(1+\frac{|x-z|+t}{\ell(Q)})^{n+\epsilon}} \le 1$ to get

$$\int_{A_0} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} \le \mu(A_0).$$

Since μ is a Carleson measure, we conclude that

$$\int_{A_0} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} \le C_\mu |Q|.$$

On each A_k, $(k = 1, 2, \ldots)$, the estimate

$$\frac{1}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} \le \frac{1}{\left(\frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}},$$

the fact that $A_k \subset T(B(z, 2^k\ell(Q)))$, and $\frac{|x-z|+t}{\ell(Q)} > 2^{k-1}$ when $(x,t) \in A_k$, imply

$$\int_{\mathbb{R}^{n+1}_+ \setminus T(B)} \frac{d\mu(x,t)}{\left(1 + \frac{|x-z|+t}{\ell(Q)}\right)^{n+\epsilon}} \le \sum_{k=1}^{\infty} \int_{A_k} \frac{d\mu(x,t)}{(2^{k-1})^{n+\epsilon}}$$

$$= C\ell(Q)^n \sum_{k=1}^{\infty} \frac{\mu(T(B(z, 2^k\ell(Q))))}{|B(z, 2^k\ell(Q))| 2^{k\epsilon}}.$$

Using the fact that μ is a Carleson measure, $\ell(Q)^n = |Q|$, and that the series $\sum_{k=1}^{\infty} 2^{-k\epsilon}$ converges, we conclude that the last line is dominated by

$$CC_\mu |Q|,$$

which proves the lemma.

■

Lemma 3.4

$$|Ta_Q(x,t)| \le C\ell(Q)^{-n/2}\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} \qquad \textit{for every } (x,t) \in \mathbb{R}^{n+1}_+.$$

Proof of Lemma 3.4: Let $(x,t) \in S(2Q^*)$. In this case we have

$$1+\frac{|x-x_Q|+t}{\ell(Q)} \le 11,$$

and consequently

$$\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} \ge 11^{-n-\epsilon}.$$

Using the cancelation property (3.1) we obtain

$$|Ta_Q(x,t)| = \left|\int_{|x-y|\le c\ell(Q)} K(x,y,t)(a_Q(y)-a_Q(x))dy\right|,$$

where c is such a constant that $supp(a_Q) \subset B(x, \frac{c}{2}\ell(Q))$. (More precisely, we can set $c = 6\sqrt{n}$.) The inequality (2.2), and the smoothness of the atom a_Q, yield

$$|Ta_Q(x,t)| \le C\ell(Q)^{-1-n/2}\int_{|x-y|\le c\ell(Q)} (|x-y|+t)^{-n}|x-y|dy,$$

$$\le C\ell(Q)^{-1-n/2}\int_0^{c\ell(Q)} r^{-n} r r^{n-1} dr = C\ell(Q)^{-n/2},$$

which together with the previous inequality implies the lemma when $(x,t) \in S(2Q^*)$.

If $(x,t) \in \mathbb{R}^{n+1}_+ \setminus S(2Q^*)$, then the fact that $\int a_Q = 0$ implies

$$|Ta_Q(x,t)| = \left|\int_{Q^*}[K(x,y,t) - K(x,x_Q,t)]a_Q(y)dy\right|.$$

Using property (2.3) (Notice that when $y \in Q^*$ we have $2|y-x_Q| \le |x-x_Q|+t$.) and the fact that $|a_Q| \le C|Q|^{-1/2}$, we get

$$|Ta_Q(x,t)| \le C\int_{Q^*} \frac{|y-x_Q|^\epsilon}{(|x-x_Q|+t)^{n+\epsilon}}|Q|^{-1/2}dy,$$

$$\leq C\ell(Q)^{\epsilon}|Q|^{-1/2}(|x-x_Q|+t)^{-n-\epsilon}|Q|,$$

and now we use the fact that $\frac{|x-x_Q|+t}{\ell(Q)} \geq 1$ to conclude

$$|Ta_Q(x,t)| \leq C|Q|^{-1/2}\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon},$$

which proves the lemma.

■

Lemma 3.5 *Let* $\alpha = \min\{\epsilon/2, 1\}$. *Then*

$$|Ta_Q(x,t) - Ta_Q(y,t)| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha} \times$$

$$\times\left\{\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} + \left(1+\frac{|y-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon}\right\}$$

for every $x, y \in \mathbb{R}^n$, *and* $t \geq 0$.

Proof of Lemma 3.5: If $|x-y| > \ell(Q)$, then Lemma 3.4 applied to each term on the left side of the inequality, and the fact that $1 < (\frac{|x-y|}{\ell(Q)})^{\alpha}$, imply the lemma.

Therefore, let $|x-y| < \ell(Q)$.

If $(x,t) \in \mathbb{R}^{n+1}_+ \backslash S(cQ)$, where $c = 4\sqrt{n}$ is chosen so that $supp(a_Q) \subset B(x_Q, \frac{c}{2}\ell(Q))$. (Notice that we could have taken $c = 3\sqrt{n}$, but we may need c to be a little bit larger.) We have

$$|Ta_Q(x,t) - Ta_Q(y,t)| = \left|\int_{Q^*}[K(x,z,t) - K(y,z,t)]a_Q(z)dz\right|.$$

For chosen x, y and t we have that $|x-y| \leq \ell(Q) \leq \frac{1}{2}(|x-z|+t)$ is true for every $z \in Q^*$, which enables us to use property (2.3), which together with the fact $|a_Q| \leq C|Q|^{-1/2}$ implies that

$$|Ta_Q(x,t) - Ta_Q(y,t)| \leq C\int_{Q^*}\frac{|x-y|^{\epsilon}}{(|x-z|+t)^{n+\epsilon}}|Q|^{-1/2}dz.$$

For every $z \in Q^*$ we have $|x - x_Q| + t \leq |x - z| + |z - x_Q| + t \leq 2(|x - z| + t)$ (The last inequality holds because if $(x,t) \in \mathbb{R}^{n+1}_+ \setminus S(cQ)$, then either $|z - x_Q| \leq |x - z|$ or $t \geq |z - x_Q|$.) which implies that the above quantity is

$$\leq C|Q|^{1/2}|x-y|^{\epsilon}(|x-x_Q|+t)^{-n-\epsilon},$$

$$= C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\epsilon}\left(\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon}.$$

Since $\frac{|x-y|}{\ell(Q)} < 1$ and $\alpha < \epsilon$, we have

$$|Ta_Q(x,t) - Ta_Q(y,t)| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}\left(\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon}.$$

The fact $|x - x_Q| + t \geq \ell(Q)$ yields

$$\frac{|x-x_Q|+t}{\ell(Q)} \geq \frac{1}{2}\left(\frac{|x-x_Q|+t}{\ell(Q)}+1\right) \quad \text{and} \quad \frac{|x-x_Q|+t}{\ell(Q)} \geq \frac{1}{3}\left(\frac{|y-x_Q|+t}{\ell(Q)}+1\right),$$

(The last inequality is true since $1 + \frac{|y-x_Q|+t}{\ell(Q)} \leq 1 + \frac{|x-x_Q|+|x-y|+t}{\ell(Q)} \leq 2 + \frac{|x-x_Q|+t}{\ell(Q)} \leq 3\frac{|x-x_Q|+t}{\ell(Q)}$.); so we conclude that

$$\left(\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} \leq C\left\{\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} + \left(1+\frac{|y-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon}\right\},$$

which proves the lemma if $(x,t) \in \mathbb{R}^{n+1}_+ \setminus S(cQ)$.

In the case $|x - y| < \ell(Q)$, and $(x,t) \in S(cQ)$, we have that $|x - x_Q| \leq \frac{c}{2}\ell(Q)$, $|y - x_Q| \leq (\frac{c}{2} + 1)\ell(Q)$, and $t \leq c\ell(Q)$. So, it is easy to conclude that there exists a positive constant $\delta = (1 + \frac{c}{2})^{-n-\epsilon} + (2 + \frac{c}{2})^{-n-\epsilon}$ such that

$$\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} + \left(1+\frac{|y-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon} \geq \delta.$$

So all we need to prove is

$$|Ta_Q(x,t) - Ta_Q(y,t)| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

Due to the fact $supp(a_Q) \subset Q^* \subset B(x, c\ell(Q))$, we can write

$$Ta_Q(x,t) - Ta_Q(y,t) = \int_{|x-z|\le c\ell(Q)} [K(x,z,t) - K(y,z,t)]a_Q(z)dz,$$

and employing the cancelation property (3.1) we can write the integral above as the sum

$$Ta_Q(x,t) - Ta_Q(y,t) = \int_{|x-z|\le 3|x-y|} K(x,z,t)(a_Q(z) - a_Q(x))dz$$
$$-\int_{|x-z|\le 3|x-y|} K(y,z,t)(a_Q(z) - a_Q(y))dz +$$
$$\int_{3|x-y|\le|x-z|\le c\ell(Q)} [K(x,z,t) - K(y,z,t)](a_Q(z) - a_Q(y))dz$$
$$-a_Q(y)\int_{|x-z|\le c\ell(Q)} K(y,z,t)dz = I + II + III + IV.$$

To estimate the term I, we use the inequality (2.2), which together with the smoothness of the atom a_Q, implies

$$|I| \le C\ell(Q)^{-1-n/2}\int_{|z-x|\le 3|x-y|} \frac{|z-x|}{(|x-z|+t)^n}dz$$

$$= C|Q|^{-1/2}\ell(Q)^{-1}\int_0^{3|x-y|} dr = C|Q|^{-1/2}\frac{|x-y|}{\ell(Q)}.$$

Having in mind that $\frac{|x-y|}{\ell(Q)} < 1$, and $\alpha \le 1$, we conclude

$$|I| \le C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

The term II is similar to the term I, now we have $|y-z| \le |x-y|+|z-x| \le 4|x-y|$ which yields a different constant C. Thus

$$|II| \le C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

In order to estimate the term III, we notice that $|z-y| \ge |x-z|-|x-y| \ge 2|x-y|$. Therefore we can use the estimate (2.3) on the kernel K, which with the smoothness of the atom a_Q produces

$$|III| \le C\ell(Q)^{-1-n/2}\int_{3|x-y|<|x-z|\le c\ell(Q)} \frac{|x-y|^{\epsilon}}{(|y-z|+t)^{n+\epsilon}}|y-z|dz$$

(Notice that $c\ell(Q) \geq 3|x-y|$, because $|x-y| \leq \ell(Q)$ and $c = 4\sqrt{n}$.)

$$\leq C|Q|^{-1/2}\frac{|x-y|^{\epsilon}}{\ell(Q)}\int_{2|x-y|}^{c\ell(Q)} r^{-\epsilon}dr.$$

If $\epsilon > 1$, we estimate the integral in the last formula in the following way

$$\int_{2|x-y|}^{c\ell(Q)} r^{-\epsilon}dr \leq \int_{2|x-y|}^{\infty} r^{-\epsilon}dr = C|x-y|^{1-\epsilon},$$

which implies

$$|III| \leq C|Q|^{-1/2}\frac{|x-y|^{\epsilon}|x-y|^{1-\epsilon}}{\ell(Q)} = C|Q|^{-1/2}\frac{|x-y|}{\ell(Q)},$$

and again, having in mind that $\frac{|x-y|}{\ell(Q)} < 1$, and $\alpha \leq 1$, we conclude

$$|III| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

If $\epsilon = 1$, then $\alpha = 1/2$, so the estimate $\ln x \leq \sqrt{x}$ when $x > 1$, yields

$$\int_{2|x-y|}^{c\ell(Q)} r^{-1}dr = C\ln\left(\frac{c\ell(Q)}{2|x-y|}\right) \leq C\sqrt{\frac{\ell(Q)}{|x-y|}},$$

and

$$|III| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

If $\epsilon < 1$, we have

$$\int_{2|x-y|}^{c\ell(Q)} r^{-\epsilon}dr \leq C\ell(Q)^{1-\epsilon},$$

which yields

$$|III| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\epsilon}.$$

Having in mind that $\frac{|x-y|}{\ell(Q)} < 1$, and $\alpha < \epsilon$, we obtain

$$|III| \leq C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha}.$$

Finally, we estimate the term IV. The cancelation property (3.1) implies

$$|IV| = |a_Q(y)|\left|\int_{|x-z|\leq c\ell(Q)} K(y,z,t)dz - \int_{|y-z|\leq c\ell(Q)} K(y,z,t)dz\right|,$$

$$\le C|Q|^{-1/2}\left|\int_{\Delta} K(y,z,t)dz\right|,$$

where $\Delta = \Delta_1 \cup \Delta_2 = \{z \in \mathbb{R}^n : |z-x| \le c\ell(Q) \text{ and } |z-y| > c\ell(Q)\} \cup \{z \in \mathbb{R}^n : |z-x| > c\ell(Q) \text{ and } |z-y| \le c\ell(Q)\}$. The estimate (2.2) implies

$$\left|\int_{\Delta_1} K(y,z,t)dz\right| \le C\int_{c\ell(Q)}^{c\ell(Q)+|x-y|} r^{-1}dr$$

$$= C\ln\left(1+\frac{|x-y|}{c\ell(Q)}\right) \le C\frac{|x-y|}{\ell(Q)},$$

because $c > 1$. Similarly

$$\left|\int_{\Delta_2} K(y,z,t)dz\right| \le C\int_{c\ell(Q)-|x-y|}^{c\ell(Q)} r^{-1}dr$$

$$= C\ln\left(1+\frac{|x-y|}{c\ell(Q)-|x-y|}\right) \le C\frac{|x-y|}{\ell(Q)},$$

because $c\ell(Q) - |x-y| \ge \ell(Q)$. When we put all of those estimates together we get

$$|IV| \le C|Q|^{-1/2}\frac{|x-y|}{\ell(Q)},$$

which due to the facts that $\alpha < 1$ and $\frac{|x-y|}{\ell(Q)} < 1$ yields:

$$|IV| \le C|Q|^{-1/2}\left(\frac{|x-y|}{\ell(Q)}\right)^{\alpha},$$

which completes the proof of the lemma.

■

The following lemma is the key ingredient in the proof of Theorem 3.2.

Lemma 3.6 *Let $\alpha = \min\{\epsilon/2, 1\}$, $\ell(P) \le \ell(Q)$, and let μ be a Carleson measure on $\mathbb{R}^{n+1}_+$. Then*

$$\left|\int Ta_P(x,t)\overline{Ta_Q(x,y)}d\mu\right| \le C\left(\frac{\ell(P)}{\ell(Q)}\right)^{n/2+\alpha}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}.$$

Proof of Lemma 3.6: The property (3.2) implies

$$\int Ta_P(x,t)\overline{Ta_Q(x,t)}d\mu = \int Ta_P(x,t)(\overline{Ta_Q(x,t) - Ta_Q(x_P,t)})d\mu.$$

(To see that the property (3.2) implies that $\int Ta_P(x,t)\overline{Ta_Q(x_P,t)}d\mu = 0$, we notice that Lemma 3.3 and Lemma 3.4 imply $\int |Ta|^2 d\mu \leq C$, where the constant C does not depend on the atom a. If we define $T_\rho a(x,t) = \int K_\rho(x,y,t)a(y)dy$, where $K_\rho(x,y,t) = K(x,y,t)\chi_{S(B(y,\frac{1}{\rho}))\setminus S(B(y,\rho))}$, we have that $T_\rho a \to Ta$ as $\rho \to 0$, where the convergence is in the $L^2(d\mu)$-sense. Thus, to prove the claim it suffices to show that for every $\rho > 0$ we have $\int T_\rho a_P(x,t)\overline{T_\rho a_Q(x_P,t)}d\mu = 0$. It follows directly from the condition (3.2) and the Fubini Theorem, since all the functions K_ρ, a_P, and a_Q are bounded over a compact set.)

Let $A = B(x_P, 3\ell(Q))$. If $x \in A$, then

$$|x_P - x_Q| \leq |x - x_Q| + |x - x_P| \leq |x - x_Q| + 3\ell(Q) + t,$$

for every $t \geq 0$. Hence

$$c\left(1 + \frac{|x - x_Q| + t}{\ell(Q)}\right)^{-n-\epsilon} \leq \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon}.$$

Lemma 3.4, applied to the first factor in the integral below, Lemma 3.5 applied to the second factor in the integral below, together with the simple fact $(1+\frac{|x_P-x_Q|+t}{\ell(Q)})^{-n-\epsilon} \leq (1 + \frac{|x_P-x_Q|}{\ell(Q)})^{-n-\epsilon}$ and the above estimate, yield

$$\left|\int_{S(A)} Ta_P(x,t)(\overline{Ta_Q(x,t) - Ta_Q(x_P,t)})d\mu\right| \leq C|P|^{-1/2}\times$$

$$\times \int_{S(A)} \left(1 + \frac{|x - x_P| + t}{\ell(P)}\right)^{-n-\epsilon} |Q|^{-1/2} \left(\frac{|x - x_P|}{\ell(Q)}\right)^{\alpha} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon} d\mu.$$

Elementary calculations yield

$$\leq C|P|^{-1/2}|Q|^{-1/2}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}\left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha}\times$$

$$\times\int_{\mathbb{R}^{n+1}_+}\left(1+\frac{|x-x_P|+t}{\ell(P)}\right)^{-n-\epsilon}\left(\frac{|x-x_P|}{\ell(P)}\right)^{\alpha}d\mu.$$

The estimate $(\frac{|x-x_P|}{\ell(P)})^\alpha \leq (1+\frac{|x-x_P|+t}{\ell(P)})^\alpha$ leads to

$$\leq C|P|^{-1/2}|Q|^{-1/2}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}\left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha}\times$$

$$\times\int_{\mathbb{R}^{n+1}_+}\left(1+\frac{|x-x_P|+t}{\ell(P)}\right)^{-n-\epsilon+\alpha}d\mu.$$

Since $\alpha < \epsilon$, Lemma 3.3 applied to the integral above, implies

$$\leq C|P|^{-1/2}|Q|^{-1/2}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}\left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha}|P|,$$

which proves the lemma in the case $(x,t)\in S(A)$.

In the case $(x,t)\in\mathbb{R}^{n+1}_+\setminus S(A)$, which means $|x-x_P|+t>3\ell(Q)$, we have

$$1+\frac{|x-x_P|+t}{\ell(P)}>\frac{|x-x_P|+t}{\ell(Q)}\frac{\ell(Q)}{\ell(P)}>\frac{1}{2}\frac{|x-x_P|+t}{\ell(Q)}\frac{\ell(Q)}{\ell(P)}+\frac{3}{2}\frac{\ell(Q)}{\ell(P)},$$

which implies

$$1+\frac{|x-x_P|+t}{\ell(P)}>\frac{1}{2}\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)\frac{\ell(Q)}{\ell(P)}.$$

Using the last inequality and Lemma 3.4, (having in mind that $(x,t)\in\mathbb{R}^{n+1}_+\setminus S(A)$) we obtain

$$|Ta_P(x,t)|\leq C|P|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon}\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)^{-n-\epsilon}. \tag{3.4}$$

Thus

$$\left|\int_{\mathbb{R}^{n+1}_+\setminus S(A)}Ta_P(x,t)(\overline{Ta_Q(x,y)-Ta_Q(x_P,t)})d\mu\right|\leq$$

$$\left|\int_{\mathbb{R}^{n+1}_+\setminus S(A)}Ta_P(x,t)\overline{Ta_Q(x,t)}d\mu\right|+\left|\int_{\mathbb{R}^{n+1}_+\setminus S(A)}Ta_P(x,t)\overline{Ta_Q(x_P,t)}d\mu\right|=I+II.$$

Let us estimate II first. The inequality (3.4), applied to the first factor in the integral II, and Lemma 3.4 applied to the second factor in the integral II, yield

$$II \leq C|P|^{-1/2}|Q|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}\times$$

$$\times\int\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)^{-n-\epsilon}d\mu.$$

Lemma 3.3 applied to the integral above produces

$$II \leq C|P|^{-1/2}|Q|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}|Q|.$$

Since $\alpha < \epsilon$ and $\frac{\ell(P)}{\ell(Q)} \leq 1$, we have

$$II \leq C\left(\frac{\ell(P)}{\ell(Q)}\right)^{n/2+\alpha}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon},$$

which is the desired estimate for II.

In order to estimate I, we need to consider the sets

$$B = \{(x,t)\in\mathbb{R}^{n+1}_+\setminus S(A) : 2(|x-x_P|+t) > |x_P-x_Q|\} \qquad \text{and}$$

$$C = \{(x,t)\in\mathbb{R}^{n+1}_+\setminus S(A) : 2(|x-x_P|+t) \leq |x_P-x_Q|\}.$$

The inequality (3.4), applied to the first factor in the integral bellow, and Lemma 3.4, applied to the second factor in the integral bellow, yield

$$\left|\int_B Ta_P(x,t)\overline{Ta_Q(x,t)}d\mu\right| \leq C|P|^{-1/2}|Q|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon}\times$$

$$\times\int\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)^{-n-\epsilon}\left(1+\frac{|x-x_Q|+t}{\ell(Q)}\right)^{-n-\epsilon}d\mu,$$

Since $(x,t)\in B$, we have

$$\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)^{-n-\epsilon} \leq C\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon},$$

which together with Lemma 3.3 implies

$$\left|\int_B Ta_P(x,t)\overline{Ta_Q(x,t)}d\mu\right| \le C\left(\frac{\ell(P)}{\ell(Q)}\right)^{n/2+\epsilon}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}.$$

Again, we use the facts that $\alpha < \epsilon$ and $\frac{\ell(P)}{\ell(Q)} \le 1$ to get the desired estimate for the integral in I over the set B.

If $(x,t) \in C$, then $|x-x_Q|+t \ge |x_P-x_Q| - (|x-x_P|+t) \ge \frac{1}{2}|x_P-x_Q|$. This fact, inequality (3.4), applied to the first factor in the integral below, and Lemma 3.4, applied to the second factor in the integral below, yield

$$\left|\int_C Ta_P(x,t)\overline{Ta_Q(x,t)}d\mu\right| \le C|P|^{-1/2}|Q|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon} \times$$

$$\times\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}\int_C\left(1+\frac{|x-x_P|+t}{\ell(Q)}\right)^{-n-\epsilon}d\mu,$$

and Lemma 3.3 implies

$$\le C|P|^{-1/2}|Q|^{-1/2}\left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\epsilon}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}|Q|,$$

which, as in two previous cases, produces the desired estimate, which proves the lemma.

■

Proof of Theorem 3.2: Let $f \in L^2(\mathbb{R}^n, dx)$ and let

$$f = \sum_P s_P a_P$$

be its atomic decomposition (The convergence of the series is in a L^2-sense.), where the summation goes over all dyadic cubes in $\mathbb{R}^n$. Then, Lemma 3.6 implies:

$$\int |Tf|^2 d\mu = \int (\sum_P s_P Ta_P)\overline{(\sum_Q s_Q Ta_Q)}d\mu$$

$$\le 2\sum_{\ell(P)\le\ell(Q)} |s_P||s_Q|\left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha+n/2}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-n-\epsilon}$$

$$= 2 \sum_{\ell(P)\leq\ell(Q)} \left\{ |s_P| \left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha/2} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-(n+\epsilon)/2} \right\} \times$$

$$\times \left\{ |s_Q| \left(\frac{\ell(P)}{\ell(Q)}\right)^{(n+\alpha)/2} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-(n+\epsilon)/2} \right\}.$$

Applying the Cauchy-Schwarz inequality we obtain

$$\int |Tf|^2 d\mu \leq 2 \left\{ \sum_{\ell(P)\leq\ell(Q)} |s_P|^2 \left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon} \right\}^{\frac{1}{2}} \times$$

$$\times \left\{ \sum_{\ell(P)\leq\ell(Q)} |s_Q|^2 \left(\frac{\ell(P)}{\ell(Q)}\right)^{n+\alpha} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon} \right\}^{\frac{1}{2}} = 2A^{\frac{1}{2}}B^{\frac{1}{2}}.$$

So, let us estimate the factor A first. We can write:

$$A = \sum_{i\in\mathbb{Z}} \sum_{\ell(P)=2^{-i}} |s_P|^2 \sum_{j=-\infty}^{i} 2^{(j-i)\alpha} \sum_{\ell(Q)=2^{-j}} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon}.$$

Notice that the function

$$g(x) = \sum_{\ell(Q)=2^{-j}} \left(1 + \frac{|x - x_Q|}{\ell(Q)}\right)^{-n-\epsilon}$$

is periodic with the period $2^j e$, where e is any of the canonical basis vectors $(1,0,0,...)$, $(0,1,0,...)$, ... in $\mathbb{Z}^n$. Thus, without loss of generality we can assume that $x \in Q_{j,0} = \{x : 0 \leq x_j \leq 2^{-j}\}$. Then we have

$$\sum_{\ell(Q)=2^{-j}} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon} \leq C \sum_{k\in\mathbb{Z}^n} (1 + |k|)^{-n-\epsilon} \leq C.$$

The last estimate implies

$$A \leq C \sum_{i\in\mathbb{Z}} \sum_{\ell(P)=2^{-i}} |s_P|^2 \sum_{j=-\infty}^{i} 2^{(j-i)\alpha} = C \sum_P |s_P|^2 = C\|f\|^2_{L^2(\mathbb{R}^n,dx)}.$$

In order to estimate the factor B, let us write B as

$$B = \sum_Q |s_Q|^2 \sum_{i=0}^{\infty} 2^{-i(n+\alpha)} \sum_{P:\ell(P)=2^{-i}\ell(Q)} \left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-n-\epsilon}.$$

Using the same argument as in the case of the factor A (the estimate on the function g), and taking into account the fact that there are 2^{ni} dyadic cubes $P \subset Q$ with $\ell(P) = 2^{-i}\ell(Q)$, we obtain

$$B \leq C \sum_Q |s_Q|^2 \sum_{i=0}^{\infty} 2^{-i(n+\alpha)} 2^{ni}$$

$$= C \sum_Q |s_Q|^2 \sum_{i=0}^{\infty} 2^{-i\alpha} = C \sum_Q |s_Q|^2 = C \|f\|^2_{L^2(\mathbb{R}^n, dx)},$$

which completes the proof of the theorem.

■

Notice that in each example in Chapter 2 the kernel K was odd. Thus condition (3.1) is clearly satisfied. Condition (3.2) is a statement that connects the Carleson measure, μ, and the kernel K. It is easy to see that when the kernel K is odd in x, and the measure μ is translation invariant in x, that condition (3.2) is satisfied.

Let $\varphi(t) = \overline{K(x', y, t)}$, for some fixed $x', y \in \mathbb{R}^n$. Then formally (assuming that all the integrals exist)

$$\int K(x,z,t)\varphi(t)d\mu(x,t) = \int_{\Pi^+} K(x,z,t)\varphi(t)d\mu(x,t) + \int_{\Pi^-} K(x,z,t)\varphi(t)d\mu(x,t),$$

where $\Pi^+ = \{x \in \mathbb{R}^n : x_1 > 0\}$, and $\Pi^- = \{x \in \mathbb{R}^n : x_1 < 0\}$. Notice that the mapping $\psi(x,t) = (-x,t)$ maps Π^+ into Π^-, and that for every set $E \subset \mathbb{R}^{n+1}_+$ we have $\mu(E) = \mu(\psi(E))$. That together with oddness of K in x implies that condition (3.2) is satisfied.

One example of such a Carleson measure is the measure defined by

$$\mu(E) = |E \cap \{t = t_0\}| \qquad \text{for every set } E \subset \mathbb{R}^{n+1}_+,$$

for any fixed $t_0 \geq 0$.

Chapter 4

Applications to Tent Spaces

In this chapter we are going to study the problem of boundedness of the singular integral operator

$$T : L^p(\mathbb{R}^n, dx) \to T^p_q(d\mu),$$

as defined in Chapter 3. The measure μ is a Carleson measure on $\mathbb{R}^{n+1}_+$, and the symbol $T^p_q(d\mu)$ denotes the tent space defined as follows.

Definition: *Let μ be a positive measure on $\mathbb{R}^{n+1}_+$, and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ a cone with vertex at the point $x \in \mathbb{R}^n$ and aperture 1. For $1 \leq q < \infty$ we set*

$$A_q f(x) = \left\{ \int_{\Gamma(x)} |f(y,t)|^q d\mu(x,t)/t^n \right\}^{1/q} \qquad x \in \mathbb{R}^n.$$

A function $f : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ belongs to the tent space $T^p_q(d\mu)$ if and only if $A_q f \in L^p(\mathbb{R}^n, dx)$, and the norm on $T^p_q(d\mu)$ is defined by

$$\|f\|_{T^p_q(d\mu)} = \|A_q f\|_{L^p(\mathbb{R}^n, dx)} \qquad \text{for} \quad 1 \leq p, q < \infty.$$

In case $q = \infty$, we define

$$A_\infty f(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)| \qquad x \in \mathbb{R}^n.$$

If the measure μ is defined by $d\mu = dxdt/t$, we obtain the tent spaces defined in [6]. We denote such tent spaces by T^p_q. A reader can find more about tent spaces in [6] or [19].

Lemma 1.5 implies that for any positive measure μ on $\mathbb{R}^{n+1}_+$, and $1 < p < \infty$ there is a constant $C > 0$ so that

$$\|Hf\|_{T^p_\infty(d\mu)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)} \qquad \text{for every} \quad f \in L^p(\mathbb{R}^n, dx).$$

As a consequence of this fact for any $1 < p < \infty$ and any admissible kernel K there is a constant $C > 0$ so that we have

$$\|K * f\|_{T^p_\infty(d\mu)} \leq C\|f\|_{L^p(\mathbb{R}^n,dx)} \qquad \text{for every} \quad f \in L^p(\mathbb{R}^n, dx).$$

The following theorem, due to F. J. Ruiz and J. L. Torrea, (see [19]), is the essential technical tool when applying the vector-valued versions of Theorem 2.1 and Theorem 2.4 to the tent spaces.

Theorem 4.1 *Let μ be a Carleson measure and T a convolution operator, associated with the kernel K, that satisfies the following two conditions.*

There exist constants $\alpha > 0$, and $C > 0$ so that

$$|K(x, y, t)| \leq \frac{Ct^\alpha}{(|x-y| + t)^{n+\alpha}} \tag{4.1}$$

for every $x, y \in \mathbb{R}^n$, with $x \neq y$, and $t \geq 0$. There exists $\epsilon > 0$ so that whenever $|x - y| + t > 2|y - y'|$ we have

$$|K(x,y,t) - K(x,y',t)| + |K(y,x,t) - K(y',x,t)| \leq \frac{C|y-y'|^\epsilon t^\alpha}{(|x-y'|+t)^{n+\epsilon+\alpha}}, \tag{4.2}$$

$x, y, y' \in \mathbb{R}^n$, and $t \geq 0$.

Let S be the operator defined by

$$Sf(x)(y,t) = Tf(y,t)\chi_{\Gamma(x)}(y,t).$$

Then the following statements are equivalent.

$$T : L^p(\mathbb{R}^n, dx) \to T_q^p(d\mu) \qquad \textit{is bounded} \tag{4.3}$$

$$S : L^p(\mathbb{R}^n, dx) \to L^p_{L^q(\mathbb{R}^{n+1}_+, d\mu/t^n)}(\mathbb{R}^n, dx) \qquad \textit{is bounded.} \tag{4.4}$$

Moreover, the operator S is a vector-valued singular integral operator whose (vector-valued) kernel $\tilde{K}$ is given by

$$\tilde{K}(x,z)(y,t) = K(y,z,t,)\chi_{\Gamma(x)}(y,t),$$

and satisfies the following two conditions. There exist positive constants C and ϵ so that

$$||\tilde{K}(x,z)||_{L^q(d\mu/t^n)} \le \frac{C}{|x-z|^n}, \tag{4.5}$$

for every $x, z \in \mathbb{R}^n$ with $x \neq z$; and whenever $2|z - z'| < |x - z|$ we have

$$||\tilde{K}(x,z) - \tilde{K}(x,z')||_{L^q(d\mu/t^n)} + ||\tilde{K}(z,x) - \tilde{K}(z',x)||_{L^q(d\mu/t^n)} \le \frac{C|z-z'|^\epsilon}{|x-z|^{n+\epsilon}}, \tag{4.6}$$

for every $x, z, z' \in \mathbb{R}^n$, with $x \neq z$.

Notice that the conditions (4.1) and (4.2) imposed on the kernel K are stronger than the conditions (2.2) and (2.3) imposed on kernel of singular integral operator defined in Chapter 3.

Proof: The equivalence is obtained by the following computations.

$$||Sf||^p_{L^p_{L^q(d\mu/t^n)}(dx)} = \int ||Tf(y,t)\chi_{\Gamma(x)}(y,t)||^p_{L^q(d\mu/t^n)}dx$$

$$= \int \left(\int_{\Gamma(x)} |Tf(y,t)|^q d\mu/t^n \right)^{p/q} dx$$

$$= \int (A_q(Tf))^p dx = ||Tf||^p_{T^p_q(d\mu)}.$$

To prove that S is a vector-valued operator whose kernel $\tilde{K}$ satisfy the conditions (4.5) and (4.6) we need the following lemma.

Lemma 4.2 *Let μ be a Carleson measure, and $\eta, b > 0$. If we set $\Gamma_b(x) = \Gamma(x) \cap \{t \leq b\}$, and $\Gamma^b(x) = \Gamma(x) \cap \{t > b\}$, then we have*

$$\int_{\Gamma^b(x)} t^{-n-\eta} d\mu \leq C b^{-\eta}$$

and

$$\int_{\Gamma_b(x)} t^{-n+\eta} d\mu \leq C b^{\eta}.$$

Proof of the lemma: For $j \geq 1$, we set

$$\Gamma_j^b(x) = \Gamma(x) \cap \{2^{j-1} b < t \leq 2^j b\}.$$

The first statement of the lemma follows from

$$\int_{\Gamma^b(x)} t^{-n-\eta} d\mu = \sum_{j=1}^{\infty} \int_{\Gamma_j^b(x)} t^{-n-\eta} d\mu$$

$$\leq \sum_{j=1}^{\infty} (2^{j-1} b)^{-n-\eta} \mu(S(B(x, 2^j b)))$$

$$= C \sum_{j=1}^{\infty} (2^j b)^{-\eta} \frac{\mu(S(B(x, 2^j b)))}{|B(x, 2^j b)|}$$

$$\leq b^{-\eta} C_\mu C \sum_{j=1}^{\infty} 2^{-j\eta}.$$

The other statement of the lemma can be proved in the same way by using

$$\Gamma_b^j(x) = \Gamma(x) \cap \{2^{-j} b < t \leq 2^{-j+1} b\},$$

which completes the proof of the lemma.

∎

Now, we go back to proving (4.5).

$$\|\tilde{K}(x,z)\|^q_{L^q(d\mu/t^n)} = \int_{\Gamma(x)} |K(z,y,t)|^q t^{-n} d\mu(y,t).$$

The estimate (4.1) produces

$$\|\tilde{K}(x,z)\|^q_{L^q(d\mu/t^n)} \leq C \int_{\Gamma(x)} \frac{t^{\alpha q-n}}{(|y-z|+t)^{(n+\alpha)q}} d\mu$$

$$= C \int_{\Gamma^{|x-z|}(x)} \frac{t^{\alpha q-n}}{(|y-z|+t)^{(n+\alpha)q}} d\mu + C \int_{\Gamma_{|x-z|}(x)} \frac{t^{\alpha q-n}}{(|y-z|+t)^{(n+\alpha)q}} d\mu$$
$$= I_1 + I_2.$$

The first statement of Lemma 4.2 (for $\eta = nq$) implies

$$I_1 \leq C \int_{\Gamma^{|x-z|}(x)} t^{-nq-n} d\mu \leq C|x-z|^{-nq}.$$

To estimate the term I_2, we use the fact that $|x-z| < t + |y-z|$, (that is because $|x-z| \leq |x-y| + |y-z|$, and $|x-y| < t$, since $(y,t) \in \Gamma(x)$) to conclude

$$I_2 \leq C \int_{\Gamma_{|x-z|}(x)} t^{\alpha q-n} |x-z|^{-(n+\alpha)q} d\mu(y,t),$$

and the second statement of Lemma 4.2 (for $\eta = -\alpha q$) to obtain

$$\leq C(|x-z|^{-(n+\alpha)q+\alpha q} = C|x-z|^{-nq},$$

which proves that $\tilde{K}$ satisfies the condition (4.5).

In a similar way we can prove that $\tilde{K}$ satisfies the condition (4.6).

■

The following lemma contains a simple, but useful observation.

Lemma 4.3 *If μ is a positive measure on $\mathbb{R}^{n+1}_+$, then*

$$T^p_p(d\mu) = L^p(\mathbb{R}^{n+1}_+, d\mu).$$

Proof: Using the Fubini theorem, we obtain

$$\|A_p f\|^p_{L^p(dx)} = \int_{\mathbb{R}^n} \left(\int_{\Gamma(x)} |f(y,t)|^p d\mu / t^n \right) dx,$$

$$= \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^p \left(\int_{\mathbb{R}^n} \chi_{\Gamma(x)} t^{-n} dx \right) d\mu(y,t)$$

$$= \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^p \left(\int_{B(y,t)} t^{-n} dx \right) d\mu(y,t)$$

$$= c_n \int_{\mathbb{R}^{n+1}_+} |f(y,t)|^p d\mu = C \|f\|^p_{L^p(d\mu)},$$

which implies the lemma.

■

Now, as we have the technique set up, we are ready to prove the main result of this chapter. We have already noticed that the kernel K that satisfies conditions (4.1), and (4.2), also satisfies the conditions (2.2), and (2.3). We also assume that the representation formula (3.3) is valid.

If we assume that for some $p_0 > 1$ the operator

$$T : L^{p_0}(\mathbb{R}^n, dx) \to L^{p_0}(\mathbb{R}^n, dm),$$

is bounded for some $p_0 > 0$, which if we assume that $K(x, y, 0) = k(x - y)$, where the function k is homogeneous with degree $-n$, would imply that the maximal singular integral operator $T_0^{\#}$, is L^p-bounded for every $p > 1$. We also assume that condition (2.4) is satisfied, i. e. there exists a constant, $B > 0$, so that for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$|Tf(x,t) - T_{0,t}f(x)| \leq BHf(x,t),$$

where the constant $B > 0$ does not depend on $x \in \mathbb{R}^n$ and $t > 0$.

Theorem 4.4 *Let μ be a Carleson measure on $\mathbb{R}^{n+1}_+$, and let T be an operator that satisfies all the conditions above. Then*

$$T : L^p(\mathbb{R}^n, dx) \to T^p_q(d\mu)$$

is bounded for every $1 < p, q < \infty$.

Proof: Using Theorem 2.4 (See Remark 2 after the theorem.) we conclude that $T : L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^{n+1}_+, d\mu)$ is a bounded operator for every $1 < p < \infty$. Lemma 4.3 implies that the operator

$$T : L^p(\mathbb{R}^n, dx) \to T^p_p(d\mu)$$

is bounded for every $1 < p < \infty$.

Applying Theorem 4.1 we obtain that the operator

$$S : L^q(\mathbb{R}^n, dx) \to L^q_{L^q(\mathbb{R}^{n+1}_+, d\mu/t^n)}(\mathbb{R}^n, dx),$$

is bounded for each fixed $q \in (1, \infty)$.

Let $E = L^q(\mathbb{R}^{n+1}_+, d\mu/t^n)$. Then Theorem 2.2 applied to the vector valued convolution operator S, on $\mathbb{R}^n$, whose vector-valued kernel satisfies condition (4.6), yields

$$S : L^p(\mathbb{R}^n, dx) \to L^p_E(\mathbb{R}^n, dx)$$

is a bounded operator for every $1 < p < \infty$. By Theorem 4.1, the last statement is equivalent to the statement that the operator

$$T : L^p(\mathbb{R}^n, dx) \to T^p_q(d\mu)$$

is bounded for every $1 < p, q < \infty$. Which proves the theorem.

■

Bibliography

[1] A. P. Calderón, *Inequalities for the maximal function relative to the metric*, Studia Math. **57** (1976) 297 — 306.

[2] A. P. Calderón and A. Zygmund, *On the existence on certain singular integrals*, Acta Mathematica, **88** (1952), 85 — 139.

[3] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math., **78** (1956), 289 — 309.

[4] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547 — 559.

[5] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 241 — 250.

[6] R. R. Coifman, Y. Meyer, and M. E. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Functional Analyses, **62** (1985), 304-335.

[7] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur certain Espaces Homogenes*, Lecture Notes in Math. **242** (1971), Springer-Verlag, Berlin.

[8] G. David and J.L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. **120** (1984), 371-397.

[9] C. Fefferman and E.M. Stein, *Some maximal inequlities*, Amer. J. Math. **93** (1971), 107 - 115.

[10] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood Paley theory and the study of function spaces*, Conference Board of Mathematical Sciences regional conference series in mathematics, 0160-7642; 79. (Based on the lectures given in

July of 1989 at a regional conference in Auburn University)

[11] J. B. Garnett, "Bounded Analytic Functions," Academic Press, New York (1981).

[12] S. Gadbois and W. T. Sledd, *Carleson measures on the spaces of homogeneous type,* Transactions of AMS, **341**, no 2, February 1994, 841-862.

[13] J. Horvath *Sur les fonctions conjuguées à plusiers variables*, Koninklingke Nederlandse Akademie van Wetenshapen. Indagationes Mathematicae ex actis Quibus Titulis, Proceedings of the section of sciences, **15**, no 1. (1953), 17 - 20.

[14] A. Korányi and S. Vági, *Singular Integrals on Homogeneous Spaces and Some Problems of Classical Analysis*, Annali della Scuola Norm. Sup. di Pisa Cl. Sci.; **XXV** (1971), 557-648.

[15] B. Muckenhoupt, *Weighted norm inequalities for the Hardy Littlewood maximal function*, Trans. Amer. Math. Soc. **165**, 207 - 226.

[16] Walter Rudin, "Functional Analysis," Springer-Verlag New York Heilderberg Berlin, (1976).

[17] Walter Rudin, "Function Theory in the Unit Ball of $\mathbb{C}^n$," Springer-Verlag New York Heidelberg Berlin, (1980).

[18] F. Ruiz and H. Torrea, *Vector-valued Calderón-Zygmund theory and Carleson measures on the spaces of homogeneous nature,* Studia Math. **88** (1988), 222-243.

[19] F. Ruiz and H. Torrea, *Weighted and vector-valued inequalities potential operators,* Adv. in Math. **62** (1986), 7-48.

[20] F. Ruiz and H. Torrea, *Weighted norm inequalities for a general maximal operators,* Ark. Math. **26** (1989), 327-340.

[21] F. Ruiz and H. Torrea, *Vector-valued Calderón-Zygmund theory applied to tent spaces,* Colloq. Math. **62** (1991), 267-277.

[22] F. Ruiz and H. Torrea, *Weighted and vector-valued inequalities for potential operators,* Trans. of the AMS, **295** (1989), 213 -232.

[23] E.M. Stein, "Singular integrals and differentiability properties of functions," Princeton Univ. Press, Princeton NJ, 1970

[24] A. Torchinsky, "Real variable methods in harmonic analysis," Academic Press, Orlando, FL, 1986